MICHAEL DOEBELI

Linear models for reductive group actions
on affine quadrics

LINEAR MODELS FOR REDUCTIVE

GROUP ACTIONS ON AFFINE QUADRICS

PAR

MICHAEL DOEBELI (*)

RÉSUMÉ. — Nous étudions les actions des groupes réductifs sur les quadriques affines complexes dont le quotient est de dimension 1. Une telle action est dite linéarisable si elle est équivalente à la restriction d'une action linéaire orthogonale dans l'espace affine ambiant de la quadrique. Une action linéaire satisfait à certaines conditions topologiques. Nous recherchons si ces conditions sont valables pour des actions générales. Si c'est le cas, il est naturel de se demander si une action donnée possède un modèle linéaire, c'est-à-dire s'il existe une action linéaire avec les mêmes types d'orbites et avec des représentations slices équivalentes. Nous montrons qu'un modèle linéaire existe si l'action a un point fixe ou si le groupe d'isotropie principal est connexe. Enfin, nous faisons une classification de toutes les actions linéaires dont le quotient est de dimension 1.

ABSTRACT. — We study reductive group actions on complex affine quadrics with 1-dimensional quotient. Such an action is called linearizable if it is equivalent to the restriction of a linear orthogonal action in the ambient affine space of the quadric. A linear action on the quadric satisfies certain topological conditions. We examine whether these condititons also hold for general actions. In case they do it is natural to ask whether a given action has a linear model, i.e., whether there is a linear action with the same orbit types and equivalent slice representations. We show that a linear model exists if the action has a fixed point or if the principal isotropy group is connected. Finally, we classify all linear actions with 1-dimensinal quotient.

1. Introduction

1.1. — Let $Q_n := \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = 1\} \subset \mathbb{C}^{n+1}$ denote the n-dimensional affine quadric over the field of complex numbers $\mathbb{C}$. Let G be a (linear) algebraic group. Every orthogonal representation $\rho : G \to O_{n+1}(\mathbb{C})$ determines an action of G on Q_n. These actions

— we call them *linear actions* — are well understood. Of course, the geometry of the situation does not change if we replace a given action by a conjugate one within the group $\mathrm{Aut}\,Q_n$ of algebraic automorphisms of Q_n. We call an action of G on Q_n *linearizable* if it is conjugate to a linear action of G on Q_n.

1.2. — The case $n = 1$ is easy. Here

$$Q_1 \cong \dot{\mathbb{C}} = \mathbb{C} \setminus \{0\} \quad \text{and} \quad \mathrm{Aut}\,Q_1 \cong \mathbb{C}^* \rtimes \mathbb{Z}_2 = \mathrm{O}_2,$$

and so every group action is linear. The situation changes dramatically for $n \geq 2$. In these cases $\mathrm{Aut}\,Q_n$ can be given the structure of an infinite dimensional algebraic group, see [11]. More is known only for $n = 2$ where this group can be written as an amalgamated product, see [10, p. 94]. The following example illustrates that $\mathrm{Aut}\,Q_n$ is indeed very big even for small n and shows that unipotent group actions need not be linearizable. Consider Q_3, which we identify with $\mathrm{SL}_2(\mathbb{C})$. Choose a U-invariant regular function f on $\mathrm{SL}_2(\mathbb{C})$, where U denotes the subgroup of matrices with 1's in the diagonal and 0 in the lower left entry. Consider the following action of the additive group $\mathbb{C}^+$ on SL_2 :

$$t \cdot h := \begin{pmatrix} 1 & tf(h) \\ 0 & 1 \end{pmatrix} \cdot h \cdot \begin{pmatrix} 1 & -tf(h) \\ 0 & 1 \end{pmatrix},$$

where $t \in \mathbb{C}^+$ and $h \in \mathrm{SL}_2$. We claim that this action is not linearizable as soon as f is not a constant. In fact, the linear actions of $\mathbb{C}^+$ on SL_2 are easily classified. Under the double cover $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{SO}_4$ the SO_4-action on Q_3 corresponds to the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on SL_2 given by $(g, g') \cdot h = ghg'^{-1}$. Thus a linear action $\mathbb{C}^+ \rightarrow \mathrm{SO}_4$ on Q_3 is given by the corresponding morphism $\mathbb{C}^+ \rightarrow \mathrm{SL}_2 \times \mathrm{SL}_2$ as an action on SL_2. It follows that such an action must be equivalent to either the trivial action, the one given by conjugation or the one given by left (or right) multiplication. It is now straightforward that of these only the trivial action or the one given by conjugation can be equivalent to the action defined above, and that such an equivalence is only possible if the function f is constant.

1.3. — Because of the previous example we restrict our attention to reductive groups G, i.e., to groups which don't have any non-trivial unipotent normal subgroups. (Equivalently, every rational representation of G is completely reducible.)

Linearization problem : *Is every action of a reductive group on an affine quadric linearizable ?*

So far no example of a non-linearizable reductive group action on Q_n is known. However, we do not believe that every such action is linearizable, except under certain « smallness » assumptions. For example, every reductive group action on Q_2 is linearizable. This follows from the structure theorem for $\operatorname{Aut} Q_2$ mentioned above. We will show among other things that linearization is possible for actions for which the only invariant regular functions on Q_n are the constants, see § 2. Therefore, the classification of these cases is achieved by classifying all orthogonal representations (V, G) for which the ring of invariant functions $\mathcal{O}(V)^G$ is generated by the invariant quadratic form.

1.4. — In case linearization holds the G-action has to satisfy certain topological conditions, e.g. the generic orbit of G on $X = Q_n$ has to be closed. Moreover, every slice representation (N_x, G_x) has to be orthogonal, where $x \in X$ is a point on a closed orbit, G_x is the stabilizer of x and $N_x = \mathrm{T}_x X / \mathrm{T}_x Gx$ is the normal space to the orbit. This follows from the fact that these properties hold for orthogonal representations, see [22, § 5]. This leads to the following

DEFINITION. — An orthogonal representation (V, G) is called a *linear model* for an action of G on the quadric $X = Q_n$ if X has the same orbit types and equivalent slice representations as the quadric

$$Q_V := \big\{ v \in V \mid (v, v) = 1 \big\} \subset V$$

with the linear G-action.

1.5. — The aim of this paper is to study the topology of a connected reductive group action on an affine quadric X under the assumption that the ring of invariants has (Krull-) dimension 1, i.e., that the algebraic quotient $X /\!\!/ G$ (see 1.9) is 1-dimensional. It turns out that for our results it is enough to assume that X is an irreducible, smooth affine variety which is homotopy equivalent to a real sphere.

PROPOSITION 1. — *Under the assumptions above we have :*

(1) $X /\!\!/ G \cong \mathbb{A}$, *the affine line.*

(2) *There are two points $y_1, y_2 \in \mathbb{A}$ such that the principal stratum is $\mathbb{A} \setminus \{y_1, y_1\}$.*

(3) *The generic fiber of the quotient map (i.e., the fiber over the principal stratum) is a G-orbit, which means that the generic orbit is closed.*

This is proved in sections 3.2 and 3.4.

1.6. — The results in PROPOSITION 1 are obvious for linear actions on quadrics, or, more generally, if there is a linear model. We believe that the assumptions in 1.5 insure the existence of a linear model. However, we have been able to prove existence only under additional hypotheses.

PROPOSITION 2. — *Under the assumptions of* 1.5 *a linear model exists in the following cases* :

(1) *The G-action on X has a fixed point.*

(2) *The principal isotropy group of the action is connected, and the dimension of the slice representations is* > 2.

This is proved in sections 4.8 and 4.9.

1.7. — The analogous situation of compact group actions on real spheres has been studied extensively. For example, BOREL, MONTGOMERY and SAMELSON classified all transitive compact group actions on spheres (see [19], [2] and [3]). The case of orbit space dimension 1 has been analyzed by WANG [25] and ASOH [1]. Their results are essential in our approach.

