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ON THE HAUSDORFF DIMENSION OF JULIA SETS OF MEROMORPHIC FUNCTIONS I

PAR

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RÉSUMÉ. — On donne dans cet article des estimations de la dimension de Hausdorff des ensembles de Julia pour trois familles de fonctions méromorphes. La dynamique de ces fonctions et la structure topologique de leurs ensembles de Julia ont été étudiées par Devaney et Keen.

ABSTRACT. — In the paper it is given a lower bound for the Hausdorff dimension of the Julia sets of three families of transcendental meromorphic functions. Dynamics of these functions and topological structure of their Julia sets have been investigated by Devaney and Keen.

0. Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ denote a meromorphic function which we shall always assume to be neither a constant nor a rational function of degree one. For $n \in \mathbb{N}$, f^n denotes the n-th iterate of f, and $f^{-n} = (f^n)^{-1}$. The Fatou set $F(f)$ is the set of the points $z \in \mathbb{C}$ such that (f^n), $n \in \mathbb{N}$, is defined, meromorphic, and forms a normal family in some neighbourhood of z. The complement of $F(f)$ in $\widehat{\mathbb{C}}$ is called the *Julia set* $J(f)$ of f. $J(f)$ is perfect and has the property of complete invariance, that is, $z \in J(f)$ if and only if $f(z) \in J(f)$.

Suppose f is transcendental meromorphic, has at least one pole and f is not of the form $f_0 = \alpha + (z-\alpha)^{-k}\exp(g(z))$, where $k \in \mathbb{N}$, with an entire g. Then $J(f)$ is the closure of the set of preimages of ∞ under all f^n. For certain maps of this type the Julia set has several properties in common with those of entire functions. For example the Julia set may

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contain Cantor bouquets, which is a typical phenomenon encountered in the study of entire functions. For some other the Julia set resembles Julia sets of rational maps, e.g. $J(f)$ is a Cantor set or a quasicircle. More details of these and other basic properties of the sets $F(f)$ and $J(f)$ can be found in [1], [2], and [3].

An upper bound for the Hausdorff dimension of the Julia set of meromorphic maps is two. The standard example when this bound is attained is a function with $J(f) = \mathbb{C}$, but this is not a unique possibility. A sharp lower bound for the dimension of the Julia set of these meromorphic functions is zero (a result announced by G. STALLARD).

In this paper we concentrate on the estimate of dimension of Julia sets of certain families of maps. Dynamics of these maps has been investigated in [4]. We prove the following theorems, where $\mathrm{HD}(J(f))$ denotes the Hausdorff dimension of $J(f)$.

THEOREM 1. — *Let* $f_\lambda(z) = \lambda/(1-\mathrm{e}^{-2z})$, $\lambda > 0$. *Then* $J(f_\lambda)$ *contains a Cantor bouquet, and* $\mathrm{HD}(J(f_\lambda))= 2$.

THEOREM 2. — *For* $f_\lambda(z) = 1/(\lambda + \mathrm{e}^{-2z})$, $\lambda > 0$, $J(f_\lambda)$ *is a Cantor set. Moreover, the asymptotic estimate*

$$\mathrm{HD}\big(J(f_\lambda)\big) \geq 1 - \frac{C}{\log|\log \lambda|}$$

holds for some $C > 0$ *and* $\lambda \to 0^+$.

Of course, the above inequality implies that our lower bound for $\mathrm{HD}(J(f_\lambda))$ tends to 1 if λ tends to 0.

THEOREM 3. — *Let* $f_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{R}$ *and* $0 < |\lambda| < \frac{3}{4}$. *Then* $J(f_\lambda)$ *is a Cantor subset of* $\mathbb{R}$ *and*

$$1 \geq \mathrm{HD}\big(J(f_\lambda)\big) \geq \frac{C}{\big|\log|\lambda|\big|}$$

for some $C > 0$ *and* $\lambda \to 0$.

The proofs of these estimates are based on the following result proved by MCMULLEN [7].

LEMMA 1. — *For each* $k \in \mathbb{N}$, *let* $\mathcal{A}_k$ *be a finite collection of disjoint compact subsets of* $\mathbb{R}^n$, *each of them has positive finite n-dimensional measure, and define*

$$\mathcal{U}_k = \bigcup_{A_k \in \mathcal{A}_k} A_k, \quad A = \bigcap_{k=1}^{\infty} \mathcal{U}_k.$$

Suppose also that, for each $A_k \in \mathcal{A}_k$, there exists $A_{k+1} \in \mathcal{A}_{k+1}$ and a unique $A_{k-1} \in \mathcal{A}_{k-1}$ such that $A_{k+1} \subset A_k \subset A_{k-1}$. If Δ_k, d_k satisfy, for each $A_k \in \mathcal{A}_k$, the conditions

$$\begin{cases} \dfrac{\operatorname{vol}(\mathcal{U}_{k+1} \cap A_k)}{\operatorname{vol}(A_k)} \geq \Delta_k, \\ \operatorname{diam} A_k \leq d_k < 1, \\ d_k \to 0 \quad as \quad k \to \infty, \end{cases}$$

then

$$\operatorname{HD}(A) \geq n - \limsup_{k \to \infty} \sum_{j=1}^{k} \frac{|\log \Delta_j|}{|\log d_k|}.$$

It is clear that if f is a homeomorphism of a domain D onto $f(D)$, then the distorsion defined by

$$L(f, D) = \sup_{z_1, z_2} \frac{|f'(z_1)|}{|f'(z_2)|}$$

satisfies $L(f, D) = L(f^{-1}, f(D))$. In the proofs we use the following Koebe's distorsion theorem, cf. [5].

LEMMA 2. — *Let $D(z, r)$ denote the disc of centre z, radius r. Then for $0 < s < r$, there is a constant $M(s/r)$ such that, for every univalent map $g : D(z, r) \to \mathbb{C}$,*

$$L\big(f, D(z, s)\big) \leq M\Big(\frac{s}{r}\Big) = \Big(\frac{r+s}{r-s}\Big)^4.$$

1. Proof of Theorem 1

Consider the family of maps

$$f_\lambda(z) = \frac{\lambda}{1 - \mathrm{e}^{-2z}}, \qquad \lambda > 0.$$

The function f_λ is periodic with period πi, and its Schwarzian derivative equals $Sf_\lambda = (f_\lambda'''/f_\lambda') - \frac{3}{2}(f_\lambda''/f_\lambda')^2 = -2$. The singularities of f_λ^{-1} are $a_1 = 0$ and $a_2 = \lambda$. They are the transcendental singularities. Recall that a point a is said to be a *transcendental singularity* of f^{-1} (or an *asymptotic value* of f) if there exists a curve Γ in $\mathbb{C}$ such that $f(z) \to a$ on $f(\Gamma)$ when $z \to \infty$ on Γ. Let

$$W_1 = \{z : \operatorname{Re} z < 0\}, \quad W_2 = \{z : \operatorname{Re} z > 0\}.$$

In each sector W_i, $i = 1, 2$, f_λ has the following behaviour : there is a disc B_i around a_i such that $f_\lambda^{-1}(B_i \setminus \{a_i\})$ contains a unique unbounded component $U_i \subset W_i$, and $f_\lambda : U_i \to B_i \setminus \{a_i\}$ is a universal covering. These U_i are called *exponential tracts.* It is seen from the graph of f_λ restricted to $\mathbb{R}$ that f_λ has two fixed points $q_i = q_i(\lambda)$, $i = 1, 2$, with $q_1 < 0 < q_2$, q_1 is repelling while q_2 is attracting. Moreover, if $\operatorname{Re} z > 0$ then ${f_\lambda}^n(z) \to q_2$ as $n \to \infty$, hence $J(f_\lambda)$ is contained in the half-plane $\{z : \operatorname{Re} z \leq 0\}$. As $\mathbb{R}^- \cup \{0\} \subset J(f_\lambda)$, hence all the preimages of $\mathbb{R}^-$ are in $J(f_\lambda)$. In particular the branches of $f_\lambda^{-2n}(\mathbb{R}^-)$ belong to, so called, Cantor bouquet. We recall its definition. Let Σ_N be the set of sequences of $s = (s_0, s_1, s_2, \ldots)$, where the s_i are integers, $|s_i| \leq N$. An invariant subset $\mathcal{C}_N$ of $J(f_\lambda)$ is called a *Cantor N-bouquet* for f_λ if there exists a homeomorphism $h : \Sigma_N \times [0\,; \infty) \to \mathcal{C}_N$ such that

$$\pi \circ h^{-1} \circ f_\lambda \circ h(s, t) = \sigma(s),$$

where $\pi : \Sigma_N \times [0\,; \infty) \to \Sigma_N$ is the projection map, σ is the shift automorphism defined by $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, \ldots)$, and $\lim h(s,t) = \infty$ if $t \to \infty$, $\lim f_\lambda^n \circ h(s,t) = \infty$ if $t \neq 0$ and $n \to \infty$. An N-bouquet $\mathcal{C}_N$ includes naturally an $(N+1)$-bouquet $\mathcal{C}_{N+1}$ by considering only sequences with entries less than or equal to N in absolute value. The set

$$\mathcal{C} = \overline{\bigcup_{N \geq 0} \mathcal{C}_N}$$

is called a *Cantor bouquet.*

Note that one of the asymptotic values 0 is also a pole of f_λ. Thus f_λ satisfies the assumptions of the following lemma proved in [4].

