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CHARACTERIZATION OF THE UNIQUE EXPANSIONS

$1 = \sum_{i=1}^{\infty} q^{-n_i}$ AND RELATED PROBLEMS

BY

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RÉSUMÉ. — On caractérise les développements uniques de 1 en bases non entières. On donne une estimation pour la longueur des chiffres 0 consécutifs dans les développements gloutons. On établit certaines relations entre ces propriétés et les nombres de Pisot.

ABSTRACT. — We characterize the unique expansions in non-integer bases. We estimate the length of consecutive 0 digits in the greedy expansions. We obtain some relations between these properties and the Pisot numbers.

0. Introduction

Consider a number $1 < q < 2$. By an expansion of a real number x we mean a representation of the form

$$x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \quad \varepsilon_i \in \{0, 1\}.$$

It is clear that x has an expansion if and only if $0 \leq x \leq 1/(q-1)$.

Let us introduce the lexicographic order $\overset{L}{<}$ between the real sequences : $(\varepsilon_i) \overset{L}{<} (\varepsilon'_i)$ if there is a positive integer m such that $\varepsilon_i = \varepsilon'_i$ for all $i < m$ and $\varepsilon_m < \varepsilon'_m$. It is easy to verify that for every fixed $0 \leq x \leq 1/(q-1)$

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in the set of all expansions of x there is a greatest and a smallest element with respect to this order : the so-called *greedy* and *lazy* expansion, cf. [4]. (The greedy expansions were studied earlier in [1] where they were called β -expansions.) A number x has a unique expansion if and only if its greedy and lazy expansions coincide.

Let us recall that the digits of these expansions may be defined recursively as follows : if $m \geq 1$ and if the digits ε_i of the greedy expansion of x are defined for all $i < m$, then we put

$$\varepsilon_m = \begin{cases} 1 & \text{if } \sum_{i < m} \varepsilon_i q^{-i} + q^{-m} \leq x, \\ 0 & \text{if } \sum_{i < m} \varepsilon_i q^{-i} + q^{-m} > x. \end{cases}$$

If $m \geq 1$ and if the digits ε_i of the lazy expansion of x are defined for all $i < m$, then we put

$$\varepsilon_m = \begin{cases} 0 & \text{if } \sum_{i < m} \varepsilon_i q^{-i} + \sum_{i > m} q^{-i} \geq x, \\ 1 & \text{if } \sum_{i < m} \varepsilon_i q^{-i} + \sum_{i > m} q^{-i} < x. \end{cases}$$

In section 1 we characterize the unique expansions of 1. This improves some earlier results in [5]. As a by-product we obtain a new proof for the characterization of the greedy expansions, obtained earlier in [2].

In [4] it was proved that for almost every $1 < q < 2$ the greedy expansion of 1 contains arbitrarily long sequences of consecutive 0 digits. In section 2 we improve this result by giving an explicit estimate on the length of these sequences. An analogous result is obtained for the lazy expansions, too.

In section 3 we generalize some other results obtained in [4]–[7].

At the end of this paper we formulate some open questions.

The authors wish to thank the referee for drawing their attention to the papers [2], [3] and [9].

1. Characterization of the greedy and the unique expansions of 1

Fix $1 < q < 2$ arbitrarily and consider an expansion of 1 :

$$(1) \quad 1 = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \quad \varepsilon_i \in \{0, 1\}.$$

THEOREM 1

a) (1) is the greedy expansion of 1 if and only if

$$(2) \quad (\varepsilon_{k+i}) \stackrel{L}{<} (\varepsilon_i) \quad \text{whenever } \varepsilon_k = 0.$$

b) (1) is the unique expansion of 1 if and only if (2) and

$$(3) \quad (1 - \varepsilon_{k+i}) \stackrel{L}{<} (\varepsilon_i) \quad \text{whenever } \varepsilon_k = 1.$$

are satisfied. \square

Remark 1. — It is easy to deduce from this theorem that if (1) is the greedy (resp. unique) expansion of 1, then (2) (resp. (2) and (3)) is satisfied for all $k \geq 1$. \square

The proof of this theorem is based on some lemmas concerning the more general expansions

$$(4) \quad x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}, \quad \varepsilon_i \in \{0, 1\}$$

for arbitrarily fixed $1 < q < 2$ and $0 \leq x \leq 1/(q - 1)$.

