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THE EXISTENCE AND THE CONTINUATION OF HOLOMORPHIC SOLUTIONS FOR CONVOLUTION EQUATIONS IN TUBE DOMAINS

BY

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RÉSUMÉ. — Pour une hyperfonction $\mu(x)$ à support compact, on considère l'équation de convolution (E) $\mu * f = g$ avec des fonctions holomorphes f et g définies sur un domaine de la forme $U \times \sqrt{-1}\,\mathbb{R}^n$. Sous une condition naturelle, dite condition (S), on démontre l'existence de solution de (E) et le prolongement de solution de l'équation homogène $\mu * f = 0$. En définissant l'ensemble caractéristique $\mathrm{Char}(\mu *)$, ceci entraîne que les directions auxquelles une solution ne se prolonge pas sont estimées par $\mathrm{Char}(\mu *)$.

ABSTRACT. — For a hyperfunction $\mu(x)$ with compact support, we consider the convolution equation (E) $\mu * f = g$ with holomorphic functions f and g defined on a domain of the form $U \times \sqrt{-1}\,\mathbb{R}^n$. Under a natural condition (the condition (S)), we will prove the existence of solution of (E) and the analytic continuation of homogeneous equation $\mu * f = 0$. Defining the characteristic set $\mathrm{Char}(\mu *)$, these imply that directions to which a solution cannot be continued are estimated by $\mathrm{Char}(\mu *)$.

0. Introduction

The convolution equation has been historically studied by many authors, especially the equation in the category of holomorphic functions defined on a complex domain. There are two most principal problems for those equations. One is the existence of holomorphic solutions, and the

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other is the continuation of holomorphic solutions of the homogeneous equation.

The existence of holomorphic solutions in a complex domain were first considered by MALGRANGE [26] and the case when $n = 1$ was completely studied by KOROBEĬNIK [21]-[23], EPIFANOV [7]-[9] and TKAČENKO [35], there they found the important connection between the existence and the property of *completely regular growth* of the *symbol* of the operator. In these directions, we cite also interesting works of NAPALKOV [32], MORZHAKOV [29]-[31] and LELONG-GRUMAN [24]. In the case of differential operators of infinite order with holomorphic coefficients, AOKI [2] gives us complete information about the local existence. And in the category of hyperfunctions or Fourier-hyperfunctions, we have fundamental works of KAWAI [19] and OKADA [33] for convolution operators (see also KANEKO [17]). There are also series of results of MEISE and VOGT for convolution equations in the categories of functions of Gevrey classes and ultra-distributions (see e.g. FRANKEN-MEISE [11]) and works of MEISE and MOMM concerning with the right inverse of convolution operators (see e.g. MOMM [28]).

The continuation of holomorphic solutions of a homogeneous equation were considered by ZERNER [37] and KISELMAN [20] for finite order differential equations and by SÉBBAR [34], VIDRAS [36] and BERENSTEIN-GAY-VIDRAS [3] for infinite order case with constant coefficients, and by AOKI [2] for infinite order case with general variable coefficients.

For these two problems for convolution equations, we also refer to BONY-SCHAPIRA [6], MÉRIL-STRUPPA [27] and BERENSTEIN-STRUPPA [4] and [5].

In the present paper, we consider the convolution operator $\mu *$ defined by a hyperfunction $\mu(x)$ with compact support, operating on holomorphic functions in tube domains invariant by any real translation. In all what follows we impose the natural condition called (S) introduced by KAWAI [19]. This condition in the $n = 1$ case is in fact equivalent to the property of completely regular growth in pure-imaginary directions. First we will prove under this condition (S), the existence of solutions in any open tube domain. And conversely, we will also prove that the existence of solutions in some special tube domain implies the condition (S), that means (S) is sufficient and almost necessary condition for the existence. Secondly, we will define the characteristic set $\mathrm{Char}(\mu *)$ of $\mu *$ and prove that all solutions of homogeneous equation $\mu * f = 0$ are extended holomorphically to any non-characteristic direction. Finally, by these results, we will conclude under the condition (S) that the *micro-support* of the complex defined by the operator $\mu *$ is included in the characteristic set $\mathrm{Char}(\mu *)$.

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1. Problem and notations

In this paper we denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of holomorphic functions on $\mathbb{C}^n$ and identify $\mathbb{R}^n \times \sqrt{-1}\,\mathbb{R}^n$ with $\mathbb{C}^n$. Denoting the projection $\mathbb{R}^n \times \sqrt{-1}\,\mathbb{R}^n \to \sqrt{-1}\,\mathbb{R}^n$ by τ, we set

$$\mathcal{O}^\tau := R\tau_* \mathcal{O}_{\mathbb{C}^n} \tag{1.1}$$

which is, in fact, concentrated at degree 0, so is considered as a sheaf on $\sqrt{-1}\,\mathbb{R}^n$. Let $\mu(x)$ be a hyperfunction on $\mathbb{R}^n$ with compact support. Then the convolution operator $P := \mu *$ operates on the sheaf $\mathcal{O}^\tau$, and we consider the complex

$$\mathcal{S} : 0 \to \mathcal{O}^\tau \xrightarrow{\mu *} \mathcal{O}^\tau \to 0. \tag{1.2}$$

Now our problem is stated as follows.

PROBLEM 1.3. — *Estimate the micro-support* $\mathrm{SS}(\mathcal{S})$ *by «the characteristic set»* $\mathrm{Char}(P)$ *of* $P = \mu *$.

We refer to KASHIWARA-SCHAPIRA [18], for terminologies above.

For an analytic functional $T \in \mathcal{O}(\mathbb{C}^n)'$, we denote by $\widehat{T}(\zeta)$ its Fourier-Borel transform

$$\widehat{T}(\zeta) = \langle T, \mathrm{e}^{z \cdot \zeta} \rangle_z,$$

which is an entire function of exponential type satisfying the following estimate (the theorem of Polyà-Ehrenpreis-Martineau). If T is supported by a compact set $K \subset \mathbb{C}^n$, for every $\varepsilon > 0$, we can take a constant $C_\varepsilon > 0$ such that

$$\big|\widehat{T}(\zeta)\big| \le C_\varepsilon \exp\big(H_K(\zeta) + \varepsilon|\zeta|\big),$$

where $H_K(\zeta) := \sup_{z \in K} \mathrm{Re}\langle z, \zeta\rangle$ is the supporting function of K. In particular, if μ is a hyperfunction with compact support, its Fourier-Borel transform $\widehat{\mu}(\zeta)$ is of infra-exponential growth on $\sqrt{-1}\,\mathbb{R}^n$.

In the sequel, we often write as $\zeta = \xi + \sqrt{-1}\,\eta$ the decomposition into the real and the imaginary parts and denote for $R > 0$ and $\zeta_0 \in \mathbb{C}^n$ by $B(\zeta_0 \,; R)$ the open ball of center ζ_0 with radius R in $\mathbb{C}^n$.