1.8. — We have been guided by the work of KRAFT, LUNA and SCHWARZ on the linearization problem for reductive group actions on affine space $\mathbb{C}^n$. In this classical setting the question is whether a given action is equivalent to a representation, and these authors have tackled the problem under the assumption that the quotient dimension is equal to 1, see [14] and [16]. (Note that actions with quotient dimension 0 are linearizable by Luna's slice theorem.) They first prove with topological methods the existence of a fixed point and then compare the tangent representation at this point with the given action. Although linearization holds in many cases, the first non-linearizable actions on $\mathbb{C}^n$ were discovered by SCHWARZ [23] in this context. Moreover, using the results of SCHWARZ, KNOP [12] proved that every non-commutative, connected reductive group has non-linearizable actions on some $\mathbb{C}^n$. Our approach to the linearization problem on quadrics is the analogon to the one taken in [16]. There the fixed point gives a linear model as the tangent representation at this point. Here we have to carry the topological analysis much further to show that a linear model exists. In section 5.1 we classify all these models, i.e., all linear actions on quadrics with 1-dimensional quotient. This classification will be used in a subsequent paper to show that the existence of a linear model suffices to prove that linearization holds.

1.9. — To conclude this introduction, we state the conventions and notation valid in this paper as well as some general facts. Our varieties

will be defined over the complex numbers. Let G be a reductive algebraic group acting on an affine variety X. We denote by $\mathcal{O}(X)$ the $\mathbb{C}$-algebra of regular functions and by $\mathcal{O}(X)^G$ the subalgebra of G-invariants. A famous theorem of Hilbert asserts that $\mathcal{O}(X)^G$ is a finitely generated $\mathbb{C}$-algebra (see [13, II.3.2]). Let $X/\!\!/G$ denote the corresponding affine variety, and let $\pi_X : X \to X/\!\!/G$ denote the morphism corresponding to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$.

PROPOSITION (see [13, II.3.2]).

(1) π_X *is surjective.*

(2) *Every fiber of π_X contains a unique closed G-orbit, hence π_X sets up a bijection between the closed orbits in X and the points of $X/\!\!/G$.*

If V is an M-representation, where M is an algebraic group, we will use the notation (V, M) to emphasize the group involved. Luna's slice theorem provides a strong link between general reductive group actions on smooth varieties and representation theory. We will often use this important result. For a detailed treatment of the slice theorem we refer the reader to the original article [18] of LUNA, or to the article [24] of SLODOWY.

To make the connection to compact group actions we will need the following well known facts (cf. [4] and [20]) :

LEMMA. — *Let G be a linear algebraic group and $H \subset G$ a closed subgroup. Let $K \subset G$ be a maximal compact subgroup such that $L :=$ $K \cap H$ is a maximal compact subgroup of H. Then $G/H = K \times^L F$, where F is an L-representation, i.e., G/H is a (differentiable) fiber bundle over K/L with fibers isomorphic to a vectorspace. In particular, the inclusion $K/L \hookrightarrow G/H$ is a homotopy equivalence. If G and H are both reductive then $\dim_{\mathbb{R}}(K/L) = \dim_{\mathbb{C}}(G/H)$. If H is connected then K/L is orientable, hence Poincaré duality holds for the cohomology ring $H^*(G/H, k)$ with coefficients in an arbitrary field k.* $\square$

This paper is the first part of my dissertation. I am indebted to my advisors Hanspeter KRAFT and Gerald SCHWARZ for their help and encouragement.

2. Linearization in case there are no invariants

2.1. — Let G be a connected reductive algebraic group, and let X be an irreducible, smooth affine variety which is homotopy equivalent to a real sphere in euclidian space. Let G act almost effectively on X, i.e., such that the kernel of the action is finite.

THEOREM. — *Suppose that the only G-invariant functions on X are constant. Then the action is transitive and we have* :

(1) *if n is even, then G is simple*;

(2) *if n is odd, then G is either simple or of the form $G = (G_1 \times R)/N$ where G_1 is a simply connected simple group, R is either trivial, isomorphic to $\mathbb{C}^*$ or to SL_2, N is a finite normal subgroup of $G_1 \times R$, and where the subgroup of G corresponding to G_1 is transitive on Q_n.*

Furthermore, the action is linearizable, i.e., equivalent to a linear action of G on some affine quadric. In particular, the variety X is isomorphic to an affine quadric.

Proof. — In the category of topological manifolds the first part of the theorem was proved for spheres in a series of papers by BOREL, MONTGOMERY and SAMELSON (cf. [2], [3] and [19]) : if K is a compact Lie group acting transitively and effectively on the n-dimensional real sphere $S^n \subset \mathbb{R}^{n+1}$, then the statements (1) and (2) hold for K (replace $\mathbb{C}^*$ by S^1 and SL_2 by SU_2). That the action is linearizable in this setting can be found in [21, § 2].

To make the translation to the algebraic category, we first show that if $\dim X /\!\!/ G = 0$, then G acts transitively on X : it follows from the slice theorem that there is a G-isomorphism

$$X \xrightarrow{\sim} G \times^H V,$$

where H is a reductive subgroup of G and V is an H-module (here $G \times^H V$ denotes the bundle with fiber V which is associated to the principal H-bundle $G \to G/H$). Since V is contractible, we get for the cohomology ring of X (with coefficients in an arbitrary field k) :

$$H^*(X) \cong H^*(G/H).$$

By the LEMMA in 1.9 we conclude that $\dim X = \dim G/H$, hence $V = \{0\}$, $X \cong G/H$, where H is the isotropy group of some point $x \in X$. Now choose maximal compact subgroups $K \subset G$, $L \subset H$, such that $L = K \cap H$. Then, again by the lemma, K/L is a homotopy sphere. It is known that K/L is therefore homeomorphic to a sphere (see [4, 4.61]), and we can apply the theorem of Borel, Montgomery and Samelson mentioned above to obtain statements (1) and (2). The fact that the action is linearizable follows by complexifying the real orthogonal representation of K in $\mathbb{R}^{n+1}$ which «realizes» the sphere $S^n = K/L$.   □

It should be mentioned that so far there is no direct proof of linearization in the compact case : one needs the classification of transitive actions on spheres, which is quite tedious. Thus, already for quotient dimension 0 linearization on affine quadrics is quite hard to obtain.

2.2. — The classification of linear transitive actions (in the sense of 1.1) on varieties X as above is given by the next result.

PROPOSITION. — *The following Table 1 classifies all representations (G, V) of connected reductive groups G with 1-dimensional quotient such that the generic fiber F of the quotient map is homotopy equivalent to a sphere. If H denotes the principal isotropy group then $F \cong G/H$. In case $\dim V > 2$ the representation (G, V) is orthogonal, and it is completely determined by the pair $G \supset H$.*

<table>
<tr><td colspan="3" align="center">Table 1</td></tr>
<tr><td>G</td><td>V</td><td>H</td></tr>
<tr><td>$\mathbb{C}^*$</td><td>$\Sigma_a \oplus \Sigma_{-b}, \ a, b > 0$</td><td>cyclic</td></tr>
<tr><td>$A_n \qquad (n \geq 1)$</td><td>$\omega_1 \oplus \omega_1^*$</td><td>$A_{n-1}$</td></tr>
<tr><td>$\mathbb{C}^* \times A_n \quad (n \geq 1)$</td><td>$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*, \ a > 0$</td><td>$\mathbb{C}^* \times A_{n-1}$</td></tr>
<tr><td>$B_n \qquad (n \geq 2)$</td><td>ω_1</td><td>D_n</td></tr>
<tr><td>$D_n \qquad (n \geq 3)$</td><td>ω_1</td><td>B_{n-1}</td></tr>
<tr><td>$C_n \qquad (n \geq 2)$</td><td>$\omega_1 \oplus \omega_1^*$</td><td>$C_{n-1}$</td></tr>
<tr><td>$\mathbb{C}^* \times C_n \quad (n \geq 2)$</td><td>$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$</td><td>$\mathbb{C}^* \times C_{n-1}$</td></tr>
<tr><td>$A_1 \times C_n \quad (n \geq 2)$</td><td>$\omega_1 \otimes \omega_1'$</td><td>$A_1 \times C_{n-1}$</td></tr>
<tr><td>$A_1 \times A_1$</td><td>$\omega_1 \otimes \omega_1'$</td><td>$A_1$</td></tr>
<tr><td>A_1</td><td>$2\omega_1$</td><td>$\mathbb{C}^*$</td></tr>
<tr><td>A_3</td><td>ω_2</td><td>C_2</td></tr>
<tr><td>B_3</td><td>ω_3</td><td>G_2</td></tr>
<tr><td>B_4</td><td>ω_4</td><td>B_3</td></tr>
<tr><td>C_2</td><td>ω_2</td><td>$A_1 \times A_1$</td></tr>
<tr><td>G_2</td><td>ω_1</td><td>A_2</td></tr>
</table>

(For the notation we refer to section 5.2.)