LEMMA 1

a) (4) is the greedy expansion of x if and only if

$$(5) \quad \sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} < 1 \quad \text{whenever } \varepsilon_k = 0.$$

b) (4) is the lazy expansion of x if and only if

$$(6) \quad \sum_{i=1}^{\infty} (1 - \varepsilon_{k+i}) q^{-i} < 1 \quad \text{whenever } \varepsilon_k = 1.$$

Proof :

a) If (5) is not satisfied for some $\varepsilon_k = 0$, then x has another expansion

$$(7) \quad x = \sum_{i=1}^{\infty} \varepsilon'_i q^{-i}, \quad \varepsilon'_i \in \{0, 1\}$$

such that $\varepsilon_i = \varepsilon'_i$ for all $i < k$ and $\varepsilon'_k = 1$. Then the expansion (4) is not greedy.

If the expansion (4) is not greedy, then there is another expansion (7) of x and there is a positive integer k such that $\varepsilon_i = \varepsilon'_i$ for all $i < k$ and $\varepsilon_k = 0$, $\varepsilon'_k = 1$. It follows that

$$\sum_{i>k} \varepsilon_i q^{-i} \geq q^{-k}$$

and therefore (5) is not satisfied.

b) The assertion follows at once from a) if we remark that the expansion (4) is lazy if and only if the expansion

$$(8) \quad 1/(q-1) - x = \sum_{i=1}^{\infty} (1 - \varepsilon_i) q^{-i}$$

is greedy. \square

LEMME 2

a) *If $x \geq 1$ and if the expansion (4) is greedy, then (2) is satisfied.*

b) *If $x \geq 1$ and if the expansion (4) is unique, then (2) and (3) are satisfied.*

Proof:

a) Assume that (2) is not satisfied for some $\varepsilon_k = 0$, then either $(\varepsilon_{k+i}) = (\varepsilon_i)$ or $(\varepsilon_{k+i}) \stackrel{L}{>} (\varepsilon_i)$. In the first case we have

$$\sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} = \sum_{i=1}^{\infty} \varepsilon_i q^{-i} = x \geq 1;$$

hence the condition (5) of LEMMA 1 is not satisfied and the expansion (4) is not greedy. In the second case there is an integer m such that $\varepsilon_{k+i} = \varepsilon_i$ for all $i < m$ and $\varepsilon_{k+m} = 1$, $\varepsilon_m = 0$. If the expansion (4) were greedy, then by LEMMA 1 we would have

$$\sum_{i=1}^{\infty} \varepsilon_{k+i} q^{-i} < 1 \leq x.$$

Therefore x would have another expansion (7) such that $\varepsilon'_i = \varepsilon_i$ for all $i < m$ and $\varepsilon'_m > \varepsilon_m$; hence $(\varepsilon'_i) \stackrel{L}{>} (\varepsilon_i)$. But this is impossible because (4) is the greedy expansion.

b) Assume that (3) is not satisfied for some $\varepsilon_k = 1$, then either $(1 - \varepsilon_{k+i}) = (\varepsilon_i)$ or $(1 - \varepsilon_{k+i}) \overset{L}{>} (\varepsilon_i)$. In the first case we have

$$\sum_{i=1}^{\infty} (1 - \varepsilon_{k+i})q^{-i} = \sum_{i=1}^{\infty} \varepsilon_i q^{-i} = x \geq 1;$$

the condition (6) of LEMMA 1 is not satisfied and the expansion (4) is not lazy.

