BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 122, nº 2 (1994), p. 209-223
<http://www.numdam.org/item?id=BSMF_1994__122_2_209_0>

Bull. Soc. math. France,
122, 1994, p. 209–223.

RIESZ MEANS ON LIE GROUPS AND RIEMANNIAN MANIFOLDS OF NONNEGATIVE CURVATURE

BY

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RÉSUMÉ. — Dans cet article, on démontre des estimations pour les sommes de Riesz associées aux sous-laplaciens invariants à gauche sur les groupes de Lie à croissance polynômiale du volume et à l'opérateur de Laplace-Beltrami sur les variétés Riemanniennes à courbure positive. On démontre aussi des estimations pour les opérateurs maximaux associés et on en déduit la convergence presque partout des sommes de Riesz.

ABSTRACT. — In this article we prove certain L^p estimates for the Riesz means associated to left invariant sub-Laplaceans on Lie groups of polynomial growth and the Laplace Beltrami operator on Riemannian manifolds of nonnegative curvature. We also prove L^p estimates for the associated maximal operators and deduce the almost everywhere convergence of the Riesz means.

0. Introduction and statement of the results

The Riesz means have already been extensively studied in the case of $\mathbb{R}^n$ (cf. [7], [8], [27], [29] as well as the book [13]) and in the case of elliptic differential operators on compact manifolds (cf. [2], [9], [16], [18], [25], [26]). Some of these results have been generalised to the case of dilation invariant sub-Laplacians on stratified nilpotent Lie groups (cf. [19], [21], [22]), to the case of compact semisimple Lie groups (cf. [10]) and more recently to the case of noncompact symmetric spaces (cf. [16]).

The goal of this aricle is to study the Riesz means associated to left invariant sub-Laplacians on connected Lie groups of polynomial volume

(*) Texte reçu le 23 septembre 1992, révisé le 18 mars 1993.
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Keywords : Riesz means, volume growth, sub-Laplacians, wave equation.
AMS classification : 22E25, 22E30, 43A80.

growth (connected nilpotent Lie groups are examples of such groups) and to the Laplace Beltrami operator on Riemannian manifolds of nonnegative curvature :

a) Lie groups of polynomial growth.

We consider a connected Lie group G and we fix a left invariant Haar measure $\mathrm{d}g$ on G. If A is a Borel measurable subset of G, then we denote by $|A|$ its $\mathrm{d}g$-measure.

We assume that G has polynomial volume growth, that is, for every compact neighborhood U of its identity element e of G, there is a constant $c > 0$ such that $|U^n| \leq cn^c$, for $n \in \mathbb{N}$.

It is easy to see that this assumption makes G unimodular. Furthermore, it can be proved (cf. [17]) that there is an integer $D \geq 0$, such that :

$$|U^n| \sim n^D, \qquad (n \to \infty).$$

By $f(t) \sim h(t)$, as $t \to t_0$ we mean that there is a constant $c > 0$ such that :

$$c^{-1} \cdot h(t) \leq f(t) \leq c \cdot h(t) \quad \text{as} \quad t \to t_0.$$

Notice that every connected nilpotent Lie group has polynomial volume growth.

We consider left invariant vector fields $X_1, ..., X_n$ on G that satisfy Hörmander's condition, i.e. they generate together with their successive Lie brackets $[X_{i_1}, [X_{i_2}, [..., X_{i_k}]...]$, at every point of G, the tangent space of G. To those vector fields is associated, in a canonical way, the control distance $d(\cdot\,,\cdot)$. This distance is left invariant and compatible with the topology of G. We put :

$$|x| = d(e,x) \quad \text{and} \quad B_r(x) = \{y \in G : d(x,y) < r\}, \quad x \in G,\ r > 0.$$

Then, we know that there is $d \in \mathbb{N}$, not depending on x (cf. [24], [30] and [33]), such that :

$$(1) \qquad |B_r(x)| \sim r^d \quad (r \to 0), \quad |B_r(x)| \sim r^D \quad (r \to \infty)$$

We call d the *local dimension* and D the *dimension at infinity* of G.

b) Riemannian manifolds of nonnegative curvature.

We consider a complete non-compact n-dimensionnal Riemannian manifold M with non-negative Ricci curvature. We denote by L the Laplace-Beltrami operator on M. Let $d(\cdot\,,\cdot)$ be the Riemannian distance on M and denote by

$$B_r(x) = \{y \in M : d(x,y) < r\}$$

the geodesic ball of radius $r > 0$ and centered at $x \in M$.