LEMMA 3. — *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic map. Suppose f has polynomial Schwarzian derivative of degree $(p-2)$ and has an asymptotic value a_j which is also a pole. Let W_j be the sector containing the exponential tract corresponding to a_j. Then for each $N > 0$, $J(f)$ contains a Cantor N-bouquet in W_j which is invariant under f^2.*

So, for $\lambda > 0$ $J(f_\lambda)$ is contained in the half-plane $\{z : \operatorname{Re} z \leq 0\}$ and contains a Cantor bouquet, while $F(f_\lambda)$ is an attractive basin of q_2. We will show that $\mathrm{HD}(J(f_\lambda)) = 2$ for $\lambda > 0$. Fix $\lambda > 0$. Further on for simplicity we omit the index λ. Denote by g the second iterate of f, $g = f^2$. Choose $p = p(\lambda)$ such that $p \le -\frac{5}{2}$, $e^{-p} > ((1+\lambda-2p)^2+\frac{1}{4}\pi^2)^{\frac{1}{2}}$ and the absolute value of $\lambda^2/(6p)$ is small enough, the meaning of this condition will appear further. Define the sets (see Fig. 1) :

$$T_n = \{z : \operatorname{Re} z < p,\ |\operatorname{Im} z - n\pi| < \tfrac{1}{4}\pi\}, \quad n \in \mathbb{Z},$$

$$T = \bigcup_{n\in\mathbb{Z}} T_n,$$

$$E = \{z : g^n(z) \in T \text{ for all } n \in \mathbb{N}\}.$$

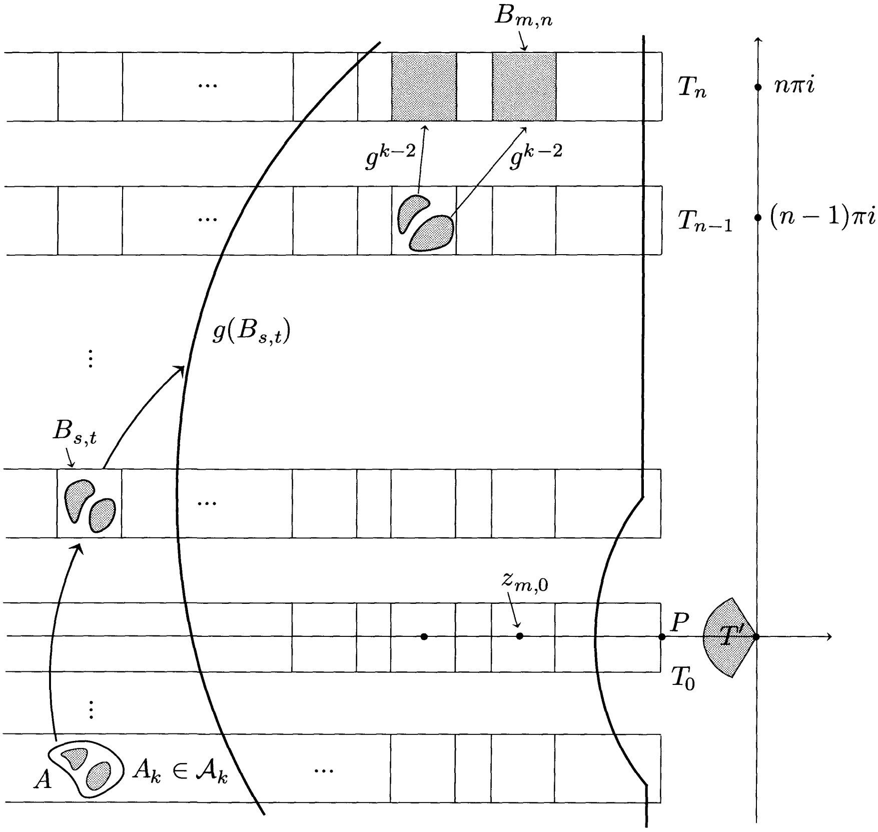

Figure 1

We will show that $E \subset J(f)$. First, we begin with the

LEMMA 4. — *Let $g^j(z) \in T$, $j = 0, \dots, n-1$, then*

$$\left|(g^n)'(z)\right| \geq 4^{-n} \exp\bigl(2(2^n - 1)|p|\bigr).$$

Proof. — A simple calculation gives

$$g(z) = \frac{\lambda}{1 - \exp(-2\lambda/(1 - \mathrm{e}^{-2z}))},$$

$$g'(z) = \bigl(2\lambda^{-1} g(z) f(z) \exp(-\lambda/(1 - \mathrm{e}^{-2z}) - z)\bigr)^2.$$

This implies

$$|g'(z)| = \Big\{ 2\lambda^{-1} |f^2(z) f(z)| \\ \exp\Big(\frac{-\lambda(1 - \mathrm{e}^{-2\operatorname{Re} z} \cos(2 \operatorname{Im} z))}{1 - 2\mathrm{e}^{-2\operatorname{Re} z} \cos(2\operatorname{Im} z) + \mathrm{e}^{-4\operatorname{Re} z}}\Big) \exp(-\operatorname{Re} z) \Big\}^2,$$

and consequently

$$\left|(g^n)'(z)\right| = 4^n \lambda^{-2n} \bigl\{ |F_n(z)|\, G_n(z) H_n(z) \bigr\}^2, \tag{1}$$

with

$$F_n(z) = \prod_{m=1}^{n} \bigl(f^{2m-1}(z) f^{2m}(z)\bigr),$$

$$G_n(z) = \exp\Bigl(-\sum_{m=0}^{n-1} \operatorname{Re} g^m(z)\Bigr),$$

$$H_n(z) = \exp\Bigl\{ \sum_{m=0}^{n-1} -\lambda\bigl[1 - \exp\bigl(-2\operatorname{Re} g^m(z)\bigr) \cos(2 \operatorname{Im} g^m(z))\bigr] \\ \times \bigl[1 - 2\exp\bigl(-2\operatorname{Re} g^m(z)\bigr) \cos\bigl(2\operatorname{Im} g^m(z)\bigr) \\ + \exp\bigl(-4\operatorname{Re} g^m(z)\bigr)\bigr]^{-1} \Bigr\}$$

First we show $|F_n(z)| > (\frac{1}{4}\lambda)^n$. Let $z \in (-\infty\,; p)$ and $z \to -\infty$, then $f(z) \in (q\,; 0)$, $q = \lambda/(1 - \mathrm{e}^{-2p})$ and $f(z) \to 0^-$. Moreover, $T' = f(T_0)$ is a domain attached to 0, contained in the half-plane $\{z : \operatorname{Re} z < 0\}$, and bounded by the three circular arcs two of them are symmetric with respect to $\mathbb{R}^-$ and pass through 0, while the third one is orthogonal to the previous two. We have $f(z) = \lambda/(1 - e^{-2z}) \approx \frac{1}{2}\lambda(1 + z^{-1})$ in the vicinity of 0. In fact, in the set T' (which is thin and contained in a small neighbourhood of 0 by the conditions imposed on p) $|f(z)| \geq \frac{1}{2}\lambda|1 + z^{-1}|$. Now, for $z \in T$, the product $|f(z)g(z)|$ is estimated from below by $|f(z)| \cdot |\frac{1}{2}\lambda(1 + f(z)^{-1})| = \frac{1}{2}\lambda|1 + f(z)|$. This implies that

$$(2)\quad |F_n(z)| = \prod_{m=1}^{n} |f^{2m-1}(z)f^{2m}(z)| \geq \prod_{m=1}^{n} \tfrac{1}{2}\lambda|1 + f^{2m-1}(z)| > (\tfrac{1}{4}\lambda)^n$$

since $f^{2m-1}(z) \in T'$ and $|f^{2m-1}(z)| < |f^{2m-3}(z)| < \cdots < |q|$.

We claim that $G_n(z) > \exp((2^n - 1)|p|)$. To prove this it is enough to show that $\operatorname{Re} g(z) < 2 \operatorname{Re} z$, which is equivalent to the inequality $\operatorname{Re} w < 2 \operatorname{Re} g^{-1}(w)$, where $w = g(z)$. As $w \in T$

$$\begin{aligned} g^{-1}(w) &= -\tfrac{1}{2} \log\bigl(1 + 2\lambda \log^{-1}(1 - \lambda/w)\bigr) \\ &= -\tfrac{1}{2} \log\bigl(1 + 2\lambda/(-\lambda w^{-1}(1 + \lambda/(2w) + \cdots))\bigr) \\ &= -\tfrac{1}{2} \log\bigl(1 - 2w(1 - \lambda/(2w) + \cdots)\bigr) \\ &\approx -\tfrac{1}{2} \log(1 + \lambda - 2w) \end{aligned}$$

and

$$\begin{aligned} \exp(-\operatorname{Re} w) &> \bigl((1 + \lambda - 2 \operatorname{Re} w)^2 + \tfrac{1}{4}\pi^2\bigr)^{\frac{1}{2}} \\ &\geq \bigl((1 + \lambda - 2 \operatorname{Re} w)^2 + 4 \operatorname{Im}^2 w\bigr)^{\frac{1}{2}}, \end{aligned}$$

we have that

$$2 \operatorname{Re} w < -\log\bigl((1 + \lambda - 2 \operatorname{Re} w)^2 + 4 \operatorname{Im}^2 w\bigr) \leq 2 \operatorname{Re} g^{-1}(w).$$

Hence we obtain $\operatorname{Re} w < 2 \operatorname{Re} g^{-1}(w)$. By induction one can prove that $\exp(-\operatorname{Re} g^m(z)) > \exp(-2^m \operatorname{Re} z)$ for all $m \leq n$. It follows that

$$(3)\qquad G_n(z) > \exp\bigl((2^n - 1)|\operatorname{Re} z|\bigr) \geq \exp\bigl((2^n - 1)|p|\bigr).$$

Now we show that

$$(4)\qquad H_n(z) \geq 1.$$

As $\operatorname{Re} g^m(z) < p$ and $|\operatorname{Im} g^m(z)| < \frac{1}{4}\pi$ we have

$$\exp\bigl(-2 \operatorname{Re} g^m(z)\bigr) \cos\bigl(2 \operatorname{Im} g^m(z)\bigr) > 2^{-\frac{1}{2}} \exp(-2p) > 1,$$

so $-\lambda(1-\exp(-2\operatorname{Re} g^m(z))\cos(2\operatorname{Im} g^m(z)))>0$. Moreover,

$$1-2\exp\bigl(-2\operatorname{Re} g^m(z)\bigr)\cos\bigl(2\operatorname{Im} g^m(z)\bigr)>1-2\exp\bigl(-2\operatorname{Re} g^m(z)\bigr)$$

and $1-2\exp\bigl(-2\operatorname{Re} g^m(z)\bigr)+\exp\bigl(-4\operatorname{Re} g^m(z)\bigr)>0$, hence we get

$$1-2\exp\bigl(-2\operatorname{Re} g^m(z)\bigr)\cos\bigl(2\operatorname{Im} g^m(z)\bigr)+\exp\bigl(-4\operatorname{Re} g^m(z)\bigr)>0,$$

and finally $H_n(z)\geq 1$. By (1)–(4) we have

$$\bigl|(g^n)'(z)\bigr|=4^n\lambda^{-2n}\bigl\{\,|F_n(z)|G_n(z)H_n(z)\bigr\}^2\geq 4^{-n}\exp\bigl(2(2^n-1)|p|\bigr).\quad\square$$

Having estimated the derivatives of iterates we can prove the

LEMMA 5. — $E\subset J(f)$.

