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KATHERINE NORRIE

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ACTIONS AND AUTOMORPHISMS OF CROSSED MODULES

BY

KATHERINE NORRIE (*)

RÉSUMÉ. — Il est bien connu que le groupe des automorphismes d'un groupe N fait partie d'un module croisé $N \to \operatorname{Aut} N$. Dans cet article, on étudie des structures plus compliquées, par exemple les *carrés croisés*, qui correspondent aux automorphismes des modules croisés. On peut ainsi étudier les actions et les produits semi-directs des modules croisés.

ABSTRACT. — It is standard that the automorphisms of a group N fit into a crossed module $N \to \operatorname{Aut} N$. This paper explores the corresponding more elaborate structures, for example *crossed squares*, into which fits the automorphism group of a crossed module. This also enables accounts of actions and semi-direct products of crossed modules.

1. Introduction

Crossed modules have been used widely, and in various contexts, since their definition by J.H.C. WHITEHEAD [24, 26] in his investigation of the algebraic structure of second relative homotopy groups. Areas in which crossed modules have been applied include the theory of group presentations (see the survey [3]), algebraic K-theory [12], and homological algebra [10, 18]. Now crossed modules can be viewed as 2-dimensionnal groups [1] and it is therefore of interest to consider counterparts for crossed modules of concepts from group theory; in this paper we shall generalize some aspects of the theory of automorphisms from groups to crossed modules. The surprise is the ease with which the theory transcribes, which in effect confirms the above view of crossed modules.

The automorphism group $\operatorname{Aut} N$ of a group N comes equipped with the canonical homomorphism $\tau : N \to \operatorname{Aut} N$ which has image $\operatorname{Inn} N$, the

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K. NORRIE, School of Mathematics, Univ. of Wales, Bangor, Gwynedd, LL57 1UT, Grande Bretagne and Dept. of Computer Sc., Royal Holloway and Bedford New College, Egham Hill, Egham, Surrey TW 20 0EX, Grande Bretagne.

group of inner automorphisms of N, and kernel $Z(N)$, the centre of N. The quotient $\operatorname{Aut} N/\operatorname{Inn} N$ is the group of outer automorphisms of N, denoted by $\operatorname{Out} N$. We note that τ is one of the standard examples of a crossed module. Now if $1 \to N \to G \to Q \to 1$ is a short exact sequence of groups, then there is a homomorphism $\theta : G \to \operatorname{Aut} N$ making commutative the following diagram

$$(*)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ 1 & \longrightarrow & \operatorname{Inn} N & \longrightarrow & \operatorname{Aut} N & \longrightarrow & \operatorname{Out} N & \longrightarrow & 1. \end{array}$$

However, in some other familiar categories, the set of structure preserving self-maps of a given object will not fulfil the role just delineated for the automorphism group $\operatorname{Aut} N$. In the category of groups, the automorphism group plays a dual role of capturing the notions of action and of structure preserving self-maps. In other categories these notions do not necessarily coincide, and for our present purposes it is the notion of action that is significant. We shall define *actor crossed modules*, and show how they provide an analogue of automorphism groups of groups. We establish that a version of (*) holds for actor crossed modules. We use this actor to define *actions* of crossed modules, and we establish the main properties of these. In particular, we construct semi-direct products of crossed modules, and the holomorph of a crossed module.

Part of the motivation for this work arose from analogies between groups and algebras. In the category of associative algebras, the appropriate replacement for the automorphism group is the bimultiplication algebra [15]; for Lie algebras we must employ the derivation algebra [14].

We shall also show that relationships between groups and crossed modules, involving the automorphism group, are mirrored in corresponding relationships between crossed modules and their three-dimensional analogues, crossed squares. Crossed squares were first defined by D. Guin-Walery and J.-L. Loday in [9], where they are applied to problems in algebraic K-theory. Some applications of crossed squares in homotopy theory may be found in [13, 4, 5]. Since the homomorphism $\tau : N \longrightarrow \operatorname{Aut} N$ is a crossed module, we might expect that a crossed module and its actor will give rise to a crossed square. We show that this is indeed the case. This allows us to relate actors of crossed modules with the equivalence between crossed squares and 2-cat-groups, due to Guin-Walery and Loday, as described in [13].

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1. The actor and centre of a crossed module

Recall that a *crossed module* (T, G, ∂) consist of a group homomorphism $\partial : T \to G$ called the *boundary map*, together with an action $(g, t) \mapsto {}^g t$ of G on T satisfying

1) $\partial({}^g t) = g\,\partial(t)\,g^{-1}$,
2) ${}^{\partial(s)}t = s\,t\,s^{-1}$,

for all $g \in G$ and $s, t \in T$. In addition to the inner automorphism map $\tau : N \to \operatorname{Aut} N$ already mentioned, other standard examples of crossed modules are :

- a G-module M with the zero homomorphism $M \to G$;
- the inclusion of a normal subgroup $N \to G$;
- and any epimorphism $E \to G$ with central kernel.

There are two canonical ways in which a group G may be regarded as a crossed module : via the identity map $G \to G$ or via the inclusion of the trivial subgroup.

We say that (S, H, ∂') is a *subcrossed module* of the crossed module (T, G, ∂) if

i) S is a subgroup of T, and H is a subgroup of G;
ii) ∂' is the restriction of ∂ to S, and
iii) the action of H on S is induced by the action of G on T.

A subcrossed module (S, H, ∂) of (T, G, ∂) is *normal* if

i) H is a normal subgroup of G;
ii) ${}^g s \in s$ for all $g \in G$, $s \in S$, and
iii) ${}^h t\, t^{-1} \in S$ for all $h \in H$, $t \in T$.