Let also $|B_r(x)|$ denote the volume of $B_r(x)$. Then there is a constant $c_x > 0$ (depending on $x \in M$) such that

$$\big|B_r(x)\big| \geq c_x r^n, \qquad 0 < r \leq 1.$$

Although we have, by the Bishop comparison theorem (cf. [3]), that there is a constant $c > 0$ independent of $x \in M$ and $r > 0$ such that $|B_r(x)| \leq cr^n$, it may happen that $|B_r(x)|$ grows much slower as $r \to \infty$. For example if M is a complete noncompact homogeneous space with nonnegative sectional curvature then $M = \mathbb{R}^k \times \overline{M}$, where $\overline{M}$ is a compact homogeneous space and $k \geq 1$ (cf. [4]). So in that case we have that $|B_r(x)| \sim r^k$ $(r \to \infty)$. In general all we can say (cf. [5]) is that there is a constant $c_x > 0$ depending on $x \in M$ such that $|B_r(x)| \geq c_x r$, where $r \geq 1$. In this article we shall only use the following inequality, which also follows from the Bishop comparison theorem (cf. [3], [5]) :

$$\frac{|B_r(x)|}{|B_t(x)|} \leq \Big(\frac{r}{t}\Big)^n, \qquad r \geq t. \tag{2}$$

We shall also put $d = D = n$.

In both of the above cases the operator L admits a spectral resolution (cf. [34]), which we denote by :

$$L = \int_0^\infty \lambda \, \mathrm{d}E_\lambda.$$

For $\alpha > 0$, the Riesz means of order α are defined to be the operators

$$m_{\alpha,R}(L) = \int_0^\infty \Big(1 - \frac{\lambda}{R}\Big)_+^\alpha \mathrm{d}E_\lambda, \qquad R > 0,$$

and the corresponding maximal operators by :

$$m_\alpha^*(L)f(x) = \sup_{R>0} \big|m_{\alpha,R}(L)f(x)\big|.$$

That $m_\alpha^*(L)f(x)$ is well defined will be shown in the proof of THEOREM 3 below.

We denote by $K_{\alpha,R}(x,y)$ the Schwartz kernel of the operator $m_{\alpha,R}(L)$.

THEOREM 1. — *There is a constant $c > 0$ such that*

(a) *if* $\alpha > \frac{1}{2}D$ *then* $\|K_{\alpha,R}(x,.)\|_1 \leq c, \quad 0 < R \leq 1$;

(b) *if* $\alpha > \frac{1}{2}\max(d,D)$ *then* $\|K_{\alpha,R}(x,\cdot)\|_1 \leq c, \quad R > 1$;

(c) *if* $\alpha = \frac{1}{2}d > \frac{1}{2}D$ *then* $\|K_{\alpha,R}(x,.)\|_1 \leq c(1 + \log R), \quad R > 1$;

(d) *if* $\frac{1}{2}d > \alpha > \frac{1}{2}D$ *then* $\|K_{\alpha,R}(x,.)\|_1 \leq c\,R^{d/4-\alpha/2}, \quad R > 1$.

THEOREM 2.

a) *If* $\alpha > \frac{1}{2}D$ *then* $m_{\alpha,R}(L)$ *is bounded on* $L^p(G)$ *for* $1 \leq p \leq \infty$.

b) *If* $0 < \alpha < \frac{1}{2}D$ *then* $m_{\alpha,R}(L)$ *is bounded on* $L^p(G)$ *for*

$$\alpha > D\Big|\frac{1}{p} - \frac{1}{2}\Big|.$$

c) *If* $0 < \alpha < \frac{1}{2}D$ *then the operators* $m_{\alpha,R}(L), R > 0$ *are uniformly bounded on* $L^p(G)$ *for* $\alpha > \Big|\frac{1}{p} - \frac{1}{2}\Big|\max(d, D)$.

THEOREM 3.

a) *If* $\alpha > \frac{1}{2}\max(d, D)$ *then* $m_\alpha^*(L)$ *is bounded on* L^p, *for* $1 < p < \infty$.

b) *If* $0 < \alpha < \frac{1}{2}\max(d, D)$ *then* $m_\alpha^*(L)$ *is bounded on* L^p, *for* $\alpha > \Big|\frac{1}{p} - \frac{1}{2}\Big|\max(d, D)$.

THEOREM 4. — *If* α *and* p *are as in theorem* 3 *above and* $f \in L^p$, *then* :

$$\big\|m_{\alpha,R}(L)f - f\big\|_p \to 0 \quad \text{as} \quad R \to \infty,$$

$$m_{\alpha,R}(L)f(x) \to f(x) \quad \text{a.e. as} \quad R \to \infty.$$

We point out that for the Laplace operator on $\mathbb{R}^n$, $n = d = D$ and the critical power in the above results is $\frac{1}{2}(n-1)$ rather than $\frac{1}{2}n$ (cf. [13], [29]).

