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ETA INVARIANTS AND COMPLEX IMMERSIONS

BY

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RÉSUMÉ. — L'objet de cet article est de calculer l'invariant êta d'un complexe qui est acyclique en dehors d'une sous-variété. On utilise pour cela le théorème d'indice d'Atiyah-Patodi-Singer et les superconnexions de Quillen.

ABSTRACT. — The purpose of this paper is to calculate the eta invariant of a chain complex of vector bundles which is acyclic off a submanifold. The main tools are the index theorem of Atiyah-Patodi-Singer and the superconnections of Quillen.

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Let $i : M' \to M$ be an embedding of complex manifolds. Let μ be a holomorphic vector bundle on M', let

$$(\xi, v) : \quad 0 \longrightarrow \xi_m \xrightarrow{v} \xi_{m-1} \xrightarrow{v} \cdots \xi_0 \longrightarrow 0$$

be a holomorphic chain complex of vector bundles on M, which provides a resolution of the sheaf $i_* \mathcal{O}_{M'}(\mu)$. In particular the complex (ξ, v) is acyclic on $M \setminus M'$.

Let Z be an odd dimensional real compact spin submanifold of M which intersects M' so that $Z' = Z \cap M'$ is a submanifold of Z. Then the complex $(\xi, v)_{|Z}$ is acyclic off Z'.

Assume that $\xi_0, \ldots, \xi_m$ are equipped with Hermitian metrics and that $T_R Z$ is equipped with a scalar product. Then for $0 \leq k \leq m$, we can construct the reduced eta function $\overline{\eta}^{\xi_k}(s)$ of ATIYAH-PATODI-SINGER [APS] associated with the Dirac operator D^k acting on the spinors of $T_R Z$ twisted by $\xi_{k|Z}$. Set

$$\overline{\eta}^{\xi}(s) = \sum_0^m (-1)^k \overline{\eta}^{\xi_k}(s).$$

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The purpose of this paper is to calculate (modulo integers) the linear combination of eta invariants $\overline{\eta}^{\xi}(0)$ in terms of a local object on Z and of an eta invariant on the submanifold Z'. In the case where Z is the boundary in M of a real submanifold Y, we give an explicit formula in THEOREM 2.9 which involves :

- A Chern-Simons current γ^Z on Z which is constructed using the results of BISMUT [B] and BISMUT-GILLET-SOULÉ [BGS1].
- A Chern-Simons form on Z' naturally associated with certain exact sequences involving the normal bundles to Z' in Z and in Y.
- An eta invariant on Z' associated with the Dirac operator on Z' acting on twisted spinors (where the twisting bundle involves the two normal bundles to Z' in Z and to M' in M explicitly).

We now make several comments on our formula.

- Our main result in THEOREM 2.9 still holds in ordinary real geometry. We work here in a complex geometric setting to be able to directly apply the results of [B]. The results of [B] have an obvious C^∞ analogue. The fact that (ξ, v) is a resolution of $\mathcal{O}_{M'}(\eta)$ implies that, by the local uniqueness of resolutions [E], [S], the complex (ξ, v) degenerates like a Koszul complex near M'. This would have to be introduced as a supplementary assumption in a C^∞ context.
- A more serious limitation of our result is that we assume the manifold Z bounds in M. However such results should hold in full generality, at the expense of more involved techniques. These will be developed elsewhere. The main interest of this paper is to give an explicit answer in a relatively simple case.
- Finally, let us point that our generalized Chern-Simons currents are directly related with the differential characters of CHEEGER and SIMONS [CSi].

In a second part of the paper, we consider a holomorphic submersion $\pi : M \to B$ which restricts to a submersion $\pi' : M' \to B$. Let $s \in S_1 \mapsto c_s \in B$ be a smooth loop. We then compare the holonomies of the direct image determinant line bundles $\lambda(\xi) = (\det R\pi_*(\xi))^{-1}$ and $\lambda(\mu) = (\det R\pi'_*(\mu))^{-1}$, when these line bundles are equipped with the holomorphic Hermitian connections associated with certain Quillen metrics [Q2], [BGS3]. Again, and for simplicity, we assume that the loop c bounds in B. Our result is then a straightforward application of the curvature calculations of BISMUT-GILLET-SOULÉ [BGS3]. The fact that such a result still holds even when c does not bound in B is now a consequence of a difficult result of BISMUT-LEBEAU [BL], whose proof is much more complicate than the one given here.

Also note that by a result of BISMUT-FREED [BF, Theorem 3.16], we know that the holonomy of certain connections on C^∞ determinant bundles of direct images can be evaluated in terms of adiabatic limits of eta invariants. Therefore our two main results, concerning eta invariants and determinants of direct images are intimately related, together with their possible extensions to the non bounding case. Also note that the relation of the holonomy theorem of [BF] to the differential characters of [CSi] has been developed in the context of direct images by GILLET and SOULÉ [GS].

This paper is organized as follows. In Section 1, we introduce our main assumptions and notations. In Section 2, we give a formula for $\overline{\eta}^{\xi}(0)$ in terms of the eta invariant of a submanifold and of a local quantity. Finally in Section 3, we give a relation which connects the holonomies of various determinant bundles over closed loops which bound.

The author is indebted to J. CHEEGER for helpful discussions concerning differential characters.

1. Complex immersions and resolutions

In this section we introduce our main assumptions and notations.

In a), we consider an immersion $i : M' \to M$ of complex manifolds, a holomorphic vector bundle μ on M' and a complex of vector bundles (ξ, v) which resolves $i_*\mu$ on M.

