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BOCHNER AND SCHOENBERG THEOREMS ON SYMMETRIC SPACES IN THE COMPLEX CASE

PAR

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RÉSUMÉ. — Dans cet article, nous trouvons une caractérisation de la transformée de Fourier sphérique des mesures positives bornées K-invariantes sur les espaces symétriques riemanniens de type non-compact dans le cas complexe.

Afin de démontrer ce théorème de Bochner dans le cas symétrique, nous utilisons les méthodes classiques de l'analyse harmonique sur les groupes de Lie semisimples ainsi que quelques propriétés intéressantes indépendamment des fonctions holomorphes liées aux fonctions de type positif et la transformée de Fourier complexe.

Nous caractérisons aussi la transformée de Fourier sphérique des mesures infiniment divisibles dans le cas symétrique complexe, ce qui correspond au théorème de Schoenberg dans le cas euclidien.

Dans la démonstration nous utilisons les formules de Lévy–Kchinchine dans le cas euclidien et symétrique.

ABSTRACT. — We find a characterization of the spherical Fourier transform of K-invariant probability measures on Riemannian symmetric spaces of noncompact type in the complex case. To do this we use some interesting properties of holomorphic functions on an n-dimensional complex linear space connected with functions of positive type and complex Fourier transform. We also characterize the spherical Fourier transform of infinitely divisible measures in the symmetric complex case.

1. Introduction

K-invariant probability measures on symmetric spaces of noncompact type (for example on the space of positive definite symmetric or hermitian

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matrices) have been studied from different points of view (cf. [7], [8], [9], [13], [14]). The main analytic tool of their study is the spherical Fourier transform.

In this paper, we characterize the spherical Fourier transform of K-invariant probability measures on a noncompact symmetric space $X = G/K$ with G a complex semisimple Lie group, that is we formulate and prove the Bochner theorem in the complex symmetric non-compact case. In the case of G compact, a characterization of the spherical Fourier transform of K-invariant bounded positive measures was proposed by BERG [1]. His characterization is also true for hyperbolic spaces, but it is not quite satisfactory in this case — one considers the spherical Fourier transform only for the positive definite spherical functions. In our paper, we consider the spherical Fourier transform on its whole domain of definition and we characterize it in a different way, using a natural differential operator on X and the classical notion of a positive definite function on a Euclidean space.

To prove our Bochner theorem, we use the explicit form of the spherical functions in the complex case. An important rôle is played by the holomorphy of the spherical Fourier transform on a tube. In the second part of the paper, we give a characterization of the spherical Fourier transform of infinitely divisible K-invariant measures on X which corresponds to the Schoenberg theorem in the Euclidean case. Besides holomorphy, we use the Lévy-Khinchine formulæ for Euclidean and symmetric spaces.

The paper is organized as follows. After the preliminaries in section 2, we prove in section 3 some properties of holomorphic functions on tubes related to the functions of positive type and the complex Fourier transform (THEOREM 1). We need this in the sequel, but these results are interesting independently. Section 4 contains the main result of the paper — the Bochner theorem in the complex symmetric non-compact case (THEOREM 2). In section 5, we deal with the example of the space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$, giving a modified proof of the THEOREM 2 in this case and presenting some concrete conclusions. In section 6, we study the spherical Fourier transform of infinitely divisible K-invariant measures on X. This section contains the Schoenberg theorem in the case of X irreducible (THEOREM 3) and some other properties of functions of negative type in the symmetric sense.

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2. Spherical functions on complex symmetric spaces

Let G be a semisimple noncompact Lie group with finite center and K a maximal compact subgroup of G. A probability measure μ on $X = G/K$ is called *K-invariant* if μ is invariant with respect to the action of K on X. Then we write $\mu \in M^\natural(X)$. One may identify such measures with K-bi-invariant measures on G.

A fundamental tool in investigating K-invariant bounded measures on X is the spherical Fourier transform, using spherical functions. We recall some basic facts concerning the spherical functions and transforms on symmetric spaces (see e.g. [10], [11], [14]).

A K-invariant function ϕ on X is said to be *spherical* if $\phi(eK) = 1$ and ϕ is an eigenfunction of all G-invariant differential operators on X. On the level of the group G, all the spherical functions are given by the Harish-Chandra formula :

$$(1) \qquad \phi_\lambda(g) = \int_K \mathrm{e}^{(i\lambda-\rho)(\mathcal{H}(gk))}\,\mathrm{d}k, \qquad g \in G,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

where $g = k \exp \mathcal{H}(g) n$ is the Iwasawa decomposition of $g \in G$, $\mathfrak{a}$ is a Cartan space of X and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ where Σ^+ denotes the set of the positive roots of multiplicity m_α (cf. [11]). The spherical function $\phi_\lambda(g)$ is holomorphic in λ for every $g \in G$ fixed; it is invariant with respect to the Weyl group W acting on $\mathfrak{a}$:

$$\phi_\lambda \equiv \phi_{w\lambda}, \qquad w \in W.$$

By the Helgason–Johnson theorem, the spherical functions are bounded if and only if λ belongs to the tube $T_\rho = \mathfrak{a}^* + iC(\rho)$, where $C(\rho)$ is the convex hull of the set $\{w\rho\,;\ w \in W\}$.

If the group G is complex, then the spherical functions may be expressed explicitly. We have

$$(2) \qquad \phi_\lambda(a\cdot x_0) = \frac{\pi(\rho)}{\pi(i\lambda)}\frac{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{iw\lambda(\log a)}}{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w\rho(\log a)}}, \qquad a \in A,$$

where $x_0 = eK$, $\pi(\lambda) = \prod_{\alpha\in\Sigma^+}\langle \alpha \mid \lambda\rangle$ with the scalar product $\langle\ ,\ \rangle$ induced by the Killing form and ϵ denotes the determinant of $w : \mathfrak{a} \mapsto \mathfrak{a}$.

In the case of the symmetric spaces $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$, the formula (2) has a more concrete form, expressing ϕ_λ as the so-called Schur functions. However, for our purposes (2) is more convenient.

