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FEYNMAN INTEGRALS AS HIDA DISTRIBUTIONS: THE CASE OF NON-PERTURBATIVE POTENTIALS

by

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Dedicated to Jean-Michel Bismut as a small token of appreciation

Abstract. — In this note the concepts of path integrals as generalized expectations of White Noise distributions is presented. Combining White Noise techniques with a generalized time-dependent Doss' formula Feynman integrands are constructed as Hida distributions beyond perturbation theory.

***Résumé* (Les intégrales de chemins comme distributions de Hida: le cas de potentiel non-perturbatif)**

Dans cette note, on introduit les intégrales de chemins comme étant des espérances de bruits blancs généralisés. On combine les techniques de bruits blancs avec une généralisation de la méthode de Doss pour construire les « intégrales » de Feynman comme distributions de Hida, au-delà de la théorie perturbative.

1. Introduction

Feynman "integrals", such as

$$J = \int d^\infty x \exp\left(i\int_0^t \left(T(\dot{x}(s)) - V\left(x(s)\right)\right) ds\right) f(x(\cdot))$$

are commonplace in physics and meaningless mathematically as they stand. Within white noise analysis [**1, 2, 9, 10, 12, 14, 15, 16, 17**] the concept of integral has a natural extension in the dual pairing of generalized and test functions and allows for the construction of generalized functions (the "Feynman integrands") for various classes of interaction potentials V, see e.g. [**5, 6, 7, 10, 11, 13, 17**], all of them by perturbative methods. This work extends this framework to the case where these fail, using complex scaling as in [**4**], see also [**3**].

In Section 2 we characterize Hida distributions. In Section 3 the U-functional is constructed, see Theorem 3.3. We prove in Section 4 that we obtain a solution of the

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Schroedinger equation, see Theorem 4.4. The strategy for a general construction of the Feynman integrand is provided in Section 5. Examples are given in Section 6.

2. White Noise Analysis

The white noise measure μ on Schwartz distribution space arises from the characteristic function

$$C(f) := \exp\Big(-\tfrac{1}{2}\|f\|_2^2\Big), \quad f \in S(\mathbb{R}),$$

via Minlos' theorem, see e.g. [**1, 9, 10**]:

$$C(f) = \int_{S'} \exp\big(i\langle \omega, f\rangle\big)\, d\mu(\omega).$$

Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $S'(\mathbb{R})$ and $S(\mathbb{R})$. We define the space

$$(L^2) := L^2(S'(\mathbb{R}), \mathcal{B}, \mu).$$

In the sense of an L^2-limit to indicator functions $\mathbf{1}_{[0,t)}, t > 0$, a version of Wiener's Brownian motion is given by:

$$B(t, \omega) := \langle \omega, \mathbf{1}_{[0,t)}\rangle = \int_0^t \omega(s)\, ds, \quad t > 0.$$

One then constructs a Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)'$$

of Hida test functions and distributions, see e.g. [**10**]. We introduce the T-transform of $\Phi \in (S)'$ by

$$(T\Phi)(g) := \langle\!\langle \Phi, \exp\big(i\langle \cdot, g\rangle\big)\rangle\!\rangle, \quad g \in S(\mathbb{R}),$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the bilinear dual pairing between $(S)'$ and (S). Expectation extends to Hida distributions Φ by

$$E_\mu(\Phi) := \langle\!\langle \Phi, 1\rangle\!\rangle.$$

Definition 2.1. — *A function $F : S(\mathbb{R}) \to \mathbb{C}$ is called U-functional if*

(i): *F is "ray-analytic": for all $g, h \in S(\mathbb{R})$ the mapping*

$$\mathbb{R} \ni y \mapsto F(g + yh) \in \mathbb{C}$$

has an analytic continuation to $\mathbb{C}$ as an entire function.