In b), we introduce various metrics on the considered vector bundles.

In c), we briefly describe the superconnection formalism of QUILLEN [Q1].

a) Complex manifolds and resolutions. — Let M be a compact connected complex manifold of complex dimension m. Let $M' = \bigcup_1^d M'_j$ be a finite union of disjoint compact connected complex submanifolds of M. Let i be the embedding $M' \to M$. Let N be the complex normal bundle to M' in M. Let

$$(\xi, v) : 0 \longrightarrow \xi_m \xrightarrow{v} \xi_{m-1} \xrightarrow{v} \cdots \xi_0 \longrightarrow 0 \tag{1.1}$$

be a holomorphic chain complex of vector bundles on the manifold M. Let μ be a holomorphic vector bundle on the manifold M'. We suppose there exists a holomorphic restriction map $r : \xi_{0|M'} \to \mu$.

We make the fundamental assumption that the sequence of sheaves

$$0 \longrightarrow \mathcal{O}_M(\xi_m) \xrightarrow{v} \cdots \xrightarrow{v} \mathcal{O}_M(\xi_0) \xrightarrow{r} i_*\mathcal{O}_{M'}(\mu) \longrightarrow 0 \tag{1.2}$$

is exact. In particular, the complex (ξ, v) is acyclic on $M \setminus M'$.

For $x \in M'$, $0 \leq k \leq m$, let $H_{k,x}$ be the k^{th} homology group of the complex $(\xi, v)_x$. Set $H_x = \bigoplus_0^m H_{k,x}$.

The following results are consequences of the local uniqueness of resolution [Se, Chapter IV, Appendix 1], [E, Theorem 8] and are proved in [B, Section 1].

• For $k = 0, \ldots, m$ the dimension of $F_{k,x}$ is constant on each M'_j, so that H_k is a holomorphic vector bundle on M'.

• For $x \in M'$, $U \in T_x M$, let $\partial_U v(x)$ be the derivative of the chain map v calculated in any given local holomorphic trivialization of (ξ, v) near x. Then $\partial_U v(x)$ acts on H_x. When acting on H_x, $\partial_U v(x)$ only depends on the image y of U in N_x. So we now write $\partial_y v(x)$ instead of $\partial_U v(x)$.

• For $x \in M'$, $y \in N$, $(\partial_y v)^2(x) = 0$. If $y \in N$, let i_y be the interior multiplication operator acting on the exterior algebra $\Lambda(N^*)$. The graded holomorphic complex $(H, \partial_y v)$ on the total space of the vector bundle N is canonically isomorphic to the Koszul complex $(\Lambda N^* \otimes \mu, i_y)$.

b) Assumption (A) on the Hermitian metrics of a chain complex. — We assume that $\xi_0, \ldots, \xi_m$ are equipped with smooth Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. We equip $\xi = \bigoplus_0^m \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Let v^* be the adjoint of v with respect ot the metric h^ξ. By finite dimensional Hodge theory, we get an identification of smooth vector bundles on M'

$$H_k \cong \{f \in \xi_k \,;\; vf = 0 \,;\; v^* f = 0\}, \qquad 0 \leq k \leq m. \tag{1.3}$$

As a smooth subvector bundle of ξ_k, the right-hand side of (1.3) inherits a Hermitian metric from the metric h^{ξ_k} on ξ_k. Therefore H_k is a holomorphic Hermitian vector bundle on M'. Let h^{H_k} denote the Hermitian metric on H_k. We equip $H = \bigoplus_0^m H_k$ with the metric h^H which is the orthogonal sum of the metrics $h^{H_0}, \ldots, h^{H_m}$.

Let g^N, g^μ be Hermitian metrics on N, η. We equip the vector bundle $\Lambda N^* \otimes \mu$ with the tensor product of the metric induced by g^N on ΛN^* and of the metric g^μ.

Definition 1.1. — Given metrics g^N, g^η on N, η, we will say that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ verify assumption (A) with respect to g^N, g^μ if the canonical identification of holomorphic chain complexes on the total space of N

$$(H, \partial_y v) \cong (\Lambda N^* \otimes \mu, i_y) \tag{1.4}$$

also identifies the metrics.

PROPOSITION 1.2. — *Given metrics g^N, g^μ on the vector bundles N, μ, there exist metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ verifying assumption (A) with respect to the metrics g^N, g^μ.*

Proof. — This result is proved in [B, Proposition 1.6]. □

From now on, we assume that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ verify assumption (A) with respect to the metrics g^N, g^μ on N, μ.

c) The superconnection formalism. — Set

$$\xi_+ = \bigoplus_{k \text{ even}} \xi_k \; ; \quad \xi_- = \bigoplus_{k \text{ even}} \xi_k. \tag{1.5}$$

Then $\xi = \xi_+ \oplus \xi_-$ is a $\mathbb{Z}_2$-graded vector bundle.

We now describe the superconnection formation of Quillen [Q1]. Let τ be the involution on ξ defining the $\mathbb{Z}_2$-grading, i.e. $\tau = \pm 1$ on $\xi_\pm$. The algebra $\operatorname{End} \xi$ is naturally $\mathbb{Z}_2$-graded, the even (resp. odd) elements in $\operatorname{End} \xi$ commuting (resp. anticommuting) with τ. TM denotes the complex holomorphic tangent bundle to M, and $T_R M$ is the real tangent bundle to M. We use a similar notation on M'.

The algebra $\Lambda(T_R^* M)$ is naturally $\mathbb{Z}_2$-graded. Therefore the algebra $\Lambda(T_R^* M) \widehat{\otimes} \operatorname{End} \xi$ is also $\mathbb{Z}_2$-graded.

