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SMOOTHNESS AND IRREDUCIBILITY OF VARIETIES OF PLANE CURVES WITH NODES AND CUSPS

BY

EUGENII SHUSTIN (*)

RÉSUMÉ. — Soit V(d,m,k) la variété des courbes projectives planes irréductibles de degré d n'ayant pour singularités que m nodes et k cusps. Nous montrons que V(d,m,k) est non vide, lisse et irréductible quand $m+2k<\alpha d^2$ où α est une constante absolue explicite. Cette inégalité est optimale quant à l'exposant de d

ABSTRACT. — Let V(d,m,k) be the variety of plane projective irreducible curves of degree d with m nodes and k cusps as their only singularities. We prove that V(d,m,k) is non-empty, non-singular and irreducible when $m+2k < \alpha d^2$, where α is some absolute explicit constant. This estimate is optimal with respect to the exponent of d

0. Introduction

In the present article we deal with plane projective algebraic curves over an algebraically closed field of characteristic 0.

It is well-known that the variety of irreducible curves of a given degree with a given number of nodes is non-singular [9], irreducible [2], and that each germ of this variety is a transversal intersection of germs of equisingular strata corresponding to all singular points [9] (from now on, speaking of a variety with the last property, we shall write T-variety, or variety with property T).

Our goal is a similar result for curves with nodes and ordinary cusps. Let V(d, m, k) denote the set of irreducible curves of degree d with m nodes and k cusps as their only singularities.

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• It is known (see [5], [6]) that $V(d, m, k) = \emptyset$ if

$$\frac{9}{8}m + 2k > \frac{5}{8}d^2.$$

• On the other hand (see [13]), $V(d, m, k) \neq \emptyset$ when

(0.1)
$$m + 2k \le \frac{1}{2}d^2 + \mathcal{O}(d).$$

Our result is

Theorem 0.2.

• If $m + 2k \le \alpha_0 d^2$, where

(0.3)
$$\alpha_0 = \frac{7 - \sqrt{13}}{81} \approx 0.0419,$$

then V(d, m, k) is a non-empty non-singular T-variety of dimension

$$\frac{1}{2}d(d+3) - m - 2k.$$

• If $m + 2k \le \alpha_1 d^2$, where

(0.4)
$$\alpha_1 = \frac{2}{225} \approx 0.0089,$$

then V(d, m, k) is irreducible.

Let us make some comments.

First, (0.3) implies (0.1), and then $V(d, m, k) \neq \emptyset$.

Let \mathbb{P}^N , with $N = \frac{1}{2}d(d+3)$, be the space of plane curves of degree d. Let z be a singular point of $F \in \mathbb{P}^N$. It is well-known (see [2, [9]) that:

- (1) if z is a node then the germ at F of the variety of curves $\Phi \in \mathbb{P}^N$, having a node in some neighbourhood of z, is smooth, has codimension 1, and its tangent space is open in $\{\Phi \in \mathbb{P}^N \mid z \in \Phi\}$;
- (2) if z is a cusp then the germ at F of the variety of curves $\Phi \in \mathbb{P}^N$, having a cusp in some neighbourhood of z, is smooth, has codimension 2, and its tangent space is open in $\{\Phi \in \mathbb{P}^N \mid (\Phi \cdot F)(z) \geq 3\}$ (here and further on the notation $(F \cdot G)(z)$ means the intersection number of the curves F, G at the point z).

Hence the property T implies the smoothness of V(d, m, k) and the expected value of its dimension given in Theorem 0.2. Further, it is well-known [15] that V(d, m, k) is a non-singular T-variety, when

$$(0.5) k < 3d.$$

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Generalizations of this fact to arbitrary singularities, given in [1], [12] are based — in fact — on the same idea. The following conditions are sufficient for the smoothness of V(d, m, k) and the property T:

$$m = 0$$
, $2k < \frac{(7 - \sqrt{13})d^2}{81} \approx 0.0418 d^2$ (see [10], [11]),

and for the irreducibility of V(d, m, k):

$$m+2k < \frac{3}{2}d$$
 (see [10], [11]), $k \le 3$ (see [7]), $\frac{1}{2}(d^2-4d+1) \le m \le \frac{1}{2}(d^2-3d+2)$ (see [8]).

The main idea of our proof is as follows. We have to prove the property T and the irreducibility for V(d, m, k). To any curve $F \in V(d, m, k)$ with nodes z_1, \ldots, z_m and cusps w_1, \ldots, w_k we assign two linear systems of curves of degree n:

$$\Lambda_1(n, F) = \{ \Phi \mid z_1, \dots, z_m \in \Phi, \ (\Phi \cdot F)(w_i) \ge 3, \ i = 1, \dots, k \},$$

$$\Lambda_2(n, F) = \{ \Phi \mid z_1, \dots, z_m \in \text{Sing}(\Phi), \ (\Phi \cdot F)(w_i) \ge 6, \ i = 1, \dots, k \}.$$

First we show the non-speciality of $\Lambda_1(d, F)$ for any $F \in V(d, m, k)$, which means according to the Riemann-Roch theorem that

(0.6)
$$\dim \Lambda_1(d,F) = \frac{1}{2}d(d+3) - m - 2k.$$

On the other hand, $\Lambda_1(d,F)$ is the intersection of the tangent spaces to germs of equisingular strata at F in the space of curves of degree d, and (0.6) gives us the transversality of this intersection, or the desired property T. Then we show that, for any F from some open dense subset $U \subset V(d,m,k)$, the system $\Lambda_2(d,F)$ is non-special. That implies the irreducibility. Indeed, first we show that an open dense subset of $\Lambda_2(d,F)$ is contained in V(d,m,k); more precisely, it consists of curves of degree d having m nodes in a fixed position and k cusps in a fixed position with fixed tangents. Then from the non-speciality we derive that $\dim \Lambda_2(d,F) = \mathrm{const}, F \in U$, and that conditions imposed by fixed singular points on curves of degree d are independent. Afterwards we represent U as an open dense subset of the space of some linear bundle, whose fibres are $\Lambda_2(d,F)$, $F \in U$, and whose base is an open dense subset of $\mathrm{Sym}^m(\mathbb{P}^2) \times \mathrm{Sym}^k(P(T\mathbb{P}^2))$, where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane.