(ii): *F is uniformly bounded of order 2, i.e., there exist some constants $0 < K, D < \infty$ and a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that for all $w \in \mathbb{C}$, $g \in S(\mathbb{R})$*

$$|F(wg)| \leq K \exp(D|w|^2\|g\|^2).$$

Theorem 2.2. — *The following statements are equivalent:*

(i): *$F : S(\mathbb{R}) \to \mathbb{C}$ is a U-functional.*

(ii): *F is the T-transform of a unique Hida distribution $\Phi \in (S)'$.*

For the proof and more see e.g. [**10**].

3. Hida distributions as candidates for Feynman Integrands

In this section we construct Hida distributions as candidates for the Feynman integrands. First we list which properties potentials must fulfill.

Assumption 3.1. — *For $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we define the set $\mathcal{D} \subset \mathbb{C}$ by*

$$\mathcal{D} := \Big\{ x + \sqrt{i}y \;\Big|\; x \in \mathcal{O} \text{ and } y \in \mathbb{R} \Big\},$$

and consider analytic functions $V_0 : \mathcal{D} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. Let $0 \leq t \leq T < \infty$. We require that there exists an $0 < \varepsilon < 1$ and a function $I : \mathcal{D} \to \mathbb{R}$ such that its restriction to $\mathcal{O}$ is measurable and locally bounded and

(3.1)

$$E\left[\left| \exp\left(-i \int_0^t V_0\Big(z+\sqrt{i}B_s\Big) ds \right) f\Big(z+\sqrt{i}B_t\Big) \right| \exp\left(\frac{\varepsilon \|B\|^2_{\sup,T}}{2} \right) \right] \leq I(z), \quad z \in \mathcal{D},$$

uniformly in $0 \leq t \leq T$. Here E denotes the expectation w.r.t. a Brownian motion B starting at 0. $\|\cdot\|_{\sup,T}$ denotes the supremum norm over $[0,T]$.

We shall consider time-dependent potentials of the form

$$\begin{aligned} V_{\dot{g}} : [0,T] \times \mathcal{D} &\to \mathbb{C} \\ (t,z) &\mapsto V_0(z) + \dot{g}(t)z \end{aligned} \tag{3.2}$$

for $g \in S(\mathbb{R})$.

Remark 3.2. — *One can show that (3.1) implies that*

$$E\left[\exp\left(-i \int_0^{t-t_0} V_{\dot{g}}\Big(t-s, z+\sqrt{i}B_s\Big) ds \right) f\Big(z+\sqrt{i}B_{t-t_0}\Big) \right],$$

is well-defined for all $g \in S(\mathbb{R})$, $0 \leq t_0 \leq t \leq T$ and $z \in \mathcal{D}$.

Theorem 3.3. — *Let $0 < T < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be Borel measurable, bounded with compact support. Moreover we assume that V_0 and f fulfill Assumption 3.1. Then for all $0 \leq t_0 \leq t \leq T$, the mapping*

$$F_{\varphi,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C}$$

$$g \mapsto \exp\left(-\frac{1}{2} \int_{[t_0,t]^c} g^2(s)\, ds \right) \int_{\mathbb{R}} \exp(-ig(t_0)x)\varphi(x) \Big(G(g,t,t_0) \exp(ig(t)\cdot) f \Big)(x)\, dx \tag{3.3}$$

is a U-functional where for $x \in \mathcal{O}$

$$(3.4)\quad \Big(G(g,t,t_0)\exp(ig(t)\cdot)f\Big)(x) := E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot{g}}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\right)\right.$$
$$\left.\times \exp\Big(ig(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)\right].$$

Proof. — F_{φ,t,f,t_0} is well-defined: (3.4) is finite because of (3.1), and the integral in (3.3) exists since φ is bounded with compact support.

To show that F_{φ,t,f,t_0} is a U-functional we must verify two properties, see Definition 2.1.

First F_{φ,t,f,t_0} must have a "ray-analytic" continuation to $\mathbb{C}$ as an entire function. I.e., for all $g,h \in S(\mathbb{R})$ the mapping

$$\mathbb{R} \ni y \mapsto F_{\varphi,t,f,t_0}(g+yh) \in \mathbb{C}$$

has an entire extension to $\mathbb{C}$.