Proof. — It is clear that all orthogonal representations with 1-dimensional quotient must be included in a list of representations as in the proposition. Such an orthogonal representation is either irreducible or of the form $(W \oplus W^*, G)$, where W is an irreducible G-representation with 0-dimensional quotient and where W^* denotes the dual representation. Using the fact that, by THEOREM 2.1, there is always a simple normal subgroup of G which is already transitive on the invariant quadric, one

obtains a list of all these orthogonal representations by examining the tables in [17]. Now suppose that we are given a representation (V', G) as in the proposition. Then the generic fiber of the quotient map is an orbit : by the slice theorem the generic fiber is of the form $G \times^H N$, where N is the nullcone of the generic slice representation, and as in the proof of THEOREM 2.1 we must have $N = \{0\}$ for cohomological reasons. Thus the representation (V', G) is stable, i.e., the generic orbit is closed, and it follows easily that V' is either irreducible or the sum of irreducible representations with 0-dimensional quotient. Since G/H is homotopy equivalent to a sphere, there is by THEOREM 2.1 an orthogonal representation (V, G) with 1-dimensional quotient and generic orbit isomorphic to G/H. Again using the fact that G has a simple normal subgroup which is transitive on G/H and by looking at the tables in [17], one can conclude that in fact the representations V and V' must be equivalent, except possibly if $G \cong \mathbb{C}^*$, i.e., if $\dim V' = 2$. In this case there are infinitely many non-equivalent representations which induce transitive actions on $G/H \cong \mathbb{C}^*$. They are given by weights $a, -b$ on $\mathbb{C}^2$, where $a, b > 0$. These representations together with all orthogonal representations with 1-dimensional quotient therefore exhaust all possibilities. They are listed in Table 1. The last statement in the proposition follows from this classification. □

3. The Leray spectral sequence of the quotient map

3.1. — Let G and X be as in § 2, and let $\pi : X \to X/\!\!/G$ be the quotient map. In the next two paragraphs we analyze the Leray spectral sequence associated to π and derive several results concerning the cohomology of the closed orbits on X as well as of the orbits in the slice representations. Ultimately, it was our goal to use the topological information to prove the existence of linear models. However, we only achieved this under additional assumptions. In any case, the topological results show that our actions look like linear actions to some extent. For instance, it is well known that for a compact group action on the real sphere S^n with 1-dimensional quotient the principal stratum consists of the quotient space minus 2 points (see [25, § 3]). But then the same holds for a linear action, since an orthogonal representation of a reductive group over $\mathbb{C}$ is always the complexification of a real representation of a compact group. Moreover, in the complexification of a compact group action the generic orbit is closed. In 3.4 we show that the same properties hold for general G-actions on X with 1-dimensional quotient.

3.2. — We first show that in our situation the quotient is isomorphic to the affine line $\mathbb{A}$.

LEMMA. — *If* $\dim X /\!\!/ G = 1$, *then* $X /\!\!/ G \cong \mathbb{A}$.

Proof. — By [15, cor. 3.4] the quotient map $\pi : X \to X /\!\!/ G$ induces a surjective map of the corresponding fundamental groups. Thus $\pi_1(X /\!\!/ G)$ is trivial since $\pi_1(X)$ is. Furthermore, $X /\!\!/ G$ is affine and normal, since X has these properties, and $X /\!\!/ G$ is therefore also non-singular (since it is 1-dimensional). Thus, $X /\!\!/ G$ is (topologically) a compact Riemannian surface with a finite, positive number of points deleted. Since $\pi_1(X /\!\!/ G) = 0$, the claim follows easily. $\square$

3.3. — For details of the results in this section we refer the reader to chapter II of [16]. Let G be a connected reductive group and Z a smooth, affine G-variety with quotient $\pi : Z \to Z /\!\!/ G$. Let k be an arbitrary field, and let $\mathcal{H}^q$ be the q-th direct image under π of the constant sheaf $k \times Z$, which we also denote by k :

$$\mathcal{H}^q := R^q \pi_\star k.$$

By definition, $\mathcal{H}^q$ is the sheaf associated to the presheaf :

$$U \longmapsto H^q(\pi^{-1}(U), k), \quad U \subseteq Z /\!\!/ G.$$

Let $Z /\!\!/ G = \bigcup_i Y_i$ be the Luna stratification of the quotient. If $y_i \in Y_i$, let O_{y_i} denote the closed orbit in the fiber $\pi^{-1}(y_i)$. It follows from the slice theorem that there is a fundamental system $\mathcal{U}$ of neighborhoods U of y_i with the following properties : For all $U \in \mathcal{U}$ and for all $y \in U \cap Y_i$ the inclusions $\mathcal{O}_y \hookrightarrow \pi^{-1}(y) \hookrightarrow \pi^{-1}(U)$ are homotopy equivalences. Using this one can show :

PROPOSITION.

(1) *The restriction of* $\mathcal{H}^q$ *to* Y_i *is locally constant for every* i.

(2) *The stalk of* $\mathcal{H}^q$ *in* $y \in Z /\!\!/ G$ *is given by* :

$$\mathcal{H}_y^q = H^q\big(\pi^{-1}(y), k\big) = H^q(O_y, k).$$

(3) *For* $U \in \mathcal{U}$, *the inclusions* $O_{y_i} \hookrightarrow \pi^{-1}(y_i) \hookrightarrow \pi^{-1}(U)$ *induce an isomorphism in cohomology*

$$H^*\big(\pi^{-1}(U), k\big) \xrightarrow{\ \sim\ } H^*(O_{y_i}, k).$$

The inverse of this isomorphism is given by a G-*invariant retraction* $\pi^{-1}(U) \to O_{y_i}$. *For every other* $y \in U$, *the inclusion*

$$O_y \hookrightarrow \pi^{-1}(y) \hookrightarrow \pi^{-1}(U)$$

gives a homomorphism $H^*(O_{y_i}, k) \to H^*(O_y, k)$ which is induced by a G-equivariant map $O_y \to O_{y_i}$.

(4) If $Z/\!\!/G \cong \mathbb{A}$, and if $\mathbb{A} \setminus \{y_1, ..., y_r\}$ is the principal stratum of $Z/\!\!/G$, then for every simply connected neighborhood V of y_i which does not contain any of the points y_j, $j \neq i$, the inclusions

$$\mathcal{O}_{y_i} \hookrightarrow \pi^{-1}(y_i) \subset \pi^{-1}(V)$$

are homotopy equivalences. In particular, the isomorphisms

$$\mathcal{H}^q(V) \xrightarrow{\sim} \mathcal{H}^q_{y_i} \quad \text{for all} \quad q.$$

These isomorphisms are given by a G-equivariant retraction

$$\pi^{-1}(V) \longrightarrow \mathcal{O}_{y_i}.$$

Proof. — (1) to (3) are proved in [16, Prop. II.1.3]. Using that the variety Z is a trivial topological G-bundle over every simply connected set contained in the principal stratum of the quotient, the generalization from (3) to neighborhoods V as in (4) is straightforward. □

3.4. — We go back to the given action of G on X.

PROPOSITION.

(1) *There are exactly two non-principal fibers of π. Furthermore, the principal fibers are closed G-orbits.*

(2) *For $1 \leq q \leq n-2$ we get an isomorphism*

$$H^q(\mathcal{O}_{y_1}, k) \oplus H^q(\mathcal{O}_{y_2}, k) \xrightarrow{\sim} H^q(G/H, k),$$

where the inclusions $H^q(\mathcal{O}_{y_i}, k) \hookrightarrow H^q(G/H, k)$, $i = 1, 2$, are induced by G-equivariant maps $\varphi_i : G/H \to \mathcal{O}_{y_i}$.