A crossed module *morphism* $\langle \alpha, \phi \rangle \,:\, (T, G, \partial) \longrightarrow (T', G', \partial')$ is a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & M \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\mu} \\ G & \xrightarrow{\phi} & P \end{array}$$

such that for all $x \in G$ and $t \in T$, we have $\alpha({}^{x}t) = {}^{\phi(x)}\alpha(t)$. We say that $\langle \alpha, \phi \rangle$ is an *isomorphism* if α and ϕ are both isomorphisms; similarly, we define *monomorphisms, epimorphisms* and *automorphisms* of crossed modules. We denote the group of automorphisms of (T, G, ∂) by $\mathrm{Aut}(T, G, \partial)$. The *kernel* of the crossed module morphism $\langle \alpha, \phi \rangle$ is the normal subcrossed module $(\ker \alpha, \ker \phi, \partial)$ of (T, G, ∂), denoted by $\ker\langle \alpha, \phi \rangle$. The *image* $\mathrm{im}\langle \alpha, \phi \rangle$ of $\langle \alpha, \phi \rangle$ is the subcrossed module $(\mathrm{im}\, \alpha, \mathrm{im}\, \phi, \mu')$ of (M, P, μ).

The *trivial* crossed module (1,1,1) will be written simply as 1. We shall occasionally supress explicit mention of the boundary map in a crossed module (T, G, ∂) and write simply (T, G).

For a crossed module (T, G, ∂), denote by $\mathrm{Der}(G, T)$ the set of all derivations from G to T, i.e. all maps $\chi : G \to T$ such that for all $x, y \in G$,

$$\chi(xy) = \chi(x)\, {}^{x}\chi(y).$$

Each such derivation χ defines endomorphisms σ $(= \sigma_\chi)$ and θ $(= \theta_\chi)$ of G, T respectively, given by

$$\sigma(x) = \partial\chi(x)x \qquad \theta(t) = \chi\partial(t)t$$

Clearly,

$$\sigma\partial(t) = \partial\theta(t), \quad \theta\chi(x) = \chi\sigma(x), \quad \theta({}^{x}t) = {}^{\sigma(x)}\theta(t).$$

Following Whitehead [25], we define a multiplication in $\mathrm{Der}(G, T)$ by the formula $\chi_1 \circ \chi_2 = \chi$, where

$$\chi(x) = \chi_1\sigma_2(x)\chi_2(x) \quad \big(= \theta_1\chi_2(x)\chi_1(x)\big).$$

This turns $\mathrm{Der}(G, T)$ into a semigroup, with identity element the derivation which maps each element of G into the identity element of T. Moreover, if $\chi = \chi_1 \circ \chi_2$ then $\sigma = \sigma_1\sigma_2$. The *Whitehead group* $D(G, T)$ is defined to be the group of units of $\mathrm{Der}(G, T)$, and the elements of $D(G, T)$ are called *regular derivations.*

The following proposition combines results from [17] and [25].

PROPOSITION — DEFINITION 1

The following statements are equivalent :

1) $\chi \in D(G,T)$;
2) $\sigma \in \operatorname{Aut} G$;
3) $\theta \in \operatorname{Aut} T$.

Moreover, $\Delta : D(G,T) \to \operatorname{Aut}(T,G,\partial)$ *defined by* $\Delta(\chi) = \langle\sigma,\theta\rangle$ *is a homomorphism of groups and there is an action of* $\operatorname{Aut}(T,G,\partial)$ *on* $D(G,T)$ *given by* ${}^{\langle\alpha,\phi\rangle}\chi = \alpha\chi\phi^{-1}$, *which makes* $(D(G,T), \operatorname{Aut}(T,G,\partial), \Delta)$ *a crossed module. This crossed module is called the* actor crossed module $\mathcal{A}(T,G,\partial)$ *of the crossed module* (T,G,∂).

There is a morphism of crossed modules

$$\langle\eta,\gamma\rangle : (T,G,\partial) \longrightarrow \mathcal{A}(T,G,\partial).$$

defined as follows. Let $t \in T$. Then $\eta_t : G \to T$ defined by $\eta_t(x) = t\,{}^{x}t^{-1}$ is a derivation, and the map $t \mapsto \eta_t$ defines a homomorphism $\eta : T \to D(G,T)$ of groups. Let $\gamma : G \to \operatorname{Aut}(T,G,\partial)$ be the homormorphism $y \mapsto \langle\alpha_y,\phi_y\rangle$, where $\alpha_y(t) = {}^{y}t$ and $\phi_y(x) = yxy^{-1}$ for $t \in T$ and $y,x \in G$.

By analogy with group theory, we define the *centre* of the crossed module (T,G,∂) to be the kernel $\xi(T,G,\partial)$ of $\langle\eta,\gamma\rangle$. Thus $\xi(T,G,\partial)$ is the crossed module $(T^G, \mathrm{st}_G(T) \cap Z(G), \partial)$ where T^G denotes the *fixed point subgroup* of T, that is,

$$T^G = \{t \in T : {}^{x}t = t \text{ for all } x \in G\};$$

$\mathrm{st}_G(T)$ is the *stabilizer* in G of T, that is,

$$\mathrm{st}_G(T) = \{x \in G : {}^{x}t = t \text{ for all } t \in T\};$$

and $Z(G)$ is the centre of G. Note that T^G is central in T. In [19] this definition is shown to be consistent with the categorical notion of centre developed by S.A. HUQ in [11].

The *inner actor* $\mathcal{I}(T,G,\partial)$ of (T,G,∂) is the image of the morphism $\langle\eta,\gamma\rangle$. It is a crossed module $(E(G,T), \overline{G}, \Delta)$, say, where the elements of $E(G,T)$ are called *principal regular derivations*. It is routine to verify that $\mathcal{I}(T,G,\partial)$ is a normal subcrossed module of $\mathcal{A}(T,G,\partial)$, see [17]. Thus we can form the quotient crossed module $\mathcal{A}(T,G,\partial)/\mathcal{I}(T,G,\partial)$; this we call the *outer actor* of (T,G,∂) and denote it by $\mathcal{O}(T,G,\partial)$.