Proof. — Suppose that $z_0\in E\cap F(f)$. Then there is a disc $D=D(z_0,r)\subset F(f)$ and a subsequence of iterates (f^{n_k}) holomorphic on D which converges to a holomorphic function g. Hence $g'(z)\neq\infty$ in D. By LEMMA 4, $g'(z_0)=\lim\limits_{k\to\infty}(f^{n_k})'(z_0)=\infty$, so we arrive at contradiction. Thus $z_0\in J(f)$. $\square$

Let $z_{m,n}=2m+n\pi i$, $m,n\in\mathbb{Z}$ and $m<p$, where p was chosen just before LEMMA 4. Define the squares

$$B_{m,n}=\bigl\{z:|\operatorname{Re}(z-z_{m,n})|<\tfrac14\pi,\ |\operatorname{Im}(z-z_{m,n})|<\tfrac14\pi\bigr\}\subset T.$$

Take a square $B_{s,t}$ such that $g^{-1}(B_{s,t})$ has at least one component in T and let A_1 be one such component. We introduce the following collection of sets :

$$\begin{aligned}
\mathcal{A}_1&=\{A_1\},\\
\mathcal{A}_2&=\bigl\{A_2:A_2\text{ is a component of }g^{-2}(B_{m,n})\text{ for some }m<p,\\
&\qquad\qquad\qquad\qquad m,n\in\mathbb{Z},\ A_2\subset A_1\bigr\},\\
&\ \ \vdots\\
\mathcal{A}_k&=\bigl\{A_k:A_k\text{ is a component of }g^{-k}(B_{m,n})\text{ for some }m<p,\\
&\qquad\qquad m,n\in\mathbb{Z},\ A_k\subset A_{k-1}\text{ for some }A_{k-1}\in\mathcal{A}_{k-1}\bigr\}.
\end{aligned}$$

Moreover, define

$$\mathcal{U}_k=\bigcup_{A_k\in\mathcal{A}_k}A_k,\qquad A=\bigcap_{k=1}^{\infty}\mathcal{U}_k.$$

Then, of course, $A\subset E$.

We will show that $\mathrm{HD}(A)=2$.

LEMMA 6. — *For each $k \in \mathbb{N}$, $A_k \in \mathcal{A}_k$, we have*

$$\operatorname{diam} A_k \leq 4^k \pi\, 2^{-\frac{1}{2}} \exp\bigl(-2(2^k-1)\,|p|\bigr).$$

Proof. — Let $A_k \in \mathcal{A}_k$. Then $g^k(A_k) = B_{m,n}$ for some $m < p$, $m, n \in \mathbb{Z}$, so $\operatorname{diam} g^k(A_k) = \pi\, 2^{-\frac{1}{2}}$. As $g^k(A_k)$ is convex and $g^j(z) \in T$, $j = 0, \ldots, k-1$, we can apply LEMMA 4 which gives

$$\operatorname{diam} A_k \leq \frac{\operatorname{diam} g^k(A_k)}{\inf\limits_{A_k} |(g^k(z))'|} \leq 4^k \pi\, 2^{-\frac{1}{2}} \exp\bigl(-2(2^k-1)\,|p|\bigr). \quad \square$$

For $A_k \in \mathcal{A}_k$ we define $G(A_k) = \{A_{k+1} \in \mathcal{A}_{k+1} : A_{k+1} \subset A_k\}$.
Then $\mathcal{U}_{k+1} \cap A_k = \bigcup\limits_{A_{k+1} \in G(A_k)} A_{k+1}$ and

$$\text{(5)} \qquad \frac{\operatorname{vol}(\mathcal{U}_{k+1} \cap A_k)}{\operatorname{vol}(A_k)} \geq \frac{1}{L(g^k, A_k)^2} \sum_{A_{k+1} \in G(A_k)} \frac{\operatorname{vol}(g^k(A_{k+1}))}{\operatorname{vol}(g^k(A_k))},$$

where $L(g^k, A_k)$ denotes the distorsion of g^k on the set A_k. So we need an upper bound of this quantity, which appears to be uniform in λ.

LEMMA 7. — *There is a constant $1 < C_1 < \infty$ such that for each $A_k \in \mathcal{A}_k$, $k \in \mathbb{N}$, the distorsion $L(g^k, A_k)$ is bounded above by C_1.*

Proof. — Recall that the singularities of f^{-1} are 0 and λ, where 0 is also a pole, while λ belongs to an attractive basin of a fixed point $q_2 > 0$. So the function $g = f^2$ has only one finite asymptotic value

$$\lambda_1 = \frac{\lambda}{1 - \mathrm{e}^{-2\lambda}} > 0.$$

Moreover, $\operatorname{Re} f^k(\lambda) > 0$ for $k \in \mathbb{N}$, so the branches of f^{-k}, and consequently the branches of g^{-k} are univalent in the half-plane $\{z : \operatorname{Re} z \leq 0\}$. Thus g^k is a homeomorphism of A_k onto $g^k(A_k)$, and

$$L(g^k, A_k) = L\bigl(g^{-k}, g(A_k)\bigr) = L(g^{-k}, B_{m,n})$$

for some $B_{m,n} \subset T$. As $B_{m,n} \subset D(z_{m,n}, 2^{-\frac{1}{2}}\pi) \subset D(z_{m,n}, |p|)$, it follows from LEMMA 2 that

$$L(g^{-k}, B_{m,n}) \leq M\Bigl(\frac{2^{-\frac{1}{2}}\pi}{|p|}\Bigr) = \Bigl(\frac{|p| + 2^{-\frac{1}{2}}\pi}{|p| - 2^{-\frac{1}{2}}\pi}\Bigr)^4.$$

So by the definition of p there is a constant C_1 such that $1 < C_1 < \infty$ and $L(g^{-k}, B_{m,n}) \leq C_1$. $\square$

Let $A_k \in \mathcal{A}_k$, $k \in \mathbb{N}$. Note that $g^k(A_{k+1})$ is a component of $g^{-1}(B_{m,n})$ for some $m < p$ and $m, n \in \mathbb{Z}$. As $g^{k+1}(A_{k+1})$ is convex subset of T, it follows from LEMMA 4 that

$$\operatorname{diam} g^k(A_{k+1}) \leq \frac{\operatorname{diam} g^{k+1}(A_{k+1})}{\inf_{A_{k+1}} \left|(g^{k+1}(z))'\right|} \leq 2^{\frac{3}{2}} \pi \exp(2p) < \frac{1}{12}\pi.$$

Thus

$$\sum_{A_{k+1} \in G(A_k)} \operatorname{vol}\bigl(g^k(A_{k+1})\bigr) > \operatorname{vol}\bigl(g^{-1}(T) \cap W_{m,n}\bigr),$$

where

$$W_{m,n} = \bigl\{z \in \mathbb{C} : |\operatorname{Re}(z - z_{m,n})| < \tfrac{1}{6}\pi,\ |\operatorname{Im}(z - z_{m,n})| < \tfrac{1}{6}\pi\bigr\} \subset B_{m,n}$$

and $g^k(A_k) = B_{m,n}$.

LEMMA 8. — *There is a positive constant C_2 such that for each $m < p$, $m, n \in \mathbb{Z}$, $\operatorname{vol}(g^{-1}(\mathcal{B}) \cap W_{m,n}) \geq C_2$, where*

$$\mathcal{B} = \bigcup_{\substack{k,\ell \in \mathbb{Z} \\ k < p}} B_{k,\ell}.$$

Proof. — By periodicity of g it is enough to consider only the sets $W_{m,0}$. Fix $m < p$, $m \in \mathbb{Z}$, and take the points $x_0 + iy$ in $W_{m,0}$, where x_0 is fixed. We want to find the maximal s such that $g^{-1}(T_k) \cap W_{m,0} \neq \emptyset$, $|k| = 0, 1, \ldots, s$. Let

$$L_k^{\pm} = \bigl\{w = u + i(k \pm \tfrac{1}{4})\pi : u < p\bigr\} \subset \partial T_k.$$

If $w \in T$, then $g^{-1}(w) \approx -\frac{1}{2}\log(1 + \lambda - 2w)$. It follows

$$\begin{aligned}(6)\quad g^{-1}(L_k^{\pm}) \approx \Bigl\{z &= -\tfrac{1}{2}\log(1 + \lambda - 2(u + i(k \pm \tfrac{1}{4})\pi)) \\ &= -\tfrac{1}{4}\log\bigl[(1 + \lambda - 2u)^2 + 4(k \pm \tfrac{1}{4})^2\pi^2\bigr] \\ &\qquad + \tfrac{1}{2} i \arctan\bigl(2(k \pm \tfrac{1}{4})\pi/(1 + \lambda - 2u)\bigr) : u < p\Bigr\}.\end{aligned}$$

As $g^{-1}(L_k^{\pm})$ intersects $x_0 + iy$ if

$$\operatorname{Re} g^{-1}(L_k^{\pm}) = x_0 \quad \text{and} \quad \bigl|\operatorname{Im} g^{-1}(L_k^{\pm})\bigr| < \tfrac{1}{6}\pi,$$

we have

$$x_0 = -\tfrac{1}{4}\log\big((1+\lambda-2u)^2 + 4(k\pm\tfrac{1}{4})^2\pi^2\big),$$

which is equivalent to $\exp(-4x_0) = (1+\lambda-2u)^2 + 4(k\pm\frac{1}{4})^2\pi^2$. Thus

$$(1+\lambda-2u) = \big(\exp(-4x_0) - 4(k\pm\tfrac{1}{4})^2\pi^2\big)^{\frac{1}{2}}. \tag{7}$$

By (6)

$$\big|\operatorname{Im} g^{-1}(L_k^{\pm})\big| = \Big|\frac{1}{2}\arctan\Big(\frac{2(k\pm\frac{1}{4})\pi}{1+\lambda-2u}\Big)\Big| < \frac{1}{6}\pi.$$

Applying (7) we have

$$2(k\pm\tfrac{1}{4})\pi\big[\exp(-4x_0) - 4(k\pm\tfrac{1}{4})^2\pi^2\big]^{-\frac{1}{2}} \le \tan(\tfrac{1}{3}\pi) = 3^{\frac{1}{2}},$$

or, equivalently,

$$16\pi^2(k\pm\tfrac{1}{4})^2 \le 3\exp(-4x_0), \quad |k\pm\tfrac{1}{4}| \le 3^{\frac{1}{2}}\big(4\pi\exp(2x_0)\big)^{-1}.$$