Recall that the sum in the denominator of (2) may be written in a different way by using the following formula :

$$\sum_{w\in W} \epsilon(w)\,\mathrm{e}^{w\rho(\log a)} = \prod_{\alpha\in\Sigma^+} 2\,\mathrm{sh}\big(\alpha(\log a)\big). \tag{3}$$

The spherical functions in the complex case may also be written by another useful formula :

$$\phi_\lambda(\operatorname{Exp} H) = J^{-1/2}(H)\int_K \mathrm{e}^{i\lambda(\mathrm{Ad}(k)H)}\,\mathrm{d}k \tag{4}$$

where $\lambda \in \mathfrak{p}^*$ and $H \in \mathfrak{p}$ with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and J is the jacobian of the diffeomorphism $\operatorname{Exp} : \mathfrak{p} \to G/K$. In the sequel we denote $J^{-1/2}(H) = j(H)$.

The *spherical Fourier transform* of $\mu \in M^\natural(X)$ is defined by

$$\widehat{\mu}(\lambda) = \int_X \phi_\lambda(x)\,\mathrm{d}\mu(x)$$

at least for $\lambda \in T_\rho$. It is holomorphic in the interior of the tube, continuous on the whole closed tube and W-invariant.

3. Holomorphic functions on tubes and positive type

As we have seen in section 2, the spherical Fourier transforms of K-invariant bounded measures are holomorphic functions on the interior of the tube $T_\rho = \mathfrak{a}^* + iC(\rho) \subset \mathfrak{a}^*_{\mathbb{C}}$. Remark that if $\lambda \in \mathfrak{a}^*$, then $\widehat{\mu}(\lambda)$ is a positive definite function on $\mathfrak{a}^*$ as the ordinary Fourier transform of the Abel transform of μ or directly by (1).

In this section, we study the relations between the (local) analyticity on a tube containing the real space, the positive type of a function on the real space and the complex Fourier transform.

THEOREM 1. — *Let V be a real vector space of finite dimension and $W = V \oplus iV$ the corresponding complex vector space. Let Ω be an open connected subset of V and $a \in \Omega$.*

Let f be a function defined on a set of the form $U \cup (V + ia)$ where U is an open subset of W containing $i\Omega$, satisfying the following conditions :

(i) *f is holomorphic on U.*

(ii) *f is of positive type on $V + ia$ (i.e. the function $t \mapsto f(t + ia)$ is positive definite on V).*

Then :

(i) *f extends uniquely to a holomorphic function $\tilde{f}$ on the tube $T = V + i\Omega$.*

(ii) *There exists a positive measure κ on V such that*

$$\tilde{f}(z) = \int_V \mathrm{e}^{i\langle z|v\rangle}\, \mathrm{d}\kappa(v) \tag{5}$$

for every $z \in T$ (in particular the integral converges).

(iii) *The functions $t \mapsto \tilde{f}(t+ib)$ are positive definite on V for every $b \in \Omega$ fixed.*

Proof. — First suppose that Ω is a ball in V centered at a and take U the ball in W of the same radius and center. Then, for $|z| \le 1$, $z \in \mathbb{C}$:

$$a + v \in \Omega \implies ia + zv \in U. \tag{6}$$

By the classical Bochner theorem, there exists a bounded positive measure κ_0 on V such that :

$$g(t) \equiv f(t+ia) = \int_V \mathrm{e}^{i\langle t|v\rangle}\, \mathrm{d}\kappa_0(v), \qquad t \in V. \tag{7}$$

Let $h \in V$ be such that $a + h \in \Omega$. By (6), we have $ia + zh \in U$ for $|z| < 1 + \epsilon$ and $\epsilon > 0$ sufficiently small. For h fixed, define :

$$\psi(z) = g(zh) = f(ia + zh), \qquad z \in \mathbb{C},\ |z| < 1 + \epsilon.$$

The function ψ is a holomorphic function which may be developed in a power series

$$\psi(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < 1 + \epsilon,$$

with the coefficients

$$c_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi(z)\Big|_{z=0}.$$

In particular :

$$\sum_{n=0}^{\infty} |c_n| < +\infty. \tag{8}$$

In order to compute the coefficients c_n, we note that as $g_{|V}$ is C^∞ on a neighbourhood of 0, the measure κ_0 has moments of all orders (see [3]). Then, by Schwarz inequality, for every $n \in \mathbb{N}$

$$\int_V \left|\langle h \mid v\rangle\right|^n \mathrm{d}\kappa_0(v) < \infty,$$

so we may differentiate in (7) with $t = zh$ with respect to z under the integral sign arbitrarily many times. We get :

$$c_n = \frac{i^n}{n!}\int_V \langle h \mid v\rangle^n \mathrm{d}\kappa_0(v).$$

It follows from (8) that :

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left|\int_V \langle h \mid v\rangle^n \mathrm{d}\kappa_0(v)\right| < \infty.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\int_V \langle h \mid v\rangle^{2n} \mathrm{d}\kappa_0(v) < \infty,$$

which implies

$$\int_V \mathrm{ch}\langle h \mid v\rangle \,\mathrm{d}\kappa_0(v) < \infty \quad \text{and} \quad \int_V \mathrm{e}^{\langle h|v\rangle}\,\mathrm{d}\kappa_0(v) < \infty$$

for every $h \in \Omega - a$. This implies that the integral $\int_V \mathrm{e}^{i\langle z|v\rangle}\,\mathrm{d}\kappa_0(v)$ converges for $z + ia \in T$. Using Morera theorem, we see that this integral defines a holomorphic function in z which is a unique analytic extension of $f(t + ia)$ to $T - ia$. Putting $\mathrm{d}\kappa(v) = \mathrm{e}^{\langle a|v\rangle}\,\mathrm{d}\kappa_0(v)$, we get (5) which implies directly (iii).