The text is divided into five parts: in section 1 there are some preliminary notions and results; in section 2 we present examples of reducible varieties or ones without property T; in section 3 we construct irreducible curves in $\Lambda_1(n, F)$, where n < d; in section 4 we prove the property T; and in section 5 — the irreducibility.

1. Preliminaries

Here we shall recall some notions and well-known classical results [3], [14], and also present some simple technical results needed below. Namely, we introduce a certain class of linear systems of plane curves and show how to compute their dimensions by means of linear series on curves.

Let $\Sigma = \bigoplus_{t \geq 0} \Sigma(t)$ be the graded ring of polynomials in three homogeneous variables over the base field. We think of the space of plane curves of degree t as the projectivization $P(\Sigma(t))$. A linear system of plane curves of degree t is a subspace of $P(\Sigma(t))$. Let

$$I = \bigoplus_{t \geq 0} I(t) \subset \Sigma$$

be a homogeneous ideal, defining a zero-dimensional subscheme $Z \subset \mathbb{P}^2$. This ideal determines a sequence of linear systems $\Lambda(t) = P(I(t)), t \geq 1$. Denote this class of linear systems by \mathcal{C} . In other words, these are linear systems defined by linear conditions associated to finite many base points. It is well-known [3] that

(1.1)
$$\dim \Lambda(t) = \dim P(\Sigma(t)) - \deg Z + i(\Lambda(t)),$$

where $i(\Lambda(t)) \geq 0$ is called the *speciality index* of $\Lambda(t)$. If $i(\Lambda(t)) = 0$ then the linear system $\Lambda(t)$ is called *non-special*. For a given ideal I, $\Lambda(t)$ is non-special when t is big enough (see [3]).

Proposition 1.2. — Let $\Lambda(t)$, $\Lambda'(t)$ belong to C, and, for all $t \geq 1$,

$$\Lambda(t) \subset \Lambda'(t)$$
.

If, for some $n \geq 1$, the system $\Lambda(n)$ is non-special then $\Lambda'(n)$ is non-special.

Proof. — The systems $\Lambda(t), \Lambda'(t)$ are non-special for t big enough. Take a straight line $L \in P(\Sigma(1))$ not intersecting the zero-dimensional schemes Z, Z' associated to our linear systems. Let us embed the space $P(\Sigma(n))$ into $P(\Sigma(t))$, multiplying by L^{t-n} . Then:

$$\Lambda(n) = \Lambda(t) \cap P(\Sigma(n)), \quad \Lambda'(n) = \Lambda'(t) \cap P(\Sigma(n)).$$

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So, the non-speciality of $\Lambda(n)$ means the transversality of the intersection of $\Lambda(t)$ and $P(\Sigma(n))$ in $P(\Sigma(t))$. But this implies that $\Lambda'(t)$ and $P(\Sigma(n))$ intersect transversally in $P(\Sigma(t))$, hence

$$\operatorname{codim}(\Lambda'(n), P(\Sigma(n))) = \operatorname{codim}(\Lambda'(t), P(\Sigma(t))) = \operatorname{deg} Z',$$

what is equivalent to the desired non-speciality.

From now on, divisor will always mean an effective Cartier divisor on a curve.

Let F be a reduced plane curve. For any divisor D on F and any component $H \subset F$ the symbol $D_{|H}$ means the restriction of D on H. For any curve G the symbol G_F means the formal expression $\sum n(P) \cdot P$, where the sum is taken over all local branches of F and n(P) is the intersection number of P and G. If F, G have no common components, G_F is the divisor on F cut out by G, otherwise we admit infinite coefficients in the above expression.

By D(F) we denote the double point divisor of the curve F. We omit its exact definition (see, for example, [14]), but only list the properties used in the sequel.

Proposition 1.3 (see [14]).

(1) The divisor D(F) can be expressed as

$$D(F) = \sum n(P) \cdot P,$$

where P runs through all the local branches of F centered at singular points, and the coefficients n(P) are positive integers. In particular, n(P) = 1 for both branches centered at a node, and n(P) = 2 for a branch centered at a cusp.

- (2) Let z be a singular point of the curve F and a non-singular point of some curve G, then
 - (i) for any singular local branch P of F centered at z,

$$(G \cdot P)(z) < n(P) + 1,$$

(ii) for any pair P_1 , P_2 of local branches of F centered at z,

$$(G \cdot P_1) \le n(P_1)$$
, or $(G \cdot P_2) \le n(P_2)$.