The proof of the above results relies on the following two ideas : assume to simplify things that $f \in C_0^\infty(\mathbb{R}^+)$ and that we want to obtain estimates of the kernel of the operator $f(L) = \int_0^\infty f(\lambda)\,\mathrm{d}E_\lambda$. Then the first idea which is due to M. TAYLOR (see for example [5]), consists of writing $f(L) = h(\sqrt{L})$ (with $h(t) = f(t^2)$, $t \in \mathbb{R}$). Then, using the fact that $h(t)$ is an even function, we have that :

$$h(\sqrt{L}) = (2\pi)^{-1/2}\int \hat{h}(t)\cos t\sqrt{L}\,\mathrm{d}t.$$

This expression allows us to take advantage of the fact that $\cos t\sqrt{L}$ is an operator bounded on L^2 as well as the fact that its kernel $G_t(x, y)$ being a fundamental solution for the wave equation

$$\Big(\frac{\partial^2}{\partial t^2} + L\Big)u(t, x) = 0, \quad u(0, x) = f(x), \quad \Big(\frac{\partial}{\partial t}u\Big)(0, x) = 0$$

propagates with finite speed, that is

$$\text{(3)} \qquad \operatorname{supp}(G_t) \subseteq \big\{(x, y) : d(x, y) \leq |t|\big\}$$

a result proved, in the case of subelliptic operators by MELROSE [23].

The second idea, which is due to HULANICKI and STEIN (cf. [14, p. 208–215]), and which has also been exploited by CHRIST [6] is to exploit the existence of very good estimates for the heat kernel $p_t(x,y)$, i.e. the fundamental solution of the associated heat equation

$$\Big(\frac{\partial}{\partial t}+L\Big)u(t,x)=0,\quad u(0,x)=f(x).$$

To do this we observe first that $p_t(x,y)$ the Schwartz kernel of the operator e^{-tL}, $t>0$. So, if $f\in C_0^\infty(\mathbb{R}^+)$ and we put $h(t)=f(t)\,\mathrm{e}^{t_0t}$, with $t_0>0$ appropriately chosen we get $f(L)=h(L)\,\mathrm{e}^{-t_0L}$. This in turn implies that the Schwartz kernel of $f(L)$ is equal to $h(L)p_{t_0}(x,y)$. This last remark is one of the basic ingredients of the proofs.

The estimate for $p_t(x,y)$, we shall use in this article, is the following (cf. [12], [20], [30], [33]) :

$$p_t(x,y)\le\frac{c}{|B_{\sqrt{t}}(x)|}\exp\Big(-\frac{d(x,y)^2}{ct}\Big),\quad t>0. \tag{4}$$

1. Proof of theorems 1 and 2

We have that

$$m_{\alpha,R}(\lambda)=\Big(1-\Big|\frac{\lambda}{R}\Big|\Big)_+^\alpha=\Big(1-\Big|\frac{\lambda}{R}\Big|\Big)_+^\alpha\mathrm{e}^{\lambda/R}\,\mathrm{e}^{-\lambda/R}.$$

Hence if we put $r=\sqrt{R}$ and

$$h_{\alpha,r}(\lambda)=\Big(1-\Big(\frac{\lambda}{r}\Big)^2\Big)_+^\alpha\mathrm{e}^{(\lambda/r)^2}$$

then

$$m_{\alpha,R}(L)=h_{\alpha,r}\big(\sqrt{L}\,\big)\,\mathrm{e}^{-1/r^2L} \tag{5}$$

The function $\psi(\lambda)=\mathrm{e}^{-\lambda^{-2}}$ is C^∞ and supported in $[0,\infty)$. Hence the function $\psi_1(\lambda)=\psi(\lambda)\psi(1-\lambda)$ is also C^∞ and supported in $[0,1]$. We put :

$$\varphi(\lambda)=\psi_1\big(\lambda+\tfrac{5}{4}\big),\quad \varphi_j(\lambda)=\varphi(2^j\big(|\lambda|-1\big).$$

Then $\varphi_j(\lambda)$ is a C^∞ function with support contained in $J_j=I_j\cup-I_j$, where $I_j=[1-5/2^{j+2},1-1/2^{j+2}]$. We put

$$\chi_j(\lambda)=\frac{\varphi_j(\lambda)}{\sum_{i\ge0}\varphi_i(\lambda)}\quad\text{and}\quad \chi_{j,r}(\lambda)=\chi_j\Big(\Big(\frac{\lambda}{r}\Big)^2\Big).$$