We note first that this is true for the expression

$$(3.5)\quad u(y) := \exp\left(-i\int_0^{t-t_0} V_{\dot{g}+y\dot{h}}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\right)$$
$$\times \exp\Big(i\,(g+yh)\,(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)$$

inside the expectation in (3.4). Hence the integral of u over any closed curve in $\mathbb{C}$ is zero. By Lebesgue dominated convergence the expectation $E[u(w)]$ is continuous in w. With Fubini

$$\oint E\,[u\,(w)]\,dw = E\left[\oint u\,(w)\,dw\right] = 0,$$

for all closed paths, hence by Morera $E(u(w))$ is entire. This extends to (3.3) since φ is bounded with compact support. Thus

$$\mathbb{C} \ni w \mapsto F_{\varphi,t,f,t_0}(g+wh) \in \mathbb{C}$$

is entire for all $0 \le t_0 \le t \le T$ and all $g,h \in S(\mathbb{R})$.

Verification is straightforward that F_{φ,t,f,t_0} is of 2nd order exponential growth, F_{φ,t,f,t_0} is a U-functional. $\square$

One can show the same result by choosing the delta distribution δ_x, $x \in \Theta$, instead of a test function φ:

Corollary 3.4. — *Let V_0 and f fulfill Assumption 3.1 and let $x \in \Theta$. Then for all $0 \le t_0 \le t \le T$ the mapping*

$$F_{\delta_x,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C}$$

$$g \mapsto \exp\left(-\frac{1}{2}\int_{[t_0,t]^c} g^2(s)\,ds\right)\exp(-ig(t_0)x)\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$$

is a U-functional, where $\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$ is defined as in Theorem 3.3.

4. Solution to time-dependent Schrödinger equation

Assumption 4.1. — *Let $V_0 : \mathcal{D} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ such that Assumption 3.1 is fulfilled and $V_{\dot g}$, $g \in S(\mathbb{R})$, as in (3.2).*

(i): *For all $u, v, r, l \in [0,T]$ and all $z \in \mathcal{D}$ we require that*

$$E^1\Bigg[\Bigg|\exp\Bigg(-i\int_0^u V_{\dot g}\Big(v-s, z+\sqrt{i}B_s^1\Big)\,ds\Bigg)$$

(4.1)

$$\times E^2\Bigg[\exp\Bigg(-i\int_0^r V_{\dot g}\Big(l-s, z+\sqrt{i}B_u^1+\sqrt{i}B_s^2\Big)\,ds\Bigg)f\Big(z+\sqrt{i}B_u^1+\sqrt{i}B_r^2\Big)\Bigg]\Bigg|\Bigg] < \infty.$$

(ii): *For all $z \in \mathcal{D}$, $0 \le t_0 \le t \le T$ and some $0 < \varepsilon \le T$ the functions*

$$\omega \mapsto \sup_{0\le h\le\varepsilon}\Bigg|\Bigg(V_{\dot g}\Big(t, z+\sqrt{i}B_h(\omega)\Big)+\int_0^h \frac{\partial}{\partial t}V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\Bigg)$$

$$\times\exp\Bigg(-i\int_0^h V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\Bigg)f\Big(z+\sqrt{i}B_h(\omega)\Big)\Bigg| \tag{4.2}$$

and

$$\omega \mapsto \sup_{h\in[0,T]}\Bigg|\Delta E^2\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_s^2\Big)\,ds\Bigg)$$

$$\times f\Big(z+\sqrt{i}B_h^1(\omega)+\sqrt{i}B_{t-t_0}^2\Big)\Bigg]\Bigg| \tag{4.3}$$

are integrable.

Here B^1 and B^2 are Brownian motions starting at 0 with corresponding expectations E^1 and E^2, respectively. Moreover Δ denotes $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial}{\partial t}$ the derivative w.r.t. the first variable.