(3) $H^{n-1}(G/H, k) = k$, *hence Poincaré duality holds for $H^*(G/H, k)$.*

Proof. — For the map $\pi : X \to \mathbb{A}$ there is a Leray spectral sequence :

$$H^p(\mathbb{A}, \mathcal{H}^q) \Longrightarrow H^{p+q}(X, k).$$

Let $\mathbb{A} \setminus \{y_1, ..., y_r\}$ be the principal stratum, and let S_i, $i = 1, ..., r$, be r open, parallel strips in $\mathbb{A} = \mathbb{R}^2$ with the following properties :

- $\mathbb{A} = \bigcup_{i=1}^r S_i$, $y_i \in S_i$.
- $S_i \cap S_{i+1}$ is a non empty strip which contains none of the points y_j for $j = 1, ..., r$.
- $S_i \cap S_j = \emptyset$ for $|j - i| \geq 2$.

By 3.3 (1), the restriction of $\mathcal{H}^q$ to S_i is locally constant along $S_i \setminus \{y_i\}$. Therefore, by [16, II.2.1], the restrictions $\mathcal{H}^q|_{S_i}$ are acyclic :

$$H^p(S_i, \mathcal{H}^q|_{S_i}) = \begin{cases} \mathcal{H}^q(S_i) & \text{for } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the cohomology of the sheaves $\mathcal{H}^q$ on $\mathbb{A}$ can be calculated as the Čech-cohomology of the covering $\mathbb{A} = \bigcup_{i=1}^r S_i$. The alternating Čech-complex has the form :

(i) $\qquad 0 \to \bigoplus_{i=1}^r H^0(S_i, \mathcal{H}^q) \longrightarrow \bigoplus_{i=1}^{r-1} H^0(S_i \cap S_{i+1}, \mathcal{H}^q) \to 0.$

For the cohomology groups $H^p(\mathbb{A}, \mathcal{H}^q)$, (i) gives the exact sequence :

(ii) $\quad 0 \to H^0(\mathbb{A}, \mathcal{H}^q) \longrightarrow \bigoplus_{i=1}^r H^0(S_i, \mathcal{H}^q) \xrightarrow{\phi} \bigoplus_{i=1}^{r-1} H^0(S_i \cap S_{i+1}, \mathcal{H}^q)$

$$\longrightarrow H^1(\mathbb{A}, \mathcal{H}^q) \to 0.$$

In particular, $H^p(\mathbb{A}, \mathcal{H}^q) = 0$ for $p \geq 2$ and for all q, i.e., the Leray spectral sequence of the quotient degenerates. Since X is homotopy equivalent to a sphere, it follows that :

(iii) $\quad H^0(\mathbb{A}, \mathcal{H}^q) \oplus H^1(\mathbb{A}, \mathcal{H}^{q-1}) \cong H^q(X, k) = \begin{cases} k & \text{if } q = 0 \text{ or } q = n, \\ 0 & \text{otherwise.} \end{cases}$

From 3.3 and by definition of the strips S_i, for all $i = 1, ..., r$ we get canonical isomorphisms :

$$H^0(S_i, \mathcal{H}^q) = \mathcal{H}^q(S_i) \cong \mathcal{H}^q_{y_i} \cong H^q(\mathcal{O}_{y_i}, k).$$

We also get an isomorphism

$$H^0(S_i \cap S_{i+1}, \mathcal{H}^q) = \mathcal{H}^q(S_i \cap S_{i+1}) \cong \mathcal{H}^q_y \cong H^q(G/H, k)$$

by fixing the principal isotropy group H and choosing a point $x \in \mathcal{O}_y$ whose stabilizer is H. Thus (ii) can be written as

(iv) $\quad 0 \to H^0(\mathbb{A}, \mathcal{H}^q) \longrightarrow \bigoplus_{i=1}^r H^q(\mathcal{O}_{y_i}, k) \xrightarrow{\phi} \bigoplus_{i=1}^{r-1} H^q(G/H, k)$

$$\longrightarrow H^1(\mathbb{A}, \mathcal{H}^q) \to 0,$$

where the components of ϕ are induced by the G-equivariant maps $G/H \to \mathcal{O}_{y_i}$. Since $\dim \mathcal{O}_{y_i} < n$, for $q = n$ we get $H^0(\mathbb{A}, \mathcal{H}^n) = 0$ from (iv), and therefore by (iii) :

$$H^1(\mathbb{A}, \mathcal{H}^{n-1}) = k.$$

By (iii) we also have $H^0(\mathbb{A}, \mathcal{H}^{n-1}) = 0$. For $q = n - 1$ we thus obtain the exact sequence :

$$\text{(v)} \quad 0 \to \bigoplus_{i=1}^{r} H^{n-1}(\mathcal{O}_{y_i}, k) \to \bigoplus_{i=1}^{r-1} H^{n-1}(G/H, k)$$
$$\longrightarrow H^1(\mathbb{A}, \mathcal{H}^{n-1}) = k \to 0.$$

Since $H^0(\mathbb{A}, \mathcal{H}^1) = 0$ by (iii), the map

$$\bigoplus_{i=1}^{r} H^1(\mathcal{O}_{y_i}, k) \longrightarrow \bigoplus_{i=1}^{r-1} H^1(G/H, k)$$

obtained from (iv) for $q = 1$ is injective. It then follows from [16, lemma II.4.3] that $\dim \mathcal{O}_{y_i} < \dim G/H$, and we therefore get from (v) :

$$\bigoplus_{i=1}^{r-1} H^{n-1}(G/H, k) = k.$$

It follows that $r = 2$ and that $H^{n-1}(G/H, k) = k$ for every field k, hence $\dim G/H = n - 1$. Since G/H has the homotopy type of a compact manifold of the same (real) dimension by LEMMA 1.9, it follows that Poincaré dualtiy holds for the cohomology ring $H^*(G/H, k)$. We have proved (1) and (3). For $1 \le q \le n - 2$ we get from (iii) and (iv)

$$0 = H^0(\mathbb{A}, \mathcal{H}^q) \longrightarrow H^q(\mathcal{O}_{y_1}, k) \oplus H^q(\mathcal{O}_{y_2}, k) \xrightarrow[\sim]{\phi} H^q(G/H, k)$$
$$\to H^1(\mathbb{A}, \mathcal{H}^q) = 0,$$

and (2) follows as well. $\square$

3.5 Remark. — Let $K \subset G$ be a maximal compact subgroup of G such that $L := K \cap H$ is a maximal compact subgroup of H. By choosing points x_i on the closed orbits $\mathcal{O}_{y_i}$ appropriately we can assume that the groups $L_i := K \cap G_{x_i}$ are maximal compact subgroups of the stabilizers G_{x_i}. The inclusions $K/L_i \subset \mathcal{O}_{y_i}$, $K/L \subset G/H$ are homotopy

equivalences by LEMMA 1.9. Moreover, there are K-equivariant retractions $\mathcal{O}_{y_i} \to K/L_i$ and $G/H \to K/L$ (see LEMMA 1.9). It follows that we can replace $\mathcal{O}_{y_i}$ by K/L_i and G/H by K/L in the second statement of the last proposition : for $1 \leq q \leq n-2$ there are isomorphisms

$$H^q(K/L_1, k) \oplus H^q(K/L_2, k) \xrightarrow{\sim} H^q(K/L, k),$$

where the inclusions $H^q(K/L_i, k) \hookrightarrow H^q(K/L, k)$ are induced by K-equivariant maps $K/L \to K/L_i$. Since these maps are K-equivariant, it is easy to see that we can assume that $L \subset L_1 \cap L_2$ and that they are just the natural projections. From now on K, L, L_1, L_2 and their complexifications G, H, H_1, H_2 will always refer to such a choice of these groups.