If $C \in \operatorname{End} \xi$, set

$$\operatorname{Tr}_s[C] = \operatorname{Tr}[\tau C]. \tag{1.6}$$

$\operatorname{Tr}_s[C]$ is called the supertrace of C. We extend $\operatorname{Tr}_s[C]$ into a linear map from $\Lambda(T_R^* M) \widehat{\otimes} \operatorname{End} \xi$ into $\Lambda(T_R^* M)$ with the convention that if $\omega \in \Lambda(T_R^* M)$, $C \in \operatorname{End} \xi$, then $\operatorname{Tr}_s[\omega C] = \omega \operatorname{Tr}[C]$.

Let ∇^{ξ_k} be the holomorphic Hermitian connection on ξ_k. Then $\nabla^\xi = \bigoplus_{k=0}^m \nabla^{\xi_k}$ is the holomorphic Hermitian connection on ξ. Set

$$V = v + v^*. \tag{1.7}$$

Then V is a smooth section of $\operatorname{End}^{\text{odd}} \xi$. For $u \geq 0$, set

$$A_u = \nabla^\xi + \sqrt{u}\, V. \tag{1.8}$$

Then A_u is a superconnection on the $\mathbb{Z}_2$-graded vector bundle ξ in the sense of Quillen [Q1].

We will consider A_u as a first order differential operator acting on smooth sections of $\Lambda(T_R^*M)\widehat{\otimes}\,\mathrm{End}\,\xi$. Its square A_u^2 is the curvature of the superconnection A_u. A_u^2 is a smooth section of $(\Lambda(T_R^*M)\widehat{\otimes}\,\mathrm{End}\,\xi)^{\mathrm{even}}$.

We fix once and for all a square root of $2i\pi$. Let φ be the linear map from $\Lambda(T_R^*M)$ into itself $\omega \in \Lambda^p(T_R^*M) \mapsto (2\pi i)^{-p/2}\ \omega \in \Lambda^p(T_R^*M)$.

The basic result of QUILLEN [Q1] asserts that for any $u \geq 0$, the even smooth form $\varphi(\mathrm{Tr}_s[\exp(-A_u^2)])$ is closed and represents in cohomology the Chern character $\mathrm{ch}(\xi) = \sum_0^m (-1)^j \,\mathrm{ch}(\xi_j)$.

2. Eta invariants and resolutions

In this Section, we calculate a linear combination of the eta invariants of the ξ_k's $(0 \leq k \leq m)$ on a submanifold Z in terms of the eta invariants of vector bundles on the submanifold $Z' = Z \cap M'$ and of a local quantity. We here make the same assumptions as in Section 1.

In a), we introduce an even dimensional submanifold Y of M with boundary Z.

In b) we calculate the eta invariant $\overline{\eta}^{\xi}(0)$ of the complex $\xi|_Z$ by a formula of ATIYAH-PATODI-SINGER [APS].

In c), we give another formula (which now depends on a parameter $u \geq 0$), for $\overline{\eta}^{\xi}(0)$.

In d), we use results of [B] to calculate the limit of certain superconnection currents.

In e), by mimicking [BGS1], we construct singular Chern-Simons currents on Z.

In f), we obtain our final formula for $\overline{\eta}^{\xi}(0)$.

a) A manifold with boundary. — To avoid endless considerations, we will assume that all the considered manifolds and vector bundles are orientable and spin. Also we assume that the complex line bundles which we will consider have square roots.

In this respect, remember that by [H, Theorem 2.2], a holomorphic vector bundle E is spin if and only if $\det E$ has a square root.

Let Y be a real compact connected oriented submanifold with boundary in M, of even dimension $n = 2\ell$. Let T_RY be the (real) tangent bundle to Y. Let $Z = \partial Y$ be the oriented boundary of Y. Set

$$\begin{aligned} Y' &= Y \cap M', \quad Z' = Z \cap M', \\ Y_j' &= Y \cap M_j', \quad Z_j' = Z \cap M_j'. \end{aligned} \tag{2.1}$$

We assume there is d', $1 \leq d' \leq d$ such that :

• for $1 \leq j \leq d'$, Y'_j is an even dimensional oriented manifold with boundary Z'_j;

• for $d'+1 \leq j \leq d$, $Z'_j = \emptyset$, and Y'_j is an even dimensional oriented manifold without boundary.

We also assume that if $y \in Y'$, $z \in Z'$

$$(2.2) \qquad \begin{aligned} (T_R Y')_y &= (T_R Y)_y \cap (T_R M')_y, \\ (T_R Z')_z &= (T_R Z)_z \cap (T_R M')_z. \end{aligned}$$

Let $N_1^{Y'}$ be the (real) normal bundle to Y' in Y. Set

$$\widetilde{N}_1^{Y'} = \frac{T_R M_{|Y'}}{(T_R Y + T_R M')_{|Y'}}.$$

Similarly let $N_1^{Z'}$ be the real normal bundle to Z' in Z. Set

$$\widetilde{N}_1^{Z'} = \frac{T_R M_{|Z'}}{(T_R Z + T_R M')_{|Z'}}.$$

Then we have the exact sequences of real vector bundles

$$(2.3) \qquad \begin{aligned} 0 \longrightarrow N_1^{Y'} \longrightarrow N_{R|Y'} \longrightarrow \widetilde{N}_1^{Y'} \longrightarrow 0\,; \\ 0 \longrightarrow N_1^{Z'} \longrightarrow N_{R|Z'} \longrightarrow \widetilde{N}_1^{Z'} \longrightarrow 0. \end{aligned}$$