Examples of actor crossed modules.

1) If N is a normal subgroup of a group G with inclusion $i : N \hookrightarrow G$ then $\mathcal{A}(N,G,i)$ is the crossed module $(D(G,N), X)$ where X is isomorphic

to the subgroup of $\operatorname{Aut} G$ consisting of those automorphisms which restrict to automorphisms of N. Thus if N is a characteristic subgroup of G, the module $\mathcal{A}(N, G, i)$ is isomorphic to the crossed module $(D(G, N), \operatorname{Aut} G)$. For example, this is the case when G is complete.

2) Special cases of 1) are that $\mathcal{A}(1, G)$ is isomorphic to $(1, \operatorname{Aut} G)$ and that $\mathcal{A}(G, G, 1)$ is isomorphic to $(\operatorname{Aut} G, \operatorname{Aut} G, 1)$.

3) Let M be a G-module. Then $(M, G, 0)$ is a crossed module, where 0 denotes the trivial homomorphism, and has actor $(\operatorname{Der}(G, M), \operatorname{Aut}(M, G, 0), 0)$. Here $\operatorname{Der}(G, M)$ is the set of all derivations $G \to M$ and is an $\operatorname{Aut}(M, G, 0)$-module.

2. Actions and semi-direct products

An *action* of a crossed module (M, P, μ) on a crossed module (T, G, ∂) is defined to be a morphism $(M, P, \mu) \to \mathcal{A}(T, G, \partial)$ of crossed modules. Thus the actor $\mathcal{A}(T, G, \partial)$ of (T, G, ∂) acts on (T, G, ∂). The inner actor $\mathcal{I}(T, G, \partial)$ also acts on (T, G, ∂) as well as on any normal subcrossed module of (T, G, ∂). Hence (T, G, ∂) acts on any of its normal subcrossed modules.

A sequence of morphisms of crossed modules

$$1 \longrightarrow (S, H, \partial') \xrightarrow{\langle i,j\rangle} (T, G, \partial) \xrightarrow{\langle \alpha,\phi\rangle} (M, P, \mu) \longrightarrow 1$$

is called a *short exact sequence* if $\langle i, j\rangle$ is a monomorphism, $\langle \alpha, \phi\rangle$ is an epimorphism and the subcrossed modules $\operatorname{im}\langle i, j\rangle$ and $\ker\langle \alpha, \phi\rangle$ of (T, G, ∂) coincide. In such case we call (T, G, ∂) an *extension* of (S, H, ∂') by (M, P, μ). Then the action of (T, G, ∂) on (S, H, ∂') induces the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & (S,H,\partial') \arrow[r] \arrow[d] & (T,G,\partial) \arrow[r] \arrow[d] & (M,P,\mu) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{I}(S,H,\partial') \arrow[r] & \mathcal{A}(S,H,\partial') \arrow[r] & \mathcal{O}(S,H,\partial') \arrow[r] & 1.
\end{tikzcd}$$

Let the crossed module (M, P, ∂) act on the crossed module (T, G, ∂), so that we are given a morphism of crossed modules

$$\begin{tikzcd} M \arrow[r, "\varepsilon"] \arrow[d, "\mu"'] & D(G,T) \arrow[d, "\Delta"] \\ P \arrow[r, "\rho"] & \mathrm{Aut}(G,T,\partial) \end{tikzcd}$$

Suppose that ρ has components $\rho_1 : P \to \mathrm{Aut}\, T$ and $\rho_2 : P \to \mathrm{Aut}\, G$, that is to say $\rho(p) = \langle \rho_1(p), \rho_2(p) \rangle$ for all $p \in P$. Then M acts on T via $\rho_1\mu$, and with this action we can form the semi-direct product of groups $T \rtimes M$. Likewise, since P acts on G via ρ_2, we can form the semi-dire t product $G \rtimes P$.

There is an action of $G \rtimes P$ on $T \rtimes M$ defined by

$$^{(g,p)}(t,m) = \Big({}^{g}({}^{p}t)\big(\varepsilon({}^{p}m)g^{-1}\big), {}^{p}m \Big).$$

Note that ${}^{p}t$ means t acted on by p, that is ${}^{p}t = \rho_1(p)(t)$. The map $\pi : T \rtimes M \to G \rtimes P$ given by $(t,m) \mapsto (\partial(t), \mu(m))$ is a homomorphism. Now it is routine to verify that, with the above action, the triple $(T \rtimes M, G \rtimes P, \pi)$ is a crossed module. We call this crossed module the *semi-direct product* of (T,G,∂) *and* (M,P,μ) *relative to* $\langle \varepsilon, \rho \rangle$ and denote it by $(T,G,\partial) \rtimes_{\langle \varepsilon,\rho \rangle} (M,P,\partial)$.

There is an equivalent internal viewpoint on semi-direct products, analogous to that for groups. So let (S,H,∂) be a crossed module with subcrossed modules (T,G,∂) and (M,P,μ) satisfying :

i) (T,G,∂) is normal in (S,H,∂);

ii) $S = TM$ and $H = GP$, and

iii) $T \cap M = 1$ and $G \cap P = 1$.

Then there is a morphism $\langle \varepsilon, \rho \rangle : (M,P,\mu) \to \mathcal{A}(T,G,\partial)$ of crossed modules defined as follows : $\varepsilon : M \to D(G,T)$ is given by $m \mapsto \varepsilon_m$ where $\varepsilon_m(g) = m\,{}^{g}m^{-1}$ for all $g \in G$, and $\rho : P \mapsto \mathrm{Aut}(T,G,\partial)$ is given by $\rho(p) = \langle \alpha_p, \phi_p \rangle$ where $\alpha_p(t) = {}^{p}t$ for all $t \in T$ and $\phi_p(g) = pgp^{-1}$ for all $g \in G$. Then the resulting semi-direct product

$$(T,G,\partial) \rtimes_{\langle \varepsilon,\phi \rangle} (M,P,\mu)$$

is isomorphic to the given crossed module (S,H,∂).