We also put :

$$h_{j,r}(\lambda) = h_{\alpha,r}(\lambda)\chi_{j,r}(\lambda).$$

Notice that there is $c > 0$ such that

$$(6) \qquad |\operatorname{supp} h_{j,r}| \leq cr2^{-j}.$$

Also, for all $k \in \mathbb{N}$ there is $c_k > 0$ such that

$$(7) \qquad \|\chi_{j,r}^{(k)}\|_\infty \leq c_k r^{-k} 2^{kj}, \quad \|h_{j,r}^{(k)}\|_\infty \leq c_k r^{-k} 2^{-(\alpha-k)j}.$$

By a simple calculation we can deduce from the estimates (6) and (7) above that for all $k \in \mathbb{N}$ there is $c_k > 0$ such that

$$(8) \qquad \int_{|t|\geq s} \big|\hat{h}_{j,r}(t)\big|\,\mathrm{d}t \leq c_k s^{-k} r^{-k} 2^{(k-\alpha)j}, \quad s > 0.$$

We consider the operator

$$m_{j,r}(L) = h_{j,r}(\sqrt{L}\,)\,\mathrm{e}^{-1/r^2 L}$$

and we denote by $K_{j,r}(x,y)$ its Schwartz kernel. Since the operators $h_{j,r}(\sqrt{L})$ and e^{-1/r^2L} are selfadjoint and commute, we have

$$(9) \qquad K_{j,r}(x,y) = h_{j,r}(\sqrt{L}\,)p_{r^{-2}}(x,y)$$

with the operator $h_{j,r}(\sqrt{L}\,)$ acting on the variable y.

LEMMA 5. — *Let $i \in \mathbb{Z}$ such that $2^{i-1} < r \leq 2^i$. Then there is a constant $c > 0$ such that*

$$\big\|K_{j,r}(x,\cdot)\big\|_1 \leq \begin{cases} c \cdot 2^{(D/2-\alpha)j} & \text{if } i \leq 0,\ j \geq 0\ ; \\ c \cdot 2^{(d/2-\alpha)j} & \text{if } i > 0,\ 0 \leq j < i\ ; \\ c \cdot 2^{(d/2-D/2)i}\, 2^{(D/2-\alpha)j} & \text{if } i \leq 0,\ j \geq i. \end{cases}$$

Proof. — It follows from (4) that

$$(10) \qquad \big\|p_t(x,\cdot)\big\|_2 \leq c \cdot \big|B_{\sqrt{2t}}(x)\big|^{-1/2}.$$

We also have

$$(11) \qquad \big\|h_{j,r}(\sqrt{L})\big\|_{2\to 2} \leq \|h_{j,r}\|_\infty \leq 2^{-\alpha j}.$$

Hence, it follows from (9) that

$$
\begin{aligned}
&\big\|K_{j,r}(x,\cdot)\big\|_{L^1(B_{2^{j-i}}(x))}\\
&\quad\le |B_{2^{j-i}}|^{1/2}\big\|K_{j,r}(x,\cdot)\big\|_2\\
&\quad\le \big|B_{2^{j-i}}(x)\big|^{1/2}\big\|h_{j,r}(\sqrt{L})\big\|_{2\to 2}\big\|p_{r^{-2}}(x,\cdot)\big\|_2\\
&\quad\le c\big|B_{2^{j-i}}(x)\big|^{1/2}\big\|h_{j,r}\big\|_\infty\big|p_{2r^{-2}}(x,x)\big|^{1/2}\\
&\quad\le c\Big(\frac{|B_{2^{j-i}}(x)|}{|B_{2^{-i}}(x)|}\Big)^{1/2}2^{-\alpha j}
\end{aligned}
$$

and from this, by using either (1) or (2), we get :

$$
\big\|K_{j,r}(x,\cdot)\big\|_{L^1(B_{2^{j-i}})}\le\begin{cases} c\cdot 2^{(D/2-\alpha)j} & \text{if } i<0,\\ c\cdot 2^{(d/2-\alpha)j} & \text{if } 0\le j\le i,\\ c\cdot 2^{(d/2-D/2)i}\,2^{(D/2-\alpha)j} & \text{if } j>i\ge 0.\end{cases}\tag{12}
$$