Note that because of our assumptions after (2.1) and by (2.2)

$$(2.4) \qquad N_1^{Z'} = N_{1\,|Z'}^{Y'}\,; \quad \widetilde{N}_1^{Z'} = \widetilde{N}_{1\,|Z'}^{Y'}.$$

Since Y and Y' are even dimensional and oriented, $N_1^{Y'}$ and $\widetilde{N}_1^{Y'}$ are also even dimensional and oriented. Also it should be pointed out that $N_{|Z'}$ only depends on $\xi_{|Z}$, $\eta_{|Z'}$, and not on the embedding $Z \to M$. In fact by (1.4)

$$(2.5) \qquad H_{|Z'} = (\Lambda N^* \otimes \mu)_{|Z'}.$$

b) The eta invariant of the chain complex on the boundary.

Let g^Z be a smooth scalar product on $T_R Z$. Let F be the Hermitian vector bundle of $T_R Z$ spinors on Z. The Levi-Civita connection $\nabla^{T_R Z}$ on

$T_R Z$ lifts into a unitary connection ∇^F on F. For $0 \leq k \leq m$, let ∇^k be the connection on $F \otimes \xi_k$

$$\nabla^k = \nabla^F \otimes 1 + 1 \otimes \nabla^{\xi_k}. \tag{2.6}$$

For $0 \leq k \leq m$, we denote by D^k the Dirac operator acting on the set of smooth sections of $F \otimes \xi_{k|Z}$ associated with the metric g^Z and the connection ∇^{ξ_k}. Recall that if $c(T_R Z)$ is the Clifford algebra of $T_R Z$, $F \otimes \xi_k$ is a $c(T_R Z)$ Clifford module. If $X \in T_R Z$, let $c(X)$ be the corresponding element in $c(T_R Z)$. If $e_1, \ldots, e_n$ is a locally defined smooth orthonormal base of $T_R Z$, then

$$D^k = \sum_1^n c(e_i) \nabla^k_{e_i}.$$

For $0 \leq k \leq m$, let $\eta^{\xi_k}(s)$ be the eta function associated with the self-adjoint operator D^k defined in ATIYAH–PATODI–SINGER [APS]. Set

$$\overline{\eta}^{\xi_k}(s) = \frac{1}{2}\left(\eta^{\xi_k}(s) + \dim \operatorname{Ker} D^k\right). \tag{2.7}$$

By [APS, Theorem 4.2] the function $\overline{\eta}^{\xi_k}(s)$ is holomorphic at $s = 0$. $\overline{\eta}^{\xi_k}(0)$ is called the reduced eta invariant of the operator D^k. Set

$$\overline{\eta}^{\xi}(s) = \sum_0^m (-1)^k \, \overline{\eta}^{\xi_k}(s). \tag{2.8}$$

Let now g^Y be a smooth metric on $T_R Y$ which has the following two properties :

- g^Y restricts to g^Z on $T_R Z$;
- g^Y is product near the boundary $Z = \partial Y$;
- the restriction of g^Y to Y' is product near the boundary $Z' = \partial Y'$.

Let $\nabla^{T_R Y}$ be the Levi-Civita connection on $T_R Y$ and let $L^{T_R Y}$ be its curvature. Let $\widehat{A}$ be the Hirzebruch polynomial $\widehat{A}(x) = \dfrac{x/2}{\sinh(x/2)}$. Then $\widehat{A}(L^{T_R Y}/(2\pi))$ is a smooth differential form on Y.

We now recall a fundamental result of ATIYAH–PATODI–SINGER [APS].

THEOREM 2.1. — *The following identity holds in* $\mathbb{R}/\mathbb{Z}$

$$\overline{\eta}^{\xi}(0) = \int_Y \widehat{A}\left(\frac{L^{T_R Y}}{2\pi}\right) \operatorname{Tr}_s\left[\exp\left(\frac{-(\nabla^{\xi})^2}{2i\pi}\right)\right]. \tag{2.9}$$

Proof. — By ATIYAH–PATODI–SINGER [APS, Theorem 4.2] for any k, $1 \leq k \leq m$

$$\overline{\eta}^{\xi_k}(0) = \int_Y \widehat{A}\Big(\frac{L^{T_R Y}}{2\pi}\Big) \operatorname{Tr}\Big[\exp\Big(\frac{-(\nabla^{\xi_k})^2}{2i\pi}\Big)\Big] \quad \text{in } \mathbb{R}/\mathbb{Z}. \tag{2.10}$$

(2.9) follows from (2.10). □

c) Eta invariants and superconnections. — We now will give another expression for $\overline{\eta}^{\xi}(0)$ in terms of Quillen's superconnections forms [Q1].

PROPOSITION 2.2. — *For any $u \geq 0$, the following identity holds in $\mathbb{R}/\mathbb{Z}$*

$$\begin{aligned}\overline{\eta}^{\xi}(0) = &\int_Y \widehat{A}\Big(\frac{L^{T_R Y}}{2\pi}\Big)\varphi\big(\operatorname{Tr}_s\big[\exp(-A_u^2)\big]\big) \\ &+ \Big(\frac{1}{2\pi}\Big)^{\dim_R Y/2} \int_Z \widehat{A}(L^{T_R Y})\Big\{\int_0^u \frac{1}{2}\operatorname{Tr}_s\big[\sqrt{s}\, V \exp(-A_s^2)\big]\frac{ds}{s}\Big\}.\end{aligned} \tag{2.11}$$

Proof. — For $u = 0$, (2.11) coincides with (2.9). Also by the transgression formula for superconnections [Q1], we know that

$$\frac{\partial}{\partial u}\operatorname{Tr}_s\big[\exp(-A_u^2)\big] = -d\ \operatorname{Tr}_s\Big[\frac{V}{2\sqrt{u}}\exp(-A_u^2)\Big]. \tag{2.12}$$

Using (2.9), (2.12) and Stokes's formula, we obtain (2.11). □

d) The limit as $u \to \infty$ of the superconnection currents.