In the general case, let us consider the following subset E of Ω :

$E = \{x \in \Omega \mid$ there exists a holomorphic function g on an open tube $V + i\widetilde{\Omega}$ with $\widetilde{\Omega} \subset V$ open connected containing a and x, such that $g = f$ on $V + ia$ and g is positive definite on $V + ib$, $b \in \widetilde{\Omega}\}$.

Remark that E is not empty because $a \in E$ taking for $\widetilde{\Omega}$ a ball with center at a contained in Ω and using the first part of the proof. The set E is open by definition. Let us prove that it is also closed.

Let (x_n) be a sequence of elements of E convergent to $x \in \Omega$. There exists m such that an open ball $B(x_m, r)$ is contained in Ω and contains x. Using the first part of the proof for x_m in the place of a we take for $\widetilde{\Omega}$ the union of the open connected set containing a and x_m having the properties given in the definition of E and the ball $B(x_m, r)$. This shows that $x \in E$ and finally $E = \Omega$.

This implies the existence of a unique function $\tilde{f}$ verifying (i) and (iii).

To complete the proof, we show (5) with the measure κ defined as in the first part of the proof. By the first part of the proof, for each ball $B \subset \Omega$ there exists a positive measure κ_B on V such that

$$\tilde{f}(\boldsymbol{z}) = \int_V \mathrm{e}^{i\langle \mathbf{z}|v\rangle} \, \mathrm{d}\kappa_B(v)$$

for $\boldsymbol{z} \in V + iB$. If for two such balls $B_1 \cap B_2 \neq \emptyset$ take $c \in B_1 \cap B_2$. Then the measures $\mathrm{d}\theta_1 = \mathrm{e}^{-\langle c|v\rangle} \, \mathrm{d}\kappa_{B_1}(v)$ and $\mathrm{d}\theta_2 = \mathrm{e}^{-\langle c|v\rangle} \, \mathrm{d}\kappa_{B_2}$ are finite and

$$\tilde{f}(t + ic) = \int_V \mathrm{e}^{i\langle t|v\rangle} \, \mathrm{d}\theta_1 = \int_V \mathrm{e}^{i\langle t|v\rangle} \, \mathrm{d}\theta_2, \qquad t \in V.$$

Hence $\theta_1 = \theta_2$ and finally $\kappa_{B_1} = \kappa_{B_2} = \kappa$. □

REMARK 1. — We have also proved the following interesting fact : *if ϕ is a positive definite function on $\mathbb{R}$ which is smooth on a neighbourhood of* 0 *and if the power series*

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} x^n$$

converges for an $x \neq 0$, then its sum equals $\phi(x)$.

REMARK 2. — THEOREM 1 implies that for any real finite dimensional vector space V the existence of some exponential moments of a probability measure on V is equivalent to the local analiticity of its Fourier transform on $W = V^{\mathbb{C}}$ (cf. [12] in the case $V = \mathbb{R}$).

4. Characterization of the spherical Fourier transform of a probability measure

In this section we prove the main result of this paper, the Bochner theorem on a symmetric space $X = G/K$ with G complex (THEOREM 2).

An important part of its proof is based on exploring the relations between the spherical Fourier transform of a K-invariant probability measure μ on X and the ordinary Fourier transform of the measure μ considered in the polar coordinates. We are going to present these relations first.

There is a natural bijective correspondence between the measures in $M^\natural(G/K)$ and the W-invariant probability measures on A (the abelian group in the Cartan decomposition of G) denoted by $M^W(A)$.

If $\mu \in M^\natural(G/K)$ is considered as a K-biinvariant measure on G, then $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(B) = \mu(KBK), \qquad B \in \mathcal{B}_A,$$

belongs to $M^W(A)$. Conversely, to get μ from $\widetilde{\mu}$ one may use the Riesz theorem.

As $A = \exp \mathfrak{a}$ we may identify $\widetilde{\mu}$ with its image on $\mathfrak{a}$ through the mapping $\log = \exp^{-1}$ and consider its ordinary Fourier transform.

In what follows we use extensively the differential operator $\mathcal{D}$ on $\mathfrak{a}^*$ defined by

$$\mathcal{D} = \pi\Big(\frac{\partial}{\partial\lambda}\Big),$$

where π is the polynomial defined in (2). It is characterized by the property

$$\mathcal{D}\,\mathrm{e}^{\langle\lambda|y\rangle} = \pi(y)\,\mathrm{e}^{\langle\lambda|y\rangle}. \tag{9}$$

PROPOSITION 1. — *If $\mu \in M^\natural(X)$, then*

$$\mathcal{D}\big[\pi(\lambda)\widehat{\mu}(\lambda)\big] = |W|\pi(\rho)\mathcal{F}\big(j(a)\widetilde{\mu}\big)(\lambda), \qquad \lambda \in \mathfrak{a}^*, \tag{10}$$

where $j(a) = \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\,\mathrm{sh}\,\alpha(\log a)}$ and $\mathcal{F}$ denotes the ordinary Fourier transform on $A \cong \mathfrak{a}$.

Proof. — By (2), (9) and the antisymetry of π with respect to the Weyl group we have :

$$\frac{1}{\pi(\rho)}\mathcal{D}\big[\pi(\lambda)\widehat{\mu}(\lambda)\big] = \int_A \sum_{w\in W} \epsilon(w)\pi\big(w(\log a)\big)\,\mathrm{e}^{iw\lambda(\log a)}\frac{\mathrm{d}\widetilde{\mu}(a)}{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w\rho(\log a)}}$$

$$= \sum_{w\in W}\int_A \mathrm{e}^{i\lambda(w\log a)}\pi(\log a)\frac{\mathrm{d}\widetilde{\mu}(a)}{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{w\rho(\log a)}}.$$

To see that the differentiation under the integral sign is allowed, we regard $\mathcal{D}[\pi\widehat{\mu}]$ as a distribution on $\mathfrak{a}^*$, we use the Fubini theorem and we integrate by parts. This is justified by the boundedness of $\phi_\lambda(a)$, $\mathrm{e}^{i\langle\lambda,\log a\rangle}$ and $j(a)$ for $\lambda \in \mathfrak{a}^*$ and $a \in A$.