4. Existence of linear models

4.1.—Let $\varphi_i : G/H \to G/H_i$ be the natural projections, $i = 1, 2$, where the H_i are as in Remark 3.5. They induce exact homotopy sequences

$$\cdots \longrightarrow \pi_1(G/H) \xrightarrow{\varphi_{i*}} \pi_1(G/H_i) \longrightarrow \pi_0(H_i/H) \to 0.$$

Here $\pi_0(H_i/H)$ is a group. This is a consequence of the more general fact that if the reductive group G acts on the affine variety Z with principal isotropy group H and such that $Z /\!\!/ G^0$ is irreducible (where G^0 denotes the connected component of the identity of G), then $\pi_0(G/H)$ is a group : the principal isotropy group M of the action of G/G^0 on $Z /\!\!/ G^0$ is the kernel of this action, because the fixed points $(Z /\!\!/ G^0)^M$ must have the same dimension as $Z /\!\!/ G^0$, hence $(Z /\!\!/ G^0)^M = Z /\!\!/ G^0$, since the latter is irreducible. Thus M is normal in G/G^0, and the claim follows from the fact that $(G/G^0)/M = \pi_0(G/H)$.

The following lemma, which is an application of the theorem of Van Kampen, will show that the generic orbits of the slice representations are connected (see also [25, 4.3]).

LEMMA. — *The amalgamated product of the diagram*

$$\pi_1(G/H_1) \xleftarrow{\varphi_{1*}} \pi_1(G/H) \xrightarrow{\varphi_{2*}} \pi_1(G/H_2)$$

is trivial. In particular, φ_{1*} and φ_{2*} are both surjective. More precisely, $\varphi_{1*}(\mathrm{Ker}\, \varphi_{2*}) = \pi_1(G/H_1)$, and similarly for φ_{2*}.

Proof. — Let S_1 and S_2 be strips which cover the quotient $\mathbb{A}$ as in the proof of 3.4, and let $T_i := \pi^{-1}(S_i)$. The inclusions $O_{y_i} \hookrightarrow T_i$ and

$O_y \hookrightarrow T_1 \cap T_2$ (for $y \in T_1 \cap T_2$) are homotopy equivalences by PROPOSITION 3.3 (4). We therefore get a commutative diagram as follows :

$$
\begin{array}{ccccc}
\pi_1(G/H_1) & \longleftarrow & \pi_1(G/H) & \longrightarrow & \pi_2(G/H_2) \\
\uparrow & & \uparrow & & \uparrow \\
\pi_1(O_{y_1}) & \longleftarrow & \pi_1(O_y) & \longrightarrow & \pi_1(O_{y_2}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(T_1) & \longleftarrow & \pi_1(T_1 \cap T_2) & \longrightarrow & \pi_1(T_2).
\end{array}
$$

Here the top horizontal maps are induced by the natural projections, and all vertical arrows are isomorphisms. It is a consequence of the theorem of Van Kampen and of $\pi_1(X) = 0$ that the amalgamated product of the bottom line of this diagram is trivial, hence the first claim. Since the $\pi_0(H_i/H)$ are groups, it follows from the exact sequences in 4.1 that the images of the φ_{i*} are normal subgroups of $\pi_1(G/H_i)$. This implies the remaining claims. □

4.2. — From the exact sequence in 4.1 we get that $\pi_0(H_i/H)$ is trivial for $i = 1, 2$, i.e. :

COROLLARY. — *H_1/H and H_2/H are both connected.* □

4.3. — Recall that all slice representations occuring in an orthogonal representation are again orthogonal. Thus the generic orbits of the slice representation of a linear action on X are quadrics (since the quotient dimension of these representations is 1). If the principal isotropy group H is connected, we can show that, at least topologically, the same holds for general actions.

PROPOSITION. — *Suppose that H is connected. Then H_1/H and H_2/H are $\mathbb{Z}$-cohomology spheres, hence homotopy equivalent to spheres.*

Proof. — For $i = 1, 2$ we consider the spectral sequences for the fibrations
$$
H_i/H \longrightarrow G/H \xrightarrow{\varphi_i} G/H_i
$$

with coefficients an arbitrary field k. Since H is connected, so are H_1 and H_2 by 4.1, and consequently these spectral sequences have ordinary coefficients
$$
E_2^{p,q} = H^p\big(G/H_i, H^q(H_i/H, k)\big) \quad \text{for} \ p, q \geq 0
$$

(see [5, 4.1 (ii)]). Note that since all the groups involved are connected, Poincaré duality holds for the cohomology rings $H^*(H_i/H, k)$ and $H^*(G/H_i, k)$, see LEMMA 1.9.

We can assume that $\dim G/H_1 \geq \dim G/H_2$ and $H_1 \neq G$. There is an $s > 0$ such that :

$$H^s(H_1/H, k) \neq 0 \quad \text{and} \quad H^j(H_1/H, k) = 0 \quad \text{for} \ 0 < j < s.$$

By PROPOSITION 3.4 (2), the induced maps $H^*(G/H_i, k) \to H^*(G/H, k)$ are injective. This means that all differentials d_r of the spectral sequences have trivial image in

$$E_r^{p,0} = E_2^{p,0} = H^p(G/H_i, k).$$

Hence all differentials of the spectral sequence for $G/H \to G/H_1$ vanish on $E_2^{0,s} = H^s(H_1/H, k)$. It follows that $H^s(G/H, k) \neq 0$, and so by Poincaré duality
$$H^{\dim G/H - s}(G/H, k) \neq 0.$$

By PROPOSITION 3.4 (2) and from $\dim G/H_1 \geq \dim G/H_2$, we obtain $H^j(G/H, k) = 0$ for $\dim G/H_1 < j < \dim G/H$. Thus

$$\dim G/H - s \leq \dim G/H_1$$

follows from $\dim G/H - s < \dim G/H$. Since $s \leq \dim H_1/H$, we get that $s = \dim H_1/H$. These arguments are valid for arbitrary coefficients k, hence H_1/H is a $\mathbb{Z}$-cohomology sphere. It also follows that the considered spectral sequence degenerates, i.e.,

$$H^*(G/H, k) \cong H^*(H_1/H, k) \otimes H^*(G/H_1, k)$$

as graded vector spaces.

If $\dim G/H_1 = \dim G/H_2$, the same conclusions hold of course for H_2/H. If $\dim G/H_1 > \dim G/H_2$, we first observe that since Poincaré duality holds for $H^*(H_2/H, k)$, there is an $\ell > 0$ such that

$$H^\ell(H_2/H, k) \neq 0 \quad \text{and} \quad H^j(H_2/H, k) = 0 \quad \text{for} \ 0 < j < \ell.$$

As before all differentials of the spectral sequence corresponding to $G/H \to G/H_2$ vanish on $E_2^{0,\ell} = H^\ell(H_2/H, k)$. It follows that

$$H^\ell(G/H_2, k) \subsetneq H^\ell(G/H, k),$$

hence by PROPOSITION 3.4 (2) that $H^\ell(G/H_1, k) \neq 0$. We also conclude that $H^j(G/H_1, k) = 0$, for $0 < j < \ell$. With Poincaré duality we get :

$$H^j(G/H_1, k) = \begin{cases} \neq 0 & \text{if } j = \dim G/H_1 - \ell, \\ 0 & \text{for } \dim G/H_1 - \ell < j < \dim G/H_1. \end{cases}$$

For the E_2-term of the spectral sequence for $G/H \to G/H_1$ we thus find the following : the term $E_2^{i,j}$ with highest degree such that $E_2^{i,j} \neq 0$ and such that $i \neq 0$ is the term $E_2^{\dim G/H_1 - \ell, \dim H_1/H}$. Since the spectral sequence degenerates, it follows from 3.4 (2) that this term corresponds to the highest degree term in $H^*(G/H_2, k)$, hence that

$$\dim G/H_1 - \ell + \dim H_1/H = \dim G/H_2.$$

Consequently :

$$\begin{aligned} \ell &= \dim G/H_1 + \dim H_1/H - \dim G/H_2 \\ &= \dim G/H - \dim G/H_2 \\ &= \dim H_2/H. \end{aligned}$$

Therefore H_2/H is also a $\mathbb{Z}$-cohomology sphere. That H_1/H and H_2/H are actually homotopy spheres follows now from the theorem of Hurewicz and the fact that the groups $\pi_1(H_i/H)$ are abelian because H is connected. $\quad\square$

4.4. — If a linear action on the variety X has a fixed point then the other exceptional closed orbit is a fixed point as well, and the generic orbit is a quadric. For general actions we have :

PROPOSITION. — *If one of the exceptional closed orbits is a fixed point, then the other one is a fixed point as well, and the generic orbit G/H is a $\mathbb{Z}$-cohomology sphere.*

Proof. — We assume that $H_1 = G$. By 3.4 we then have

(*) $H^p(G/H, k) = H^p(G/H_2, k)$

for all $p = 0, 1, \ldots, d - 1$, where $d = \dim G/H$ and k is an arbitrary field. To prove the proposition it is therefore enough to show that $H_2 = G$.