In this paper, we suppose the following condition (S) due to T. KAWAI [19] for the entire function $f(\zeta) = \widehat{\mu}(\zeta)$.

$$\text{(S)} \qquad \begin{cases} \textit{For every } \varepsilon > 0, \textit{ there exists } N > 0 \textit{ such that for any} \\ \sqrt{-1}\,\eta \in \sqrt{-1}\,\mathbb{R}^n \textit{ with } |\eta| > N, \textit{ we can find } \zeta \in \mathbb{C}^n, \\ \textit{which satisfies } |\sqrt{-1}\eta - \zeta| < \varepsilon|\eta|,\ |f(\zeta)| \geq \mathrm{e}^{-\varepsilon|\eta|}. \end{cases}$$

In the next section, we will investigate the meaning of this condition (S).

2. A division lemma

First we prove the following lemma which ensures the division of analytic functionals under the condition (S).

LEMMA 2.1. — *Let f, g and h be entire functions satisfying $fg = h$, and M and K be two compact convex sets in $\mathbb{R}^n$ and in $\mathbb{C}^n$ respectively. We suppose that for every $\varepsilon > 0$, f and h satisfy the following estimates* (2.1) *and* (2.2) *with constants $A_\varepsilon > 0$ and $B_\varepsilon > 0$,*

$$\log\big|f(\zeta)\big| \leq A_\varepsilon + H_M(\zeta) + \varepsilon|\zeta|, \tag{2.1}$$

$$\log\big|h(\zeta)\big| \leq B_\varepsilon + H_K(\zeta) + \varepsilon|\zeta|. \tag{2.2}$$

We also assume that f satisfies the condition (S)*. Then for any $\varepsilon > 0$, there exists a compact set $L = L_\varepsilon \subset \mathbb{C}^n$ and $C_\varepsilon > 0$ such that*

$$\begin{cases} \tau(L) \subset \tau\big(K + B(0;\varepsilon)\big), \\ \log\big|g(\zeta)\big| \leq C_\varepsilon + H_L(\zeta). \end{cases} \tag{2.3}$$

Proof. — For any ε with $0 < \varepsilon < 1$, we set

$$\Gamma_\varepsilon := \Big\{\zeta \in \mathbb{C}^n;\ \text{there exists } \sqrt{-1}\,\eta \in \sqrt{-1}\,\mathbb{R}^n \quad \text{such that } \big|\zeta - \sqrt{-1}\,\eta\big| < \varepsilon|\eta|\Big\}.$$

First we prove the estimate (2.3) for $\zeta \in \Gamma_\varepsilon$. By the condition (S), there exists $N > 0$ so that if $|\zeta| > N$, we can find $\zeta' \in \mathbb{C}^n$ satisfying $|\zeta' - \sqrt{-1}\eta| < \varepsilon|\eta|$ and $|f(\zeta')| \geq \mathrm{e}^{-\varepsilon|\eta|}$. We recall a lemma of HARNACK-MALGRANGE-HÖRMANDER [12, lemma 3.1].

LEMMA 2.2. — *Let $F(\zeta)$, $H(\zeta)$ and $G(\zeta) = H(\zeta)/F(\zeta)$ be three holomorphic functions in the open ball $B(0;R)$. If the inequalities $|F(\zeta)| < A$ and $|H(\zeta)| < B$ hold on $B(0;R)$, then the estimate*

$$\big|G(\zeta)\big| \leq B \cdot A^{\frac{2|\zeta|}{R-|\zeta|}} \cdot \big|F(0)\big|^{-\frac{R+|\zeta|}{R-|\zeta|}}$$

holds for all $\zeta \in B(0;R)$.

We apply this lemma to the ball $B(\zeta'\,;3\varepsilon|\eta|)$. By (2.2), we have

$$\sup_{\zeta''\in B(\zeta';3\varepsilon|\eta|)} \log\bigl|h(\zeta'')\bigr| \le B_\varepsilon + \sup_{|\zeta''-\zeta'|\le 3\varepsilon|\eta|} \bigl(H_K(\zeta'')+\varepsilon|\zeta''|\bigr).$$

Since $|\zeta''-\zeta'|\le 3\varepsilon|\eta|$ implies $|\zeta''-\sqrt{-1}\,\eta|\le 4\varepsilon|\eta|$, setting

$$k := \sup_{z\in K}|z|,$$

the right hand side is estimated by

$$\begin{aligned} B_\varepsilon + \sup_{|\zeta''-\sqrt{-1}\,\eta|\le 4\varepsilon|\eta|} &\bigl(H_K(\zeta''-\sqrt{-1}\,\eta)+\varepsilon|\zeta''-\sqrt{-1}\,\eta|\bigr)\\ &\qquad + H_K(\sqrt{-1}\,\eta)+\varepsilon|\eta|\\ &\le B_\varepsilon + 4(k+\varepsilon)\varepsilon|\eta| + H_K\bigl(\sqrt{-1}\,\eta\bigr)+\varepsilon|\eta|\\ &\le B_\varepsilon + (4k+5)\varepsilon|\eta| + H_K\bigl(\sqrt{-1}\,\eta\bigr). \end{aligned}$$

By the same way, by setting

$$m := \sup_{x\in M}|x|$$

and noting that $M\subset\mathbb{R}^n$, using (2.1) we have :

$$\begin{aligned} \sup_{\zeta''\in B(\zeta';3\varepsilon|\eta|)} \log\bigl|f(\zeta'')\bigr| &\le A_\varepsilon + (4m+5)\varepsilon|\eta| + H_M\bigl(\sqrt{-1}\,\eta\bigr)\\ &= A_\varepsilon + (4m+5)\varepsilon|\eta|. \end{aligned}$$

Remarking that $\zeta\in B(\zeta'\,;2\varepsilon|\eta|)$ (which implies $\zeta\in B(\zeta'\,;3\varepsilon|\eta|)$), by the LEMMA 2.2, we have :

$$\begin{aligned} \log\bigl|g(\zeta)\bigr| &\le B_\varepsilon + (4k+5)\varepsilon|\eta| + H_K\bigl(\sqrt{-1}\,\eta\bigr)\\ &\qquad + \frac{4\varepsilon|\eta|}{3\varepsilon|\eta|-2\varepsilon|\eta|}\bigl(A_\varepsilon+(4m+5)\varepsilon|\eta|\bigr) + \frac{3\varepsilon|\eta|+2\varepsilon|\eta|}{3\varepsilon|\eta|-2\varepsilon|\eta|}\varepsilon|\eta|\\ &= B_\varepsilon + 4A_\varepsilon + H_K\bigl(\sqrt{-1}\,\eta\bigr) + (4k+16m+30)\varepsilon|\eta|. \end{aligned}$$

So for any $\varepsilon>0$, we can find a constant $C'_\varepsilon>0$ such that

$$\log\bigl|g(\zeta)\bigr| \le C'_\varepsilon + H_K\bigl(\sqrt{-1}\,\eta\bigr)+\varepsilon|\eta|$$

for $\zeta\in\Gamma_\varepsilon$.