Thus

$$s \ge 3^{\frac{1}{2}}\big(4\pi\exp(2x_0)\big)^{-1} - \tfrac{3}{4}. \tag{8}$$

Let J_k be a subinterval of the line $\{z : \operatorname{Re} z = x_0\}$ lying in $W_{m,0}$ such that $y \in J_k$ if and only if $f(x_0+iy) \in T_k$. Denote

$$\begin{aligned} y_k^+ &= -\tfrac{1}{2}\arctan\Big\{2\pi(k+\tfrac{1}{4})\big[\exp(-4x_0) - 4(k+\tfrac{1}{4})^2\pi^2\big]^{-\frac{1}{2}}\Big\},\\ y_k^- &= -\tfrac{1}{2}\arctan\Big\{2\pi(k-\tfrac{1}{4})\big[\exp(-4x_0) - 4(k-\frac{1}{4})^2\pi^2\big]^{-\frac{1}{2}}\Big\}. \end{aligned}$$

We want to estimate the length of the interval J_k, $k \ge 0$, as $|J_k| = |J_{-k}|$, that is

$$|y_k^+ - y_k^-| = \Big|\frac{1}{2}\arctan\Big(2\pi\frac{(k+\frac{1}{4})K_1^{\frac{1}{2}} - (k-\frac{1}{4})K_2^{\frac{1}{2}}}{K_1^{\frac{1}{2}}K_2^{\frac{1}{2}} + 4\pi^2(k^2-\frac{1}{16})}\Big)\Big|,$$

where

$$K_1 = \exp(-4x_0) - 4(k-\tfrac{1}{4})^2\pi^2, \quad K_2 = \exp(-4x_0) - 4(k+\tfrac{1}{4})^2\pi^2.$$

We obtain

$$|J_k| = \left| \tfrac{1}{2} \arctan \left(2\pi \frac{4k^2\pi^2 (K_1^{1/2} + K_2^{1/2})^{-1} + \frac{1}{4}(K_1^{1/2} + K_2^{1/2})}{K_1^{1/2} K_2^{1/2} + 4\pi^2 (k^2 - \frac{1}{16})} \right) \right|$$

$$\geq \left| \tfrac{1}{2} \arctan \left(2\pi \frac{(K_1^{1/2} K_2^{1/2})^{1/2}}{16K_1 + 4\pi^2 (k^2 - \frac{1}{16})} \right) \right|.$$

As $(k \pm \frac{1}{4})^2 \leq 3(16\pi^2 \exp(4x_0))^{-1}$ we have, for $x_0 < p$

$$|J_k| \geq \left| \tfrac{1}{2} \arctan\{\pi(\tfrac{1}{2} \exp(-2x_0))[\exp(-4x_0) + 2\pi \exp(-2x_0)]^{-1}\} \right|$$

$$\geq \tfrac{1}{2} \arctan\bigl(\tfrac{1}{2}\pi \exp(2x_0)\bigr) \geq \tfrac{1}{8}\pi \exp(2x_0).$$

The above inequality together with (8) imply that

$$\sum_{k=0}^{s} |J_k| \geq \tfrac{1}{8}\pi \exp(2x_0)\bigl[3^{\frac{1}{2}} (4\pi \exp(2x_0))^{-1} - \tfrac{3}{4}\bigr]$$

$$\geq 0.0541 - 0.2945 \exp(2p) > 0.052.$$

Thus $\mathrm{vol}(g^{-1}(T) \cap W_{m,n}) \geq 2(\frac{1}{3}\pi) 0.052 > 0.1 =: C_2'$ and hence

$$\mathrm{vol}(g^{-1}(\mathcal{B}) \cap W_{m,n}) \geq C_2$$

for some $C_2 > 0$. (Observe that $\lim_{M \to -\infty} \mathrm{vol}(\mathcal{B}_M)/\mathrm{vol}(T \cap \{z : \mathrm{Re}\, z \leq M\}) = \frac{1}{4}\pi$, where $\mathcal{B}_M = \bigcup_{k,\ell \in \mathbb{Z}, k < M < p} B_{k,\ell}$). □

The sets A_k satisfy the conditions of LEMMA 1. By LEMMA 6

$$\mathrm{diam}\, A_k \leq d_k = 4^k \pi 2^{-\frac{1}{2}} \exp\bigl(-2(2^k - 1)\,|p|\bigr).$$

Moreover, LEMMA 7 and LEMMA 8 together with (5) imply that

$$\frac{\mathrm{vol}(\mathcal{U}_{k+1} \cap A_k)}{\mathrm{vol}(A_k)} \geq \frac{4C_2}{(\pi C_1)^2} = \Delta_k,$$

independently of k. Thus we have

$$2 \geq \mathrm{HD}(A) \geq 2 - \limsup_{k \to \infty} \frac{k \bigl|\log(4C_2/\pi^2 C_1^2)\bigr|}{|\log d_k|}$$

$$= 2 - \limsup_{k \to \infty} \frac{k \bigl|\log 4 + \log C_2 - 2\log \pi - 2 \log C_1\bigr|}{\bigl|\log \pi + k \log 4 - \frac{1}{2}\log 2 - 2(2^k - 1)|p|\bigr|}$$

$$= 2,$$

since C_1, C_2 do not depend on k. As $A \subset E \subset J(f_\lambda)$, we obtain $\mathrm{HD}(J(f_\lambda)) = 2$ for all $\lambda > 0$, hence THEOREM 1.

2. Proof of Theorem 2

Let $f_\lambda(z) = 1/(\lambda + e^{-2z})$, $\lambda > 0$. We remind to the reader some of the essential properties of this family described in [4]. The function f_λ maps $\mathbb{R}$ diffeomorphically onto the interval with asymptotic values 0, $1/\lambda$ as the endpoints, so f_λ has an attracting fixed point $p_\lambda \in (0\,;\,1/\lambda)$, and the entire real axis lies in the immediate basin of p_λ. In particular, both asymptotic values lie in the immediate basin of p_λ, and so there are discs about these points which lie in the basin. Taking preimages of these discs, it follows that there are half planes of the form

$$H_1 = \{z : \operatorname{Re} z < \nu_1 = \nu_1(\lambda)\} \text{ and } H_2 = \{z : \operatorname{Re} z > \nu_2 = \nu_2(\lambda)\}$$

with $\nu_1 < p_\lambda < \nu_2$, which lie in the immediate basin of p_λ. Let

$$S_\mu = H_1 \cup H_2 \cup \{z : \nu_1 \le \operatorname{Re} z \le \nu_2,\ |\operatorname{Im} z - n\pi| < \mu,\ n \in \mathbb{Z}\},$$

where $\mu = \mu(\lambda)$. Then $f_\lambda : S_\mu \to S_\mu$, so $S_\mu \subset F(f_\lambda)$. The complement of S_μ consists of infinitely many congruent rectangles R_m, where $m \in \mathbb{Z}$ and R_m are indexed according to increasing imaginary parts. In each R_m f_λ has exactly one pole $s_m = -\frac{1}{2}\log\lambda + i(m+\frac{1}{2})\pi$, $m \in \mathbb{Z}$, and maps each R_m diffeomorfically onto $\widehat{\mathbb{C}} \setminus f_\lambda(S_\mu)$. Thus $J(f_\lambda) = \bigcap_{n \ge 0} f^{-n}(\widehat{\mathbb{C}} \setminus S_\mu)$, so $J(f_\lambda)$ is a planar Cantor set, while $F(f_\lambda)$ is the attractive basin of p_λ.

Fix λ and assume that $0 < \lambda < \varepsilon$ with ε small enough (we can take e.g. $\varepsilon = 10^{-10}$). As before we omit the index λ. Let s_K be a pole such that

$$-\tfrac{3}{2}\log\lambda \le (K+\tfrac{1}{2})\pi \le -\tfrac{3}{2}\log\lambda + \pi,$$

w_K be the preimage of s_K with $0 < \operatorname{Im} w_K < \frac{1}{2}\pi$, that is,

$$\begin{aligned}
w_K &= -\tfrac{1}{2}\log\big((1/s_K) - \lambda\big),\\
\operatorname{Re} w_K &\approx -\tfrac{1}{2}\log\big((\tfrac{5}{2}\log^2\lambda + \tfrac{5}{2}\lambda\log^3\lambda + \tfrac{25}{4}\lambda^2\log\lambda)^{\frac{1}{2}}/(\tfrac{5}{2}\log^2\lambda)\big)\\
&\approx -\tfrac{1}{2}\log\big((\tfrac{5}{2}\log^2\lambda)^{-\frac{1}{2}}\big) = \tfrac{1}{2}\log\big(-(2.5)^{\frac{1}{2}}\log\lambda\big),\\
\operatorname{Im} w_K &\approx -\tfrac{1}{2}\arctan\big(\tfrac{3}{2}\log\lambda(-\tfrac{1}{2}\log\lambda - \tfrac{5}{2}\lambda\log^2\lambda)^{-1}\big)\\
&\approx \tfrac{1}{2}\arctan 3 \approx 0.6245.
\end{aligned}$$

Above and in the sequel $G(\lambda) \approx H(\lambda)$ means

$$\lim_{\lambda\to 0^+} G(\lambda)/H(\lambda) = 1.$$

Let

$$\begin{aligned}\xi &= -\tfrac{1}{2}\log\lambda - \operatorname{Re} w_K \approx -\tfrac{1}{2}\log\lambda - \tfrac{1}{2}\log\big(-(2.5)^{\frac{1}{2}}\log\lambda\big)\\ &= -\tfrac{1}{2}\log\big(-(2.5)^{\frac{1}{2}}\lambda\log\lambda\big),\\ \eta &= \tfrac{1}{2}\pi - \operatorname{Im} w_K \approx 0.9463.\end{aligned}$$

Define the sets (see Fig. 2) :

$$\begin{aligned}B_m &= \big\{z : \operatorname{Re} w_K < \operatorname{Re} z < -\tfrac{1}{2}\log\lambda,\ |\operatorname{Im}(z - s_m)| < \eta\big\},\\ T &= \bigcup_{m\in I} B_m,\\ I &= \mathbb{Z}\setminus\{-K,\ldots,0,\ldots,K-1\},\\ E &= \big\{z : f^n(z) \in T \text{ for all } n \in \mathbb{N}\big\}.\end{aligned}$$