We define $H(\mathcal{D})$ to be the set of holomorphic functions from $\mathcal{D}$ to $\mathbb{C}$. As pointed out by H. Doss, see [4], under specified assumptions (similar to Assumption 3.1 and Assumption 4.1 (ii)) there is a solution $\psi : [0,T] \times \mathcal{D} \to \mathbb{C}$ to the time-independent Schrödinger equation, i.e., for all $t \in [0,T]$ and $x \in \mathcal{O}$

$$\begin{cases} i\frac{\partial}{\partial t}\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V_0(x)\psi(t,x) \\ \qquad\psi(0,x) = f(x), \end{cases}$$

which is given by

$$\psi(t,x) = E\Bigg[\exp\Bigg(-i\int_0^t V_0\Big(x+\sqrt{i}B_s\Big)ds\Bigg)f\Big(x+\sqrt{i}B_t\Big)\Bigg].$$

Remark 4.2. — *Let us consider the case of the free motion, i.e.,* $V_0 \equiv 0$. *We assume that* $f : \mathcal{D} \to \mathbb{C}$ *is an analytic function, such that* $E\Big[f\Big(z+\sqrt{i}B_t\Big)\Big]$, $z \in \mathcal{D}$, $0 \le t \le T$, *exists and is uniformly bounded on* $[0,T]$. *Moreover let*

$$\omega \mapsto \sup_{h\in[0,T]} \Big|\Delta f\Big(z+\sqrt{i}B_h(\omega)\Big)\Big|$$

be integrable, then

$$\frac{\partial}{\partial t}E\Big[f\Big(x+\sqrt{i}B_t\Big)\Big] = -i\frac{1}{2}\Delta E\Big[f\Big(x+\sqrt{i}B_t\Big)\Big],$$

for $x \in \mathcal{O}$, $0 \le t \le T$, *see* [4].

For our purpose a generalization to the time-dependent case

$$\text{(4.4)} \qquad \begin{cases} i\frac{\partial}{\partial t}\big(U(t,t_0)f\big)(x) = \big(H(t)U(t,t_0)f\big)(x) \\ \big(U(t_0,t_0)f\big)(x) = f(x), \end{cases} \qquad x \in \mathcal{O},\ 0 \le t_0 \le t \le T,$$

where $H(t) := -\frac{1}{2}\Delta + V_{\dot g}(t,\cdot)$ for $g \in S(\mathbb{R})$ and $0 \le t \le T$, is necessary. In the following we show that the operator $U(t,t_0) : D(t,t_0) \subset H(\mathcal{D}) \to H(\mathcal{D})$, $0 \le t_0 \le t \le T$, given by

(4.5)

$$U(t,t_0)f(z) := E\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\Bigg) f\Big(z+\sqrt{i}B_{t-t_0}\Big)\Bigg],\ z \in \mathcal{D},$$

provides us with a solution to (4.4). Here by $D(t,t_0)$ we denote the set of functions $f \in H(\mathcal{D})$ such that the expectation in (4.5) is a well-defined object in $H(\mathcal{D})$.

Lemma 4.3. — *Let* V_0 *and* f *fulfill the Assumptions 3.1 and 4.1 then the operator* $U(t,t_0)$, $0 \le t_0 \le t \le T$, *as in (4.5), maps from* $D(t,t_0)$ *to* $H(\mathcal{D})$. *Moreover* $U(r,t_0)f \in D(t,r)$ *and one gets that*

$$U(t,t_0)f(z) = U(t,r)(U(r,t_0)f)(z),$$

for all $0 \le t_0 \le r \le t \le T$ *and* $z \in \mathcal{D}$.