In the second case there is an integer m such that $1 - \varepsilon_{k+i} = \varepsilon_i$ for all $i < m$ and $1 - \varepsilon_{k+m} = 1, \varepsilon_m = 0$. If the expansion (4) were unique, then by LEMMA 1 we would have

$$\sum_{i=1}^{\infty} (1 - \varepsilon_{k+i})q^{-i} < 1 \leq x.$$

Therefore x would have another expansion (7) such that $\varepsilon'_i = \varepsilon_i$ for all $i < m$ and $\varepsilon_m < \varepsilon'_m$; hence $(\varepsilon_i) \overset{L}{<} (\varepsilon'_i)$. But this is impossible because (4) is the unique expansion. \square

LEMMA 3. — Assume that $x \leq 1$ and that there is an expansion

$$(9) \quad y = \sum_{i=1}^{\infty} \delta_i q^{-i}, \quad \delta_i \in \{0, 1\}$$

for some $0 \leq y \leq 1/(q - 1)$ such that

$$(10) \quad (\delta_{k+i}) \overset{L}{<} (\varepsilon_i) \quad \text{whenever } \delta_k = 0.$$

Assume that either the expansion (4) is infinite (i.e. it contains infinitely many digits 1) or the expansion (9) is finite (i.e. it contains only finitely many digits 1). Then (9) is the greedy expansion of y .

Proof: It is sufficient to verify the condition (5). Fix k such that $\delta_k = 0$. We shall construct a sequence of positive integers $k =: k_0 < k_1 < \dots$ such that

$$(11) \quad \sum_{j=1+k_i}^{k_{i+1}} \delta_j q^{-j} \leq q^{-k_i} - q^{-k_{i+1}}, \quad i = 0, 1, \dots$$

Furthermore, if the expansion (4) is infinite, then we will also have the strict inequalities

$$(12) \quad \sum_{j=1+k_i}^{k_{i+1}} \delta_j q^{-j} < q^{-k_i} - q^{-k_{i+1}}, \quad i = 0, 1, \dots$$

The property (5) hence will follow easily. Indeed, if the expansion (4) is infinite, then we conclude from (12) that

$$\begin{aligned} \sum_{j>k} \delta_j q^{-j} &= \sum_{i=0}^{\infty} \sum_{j=1+k_i}^{k_{i+1}} \delta_j q^{-j} \\ &< \sum_{i=0}^{\infty} (q^{-k_i} - q^{-k_{i+1}}) = q^{-k} \end{aligned}$$

which implies (5). If the expansion (9) is finite, then choosing m such that $\delta_j = 0$ for all $j > k_m$, we conclude from (11) that

$$\begin{aligned} \sum_{j>k} \delta_j q^{-j} &= \sum_{i=0}^{m-1} \sum_{j=1+k_i}^{k_{i+1}} \delta_j q^{-j} \\ &\leq \sum_{i=0}^{m-1} (q^{-k_i} - q^{-k_{i+1}}) < q^{-k} \end{aligned}$$

and (5) follows again.

We define the sequence (k_i) by induction. We set $k_0 = k$. If k is already defined for some $i \geq 0$ and if $\delta_{k_i} = 0$, then applying (10) there exists $m \geq 1$ such that $\delta_{j+k_i} = \varepsilon_j$ for $j < m$ and $\delta_{m+k_i} < \varepsilon_m$. Setting $k_{i+1} = k_i + m$ we have $\delta_{k_{i+1}} = 0$ and

$$\begin{aligned} \sum_{j=1+k_i}^{k_{i+1}} \delta_j q^{-j} &= q^{-k_i} \sum_{j=1}^m \delta_{j+k_i} q^{-j} \\ &= q^{-k_i} \left(\sum_{j=1}^m \varepsilon_j q^{-j} - q^{-m} \right) \leq q^{-k_i} (x - q^{-m}) \\ &\leq q^{-k_i} (1 - q^{-m}) = q^{-k_i} - q^{-k_{i+1}} \end{aligned}$$

i.e. (11) is satisfied. If the expansion (4) is infinite, then the first inequality in this chain is strict and therefore (12) is also satisfied. \square

LEMMA 4

- a) If $x < 1$ and if (2) is satisfied, then (4) is the greedy expansion.
- b) If $x < 1$ and if (3) is satisfied, then (4) is the lazy expansion.

Proof :

a) It is sufficient to apply LEMMA 3 with $y := x$ and $\delta_i := \varepsilon_i$.

b) The assertion is trivial if $x = 0$ because the expansion of 0 is unique.