(3) If F is an irreducible curve of degree d and geometric genus g(F) then:

$$\deg D(F) = d(d-3) + 2 - 2g(F).$$

(4) If a reduced curve G has no common components with F then:

$$D(FG)|_F = D(F) + G_F.$$

For any divisor D on F, the symbol $\mathcal{L}_F(n,D)$ denotes the linear system of plane curves of degree n

$$\{\Phi \mid \Phi_F \geq D + D(F)\}.$$

It is clear from the definition and Proposition 1.3 that $\Lambda_1(n, F)$, $\Lambda_2(n, F)$ belong to this class. Also these systems belong to C.

THEOREM 1.4 (Brill-Noether (see [14])). — If F is irreducible then curves from $\mathcal{L}_F(n,D)$ cut out on F the linear series $|nL_F - D - D(F)|$, where L is a general straight line.

Theorem 1.5 (Noether (see [14])). — Let F_1, \ldots, F_k be different irreducible curves of degrees n_1, \ldots, n_k , and $F = F_1 \cdots F_k$, deg F = d. Then:

(1.6)
$$\mathcal{L}_F(n,D) = \sum_{i=1}^k \mathcal{L}_{F_i}(n+n_i-d,D_{|F_i}) \cdot F_1 \cdots F_{i-1}F_{i+1} \cdots F_k.$$

Theorem 1.7 (Riemann-Roch for curves (see [3], [14])). — For any divisor D on an irreducible curve F the dimension of the linear series |D| is

$$\dim |D| = \deg D - g(F) + i(D),$$

where i(D) is non-negative. If $\deg D > 2g(F) - 2$ then i(D) = 0.

Proposition 1.8. — For any reduced curve F of degree $d \leq n$,

(1.9)
$$\dim \mathcal{L}_F(n,D) \ge \frac{1}{2}n(n+3) - \frac{1}{2}\deg D(F) - \deg D.$$

The non-speciality of $\mathcal{L}_F(n,D)$ is equivalent to the equality in (1.9).

Proof. — Assume that F is irreducible. Representing $\mathcal{L}_F(n,D)$ as the span of $|nL_F - D - D(F)|$ and $F \cdot P(\Sigma(n-d))$, we obtain

$$\dim \mathcal{L}_F(n,D) = \dim |nL_F - D - D(F)| + \dim \Sigma(n-d),$$

hence according to Theorem 1.7 and Proposition 1.3,

$$\dim \mathcal{L}_F(n,D) \ge nd - \deg D - \deg D(F) - g(F)$$

$$+ \frac{1}{2}(n-d+1)(n-d+2)$$

$$= nd - \frac{1}{2}d(d-3) - 1 - \deg D - \frac{1}{2}\deg D(F)$$

$$+ \frac{1}{2}(n-d+1)(n-d+2),$$

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which is equivalent to (1.9). Also we obtain that the equality in (1.9) means i(D) = 0. Therefore, for all $t \ge d$,

(1.10)
$$\operatorname{codim}(\mathcal{L}_F(t,D), P(\Sigma(t))) = \frac{1}{2} \operatorname{deg} D(F) + \operatorname{deg} D.$$

On the other hand, for t big enough, $\mathcal{L}_F(t, D)$ is non-special. Comparing this with (1.1) and (1.10), we get that the equality in (1.9) means the non-speciality of $\mathcal{L}_F(n, D)$.

If F is reducible, combine the previous computation with (1.6).

Proposition 1.11. — Let $F \in V(d, m, k)$.

(1) If $G \in \Lambda_1(n, F)$ is reduced, then there is a divisor D on G of degree $\leq m + 2k$ such that, for all $t \geq 1$,

(1.12)
$$\Lambda_1(t,F) \supset \mathcal{L}_G(t,D).$$

(2) Let $G \in \Lambda_1(n, F)$ be irreducible, let S be a subset of $\operatorname{Sing}(F)$, and let H be a reduced curve containing S but not G. Let $\Lambda_3(t, F, S)$ be a linear system of curves $\Phi \in \Lambda_1(t, F)$ such that $S \subset \operatorname{Sing}(\Phi)$, and Φ meets F at each cusp from S with multiplicity ≥ 5 . Then there is a divisor D on GH such that

$$\deg D_{|G} \le m + 2k, \quad \deg D_{|K} \le \operatorname{card}(S \cap K),$$

for each component $K \subset H$, and, for all t > 1,

$$\Lambda_3(t, F, S) \supset \mathcal{L}_{GH}(t, D).$$

Proof. — We will construct the divisor $D = \sum n(P) \cdot P$ explicitly.

(1) We have to find a divisor D on G such that any curve from $\mathcal{L}_G(t,D)$ goes through each node and each cusp of F, and intersects a tangent line to F at any cusp with multiplicity ≥ 2 .

Let z be a node of F. Since $G \in \Lambda_1(n, F)$, then G goes through z. If G is non-singular at z we can put n(P) = 1 for the local branch P of G centered at z. If G is singular at z then we can put n(P) = 0 for all local branches of G centered at z, because in this case, according to Proposition 1.3, curves from $\mathcal{L}_G(t,0)$ go through z.

Let z be a cusp of F. Analogously, G goes through z. If G is non-singular at z, then the local branch P of G at z is tangent to the tangent line L to the curve F at z. Put n(P) = 2. Now, since any curve from $\mathcal{L}_G(p, D)$ intersects P with multiplicity ≥ 2 , the same holds for L. If G is singular

at z, then either there is a singular local branch P of G centered at z, or there are at least two local branches P_1, P_2 of G centered at z. In the first case we put n(P) = 2, in the second case we put $n(P_1) = n(P_2) = 1$. According to Proposition 1.3 any curve from $\mathcal{L}_G(t,D)$ is singular at z, and thereby intersects L with multiplicity ≥ 2 .