Proof. — The property that $U(t,t_0)$, $0 \le t_0 \le t \le T$, as in (4.5), maps from $D(t,t_0)$ to $H(\mathcal{D})$ follows by using Morera and Assumption 3.1. The fact that $U(r,t_0)f \in D(t,r)$ follows by Assumption 4.1 (i). Let $0 \le t_0 \le r \le t \le T$ and $z \in D$, then one gets with the Markov property and the time-reversibility of Brownian motion that

$$\begin{aligned} \text{(4.6)} \quad U(t,t_0)f(z) &= E\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\Bigg) f\Big(z+\sqrt{i}B_{t-t_0}\Big)\Bigg] \\ &= E\Bigg[\exp\Bigg(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\Bigg) \\ &\quad \times \exp\Bigg(-i\int_{t-r}^{t-r+r-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\Bigg) f\Big(z+\sqrt{i}B_{t-t_0}\Big)\Bigg] \end{aligned}$$

$$= E\Bigg[\exp\bigg(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B_s\Big)ds\bigg)$$

$$\times \exp\bigg(-i\int_0^{r-t_0} V_{\dot g}\Big(r-s+t-r, z+\sqrt{i}B_{s+t-r}\Big)ds\bigg) f\Big(z+\sqrt{i}B_{t-r+r-t_0}\Big)\Bigg]$$

$$= E^1\Bigg[\exp\bigg(-i\int_0^{t-r} V_{\dot g}\Big(t-s, z+\sqrt{i}B^1_s\Big)ds\bigg)$$

$$\times E^2\Bigg[\exp\bigg(-i\int_0^{r-t_0} V_{\dot g}\Big(r-s, z+\sqrt{i}B^1_{t-r}+\sqrt{i}B^2_s\Big)ds\bigg) f\Big(z+\sqrt{i}B^1_{t-r}+\sqrt{i}B^2_{r-t_0}\Big)\Bigg]\Bigg]$$

$$= U(t,r)\big(U(r,t_0)f\big)(z). \quad \square$$

One can show that by $U(t,t_0)$, $0 \le t_0 \le t \le T$, a pointwise-defined (unbounded) evolution system is given.

Theorem 4.4. — *Let $0 < T < \infty$, V_0, $V_{\dot g}$, $g \in S(\mathbb{R})$, as in (3.2), and f such that Assumption 3.1 and 4.1 are fulfilled. Then $U(t,t_0)f(x)$, $0 \le t_0 < t \le T$, $x \in \mathcal{O}$, given in (4.5) solves the Schrödinger equation (4.4).*

Proof. — Let $0 \le t_0 < t \le T$, $x \in \mathcal{O}$ and $g \in S(\mathbb{R})$. If we have a look at the difference quotient from the right side, we get with Lemma 4.3 that

$$\frac{\partial}{\partial t}^{+} U(t,t_0)f(x) = \lim_{h\searrow 0} \frac{U(t+h,t_0)-U(t,t_0)}{h} f(x)$$
$$= \lim_{h\searrow 0} \frac{U(t+h,t)-U(t,t)}{h} U(t,t_0)f(x).$$

Hence it is left to show that

$$\lim_{h\searrow 0} \frac{U(t+h,t)k(x)-U(t,t)k(x)}{h} = H(t)k(x),$$

for $k = U(t,t_0)f$. Note that

(4.7)

$$\lim_{h\searrow 0}\frac{1}{h}E\Bigg[\exp\bigg(-i\int_0^{t+h-t} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg) k\Big(x+\sqrt{i}B_h\Big) - k\Big(x+\sqrt{i}B_0\Big)\Bigg]$$

$$= \lim_{h\searrow 0} E\Bigg[\frac{1}{h}\exp\bigg(-i\int_0^{h} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg) k\Big(x+\sqrt{i}B_h\Big) - \frac{1}{h}k\Big(x+\sqrt{i}B_h\Big)\Bigg]$$

$$+ \lim_{h\searrow 0} E\Bigg[\frac{1}{h}k\Big(x+\sqrt{i}B_h\Big) - \frac{1}{h}k\Big(x+\sqrt{i}B_0\Big)\Bigg].$$

The integrand of the first summand yields

$$\lim_{h\searrow 0}\frac{1}{h}\Bigg(\exp\bigg(-i\int_0^{h} V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\bigg) - 1\Bigg) k\Big(x+\sqrt{i}B_h\Big)$$

$$= -\, iV_{\dot g}\Big(t, x+\sqrt{i}B_0\Big)k\Big(x+\sqrt{i}B_0\Big) = -iV_{\dot g}(t,x)k(x).$$