Let $A_p(x)=\{y: 2^p\le d(x,y)<2^{p+1}\}$, where $p\ge j-i$. Then, it follows from (3) that, if $z\in A_p(x)$, then

$$
\begin{aligned}
&K_{j,r}(x,z)\\
&\quad=\big[h_{j,r}(\sqrt{L})p_{r^{-2}}(x,\cdot)\big](z)\\
&\quad=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat h_{j,r}(t)\big[\cos t\sqrt{L}\,p_{r^{-2}}(x,\cdot)\big](z)\,\mathrm{d}t\\
&\quad=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat h_{j,r}(t)\big\{\cos t\sqrt{L}\,[p_{r^{-2}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\le 2^{p-1}\}}\\
&\qquad\qquad\qquad+p_{r^{-2}}(x,.)\mathbf{1}_{\{y:d(x,y)>2^{p-1}\}}]\big\}(z)\,\mathrm{d}t\\
&\quad=(2\pi)^{-1/2}\int_{|t|\ge 2^{p-1}}\hat h_{j,r}(t)\big\{\cos t\sqrt{L}\,[p_{r^{-2}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)\le 2^{p-1}\}}]\big\}(z)\,\mathrm{d}t\\
&\qquad+(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat h_{j,r}(t)\big\{\cos t\sqrt{L}\,[p_{r^{-2}}(x,y)\mathbf{1}_{\{y:d(x,y)>2^{p-1}\}}]\big\}(z)\,\mathrm{d}t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big\|K_{j,r}(x,\cdot)\big\|_{L^1(A_p(x))}\\
&\quad\le\big|A_p(x)\big|^{1/2}(2\pi)^{-1/2}\int_{|t|\ge 2^{p-1}}\big|\hat h_{j,r}(t)\big|\cdot\big\|p_{r^{-2}}(x,\cdot)\big\|_2\\
&\qquad+\big|A_p(x)\big|^{1/2}\|h_{j,r}\|_\infty\|[p_{r^{-2}}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p-1}\}}\|_2.
\end{aligned}\tag{13}
$$

Now it follows from (3) and (12) that there are constants c and $C > 0$ such that

$$\left|A_p(x)\right|^{1/2}\|h_{j,r}\|_\infty \left\|p_{1/r^2}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p-1}\}}\right\|_2 \leq c\left\{\frac{|B_{2^p}(x)|}{|B_{2^{-i}}(x)|}\right\}^{1/2} 2^{-\alpha j}\,\mathrm{e}^{-C2^{i+p}}$$

and from this, by using either (1) or (2), we get that there is $c > 0$ such that

$$\sum_{p\geq j-i} \left|A_p(x)\right|^{1/2}\|h_{j,r}\|_\infty \left\|p_{1/r^2}(x,\cdot)\mathbf{1}_{\{y:d(x,y)>2^{p-1}\}}\right\|_2 \leq c\cdot 2^{-\alpha j}. \tag{14}$$

On the other hand if we put

$$I_p(x) = \left|A_p(x)\right|^{1/2}(2\pi)^{-1/2}\int_{|t|\geq 2^{p-1}} \left|\hat{h}_{j,r}(t)\right|\mathrm{d}t\,\left\|p_{r^{-2}}(x,\cdot)\right\|_2,$$

then it follows from (10) that there is $c > 0$ such that

$$I_p(x) \leq c\left\{\frac{|B_{2^p}(x)|}{|B_{2^{-i}}(x)|}\right\}^{1/2}\int_{|t|\geq 2^{p-1}} \left|\hat{h}_{j,r}(t)\right|\mathrm{d}t.$$

Hence, if we chose $k \in \mathbb{N}$, $k > \frac{1}{2}\max(d,D)$, then it follows from (8) (as well as either (1) or (2)) that there is $c > 0$ such that

$$I_p(x) \leq \begin{cases} c\cdot 2^{(D/2-k)p}\,2^{(d/2-k)i}2^{(k-\alpha)j} & \text{if } \ i\leq 0,\\ c\cdot 2^{(d/2-k)p}\,2^{(d/2-k)i}2^{(k-\alpha)j} & \text{if } \ i>0,\ \min(0,j-i)\leq p\leq 0,\\ c\cdot 2^{(D/2-k)p}\,2^{(d/2-k)i}2^{(k-\alpha)j} & \text{if } \ i>0,\ p\geq \max(0,j-i)\end{cases}$$

and from this

$$\sum_{p\geq j-i} I_p(x) \leq \begin{cases} c\cdot 2^{(D/2-\alpha)j} & \text{if } \ i\leq 0,\\ c\cdot 2^{(d/2-\alpha)j} & \text{if } \ i>0,\ j<i,\\ c\cdot 2^{(D/2-d/2)i}\,2^{(D/2-\alpha)j} & \text{if } \ i>0,\ j\geq i,\end{cases}$$

which together with (12), (13) and (14) prove the lemma.