Let ∇^N be the holomorphic Hermitian connection on N.

We identify $\widetilde{N}_1^{Y'}$ with the subvector bundle of $N_{R|Y'}$ orthogonal to $N_1^{Y'}$ in $N_{R|Y'}$. Then $N_{R|Y'} = N_1^{Y'} \oplus \widetilde{N}_1^{Y'}$. Moreover $N_1^{Y'}$ and $\widetilde{N}_1^{Y'}$ are now equipped with the metrics induced by the metric g^N on $N_{R|Y'}$.

Let $P^{N_1^{Y'}}$, $P^{\widetilde{N}_1^{Y'}}$, be the orthogonal projection operators from $N_{R|Y}$ on $N_1^{Y'}$, $\widetilde{N}_1^{Y'}$ respectively. Let $\nabla^{N_1^{Y'}}, \nabla^{\widetilde{N}_1^{Y'}}$ be the Euclidean connections which are the orthogonal projections of the connection $\nabla^{N_R}{}_{|Y'}$ on $N_1^{Y'}$, $\widetilde{N}_1^{Y'}$ respectively. Let R^N, $R^{N_1^{Y'}}$, $R^{\widetilde{N}_1^{Y'}}$ denote the curvatures of the connections ∇^N, $\nabla^{N_1^{Y'}}$, $\nabla^{\widetilde{N}_1^{Y'}}$.

Because of (2.4), note that $N_1^{Z'}$, $\widetilde{N}_1^{Z'}$ are also equipped with metrics and connections, which are the orthogonal projections of $\nabla^{N_R}{}_{|Z'}$.

In the sequel, we will use the notation $\operatorname{ch}(\xi)$, $\widehat{A}(T_R Y)$, $\widehat{A}(N_1^Y)$, $\widehat{A}(\widetilde{N}_1^Y)$,... instead of

$$\operatorname{Tr}_s\Big[\exp\Big(\frac{-(\nabla^{\xi})^2}{2i\pi}\Big)\Big],\ \widehat{A}\Big(\frac{L_R^T Y}{2\pi}\Big),\ \widehat{A}\Big(\frac{R^{N_1^{Y'}}}{2\pi}\Big),\ \widehat{A}\Big(\frac{R^{\widetilde{N}_1^{Y'}}}{2\pi}\Big),\ldots$$

These will in fact be closed differential forms.

Let $\widetilde{F}_1^{Y'} = \widetilde{F}_{1,+}^{Y'} \oplus \widetilde{F}_{1,-}^{Y'}$ be the Hermitian $\mathbb{Z}_2$-graded vector bundle of $\widetilde{N}_1^{Y'}$ spinors. The connection $\widetilde{\nabla}^{N_1^{Y'}}$ lifts into a unitary connection on $\widetilde{F}_1^{Y'}$. Let $\mathrm{ch}(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*})$ denote the corresponding Chern-Weil representative of the Chern character of $\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}$. Remember that the symmetric functions associated with $\mathrm{ch}(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*})$ is given by

$$\prod \left(e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}\right). \tag{2.13}$$

If $\widetilde{N}_1^{Y'} = \{0\}$, $\mathrm{ch}(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*})$ should be replaced by one.

THEOREM 2.3. — *The following identity holds*

$$\lim_{u \to +\infty} \int_Y \widehat{A}(T_R Y) \varphi\left(\mathrm{Tr}_s\left[\exp(-A_u^2)\right]\right) = \int_{Y'} \widehat{A}(T_R Y) \frac{\widehat{A}(\widetilde{N}_1^{Y'})}{\widehat{A}(N_R|_{Y'})} e^{-\frac{1}{2}c_1(N|_{Y'})} \mathrm{ch}\left(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\right) \mathrm{ch}(\mu). \tag{2.14}$$

Proof. — To make the references simpler, we first assume that Y is itself a complex manifold. $N_1^{Y'}$, $\widetilde{N}_1^{Y'}$ are temporarily considered as complex bundles instead of real bundles. Then by BISMUT [B, Theorem 5.1], we know that

$$\lim_{u \to +\infty} \int_Y \widehat{A}(T_R Y) \varphi\left(\mathrm{Tr}_s\left[\exp(-A_u^2)\right]\right) = \int_{Y'} \widehat{A}(T_R Y)\, \mathrm{Td}^{-1}(N) c_{\max}(\widetilde{N}_1^{Y'})\, \mathrm{ch}(\mu). \tag{2.15}$$

Now using (2.13), it is clear that

$$\begin{aligned} c_{\max}(\widetilde{N}_1^{Y'}) &= \widehat{A}(\widetilde{N}_1^{Y'})\, \mathrm{ch}\left(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\right), \\ \mathrm{Td}(N) &= \widehat{A}(N_R) e^{\frac{1}{2}c_1(N)}. \end{aligned} \tag{2.16}$$

From (2.15), (2.16), we get (2.14).