If $x > 0$, then the expansion (4) cannot be finite. Indeed, if (4) were a finite expansion, then there would exist a positive integer k such that $\varepsilon_k = 1$ and $\varepsilon_i = 0$ for all $i > k$, contradicting (3). The assertion now follows by applying LEMMA 3 with $y := 1/(q - 1)$ and $\delta_i = 1 - \varepsilon_i$. \square

THEOREM 1 now follows at once from LEMMAS 2 and 4.

2. Distribution of the digits

We recall from [4] that for every fixed $1 < q < 2$ the greedy expansion of 1 does not contain arbitrarily long sequences of consecutive 1 digits; on the other hand, the set of those $1 < q < 2$ for which the greedy expansion of 1 contains arbitrarily long sequences of consecutive 0 digits, is residual and has full measure in (1,2).

Similarly, for every fixed $1 < q < 2$ the lazy expansion of 1 does not contain arbitrarily long sequences of consecutive 0 digits; on the other hand, the set of those $1 < q < 2$ for which the greedy expansion of 1 contains arbitrarily long sequences of consecutive 1 digits, is residual and has full measure in (1,2).

Now we prove the following stronger statements :

THEOREM 2

a) *Let G denote the set of those $1 < q < 2$ for which the greedy expansion of 1 has the following property : there are arbitrarily large integers m such that the sequence $\varepsilon_1, \dots, \varepsilon_m$ contains more than $\log_2 m$ consecutive 0 digits. Then G is residual and has full measure in (1,2).*

b) *Let L denote the set of those $1 < q < 2$ for which the lazy expansion of 1 has the following property : there are arbitrarily large integers m such that the sequence $\varepsilon_1, \dots, \varepsilon_m$ contains more than $\log_2 m$ consecutive 1 digits. Then L is residual and has full measure in (1,2). \square*

We need two lemmas; the first of them was proved implicitly in [4].

LEMMA 5. — *For every $0 < \delta < 1$ there is a positive constant $C(\delta)$ and a positive integer $N(\delta)$ having the following property : for any positive integers $n > m \geq N(\delta)$ and for any $\eta_1, \dots, \eta_m \in \{0, 1\}$, if we denote A (resp. by B) the interval of those $1 + \delta < q < 2$ for which the greedy expansion of 1 satisfies $\varepsilon_i = \eta_i, i = 1, \dots, m$ (resp. $\varepsilon_i = \eta_i, i = 1, \dots, m$*

and $\varepsilon_{m+1} = \dots = \varepsilon_n = 0$), then the lengths $|A|$, $|B|$ of these intervals satisfy the inequalities

$$|B| \geq C(\delta)2^{m-n}|A|. \quad \square$$

LEMMA 6. — *There exists a sequence n_1, n_2, \dots of natural numbers such that*

$$(13) \quad n_k > \log_2(n_1 + \dots + n_k), \quad k = 1, 2, \dots$$

and

$$(14) \quad \sum_{k=1}^{\infty} 2^{-n_k} = \infty.$$

Proof: Setting

$$a_j = [\log j + \log \log j + \log \log \log j], \quad j = 4, 5, \dots$$

(we write for brevity \log instead of \log_2), for $k \rightarrow \infty$ we have the following estimates (we apply the Stirling formula) :

$$\begin{aligned} & \log(a_4 + \dots + a_k) \\ & \leq \log \sum_{j=4}^k (\log j + \log \log j + \log \log \log j) \\ & = \log \left\{ (1 + o(1)) \sum_{j=4}^k \log j \right\} \\ & \leq \log(1 + o(1)) + \log \log(k!) \\ & = o(1) + \log \log \{ O(1)(k/e)^{k+1/2} \} \\ & = o(1) + \log \{ O(1) + (k+1/2) \log k - (k+1/2) \log e \} \\ & \leq o(1) + \log(k \log k) \\ & = o(1) + \log k + \log \log k. \end{aligned}$$

It follows that $\log(a_4 + \dots + a_k) < a_k$ if k is sufficiently large, say $k > K$ for some $K > 3$. Then

$$\log(a_K + \dots + a_k) < a_k \quad \text{for all } k > K$$

and the lemma follows by taking $n_k = a_{K+k}$, $k = 1, 2, \dots$ \square

Proof of THEOREM 2 : We restrict ourselves to the proof of assertion a) ; the proof of assertion b) is analogous, by making the modifications of the same type as in [4].