Proof of theorem 1. — This follows immediately from LEMMA 5 and the inequality

$$\left\|K_{\alpha,R}(x,\cdot)\right\|_1 \leq \sum_{j\geq 0}\left\|K_{j,r}(x,\cdot)\right\|_1. \quad \square$$

Proof of theorem 2. — We observe that (a) follows immediately from theorem 1 and that it is enough to prove (b) and (c) for those p for which we also have $p < 2$. Then, since $m_{\alpha,R}(L)$ is self adjoint, by duality, we shall also have these results for those p for which we also have $p > 2$.

Now, if $0 < t < 1$,

$$\frac{1}{p} = \frac{t}{1} + \frac{1-t}{2}, \quad \text{i.e.} \quad t = \frac{2}{p} - 1,$$

then, by interpolation, we have

$$\begin{aligned}\big\|m_{j,r}(L)\big\|_{p\to p} &\le \big\|m_{j,r}(L)\big\|_{1\to 1}^{t}\big\|m_{j,r}(L)\big\|_{2\to 2}^{1-t}\\ &\le \big(\sup_x \|K_{j,r}(x,\cdot)\|_1\big)^t \big\|h_{j,r}(\lambda)\big\|_\infty^{1-t}.\end{aligned}$$

Hence it follows from (11) and Lemma 5 that there is $c > 0$ such that

$$\big\|m_{j,r}(L)\big\|_{p\to p} \le \begin{cases} c\cdot 2^{-[\alpha - D(1/p-1/2)]j} & \text{if } 0 < R \le 1,\\ c\cdot 2^{-[\alpha - d(1-1/p)]j} & \text{if } R > 1,\ 0 \le j < i,\\ c\cdot 2^{-[\alpha - D(1-1/p)]j}\, 2^{(d-D)(1/p-1/2)i} & \\ & \text{if } R > 1,\ 0 < i \le j.\end{cases}$$

Assertions (b) and (c) of Theorem 1 follow from the above estimates, by taking the sums over j.

2. Proof of theorem 3

We shall prove first the following

Lemma 6. — *If $f \in L^p$, $1 < p < \infty$, then $\gamma \mapsto L^{i\gamma}f$ is a strongly continuous L^p-valued function.*

Proof. — If $\epsilon, \delta > 0$ then

$$\begin{aligned}\big\|L^{i(\gamma+\epsilon)}f - L^{i\gamma}f\big\|_p \le \big\|L^{i(\gamma+\epsilon)}&(f - \mathrm{e}^{-\delta L}f)\big\|_p\\ &+ \big\|(L^{i(\gamma+\epsilon)} - L^{i\gamma})\mathrm{e}^{-\delta L}f\big\|_p\\ &+ \big\|L^{i\gamma}(\mathrm{e}^{-\delta L}f - f)\big\|_p.\end{aligned}$$

Now since, by the multiplier theorem of Stein [28], the operators $L^{i(\gamma+\epsilon)}$, $0 \le \epsilon \le 1$ are uniformly bounded on L^p and since $\|\mathrm{e}^{-\delta L}f - f\|_p \to 0$, as $\delta \to 0$ we have

$$\big\|L^{i(\gamma+\epsilon)}(f - \mathrm{e}^{-\delta L}f)\big\|_p + \big\|L^{i\gamma}(\mathrm{e}^{-\delta L}f - f)\big\|_p \to 0, \quad (\delta \to 0).$$

On the other hand, since

$$\big\|(L^{i(\gamma+\epsilon)} - L^{i\gamma})\,\mathrm{e}^{-\delta L}\big\|_{2\to 2} \le \big\|(\lambda^{i(\gamma+\epsilon)} - \lambda^{i\gamma})\,\mathrm{e}^{-\delta\lambda}\big\|_{\infty} \to 0, \quad (\epsilon \to 0),$$

and since again by the multiplier theorem of Stein [28], the operators $(L^{i(\gamma+\epsilon)} - L^{i\gamma})\,\mathrm{e}^{-\delta L}$, for $0 \le \epsilon \le 1$, are uniformly bounded on L^p, it follows by interpolating with L^2 that

$$\big\|(L^{i(\gamma+\epsilon)} - L^{i\gamma})\,\mathrm{e}^{-\delta L} f\big\|_p \to 0, \quad (\epsilon \to 0)$$

and the lemma follows. □

Now, we continue with the proof of THEOREM 3. Following [21] we write

$$m_{\alpha,1}(\lambda) = M(\lambda) + \mathrm{e}^{-\lambda} \quad \text{with} \quad M(\lambda) = m_{\alpha,1}(\lambda) - \mathrm{e}^{-\lambda}.$$

Then we have that

$$m_*(L)f(x) \le \sup_{t>0}\big|M(tL)f(x)\big| + \sup_{t>0}\big|\,\mathrm{e}^{-tL}f(x)\big|.$$

Now we know that the heat maximal operator $\sup_{t>0}|\mathrm{e}^{-tL}f(x)|$ is bounded on L^p, $1 < p < \infty$ (cf. [28]).