Now we prove the estimate (2.3) for all $\zeta \in \mathbb{C}^n$. Recall that if the quotient of two entire functions of exponential type is entire, then it is also of exponential type (see for example AVANISSIAN [1] or ISHIMURA [14, Lemma 8]). Thus g satisfies the estimate, for constants $\sigma > 0$ and $C > 0$,

$$\log\bigl|g(\zeta)\bigr| \leq \sigma|\zeta| + C.$$

We remark that in the case $n = 1$, by the Phragmén-Lindelöf principle and by the estimate in Γ_ε, we have the estimate (2.3) for all $\zeta \in \mathbb{C}$. But in the general case when $n \geq 1$, we prove it for $\zeta = \xi + \sqrt{-1}\,\eta \notin \Gamma_\varepsilon$ directly. So we may suppose $|\xi| \geq \varepsilon|\eta|$ and this implies :

$$|\zeta| \leq |\xi| + |\eta| \leq \Bigl(1 + \frac{1}{\varepsilon}\Bigr)|\xi|.$$

Then we can take another constant $\sigma_\varepsilon > 0$ which satisfies

$$\log\bigl|g(\zeta)\bigr| \leq \sigma_\varepsilon|\xi| + C.$$

Finally put

$$L_\varepsilon := \text{ the convex hull of } \bigl(K + B(0\,;\varepsilon)\bigr) \cup \bigl\{\xi \in \mathbb{R}^n\,;\ |\xi| \leq \sigma_\varepsilon\bigr\},$$

$$C_\varepsilon := \max(C'_\varepsilon, C),$$

and we conclude the desired estimate (2.3).

3. The existence of holomorphic solutions

For the first cohomology group of the complex $\mathcal{S}$, we will prove the surjectivity theorem under the condition (S).

THEOREM 3.1. — *Let $\mu(x)$ be a hyperfunction with compact support. Assume that $\widehat{\mu}(\zeta)$ satisfies the condition* (S). *Then for any open set $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\mathbb{R}^n$, the operator*

$$\mu* : \mathcal{O}^\tau\bigl(\sqrt{-1}\,\omega\bigr) \longrightarrow \mathcal{O}^\tau\bigl(\sqrt{-1}\,\omega\bigr)$$

is surjective.

Proof. — The transpose of

$$P := \mu* : \mathcal{O}^\tau\bigl(\sqrt{-1}\,\omega\bigr) \longrightarrow \mathcal{O}^\tau\bigl(\sqrt{-1}\,\omega\bigr)$$

is

$$ {}^tP = \check{\mu}* : \mathcal{O}\big(\mathbb{R}^n \times \sqrt{-1}\,\omega\big)' \longrightarrow \mathcal{O}\big(\mathbb{R}^n \times \sqrt{-1}\,\omega\big)' $$

with $\check{\mu}(\zeta) = \mu(-\zeta)$. By the standard argument, it is enough to prove that ${}^tP$ is injective and that the image of ${}^tP$ is (weakly) closed. In fact, the injectivity of ${}^tP$ shows that the image of P is dense, and the closedness of the image of ${}^tP$ shows the closedness of the image of P. Because μ is not 0, the injectivity of ${}^tP$ is clear. Then we will show that the image of ${}^tP$ is weakly closed.

Let (T_ν) be a sequence in $\mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$ and assume that $({}^tPT_\nu)$ converges to $S \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$ in $\mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$. By taking the Fourier-Borel transformation, $v_\nu(\zeta) := \widehat{\mu}(-\zeta)\widehat{T}_\nu(\zeta)$ converges to $\widehat{S}(\zeta)$.

Then $G(\zeta) := \widehat{S}(\zeta)/\widehat{\mu}(-\zeta)$ becomes an entire function (cf. MALGRANGE [26]). By LEMMA 2.1 and POLYÀ-EHRENPREIS-MARTINEAU (see [12, thm 4.5.3]), there exists $T \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$ such that $\widehat{T} = G$ and so $\check{\mu} * T = S$, i.e. $S \in$ the image of ${}^tP$. □

COROLLARY 3.2. — *$H^1(\mathcal{S}) = 0$ and so* $\mathrm{SS}(H^1(\mathcal{S})) = \emptyset$.

Conversely by means of FAVOROV [10] and MORZHAKOV [30] (see also LELONG-GRUMAN [24]), we also note that the condition (S) is the necessary condition for the surjectivity in the following sense.

THEOREM 3.3. — *Let $\sqrt{-1}\,\omega$ be a non-empty strictly convex bounded open domain with C^2-boundary in $\sqrt{-1}\,\mathbb{R}^n$. Then if*

$$ \mu* : \mathcal{O}^\tau\big(\sqrt{-1}\,\omega\big) \longrightarrow \mathcal{O}^\tau\big(\sqrt{-1}\,\omega\big) $$

is surjective, then $f(\zeta) := \widehat{\mu}(\zeta)$ satisfies the condition (S).

Proof. — Assume that $\widehat{\mu}(\zeta)$ does not satisfy the condition (S). Then by the compactness of $S := \{\sqrt{-1}\,\eta \in \sqrt{-1}\,\mathbb{R}^n\,;\ |\eta| = 1\}$, there exist $\varepsilon > 0$, $\sqrt{-1}\,\eta_0 \in \sqrt{-1}\,\mathbb{R}^n$ and the increasing sequence (t_j) of positive numbers tending to ∞ such that $|\eta_0| = 1$ and that

$$ |f(t_j\zeta)| < \mathrm{e}^{-\varepsilon t_j} \quad \text{if} \quad |\zeta - \sqrt{-1}\,\eta_0| < \varepsilon. \tag{3.1} $$

There is no loss of generality if we assume that t_j's also satisfy $2t_j < t_{j+1}$. From this, we derive that for any $\delta \in \,]0, \min(\frac{1}{3}\varepsilon, \frac{1}{36})[$,

$$ |f(t\zeta)| < \mathrm{e}^{-\frac{1}{2}\varepsilon t} \quad \text{if} \quad \zeta \in B\big(\sqrt{-1}\,\eta_0\,;\,\delta\big),\ \ t \in \big[(1-\delta)t_j, (1+\delta)t_j\big]. \tag{3.2} $$

Denoting $U = \mathbb{R}^n \times \sqrt{-1}\,\omega$, we may take $\sqrt{-1}\,y_0 \in \partial U = \mathbb{R}^n \times \sqrt{-1}\,\partial\omega$ and a non-empty ball $B(z_1, s) \subset U$ so that

$$ H_{\overline{U}}\big(\sqrt{-1}\,\eta_0\big) = \mathrm{Re}\big\langle\sqrt{-1}\,\eta_0, \sqrt{-1}\,y_0\big\rangle, $$

$$ \overline{B(z_1, s)} \cap \partial U = \big\{\sqrt{-1}\,y_0\big\} $$

because U is strictly convex and has C^2-boundary. By a translation we may also assume that $z_1 = 0$. In this situation, we have :

$$H_{\overline{B(0\,;s)}}(\zeta) \quad (= s|\zeta|) \quad \le H_{\overline{U}}(\zeta), \tag{3.3}$$

$$H_{\overline{B(0\,;s)}}(\zeta) < H_{\overline{U}}(\zeta) \quad \text{if} \quad \zeta \notin B\big(\sqrt{-1}\,\eta_0\,;\tfrac{1}{3}\delta\big),\ |\zeta| = 1. \tag{3.4}$$