Let us now assume that Y is not complex. Observe that in the proof of [B, Theorem 5.1], the only stage where the complex structure of $N_1^{Y'}$, $\widetilde{N}_1^{Y'}$ is used is when expressing Pfaffians of complex endomorphisms as complex

determinants. Inspection of the proof shows that if $e(\widetilde{N}_1^{Y'})$ denotes the Chern-Weil representative of the Euler class of $\widetilde{N}_1^{Y'}$, then

$$\lim_{u\to+\infty}\int_Y \widehat{A}(T_RY)\,\varphi\big(\mathrm{Tr}_s\big[\exp(-A_u^2)\big]\big) \int_{Y'} \widehat{A}(T_RY)\ \mathrm{Td}^{-1}(N)\,e(\widetilde{N}_1^{Y'})\,\mathrm{ch}(\mu). \tag{2.17}$$

(2.14) follows in full generality. □

Observe that on Z', we have the exact sequences of vector bundles

$$0 \longrightarrow N_1^{Z'} \longrightarrow N_{R|Z'} \longrightarrow \widetilde{N}_1^{Z'} \longrightarrow 0\,;$$

$$0 \longrightarrow T_RZ' \longrightarrow T_RZ \longrightarrow N_1^{Z'} \longrightarrow 0.$$

Let B be a Chern-Simons form on the manifold Z' such that

$$\widehat{A}(T_RZ) - \widehat{A}(T_RZ')\,\frac{\widehat{A}(N_{R|Z'})}{\widehat{A}(\widetilde{N}_1^{Z'})} = dB.$$

B is unambiguously defined modulo coboundaries.

For $1 \leq j \leq d'$, let $\overline{\eta}^{\mu}_{Z'_j}(0)$ be the reduced eta invariant of the Dirac operator D'_{Z_j} on Z'_j associated with :

- the metric of $T_RZ'_j$ and the corresponding spinors of $T_RZ'_j$;
- the Hermitian vector bundle with connection $\mu \otimes (\det N_{Z'_j})^{-1/2} \otimes (\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*})$.

Here $\overline{\eta}^{\mu}_{Z'_j}(0)$ is the difference of the reduced eta invariants associated respectively with $\mu\otimes(\det N_{|Z'_j})^{-1/2}\otimes\widetilde{F}_{1,+}^{Y',*}$ and $\mu\otimes(\det N_{|Z'_j})^{-1/2}\otimes\widetilde{F}_{1,-}^{Y',*}$. If Y and M'_j- i.e. if Z and M'_j- intersect transversally, $\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}$ is equal to the trivial Hermitian line bundle $\mathbb{C}$. Set

$$\overline{\eta}^{\mu}(0) = \sum_1^{d'} \overline{\eta}^{\mu}_{Z'_j}(0) \tag{2.18}$$

Remember that since the form B is well defined modulo coboundaries, B can be unambiguously paired with closed forms on Z'.

THEOREM 2.4. — *The following identity holds in* $\mathbb{R}/\mathbb{Z}$

$$\lim_{u\to+\infty}\int_Y \widehat{A}(T_RY)\,\varphi\big(\mathrm{Tr}_s\big[\exp(-A_u^2)\big]\big) = \overline{\eta}^{\mu}(0) + \int_{Z'} B\,\frac{\widehat{A}(\widetilde{N}_1^{Z'})}{\widehat{A}(N_{R|Z'})}\,e^{-c_1(N)/2}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}(\mu). \tag{2.19}$$

Proof. — Let B' be a Chern-Simons forms on Y' such that

$$\widehat{A}(T_RY) - \widehat{A}(T_RY')\,\frac{\widehat{A}(N_{R|Y'})}{\widehat{A}(\widetilde{N}_1^{Y'})} = dB'. \tag{2.20}$$

Observe that since the metric g^Y is a product near $Z = \partial Y$, and since its restriction to Y' is product near $Z' = \partial Y'$, on Z, $\widehat{A}(T_RY)$ restricts to $\widehat{A}(T_RZ)$ and on Z', $\widehat{A}(T_RY')$ restricts to $\widehat{A}(T_RZ')$. Therefore we may assume that the restriction of B' to Z' coincides with B. Clearly

$$\begin{aligned}\int_{Y'} \widehat{A}(T_RY)\,\frac{\widehat{A}(\widetilde{N}_1^{Y'})}{\widehat{A}(N_{R|Y'})}\,e^{-\frac{1}{2}c_1(N_{|Y'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu \\ = \sum_1^{d'}\int_{Y_j'} \widehat{A}(T_RY')\,e^{-\frac{1}{2}c_1(N_{|Y'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu \\ + \int_{Z'} B\,\frac{\widehat{A}(\widetilde{N}_1^{Z'})}{\widehat{A}(N_{R|Z'})}\,e^{-\frac{1}{2}c_1(N_{|Z'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu \\ + \sum_{d'+1}^{d}\int_{Y_j'} \widehat{A}(T_RY')\,e^{-\frac{1}{2}c_1(N_{|Y'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu.\end{aligned} \tag{2.21}$$

Now for $1 \leq j \leq d'$, the metric of Y_j' is product near $Z_j' = \partial Y_j'$. Therefore by ATIYAH–PATODI–SINGER [APS1, Theorem 4.2], we have in $\mathbb{R}/\mathbb{Z}$

$$\overline{\eta}_{Z_j'}^{\mu}(0) = \int_{Y_j'} \widehat{A}(T_RY')\,e^{-\frac{1}{2}c_1(N_{|Y'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu. \tag{2.22}$$

Moreover since for $d' + 1 \leq j \leq d$, the $Y_{j'}$'s are closed orientable spin manifolds, by the Atiyah-Singer Index Theorem, we know that

$$\int_{Y_j'} \widehat{A}(T_RY')\,e^{-\frac{1}{2}c_1(N_{|Y'})}\,\mathrm{ch}\big(\widetilde{F}_{1,+}^{Y',*} - \widetilde{F}_{1,-}^{Y',*}\big)\,\mathrm{ch}\,\mu$$

is an integer. Identity (2.19) now follows from THEOREM 2.3 and from (2.21), (2.22). □

Remark 2.5. — It is very important to observe that the right-hand side of (2.19) only depends on Z' and not on Y'.

e) A Chern-Simons current.