To deal with the maximal operator $\sup_{t>0}|M(tL)f(x)|$, we proceed as in [11], that is we consider the Mellin inversion formula

$$M(t\lambda) = \int_{-\infty}^{\infty} \mathcal{M}(\gamma)(t\lambda)^{i\gamma}\,\mathrm{d}\gamma,$$

where $\mathcal{M}(\gamma)$ is the Mellin transform of $M(\lambda)$

$$\mathcal{M}(\gamma) = (2\pi)^{-1}\int_0^{\infty} M(\lambda)\lambda^{-i\gamma}\frac{\mathrm{d}\lambda}{\lambda}.$$

This formula gives :

$$M(tL)f = \int_{-\infty}^{\infty} \mathcal{M}(\gamma)t^{i\gamma}L^{i\gamma}f\,\mathrm{d}\gamma.$$

From this we have

$$\begin{aligned}\sup_{t>0}\big|M(tL)f\big| &= \sup_{t>0}\Big|\int_{-\infty}^{\infty} \mathcal{M}(\gamma)t^{i\gamma}L^{i\gamma}f\,\mathrm{d}\gamma\Big| \\ &\le \int_{-\infty}^{\infty}\big|\mathcal{M}(\gamma)\big|\cdot|L^{i\gamma}f|\,\mathrm{d}\gamma,\end{aligned}$$

which in turn implies :

$$\left\|\sup_{t>0} M(tL)f\right\|_p \leq \int_{-\infty}^{\infty} \left|\mathcal{M}(\gamma)\right| \cdot \left\|L^{i\gamma}\right\|_{p\to p} \|f\|_p \, \mathrm{d}\gamma.$$

The above formal calculations are justified by the fact that as was proved in LEMMA 6, $\gamma \mapsto L^{i\gamma} f$ is a strongly continuous, hence strongly measurable, L^p-valued function. So if

$$\int_0^{\infty} \left|\mathcal{M}(\gamma)\right| \cdot \left\|L^{i\gamma}\right\|_{p\to p} \mathrm{d}\gamma < \infty, \tag{15}$$

then

$$\int_0^{\infty} \mathcal{M}(\gamma) t^{i\gamma} L^{i\gamma} f \, \mathrm{d}\gamma$$

is a convergent L^p-valued integral. This integral defines a continuous function of t, which implies that $\sup_{t>0} |M(tL)f|$ is well defined in L^p.

Now, it has been proved in [21] that

$$\left|\mathcal{M}(\gamma)\right| \leq c\big(1 + |\gamma|\big)^{-(\alpha+1)}. \tag{16}$$

Furthermore, we have that $\|L^{i\gamma}\|_{2\to 2} = 1$ and it follows from the proof of the main result of [1] (that result is proved only for left invariant sub-Laplaceans on Lie groups of polynomial growth, but it is also true for the Laplace-Beltrami operator on a Riemannian manifold of non-negative curvature; the proof is exactly the same) that for every $\epsilon > 0$

$$\|L^{i\gamma}\|_{L^1 \to \text{weak-}L^1} \leq c\big(1 + |\gamma|\big)^{\max(d/2, D/2)+\epsilon}.$$

So, by interpolation and duality if necessary, we have that

$$\|L^{i\gamma}\|_{p\to p} \leq c\big(1 + |\gamma|\big)^{(\max(d/2,D/2)+\epsilon)|2/p-1|}, \quad 1 < p < \infty. \tag{17}$$

Now, it follows from (16) and (17) that when

$$\alpha > \max\Big(\frac{d}{2}, \frac{D}{2}\Big)\Big|\frac{2}{p} - 1\Big| = \max(d, D)\Big|\frac{1}{p} - \frac{1}{2}\Big|,$$

then (15) holds and THEOREM 3 follows. □

3. Proof of theorem 4

It is enough to prove this theorem for functions f belonging to some space A which is dense to all spaces L^p, $1 < p < \infty$. Then THEOREM 4 will follow from THEOREM 3 by well known measure theoretic arguments.

The space A we shall consider is

$$A = \{\varphi_t(L)\,\mathrm{e}^{-sL}f\,;\ f \in C_0^\infty,\ t \geq 1,\ 0 < s \leq 1\},$$

where $\varphi_t(\lambda) = \varphi(\lambda/t)$ and $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$.