In the holomorph $\operatorname{Hol} X$ $(= X \rtimes \operatorname{Aut} X)$ of a group X, the action of $\operatorname{Aut} X$ on X becomes conjugation. Thus there is a one to one correspondence between characteristic subgroups of X and normal subgroups of $\operatorname{Hol} X$ contained in X, see [22].

We now show that the analogous situation holds for crossed modules. The *holomorph* $\operatorname{Hol}(T,G,\partial)$ of a crossed module (T,G,∂) is the semi-direct product

$$(T,G,\partial) \rtimes \mathcal{A}(T,G,\partial) = (T \rtimes D(G,T), G \rtimes \operatorname{Aut}(T,G,\partial), \partial \times \Delta),$$

with action given by

$${}^{(g,\langle\alpha,\phi\rangle)}(t,\chi) = \left({}^{g}\alpha\big(t\chi\phi^{-1}(g^{-1})\big), \alpha\chi\phi^{-1}\right)$$

for all $g \in G$, $t \in T$, $\langle\alpha,\phi\rangle \in \operatorname{Aut}(T,G,\partial)$ and $\chi \in D(G,T)$.

A subcrossed module (T',G',∂) of the crossed module (T,G,∂) is characteristic in (T,G,∂) if restriction defines a morphism $\mathcal{A}(T,G,\partial) \to \mathcal{A}(T',G',\partial)$. One then finds, analogously to the group case, that a subcrossed module of (T,G,∂) is characteristic if and only if its image in $\operatorname{Hol}(T,G,\partial)$ is a normal subcrossed module.

3. Crossed squares

We recall the following definition from [9,13], in the form given in [4]. A *crossed square* is a commutative diagram of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow{\nu} & P \end{array}$$

together with actions of the group P on L, M and N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$, such that the following axioms are satisfied :

i) the maps λ, λ' preserve the actions of P. Further, with the given actions the maps μ, ν and $\kappa = \mu\lambda = \nu\lambda'$ are crossed modules;

ii) $\lambda h(m,n) = m\,{}^{n}m^{-1}$, $\lambda' h(m,n) = {}^{m}n\,n^{-1}$;

iii) $h(\lambda\ell, n) = \ell\,{}^{n}\ell^{-1}$, $h(m, \lambda'\ell) = {}^{m}\ell\,\ell^{-1}$;

iv) $h(mm',n) = {}^{m}h(m',n)h(m,n)$, $h(m,nn') = h(m,n)\,{}^{n}h(m,n')$;
v) $h({}^{p}m, {}^{p}n) = {}^{p}h(m,n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

Note that in these axioms a term such as ${}^{m}\ell$ is ℓ acted on by m, and so ${}^{m}\ell = {}^{\mu(m)}\ell$. It is a consequence of i) that λ, λ' are crossed modules. Further, by iii), M acts trivially on $\ker\lambda$ and N acts trivially on $\ker\lambda$.

The following result is the analogue of the fact that a group N gives rise to a crossed module $\tau : N \to \operatorname{Aut} N$.

THEOREM 2. — *Let (T, G, ∂) be a crossed module. Then the natural morphism $\langle\eta, \gamma\rangle : (T, G, \partial) \to \mathcal{A}(T, G, \partial)$ gives rise to the crossed square*

$$\begin{tikzcd} T \arrow[r, "\eta"] \arrow[d, "\partial"'] & D(G,T) \arrow[d, "\Delta"] \\ G \arrow[r, "\gamma"] & \operatorname{Aut}(G,T,\partial) \end{tikzcd}$$

with function $h : D(G,T) \times G \to T$ given by $(\chi, g) \mapsto \chi(g)$ and where $\operatorname{Aut}(T,G,\partial)$ acts on T and G via the appropriate projections.

The proof is quite straightforward and is omitted.

As explained in [13], the crossed square of THEOREM 4 has associated to it the complex of non-abelian groups :

$$T \longrightarrow D(G,T) \rtimes G \longrightarrow \operatorname{Aut}(T,G,\partial).$$

It has been shown by D. CONDUCHÉ (private communication) that this complex may be given the structure of a 2-crossed module, as defined by CONDUCHÉ in [7]. This 2-crossed module is one of the automorphism structures of (T, G, ∂) derived by BROWN and GILBERT in [2] from a monoidal closed structure on the category of crossed modules (over groupoids).

We now show that a crossed square gives rise to two actions of crossed modules.

THEOREM 3. — *Let*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow{\nu} & P \end{array}$$

be a crossed square with function $h : M \times N \to L$. *Then* $\langle \lambda, \nu \rangle : (L, N, \lambda') \to (M, P, \mu)$ *is a morphism of crossed modules, and* (M, P, μ) *acts on* (L, N, λ').

Proof. — Recall that the actor $\mathcal{A}(L, N, \lambda')$ of (L, N, λ') is the crossed module $(D(N, L), \mathrm{Aut}(L, N, \lambda'), \Delta)$. Consider the map $\beta : M \to D(N, L)$ defined by

$$\beta(m)(n) = h(m, n) \quad \text{for } m \in M,\ n \in N\ .$$

Let be the action of P on L and N be given by the following homomorphisms :

$$\omega : P \longrightarrow \mathrm{Aut}\, L, \quad \omega' : P \longrightarrow \mathrm{Aut}\, N.$$

Now define $\psi : P \to \mathrm{Aut}(L, N, \lambda')$ by

$$p \longmapsto \big\langle \omega(p), \omega'(p) \big\rangle \quad \text{for all } p \in P.$$

Then the pair $\langle \beta, \psi \rangle$ is a morphism $(M, P, \mu) \to \mathcal{A}(L, N, \lambda')$ of crossed modules.