Then there exists small $a > 0$ such that :

$$s|\zeta| + 4sa \le H_{\overline{U}}(\zeta) \quad \text{if} \quad \zeta \notin B\big(\sqrt{-1}\,\eta_0\,;\tfrac{1}{3}\delta\big),\ |\zeta| = 1. \tag{3.5}$$

Now we follow the argument of the theorem 7 of MORZHAKOV [30] (or it is possible to follow the argument of the Theorem 9.35 in LELONG-GRUMAN [24]). By the proof of theorem 6 in FAVOROV [10] (or MORZHAKOV [30, thm 5]), we can take an entire function $g(\zeta)$ on $\mathbb{C}^n$ satisfying with large $R_0 > 0$,

$$h_g^*(\zeta) \le (1+a)s|\zeta|, \tag{3.6}$$

$$h_g^*(\zeta) \ge s|\zeta| \quad \text{if} \quad \zeta \in B\big(\sqrt{-1}\,\eta_0\,;\tfrac{1}{12}\delta\big), \tag{3.7}$$

$$\begin{cases} \log\big|g(t\zeta)\big| \le (1-2a)s|\zeta|t \\ \text{if } \zeta \in B(\sqrt{-1}\,\eta_0\,;\delta), \quad t \notin \displaystyle\bigcup_{j=1}^{\infty}\big[(1-\delta)t_j,(1+\delta)t_j\big] \cup [0,R_0]. \end{cases} \tag{3.8}$$

Here $h_g^*(\zeta)$ is the regularized radial growth indicator, (see for example, LELONG–GRUMAN [24]). Taking $a < \dfrac{\delta\varepsilon}{288s}$, for every r with $\dfrac{1}{1+2a} < r \le 1$ we define entire function $\phi_r(\zeta) := g(r\zeta)$ in ζ. For any $\tau > 0$, there exists $R_0 = R_0(\tau) > 0$ such that

$$\frac{\log|\phi_r(t\zeta)|}{t} \le H_{\overline{B(0\,;sr)}}(\zeta) + asr + \tau \quad \text{if} \quad t \ge R_0,\ |\zeta| = 1. \tag{3.9}$$

Because we have $rB(\sqrt{-1}\,\eta_0\,;\frac{1}{3}\delta) \subset B(\sqrt{-1}\,\eta_0\,;\delta)$, using (3.8) there exists $R_1 > 0$ such that :

$$\frac{\log|\phi_r(t\zeta)|}{t} \le (1-2a)sr|\zeta| \quad \text{if } \zeta \in B\big(\sqrt{-1}\,\eta_0\,;\tfrac{1}{3}\delta\big), \tag{3.10}$$

$$t \notin \bigcup_{j=1}^{\infty}\big[(1-\delta)t_j,(1+\delta)t_j\big] \cup [0,R_1].$$

By the fact $\operatorname{supp}\mu = M$, there exists also $R_2 > 0$ such that :

$$\frac{\log|f(t\zeta)|}{t} \leq H_{-M}(\zeta) + \tau \quad \text{if} \quad t \geq R_2, \ |\zeta| = 1. \tag{3.11}$$

(I) Suppose $\zeta \notin B(\sqrt{-1}\,\eta_0\,;\,\frac{1}{3}\delta)$ with $|\zeta| = 1$ and

$$t \geq R := \max(R_0, R_1, R_2).$$

Then we have :

$$\begin{aligned} \frac{\log|f(t\zeta)\phi_r(t\zeta)|}{t} &\leq H_{-M}(\zeta) + (1+a)sr + \tau \\ &\leq H_{-M}(\zeta) + H_{\overline{B(0\,;sr)}}(\zeta) + asr + \tau. \end{aligned} \tag{3.12}$$

(II) Suppose $\zeta \in B(\sqrt{-1}\,\eta_0\,;\,\frac{1}{3}\delta)$ with $|\zeta| = 1$ and $t \geq R$.

(i) In addition, if $t \in \bigcup_{j=1}^{\infty}[(1-\delta)t_j, (1+\delta)t_j]$, then by (3.2), (3.5) and (3.9) we have

$$\begin{aligned} \frac{\log|f(t\zeta)\phi_r(t\zeta)|}{t} &\leq H_{\overline{B(0\,;sr)}}(\zeta) + asr - \tfrac{1}{2}\varepsilon + \tau \\ &\leq H_{\overline{B(0\,;sr)}}(\zeta) - 3asr + \tau \end{aligned} \tag{3.13}$$

because $as < \delta\varepsilon/288 < \frac{1}{8}\varepsilon$.

(ii) If $t \notin \bigcup_{j=1}^{\infty}[(1-\delta)t_j, (1+\delta)t_j]$, then by (3.10) and (3.11), we have :

$$\begin{aligned} \frac{\log|f(t\zeta)\phi_r(t\zeta)|}{t} &\leq H_{-M}(\zeta) + (1-2a)sr + \tau \\ &\leq H_{-M} + H_{\overline{B(0\,;sr)}}(\zeta) - 2asr + \tau. \end{aligned} \tag{3.14}$$

Finally setting $\tau = as$, by (3.12), (3.13) and (3.14) we have :

$$\frac{\log|f(t\zeta)\phi_r(t\zeta)|}{t} \leq H_{-M}(\zeta) + (1-2a)sr + \tau \qquad \text{if} \ \ t \geq R, \ \ |\zeta| = 1. \tag{3.15}$$

Now we are ready to prove that $\mu* : \mathcal{O}^\tau(\sqrt{-1}\,\omega) \to \mathcal{O}^\tau(\sqrt{-1}\,\omega)$ is not surjective. It is enough to prove that the image of $\check{\mu} : \mathcal{O}(U)' \to \mathcal{O}(U)'$ is not closed : for any $r > 0$ with $\dfrac{1}{1+2a} < r \leq 1$, there is $v_r \in \mathcal{O}(U)'$ so that $\hat{v}_r = \phi_r(\zeta)$. By (3.9) and (3.15), if $r < 1$, we know $v_r, \check{\mu} * v_r \in \mathcal{O}(U)'$.