Using the W-invariance of $\widetilde{\mu}$ and $j(a)$ we get

$$\frac{1}{\pi(\rho)}\mathcal{D}\big[\pi(\lambda)\widehat{\mu}(\lambda)\big] = |W| \int_A \mathrm{e}^{i\lambda(\log a)} j(a)\,\mathrm{d}\widetilde{\mu}(a),$$

which gives (10). □

REMARK 1. — In (10) the measure $\widetilde{\mu}$ is multiplied by $j(a)$ which may be understood as the jacobian of changing to polar coordinates.

REMARK 2. — Let us notice that using (4) we get for $\mu \in M^\natural(G/K)$ and $\lambda \in \mathfrak{p}^*$

$$\widehat{\mu}(\lambda) = \int_{\mathfrak{p}} \mathrm{e}^{i\lambda(H)} j(H)\,\mathrm{d}(\widetilde{\widetilde{\mu}}),$$

where the Ad_K-invariant measure $\widetilde{\widetilde{\mu}}$ on $\mathfrak{p}$ is the image of μ on $\mathfrak{p}$ by the mapping Exp^{-1} (cf. [4]). This shows that in the complex case the spherical Fourier transform of μ may be interpreted as the ordinary Fourier transform of the Ad_K-invariant measure $\nu(\mathrm{d}H) = j(H)\widetilde{\widetilde{\mu}}(\mathrm{d}H)$ on the vector space $\mathfrak{p}$. The problem of characterization of the spherical Fourier transform of K-invariant bounded positive measures on X in the complex case is also related to the characterization of the ordinary Fourier transform of Ad_K-invariant measures on $\mathfrak{p}$.

In the sequel we will use the following uniqueness property of the differential operator $\mathcal{D}$.

PROPOSITION 2. — *Let U be a domain in $\mathfrak{a}^*_{\mathbb{C}}$ containing* 0. *If F is holomorphic on U and if*

$$(11) \qquad \mathcal{D}\big[\pi(\lambda)F(\lambda)\big] \equiv 0, \qquad \lambda \in U,$$

then $F \equiv 0$ on U.

Proof. — F is holomorphic so it may be represented as a series

$$F(\lambda) = \sum_{k=0}^{\infty} f_k(\lambda),$$

where f_k are homogeneous polynomials of degree k. Then, for each $k \in \mathbb{N}$, we have :

$$\mathcal{D}\big[\pi(\lambda)f_k(\lambda)\big] = 0.$$

Using the definition of $\mathcal{D}$ and the Fischer inner product on the space of polynomials

$$(P \mid Q) = P\Big(\frac{\partial}{\partial \lambda}\Big)\bar{Q}_{|\lambda=0}, \quad \text{with} \quad \bar{Q}(\lambda) = \overline{Q(\bar{\lambda})},$$

we infer that $\pi f_k = 0$, then $f_k = 0$. Hence $F \equiv 0$ on U. $\square$

REMARK. — By analytic continuation it is enough to assume (11) for $\lambda \in U \cap \mathfrak{a}^*$.

Before stating the main theorem let us prove a technical lemma.

LEMMA 1. — *Suppose that κ is a W-invariant bounded positive measure on $\mathfrak{a}$ and that its Fourier transform is defined on* $\operatorname{Int} T_\rho$. *Then the function*

$$k(\lambda) = \int_{\mathfrak{a}} \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\,\mathrm{e}^{iw\lambda(H)}}{\pi(H)}\,\mathrm{d}\kappa(H) \tag{12}$$

is well-defined and holomorphic on $\operatorname{Int} T_\rho$ *and*

$$\mathcal{D}[\pi k](\lambda) = \pi(\rho)|W| \int_{\mathfrak{a}} \mathrm{e}^{i\lambda(H)}\,\mathrm{d}\kappa(H) \tag{13}$$

for $\lambda \in \operatorname{Int} T_\rho$.

Proof. — To show that the integral (12) converges for λ fixed, we take $0 < r < 1$ such that $(\lambda/r) \in \operatorname{Int} T_\rho$ and we note that the function under the integral in (12) may be written as

$$\phi_\lambda(H) j(H)^{-1} = \phi_{\lambda/r}(rH) \frac{\prod_{\alpha\in\Sigma^+} 2\operatorname{sh} r\alpha(H)}{\pi(rH)}, \tag{14}$$

where we define the function j on $\mathfrak{a}$ by $j(H) = j(\exp H)$. As $\phi_{\lambda/r}(rH)$ is bounded in H and $\operatorname{sh} x/x \le \mathrm{e}^x$ for $x \ge 0$ we see that on the positive Weyl chamber

$$\big|\phi_\lambda(H) j(H)^{-1}\big| \le c\,\mathrm{e}^{r\rho(H)}$$

for a positive constant c. Using the hypothesis on κ we infer that (12) converges.

Substituting (14) into (12) we get

$$k(\lambda) = \int_{\mathfrak{a}} \phi_{\lambda/r}(rH)\,\mathrm{d}\kappa_r(H), \tag{15}$$

where $\mathrm{d}\kappa_r(H) = j(rH)^{-1}\,\mathrm{d}\kappa(H)$. By (15) and the holomorphy and boundedness properties of the spherical functions on $\operatorname{Int} T_\rho$, it follows that $k(\lambda)$ is continuous and by Morera theorem holomorphic on $\operatorname{Int} T_\rho$.

Finally, to prove (13) it is enough to prove it for $\lambda \in \mathfrak{a}^*$. It suffices to justify differentiating under the integral sign in (12). We proceed as in the proof of the PROPOSITION 1 using (15). $\square$

Now we may formulate and prove the Bochner theorem for a symmetric space in the complex case.