Hence by Assumption 4.1 (ii), the mean value theorem and Lebesgue dominated convergence

$$\lim_{h\searrow 0} E\left[\frac{1}{h}\left(\exp\left(-i\int_0^h V_{\dot g}\Big(t+h-s, x+\sqrt{i}B_s\Big)ds\right)-1\right)k\Big(x+\sqrt{i}B_h\Big)\Big)\right] = -iV_{\dot g}(t,x)k(x).$$

Moreover we know by Remark 4.2 and Assumption 4.1 (ii) that $E\Big[k(x+\sqrt{i}B_t)\Big]$ solves the free Schrödinger equation, hence

$$\lim_{h\searrow 0} E\Big[k\Big(x+\sqrt{i}B_h\Big) - k\Big(x+\sqrt{i}B_0\Big)\Big] = -\frac{i}{2}\Delta k(x).$$

Similar with

$$\begin{aligned}\frac{\partial}{\partial t}^{-} U(t,t_0)f(x) &= \lim_{h\searrow 0}\frac{U(t-h,t_0)-U(t,t_0)}{h}f(x)\\ &= \lim_{h\searrow 0}\frac{U(t-h,t-h)-U(t,t-h)}{h}U(t-h,t_0)f(x)\end{aligned}$$

one can show the same for the difference quotient from the left side. □

5. General construction of the Feynman integrand

Of course one is interested in the Feynman integrand I_{V_0} for a general class of potentials $V_0 : \mathcal{O} \to \mathbb{C}$, where $\mathbb{R}\setminus\mathcal{O}$ is of measure zero, having an analytic continuation to $\mathcal{D}$. I.e., we are interested in the Feynman integrand corresponding to the Hamiltonian

$$H = -\frac{1}{2}\Delta + V_0(q),$$

where q is the position operator, i.e.,

$$H\varphi(x) = -\frac{1}{2}\Delta\varphi(x) + V_0(x)\varphi(x), \qquad x\in\mathcal{O},$$

for suitable $\varphi : \mathbb{R}\to\mathbb{R}$ (see the introduction for a comprehensive list of references). In all cases it turned out that for a test function $g \in S(\mathbb{R})$ and $0 \le t_0 < t \le T$ we have that

$$(TI_{V_0})(g) = \exp\left(-\frac{1}{2}\left\|g\mathbf{1}_{[t_0,t]^c}\right\|^2 + ig(t)x - ig(t_0)x_0\right)K_{V_0}^{(\dot g)}(x,t|x_0,t_0), \tag{5.1}$$

where $K_{V_0}^{(\dot g)}(x,t|x_0,t_0)$ denotes the Green's function corresponding to the potential $V_{\dot g}$ (see [8] for a justification of (5.1) under natural assumptions on I_{V_0}). This leads us to the following definition (see e.g. [6]).

Definition 5.1. — *Let $V_0 : \mathcal{D} \to \mathbb{C}$ be an analytic potential, $f : \mathbb{C} \to \mathbb{C}$ an analytic initial state, $V_{\dot{g}}$, $g \in S(\mathbb{R})$, as in (3.2), and $\varphi : \mathbb{R} \to \mathbb{R}$, Borel measurable, bounded with compact support. Assume that V_0, $V_{\dot{g}}$ and f fulfill Assumption 3.1 and Assumption 4.1. Then by Theorem 3.3 one has that for all $0 \le t_0 \le t \le T$, the function F_{φ,t,f,t_0} exists and forms a U-functional. Moreover by Theorem 4.4 it follows that for all $x \in \mathcal{D}$ and all $0 \le t_0 \le t \le T$*

$$U_{\dot{g}}(t,t_0)f(x) = E\left[\exp\left(-i\int_0^{t-t_0} V_{\dot{g}}\left(s, x+\sqrt{i}B_s\right)ds\right) f\left(x+\sqrt{i}B_{t-t_0}\right)\right]$$