Thus a crossed square as in Theorem 3

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow{\nu} & P \end{array}$$

gives rise to the crossed module $(L, N, \lambda') \rtimes_{\langle \beta, \psi \rangle} (M, P, \mu)$. Of course, we obtain a second semi-direct product crossed module $(L, M, \lambda) \rtimes_{\langle \beta', \psi' \rangle} (N, P, \nu)$ by interchanging the roles of M and N.

We note here that the construction of a semi-direct product of crossed modules from a crossed square is implicit in the equivalence established by GUIN-WALERY and LODAY between crossed squares and *2-cat-groups,* see [9, 13]. A 2-cat-group gives rise to two crossed modules and these are the two semi-direct products of crossed modules obtained from its equivalent crossed square. (The other name for the 2-cat-groups of [9] is cat^2-groups as in [4].)

It would be interesting to interpret the results of sections 2 and 3 in terms of 2-cat-groups.

4. The actor tower and complete crossed modules

By the definition of the centre of a crossed module (T, G, ∂) we see that (T, G, ∂) embeds in its actor $\mathcal{A}(T, G, \partial)$ if the centre $\xi(T, G, \partial)$ of (T, G, ∂) is trivial. The analogy with group theory continues by virtue of the following result.

PROPOSITION 4. — *If (T, G, ∂) has trivial centre then its actor $\mathcal{A}(T, G, \partial)$ also has trivial centre.*

Proof. — Let us assume that $\xi(T, G, \partial) = 1$, so that $T^G = 1$ and $Z(G) \cap st_G(T) = 1$. Now the centre of $\mathcal{A}(T, G, \partial)$ is the crossed module

$$\Big(D(G, T)^{\mathrm{Aut}(T,G,\partial)}, Z(\mathrm{Aut}(T, G, \partial))\Big) \cap \mathrm{st}_{\mathrm{Aut}(T,G,\partial)}\big(D(G, T), \Delta\big).$$

So assume that $\chi \in D(G, T)^{\mathrm{Aut}(T,G,\partial)}$. Then for all $\langle \alpha, \phi \rangle \in \mathrm{Aut}(T, G, \partial)$ we have ${}^{\langle \alpha, \phi \rangle}\chi = \chi$. In particular this is true for all $\langle \alpha_y, \phi_y \rangle$ where $y \in G$. But

$${}^{\langle \alpha_y, \phi_y \rangle}\chi = \eta_{\chi(y)^{-1}} \circ \chi$$

so that ${}^{\langle \alpha_y, \phi_y \rangle}\chi = \chi$ implies that $\eta_{\chi(y)^{-1}} = 1$ for all $y \in G$, that is $\chi(y)^{-1}$ ${}^{x}\chi(y) = 1$ for all $x, y \in G$. Now since $T^G = 1$, χ is the trivial derivation and so $D(G, T)^{\mathrm{Aut}(T,G,\partial)} = 1$.

Now suppose that $\langle \alpha, \phi \rangle \in Z(\mathrm{Aut}(T, G, \partial)) \cap st_{\mathrm{Aut}(T,G,\partial)}(D(G, T))$. Then ${}^{\langle \alpha, \phi \rangle}\chi = \chi$ for all $\chi \in D(G, T)$. In particular, ${}^{\langle \alpha, \phi \rangle}\eta_t = \eta_t$ for all $t \in T$, that is $\eta_{\alpha(t)} = \eta_t$ which implies that $t^{-1}\alpha(t) \in T^G = 1$ for all $t \in T$. Thus $\alpha = 1_T$, the identity automorphism of T. Now $\langle \alpha, \phi \rangle \in Z(\mathrm{Aut}(T, G, \partial))$ and hence for all $y \in G$,

$$\langle \alpha, \phi \rangle \langle \alpha_y, \phi_y \rangle = \langle \alpha_y, \phi_y \rangle \langle \alpha, \phi \rangle$$

implying that $\phi\phi_y = \phi_y\phi$ for all $y \in G$. So $\phi(xyx^{-1}) = y\phi(x)y^{-1}$ for all $x, y \in G$. Since ϕ is an automorphism of G, we have $y^{-1}\phi(y) \in Z(G)$ for all $y \in G$.

Now since $\langle \alpha, \phi \rangle$ is a crossed module morphism, $\alpha({}^{y}t) = {}^{\phi(y)}\alpha(t)$; but $\alpha = 1_T$ so that ${}^{y}t = {}^{\phi(y)}t$ for all $y \in G$ and $t \in T$. Therefore

$y^{-1}\phi(y) \in Z(G) \cap st_G(T) = 1$ so that $\phi = 1_G$. Therefore $\langle \alpha, \phi \rangle = \langle 1_T, 1_G \rangle$ completing the proof.

In view of the above result we see that given a crossed module (T, G, ∂) with trivial centre, then a sequence of crossed modules can be constructed :

$$(T, G, \partial),\ \mathcal{A}(T, G, \partial),\ \mathcal{A}(\mathcal{A}(T, G, \partial)), \ldots$$

in which each term embeds in its successor. We call this sequence the *actor tower* of (T, G, ∂). It is therefore natural to define the crossed module (T, G, ∂) to be *complete* if $\xi(T, G, \partial) = 1$ and the canonical morphism $\langle \eta, \gamma \rangle : (T, G, \partial) \to \mathcal{A}(T, G, \partial)$ is an isomorphism. Thus an actor tower stops when it reaches a complete crossed module. We note that in [17] LUE defined a crossed module (T, G, ∂) to be *semi-complete* if $\langle \eta, \gamma \rangle$ is an epimorphism. Thus a semicomplete crossed module with trivial centre is complete.

Examples.