By (3.15) we have $\check{\mu} * v_r \to \check{\mu} * v_1$ in $\mathcal{O}(U)$ when $r \to 1$. But by (3.7) we have :

$$\log\left|\hat{v}_1(\sqrt{-1}\,\eta_0)\right| \geq tH_{\overline{B(0\,;sr)}}(\sqrt{-1}\,\eta_0) = t\,\mathrm{Re}\langle\sqrt{-1}\,y_0, \sqrt{-1}\,\eta_0\rangle.$$

If $L \subset U$ is a compact convex set supporting v_1, we have for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\log\left|\hat{v}_1(t\sqrt{-1}\,\eta_0)\right| \leq C_\varepsilon + H_{L+\overline{B(0\,;\varepsilon)}}(t\sqrt{-1}\,\eta_0),$$

so we have

$$\frac{C_\varepsilon}{t} + H_{L+\overline{B(0\,;\varepsilon)}}(\sqrt{-1}\,\eta_0) \leq \mathrm{Re}\langle\sqrt{-1}\,y_0, \sqrt{-1}\,\eta_0\rangle \tag{3.16}$$

for any $t \gg 1$. But by taking ε so small that $L + \overline{B(0\,;\varepsilon)} \Subset U$ and taking t large enough, as $\sqrt{-1}\,y_0 \in \partial U$ this gives a contradiction. So v_1 is not in $\mathcal{O}(U)'$ i.e. $\check{\mu} * v_1$ is not an element of the image of $\check{\mu} : \mathcal{O}(U)' \to \mathcal{O}(U)'$ while $\check{\mu} * v_r$ converges to $\check{\mu} * v_1$ in $\mathcal{O}(U)'$. This means that the image of $\check{\mu} : \mathcal{O}(U)' \to \mathcal{O}(U)'$ is not closed. $\square$

PROPOSITION 3.4. — *In the case $n = 1$, the condition (S) is nothing but to say that $f(\zeta)$ is of (completely) regular growth (see for example* LEVIN [25]) *in the pure imaginary directions $\sqrt{-1}\,\mathbb{R}^1$.*

Proof. — In fact if $f(\zeta)$ is of regular growth, then it is easy to see that $f(\zeta)$ satisfies (S). Conversely, if $f(\zeta)$ satisfies (S), then by the THEOREM 3.1, for any bounded open interval $\sqrt{-1}\,\omega \Subset \sqrt{-1}\,\mathbb{R}^1$, $\mu* : \mathcal{O}^\tau(\sqrt{-1}\,\omega) \to \mathcal{O}^\tau(\sqrt{-1}\,\omega)$ is surjective. But by the proof of the theorem 9.35 of LELONG-GRUMAN [24], we can conclude that $f(\zeta)$ is of regular growth (in their meanings, but in the case $n = 1$ they coincide with the classical one).

Remark 3.5. — In the case $n \geq 1$, we can verify that the property of the regular growth in the pure imaginary directions $\sqrt{-1}\,\mathbb{R}^n$ of LELONG-GRUMAN sense [24] or of the sense of AGRANOVICH-RONKIN (see FAVOROV [10]) implies the condition (S), but we do not know whether the converse is true.

4. The characteristic set and the continuation of homogeneous solutions

For the 0-th cohomology group of the complex $\mathcal{S}$, under the condition (S) for $\mu \in \mathcal{B}_c(\mathbb{R}^n)$ a hyperfunction with compact support, we shall now solve the problem of continuation for $\mathcal{O}^\tau$-solutions of the homogeneous equation $\mu * g = 0$ by the method of KISELMAN [20]–SÉBBAR [34]. We denote the sheaf of $\mathcal{O}^\tau$-solutions by $\mathcal{N}$, that is, for any open set $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\mathbb{R}^n$, we set :

$$\mathcal{N}\big(\sqrt{-1}\,\omega\big) := \big\{g \in \mathcal{O}^\tau(\sqrt{-1}\,\omega)\,;\ \mu * g = 0\big\}.$$

For an open set $\sqrt{-1}\,\Omega \subset \sqrt{-1}\,\mathbb{R}^n$ with $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\Omega$, the problem is formulated as to get the condition so that the restriction map

$$r : \mathcal{N}\big(\sqrt{-1}\,\Omega\big) \longrightarrow \mathcal{N}\big(\sqrt{-1}\,\omega\big)$$

is surjective.

We note that the subspace $\mathcal{N}(\sqrt{-1}\,\omega)$ of the (FS) space $\mathcal{O}^\tau(\sqrt{-1}\,\omega)$, endowed with the induced topology is a closed subspace. We have :

PROPOSITION 4.1. — *Let $\sqrt{-1}\,\omega$ and $\sqrt{-1}\,\Omega$ be two open sets of $\sqrt{-1}\,\mathbb{R}^n$ with $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\Omega$. Assume that μ satisfies (S), then the image $r(\mathcal{N}(\sqrt{-1}\,\Omega))$ of the restriction map r is dense in $\mathcal{N}(\sqrt{-1}\,\omega)$.*

Proof. — Let E be the space of exponential-polynomial solutions and E° its polar set in $\mathcal{O}(\mathbb{C}^n)'$, that is,

$$E := \Big\{g(z) = \sum_{j=1}^{\ell} p_j(z)\,\mathrm{e}^{\alpha_j\cdot z}\,;\ \mu * g = 0,\ p_j\text{'s are polynomials in } z \text{ and } \alpha_j\text{'s are constants}\Big\},$$

$$E^\circ := \big\{T \in \mathcal{O}(\mathbb{C}^n)'\,;\ \langle T, g\rangle = 0 \text{ for any } g \in E\big\}.$$

We shall show that E is dense in $\mathcal{N}(\sqrt{-1}\,\omega)$. To do this, it is sufficient, by Hahn-Banach's theorem, to prove that any functional

$$T \in E^\circ \cap \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$$

is orthogonal to $\mathcal{N}(\sqrt{-1}\,\omega)$. With regard to functionals in E°, we quote the following lemma.

LEMMA 4.2. (B. MALGRANGE [26]). — *$T \in \mathcal{O}(\mathbb{C}^n)'$ belongs to E° if and only if there exists an entire function $s(\zeta)$ satisfying*

$$\widehat{T}(\zeta) = s(\zeta)\widehat{\mu}(-\zeta).$$

We note that the above s in the lemma is of exponential type. Moreover if T is in $\mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$, then by the LEMMA 2.1, we can take a functional $S \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$ with $s(\zeta) = \widehat{S}(\zeta)$. Finally we get the desired result,

$$\langle T, g\rangle = \langle \check{\mu} * S, g\rangle = \langle S, \mu * g\rangle = 0, \text{ for any } \ g \in \mathcal{N}\big(\sqrt{-1}\,\omega\big),$$

where we have used the convention $\check{\mu}(\zeta) = \mu(-\zeta)$. □

In order to describe the theorem of continuation, we will prepare the notion of characteristics which is a natural generalization of the case of usual differential operators of finite order with constant coefficients. We define the sphere at infinity S^{2n-1}_∞ by $(\mathbb{C}^n \setminus \{0\})/\,\mathbb{R}_+$ and consider the compactification with directions $\mathbb{D}^{2n} = \mathbb{C}^n \sqcup S^{2n-1}_\infty$ of $\mathbb{C}^n$. For $\zeta \in \mathbb{C}^n \setminus \{0\}$, we write $\zeta\infty \in S^{2n-1}_\infty$ the class represented by ζ, i.e.

$$\{\zeta\infty\} = \big(\text{the closure of } \{t\zeta\,;\ t > 0\} \text{ in } \mathbb{D}^{2n}\big) \cap S^{2n-1}_\infty.$$

We denote by $\sqrt{-1}\,S^{n-1}_\infty$ the pure imaginary sphere at infinity

$$\big\{(\xi + \sqrt{-1}\,\eta)\infty \in S^{2n-1}_\infty\,;\ \xi = 0\big\},$$

which is a closed subset of S^{2n-1}_∞.