exists and solves the Schrödinger equation (4.4) corresponding to the Hamiltonian

$$H(t) = -\frac{1}{2}\Delta + V_0(q) + \dot{g}(t)q,$$

for all $g \in S(\mathbb{R})$. Then by Theorem 2.2 we define the Feynman integrand

$$I_{V_0,\varphi,f} := T^{-1}F_{\varphi,t,f,t_0} \in (S)'.$$

Definition 5.2. — *Again let $V_0 : \mathcal{D} \to \mathbb{C}$ be an analytic potential, $f : \mathbb{C} \to \mathbb{C}$ an analytic initial state, $V_{\dot{g}}$, $g \in S(\mathbb{R})$, as in (3.2) and $x \in \mathcal{O}$. Analogously with Theorem 2.2, Corollary 3.4 and Theorem 4.4 we define the Feynman integrand*

$$I_{V_0,\delta_x,f} := T^{-1}F_{\delta_x,t,f,t_0} \in (S)'.$$

Remark 5.3. — *Note that the Green's function $K_{V_0}^{(\dot{g})}(x,t|x_0,t_0)$, if it exists, is the integral kernel of the operator $U_{\dot{g}}(t,t_0)$.*

6. Examples

To show the existence of the Feynman integrand for concrete examples one only has to verify Assumption 3.1 and 4.1. In this section we look at analytic potentials V_0 which are already considered in [**4**]. First we introduce the set of initial states f. For $m \in \mathbb{N}$ we choose the function

$$f_m : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto (2^m m!)^{-\frac{1}{2}}(-1)^m \pi^{-\frac{1}{4}} e^{\frac{1}{2}z^2}\left(\frac{\partial}{\partial z}\right)^m e^{-z^2}. \tag{6.1}$$

Note that the set of functions given by the restrictions of f_m, $m \in \mathbb{N}$, to $\mathbb{R}$ are the Hermite functions, whose span is a dense subset of $L^2(\mathbb{R})$.

Lemma 6.1. — *Let $k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a measurable function and B a real-valued Brownian motion, then*

$$E\left[k(\|B\|_{\sup,T})\right] \le 2\left(\frac{2}{\pi T}\right)^{1/2}\int_0^\infty k(u)e^{-\frac{u^2}{2T}}du.$$

For the proof see [**4**, Sec.1, Lem.1].

Lemma 6.2. — *Let f_m, $m \in \mathbb{N}$, be as in (6.1). Then for all $l \in \mathbb{N}_0$ and $\varepsilon > 0$ there exists a locally bounded measurable function $c_{m,l} : \mathbb{C} \to \mathbb{R}^+$ such that*

$$\left|f_m^{(l)}\left(z+\sqrt{i}y\right)\right| \leq c_{m,l}(z)|y|^{m+l} \exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right) \exp\left(\frac{\varepsilon}{2}|y|^2\right) \quad \textit{for all } z \in \mathbb{C}, y \in \mathbb{R},$$

where $f_m^{(l)}$ denotes the l-th derivative of f.

6.1. The Feynman integrand for polynomial potentials. — Here for $n \in \mathbb{N}_0$ we have a look at the potential

$$V_0 : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto (-1)^{n+1} a_{4n+2} z^{4n+2} + \sum_{j=1}^{4n+1} a_j z^j, \tag{6.2}$$

for $a_0, \dots, a_{4n+1} \in \mathbb{C}$ and $a_{4n+2} > 0$. If we have a look at the function

$$y \mapsto -iV_{\dot{g}}\left(t, x+\sqrt{i}y\right)$$

for $g \in S(\mathbb{R})$, $x \in \mathbb{C}$ and $t \in [0,T]$, then it is easy to see that the term of highest order of the real part is given by $-a_{4n+2}y^{4n+2}$. So it follows that for all compact sets $K \subset \mathbb{C}$ there exists a constant $C_K > 0$ such that

$$\sup_{z \in K} \sup_{t \in [0,T]} \sup_{y \in \mathbb{R}} \left|\exp\left(|g(t)|\,(|z|+|y|) - iV_0\left(z+\sqrt{i}y\right)\right)\right| \leq C_K. \tag{6.3}$$