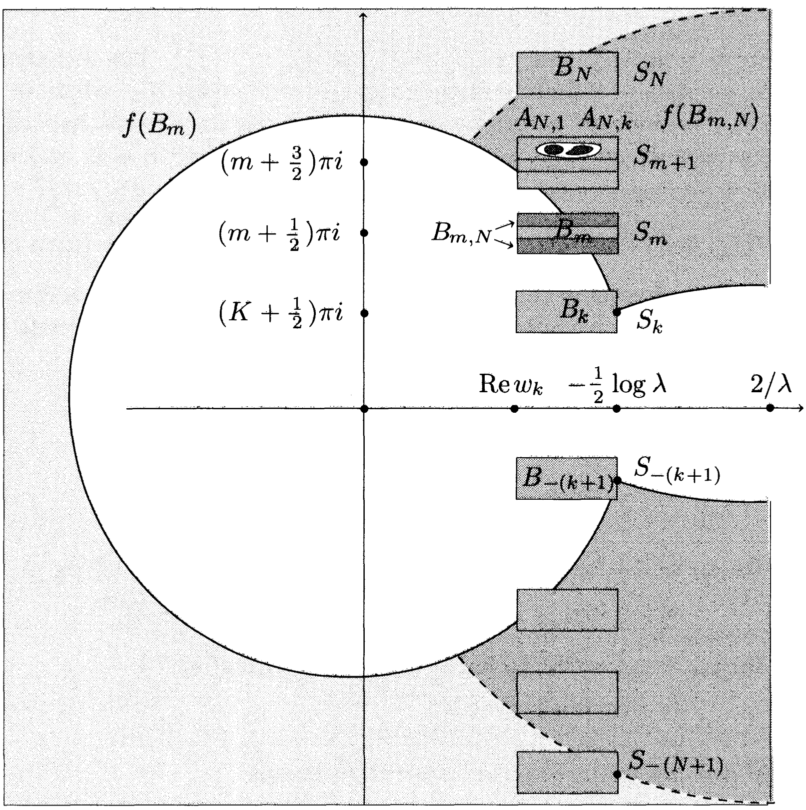

Figure 2

We want to show that $E \subset J(f)$ and estimate $\mathrm{HD}(E)$ from below.

LEMMA 9. — *If* $f^j(z) \in T$, $j = 0, \dots, n-1$, *then*

$$\left|(f^n)'(z)\right| \geq C_1^n,$$

for a constant $C_1 > 1$.

Proof. — Note that $|f'(z)| = 2|(\lambda + e^{-2z})e^z|^{-2}$. Taking the points $x_0 + iy$ in $B_m \subset T$, where x_0 is fixed, we have then

$$\left|f'\left(x_0 + iy\right)\right| \geq \left|f'\left(x_0 + i(\operatorname{Im} w_K + m\pi)\right)\right|.$$

Now, take $x + iy_0$ in B_m with $y_0 = \operatorname{Im} w_K + m\pi$. Then

$$\left|f'(x + iy_0)\right| \geq \left|f'(\operatorname{Re} w_K + iy_0)\right|,$$

and consequently $|f'(z)| \geq |f'(w_K + im\pi)|$ for all $z \in B_m$. As

$$\begin{aligned}\left|f'(w_K + im\pi)\right| &= \left|f'\left(f^{-1}(s_K)\right)\right| = 2\left|s_K(1 - \lambda s_K)\right| \\ &= (-\log\lambda)\left[10\left(1 + \lambda\log\lambda + \tfrac{5}{2}\lambda^2\log^2\lambda\right)\right]^{\frac{1}{2}} =: C_1,\end{aligned}$$

$C_1 > 1$, then $|(f^n)'(z)| \geq C_1^n$ for $z \in B_m$, and by periodicity of f, for all $z \in T$. $\square$

Take the smallest $M \in \mathbb{N}$ satisfying $M \geq 10^{1/2}\pi^{-1}(-\log\lambda) + 1$. Note that $M > K$. Fix $N \in \mathbb{N}$ such that $N \geq M$. For a pole s_N, consider the preimage $f^{-1}(s_N)$ lying in B_0, and define $\eta_N = |\operatorname{Im}(s_0 - f^{-1}(s_N))|$. We introduce auxiliary sets :

$$\begin{aligned} B_{m,N} &= \left\{z : \operatorname{Re} w_K < \operatorname{Re} z < -\tfrac{1}{2}\log\lambda,\ \eta_N < |\operatorname{Im}(z - s_m)| < \eta\right\}, \\ T_N &= \bigcup_{m \in I} B_{m,N}, \\ E_N &= \left\{z : f^n(z) \in T_N \ \text{ for all } \ n \in \mathbb{N}\right\}. \end{aligned}$$

Clearly $B_{m,N} \subset B_{m,N+1}$, $T_N \subset T_{N+1}$, $E_N \subset E_{N+1}$ for each $N \geq M$, $N \in \mathbb{N}$. Thus

$$\bigcup_{N \geq M} B_{m,N} = B_m, \qquad \bigcup_{N \geq M} T_N = T, \qquad \bigcup_{N \geq M} E_N = E.$$

To estimate from below the Hausdorff dimension of the set E, it is enough to find a lower bound for the dimension of each E_N.

LEMMA 10. — *For each $N \geq M$, $N \in \mathbb{N}$, the set E_N is contained in $J(f)$, so $E \subset J(f)$.*

The proof is analogous as that of LEMMA 5.

Note that $B_{m,N}$ has two components and $f(B_{m,N}) \cap B_{n,N} \neq \emptyset$ for

$$m, n \in I_1 = \{n \in \mathbb{Z} : -(N+1) \leq n \leq -(K+1) \text{ or } K \leq n \leq N\}.$$

Take a specific set $B_{s,N}$, $s \in I_1$, and let $A_{N,1}$ be a component of $f^{-1}(B_{s,N})$ contained in T_N. Define the following collection of the sets inductively :

$$\begin{aligned}
\mathcal{A}_{N,1} &= \{A_{N,1}\},\\
\mathcal{A}_{N,2} &= \big\{A_{N,2} : A_{N,2} \text{ is a component of } f^{-2}(B_{m,N})\\
&\qquad\qquad\qquad\qquad \text{for some } m \in I_1, \text{ and } A_{N,2} \subset A_{N,1}\big\},\\
&\ \ \vdots\\
\mathcal{A}_{N,k} &= \big\{A_{N,k} : A_{N,k} \text{ is a component of } f^{-k}(B_{m,N}) \text{ for some } m \in I_1,\\
&\qquad\qquad \text{and } A_{N,k} \subset A_{N,k-1} \text{ for some } A_{N,k-1} \in \mathcal{A}_{N,k-1}\big\}.
\end{aligned}$$

Thus $\mathcal{A}_{N,k}$ consists of $(2(N-K+1))^{k-1}$ sets $A_{N,k}$, $k \in \mathbb{N}$. Let

$$\mathcal{U}_{N,k} = \bigcup_{A_{N,k} \in \mathcal{A}_{N,k}} A_{N,k}, \quad A_N = \bigcap_{k=1}^{\infty} A_{N,k}.$$

Thus $A_N \subset E_N$ and so it is sufficient to find a lower bound for $\mathrm{HD}(A_N)$.

LEMMA 11. — *For each $A_{N,k} \in \mathcal{A}_{N,k}$, $k \in \mathbb{N}$,* $\operatorname{diam} A_{N,k} \leq DC_1^{-k}$, *where* $D = (\frac{1}{4}\log^2(-(2.5)^{\frac{1}{2}}\lambda\log\lambda) + 4\eta^2)^{\frac{1}{2}}$.

Proof. — By the definition of $A_{N,k}$ $f^k(A_{N,k})$ is a connected component of $B_{m,N}$ for some $m \in I_1$. As $B_{m,N} \subset B_m$

$$\begin{aligned}
\operatorname{diam} f^k(A_{N,k}) &\leq (\xi^2 + 4\eta^2)^{1/2}\\
&= \big(\tfrac{1}{4}\log^2\big(-(2.5)^{1/2}\lambda\log\lambda\big) + 4\eta^2\big)^{\frac{1}{2}} = D.
\end{aligned}$$

Moreover, $f^k(A_{N,k})$ is convex subset of $T_N \subset T$, so by LEMMA 9 $\operatorname{diam} A_{N,k} \leq \operatorname{diam} f^k(A_{N,k}) / \inf_{A_{N,k}} |(f^k)'| \leq DC_1^{-k}$. □

The sets $A_{N,k}$ and constants $d_k = DC_1^{-k}$, $k \in \mathbb{N}$, satisfy the conditions of LEMMA 1. To use this lemma we should find $\Delta_{N,k}$ such that for each $A_{N,k} \in \mathcal{A}_{N,k}$

$$\frac{\operatorname{vol}(\mathcal{U}_{N,k+1} \cap A_{N,k})}{\operatorname{vol}(A_{N,k})} \geq \Delta_{N,k}. \tag{9}$$

Note that $\mathcal{U}_{N,k+1} \cap A_{N,k} = \bigcup_{A_{N,k+1} \in G(A_{N,k})} A_{N,k+1}$, where

$$G(A_{N,k}) = \{A_{N,k+1} : A_{N,k+1} \in \mathcal{A}_{N,k+1},\ A_{N,k+1} \subset A_{N,k}\},$$

so

$$\frac{\operatorname{vol}(\mathcal{U}_{N,k+1} \cap A_{N,k})}{\operatorname{vol}(A_{N,k})} \geq L(f^k, A_{N,k})^{-2} \sum_{A_{N,k+1} \in G(A_{N,k})} \frac{\operatorname{vol}(f^k(A_{N,k+1}))}{\operatorname{vol}(f^k(A_{N,k}))}. \tag{10}$$

LEMMA 12. — *For each $A_{N,k} \in \mathcal{A}_{N,k}$, $k \in \mathbb{N}$, the distortion $L(f^k, A_{N,k})$ is bounded above by a constant $C_2 < 4$.*