For a hyperfunction μ with compact support, using the terms of the modulus of the Fourier-Borel transform $f = \widehat{\mu}$ of μ, we define the characteristics $\mathrm{Char}_\infty(\mu*)$ as follows. For $\varepsilon > 0$ we set :

$$V_f(\varepsilon) := \big\{\zeta \in \mathbb{C}^n\,;\ \mathrm{e}^{\varepsilon|\zeta|}|f(\zeta)| < 1\big\},$$

$$W_f(\varepsilon) := \sqrt{-1}\,S^{n-1}_\infty \cap (\text{the closure of } V_f(\varepsilon) \text{ in } \mathbb{D}^{2n}).$$

Now we define the characteristic set of $\mu*$.

Definition 4.3. — With the above notation, we define the *characteristics* of $\mu*$ (at infinity)

$$\mathrm{Char}_\infty(\mu*) := \text{ the closure of } \bigcup_{\varepsilon>0} W_f(\varepsilon),$$

a closed set in $\sqrt{-1}\,S^{n-1}_\infty$.

We remark that if $P = \mu *$ defines a finite order differential operator with constant coefficients, the characteristic set of $\mu *$ coincides with the usual characteristics of P defined as zeros of its principal symbol. Even if P is a differential operator of infinite order, it can be shown by a similar argument to the proof of LEMMA 2.1, that $\mathrm{Char}_\infty(\mu *)$ coincides with accumulating directions of zeros of the total symbol. (See for example T. KAWAI [19].)

We note that the direction $\sqrt{-1}\,\rho\infty \in \sqrt{-1}\,S^{n-1}_\infty$ does not belong to $\mathrm{Char}_\infty(\mu *)$ if and only if for any $\varepsilon > 0$, there exist a conical neighborhood $\Gamma \subset \mathbb{C}^n$ of $\sqrt{-1}\,\rho$ and a positive N such that $f(\zeta)$ satisfies the following estimate on $\Gamma \cap \{|\zeta| > N\}$:

$$|f(\zeta)| > \mathrm{e}^{-\varepsilon|\zeta|} \tag{4.1}$$

For this $\mathrm{Char}_\infty(\mu *)$ and an open convex set $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\mathbb{R}^n$, now we put

$$\sqrt{-1}\,\Omega := \text{ the interior of } \bigcap_{\sqrt{-1}\,\eta\infty\in\mathrm{Char}_\infty(\mu *)^a} \Big\{\sqrt{-1}\,y \in \sqrt{-1}\,\mathbb{R}^n\,; \; - y\cdot\eta \le H_{\sqrt{-1}\,\omega}(\sqrt{-1}\,\eta)\Big\},$$

here «$\cdot^a$» means the antipodal. Then $\sqrt{-1}\,\Omega$ becomes an open convex subset in $\sqrt{-1}\,\mathbb{R}^n$ containing $\sqrt{-1}\,\omega$. Moreover for any compact set L in $\mathbb{R}^n \times \sqrt{-1}\,\Omega$, we can take compact set K in $\mathbb{R}^n \times \sqrt{-1}\,\omega$ such that

$$H_L(\sqrt{-1}\,\eta) \le H_K(\sqrt{-1}\,\eta) \quad \text{for any } \sqrt{-1}\,\eta\infty \in \mathrm{Char}_\infty(\mu *)^a. \tag{4.2}$$

Under the above situation we are ready to state the main theorem of this section.

THEOREM 4.4. — *Let μ be a hyperfunction with compact support, and a pair of open sets $\sqrt{-1}\,\omega \subset \sqrt{-1}\,\Omega$ as above. Assume that $\widehat{\mu}$ satisfies the condition* (S). *Then the restriction*

$$r : \mathcal{N}(\sqrt{-1}\,\Omega) \longrightarrow \mathcal{N}(\sqrt{-1}\,\omega)$$

is surjective.

Proof. — Remarking that the restriction $r : \mathcal{N}(\sqrt{-1}\,\Omega) \to \mathcal{N}(\sqrt{-1}\,\omega)$ has a dense image, the transpose ${}^t r : \mathcal{N}(\sqrt{-1}\,\Omega)' \leftarrow \mathcal{N}(\sqrt{-1}\,\omega)'$ is injective and we must show only its surjectivity. By Hahn-Banach's theorem, any functional $T \in \mathcal{N}(\sqrt{-1}\,\Omega)'$ has an extension

$$T_1 \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\Omega)',$$

and we prove that we can choose $R \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$ so that T_1 and R coincide on $\mathcal{N}(\sqrt{-1}\,\Omega)$. To do this, we use the following division lemma.

LEMMA 4.5. — *Let L and K be a pair of compact subsets of $\mathbb{C}^n$ satisfying the estimate* (4.2) *on* $\mathrm{Char}_\infty(\mu*)$ *and $p(\zeta)$ be an entire function satisfying the estimate*

$$\log|p(\zeta)| \leq H_L(\zeta).$$

Then for any $\varepsilon > 0$, there exist constants $C_\varepsilon > 0$, $k_\varepsilon > 0$ and entire functions $q(\zeta)$ and $r(\zeta)$ which satisfy :

$$p(\zeta) = \widehat{\mu}(-\zeta)q(\zeta) + r(\zeta),$$

$$\log|q(\zeta)| \leq H_{L\cup K}(\zeta) + k_\varepsilon|\mathrm{Re}\,\zeta| + \varepsilon|\zeta| + C_\varepsilon,$$

$$\log|r(\zeta)| \leq H_K(\zeta) + k_\varepsilon|\mathrm{Re}\,\zeta| + \varepsilon|\zeta| + C_\varepsilon.$$

Proof of the lemma. — In this proof, we write $f(\zeta) := \widehat{\mu}(-\zeta)$. For any constant $\delta > 0$, consider the following set

$$\Lambda_1 := \{\zeta\infty \in S_\infty^{2n-1}\,;\ \ H_L(\zeta) < H_K(\zeta) + \delta|\zeta|\},$$

which is a neighborhood of $\mathrm{Char}(\mu*)^a$ in S_∞^{2n-1}. Since $\sqrt{-1}\,S_\infty^{n-1} \setminus \Lambda_1$ is compact and does not meet $\mathrm{Char}(\mu*)^a$, we can take a neighborhood Λ_2 of $\sqrt{-1}\,S_\infty^{n-1} \setminus \Lambda_1$ in S_∞^{2n-1} and a constant $N > 1$ such that

$$\zeta\infty \in \Lambda_2,\ \ |\zeta| > N \ \text{ implies } \ |f(\zeta)| > \mathrm{e}^{-\delta|\zeta|}.$$