THEOREM 2.6. — *There is* $C > 0$ *such that if* μ *is any smooth differential form on* Z, *if* $u \geq 1$, *then*

$$\left| \int_Z \mu \operatorname{Tr}_s\left[\sqrt{u}\, V \exp(-A_u^2)\right] \right| \leq \frac{C\|\mu\|_{C^1(Z)}}{\sqrt{u}}. \tag{2.23}$$

Proof. — This result is proved in [B, Theorem 4.1] if Z is instead the manifold M itself, and in [B, Section 5] if, more generally, Z is a complex submanifold of M transversal to M'. The same arguments as in B lead easily to (2.23) in the case which is considered here. □

As in [BGS1, Definition 2.4] we now construct a current γ^Z on Z associated with the immersion $Z' \to Z$ and with the complex $(\xi, v)_{|Z}$.

Definition 2.7. — Let γ^Z be the current on Z

$$\gamma^Z = \frac{1}{\sqrt{2i\pi}} \int_0^{+\infty} \varphi\Big(\operatorname{Tr}_s\left[\sqrt{u}\, V \exp(-A_u^2)\right]\Big) \frac{du}{2u}. \tag{2.24}$$

By Theorem 2.6, the current γ^Z is well-defined. More precisely, we have the following result.

THEOREM 2.8. — *The wave front set of* γ^Z *is included in* $N_1^{Z'}$. *Also*

$$d\gamma^Z = \operatorname{ch}(\xi) - \operatorname{Td}^{-1}(N_{|Z'})\, e(\widetilde{N}_1^{Z'}) \operatorname{ch}(\mu) \delta_{\{Z'\}}. \tag{2.25}$$

Proof. — The wave front set properties of the current γ^Z follow from [BGS1, Theorem 2.5]. (2.25) follows from (2.12), (2.23) and from [B, Theorem 5.1]. □

f) A formula for $\overline{\eta}^{\xi}(0)$. — We here obtain our final formula for $\overline{\eta}^{\xi}(0)$ in terms of $\overline{\eta}^{\mu}(0)$, of B and γ^Z.

THEOREM 2.9. — *The following identity holds in* $\mathbb{R}/\mathbb{Z}$

$$\overline{\eta}^{\xi}(0) = \overline{\eta}^{\mu}(0) + \int_Z \widehat{A}(T_R Z)\, \gamma^Z + \int_{Z'} B \operatorname{Td}^{-1}(N_{|Z'})\, e(\widetilde{N}_1^{Z'}) \operatorname{ch}(\mu). \tag{2.26}$$

Proof. — (2.26) follows from PROPOSITION 2.2 and from THEOREMS 2.4, 2.6 and from DEFINITION 2.7. □

Remark 2.10. — As should be the case, formula (2.26) only involves quantities which are calculated over Z. Also observe that $\overline{\eta}^{\xi}(0)$ does not depend on the chain map v. Therefore the right-hand side of (2.26) does not depend either on the chain map v. It has all the properties of a differential character in the sense of CHEEGER-SIMONS [CS], which is here evaluated on the manifold Z.

3. The holonomy of the determinant bundle of a direct image

We make the same assumptions as in Section 1. Let $\pi : M \to B$ be a holomorphic submersion with compact connected fiber Y, which restricts to a submersion $\pi' : M' \to B$, with compact fiber Y'.

We assume that $\pi : M \to B$ is locally Kähler in the sense of [BGS3, Definition 1.25], i.e. there exists an open cover $\mathcal{U}$ of B such that if $U \in \mathcal{U}$, $\pi^{-1}(U)$ is Kähler. Then clearly $\pi' : M' \to B$ is also locally Kähler.

Let g^{TY} be a Hermitian metric on the relative tangent space TY, which is Kähler on each fiber Y. Let $g^{TY'}$ be the restriction of g^{TY} to the relative tangent space TY'. Then $g^{TY'}$ induces a Kähler metric on each fiber Y'. Set

$$
\begin{aligned}
&\lambda(\xi_i) = (\det R\pi_*\xi_i)^{-1}, \quad 0 \leq i \leq m\,; \\
&\lambda(\xi) = \bigotimes_{i=1}^{m} (\lambda(\xi_i))^{(-1)^i}\,; \\
&\lambda(\mu) = (\det R\pi'_*\mu)^{-1}.
\end{aligned} \tag{3.1}
$$

Then by [KM], $\lambda(\xi_i), \lambda(\xi), \lambda(\mu)$ are holomorphic line bundles. Also $\lambda(\xi)$ and $\lambda(\eta)$ are canonically isomorphic.

We equip $\lambda(\xi_i)_{0\leq i\leq m}$ and $\lambda(\eta)$ with the Quillen metrics constructed in [BGS3]. Then by [BGS3, Corollary 3.9], the Quillen metrics are smooth on $\lambda(\xi_i)_{0\leq i\leq m}$ and $\lambda(\eta)$. We equip $\lambda(\xi)$ with the tensor product of the metrics on the $(\lambda(\xi_i))^{(-1)^i}$'s.