We first claim that $\dim H_2/H > 1$. For if $\dim H_2/H = 1$, then one easily sees that $H_2/H \cong \mathbb{C}^*$ (use LEMMA 1.9 and the fact that H_2/H is connected by COROLLARY 4.2). It is then clear that the spectral sequence for

$$H_2/H \longrightarrow G/H \longrightarrow G/H_2$$

has ordinary coefficients if we choose $k = \mathbb{Z}_2$. Then $E_2^{0,1} \neq 0$ for the considered spectral sequence, and as in the second part of the proof of PROPOSITION 4.3 we can now conclude that $H^1(G/H_1, \mathbb{Z}_2) \neq 0$, which is absurd since $H_1 = G$. Thus $\dim H_2/H > 1$.

From this in turn it follows that $H^1(G/H, k) = 0$ for all fields k. For if $H^1(G/H, k) \neq 0$, then, by Poincaré duality (see 3.4 (3)),

$$H^{d-1}(G/H, k) \neq 0,$$

and by (*) above also $H^{d-1}(G/H_2, k) \neq 0$, hence $\dim H_2/H = 1$, a contradiction. In particular, $\pi_1(G/H)$ must be a perfect group by the theorem of Hurewicz, i.e., a group which is equal to its commutator subgroup. It now follows from the exact sequence in homotopy for the fibration

$$H \longrightarrow G \longrightarrow G/H$$

that $\pi_0(H)$ is a perfect group as well, hence so is $\pi_0(H_2)$, since H_2/H is connected and thus $\pi_0(H) \to \pi_0(H_2)$ is surjective. It is well known that therefore Poincaré duality holds for $H^*(G/H_2, k)$ (see e.g. [6, chap.III, lemma 2.3]). Using this as well as duality in $H^*(G/H, k)$, one easily concludes the following from formula (*) above. Let $m := \dim H_2/H$. Then m divides d and we have :

$$H^j(G/H_2, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } j = 0, m, 2m, \ldots, d - m, \\ 0 & \text{otherwise}; \end{cases}$$

$$H^j(G/H, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } j = 0, m, 2m, \ldots, d, \\ 0 & \text{otherwise}. \end{cases}$$

For the Euler characteristic we get

$$(1) \qquad\qquad \chi(G/H) = \chi(G/H_2) \pm 1,$$

depending on whether m is even or odd. On the other hand, we have

$$(2) \qquad\qquad \chi(G/H) = \chi(G/H_2) \cdot \chi(H_2/H),$$

as it follows from the multiplicative properties of χ for fibrations. It is well known that the Euler characteristic of a homogenous space is ≥ 0 (see [2, § 2]). From this and from equations (1) and (2) above one easily concludes that $\chi(G/H_2) = 1$. Again it is well known (loc. cit.) that this can only happen if $\operatorname{rank} H_2 = \operatorname{rank} G$. In [5, proof of 26.1] it is shown

that the space G/H_2^0 has cohomology only in even degrees, i.e., that $H^j(G/H_2^0, \mathbb{R}) = 0$ for odd j. By [6, chap.III, Thm. 2.1] the natural map

$$H^*(G/H_2, \mathbb{R}) \longrightarrow H^*(G/H_2^0, \mathbb{R})$$

is an inclusion, hence $H^j(G/H_2, \mathbb{R})$ has cohomology only in even degrees as well. But then $\chi(G/H_2) = 1$ can only happen if $H_2 = G$, since we know that at least $H^{\dim G/H_2}(G/H_2, \mathbb{R}) = \mathbb{R}$. $\square$

4.5. — We use a result of Bredon to show that in the fixed point case the generic orbit is in fact a homotopy sphere.

PROPOSITION. — *If there is a fixed point the generic fiber G/H is homotopy equivalent to a sphere.*

Proof. — Let $K \subset G$ be a maximal compact subgroup of G such that $L := K \cap H$ is a maximal compact subgroup of H. By 4.4 K/L is a $\mathbb{Z}$-cohomology sphere. By theorem 1.1 in [7], K/L is therefore either a sphere or the Poincaré sphere $SO_3(\mathbb{R})/I$, where I is the icosahedral subgroup of $SO_3(\mathbb{R})$. But if the latter were the case then the generic orbit of the slice representation of G at the fixed points would be $SO_3(\mathbb{C})/I$, which is impossible, because this homogenous space does not occur as generic orbit of a 4-dimensional representation with 1-dimensional quotient. $\square$

4.6 Remark. — If instead of requiring that H is connected we assume in 4.3 that both exceptional isotropy groups H_1 and H_2 are connected, then the spectral sequences of the fibrations

$$H_i/H \longrightarrow G/H \longrightarrow G/H_i$$

still have ordinary coefficients, and the proof of 4.3 shows that under this assumption the H_i/H are $\mathbb{Z}$-cohomology spheres as well. We can now proceed as in the proof of 4.5 to conclude that H_i/H are homotopy equivalent to spheres. In particular, H is connected if $\dim H_1/H > 1$ or $\dim H_2/H$ is > 1. Thus H being connected is equivalent to both H_1 and H_2 being connected if $\dim H_i/H > 1$ for $i = 1$ or $i = 2$.

4.7. — We can now prove the existence of a linear model in the fixed point case provided that the dimension n of the quadric X is > 2.

PROPOSITION. — *Assume $\dim X > 2$. If the given G-action on X has a fixed point then the action has a linear model.*

Proof. — By PROPOSITION 4.4 the two exceptional closed orbits are fixed points. By PROPOSITION 4.5 the generic fibers of the quotients of the slice representations, which are just the tangent space representations at the fixed points, are homotopy equivalent to spheres. It now follows

from PROPOSITION 2.2 and from $\dim X > 2$ that both of the slice representations are therefore equivalent to a fixed, orthogonal representation (W, G) with 1-dimensional quotient. Let Σ denote a 1-dimensional trivial G-module, and consider the G-module $V := W \oplus \Sigma$. This is clearly an orthogonal representation of G with 2-dimensional quotient. Let f be a quadratic homogenous invariant on W, and let z be a coordinate on Σ. Then $Q_V := \{(w, z) \mid f(w) + z^2 = 1\} \subset V$ is a G-invariant quadric such that the induced linear G-action on Q_V has the following properties : $Q_V /\!\!/ G \cong \mathbb{A}$, there are exactly two fixed points on Q_V, and the slice representations at the fixed points are equivalent to (W, G). Thus (V, G) is a linear model for the given action. $\quad\square$

4.8 Remark. — The reason for the assumption $\dim X > 2$ is that there are infinitely many non-equivalent representations on $\mathbb{C}^2$ inducing a transitive action on $\mathbb{C}^*$, see PROPOSITION 2.2. It is easy to see that if $\dim X = 2$ and $\dim X /\!\!/ G = 1$ then $G \cong \mathbb{C}^*$. Such an action must have two fixed points and a generic orbit isomorphic to $\mathbb{C}^*$, but the slice representations are not determined by these properties. However, one can show (FIESELER, personal communication) with topological arguments, using the first homology group at infinity, that X is isomorphic to a quadric if and only if the two slice representations are orthogonal, i.e., given by weights 1 and -1 after factoring out the kernel of the action. In this case the action has a linear model as in the last proposition.

4.9. — Using results of WANG [25] and ASOH [1] about compact group actions on spheres with orbit space dimension 1 we now deal with the case of connected principal isotropy groups H.