Moreover if we remark that $\Lambda_1 \cup \Lambda_2$ is a neighborhood of $\sqrt{-1}\,S_\infty^{n-1}$ in S_∞^{2n-1}, we can take a positive δ' such that

$$\zeta \in \mathbb{C}^n,\ \ |\mathrm{Re}\,\zeta| < \delta'|\mathrm{Im}\,\zeta| \ \text{ implies } \ \zeta\infty \in \Lambda_1 \cup \Lambda_2.$$

According to the above, we get the covering $\bigcup_j \gamma_j$ of $\mathbb{C}^n$ by :

$$\gamma_1 = \{\zeta \in \mathbb{C}^n\,;\ \ \zeta\infty \in \Lambda_1, |\zeta| > N\},$$

$$\gamma_2 = \{\zeta \in \mathbb{C}^n\,;\ \ \zeta\infty \in \Lambda_2, |\zeta| > N\},$$

$$\gamma_3 = \{\zeta \in \mathbb{C}^n\,;\ \ |\mathrm{Re}\,\zeta| \geq \delta'|\mathrm{Im}\,\zeta|,\ \ |\zeta| > N\},$$

$$\gamma_4 = \{\zeta \in \mathbb{C}^n\,;\ \ |\zeta| \leq N\}.$$

Now we construct $q(\zeta)$ and $r(\zeta)$ as following :

$$\begin{cases} q(\zeta) := \dfrac{p(\zeta)}{f(\zeta)}\big(1 - \varphi(\mathrm{e}^{\lambda(|\zeta|)}f(\zeta))\big) + v(\zeta), \\ r(\zeta) := p(\zeta)\varphi\big(\mathrm{e}^{\lambda(|\zeta|)}f(\zeta)\big) - f(\zeta)v(\zeta), \end{cases}$$

by choosing a suitable C^∞ function $v(\zeta)$ on $\mathbb{C}^n$, with fixed real-valued functions $\varphi(\tau) \in C^\infty(\mathbb{C})$ and $\lambda(t) \in C^\infty([0,\infty[)$ so that $0 \le \varphi(\tau) \le 1$, $0 \le \lambda(t) \le \delta t$, $0 \le \lambda'(t) \le 1$ and that

$$\varphi(\tau) = \begin{cases} 1 & \text{if } |\tau| \le \frac{1}{2}, \\ 0 & \text{if } |\tau| \ge 1, \end{cases} \qquad \lambda(t) = \begin{cases} 0 & \text{if } |\tau| \le \frac{1}{2}, \\ \delta t & \text{if } |\tau| \ge 1. \end{cases}$$

The condition for q and r to be holomorphic, is given as

$$\text{(4.3)} \quad \bar\partial v = \frac{p(\zeta)}{f(\zeta)}\, \mathrm{e}^{\lambda(|\zeta|)} \Big\{ \Big(\frac{\partial\varphi}{\partial\tau} f(\zeta) + \frac{\partial\varphi}{\partial\bar\tau} \bar f(\zeta) \Big) \frac{\lambda'(|\zeta|)}{2|\zeta|} \zeta \cdot \mathrm{d}\bar\zeta + \frac{\partial\varphi}{\partial\bar\tau} \bar\partial \bar f \Big\}.$$

We denote the right hand side by w which satisfies $\bar\partial w = 0$. Let us estimate w.

- In the outside of $\{\frac{1}{2} < \mathrm{e}^{\lambda(|\zeta|)}|f(\zeta)| < 1\}$, especially in γ_2, we have $w = 0$ since $\varphi(\mathrm{e}^{\lambda(|\zeta|)} f(\zeta))$ is constant there.
- In $\{\frac{1}{2} < \mathrm{e}^{\lambda(|\zeta|)}|f(\zeta)| < 1\}$, we have :

$$\begin{aligned} |w| &\le |p|\,\mathrm{e}^{\lambda(|\zeta|)} \Big\{ \tfrac{1}{2} \Big(\max_{|\tau|\le 1} \Big| \frac{\partial\varphi}{\partial\tau} \Big| + \max_{|\tau|\le 1} \Big| \frac{\partial\varphi}{\partial\bar\tau} \Big| \Big) + \max_{|\tau|\le 1} \Big| \frac{\partial\varphi}{\partial\bar\tau} \Big| \cdot \Big| \frac{\bar\partial \bar f}{f} \Big| \Big\} \\ &\le \mathrm{e}^{H_L(\zeta)+\lambda(|\zeta|)} \max_{|\tau|\le 1} \Big(\Big| \frac{\partial\varphi}{\partial\tau} \Big|, \Big| \frac{\partial\varphi}{\partial\bar\tau} \Big| \Big) \big(1 + 2\mathrm{e}^{\lambda(|\zeta|)} |\partial f| \big). \end{aligned}$$

Here we employed the norm

$$|w| := \sup |w_{I,J}|$$

of a (p,q)-form

$$w = \sum w_{I,J}\, \mathrm{d}\zeta^I \wedge \mathrm{d}\bar\zeta^J.$$

Since $|\partial f| \exp(-H_{\operatorname{supp}\check\mu}(\zeta) - \delta|\zeta|)$ is bounded, we have, with some constant $C > 0$,

$$\log|w| \le H_L(\zeta) + H_{\operatorname{supp}\check\mu}(\zeta) + 3\delta|\zeta| + C.$$

In γ_1, from the estimate $H_L(\zeta) < H_K(\zeta) + \delta|\zeta|$, we get :

$$\log|w| \le H_K(\zeta) + H_{\operatorname{supp}\check\mu}(\zeta) + 4\delta|\zeta| + C.$$

In γ_3, setting $\ell := \max_{z\in L} |z|$, we have :

$$H_L(\zeta) \le \ell|\zeta| \le \ell\big(|\operatorname{Re}\zeta| + |\operatorname{Im}\zeta|\big) \le \ell\Big(1 + \frac{1}{\delta'}\Big)|\operatorname{Re}\zeta|.$$

Thus we get :

$$\log|w| \leq H_{\operatorname{supp}\check{\mu}}(\zeta) + \ell\Big(1+\frac{1}{\delta'}\Big)|\operatorname{Re}\zeta| + 3\delta|\zeta| + C.$$

Remarking that γ_4 is compact, we can take another constant $C_1 > 0$ such that the estimate

$$\log|w| \leq H_K(\zeta) + \Big\{\ell\Big(1+\frac{1}{\delta'}\Big)+m\Big\}|\operatorname{Re}\zeta| + 4\delta|\zeta| + C_1,$$

holds in the whole $\mathbb{C}^n$, where $m := \max_{x\in\operatorname{supp}\check{\mu}}|x|$.

By the same argument as SÉBBAR [34, p. 205–206], we conclude by the theorem 4.4.2 and the proof of the theorem 4.2.5 in HÖRMANDER [13] that there exists a solution $v \in C^\infty(\mathbb{C}^n)$ of (4.3) which satisfies the following estimate

$$\log|v| \leq H_K(\zeta) + \Big\{\ell\Big(1+\frac{1}{\delta'}\Big)+m\Big\}|\operatorname{Re}\zeta| + 5\delta|\zeta| + C_2 \tag{4.4}$$

for some $C_2 > 0$.