Let $\nabla^{\lambda(\xi)}$, $\nabla^{\lambda(\mu)}$ be the holomorphic Hermitian connections on $\lambda(\xi)$, $\lambda(\mu)$ respectively.

Let $s \in S_1 = \mathbb{R}/\mathbb{Z} \mapsto c_s \in B$ be a smooth loop which bounds a disk D. Let $\tau^{\lambda(\xi)}$, $\tau^{\lambda(\mu)}$ be the holonomy of the connections $\nabla^{\lambda(\xi)}$, $\nabla^{\lambda(\mu)}$ on the loop $c \cdot \tau^{\lambda(\xi)}$ and $\tau^{\lambda(\mu)}$ are complex numbers of module 1, which represent the parallel transport operators from c_0 into $c_1 = c_0$ along $s \in [0,1] \to c_s$.

Let Q be a Chern-Simons form on M' such that

$$i^*Td(TY) - Td(TY')\,Td(N) = dQ. \tag{3.2}$$

Let γ^M be the current on M

$$\gamma^M = \frac{1}{\sqrt{2i\pi}} \int_0^{+\infty} \varphi\Big(\mathrm{Tr}_s\big[\sqrt{u}\,V \exp(-A_u^2)\big]\Big)\frac{du}{2u}. \tag{3.3}$$

Using THEOREM 2.8, we know that the wave front set of γ^M is included in N_R^* and moreover

$$d\gamma^M = \mathrm{ch}(\xi) - Td^{-1}(N)\,\mathrm{ch}(g^\eta)\,\delta_{\{M'\}}. \tag{3.4}$$

By [BGS 1, Section 2b], the restriction of the current γ^M to the submanifold $\pi^{-1}(c)$ of M or to the fibers Y is well-defined. Note that except when Z and M' are transversal the current γ^M has no natural restriction to Z. The current γ^Z considered in Definition 2.7 is not in general the restriction of γ^M to Z.

THEOREM 3.1. — *The following identity holds*

$$\tau^{\lambda(\xi)} = \tau^{\lambda(\eta)} \exp\left\{-2i\pi \int_c \left(\int_Y \mathrm{Td}(TY)\,\gamma^M + \int_{Y'} Q\ \mathrm{Td}^{-1}(N)\,\mathrm{ch}(\eta)\right)\right\}. \tag{3.5}$$

Proof. — By BISMUT–GILLET–SOULÉ [BGS3, Theorems 1.27 and 3.14], the curvature of $\nabla^{\lambda(\xi)}$ is given by

$$2i\pi\left[\int_Y \mathrm{Td}(TY)\,\mathrm{ch}(\xi)\right]^{(2)}. \tag{3.6}$$

Using (3.6), we find that

$$\tau^{\lambda(\xi)} = \exp\left\{-2i\pi \int_{\pi^{-1}(D)} \mathrm{Td}(TY)\,\mathrm{ch}(\xi)\right\}. \tag{3.7}$$

On the other hand by (3.4), we find easily that

$$\int_{\pi^{-1}(D)} \mathrm{Td}(TY)\,\mathrm{ch}(\xi) = \int_{\pi'^{-1}(D)} \mathrm{Td}(TY)\ \mathrm{Td}^{-1}(N)\,\mathrm{ch}(\eta) + \int_{\pi^{-1}(c)} \mathrm{Td}(TY)\,\gamma^M. \tag{3.8}$$

Also by (3.2), we find that

$$\int_{\pi'^{-1}(D)} \mathrm{Td}(TY)\,\mathrm{Td}^{-1}(N)\,\mathrm{ch}(\eta) = \int_{\pi'^{-1}(D)} \mathrm{Td}(TY')\,\mathrm{ch}(\eta) + \int_{\pi'^{-1}(c)} Q\,\mathrm{Td}^{-1}(N)\,\mathrm{ch}(\eta). \tag{3.9}$$

Using [BGS3, Theorem 1.27 and 3.14] again, we know that

$$\tau^{\eta} = \exp\left\{-2i\pi \int_{\pi'^{-1}(D)} \mathrm{Td}(TY')\,\mathrm{ch}(\eta)\right\}. \tag{3.10}$$

From (3.7)–(3.10), we get (3.5). □

Remark 3.2. — A result of BISMUT–FREED [BF, Theorem 3.16] calculates the holonomy of closed (not necessary bounding) loops of determinant bundles of C^{∞} direct images as the adiabatic limit of eta invariants of the cylinder constructed over the loop.

As explained in [BBos, Section 6], the result of [BF] can be used only if the metric g^{TY} is the restriction to TY of a Kähler metric on a neighborhood of $\pi^{-1}(c)$ in M. Another easy deformation argument is needed in the general case. So Theorem 3.1 can be considered as a consequence of Theorem 2.9.

Observe that the relation of the holonomy theorem of [BF] to the differential characters of CHEEGER–SIMONS [CSi] has been considered in detail in GILLET–SOULÉ [GS].

Finally note that a recent result of BISMUT–LEBEAU [BL] calculates the Quillen norm of the canonical isomorphism between $\lambda(\mu)$ and $\lambda(\xi)$. From [BL], one easily deduces THEOREM 3.1 for arbitrary (i.e. non necessarily contractible) loops c. However the complexity of the arguments in [BL] gives some value to the shortness of the proof of THEOREM 3.1, even if this Theorem is only proved to hold for contractible loops.

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