Proof. — The branches of f^{-k}, $k \in \mathbb{N}$, are univalent in T since $f^k(0)$ and $f^k(1/\lambda)$ are contained in $\mathbb{R}$. Thus f^k is a homeomorphism of $A_{N,k} \in \mathcal{A}_{N,k}$ onto $f^k(A_{N,k})$ and

$$L(f^k, A_{N,k}) = L\big(f^{-k}, f^k(A_{N,k})\big) = L(f^{-k}, B_{m,N}).$$

As $B_{m,N}$ is contained in a disc D of diameter $2s = \operatorname{diam} B_m$, and $D \subset D(s_m, r)$ with $r = (K + \frac{1}{2})\pi$, it follows from LEMMA 2 that

$$\begin{aligned} L(f^{-k}, B_{m,N}) &\leq M\Big(\frac{s}{r}\Big) = \Big(\frac{r+s}{r-s}\Big)^4 \\ &= \left(\frac{-\frac{3}{2}\log\lambda + \frac{1}{2}[\frac{1}{4}\log^2(-(2.5)^{\frac{1}{2}}\lambda\log\lambda) + 4\eta^2]^{\frac{1}{2}}}{-\frac{3}{2}\log\lambda - \frac{1}{2}[\frac{1}{4}\log^2(-(2.5)^{\frac{1}{2}}\lambda\log\lambda) + 4\eta^2]^{\frac{1}{2}}}\right)^4 \\ &\leq \left(\frac{1 + (0.0277 + \varepsilon(\lambda))^{\frac{1}{2}}}{1 - (0.0277 + \varepsilon(\lambda))^{\frac{1}{2}}}\right)^4, \end{aligned}$$

where $\varepsilon(\lambda) = (\frac{1}{4}\log^2(-(2.5)^{\frac{1}{2}}\log\lambda) + 3.581)/(9\log^2\lambda)$. Thus

$$M\Big(\frac{s}{r}\Big) \leq (1.414)^4 \leq 3.9976 =: C_2 < 4$$

if $\lambda < 10^{-10}$. $\square$

LEMMA 13. — *There exist constants $0 < \alpha_N$, $\beta_N < 1$, and C_3, $C_4 > 0$ such that for each $A_{N,k} \in \mathcal{A}_{N,k}$, $k \in \mathbb{N}$*

$$C_3 - \alpha_N \leq \sum_{A_{N,k+1} \in G(A_{N,k})} \operatorname{vol}\big(f^k(A_{N,k+1})\big) \leq C_4 - \beta_N.$$

Proof. — Let $W_m \subset B_m$ be defined by

$$W_m = \{z : \operatorname{Re} w_{K+2} < \operatorname{Re} z < -\tfrac{1}{2}\log\lambda,\ \eta_N < |\operatorname{Im}(z - s_m)| < \eta\},$$

$$W_{m,N} = B_{m,N} \cap W_m,$$

where $w_{K+2} = f^{-1}(s_{K+2})$, $m \in I$. Then for each $A_{N,k} \in \mathcal{A}_{N,k}$

$$\sum_{A_{N,k+1} \in G(A_{N,k})} \operatorname{vol}\big(f^k(A_{N,k+1})\big) \geq \tfrac{1}{2}\operatorname{vol}\big(f^{-1}(T_N) \cap W_{m,N}\big), \tag{11}$$

where $f^k(A_{N,k})$ is a component of $B_{m,N}$ for some $m \in I$. Note that $f(B_m)$ intersects $B_{n,N}$ for $n \in I_2$ and covers $B_{n,N}$ for $n \in I_3$ where

$$I_2 = \{n \in \mathbb{Z} : -\tfrac{1}{2}(N-5) \leq n \leq -(K+1) \ \text{ or }\ K \leq n \leq \tfrac{1}{2}(N-3)\}$$

and $I_3 = I \setminus I_2$. Indeed, f maps $\{z : \operatorname{Re} z = \operatorname{Re} w_K\}$ onto the circle of radius $r_0 = r/(r^2 - \lambda^2)$ and centre $z_0 = -\lambda/(r^2 - \lambda^2)$, where $r = \exp(-2\operatorname{Re} w_K)$. For $0 < \lambda < \varepsilon$, $r_0 \approx -(2.5)^{\frac{1}{2}}\log\lambda$, $z_0 \approx -(2.5)\lambda\log^2\lambda$, so $r_0/\pi < \frac{1}{2}(N-1)$. The lines

$$\{z : \operatorname{Im} z = (m + \tfrac{1}{2})\pi - \eta\}, \quad \big(\text{resp. } \{z : \operatorname{Im} z = (m + \tfrac{1}{2})\pi + \eta\}\big)$$

are mapped by f onto circles passing through 0, $1/\lambda$ and the pole s_K (resp. $s_{-(K+1)}$), while $f(\{z : \operatorname{Re} z = -\frac{1}{2}\log\lambda\})$ is the line $\{z : \operatorname{Re} z = \frac{2}{\lambda}\}$.

As

$$\operatorname{vol}\big(f^{-1}(B_m) \cap B_n\big) = \int_{B_m \cap f(B_n)} |(f^{-1})'(z)|^2$$

and $(f^{-1})'(z) = (2z(1 - \lambda z))^{-1}$, we have

$$\max_{B_m} |(f^{-1})'| \leq \Big(4\big[\tfrac{1}{4}\log^2\big(-(2.5)^{\frac{1}{2}}\log\lambda\big) + (m\pi)^2\big]\big(1 + (\lambda m\pi)^2\big)\Big)^{-1},$$

$$\min_{B_m} |(f^{-1})'| \geq \Big(4\big[\tfrac{1}{4}\log^2\lambda + (m + \tfrac{1}{2} + \eta/\pi)^2\pi^2\big]\big(1 + (\lambda m\pi)^2\big)\Big)^{-1}.$$

In these estimates we take simply $|1 - \lambda z|^2 \approx 1 + (\lambda m\pi)^2$, which is a sufficient approximation for our purposes.

Now, we estimate $\operatorname{vol}(f^{-1}(\bigcup B_m) \cap W_n)$ for $m \in I_3$ and $n \in I$:

$$\begin{aligned}
&\operatorname{vol}\bigl(f^{-1}(\textstyle\bigcup B_m) \cap W_n\bigr)\\
&\quad \geq 2 \sum_{m \geq (N-1)/2} \operatorname{vol}(B_m)\bigl(\min_{B_m} |(f^{-1})'|^2\bigr)\\
&\quad \geq 2 \sum_{m \geq (N-1)/2} \frac{(-\frac{1}{2}\log(-(2.5)^{\frac{1}{2}}\lambda \log\lambda))2\eta}{4[\frac{1}{4}\log^2\lambda + (m+1/2+\eta/\pi)^2\pi^2](1+(\lambda m\pi)^2)}\\
&\quad \geq \tfrac{1}{8}\pi\bigl(-\log(-(2.5)^{\frac{1}{2}}\lambda\log\lambda)\bigr)\int_a^\infty (\lambda^2\pi^4)^{-1}\bigl((x^2+D_2^2)(x^2+D_1^2)\bigr)^{-1}\,\mathrm{d}x,
\end{aligned}$$

where $2\eta > \frac{1}{2}\pi$, $a = \frac{1}{2}N + \eta/\pi$, $D_1 = 1/(\lambda\pi)$, $D_2 = (-\log\lambda)/(2\pi)$,

$$\begin{aligned}
&\operatorname{vol}\bigl(f^{-1}(\textstyle\bigcup B_m) \cap W_n\bigr)\\
&\quad = \tfrac{1}{8}\pi\bigl(-\log(-(2.5)^{\frac{1}{2}}\lambda\log\lambda)\bigr)\int_a^\infty \frac{(x^2+D_2^2)^{-1}-(x^2+D_1^2)^{-1}}{\lambda^2\pi^4(D_1^2-D_2^2)}\,\mathrm{d}x\\
&\quad = \frac{-\log(-(2.5)^{\frac{1}{2}}\lambda\log\lambda)}{8\pi(1-\frac{1}{4}\lambda^2\log^2\lambda)}\Bigl[D_2^{-1}\arctan\Bigl(\frac{x}{D_2}\Bigr) - D_1^{-1}\arctan\Bigl(\frac{x}{D_1}\Bigr)\Bigr]_a^\infty\\
&\quad = \frac{1}{8\pi}\,\frac{1+G_1(\lambda)}{G_3(\lambda)}\Bigl[\pi^2 - 2\pi\arctan\Bigl(\frac{N\pi+2\eta}{-\log\lambda}\Bigr) + G_2(\lambda)\Bigr]
\end{aligned}$$

with

$$\begin{aligned}
G_1(\lambda) &= -\log(-(2.5)^{\frac{1}{2}}\log\lambda)/(-\log\lambda),\\
G_2(\lambda) &= \pi\lambda\log\lambda\bigl(\tfrac{1}{2}\pi - \arctan(\tfrac{1}{2}\lambda\pi N + \eta\lambda)\bigr),\\
G_3(\lambda) &= 1 - \tfrac{1}{4}\lambda^2\log^2\lambda;
\end{aligned}$$

we get

$$\begin{aligned}
&\operatorname{vol}\bigl(f^{-1}(\textstyle\bigcup B_m) \cap W_n\bigr)\\
&\quad \geq \frac{1}{8\pi}\bigl(1+G_1(\varepsilon)\bigr)\Bigl\{\pi^2 - 2\pi\arctan\bigl(10^{\frac{1}{2}} + 2(\pi+1)(-\log^{-1}\varepsilon)\bigr)\\
&\qquad\qquad + \pi^2\varepsilon\log\tfrac{1}{2}\varepsilon\Bigr\}\\
&\quad \geq 0.0569 =: C_3'.
\end{aligned}$$

Thus for $n \in I$

$$\operatorname{vol}\Bigl(f^{-1}\bigl(\bigcup_{m\in I_3} B_m\bigr) \cap W_n\Bigr) \geq C_3'. \tag{12}$$

Analogously we can find an upper bound for $\operatorname{vol}(f^{-1}(\bigcup_{m\in I_3} B_m)\cap B_n)$, $n\in I$, namely

$$\operatorname{vol}\Big(f^{-1}\big(\bigcup_{m\in I_3} B_m\big)\cap W_n\Big) \le 2\sum_{m\ge (N-1)/2} \operatorname{vol}(B_m)\big(\max_{B_m} |(f^{-1})'|^2\big)$$
$$\le 2\sum_{m\ge (N-1)/2} \frac{(-\frac12\log\lambda)\pi}{4\big[\frac14\log^2(-(2.5)^{\frac12}\log\lambda)+(m\pi)^2\big]\big(1+(\lambda m\pi)^2\big)}$$
$$\le \tfrac14\pi(-\log\lambda)\int_{(N-3)/2}^{\infty} \frac{(x^2+D_3^2)^{-1}-(x^2+D_1^2)^{-1}}{\lambda^2\pi^4(D_1^2-D_3^2)}\,\mathrm{d}x,$$

where $D_3=\log(-(2.5)^{\frac12}\log\lambda)/(2\pi)$, and therefore

$$\le \frac{-\log\lambda}{4\pi}\big[1-\tfrac14\lambda^2\log^2(-(2.5)^{\frac12}\log\lambda)\big]^{-1}\Big[D_3^{-1}\arctan\big(\frac{x}{D_3}\big)\Big]_{\frac12(N-3)}^{\infty}$$
$$\le \frac{-\log\lambda}{2\pi\big[10^{\frac12}(-\log\lambda)/\pi-2\big]} < \frac{0.55}{\pi} =: C_4'.$$