1. — If N is a normal subgroup of a group G, then the crossed module $N \hookrightarrow G$ is complete if and only if G has trivial centre and the only automorphisms of G which restrict to automorphisms of N are inner automorphisms of G. Therefore if G is complete the crossed module $N \hookrightarrow G$ is complete for any normal subgroup N of G. So putting $N = 1$ or $N = G$ we see that a complete group regarded as a crossed module in either canonical way is a complete crossed module. More generally, if N is a characteristic subgroup of G, then the crossed module $N \hookrightarrow G$ is complete if and only if G is complete.

2. — The main result of [17] is that if T is a group with trivial centre, then the crossed module $T \longrightarrow \operatorname{Aut} T$ is semicomplete if and only if the group $\operatorname{Out} T$ of outer automorphisms of T has trivial centre. However, the centre of the crossed module $T \to \operatorname{Aut} T$ is $T^{\operatorname{Aut} T} \to 1$, and so since $T^{\operatorname{Aut} T}$ is a subgroup of $Z(T)$, if T has trivial centre then the crossed module $T \to \operatorname{Aut} T$ has trivial centre. Thus the above result can be restated as : if T is a group with trivial centre then the crossed module $T \to \operatorname{Aut} T$ is complete if and only if $\operatorname{Out} T$ has trivial centre. So any group whose centre is trivial, and whose group of outer automorphisms also has trivial centre will give rise to a complete crossed module. One such class of groups is the class of free groups F_n of finite rank n, where $n \geq 3$ (cf. [16]).

3. — Let (T, G, ∂) be a crossed module with ∂ surjective. Then (T, G, ∂) is complete if and only if T is isomorphic to G and is a complete group.

In order to provide further evidence that the actor of a crossed module is the correct generalisation of the automorphism group of a group, we give an analogue of a well known result of ROSE [20].

THEOREM 5. — *Let G be a group with a characteristic subgroup H with trivial centralizer. Then G is naturally embedded in* $\operatorname{Aut} H$ *by means of conjugation of H by elements of G, and restriction defines a natural monomorphism* $\operatorname{Aut} G \to \operatorname{Aut} H$ *whose image is the normalizer of G in* $\operatorname{Aut} H$.

A corollary is a famous theorem of W. BURNSIDE [6].

THEOREM 6. — *If G is a group with trivial centre and if Inn G is characteristic in* $\operatorname{Aut} G$ *then* $\operatorname{Aut} G$ *is complete.*

The translation of the group theoretic terms in the statement of ROSE's result into their counterparts for crossed modules is straightforward. So, suppose that (S, H, ∂) is a subcrossed module of the crossed module (T, G, ∂). Analogously to the definition of the centre of a crossed module, we define the *centralizer* $C_{(T,G)}(S, H)$ of (S, H) in (T, G) to be the subcrossed module $(T^H, C_G(H) \cap \mathrm{st}_G(S))$ of (T, G), where $C_G(H)$ denotes the centralizer of H in G. In view of the definition of a normal subcrossed module it is natural to define the *normalizer* $N_{(T,G)}(S, H)$ of (S, H) in (T, G) to be the subcrossed module

$$\left(T^{(S,H)}, N_G(H) \cap \mathrm{st}_G\{S\}\right)$$

of (T, G), where $N_G(H)$ is the normalizer of H in G, where $\mathrm{st}_G\{S\}$ is the stabilizer in G of S as a set, that is

$$\begin{aligned}\mathrm{st}_G\{S\} &= \{g \in G : {}^g s \in S \text{ for all } s \in S\} \text{ and}\\ T^{(S,H)} &= \{t \in T : t\,{}^h t^{-1} \in S \text{ for all } h \in H\}.\end{aligned}$$

Note that this ties in with previous notation in that $T^{(1,H)} = T^H$. We can now state the analogue of ROSE's result.

THEOREM 7. — *Let (S, H, ∂) be a characteristic subcrossed module of (T, G, ∂) and suppose that $C_{(T,G)}(S, H) = 1$. Then the morphism*

$$\mathrm{Res} : \mathcal{A}(T, G) \longrightarrow \mathcal{A}(S, H)$$

is a monomorphism and its image is

$$N_{\mathcal{A}(S,H)}\big(\mathrm{Res}(\mathcal{I}(T, G))\big).$$

Proof. — Let $\mathrm{Res} : \mathcal{A}(T, G) \to \mathcal{A}(S, H)$ have components $\langle \mathrm{Res}_1, \mathrm{Res}_2 \rangle$. We firstly show that $\mathrm{Res}_1 : D(G, T) \to D(H, S)$ and $\mathrm{Res}_2 : \mathrm{Aut}(T, G) \to$

$\mathrm{Aut}(S,H)$ are both injective. So let $\chi \in D(G,T)$ be such that $\mathrm{Res}_1(\chi) = 1$. Then since H is a normal subgroup of G, for all $g \in G$, $h \in H$,

$$1 = \chi(ghg^{-1}) = \chi(g)\,{}^{g}\big(\chi(h)\,{}^{h}\chi(g^{-1})\big) = \chi(g)\,{}^{gh}\chi(g^{-1}),$$

so that ${}^{gh}\chi(g^{-1}) = \chi(g)^{-1} = {}^{g}\chi(g^{-1})$. Now acting with g^{-1} we see that for all $g \in G$, $\chi(g^{-1}) \in T^H = 1$, proving that $\chi = 1$.

Now suppose that $\langle \alpha, \phi \rangle$ is such that $\mathrm{Res}_2(\langle \alpha, \phi \rangle) = \langle 1_S, 1_H \rangle$. Then for all $g \in G$, $s \in S$,

$$^{g}s = \alpha(^{g}s) = {}^{\phi(g)}\alpha(s) = {}^{\phi(g)}s.$$

Further for all $g \in G$, $h \in H$,

$$ghg^{-1} = \phi(ghg^{-1}) = \phi(g)h\phi(g^{-1}).$$

Therefore, for all $g \in G$, we have $g^{-1}\phi(g) \in C_G(H) \cap \mathrm{st}_G(H)$ proving that $\phi = 1_G$.