THEOREM 2. — *A W-invariant function Ψ on $T_\rho \subset \mathfrak{a}^*_{\mathbb{C}}$ is the spherical Fourier transform of a K-invariant probability measure on $X = G/K$ if and only if the following conditions are satisfied :*

(i) *Ψ is holomorphic on* $\operatorname{Int}T_\rho$ *and continuous on* T_ρ ;

(ii) $\Psi(i\rho) = 1$;

(iii) *the function $\Phi(\lambda) = \mathcal{D}[\pi(\lambda)\Psi(\lambda)]$ is positive definite for $\lambda \in \mathfrak{a}^*$.*

Proof. — The conditions (i)–(iii) are necessary by general properties of the spherical Fourier transform quoted in section 2 and by the PROPOSITION 1. To prove the sufficiency, let κ be a bounded and W-invariant positive measure on A such that :

$$\Phi(\lambda) = \mathcal{F}\big(\pi(\rho)|W|\kappa\big) = \pi(\rho)|W| \int_A \mathrm{e}^{i\lambda(\log a)}\,\mathrm{d}\kappa(a), \qquad \lambda \in \mathfrak{a}^*.$$

By the THEOREM 1, the integral

$$\int_A \mathrm{e}^{i\lambda(\log a)}\,\mathrm{d}\kappa(a)$$

converges absolutely for each $\lambda \in \operatorname{Int} T_\rho$. By the LEMMA 1, we may define for such λ

$$\widetilde{\Psi}(\lambda) = \int_A \frac{\pi(\rho)}{\pi(i\lambda)}\,\frac{\sum_{w\in W}\epsilon(w)\,\mathrm{e}^{iw\lambda(\log a)}}{\pi(\log a)}\,\mathrm{d}\kappa(a), \tag{16}$$

the function $\widetilde{\Psi}$ being holomorphic on $\operatorname{Int} T_\rho$ and

$$\mathcal{D}\big[\pi(\lambda)\widetilde{\Psi}(\lambda)\big] = \Phi(\lambda).$$

Then PROPOSITION 2 implies that $\widetilde{\Psi} = \Psi$ on $\operatorname{Int} T_\rho$.

For $0 < r < 1$ put $\lambda = -ir\rho$. We have then :

$$\widetilde{\Psi}(-ir\rho) = \int_A \prod_{\alpha\in\Sigma^+} \frac{2\,\mathrm{sh}\, r\alpha(\log a)}{r\alpha(\log a)}\,\mathrm{d}\kappa(a).$$

As $\lim_{r\to 1}\widetilde{\Psi}(-ir\rho) = \lim_{r\to 1}\Psi(-ir\rho) = 1$ exists, the monotone convergence theorem implies that :

$$\int_A j(a)^{-1}\,\mathrm{d}\kappa(a) = \int_A \prod_{\alpha\in\Sigma^+} \frac{2\,\mathrm{sh}\,\alpha(\log a)}{\alpha(\log a)}\,\mathrm{d}\kappa(a) < \infty.$$

Defining $\mathrm{d}\widetilde{\mu}(a) = j(a)^{-1}\,\mathrm{d}\kappa(a)$, we have then $\widetilde{\mu} \in M^W(A)$ and by (16)

$$\Psi(\lambda) = \widetilde{\Psi}(\lambda) = \int_A \phi_\lambda(a)\,\mathrm{d}\widetilde{\mu}(a), \qquad \lambda \in T_\rho,$$

so $\Psi = \widehat{\mu}$ for the corresponding measure $\mu \in M^\natural(X)$. $\square$

COROLLARY 1. — *If Ψ is a W-invariant function on T_ρ satisfying the conditions* (i)–(iii) *of Theorem* 2, *then Ψ is positive definite on $\mathfrak{a}^*$ and on $\mathfrak{a}^* + i\lambda$ for $\lambda \in C(\rho)$ fixed.*

COROLLARY 2. — *If Ψ is the spherical Fourier transform of a probability measure $\mu \in M^\natural(X)$, then $\mathrm{e}^{\Psi-1}$ is the spherical Fourier transform of the compound Poisson measure*

$$\mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{\mu^{*k}}{k!}.$$

In particular the function $\mathcal{D}\big[\pi(\lambda)\,\mathrm{e}^{\Psi(\lambda)}\big]$ is positive definite on $\mathfrak{a}^*$.

5. Example of $\mathbf{SL(2,\mathbb{C})/SU(2)}$

In the case of the symmetric space $X = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$, THEOREM 2 becomes more explicit and its proof may be modified. The spherical functions are of the form

$$\phi_\lambda\begin{pmatrix}\mathrm{e}^t & 0\\ 0 & \mathrm{e}^{-t}\end{pmatrix} = \frac{\sin\lambda t}{\lambda\,\mathrm{sh}\,t}, \qquad \lambda\in\mathbb{C}, \tag{17}$$

and ρ corresponds to $\lambda = i$. The THEOREM 2 may be formulated as follows.

THEOREM 2′. — *An even function Ψ on the strip $\{|\operatorname{Im}\lambda| \le 1\}$ is the spherical Fourier transform of a K-invariant probability measure on* $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ *if and only if*:

(i) *Ψ is holomorphic on $\{|\operatorname{Im}\lambda| < 1\}$ and continuous on $\{|\operatorname{Im}\lambda| \le 1\}$*;

(ii) $\Psi(i) = 1$;

(iii) *the function $\Phi(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[t\,\Psi(t)]$, $t\in\mathbb{R}$ is positive definite on $\mathbb{R}$.*

An important modification of the proof of the sufficiency of the above conditions consists in omitting the PROPOSITION 2 and using the fact that if $\Phi = 2\mathcal{F}(\kappa)$, then Φ is the complex Fourier transform of 2κ for $|\operatorname{Im}\lambda| < 1$ and

$$\int_0^1 \Phi(i\lambda)\,\mathrm{d}\lambda = \int_{-\infty}^{\infty}\frac{2\,\mathrm{sh}\,x}{x}\,\mathrm{d}\kappa(x).$$