(2) We can obtain the second statement easily by combining the previous arguments with the Noether theorem. \square

2. Non-transversality and reducibility

The upper bounds in the sufficient conditions (0.3), (0.4) are the best possible as far as the exponent of d is concerned. The slightly modified classical examples [15] presented below give an upper bound for the allowable coefficient of d^2 in (0.3), (0.4).

THEOREM 2.1. — The set $V(6p, 0, 6p^2)$ is reducible if p = 1, 2, and has components with different dimensions if $p \ge 3$.

Proof. — The case p=1 is well-known [15]. Let $p\geq 2$. It is easy to see that the curves

$$H = F_{2p}^3 + G_{3p}^2$$

belong to $V(6p, 0, 6p^2)$, where F_{2p} , G_{3p} are general curves of degrees 2p, 3p respectively. A simple computation gives us:

(2.2)
$$\dim \left\{ H \in V(6p, 0, 6p^2) \mid H = F_{2p}^3 + G_{3p}^2 \right\} \\ = \frac{1}{2} 6p(6p+3) - 12p^2 + \frac{1}{2}(p-1)(p-2).$$

According to [13], for $p \geq 2$, there is a component of $V(6p,0,6p^2)$ with dimension :

$$\frac{1}{2}6p(6p+3) - 12p^2.$$

If $p \geq 3$ we obtain at least two components of $V(6p,0,6p^2)$ with different dimensions.

Let p=2. According to (0.5), V(12,0,24) is a T-variety, and hence has dimension 42. According to (2.2) curves $H=F_4^3+G_6^2$ form a component \widetilde{V} of V(12,0,24). Assume that $\widetilde{V}=V(12,0,24)$.

Let J be an irreducible curve of degree 12 with 28 cusps constructed in [13]. Since V(12,0,28) is a T-variety (see (0.5)), we can smooth out any four cusps of J, preserving the others, by means of a variation of J in the space $P(\Sigma(12))$. Indeed, since all 28 equisingular strata intersect transversally at J, we can leave four of them by moving J along the intersection of the others. So we obtain that J belongs to the closure of \widetilde{V} ,

and hence to any set s_{24} of 24 cusps of J there correspond a quartic F_4 and a sextic G_6 , passing through s_{24} . Distinct 24-tuples of cusps correspond to distinct quartics, because, according to Bézout's theorem, a quartic cannot contain more than 24 cusps of J. On the other hand, two 24-tuples s_{24} , s'_{24} with 23 common cusps give quartics F_4 , F'_4 with 23 common points. Therefore F_4 , F'_4 have a common component C_i of degree i=1, 2, or 3. If i=3 then $F_4=C_3C_1$, $F'_4=C_3C'_1$. Since C_3 passes through at most 18 cusps of J, then the straight lines C_1 , C'_1 have at least 5 common points, that means they coincide. The cases i=1 or 2 lead analogously to contradictions, which prove that V(12,0,24) is reducible.

Theorem 2.3. — The set $V(7p-3,0,6p^2)$ contains a component without property T when $p \geq 3$.

Proof. — Obviously, the curve $H=A_{p-3}F_{2p}^3+B_{p-3}G_{3p}^2$ belongs to $V(7p-3,0,6p^2)$, if A_{p-3} , F_{2p} , B_{p-3} , G_{3p} are general curves of degrees p-3, 2p, p-3, 3p respectively. The property T is equivalent to the nonspeciality of $\Lambda_1(7p-3,H)$. From Theorem 1.5 it is not difficult to deduce that

$$\Lambda_1(7p-3, H) = \left\{ \Phi \mid \Phi = R_{3p-3}F_{2p}^2 + S_{4p-3}G_{3p} \right\}$$

with arbitrary curves R_{3p-3} , S_{4p-3} of degrees 3p-3, 4p-3. Further, a trivial computation gives

$$\dim \Lambda_1(7p-3, H) = \frac{1}{2}(7p-3) \cdot 7p - 12p^2 + 1,$$

that means $\Lambda_1(7p-3, H)$ is special.

COROLLARY 2.4. — The allowable coefficient at d^2 in the right hand side of (0.3) cannot exceed $\frac{12}{49}$, and in the right hand side of (0.4) cannot exceed $\frac{1}{3}$.

3. Main lemma

Lemma 3.1. — For any curve $F \in V(d, m, k)$ and real $\alpha \geq (m+2k)/d^2$, there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, where $n = \left[(\sqrt{2\alpha} + 2/3)d \right]$.

Proof. — Let z_1, \ldots, z_m be the nodes of F, and let w_1, \ldots, w_k be the cusps of F. Let h be the minimal integer such that $\Lambda_1(h, F) \neq \emptyset$. Then, $\Lambda_1(h-1, F) = \emptyset$ implies

$$m + 2k > \frac{1}{2}(h-1)(h+2),$$

and hence

$$(3.2) h < \sqrt{2(m+2k)} \le \sqrt{2\alpha} d.$$

Take a general curve $H \in \Lambda_1(h, F)$. Assume that $H = H_1^{i_1} \cdots H_r^{i_r}$, where H_1, \ldots, H_r are irreducible components of degrees h_1, \ldots, h_r respectively. Since h is minimal,

$$\max\{i_1,\ldots,i_r\}\leq 2.$$

We shall construct the curve Φ as follows. First we will construct, for each $s=1,\ldots,r$, a curve C_s of degree $\ell_s \leq i_s h_s + \frac{2}{3}d$ such that C_s does not contain H_s and the curve

(3.3)
$$R_s \stackrel{\text{def}}{=} H_1^{i_1} \cdots H_{s-1}^{i_{s-1}} C_s H_{s+1}^{i_{s+1}} \cdots H_r^{i_r}$$

belongs to $\Lambda_1(h + \ell_s - i_s h_s, F)$. After that we obtain the desired curve Φ in the form

$$G_0H + G_1R_1 + \cdots + G_rR_r$$

where G_0, G_1, \ldots, G_r are generic curves of suitable degrees.