Now for all $t \in T$, $h \in H$, $t\,{}^{h}t^{-1} \in S$, and so

$$t\,{}^{h}t^{-1} = \alpha(t\,{}^{h}t^{-1}) = \alpha(t)\,{}^{h}\alpha(t^{-1}).$$

Therefore for all $t \in T$, we have $t^{-1}\alpha(t) \in T^H = 1$, so that $\alpha = 1_T$, completing the proof that $\mathrm{Res} : \mathcal{A}(T,G) \to \mathcal{A}(S,H)$ is a monomorphism.

We now show that $\mathrm{Res}(\mathcal{A}(T,G)) = N_{\mathcal{A}(S,H)}(\mathrm{Res}(\mathcal{I}(T,G)))$. Since $\mathcal{I}(T,G)$ is a normal subcrossed module of $\mathcal{A}(T,G)$ it is clear that $\mathrm{Res}(\mathcal{A}(T,G))$ is a subcrossed module of $N_{\mathcal{A}(S,H)}(\mathrm{Res}(\mathcal{I}(T,G)))$, and so it remains to prove the reverse inclusion. Now $N_{\mathcal{A}(S,H)}(\mathrm{Res}(\mathcal{I}(T,G)))$ is the crossed module

$$\Big(D(H,S)^{(\mathrm{Res}_1(E(G,T)),\mathrm{Res}_2(\overline{G}))}, \\ N_{\mathrm{Aut}(S,H)}\big(\mathrm{Res}_2(\overline{G})\big) \cap \mathrm{st}_{\mathrm{Aut}(S,H)}\big(\mathrm{Res}_1 E(G,T)\big)\Big).$$

In the following discussion we write $|$ either for restriction to $\mathrm{Aut}(S,H)$ or for restriction to $D(H,S)$. It will be clear from the context which is intended.

Suppose then that $\chi \in D(H,S)^{(\mathrm{Res}_1(E(G,T)))}$. Then for all $\chi \in G$, $\chi \circ {}^{\gamma(\chi)|}\chi^{-1} \in \mathrm{Res}_1(E(G,T))$. Therefore we can define a map $\chi^* : G \to T$ by $\eta_{\chi^*(x)}| = \chi \circ {}^{\gamma(x)|}\chi^{-1}$. Now χ^* is well-defined and is a derivation which restricts to χ on H : this is a straightforward check and we omit the details. Thus

$$D(H,S)^{(\mathrm{Res}_1(E(G,T)),\mathrm{Res}_2(\overline{G}))} \leq \mathrm{Res}_1\big(D(G,T)\big).$$

Now suppose that

$$\langle\alpha,\phi\rangle \in N_{\mathrm{Aut}(S,H)}\big(\mathrm{Res}_2(\overline{G})\big) \cap \mathrm{st}_{\mathrm{Aut}(S,H)}\big(\mathrm{Res}_1 E(G,T)\big).$$

Then :

1) ${}^{\langle\alpha,\phi\rangle}\eta_t| \in \mathrm{Res}_1(E(G,T))$ for all $t \in T$ and
2) $\langle\alpha,\phi\rangle\gamma(x)\langle\alpha,\phi\rangle^{-1} \in \mathrm{Res}_2(\overline{G})$ for all $g \in G$.

Using 1) we can define a map $\widetilde{\alpha} : T \to T$ by $\eta_{\widetilde{\alpha}(t)}| = {}^{\langle\alpha,\phi\rangle}\eta_t|$. Then it is easily verified that $\widetilde{\alpha}$ is an automorphism of T which extends α. Likewise using 2) we can define a map $\widetilde{\phi} : G \to G$ by

$$\gamma(\widetilde{\phi}(x))| = \langle\alpha,\phi\rangle\gamma(x)|\langle\alpha,\phi\rangle^{-1}.$$

Similarly $\widetilde{\phi}$ is an automorphism of G extending ϕ. Moreover, the pair $\langle\widetilde{\alpha},\widetilde{\phi}\rangle$ forms an automorphism of (T,G,∂) which clearly restricts to $\langle\alpha,\phi\rangle$ on (S,H,∂). Therefore $N_{\mathrm{Aut}(S,H)}(\mathrm{Res}_2(\overline{G})) \cap \mathrm{st}_{\mathrm{Aut}(S,H)}(\mathrm{Res}_1 E(G,T))$ is a subgroup of $\mathrm{Res}_2(\mathrm{Aut}(T,G))$, which proves that $N_{\mathcal{A}(S,H)}(\mathrm{Res}(\mathcal{I}(T,G))$ is a subcrossed module of $\mathrm{Res}(\mathcal{A}(T,G))$, so that there is in fact equality, completing the proof of the theorem.

COROLLARY 8. — *If (T,G) is a crossed module with trivial centre and $\mathcal{I}(T,G)$ is characteristic in $\mathcal{A}(T,G)$, then $\mathcal{A}(T,G)$ is complete.*

Proof. — It is easily verified that $\xi(T,G) = 1$ implies that

$$C_{\mathcal{A}(T,G)}\big(\mathcal{I}(T,G)\big) = 1.$$

Thus we can apply THEOREM 7 which tells us that $\mathrm{Res} : \mathcal{A}(\mathcal{A}(T,G)) \to \mathcal{A}(\mathcal{I}(T,G))$ is a monomorphism, and

$$\mathrm{im}(\mathrm{Res}) = N_{\mathcal{A}(\mathcal{I}(T,G))}\big(\mathrm{Res}\,\mathcal{I}\big(\mathcal{A}(T,G)\big)\big).$$

Now the canonical morphism $\langle\eta,\gamma\rangle : (T,G) \to \mathcal{I}(T,G)$ is an isomorphism, and so there is an induced isomorphism $\mathcal{A}(\mathcal{I}(T,G)) \to \mathcal{A}(T,G)$; $\langle\beta,\tau\rangle$ say. Specifically, $\beta : D(\overline{G}, E(G,T)) \to D(G,T)$ is given by $\chi \mapsto \eta^{-1}\chi\gamma$ and $\tau : \mathrm{Aut}(\mathcal{I}(T,G)) \to \mathrm{Aut}(T,G)$ is given by $\langle\varepsilon,\mu\rangle \mapsto \langle\eta^{-1}\varepsilon\eta, \gamma^{-1}\mu\gamma\rangle$.