PROPOSITION. — *A linear model exists if H is connected and if the dimension of the slice representations is > 2.*

Proof. — By Remark 3.5 we can find maximal compact subgroups K, L, L_1 and L_2 in G, H, H_1 and H_2 respectively such that the following is satisfied : for an arbitrary field k and for $p = 1, \ldots, \dim K/L - 1$

$$H^p(K/L, k) = H^p(K/L_1, k) \oplus H^p(K/L_2, k).$$

Moreover, by 4.4 the spaces H_i/H are homotopy spheres, $i = 1, 2$. PROPOSITION 2.2 then shows that H_i/H must be a quadric, and consequently the spaces L_i/L are spheres. Precisely such quadruples K, L, L_1, L_2 of compact groups were classified by WANG in [25] and by ASOH in [1], who completed Wang's results. They studied compact group actions on spheres and $\mathbb{Z}_2$-cohomology spheres respectively. Each such action gives

rise to a quadruple as above, and the classsification of these actions is achieved through the classification of the corresponding quadruples. The assumption that the group L is connected is not necessary to get the results, which are summarized in [1, thm 6.1]. However, if L is connected as in our situation it follows that there is a linear action of K on $\mathbb{R}^{n+1}$ such that the induced action on the sphere $S^n \subset \mathbb{R}^{n+1}$ has 1-dimensional orbit space and isotropy groups L, L_1 and L_2, see [25, §13]. Consider the complexification of this action. It is a linear action of G on an affine quadric with isotropy groups H, H_1 and H_2 and with orthogonal slice representations. If the dimension of these representations is > 2 it follows from PROPOSITION 2.2 that they must be equivalent to the original ones given by the G-action on X, for which the linear action of G on the quadric is consequently a linear model. □

4.10 Remark. — Remark 4.6 shows that the last proposition remains valid if we assume that both exceptional isotropy groups are connected instead of assuming that the principal isotropy group is connected.

4.11 Remark. — If we drop the assumption about the dimension of the slice representations, we get a linear model in a weaker sense than that of definition 1.4. Indeed, if, say, the dimension of the slice representation of H_1 is 2, then $\dim H_1/H = 1$, and as already mentioned in the proof of 4.4 it follows that $H_1/H \cong \mathbb{C}^*$. Since H is connected, we have $H_1 = S' \times H'$ up to a finite covering, where $S' \cong \mathbb{C}^*$. The 2-dimensional slice representation is given by weights a and $-b$ for S', $a, b > 0$, and by the trivial action of H'. It then follows again from the results of Wang or by direct arguments using PROPOSITION 3.4 that, up to a finite covering, G is of the form $G = S \times H_2$, where $S \cong \mathbb{C}^*$.

Let (W_2, H_2) be the slice representation, and consider the G-representation $W_1 \oplus W_2$, where the 2-dimensional representation W_1 is given by the same weights a and $-b$ as above for S and by the trivial action of H_2, and where S acts trivially on W_2. It is easy to see that if x and y are coordinates on W_1, and if f is a homogenous generator of the G-invariants on W_2, then the restriction of the linear G-action on $W_1 \oplus W_2$ to the invariant affine smooth subvariety Q'_n given as the zero set of the polynomial $x^b y^a + f - 1$ has the following properties : the quotient is isomorphic to the affine line $\mathbb{A}$, the principal isotropy group is H, and there are exactly two exceptional closed orbits, whose corresponding isotropy groups are H_1 and H_2 respectively. Thus the linear action of G on Q'_n has the same local data as the given action on X. If $a = b$ and if W_2 is an orthogonal G-module (which it is if its dimension is > 2), then this construction gives a linear model in the sense of 1.4, see also Remark 5.3.

In connection with the question of whether or not the G-actions on X and Q'_n are equivalent the following problem arises. For which positive integers a and b is the variety

$$Q'_m(a,b) := \left\{ (x,y,z_1,\ldots,z_m) \in \mathbb{C}^{m+2} \mid x^b y^a + \sum_{i=1}^{m} z_i^2 = 1 \right\} \subset \mathbb{C}^{m+2}$$

isomorphic to a quadric? For example, the homological results mentioned in Remark 4.8 imply that $Q'_1(a,b)$ is isomorphic to the quadric Q_2 only if $a = b = 1$.

5. Classification of linear models

5.1. — According to the definition in 1.4, to obtain a list of all possible linear models in our situation we have to classify all orthogonal representations (V,G) of connected reductive groups G such that $\dim V /\!/ G = 2$. We will consider almost faithful actions, i.e., actions with a finite kernel.

PROPOSITION. — *If (V,G) is an almost faithful orthogonal representation with 2-dimensional quotient, then (V,G) is equivalent to one of the representations in Tables 2–5.*

Proof. — Since (V,G) is orthogonal, we can find a maximal compact subgroup $K \subset G$ and a real representation (W,K) such that (V,G) is the complexification of $(W,K) : (V,G) = (W_\mathbb{C}, K_\mathbb{C})$ (see [22, Prop. 5.7]). Suppose first that W is a reducible K-representation : $W = U \oplus T$. Then $V = U_\mathbb{C} \oplus T_\mathbb{C}$ as G-representation, where both $U_\mathbb{C}$ and $T_\mathbb{C}$ are orthogonal with 1-dimensional quotient. Such representations are, up to adding trivially acting factors to G, listed in Table 1 in 2.2. One can assume that $G = G_1 \times \cdots \times G_s \times S$ where $S \cong (\mathbb{C}^*)^r$. The simplest possibility for a representation $(V = U_\mathbb{C} \oplus T_\mathbb{C}, G)$ as above is now that we can arrange the factors of G into two groups G'_1 and G'_2 such that the following holds : $G = G'_1 \times G'_2$, $(U_\mathbb{C}, G'_1)$ and $(T_\mathbb{C}, G'_2)$ are from Table 1, G'_1 acts trivially on $T_\mathbb{C}$ and G'_2 acts trivially on $U_\mathbb{C}$. All such representations are listed in Table 2. Furthermore, there are some «mixed» cases, i.e., cases where the effective part of G on $U_\mathbb{C}$ doesn't act trivially on $T_\mathbb{C}$. This corresponds to arranging the factors of G into two groups G'_1 and G'_2 as above, but these groups now «overlap» in the sense that some factors of G'_1 are also factors of G'_2. However, since Table 1 is fairly short, it is easy to sort out these remaining possibilities using the fact that both representations $(U_\mathbb{C}, G)$ and $(T_\mathbb{C}, G)$ are orthogonal, which makes it easy to calculate principal isotropy groups and hence quotient dimensions.

The resulting representations are listed in Table 3. This concludes the case where (W, K) is reducible, so assume that W is irreducible as K-representation.

To do the classification in this case we use results from the theory of polar representations as worked out in [9] and [8]. First of all, it is easy to see that (V, G) is a polar representation. Indeed, if H, H_1 and H_2 are the principal and the two exceptional isotropy groups, and if $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the respective Lie algebras, then, as an H-representation, V is isomorphic to $\mathfrak{g}/\mathfrak{h} \oplus \Theta \oplus \Sigma$, where Θ and Σ are H_1-invariant (resp. H_2-invariant) 1-dimensional subspaces of V, and where all subspaces are mutually orthogonal. This follows from the slice theorem and the fact that V is an orthogonal representation. If $0 \neq \theta \in \Theta$ and $0 \neq \sigma \in \Sigma$, we have, since $(Xv, w) = -(v, Xw)$ for all $X \in \mathfrak{g}$ and $v, w \in V$,

$$\mathfrak{g}.\theta = \mathfrak{g}/\mathfrak{h}_1 \subset \mathfrak{g}/\mathfrak{h}, \quad \mathfrak{g}.\sigma = \mathfrak{g}/\mathfrak{h}_2 \subset \mathfrak{g}/\mathfrak{h}, \quad \mathfrak{g}.(\theta, \sigma) = \mathfrak{g}/\mathfrak{h},$$

and the claim follows by definition of polar representations. Now if (V, G) is irreducible, too, then G is semisimple. Up to castling transformations (see [17] for details), these representations must occur in the tables in [17]. These tables list for each castling class the lowest dimensional representation in this class. But by lemma 3 in [17], at most the lowest dimensional representation in each class is polar. Since we already know that the representations we are looking for are polar, they must occur in the tables of [17] if they are irreducible. This leads to Table 4. It remains to get the list of all reducible representations (V, G) for which (W, K) is irreducible. In this case, W admits a complex structure and gives an irreducible representation (W, G). Furthermore, we have

$$(V, G) = (W_{\mathbb{C}}, K_{\mathbb{C}}) = (W \oplus W^*, G)$$

(for details see [8]). Since (W, K) is polar and irreducible, we can use the tables in [8] and some easy dimension arguments to determine all the possibilities. In fact there are only four, and they appear in Table 5. This completes the proof of the proposition. $\quad[]$

5.2. — We want to explain the contents of the tables. Tables 2 to 5 contain the orthogonal representations of connected reductive groups with 2-dimensional quotient according to the distinctions made in the proof of 5.1 :

• Table 2 contains the representations of the form $(V = V_1 \oplus V_2, G = G_1 \times G_2)$, where (V_i, G_i), $i = 1, 2$, are orthogonal representations with 1-dimensional quotient (see Table 1 in section 2), and where G_1 acts trivially on V_2 and G_2 acts trivially on V_1.