The rest of the proof is divided into five steps: in steps 1, 2, 3 and 4 we construct the curves C_1, \ldots, C_r , in the fifth step we construct the curve Φ .

Let us do the construction of C_1 . Let H_1 pass through z_1, \ldots, z_p , w_1, \ldots, w_q and meet F at w_{q+1}, \ldots, w_{q+t} with multiplicities ≥ 3 . Let deg $H_1 = h_1$. The Bézout theorem gives:

$$(3.4) 2p + 2q + 3t \le h_1 d.$$

Step 1. — Assume $i_1 = 1$. Let us find a curve C_1 passing through $z_1, \ldots, z_p, w_1, \ldots, w_q$, meeting F at w_{q+1}, \ldots, w_{q+t} with multiplicities ≥ 3 , and not containing H_1 . This can be done under the following sufficient condition on $\ell_1 = \deg C_1$

$$\frac{1}{2}\ell_1(\ell_1+3) - \frac{1}{2}(\ell_1-h_1)(\ell_1-h_1+3) > p+q+2t,$$

which is equivalent to

$$\ell_1 > \frac{1}{2}h_1 - \frac{3}{2} + \frac{p+q+2t}{h_1},$$

and, using (3.4), we can take

$$(3.5) \qquad \ell_1 \le \frac{1}{2}h_1 + \frac{p+q+2t}{h_1} \le \frac{1}{2}h_1 + \frac{p+q+2t}{2p+2q+3t}d \le \frac{1}{2}h_1 + \frac{2}{3}d.$$

Step 2. — From now on assume $i_1 = 2$. First we look for a curve C'_1 of degree ℓ'_1 passing through z_1, \ldots, z_p , meeting F at w_{q+1}, \ldots, w_{q+t} with

multiplicities ≥ 3 and not containing H_1 . As in the first step we have the following sufficient condition for the existence of such a curve

$$\ell_1' > \frac{1}{2}h_1 - \frac{3}{2} + \frac{p+2t}{h_1},$$

and then we can take

(3.6)
$$\ell_1' \le \frac{1}{2}h_1 + \frac{p+2t}{h_1}.$$

Step 3. — Assume that there is a curve $H'_1 \neq H_1$ of degree $h'_1 \leq h_1$, passing through w_1, \ldots, w_q , and that h'_1 is the minimal such degree. This minimality implies:

$$(3.7) h_1'^2 \le 2q.$$

Then we put $C_1 = C'_1(H'_1)^2$. According to (3.6)

$$\ell_1 = \deg C_1 \le \frac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h_1'.$$

Hence, using (3.4), (3.7) and the initial assumption $h'_1 \leq h_1$, it is easy to compute

(3.8)
$$\ell_{1} \leq \frac{1}{2}h_{1} + \frac{p+q+2t}{2p+2q+3t}d - \frac{q}{h_{1}} + 2h'_{1} \\ \leq \frac{1}{2}h_{1} + \frac{2}{3}d - \frac{q}{h_{1}} + 2h'_{1} \leq \frac{2}{3}d + 2h_{1}.$$

Step 4. — Now assume that H_1 is the unique curve of degree $\leq h_1$, passing through w_1, \ldots, w_q . In particular, that means

$$(3.9) \frac{1}{2}h_1(h_1+3) \le q.$$

Let C_1' be the curve of degree ℓ_1' constructed in step 2. Consider two situations.

• Assume first $q \leq h_1^2$. Then $2q < 2h_1(2h_1+3)/2$. That means the set M of curves of degree $2h_1$, meeting F at w_1, \ldots, w_q with multiplicities ≥ 3 , is infinite. The only curve in M containing H_1 is H_1^2 . Now take $H_1' \in M$ different from H_1^2 , and put $C_1 = C_1'H_1'$. Here:

$$\ell_1 = \deg C_1 \le \frac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h_1.$$

Finally, from (3.4), (3.9) we get:

$$(3.10) \ell_1 \le \frac{2}{3}d + 2h_1.$$

• Now assume:

$$(3.11) q \ge h_1^2 + 1.$$

Introduce the integer

$$h'_1 = \max\{\nu \in \mathbb{N} \mid h_1(\nu - h_1) + 1 \le q\}.$$

According to (3.11), $h'_1 \geq 2h_1$. Denote by M the set of curves of degree h'_1 meeting F with multiplicity ≥ 3 at each point w_1, \ldots, w_{π} , where $\pi = h_1(h'_1 - h_1) + 1$. If $G \in M$ contains H_1 then $G = G_1H_1$, and G_1 goes through w_1, \ldots, w_{π} , because

$$(G_1 \cdot F)(w_i) = (G \cdot F)(w_i) - (H_1 \cdot F)(w_i) \ge 3 - 2 = 1, \ i = 1, \dots, \pi.$$

Hence G_1 contains H_1 , because these curves meet at $\pi > \deg H_1 \cdot \deg G_1$ points. Then there is a curve $H'_1 \in M$ not containing H_1 as component, because the sufficient condition for this existence is

$$\frac{1}{2}h_1'(h_1'+3) - 2(h_1(h_1'-h_1)+1) > \frac{1}{2}(h_1'-2h_1)(h_1'-2h_1+3),$$

which is equivalent to

$$3h_1 > 2$$
.