Let Res^* denote the composite morphism

$$\mathcal{A}\big(\mathcal{A}(T,G)\big) \longrightarrow \mathcal{A}\big(\mathcal{I}(T,G)\big) \xrightarrow{\cong} \mathcal{A}(T,G).$$

In order to prove that $\mathcal{A}(T,G)$ is complete, we must prove that the canonical morphism $\mathcal{A}(T,G) \to \mathcal{A}(\mathcal{A}(T,G))$ (which we denote by $\langle\tau_1,\tau_2\rangle$)

is an isomorphism. We shall show that Res^* is its inverse. We consider how Res^* acts on $\mathcal{I}(\mathcal{A}(T,G))$, which is the crossed module

$$\Big(E\big(\mathrm{Aut}(T,G), D(G,T)\big), \overline{\mathrm{Aut}(T,G)}\Big).$$

Recall that $\overline{\mathrm{Aut}(T,G)}$ is the image of $\widetilde{\gamma}$, where

$$\widetilde{\gamma} : \mathrm{Aut}(T,G) \to \mathrm{Aut}(\mathcal{A}(T,G))$$

is given by $\langle \alpha, \phi \rangle \mapsto \langle \widetilde{\alpha}_{\langle \alpha,\phi\rangle}, \widetilde{\phi}_{\langle \alpha,\phi\rangle}\rangle$, with $\widetilde{\alpha}_{\langle \alpha,\phi\rangle} \in \mathrm{Aut}(D(G,T))$ defined by $\widetilde{\alpha}_{\langle \alpha,\phi\rangle}(\eta) = \alpha\eta\phi^{-1}$ and $\widetilde{\phi}_{\langle \alpha,\phi\rangle} \in \mathrm{Aut}(\mathrm{Aut}(T,G))$ the inner automorphism determined by $\langle \alpha, \phi\rangle$.

Let $\mathrm{Res} : \mathcal{A}(\mathcal{A}(T,G)) \to \mathcal{A}(\mathcal{I}(T,G))$ have components R_1 and R_2 and $\mathrm{Res}^* : \mathcal{A}(\mathcal{A}(T,G)) \to \mathcal{A}(T,G)$ have components R_1^* and R_2^*. Consider $\chi_d \in E(\mathrm{Aut}(T,G), D(G,T))$ for arbitrary $d \in D(G,T)$. Then $\chi_d|\overline{G}$ is a derivation $\overline{G} \to E$ taking $\langle \alpha,\phi\rangle \mapsto d^{\langle \alpha_y,\phi_y\rangle} d^{-1} = \eta_{d(y)}$ for any $y \in G$. Therefore $R_1^*(\chi_d) = \eta\chi_d|\overline{G}\gamma^{-1} = d$, and we have the commutative diagram :

$$\begin{tikzcd}
\chi_d \arrow[r, mapsto, "R_1"] \arrow[dr, "\tau_1"'] & \chi_d|\overline{G} \arrow[d, "\beta"] \\
 & d
\end{tikzcd}$$

Then clearly R_1^* is surjective and is therefore an isomorphism. Now τ_1 is a right inverse and since R_1^* is an isomorphism τ_1 must also be an isomorphism.

Now let $\langle \widetilde{\alpha}_{\langle \alpha,\phi\rangle}, \widetilde{\phi}_{\langle \alpha,\phi\rangle}\rangle \in \overline{\mathrm{Aut}(T,G)}$ for arbitrary $\langle \alpha,\phi\rangle \in \mathrm{Aut}(T,G)$. Then

$$R_2^*\big(\langle \widetilde{\alpha}_{\langle \alpha,\phi\rangle}, \widetilde{\phi}_{\langle \alpha,\phi\rangle}\rangle\big) = \langle \eta^{-1}\widetilde{\alpha}_{\langle \alpha,\phi\rangle}|E(G,T)\,\eta, \gamma^{-1}\widetilde{\phi}_{\langle \alpha,\phi\rangle}|\overline{G}\,\gamma\rangle = \langle \alpha,\phi\rangle,$$

giving the following commutative diagram

$$\begin{tikzcd}
\big(\widetilde{\alpha}_{\langle \alpha,\phi\rangle}, \widetilde{\phi}_{\langle \alpha,\phi\rangle}\big) \arrow[r, mapsto, "R_1"] & \big(\widetilde{\alpha}_{\langle \alpha,\phi\rangle}|E(G,T), \widetilde{\phi}_{\langle \alpha,\phi\rangle}|G\big)_{\zeta} \arrow[d, "\psi"] \\
 & \langle \alpha,\phi\rangle \arrow[ul, "\tau_2"]
\end{tikzcd}$$

This clearly implies that R_2^* is surjective, and is therefore an isomorphism. Therefore Res^* is an isomorphism of crossed modules, and $\langle \tau_1, \tau_2 \rangle$ is its inverse.

We conclude with an indication of further applications of the results above. Recall that extensions of crossed modules are defined in section 2. The actor crossed module can be used to give partial solutions to the problem of determining when every extension of (S,H) by (M,P) is necessarily isomorphic to the direct product $(S \times M, H \times P)$. The results are parallel to the treatment of group extensions in [21], pp. 228–230. A specific instance is the following.

PROPOSITION 9. — *If (S,H) is a complete crossed module then for any crossed module (M,P) every extension of (S,H) by (M,P) is isomorphic to $(S,H) \times (M,P)$.*

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