Moreover, if

$$m\in I_4=\big\{n\in\mathbb{Z} : -\tfrac12(N-5)\le n\le -(K+2)\ \text{ or } (K+1)\le n\le \tfrac12(N-3)\big\},$$

then using

$$\operatorname{vol}(B_m\cap f(B_n))\le \pi(-\log\lambda)\big(-\frac12-\big[2.5-(k\pi/\log\lambda)^2\big]^{\frac12}\big)$$

we obtain

$$\operatorname{vol}\Big(f^{-1}\big(\bigcup_{m\in I_4} B_m\big)\cap B_n\Big)$$
$$\le 2\sum_{K+1}^{(N-3)/2} \frac{\pi(-\log\lambda)\big(-\frac12-\big[2.5-(k\pi/\log\lambda)^2\big]^{\frac12}\big)}{4\big[\frac14\log^2\big(-(2.5)^{\frac12}\log\lambda\big)+(m\pi)^2\big]\big[1+(\lambda m\pi)^2\big]}$$
$$\le \tfrac12\pi(-\log\lambda)\int_K^{\frac12(N-1)} \frac{\big(\frac12-\big[2.5-(x\pi/\log\lambda)^2\big]^{\frac12}\big)\,\mathrm{d}x}{\big[\frac14\log^2\big(-(2.5)^{\frac12}\log\lambda\big)+(m\pi)^2\big]\big[1+(\lambda m\pi)^2\big]}$$
$$\le \tfrac12\pi(-\log\lambda)\int_K^{\frac12(N-1)} \big(\tfrac12-\big[2.5-(x\pi/\log\lambda)^2\big]^{\frac12}\big)(\pi x)^{-2}\,\mathrm{d}x$$
$$\le \tfrac12\int_{\frac32}^{(2.5)^{\frac12}} \big(\tfrac12-(2.5-y^2)^{\frac12}\big)y^{-2}\,\mathrm{d}y < 0.0091 =: C_4''.$$

As $\mathrm{vol}(f^{-1}(B_{-(K+1)} \cup B_K) \cap B_n) < 0.0001 =: C_4'''$ we have for $n \in I$

$$\mathrm{vol}\Big(f^{-1}\big(\bigcup_{m\in I} B_m\big) \cap B_n\Big) \le C_4' + C_4'' + C_4''' \le 0.177 =: C_4. \tag{13}$$

Note that for $n \in I$

$$\lim_{N\to\infty} \mathrm{vol}\Big(f^{-1}\big(\bigcup_{m\in I} B_{m,N}\big) \cap W_{n,N}\Big) = \mathrm{vol}\Big(f^{-1}\big(\bigcup_{m\in I} B_m\big) \cap W_n\Big), \tag{14}$$

so by (11)–(12) and (13)–(14) there are constants $C_3 = \frac{1}{2}C_3'$ and α_N, β_N such that

$$C_3 - \alpha_N \le \sum_{A_{N,k+1}\in G(A_{N,k})} \mathrm{vol}\big(f^k(A_{N,k+1})\big) \le C_4 - \beta_N,$$

and $\alpha_N, \beta_N \to 0$ if $N \to \infty$. LEMMA 12 and LEMMA 13 together with (10) imply that

$$\frac{\mathrm{vol}(\mathcal{U}_{N,k+1} \cap A_{N,k})}{\mathrm{vol}(A_{N,k})} \ge \frac{C_3 - \alpha_N}{C_2^2 C_5}$$

with $C_5 = -\log(-(2.5)^{\frac{1}{2}}\lambda\log\lambda) \ge \mathrm{vol}(B_m) \ge \mathrm{vol}(B_{m,N})$, that is, $\Delta_{N,k} = (C_3 - \alpha_N)/(C_2^2C_5)$ by (9).

Now we can apply LEMMA 1 to the sets $A_{N,k}$. Thus

$$\begin{aligned}\mathrm{HD}(A_N) &\ge 2 - \limsup_{k\to\infty} \frac{k\big|\log[(C_3-\alpha_N)/(C_2^2C_5)]\big|}{|\log(DC_1^{-k})|}\\ &= 2 - \frac{\big|\log[(C_3-\alpha_N)/(C_2^2C_5)]\big|}{|\log C_1|}.\end{aligned}$$

As $A_N \subset A_{N+1}$ and $\bigcup_{N\ge M} A_N = A$ we have

$$\mathrm{HD}(A) \ge \lim_{N\to\infty} \mathrm{HD}(A_N) \ge 2 - \frac{\big|\log(C_3/(C_2^2C_5))\big|}{|\log C_1|}.$$

By the definition $A_N \subset E_N$ for each $N \ge M$ and $\bigcup_{N\ge M} E_N = E \subset J(f_\lambda)$, so

$$\begin{aligned}\mathrm{HD}(J(f_\lambda)) &\ge 2 - \frac{|\log(C_3/(C_2^2C_5))|}{|\log C_1|}\\ &= 2 - \frac{\big|\log 0.0284 - \log 3.998^2 - \log\big(-\log(-(2.5)^{\frac{1}{2}}\lambda\log\lambda)\big)\big|}{\big|\log(-\log\lambda) + \frac{1}{2}\log\big(10(1-\lambda\log\lambda+\frac{5}{2}\lambda^2\log^2\lambda)\big)\big|}\\ &\ge 2 - \frac{\log\big(|\log\lambda| - \log(1.581\,|\log\lambda|)\big) + 6.331}{\log(|\log\lambda|) + 1.152}\\ &\ge 1 - \frac{C}{\log|\log\lambda|}\end{aligned}$$

for some $C > 0$.

Our lower bound for $\mathrm{HD}(J(f_\lambda))$ tends to 1 if $\lambda \to 0^+$, as follows from the above inequality valid for $0 < \lambda < 10^{-10}$. $\square$

3. Proof of Theorem 3

Let $f_\lambda(z) = \lambda \tan z$. Suppose $\lambda \in \mathbb{R}$ and $0 < |\lambda| < 1$. Then 0 is an attracting fixed point for f_λ. Let I denote the component of the basin of attraction of 0 in $\mathbb{R}$. I is an open interval of the form $(-p\,; p)$, where $f_\lambda(\pm p) = \pm p$. The points $\pm p$ lie on a repelling periodic orbit of period two if $-1 < \lambda < 0$, or are repelling fixed points if $0 < \lambda < 1$. The set $\mathbb{R} \setminus f_\lambda^{-1}(I)$ consists of infinitely many disjoint open intervals I_m, $m \in \mathbb{Z}$. Each of them contains exactly one pole $s_m = (m + \frac{1}{2})\pi$. Thus $f_\lambda : I_m \to (\mathbb{R} \cup \{\infty\}) \setminus I$ and $|f_\lambda{}'(x)| > 1$ for each $x \in I_m$. Thus

$$J(f_\lambda) \setminus \{\infty\} = \bigcap_{n=0}^{\infty} f_\lambda{}^{-n}\Big(\bigcup_{m\in\mathbb{Z}} I_m\Big)$$

is a Cantor subset of $\mathbb{R}$.

As before we fix λ and assume additionally $0 < |\lambda| < \frac{3}{4}$. Let

$$\begin{aligned} w_1 &= f^{-1}(s_1) = \arctan(s_1/\lambda) \approx \tfrac{1}{2}\pi - 2|\lambda|/(3\pi), \\ \delta &= \delta_1 = |s_0 - w_1| \approx 2|\lambda|/(3\pi). \end{aligned}$$

Define the sets :

$$\begin{aligned} B_m &= \big\{z \in \mathbb{R} : |z - s_m| < \delta\big\}, \\ T &= \bigcup_{m\in J} B_m, \\ J &= \mathbb{Z} \setminus \{-1, 0\}, \\ E &= \big\{z \in \mathbb{R} : f^n(z) \in T \text{ for all } n \in \mathbb{N}\big\}. \end{aligned}$$

Fix $N \in \mathbb{N}$, $N \geq M \geq 10$, and define $\delta_N = |s_0 - w_N|$, where $w_N = f^{-1}(s_N) \in (0, \pi)$. Let

$$\begin{aligned} B_{m,N} &= \big\{z \in \mathbb{R} : \delta_N < |z - s_m| < \delta\big\}, \\ T_N &= \bigcup_{m\in J} B_{m,N}, \\ E_N &= \big\{z \in \mathbb{R} : f^n(z) \in T_N \text{ for all } n \in \mathbb{N}\big\}. \end{aligned}$$

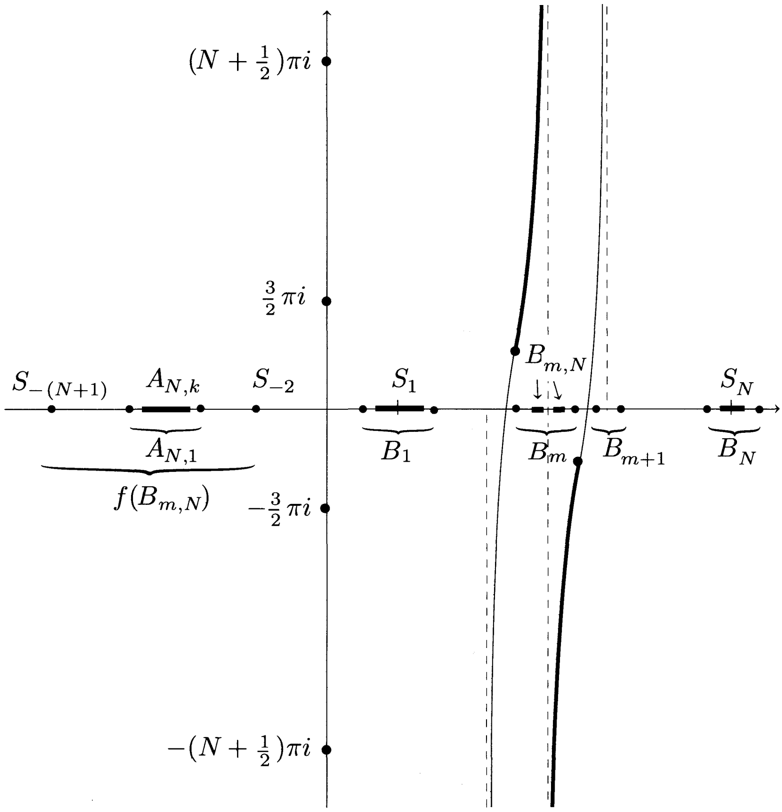

Figure 3

Then $B_{m,N} \subset B_{m,N+1}$, $T_N \subset T_{N+1}$, $E_N \subset E_{N+1}$ for each $N \geq M$, $N \in \mathbb{N}$, and thus

$$\bigcup_{N\geq M} B_{m,N} = B_m, \qquad \bigcup_{N\geq M} T_N = T, \qquad \bigcup_{N\geq M} E_N = E.$$

We show that $E \subset J(f)$ and estimate $\mathrm{HD}(E)$ from below.