Let us fix a sequence (n_k) of natural numbers satisfying (13), (14) and let us denote by C_k the set of those $1 < q < 2$ for which the greedy expansion of 1 does not satisfy

$$\varepsilon_i = 0$$

for all $n_1 + \dots + n_{k-1} < i \leq n_1 + \dots + n_k, (k = 1, 2, \dots)$. Since

$$\bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} C_j \supset (1, 2) \setminus G,$$

it is sufficient to prove that for every $k \geq 1$ the set $B_k := \bigcap_{j=k}^{\infty} C_j$ has measure 0 and that the closure \bar{B}_k of this set has no interior points.

The first property easily follows from LEMMA 5 and from (14). Indeed, applying LEMMA 5 we obtain that for every $0 < \delta < 1$ the measure of $B_k \cap (1 + \delta, 2)$ is less than or equal to

$$\prod_{j=k'}^{\infty} (1 - C(\delta)2^{-n_j})$$

with $k' := \max\{k, N(\delta)\}$, and this product is equal to zero by (14). Taking $\delta \rightarrow 0$ we conclude that B_k has measure zero.

Since C_j is the union of a finite number of intervals, its boundary is a finite set. Hence \bar{B}_k is the union of B_k and of a countable set. Therefore \bar{B}_k has also measure zero and hence it has no interior points. \square

3. Miscellaneous Results

It follows from the results of [4] cited in the preceding section that the set of those $1 < q < 2$ for which the expansion of 1 is unique, has measure 0. On the other hand, it was shown in [5] that this set has 2^{χ_0} elements. Finally, this set belongs to $(A, 2)$ where we write $A := (1 + \sqrt{5})/2$, because it was proved in [5] that for $q = A$ there are χ_0 different expansions of 1 and for every $1 < q < A$ there are 2^{χ_0} different expansions of 1.

First we show that this last result remains valid for the expansions of every $0 < x < 1/(q - 1)$.

THEOREM 3. — *If $1 < q < A$ and $0 < x < 1/(q - 1)$, then x has 2^{\aleph_0} different expansions.*

Proof: Thanks to the choice of q there is a natural number k such that

$$(15) \quad 1 < q^{-2} + q^{-3} + \dots + q^{-k}.$$

Let $m_1 < m_2 < \dots$ denote the sequence of the elements of $\mathbb{N} \setminus k\mathbb{N}$, then (15) implies that for $\ell = 1, 2, \dots$

$$(16) \quad q^{-m_\ell} < \sum_{j=\ell+1}^{\infty} q^{-m_j}.$$

If we choose k sufficiently large, then we have also

$$(17) \quad \sum_{i=1}^{\infty} q^{-k_i} < x \quad \text{and} \quad \sum_{i=1}^{\infty} q^{-m_i} > x.$$

It follows from (16) and (17) that for every sequence $(\delta_i) \in \{0, 1\}^{\mathbb{N}}$ there exists an expansion (4) of x satisfying $\varepsilon_{k_i} = \delta_i$, $i = 1, 2, \dots$ \square

Now fix $1 < q < 2$ and let $0 =: y_1 < y_2 < \dots$ be the increasing sequence of those real numbers y which have at least one representation of the form

$$y = q^{n_1} + q^{n_2} + \dots + q^{n_k}$$

with finitely many different nonnegative integers n_j . It is clear that $y_n \rightarrow \infty$. We study here the behavior of the difference sequence $y_{n+1} - y_n$.