- Table 3 contains the mixed cases, which are as above but without the assumption that G_1 and G_2 act trivially on V_2 and V_1 respectively.

- Tables 4 and 5 contain those representations which are complexifications of real irreducible representations. Table 4 contains the ones which remain irreducible after complexification, the others appear in Table 5.

In the tables G will denote the group that acts according to standard notation. In the column V we list the representation using the notation in [17] : ω_i denotes the irreducible representation corresponding to the fundamental weight ω_i, $n\omega_i$ the one with highest weight $n\omega_i$ (where $n \in \mathbb{N}$). Σ_a, $a \in \mathbb{Z}$, denotes the 1-dimensional representation of $\mathbb{C}^*$ with weight a. If G has more than one simple factor, the fundamental weights of the second factor (and the corresponding representation) are denoted by ω_i', the one of the third factor by ω_i''. ω_i^* denotes the dual representation of ω_i, etc. Under H, H_1 and H_2 we list the principal as well as the two exceptional isotropy groups.

In Table 2, (ω, M) and (ω', N) are representations from Table 1, Σ and Σ' denote 1-dimensional trivial representations of M and N respectively, and M' and N' denote the principal isotropy groups of (ω, M) and (ω', N).

5.3 Remark. — From the representations in Tables 2 and 3 containing a direct summand $\Sigma_a \oplus \Sigma_a^*$ one gets the linear models from Remark 4.11 by replacing this summand by a $\mathbb{C}^*$-representation of the form $\Sigma_a \oplus \Sigma_{-b}$ with $a, b > 0$.

Table 2					
G	V	H	H_1	H_2	
M	$\omega \oplus \Sigma$	M'	M	M	
$M \times N$	$\omega \otimes \Sigma' \oplus \Sigma \otimes \omega'$	$M' \times N'$	$M' \times N$	$M \times N'$	

Table 3

G		V	H	H_1	H_2
A_3		$\omega_1 \oplus \omega_1^* \oplus \omega_2$	A_1	A_2	C_2
$\mathbb{C}^* \times A_3$		$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^* \oplus \omega_2'$	$\mathbb{C}^* \times A_1$	$\mathbb{C}^* \times A_2$	$\mathbb{C}^* \times C_2$
B_3		$\omega_1 \oplus \omega_3$	A_2	G_2	A_3
D_4		$\omega_1 \oplus \omega_3$	G_2	B_3	B_3
$A_1 \times C_n$	(*)	$2\omega_1 \oplus (\omega_1 \otimes \omega_1')$	$\mathbb{C}^* \times C_{n-1}$	$A_1 \times C_{n-1}$	$\mathbb{C}^* \times C_n$
$A_1 \times C_n$	(*)	$\omega_1 \oplus \omega_1^* \oplus (\omega_1 \otimes \omega_1')$	C_{n-1}	$A_1 \times C_{n-1}$	C_n
$A_1 \times A_1 \times C_n$	(*)	$(\omega_1 \otimes \omega_1') \oplus (\omega_1' \otimes \omega_1'')$	$A_1 \times C_{n-1}$	$(A_1)^2 \times C_{n-1}$	C_n
$\mathbb{C}^* \times A_1 \times C_n$	(*)	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$ $\oplus (\omega_1' \otimes \omega_1'')$	$\mathbb{C}^* \times C_{n-1}$	$\mathbb{C}^* \times A_1 \times C_{n-1}$	$\mathbb{C}^* \times C_n$
$\mathbb{C}^* \times A_n$	(*)	$\Sigma_a \oplus \Sigma_a^* \oplus (\Sigma_a \otimes \omega_1')$ $\oplus (\Sigma_a \otimes \omega_1')^*$	A_{n-1}	$\mathbb{C}^* \times A_{n-1}$	A_n
$\mathbb{C}^* \times A_n \times A_m$	(*)	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$ $\oplus (\Sigma_a \otimes \omega_1'') \oplus (\Sigma_a \otimes \omega_1'')^*$	$\mathbb{C}^* \times A_{n-1} \times A_{m-1}$	$\mathbb{C}^* \times A_{n-1} \times A_m$	$\mathbb{C}^* \times A_n \times A_{m-1}$
$\mathbb{C}^* \times C_n$	(**)	$\Sigma_a \oplus \Sigma_a^* \oplus (\Sigma_a \otimes \omega_1')$	C_{n-1}	$\mathbb{C}^* \times C_{n-1}$	C_n
$\mathbb{C}^* \times C_n \times C_m$	(**)	$(\Sigma_a \otimes \omega_1') \oplus (\Sigma_a \otimes \omega_1')^*$ $\oplus (\Sigma_a \otimes \omega_1'') \oplus (\Sigma_a \otimes \omega_1'')^*$	$\mathbb{C}^* \times C_{n-1} \times C_{m-1}$	$\mathbb{C}^* \times C_{n-1} \times C_{m-1}$	$\mathbb{C}^* \times C_n \times C_{m-1}$

(*) indices ≥ 1, (**) indices ≥ 2

Table 4				
G	V	H	H_1	H_2
A_1	$4\omega_1$	$\mathcal{D}_4$	$\mathbb{C}^* \rtimes \mathbb{Z}_2$	$\mathbb{C}^* \rtimes \mathbb{Z}_2$
A_2	$\omega_1\omega_2$	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$
B_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$
D_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$
C_3	ω_2	$(A_1)^3$	$C_2 \times A_1$	$A_1 \times C_2$
F_4	ω_1	D_4	B_4	B_4
G_2	ω_2	$\mathbb{C}^* \times \mathbb{C}^*$	$A_1 \times \mathbb{C}^*$	$\mathbb{C}^* \times A_1$
$A_1 \times A_1$	$3\omega_1 \otimes \omega_1'$	$\mathcal{D}_4$	$\mathbb{C}^* \rtimes \mathbb{Z}_2$	$\mathbb{C}^* \rtimes \mathbb{Z}_2$
$C_2 \times C_m$ $(m \geq 2)$	$\omega_1 \otimes \omega_1'$	$C_{m-2} \times (A_1)^2$	$C_{m-2} \times C_2$	$C_{m-1} \times (A_1)^2$

Table 5				
G	V	H	H_1	H_2
$\mathbb{C}^* \times \mathrm{SO}_n$ $(n \geq 3)$	$(\Sigma_a \otimes \omega_1')$ $\oplus(\Sigma_a \otimes \omega_1')^*$	$\mathrm{SO}_{n-2} \times \mathbb{Z}_2$	$\mathrm{SO}_{n-2} \times \mathbb{C}^*$	$\mathrm{SO}_{n-1} \rtimes \mathbb{Z}_2$
D_5	$\omega_4 \oplus \omega_5$	A_3	A_4	B_3
A_4	$\omega_2 \oplus \omega_2^*$	$A_1 \times A_1$	$A_2 \times A_1$	C_2
$\mathbb{C}^* \times A_4$	$(\Sigma_a \otimes \omega_2')$ $\oplus(\Sigma_a \otimes \omega_2')^*$	$\mathbb{C}^* \times A_1 \times A_1$	$\mathbb{C}^* \times A_1 \times A_2$	$\mathbb{C}^* \times C_2$

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