Finally, let H_1'' be a curve of the smallest degree h_1'' meeting F at w_i , $i = h_1(h_1' - h_1) + 2, \ldots, q$, with multiplicities ≥ 3 . By definition of h_1' , the number of these points is $\leq h_1 - 1$. If $h_1 \leq 3$ then, obviously, $h_1'' \leq h_1 - 1$. If $h_1 \geq 4$, since

$$\frac{1}{2}(h_1''-1)(h_1''+2) \le 2h_1 - 2,$$

we have:

$$(3.12) h_1'' \le \frac{5}{6}h_1.$$

The last inequality is true in the case $h_1 \leq 3$ too. In particular, H_1'' does not contain H_1 . Now put $C_1 = C_1'H_1'H_1''$. Here we have from (3.4), (3.6), and (3.12)

(3.13)
$$\ell_{1} = \deg C_{1} \leq \frac{1}{2}h_{1} + \frac{p+2t}{h_{1}} + h'_{1} + h''_{1}$$

$$= \frac{1}{2}h_{1} + \frac{p+\frac{4}{3}q+2t}{h_{1}} + h'_{1} + h''_{1} - \frac{4q}{3h_{1}}$$

$$\leq \frac{1}{2}h_{1} + \frac{2}{3}d + h'_{1} + h''_{1} - \frac{4}{3}(h'_{1} - h_{1})$$

$$\leq \frac{8}{3}h_{1} + \frac{2}{3}d - \frac{1}{3}h'_{1} \leq 2h_{1} + \frac{2}{3}d,$$

because $h'_1 \geq 2h_1$ as it was mentioned above.

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Step 5. — Inequalities (3.2), (3.5), (3.8), (3.10), (3.13) and the definitions of n and α imply that the degree of any curve R_j , $j = 1, \ldots, r$, defined by (3.3), is less than n. Now consider the linear system

$$(3.14) \lambda_0 G_0 H + \lambda_1 G_1 R_1 + \dots + \lambda_r G_r R_r, \quad (\lambda_0, \dots, \lambda_r) \in \mathbb{P}^r,$$

where G_0, \ldots, G_r are generic curves of positive degrees n-h, $n-\deg R_1$, $\ldots, n-\deg R_r$ respectively. According to the construction of H, C_1, \ldots, C_r , this is a subsystem of $\Lambda_1(n, F)$. Also note that the curves G_0, \ldots, G_r do not go through base points of our linear system. Then we obtain immediately from the Bertini theorem (see [3], [14]) that a generic member in the linear system (3.14) is reduced and irreducible. Thereby we can take this generic member as the desired curve Φ .

4. The property T

Let $F \in V(d, m, k)$. As said above, the property T and the smoothness of V(d, m, k) at F follow from :

PROPOSITION 4.1. — Under condition (0.3) the linear system $\Lambda_1(d, F)$ is non-special.

Proof. — According to Lemma 3.1, under condition (0.3) there is an irreducible curve $\Phi \in \Lambda_1(n,F)$, $n = [\sqrt{2\alpha_0} + \frac{2}{3}d]$. According to Proposition 1.11 there is a divisor D on Φ of degree

$$(4.2) \deg D \le m + 2k$$

such that

$$\Lambda_1(p,F) \supset \mathcal{L}_{\Phi}(p,D), \quad p > 1.$$

Therefore, according to Proposition 1.2, it is enough to establish the non-speciality of $\mathcal{L}_{\Phi}(d, D)$, which will follow from

(4.3)
$$\deg(G_{\Phi} - D(\Phi) - D) > 2g(\Phi) - 2$$

where $G \in \mathcal{L}_{\Phi}(d, D)$, and $g(\Phi)$ is the geometric genus of Φ . Indeed, we have by Proposition 1.3 and (4.2):

$$\deg(G_{\Phi} - D(\Phi) - D) = nd - n(n-3) - 2 + 2g(\Phi) - \deg D$$

$$\geq n(d-n+3) - (m+2k) + 2g(\Phi) - 2$$

$$> n(d-n+3) - \alpha_0 d^2 + 2g(\Phi) - 2.$$

Since $n = \left[\left(\sqrt{2\alpha_0} + \frac{2}{3} \right) d \right]$ and α_0 is the positive root of the equation

$$\left(\sqrt{2\alpha} + \frac{2}{3}\right)\left(\frac{1}{3} - \sqrt{2\alpha}\right) = \alpha,$$

hence

$$n(d-n+3) \ge \alpha_0 d^2.$$

Then (4.4) implies (4.3) and completes the proof.

5. Irreducibility

We prove the irreducibility along the plan mentioned in introduction.

PROPOSITION 5.1. — For any $F \in V(d, m, k)$, the intersection of V(d, m, k) with $\Lambda_2(d, F)$ contains an open dense subset of $\Lambda_2(d, F)$, and consists exactly of curves from V(d, m, k) with the same nodes, and the same cusps with the same tangents as F.