That A is dense to all spaces L^p, $1 < p < \infty$, follows from the fact that $\|\,\mathrm{e}^{-sL}f - f\|_p \to 0$ as $s \to 0$ for all $f \in C_0^\infty(G)$ and $1 < p < \infty$ and the observation that for all $k \in \mathbb{N}$

$$\sup_{\lambda>0}\Big|\lambda^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big[\mathrm{e}^{-s\lambda} - \varphi_t(\lambda)\,\mathrm{e}^{-s\lambda}\big]\Big| \to 0, \quad (t \to \infty),$$

which together with the proof of the main result of [1] (we repeat that the main result of [1], although is proved only for left invariant sub-Laplaceans on Lie groups of polynomial growth, but it is also true for the Laplace-Beltrami operator on a Riemannian manifold of non-negative curvature; the proof is exactly the same) imply that :

$$\big\|\,\mathrm{e}^{-sL}f - \varphi_t(L)\,\mathrm{e}^{-sL}f\big\|_p \to 0, \quad \text{as} \quad t \to \infty.$$

Let us now fix some $h = \varphi_t(L)\,\mathrm{e}^{-sL}f \in A$. Let us also consider a function $\psi \in C^\infty(\mathbb{R})$ such that

$$\psi(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq \frac{1}{4}, \\ 0 & \text{for } |\lambda| \geq \frac{1}{2}, \end{cases}$$

and put $\psi_R(\lambda) = \psi(\lambda/R)$, $R > 0$. Then for R large enough we have that

$$m_{\alpha,R}(L)h = \psi_R(L)m_{\alpha,R}(L)\varphi_t(L)\,\mathrm{e}^{-sL}f$$

and therefore

$$m_{\alpha,R}(L)h - h = \big[\psi_R(L)m_{\alpha,R}(L) - 1\big]\varphi_t(L)\,\mathrm{e}^{-sL}f.$$

Now since for all $k \in \mathbb{N}$

$$\sup_{\lambda>0}\Big|\lambda^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big\{[\psi_R(\lambda)m_{\alpha,R}(\lambda) - 1]\varphi_t(\lambda)\,\mathrm{e}^{-s\lambda}\big\}\Big| \to 0, \quad (R \to \infty),$$

it follows from the proof of the main result of [1] that

$$\big\|m_{\alpha,R}(L)h - h\big\|_p = \Big\|\big[\psi_R(L)m_{\alpha,R}(L) - 1\big]\varphi_t(L)\,\mathrm{e}^{-sL}f\Big\|_p \to 0, \quad (R \to 0),$$

which proves the first part of THEOREM 4.

The second part of the theorem follows from the observation that

$$\begin{aligned}
\left|m_{\alpha,R}(L)h(x)-h(x)\right| &= \left|\left[\psi_R(L)m_{\alpha,R}(L)-1\right]\varphi_t(L)\,\mathrm{e}^{-sL}f(x)\right| \\
&\le \left\|\psi_R(L)m_{\alpha,R}(L)-1]\varphi_t(L)p_s(x,\cdot)\right\|_2\cdot\|f\|_2, \\
&\le \sup_{\lambda>0}\left[\psi_R(\lambda)m_{\alpha,R}(\lambda)-1\right]\cdot\left|\varphi_t(\lambda)\right|\cdot\left\|p_s(x,\cdot)\right\|_2\cdot\|f\|_2,
\end{aligned}$$

which together with the fact that

$$\sup_{\lambda>0}\left[\psi_R(\lambda)m_{\alpha,R}(\lambda)-1\right]\cdot\left|\varphi_t(\lambda)\right|\to 0,\quad (R\to\infty),$$

imply that

$$\left|m_{\alpha,R}(L)h(x)-h(x)\right|\to 0,\quad (R\to\infty).$$

This completes the proof of THEOREM 4. □

4. Final remarks

We point out that that our method also works when L is a self-adjoint non-negative real subelliptic differential operator on a compact manifold X, since, in that case, the finite propagation speed (3) for the wave operator has already been proved in [23] and the gaussian estimates (4) for the associated heat kernel have been proved in [31], [32]. The results that we shall obtain are similar. The only change is that as dimension at infinity D we shall put $D=0$ and as local dimension d we shall put the best constant b for which we have that

$$\frac{|B_r(x)|}{|B_t(x)|}\le c\left(\frac{r}{t}\right)^b,\quad r\ge t$$

with the $c>0$ independent of $x\in X$ (cf. [24]). For example when L is a sum of squares of vector fields that satisfy Hörmanders condition in a uniform way, then there are constants $c>0$ and $k\in\mathbb{N}$, independent of $x\in X$, such that (cf. [24], [30], [33])

$$c^{-1}t^k\le\left|B_t(x)\right|\le ct^k,\quad 0<t\le 1$$

and then, of course, we take $d=k$.

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