On the other hand,

$$\int_{-1}^1 \Phi(i\lambda)\,\mathrm{d}\lambda < \infty$$

by the properties of the primitive $\lambda\Psi$ of Φ, so the measure

$$\mathrm{d}\widetilde{\mu}(x) = \frac{2\,\mathrm{sh}\,x}{x}\,\mathrm{d}\kappa(x)$$

is finite. $\square$

In the case of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$, the remark 2 after the PROPOSITION 1 says that the spherical Fourier transforms of K-invariant measures on X are the ordinary Fourier transforms of radial measures on $\mathbb{R}^3$, having a holomorphic continuation to $\{|\operatorname{Im} z| < 1\}$, continuous up to the boundary of the strip. This may be seen directly using the well-known formula for the ordinary Fourier transform of a radial measure ν on $\mathbb{R}^n$; in the case of $\mathbb{R}^3$ we have (see e.g. [3])

$$\mathcal{F}(\nu)(\zeta) = \int_0^\infty \frac{\sin \lambda x}{\lambda x}\,\mathrm{d}\tilde{\nu}(x),$$

where $\tilde{\nu}$ denotes the measure ν in the polar coordinates and $\lambda^2 = \|\zeta\|^2$. We see directly using (17) that :

$$\mathcal{F}(\nu)(\zeta) = \mathcal{F}_{\mathrm{sph}}\Big(\frac{\operatorname{sh} x}{2x}\tilde{\nu}\Big)(\lambda).$$

Remark that in this case one verifies easily that the condition

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\big[\lambda\Psi(\lambda)\big] \gg 0, \qquad \lambda \in \mathbb{R},$$

with $\Psi(\lambda)$ defined by $\Psi(\lambda) = \Psi(\zeta)$ characterizes the Fourier transforms of positive bounded radial measures on $\mathbb{R}^3$, without the assumption of the analytic continuation of Ψ on a strip.

Let us give now some examples of using the Bochner theorem in the complex case and its corollaries for the space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$.

EXAMPLES.

(i) The positive type on $\mathbb{R}$ of the function

$$\Phi(\lambda) = \frac{1}{1+c^2\lambda^2}, \qquad 0 < c < 1,$$

implies that the function

$$\Psi(\lambda) = \frac{\operatorname{arctg}(c\lambda)}{c\lambda}$$

is positive definite on $\mathbb{R}$ and on the horizontal lines $\operatorname{Im}\lambda = a$, $|a| < 1$. The function Ψ is the spherical Fourier transform of a *positive* K-invariant measure on X.

(ii) Similarly, putting $\Phi(\lambda) = \mathrm{e}^{-\lambda^2}$ we get that

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n+1)!}$$

is the spherical Fourier transform of a *positive* K-invariant measure on X.

(iii) On the other hand $\Psi(\lambda) = \cos\lambda$ is not the spherical Fourier transform of a K-invariant positive measure on X because

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda\cos\lambda) = \cos\lambda - \lambda\sin\lambda$$

is not positive definite. This example shows that a positive definite function on $\mathbb{R}$ which has a holomorphic continuation even to the complex plane may not be the spherical Fourier transform of a K-invariant measure on X.

(iv) By COROLLARY 2, taking $\Psi(\lambda) = \mathrm{e}^{-\lambda^2}$ we get that the function

$$\mathrm{e}^{\mathrm{e}^{-\lambda^2}}\big[1 - 2\lambda^2 \mathrm{e}^{-\lambda^2}\big]$$

is positive definite on $\mathbb{R}$.

6. Spherical Fourier transform of a semigroup of measures

Let us consider a continuous semigroup $(\mu_t)_{t>0}$ of K-invariant probability measures on X. One may consider equivalently an infinitely divisible measure $\mu \in M^\natural(G/K)$ (cf. [5]). The spherical Fourier transform of the measure μ_t is given by

$$\widehat{\mu}_t(\lambda) = \mathrm{e}^{-t\eta(\lambda)}, \qquad t > 0, \tag{18}$$

where η is a holomorphic function on $\operatorname{Int} T_\rho$, continuous on T_ρ. In this section we study properties of the functions η in (18), which we will call *negative definite in the symmetric sense.*

Recall that, by definition, a function θ on $\mathbb{R}^n$ is called *negative definite* if it is continuous and for all $z_1, \dots, z_n \in \mathbb{C}$, $x_1, \dots, x_n \in \mathbb{R}^n$

$$\sum_{i,j}\big(\theta(x_i) + \overline{\theta(x_j)} - \theta(x_i - x_j)\big) z_i \bar{z}_j \geq 0.$$

We recall two properties of negative definite functions on $\mathbb{R}^n$ - the connection with positive definite functions and the Schoenberg theorem. For these and other properties of negative definite functions see e.g. [2].

PROPOSITION 3. — *A continuous function θ is negative definite on $\mathbb{R}^n$ if and only if there exists a sequence $(\psi_n)_{n\geq 1}$ of positive definite functions and a sequence of constants $(c_n)_{n\geq 1}$ with $c_n \geq \psi_n(0)$ such that*

$$\theta = \lim_{n\to\infty} (c_n - \psi_n).$$

THEOREM OF SCHOENBERG. — *A family $(\nu_t)_{t>0}$ is a continuous semigroup of bounded positive measures on $\mathbb{R}^n$ if and only if its Fourier transform is of the form*

$$\hat{\nu}_t(x) = \mathrm{e}^{-t\theta(x)}$$

with θ negative definite.

We are going now to give a characterization of functions of negative type in the symmetric sense, that is to formulate and prove the Schoenberg theorem in the symmetric complex irreducible case. The essential role is played, as in the THEOREM 2, by the operator $\mathcal{D} = \pi(\partial/\partial\lambda)$.