Proof. — Since $F \in \Lambda_2(d, F)$ and any curve $G \in \Lambda_2(d, F)$ is singular at z_1, \ldots, z_m , then the Bertini theorem implies that almost all curves in $\Lambda_2(d, F)$ have nodes at z_1, \ldots, z_m , and are non-singular outside $\operatorname{Sing}(F)$. Consider the cusp $w_1 \in F$. In some affine neighbourhood of w_1 we fix an affine coordinate system (x, y) such that $w_1 = (0; 0)$, and y = 0 is a tangent to F at w_1 . Then in this neighbourhood, F is defined by polynomial

(5.2)
$$Ay^2 + Bx^3 + \sum_{2i+3j>6} A_{ij}x^iy^j, \quad AB \neq 0,$$

and can be locally parametrized analytically (see [3], [14]) by

(5.3)
$$x = \tau^2, \quad y = \lambda \tau^3 + O(\tau^4).$$

To determine $(G \cdot F)(w_1)$ we plug (5.3) into the affine equation

$$\sum a_{ij}x^iy^j = 0$$

of G, and then compute the order of vanishing at $\tau=0$ (see [14]). Thus we obtain for curves $G\in\Lambda_2(d,F)$ that :

$$a_{00} = a_{01} = a_{10} = a_{11} = a_{20} = 0$$
.

Since $F \in \Lambda_2(d, F)$, almost all curves in $\Lambda_2(d, F)$ have affine equations like (5.2), that means they have a cusp at w_1 with the tangent y = 0. Now to complete the proof we should note that any curve $G \in V(d, m, k)$ with nodes z_1, \ldots, z_m , cusps w_1, \ldots, w_k and tangents $T_{w_1}F, \ldots, T_{w_k}F$, belongs to $\Lambda_2(d, F)$.

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PROPOSITION 5.4. — Under condition (0.4), V(d, m, k) contains an open dense subset \widetilde{V} consisting of curves F with non-special linear system $\Lambda_2(d, F)$.

REMARK 5.5. — Even under condition (0.4), V(d, m, k) may contain curves G with special system $\Lambda_2(d, G)$. This holds for $V(2p, 3p, 0), p \geq 3$ (see [12]).

Proof of Proposition 5.4. — It is enough to prove the statement for any irreducible component V_0 of V(d, m, k). For any curve $G \in V_0$, let $\mathcal{H}(G)$ denote the linear system of curves of the smallest degree h, passing through $\mathrm{Sing}(G)$. Evidently, any $H \in \mathcal{H}(G)$ is reduced, and

(5.6)
$$h = \deg H < \sqrt{2(m+k)} < \sqrt{2\alpha_1} d.$$

Denote by V_1 the set of curves $G \in V_0$ with maximal h. This is an open dense irreducible subset of V_0 (see [3]). Now denote by V_2 the set of curves $G \in V_1$ with minimal dim $\mathcal{H}(G)$. Similarly this is an open dense irreducible subset of V_1 . Then

$$W = \bigcup_{G \in V_2} \mathcal{H}(G)$$

is irreducible as the image in $P(\Sigma(h))$ of the space of a projective bundle with base V_2 and fibres $\mathcal{H}(G)$, $G \in V_2$. Denote by W_0 the set of curves $H \in W$ with minimal number of irreducible components. It is irreducible, open and dense in W. In particular, that means all the curves $H \in W_0$ determine the same sequence of degrees of their components (up to permutation). Moreover, if H runs through W_0 , then any of its irreducible components K runs through some irreducible set

$$W(K) \subset P(\Sigma(\deg K)).$$

For $H \in W_0 \cap \mathcal{H}(G)$ define N(H) to be $\sum N(K, H)$, where K runs through all components of H, and $N(K, H) = \operatorname{card}(K \cap \operatorname{Sing}(G))$. Denote by W_1 the set of curves $H \in W_0$ with minimal N(H). First, it is an open dense irreducible subset of W_0 , and, second, for any component K of $H \in W_1$,

$$N(K,H) = \min_{K' \in W(K)} N(K',H').$$

At last, introduce

$$V_3 = \{G \in V_2 \mid \mathcal{H}(G) \cap W_1 \neq \emptyset\}.$$

According to the above construction, this is an open dense subset of V_0 . Now we will show that V_3 contains an open subset consisting of curves satisfying the conditions of Proposition 5.4, in three steps.

Step 1. — Fix $G \in V_3$ and $H \in \mathcal{H}(G) \cap W_1$. Show that any component K of H of degree δ contains at most $\frac{1}{2}\delta(\delta+3)$ points from $\mathrm{Sing}(G)$.

Indeed, let K contain $\frac{1}{2}\delta(\delta+3)+1$ points from Sing(G). Denote the set of these points by S, and consider the linear system $\Lambda_3(d,G,S)$. Let us take an irreducible curve $\Phi \in \Lambda_1(n,G), n = [(\frac{2}{3} + \sqrt{2\alpha_1})d]$, from Lemma 3.1. According to Proposition 1.11, for all $t \geq 1$,

$$\Lambda_3(t,G,S)\supset \mathcal{L}_{\Phi K}(t,D),$$

where

(5.7)
$$\deg D_{|\Phi} \le m + 2k, \quad \deg D_{|K} \le \frac{1}{2}\delta(\delta + 3) + 1.$$

We shall show that $\mathcal{L}_{\Phi K}(d,D)$ is non-special. According to Proposition 1.8 and arguments from its proof, this is equivalent to

$$i((d-\delta)L_{\Phi} - D(\Phi) - D_{|\Phi}) = i((d-n)L_K - D(K) - D_{|K}) = 0.$$

According to Theorem 1.7 these equalities follow from :