THEOREM 4

- a) $y_{n+1} - y_n \leq 1$ for all $n \geq 1$.
- b) If $q > A$, then $y_{n+1} - y_n = 1$ for infinitely many n .
- c) If $y_{n+1} - y_n \rightarrow 0$, then 1 has an infinite expansion containing arbitrarily long sequences of consecutive 0 digits.
- d) There exists $1 < q < A$ for which $y_{n+1} - y_n \not\rightarrow 0$. \square

Remark 2. — In [6] the assertion c) was proved under the additional assumption $q < \sqrt{2}$. We remark that the relation $y_{n+1} - y_n \rightarrow 0$ is satisfied for example if $q = \sqrt[m]{2}$ for some integer $m \geq 2$. As a consequence we obtain that for $q = \sqrt{2}$ 1 has an expansion which contains arbitrarily long sequences of consecutive 0 digits. This was proved by another method in [7]. \square

Proof :

a) We apply induction by n . For $n = 1$ the assertion is true because $y_1 = 0$ and $y_2 = 1$. Let $n \geq 1$ and assume that $y_{k+1} - y_k \leq 1$ for $k = 1, \dots, n$. We have to prove that $y_{n+2} - y_{n+1} \leq 1$.

Let $y_{n+1} = \varepsilon_0 + \varepsilon_1 q + \dots + \varepsilon_k q^k$ ($\varepsilon_0, \dots, \varepsilon_k \in \{0, 1\}$) be a representation of y_{n+1} . If $\varepsilon_0 = 0$, then

$$y_{n+2} \leq 1 + \varepsilon_1 q + \dots + \varepsilon_k q^k = 1 + y_{n+1}.$$

If $\varepsilon_0 = 1$, then let $\ell \geq 0$ be the largest integer such that $\varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_\ell = 1$. It is sufficient to find $\delta_0, \dots, \delta_\ell \in \{0, 1\}$ such that

$$(18) \quad \begin{aligned} 1 + q + \dots + q^\ell &< \delta_0 + \delta_1 q + \dots + \delta_\ell q^\ell + q^{\ell+1} \\ &\leq 2 + q + \dots + q^\ell. \end{aligned}$$

Indeed, (18) implies that

$$y_{n+2} < \delta_0 + \dots + \delta_\ell q^\ell + q^{\ell+1} + \sum_{i=\ell+2}^k \varepsilon_i q^i \leq 1 + y_{n+1}.$$

For the proof of (18) first we observe that $q^{\ell+1} < 2 + q + \dots + q^\ell$ because

$$2 + q + \dots + q^\ell - q^{\ell+1} = (2 - q)(1 + q + \dots + q^\ell) > 0.$$

If $q^{\ell+1} > 1 + q + \dots + q^\ell$, then (18) hence follows by taking $\delta_0 = \dots = \delta_\ell = 0$.

If $q^{\ell+1} \leq 1 + q + \dots + q^\ell$, then, taking into account that $y_1 = 0$ and that

$$2 + q + \dots + q^\ell - q^{\ell+1} < 1 + q + \dots + q^\ell \leq y_{n+1},$$

we obtain by the induction hypothesis that

$$\begin{aligned} 1 + q + \dots + q^\ell - q^{\ell+1} &< \delta_0 + \delta_1 q + \dots + \delta_\ell q^\ell \\ &\leq 2 + q + \dots + q^\ell - q^{\ell+1} \end{aligned}$$

for suitable $\delta_0, \dots, \delta_\ell \in \{0, 1\}$, which is equivalent to (18).

b) It is sufficient to prove that, for $m = 0, 1, \dots$ the open intervals $]q^2 + q^4 + \dots + q^{2m}, 1 + q^2 + \dots + q^{2m}[$ do not contain any element of the sequence y_n .

Assume on the contrary that there exist $m \geq 0$ and $y_n = \varepsilon_0 + \dots + \varepsilon_k q^k$ such that

$$q^2 + \dots + q^{2m} < \varepsilon_0 + \dots + \varepsilon_k q^k < 1 + q^2 + \dots + q^{2m}.$$