THEOREM 3. — *Let X be irreducible. A W-invariant function η on $T_\rho \subset \mathfrak{a}^*_{\mathbb{C}}$ is the logarithm of the spherical Fourier transform of an infinitely divisible measure $\mu \in M^\natural(G/K)$ if and only if the following conditions are satisfied :*

(i) *η is holomorphic on* $\operatorname{Int} T_\rho$ *and continuous on T_ρ ;*

(ii) $\eta(i\rho) = 0$;

(iii) *the function $\theta(\lambda) = \mathcal{D}[\pi(\lambda)\eta(\lambda)]$ is negative definite for $\lambda \in \mathfrak{a}^*$.*

Proof. — The necessity of (i) and (ii) follows from the analogous properties of $\widehat{\mu}(\lambda)$ on T_ρ. To prove that the condition (iii) is necessary, we use the PROPOSITION 1 applied to the continuous semigroup $(\mu_t)_{t\geq 0}$ with $\mu = \mu_1$. The functions

$$\Phi_t(\lambda) = \mathcal{D}\big[\pi(\lambda)\widehat{\mu}_t(\lambda)\big]$$

are positive definite on $\mathfrak{a}^*$ and

$$\theta(\lambda) = \mathcal{D}\big[\pi(\lambda)\eta(\lambda)\big] = \lim_{t\to 0+} \frac{1}{t}\big(\Phi_0(\lambda) - \Phi_t(\lambda)\big). \tag{19}$$

By homogeneity of $\pi(\lambda)$, we have $\Phi_0(\lambda) = \mathcal{D}[\pi](\lambda) \equiv \text{const}$. On the other hand, by (10)

$$\Phi_t(0) = |W|\pi(\rho)\int_A j(a)\,\mathrm{d}\widetilde{\mu}_t(a), \qquad t \geq 0.$$

As $\widetilde{\mu}_0 = \delta_e$ and $j(e) = \max_{a\in A} j(a)$, we see that $\Phi_0(0) \geq \Phi_t(0)$ for $t > 0$. Thus by (19) and the PROPOSITION 3, the function θ is negative definite on $\mathfrak{a}^*$.

To prove the sufficiency of (i)–(iii), we write the function $\theta(\lambda)$ using the classical Lévy–Khinchine formula on $\mathfrak{a} \cong \mathbb{R}^n$. By the W-invariance of θ and the irreducibility of X

$$\begin{aligned}\theta(\lambda) = b\|\lambda\|^2 &+ \int_{\{\|H\|\leq\delta\}} \Big[1 - \mathrm{e}^{i\lambda(H)} - \frac{i\lambda(H)}{1+\|H\|^2}\Big] \, \mathrm{d}L(H) \\ &+ \int_{\{\|H\|>\delta\}} \big[1 - \mathrm{e}^{i\lambda(H)}\big] \, \mathrm{d}L(H),\end{aligned} \tag{20}$$

for $\lambda \in \mathfrak{a}^*$, a nonnegative constant b, a W-invariant Lévy measure L on $\mathfrak{a}$ and any $\delta > 0$. We denote

$$L_\delta = L_{|\{\|H\|>\delta\}}.$$

As $\int_{\{\|H\|\leq\delta\}} \|H\|^2 \, \mathrm{d}L(H) < \infty$, we see that the first two terms on the right-hand side of (20) define a holomorphic function on $\mathfrak{a}^*_{\mathbb{C}}$. Using the boundedness of L_δ and the THEOREM 1, we deduce that the Fourier transform of L_δ is defined and holomorphic on $\operatorname{Int} T_\rho$. By the LEMMA 1, the function

$$\beta_\delta(\lambda) = \int_{\mathfrak{a}} \Big(\frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\,\mathrm{e}^{iw\lambda(H)}}{\pi(H)} \Big) \, \mathrm{d}L_\delta(H) \tag{21}$$

is well-defined and holomorphic on $\operatorname{Int} T_\rho$ and

$$\mathcal{D}\big[\pi\beta_\delta\big](\lambda) = \int_{\mathfrak{a}} \big[1 - \mathrm{e}^{i\lambda(H)}\big] \, \mathrm{d}L_\delta(H), \qquad \lambda \in \mathfrak{a}^*. \tag{22}$$

Next we show that $\beta_\delta(\lambda)$ converges if $\delta \to 0$ for λ fixed. The function under the integral in (21) equals

$$\frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(\rho)} \phi_\lambda(H) \jmath(H)^{-1}.$$

The Taylor expansion of order 2 of the W-invariant function $\phi_\lambda(H)\jmath(H)^{-1}$ at $H = 0$ may be written as (see [8])

$$\phi_\lambda(H)\jmath(H)^{-1} = \jmath(0)^{-1} + p(\lambda)\|H\|^2 + \|H\|^2 R(\lambda, H),$$

where $p(\lambda)$ is a polynomial of order 2 and $R(\lambda, H)$ is continuous in λ and H with $\lim_{H\to 0} R(\lambda, H) = 0$ for all λ. By (10) applied to the measure δ_{eK}

$$\mathcal{D}[\pi] = |W|\pi(\rho)\jmath(0), \tag{23}$$

so that

$$(24)\quad \frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{iw\lambda(H)}}{\pi(H)} = p(\lambda)\|H\|^2 + \|H\|^2 R(\lambda, H).$$

It follows that the integral

$$(25)\quad \int_{\{\|H\|\le 1\}} \Big| \frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{iw\lambda(H)}}{\pi(H)} \Big| \,\mathrm{d}L(H) < \infty$$

for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the limit $\lim_{\delta\to 0} \beta_\delta(\lambda) = \beta(\lambda)$ exists for each $\lambda \in \operatorname{Int} T_\rho$ and

$$(26)\qquad \beta(\lambda) = \int_{\mathfrak{a}} \Big[\frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{iw\lambda(H)}}{\pi(H)} \Big] \,\mathrm{d}L(H).$$