Take $z \in B_m$, $z \neq s_m$. As $f'(z) = \lambda(1+\tan^2 z)$, it is easy to see that $|f'(z)| \geq |f'(s_m - \delta)| = |f'(s_m+\delta)|$. But

$$\big|f'(s_m-\delta)\big| = \big|f'(f^{-1}(s_{\pm 1}))\big| = \big|\lambda(1+(s_1/\lambda)^2)\big| =: C_1 > 1.$$

Thus $|f'(z)| \geq C_1$, and if additionally $f^j(z) \in T$, $j = 0, \dots, n-1$, then

$$\min_T \big|(f^n)'\big| \geq C_1^n. \tag{15}$$

Using this property one can prove that $E_N \subset J(f)$, and consequently $E = \bigcup_{N \geq M} E_N \subset J(f)$ (compare LEMMA 5).

The set $B_{m,N}$ consists of two intervals, $f(B_{m,N}) \cap B_{n,N} \neq \emptyset$ if

$$m, n \in J_1 = \{n \in \mathbb{Z} : -(N+1) \leq n \leq -2 \text{ or } 1 \leq n \leq N\},$$

and $f(B_{m,N}) \supset B_{n,N}$ if

$$n \in J_2 = \{n \in \mathbb{Z} : -N \leq n \leq -3 \text{ or } 2 \leq n \leq N-1\},, \quad m \in J_1.$$

Let $s \in J_2$. Take a component of $f^{-1}(B_{s,N})$ contained in T_N and denote it by $A_{N,1}$. Define the following family of sets :

$$\begin{aligned}
\mathcal{A}_{N,1} &= \{A_{N,1}\},\\
\mathcal{A}_{N,2} &= \{A_{N,2} : A_{N,2} \text{ is a component of } f^{-2}(B_{m,N}) \text{ for some } m \in J_2,\\
&\qquad\qquad \text{and } A_{N,2} \subset A_{N,1}\},\\
&\ \vdots\\
\mathcal{A}_{N,k} &= \{A_{N,k} : A_{N,k} \text{ is a component of } f^{-k}(B_{m,N}) \text{ for some } m \in J_2,\\
&\qquad\qquad \text{and } A_{N,k} \subset A_{N,k-1} \text{ for some } A_{N,k-1} \in \mathcal{A}_{N,k-1}\}.
\end{aligned}$$

Let $\mathcal{U}_{N,k} = \bigcup_{A_{N,k} \in \mathcal{A}_{N,k}} A_{N,k}$ and $A_N = \bigcap_{k=1}^{\infty} \mathcal{U}_{N,k}$, so clearly $A_N \subset E_N$ for each $N \geq M$. To estimate $\mathrm{HD}(A_N)$ we should find the constants d_k and $\Delta_{N,k}$ such that

$$\frac{\mathrm{vol}(\mathcal{U}_{N,k+1} \cap A_{N,k})}{\mathrm{vol}(A_{N,k})} \geq \Delta_{N,k}, \tag{16}$$

and $\operatorname{diam} A_{N,k} \leq d_k < 1$. Let $A_{N,k} \in \mathcal{A}_{N,k}$, then $f^k(A_{N,k})$ is connected component of $B_{m,N}$ for some $m \in J_2$. By (15)

$$\operatorname{diam}(A_{N,k}) \leq \frac{\operatorname{diam} f^k(A_{N,k})}{\inf_{A_{N,k}} |(f^k)'|} \leq L C_1^{-k}, \tag{17}$$

where $L = 2(\delta - \delta_N)$. As before

$$G(A_{N,k}) = \{A_{N,k+1} \in \mathcal{A}_{N,k+1} : A_{N,k+1} \subset A_{N,k}\}.$$

Then

$$\frac{\mathrm{vol}(\mathcal{U}_{N,k+1} \cap A_{N,k})}{\mathrm{vol}(A_{N,k})} \geq \frac{1}{L(f^k, A_{N,k})^2} \sum_{A_{N,k+1} \in G(A_{N,k})} \frac{\mathrm{vol}(f^k(A_{N,k+1}))}{\mathrm{vol}(f^k(A_{N,k}))}. \tag{18}$$

First we find a bound for $L(f^k, A_{N,k})$. The branches of f^{-k}, $k \in \mathbb{N}$, are univalent in T, since both asymptotic values λi, $-\lambda i$ of f tend to the attracting fixed point 0 along the imaginary axis. Thus

$$L(f^k, A_{N,k}) = L(f^{-k}, f^k(A_{N,k})) = L(f^{-k}, B_{m,N}).$$

As $B_{m,N} \subset D(s_m, \delta) \subset D(s_m, \frac{3}{2}\pi)$, and it follows from LEMMA 2 that

$$L(f^k, A_{N,k}) \leq M\Big(\frac{2|\lambda|/(3\pi)}{3\pi/2}\Big) =: C_2. \tag{19}$$

Let $W_m = \{z \in \mathbb{R} : |z - s_m| < \eta\}$, $W_{m,N} = \{z \in \mathbb{R} : \eta_N < |z - s_m| < \eta\}$ where $\eta = \eta_1 = |s_0 - f^{-1}(s_1 + \delta)|$, $f^{-1}(s_1 + \delta) \in (0\,; \pi)$, and $\eta_N = |s_0 - f^{-1}(s_N + \delta)|$, $f^{-1}(s_N + \delta) \in (0\,; \pi)$ for $N \geq M$. Then $W_m \subset B_m$, $W_{m,N} \subset B_{m,N}$. Note that

$$\sum_{A_{N,k+1} \in G(A_{N,k})} \operatorname{vol}\big(f^k(A_{N,k+1})\big) \geq \tfrac{1}{2}\operatorname{vol}\big(f^{-1}(T) \cap W_{m,N}\big), \tag{20}$$

if $f^k(A_{N,k})$ is a connected component of $B_{m,N}$, $m \in J_2$, and

$$\lim_{N\to\infty} \operatorname{vol}\big(f^{-1}(T_N) \cap W_{m,N}\big) = \operatorname{vol}\big(f^{-1}(T) \cap W_m\big). \tag{21}$$

Consider the quotient $\operatorname{vol}(f^{-1}(T) \cap W_m)/\operatorname{vol}(B_m)$. The uniform bound for it from below for $m \in J$ is obtained from the inequalities

$$\left\{\begin{aligned} & 2\sum_{m=2}^{\infty} \frac{\operatorname{vol}(B_m)\big|(f^{-1})'(s_m + \delta)\big|}{\operatorname{vol}(B_m)} \\ & = 2\sum_{m=2}^{\infty} \frac{|\lambda|}{\lambda^2 + ((m + \frac{1}{2})\pi + \delta)^2} \\ & \geq 2\sum_{m=2}^{\infty} \frac{|\lambda|}{\lambda^2 + (m+1)^2\pi^2} = 2\sum_{m=3}^{\infty} \frac{|\lambda|}{\lambda^2 + m^2\pi^2} \\ & = \Big|\coth|\lambda| - |\lambda|^{-1} - 2|\lambda|\big[(\lambda^2 + \pi^2)^{-1} + (\lambda^2 + 4\pi^2)^{-1}\big]\Big| =: C_3, \end{aligned}\right. \tag{22}$$

since $1/\lambda - \sum\limits_{n=1}^{\infty} 2\lambda/(\lambda^2 + n^2\pi^2) = \coth\lambda$ from the well known expansion of coth in fractions.

As $f^k(A_{N,k})$ is a connected component of $B_{m,N} \subset B_m$ for some $m \in J_2$, this implies together with (20)–(22), that there are constants $0 < \alpha_N < 1$ such that $\alpha_N \to 0$, and

$$\sum_{A_{N,k+1} \in G(A_{N,k})} \frac{\operatorname{vol}(f^k(A_{N,k+1}))}{\operatorname{vol}(f^k(A_{N,k}))} \geq C_3 - \alpha_N. \tag{23}$$

By (16), (18)–(19) and (23) $\Delta_{N,k} = (C_3 - \alpha_N)/C_2^2$. Applying LEMMA 1 to the sets $A_{N,k}$ we have

$$\begin{aligned} 1 \geq \operatorname{HD}(A_n) &\geq 1 - \limsup_{k \to \infty} \frac{k \big|\log[(C_3 - \alpha_N)/C_2^2]\big|}{|\log(LC_1^{-k})|} \\ &= 1 - \frac{\big|\log[(C_3 - \alpha_N)/C_2^2]\big|}{|\log C_1|} \end{aligned}$$

As $A_N \subset E_N$ and $\bigcup_{N \geq M} A_N = A$, we obtain

$$1 \geq \operatorname{HD}(A) \geq \lim_{N \to \infty} \operatorname{HD}(A_N) = 1 - \frac{|\log(C_3/C_2^2)|}{|\log C_1|}.$$

Moreover, for each $N \geq M$, $A_N \subset E_N$ and $\bigcup_{N \geq M} E_N = E \subset J(f_\lambda) \subset \mathbb{R}$, so

$$\begin{aligned} 1 \geq \operatorname{HD}(J(f_\lambda)) &\geq 1 - \frac{|\log(C_3/C_2^2)|}{|\log C_1|} \\ &= 1 - \frac{\Big|\log\Big\{\Big|\coth|\lambda| - \frac{1}{|\lambda|} - \Big[\frac{2|\lambda|}{\lambda^2+\pi^2} + \frac{2|\lambda|}{\lambda^2+4\pi^2}\Big]\Big|\Big\} - \log\Big\{\frac{\frac{3}{2}\pi + 2|\lambda|/(3\pi)}{\frac{3}{2}\pi - 2|\lambda|/(3\pi)}\Big\}^8\Big|}{\big|\log|\lambda| + \log\big(1 + 9\pi^2/(4\lambda)^2\big)\big|} \\ &\geq \frac{C}{|\log|\lambda||} \end{aligned}$$

for some $C > 0$, which concludes the proof of THEOREM 3.

In the second part (see [6]) we give the global formula for a lower bound for the Hausdorff dimension of the Julia set of meromorphic function. To prove it we construct a limit Hausdorff measure of the computable dimension, supported on hyperbolic subsets of the Julia set.

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