(5.8)
$$(d - \delta)n - \deg D(\Phi) - \deg D_{|\Phi} > 2g(\Phi) - 2,$$

(5.9)
$$\delta(d-n) - \deg D(K) - \deg D_{|K|} > 2g(K) - 2.$$

According to Proposition 1.3 and (5.7), inequality (5.8) follows from:

$$m + 2k < n(d - \delta - n + 3).$$

This inequality can be easily deduced from the definition of n and (5.6), because $\alpha_1 = \frac{2}{225}$ satisfies the inequality:

$$\alpha_1 < \left(\frac{2}{3} + \sqrt{2\alpha_1}\right)\left(\frac{1}{3} - 2\sqrt{2\alpha_1}\right).$$

Analogously, by Proposition 1.3 and (5.7) the inequality (5.9) follows from :

$$d - n \ge \frac{3}{2}\delta.$$

This can be easily deduced from the definition of n and (5.6), because α_1 is the root of the equation

$$1 - \left(\frac{2}{3} + \sqrt{2\alpha}\right) = \frac{3}{2}\sqrt{2\alpha}.$$

So, according to Proposition 1.2, $\Lambda_3(d,G,S)$ is non-special, and according to Propositions 1.3 and 1.8

(5.10)
$$\dim \Lambda_3(d, G, S) = \frac{1}{2}d(d+3) - m - 2k - 2 \cdot \operatorname{card} S.$$

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If G runs through V_3 , then S runs through some subset of

$$\operatorname{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

thereby defining a morphism

$$\nu: \widetilde{V}_3 \longrightarrow \operatorname{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

where \widetilde{V}_3 is the finite covering of V_3 corresponding to different choices of $\frac{1}{2}\delta(\delta+3)+1$ points from the set $\mathrm{Sing}(G)\cap K$ (which might contain more than $\frac{1}{2}\delta(\delta+3)+1$ points). The tangent space to the fibre $\nu^{-1}(S)$ at the point $(G,S)\in \widetilde{V}_3$ is contained in $\Lambda_3(d,G,S)$ (see [2], [12]). Therefore (5.10) implies:

$$\dim \nu^{-1}(S) = \dim V_3 - 2(\frac{1}{2}\delta(\delta+3) + 1),$$

hence $\nu(\widetilde{V}_3)$ is dense in $\operatorname{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2)$. Therefore there is a curve $G \in V_3$ such that, for any set S of $\frac{1}{2}\delta(\delta+3)+1$ points in $\operatorname{Sing}(G) \cap K$, no more than $\frac{1}{2}\delta(\delta+3)$ points of S lie on a curve of degree δ . But this contradicts the definition of the set V_3 and the initial assumption that $K \cap \operatorname{Sing}(G)$ contains more than $\frac{1}{2}\delta(\delta+3)$ points, and thus completes the proof.

Step 2. — Consider the linear system $\Lambda_3(d, G, \operatorname{Sing}(G))$. As in the previous step, for any curve $H \in \mathcal{H}(G) \cap W_1$ and all $t \geq 1$ we have

$$\Lambda_3(t, G, \operatorname{Sing}(G)) \supset \mathcal{L}_{\Phi H}(t, D),$$

where D satisfies (5.7) for any component $K \subset H$, hence $\Lambda_3(d, G, \operatorname{Sing}(G))$ is non-special.

Step 3. — As in the first step, the non-speciality of $\Lambda_3(d, G, \operatorname{Sing}(G))$, $G \in V_3$, implies that the image of V_3 by the morphism

$$\mu: V_3 \to \operatorname{Sym}^{m+k}(\mathbb{P}^2), \quad \mu(G) = \operatorname{Sing}(G),$$

contains an open dense subset U of $\operatorname{Sym}^{m+k}(\mathbb{P}^2)$. According to [4], under condition (0.4), for any Z from some open subset $U' \subset U$ the linear system $\Lambda(d,Z)$ of curves of degree d, having multiplicity ≥ 3 at each point $z \in Z$, is non-special. It is easy to see that for any $G \in V(d,m,k)$ and $t \geq 1$:

$$\Lambda_2(t,G) \supset \Lambda(t,\mathrm{Sing}(G)).$$

Therefore $\Lambda_2(d, F)$ is non-special for any curve $F \in \mu^{-1}(U') \cap V_3$.

Now we can finish the proof of the irreducibility of V(d, m, k), showing that \widetilde{V} is irreducible. To any curve $F \in \widetilde{V}$ we assign the set $\operatorname{Sing}(F)$ and the set of tangents at its cusps. Thereby we obtain a morphism

$$\pi: \widetilde{V} \longrightarrow \operatorname{Sym}^m(\mathbb{P}^2) \times \operatorname{Sym}^k(P(T\mathbb{P}^2)),$$

where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane. According to Proposition 5.1 any fibre of π is an open subset of some linear system $\Lambda_2(d,F)$, $F \in \widetilde{V}$, hence is irreducible. The non-speciality of these linear systems, Propositions 1.2 and 1.8 imply immediately that all the fibres have the same dimension:

$$\frac{1}{2}d(d+3)-3m-5k=\dim \widetilde{V}-\dim \left(\operatorname{Sym}^m(\mathbb{P}^2)\times \operatorname{Sym}^k(P(T\mathbb{P}^2))\right).$$

Finally, this equality means that $\pi(\widetilde{V})$ is dense in

$$\operatorname{Sym}^m(\mathbb{P}^2) \times \operatorname{Sym}^k(P(T\mathbb{P}^2)),$$

hence is irreducible. This completes the proof.

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