By (24) and (25) the functions $\beta_\delta(\lambda)$ are uniformly bounded on any compact subset of $\operatorname{Int} T_\rho$. Hence $\beta(\lambda)$ is holomorphic on $\operatorname{Int} T_\rho$ and $\lim_{\delta\to 0} \mathcal{D}[\pi\beta_\delta] = \mathcal{D}[\pi\beta]$. Let b' be such that $b'\mathcal{D}[\pi\|\lambda\|^2] = b\|\lambda\|^2$. Define :

$$\tilde{\eta}(\lambda) = b'\|\lambda\|^2 + \beta(\lambda).$$

By (20) and (25) we have finally

$$\mathcal{D}\big[\pi\tilde{\eta}(\lambda)\big] = \theta(\lambda), \qquad \lambda \in \mathfrak{a}^*,$$

so by the PROPOSITION 2 and the holomorphy of $\tilde{\eta}$ on $\operatorname{Int} T_\rho$:

$$(27)\qquad \eta = \tilde{\eta} \quad \text{on} \quad \operatorname{Int} T_\rho.$$

In particular, using (25) and (27) and arguing as in the proof of the THEOREM 2, we see that

$$(28)\qquad \int_{\mathfrak{a}} j(H)^{-1} \,\mathrm{d}L_\delta(H) < \infty$$

for any $\delta > 0$. This allows to write (26) in the following form :

$$\beta(\lambda) = \frac{1}{\pi(\rho)|W|} \int_{\mathfrak{a}} \big[1 - \phi_\lambda(H)\big] j(H)^{-1} \,\mathrm{d}L(H) + C$$

with

$$(29)\qquad C = \int_{\mathfrak{a}} \Big[\frac{1}{\mathcal{D}[\pi]} - \frac{1}{|W|\pi(\rho)} j(H)^{-1} \Big] \,\mathrm{d}L(H).$$

To see that (29) converges we use (28), the Taylor expansion of $j(H)^{-1}$ and (23). Finally

$$(30)\qquad \eta(\lambda) = C + b'\|\lambda\|^2 + \int_{\mathfrak{a}} \big[1 - \phi_\lambda(H)\big] \,\mathrm{d}\widetilde{L}(H), \qquad \lambda \in \mathfrak{a}^*,$$

where $\mathrm{d}\widetilde{L} = (j(H)\pi(\rho)|W|)^{-1}\,\mathrm{d}L$. Comparing (30) with the Gangolli formula for the spherical Fourier transform of an infinitely divisible K-invariant measure on X (see [4]), we conclude that $\mathrm{e}^{-\eta}$ is the spherical Fourier transform of such a measure, with the Lévy measure $\widetilde{L}$. $\square$

We complete the study of functions of negative type in the symmetric sense by giving two more properties which are true for all symmetric spaces of non-compact type. The first of them is analogous to the PROPOSITION 3.

PROPOSITION 4. — *A continuous function η on T_ρ is of negative type in the symmetric sense if and only if there exists a sequence $(\Psi_n)_{n\geq 1}$ of spherical Fourier transforms of K-invariant bounded positive measures on X and a sequence $(c_n)_{n\geq 1}$ with $c_n \geq \Psi_n(0)$ such that*

$$\eta = \lim_{n\to\infty} (c_n - \Psi_n)$$

on $\mathfrak{a}^*$.

Proof. — If η satisfies (18), then there exists a sequence $(L_n)_{n\geq 1}$ of K-invariant bounded positive measures on X such that

$$\eta_n(\lambda) = \int_X [1 - \phi_\lambda(x)]\,\mathrm{d}L_n(x) \longrightarrow \eta(\lambda)$$

on $\mathfrak{a}^*$ (cf.[5]) and $\eta_n = L_n(X) - \widehat{L}_n$.

Conversely, write $\eta_n = c_n - \Psi_n$. Then $\mathrm{e}^{t(c_n - \eta_n)}$ is the spherical Fourier transform of a K-invariant bounded positive measure for any $t > 0$ and $n \in \mathbb{N}$ and so is $\mathrm{e}^{-t\eta_n}$. By the Lévy continuity theorem on symmetric spaces (cf. [5]) it follows that η is of negative type in the symmetric sense. $\square$

PROPOSITION 5. — *Suppose that η be of negative type in the symmetric sense with $\eta > 0$ on $\mathfrak{a}^*$. Then, for any $0 < \alpha < 1$ the function η^α on $\mathfrak{a}^*$ has an analytic continuation on* $\operatorname{Int} T_\rho$, *continuous on T_ρ and it is of negative type in the symmetric sense.*

Proof. — We have

$$\mathrm{e}^{-\eta^\alpha(\lambda)} = \lim_{t\to 0} \exp\Big(-\frac{1}{t^\alpha}\big[1 - \mathrm{e}^{-t\eta(\lambda)}\big]^\alpha\Big)$$

for $\lambda \in \mathfrak{a}^*$. The proposition follows by using the generalized binomial formula and the Lévy continuity theorem on symmetric spaces (cf. [5]). $\square$

REMARK. — If the K-invariant measure with the spherical Fourier transform equal to $\mathrm{e}^{-\eta}$ is Gaussian, the measures $\mu^{(\alpha)}$ such that

$$\widehat{\mu}^{(\alpha)} = \mathrm{e}^{-\eta^\alpha}$$

may be thought of as an analogue of *stable* measures on X. GETOOR [6] gives directly the relation between the densities of $\mu^{(\alpha)}$ and the Gaussian density in the case of the hyperbolic plane. This may be done on any symmetric space.

COROLLARY 3. — *If η is given by* (18) *and $\eta(\lambda) > 0$ for $\lambda \in \mathfrak{a}^*$ then the function*

$$\mathcal{D}\big[\pi(\lambda)\,\mathrm{e}^{-t\eta^{\alpha}(\lambda)}\big]$$

is positive definite on $\mathfrak{a}^*$ for $0 < \alpha < 1$.

EXAMPLE. — COROLLARY 3 applied to the Gaussian semigroup on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ with $\eta(\lambda) = \lambda^2 + 1$ implies that the function

$$\Phi(\lambda) = \mathrm{e}^{-t(\lambda^2+1)^{\alpha}}\big[1 - 2t\alpha\lambda^2(\lambda^2+1)^{\alpha-1}\big]$$

is of positive type on $\mathbb{R}$ for any $0 < \alpha < 1$ and $t > 0$. This example shows that the function Φ in the THEOREM 2 may be discontinuous on the boundary of T_ρ.

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