If $m \geq 1$, then $k \leq 2m$ because

$$q^{2m+1} > 1 + q^2 + \dots + q^{2m}$$

and $\varepsilon_{2m} = 1$ because

$$1 + q + \dots + q^{2m-1} < q^2 + q^4 + \dots + q^{2m};$$

finally $\varepsilon_{2m-1} = 0$ because

$$q^{2m-1} + q^{2m} > 1 + q^2 + q^4 + \dots + q^{2m}.$$

Hence

$$q^2 + \dots + q^{2m-2} < \varepsilon_0 + \dots + \varepsilon_{2m-2} q^{2m-2} < 1 + q^2 + \dots + q^{2m-2}.$$

Repeating this reasoning $(m-1)$ times we obtain $0 < \varepsilon_0 < 1$ which is impossible.

c) It is sufficient to construct two strictly increasing sequences of integers, $0 = n_0 < n_1 < \dots$ and $0 = i_0 < i_1 < \dots$ satisfying for all $k \geq 0$ the inequalities

$$(19) \quad 0 < 1 - \sum_{i=1}^{i_k} q^{-n_i} \leq q^{-k-n_{i_k}}.$$

Indeed, then $\sum_{i=1}^{\infty} q^{-n_i}$ is an expansion of 1 which has also the desired property because $n_{1+i_k} \geq n_{i_k} + k$ for all $k \geq 1$.

We construct these sequences by recursion. Taking $n_0 = i_0 = 0$ (19) is satisfied for $k = 0$.

Assume that (19) is valid for some $k \geq 0$ and consider for some integer $m \geq n_{i_k}$ (to be chosen later) the number

$$q^m \left(1 - \sum_{i=1}^{i_k} q^{-n_i} \right).$$

It is equal to y_n for some $n \geq 1$. Write y_{n-1} in the form

$$y_{n-1} = q^m \sum_{i=1+i_k}^{\iota_{k+1}} q^{-n_i}, n_{1+i_k} < \dots < n_{i_{k+1}}.$$

It follows from (19) that $y_{n-1} < y_n \leq q^{m-k-n_{i_k}}$ and therefore $n_{1+i_k} > k + n_{i_k} \geq n_{i_k}$.

Since $m \rightarrow \infty$ implies $n \rightarrow \infty$ and hence also $y_{n+1} - y_n \rightarrow 0$, we may choose m such that

$$0 < y_n - y_{n-1} \leq q^{-k-1}$$

whence

$$0 < 1 - \sum_{i=1}^{i_{k+1}} q^{-n_i} \leq q^{-m-k-1}.$$

Since $m \geq n_{i_{k+1}}$ by construction, hence (19) follows for $k + 1$ instead of k .

d) We recall from [8], [9] that if $1 < q < 2$ is a Pisot number, then no expansion of 1 contains arbitrarily long sequences of consecutive 0 digits unless it is a finite expansion (i.e. only finitely many digits are different from 0). It is easy to see that the real zero of the polynomial $q^3 - q^2 - 1$ is a Pisot number satisfying $1 < q < A$, cf. [8]. It follows that 1 has a finite expansion, namely $1 = q^{-1} + q^{-3}$, but 1 has 2^{x_0} different expansions (because $q < A$) and therefore 1 has infinite expansions, too. Applying the assertion c) we conclude that $y_{n+1} - y_n \not\rightarrow 0$. \square

Remark 3. — The example in the proof of assertion d) solves also PROBLEM 8 in [5]. \square

Finally we formulate some open problems.

Problem 1 : Characterize the lazy expansions of 1.

Problem 2 : Does there exist $A < q < 2$ for which 1 has a finite expansion and also 2^{x_0} other expansions?

Problem 3 : Is the value $\log_2 m$ in THEOREM 2 optimal?

Problem 4 : Characterize the set of those $1 < q < 2$ for which $y_{n+1} - y_n \rightarrow 0$. Is it true that every q which is sufficiently close to 1 has this property?

Problem 5 : Is the following strenghtening of THEOREM 2 true : for almost every $1 < q < 2$ there exists an integer m_q such that for every integer $m > m_q$ the sequence of the first m digits of the greedy expansion contains more than $\log_2 m$ consecutive 0 digits?

Problem 6 : Is it true that if for some $1 < q < 2$ 1 has an expansion which contains arbitrarily long sequences of consecutive 0 digits (resp. 1

digits), then the greedy (resp. the lazy) expansion also has this property?

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