Finally let us estimate the growth of q and r constructed from this solution v of $\bar\partial v = w$. About q, we have

$$|q-v| = \Big|\frac{(1-\varphi)p}{f}\Big| \leq \begin{cases} 0 & \text{if } \mathrm{e}^{\lambda(|\zeta|)}\big|f(\zeta)\big| \leq \frac{1}{2}, \\ 2\exp\big(H_L(\zeta)+\lambda(|\zeta|)\big) & \text{if } \mathrm{e}^{\lambda(|\zeta|)}\big|f(\zeta)\big| > \frac{1}{2}. \end{cases}$$

Thus we have

$$\begin{aligned} |q| &\leq 2\max\big(|q-v|,|v|\big) \\ &\leq \exp\Big(H_{L\cup K}(\zeta) + \Big\{\ell\Big(1+\frac{1}{\delta'}\Big)+m\Big\}|\operatorname{Re}\zeta| + 5\delta|\zeta| + C_3\Big) \end{aligned}$$

by a constant $C_3 > 0$. About r, we have

$$\begin{aligned} &|p\varphi| \leq \exp\big(H_L(\zeta)\big) \leq \exp\big(H_K(\zeta)+\delta|\zeta|\big) \quad \text{in } \gamma_1, \\ &|p\varphi| = 0 \quad \text{in } \gamma_2, \\ &|fv| \leq \exp\Big(H_K(\zeta) + \Big\{\ell\Big(1+\frac{1}{\delta'}\Big)+2m\Big\}|\operatorname{Re}\zeta| + 6\delta|\zeta| + C_4\Big) \end{aligned}$$

by a constant $C_4 > 0$. If we take sufficiently small δ satisfying $6\delta < \varepsilon$, we get the desired results. (End of LEMMA 4.5.)

End of the proof of the theorem. — We can choose a compact

$$L \subset \mathbb{R}^n \times \sqrt{-1}\,\Omega$$

with $\widehat{T}_1(\zeta) \leq \mathrm{Const} \cdot \mathrm{e}^{H_L(\zeta)}$. For this L, take a compact $K \subset \mathbb{R}^n \times \sqrt{-1}\,\omega$ and a constant $\varepsilon > 0$ so that :

$$\begin{cases} K \text{ satisfies (4.2)}, \\ K + B(0\,;\varepsilon) \subset \mathbb{R}^n \times \sqrt{-1}\,\omega, \\ (L \cup K) + B(0\,;\varepsilon) \subset \mathbb{R}^n \times \sqrt{-1}\,\Omega. \end{cases}$$

Applying the lemma to $p = \widehat{T}_1$, L and K, we obtain $Q \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\Omega)'$ and $R \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\,\omega)'$, which satisfy $\widehat{T}_1 = \widehat{\check{\mu}} \cdot \widehat{Q} + \widehat{R}$. Then for any $g \in \mathcal{N}(\sqrt{-1}\,\Omega)$, we get :

$$\langle T, g\rangle = \langle T_1, g\rangle = \langle Q, \mu * g\rangle + \langle R, g\rangle = \langle R, g\rangle.$$

It follows that $T = {}^t r R$. $\square$

5. Main theorem

In this section, we define *the characteristic set* $\mathrm{Char}(\mu *)$ of $\mu *$ as the closed conic set in $T^*(\sqrt{-1}\,\mathbb{R}^n)$ defined by $\mathrm{Char}_\infty(\mu *)$ i.e.

Definition 5.1.

$$\mathrm{Char}(\mu *) := \Big\{ \big(\sqrt{-1}\,y, \sqrt{-1}\,\eta\big) \in T^*\big(\sqrt{-1}\,\mathbb{R}^n\big)\,; \\ \sqrt{-1}\,\eta\infty \in \mathrm{Char}_\infty(\mu *) \text{ or } \sqrt{-1}\,\eta = 0 \Big\}.$$

Now we will resume our results of preceding sections as following final theorem.

THEOREM 5.2. — *Under the condition* (S), *we have*

$$\mathrm{SS}(\mathcal{S}) \subset \mathrm{Char}(\mu *).$$

Proof. — We can prove our theorem as same way as the proof of the theorem 11.3.3 in KASHIWARA-SCHAPIRA [18] : take any

$$q = \big(\sqrt{-1}\,y_0, \sqrt{-1}\,\eta_0\big) \in T^*\big(\sqrt{-1}\,\mathbb{R}^n\big) \setminus \mathrm{Char}(\mu *).$$

By the DEFINITION 5.1, we have $\eta_0 \neq 0$ and then there exists a closed convex neighborhood G of $\sqrt{-1}\,\eta_0\infty \in \sqrt{-1}\,S^{n-1}_\infty$ which does not intersect $\mathrm{Char}_\infty(\mu *)$. Set $\gamma := (\{\sqrt{-1}\,v; \sqrt{-1}\,v\infty \in G \text{ or } v = 0\}^\circ)^a$, then we have :

$$\gamma \setminus \{0\} \subset \{\sqrt{-1}\,v\,;\ \langle\sqrt{-1}\,v, \sqrt{-1}\,\eta_0\rangle < 0\}.$$

For any $\varepsilon > 0$, we set :

$$H := \{\sqrt{-1}\,y \in \sqrt{-1}\,\mathbb{R}^n\,;\ \langle\sqrt{-1}\,(y-y_0), \sqrt{-1}\,\eta_0\rangle \geq -\varepsilon\},$$
$$L := \{\sqrt{-1}\,y \in \sqrt{-1}\,\mathbb{R}^n\,;\ \langle\sqrt{-1}\,(y-y_0), \sqrt{-1}\,\eta_0\rangle = -\varepsilon\}.$$

Remark that $R\Gamma(\sqrt{-1}\,K, \mathcal{O}^\tau) \simeq \mathcal{O}^\tau(\sqrt{-1}\,K)$ for any compact convex set $\sqrt{-1}\,K \subset \sqrt{-1}\,\mathbb{R}^n$, because $\sqrt{-1}\,K$ has a fundamental system of neighborhood which consists of open convex sets in $\sqrt{-1}\,\mathbb{R}^n$. Now in virtue of the prop. 5.1.1, (3) in [18], using the THEOREM 3.1, it suffices to prove for any $\sqrt{-1}\,y$ (near $\sqrt{-1}\,y_0$),

(5.3) $f \in \mathcal{O}^\tau\big((\sqrt{-1}\,y+\gamma)\cap L\big), Pf = 0$ implies $f \in \mathcal{O}^\tau\big((\sqrt{-1}\,y+\gamma)\cap H\big)$.

In the notation of the THEOREM 4.4, the open convex set $\sqrt{-1}\,\Omega$ for $\mathrm{Char}(\mu *)$ and for $\sqrt{-1}\,\omega :=$ (any small open convex neighborhood of $(\sqrt{-1}\,y + \gamma) \cap L$) contains, by construction, $(\sqrt{-1}\,y + \gamma) \cap H$ and this implies (5.3) by the THEOREM 4.4.

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