Hence the function

$$\omega \mapsto \exp\left(-i\int_0^t V_{\dot{g}}\left(s, z+\sqrt{i}B_s(\omega)\right) ds\right) \tag{6.4}$$

is bounded uniformly in $0 \leq t \leq T$ and locally uniformly in $z \in \mathbb{C}$.

Theorem 6.3. — *Let $0 < T < \infty$, V_0 as in (6.2) and f_m, $m \in \mathbb{N}$, as in (6.1). Then it is possible to define the corresponding Feynman integrand I_{V_0,φ,f_m}, φ Borel measurable, bounded with compact support, and I_{V_0,δ_x,f_m}, $x \in \mathbb{R}$, as in Definition 5.1 and Definition 5.2, respectively.*

Proof. — As discussed above V_0 and f_m are analytic. Moreover with Lemma 6.1 and

$$k_{z,l} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

$$u \mapsto c_{m,l}(z) u^{m+l} \exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right) \exp\left(\frac{\varepsilon}{2}u^2\right), \tag{6.5}$$

$l \in \mathbb{N}_0$, we get that

(6.6)

$$E\left[\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|\right] \leq E\left[\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)k_{z,0}(\|B\|_{\sup,T})\right] < \infty,$$

for $0 < \varepsilon < \frac{1}{2T}$, $z \in \mathbb{C}$ and $c_{m,l}$ as in Lemma 6.2. If we multiply the integrand in (6.6) with the bounded function in (6.4) we still have an integrable function for all $z \in \mathbb{C}$ and all $0 \le t \le T$. So for showing Assumption 3.1 one has to check whether there exists a function $I : \mathbb{C} \to \mathbb{R}^+$ whose restriction to $\mathbb{R}$ is locally bounded and measurable, such that relation (3.1) holds. It is easy to see that this is true for the function

$$\begin{aligned} I &: \mathbb{C} \to \mathbb{R}^+ \\ z &\mapsto E\left[\left|\exp\left(Re\left(-i\int_0^t V_0\left(z+\sqrt{i}B_s\right)ds\right)\right)\right| \exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right) k_{z,0}(\|B\|_{\sup,T})\right]. \end{aligned}$$

The locally boundedness of the restriction of I to $\mathbb{R}$ follows from (6.3) and the fact that $c_{m,l}$ is locally bounded. Since $\mathcal{O} = \mathbb{R}$ one can choose an arbitrary φ, Borel measurable, bounded with compact support, to apply Theorem 3.3. Moreover if we omit the integration the assumptions of Corollary 3.4 are also fulfilled.

So it is only left to check whether Assumption 4.1 is fulfilled. To show (4.1) again by the boundedness of (6.4) one only has to show that

$$E^1\left[\left|E^2\left[f_m\left(z+\sqrt{i}B^1_{t-r}+\sqrt{i}B^2_{r-t_0}\right)\right]\right|\right] < \infty, \quad z \in \mathbb{C}, \quad 0 \le t_0 \le r \le t \le T.$$

But this follows directly by Lemma 6.1 and Lemma 6.2. To show Assumption 4.1 (ii) note first that differentiation and integration in (4.3) can be interchanged since the integrand is analytic and its derivatives are integrable. Since V is polynomial, using the functions $k_{z,0}$, $k_{z,1}$ and $k_{z,2}$, see (6.5), Lemma 6.1 and Lemma 6.2 one can show a estimate similar to (6.6) for (4.2) and (4.3), respectively. Hence they are integrable. □

Remark 6.4. — *For $n = 0$ we are not dealing with the harmonic oscillator. Nevertheless it is possible to handle a potential of the form*

$$x \mapsto a_0 + a_1 x + a_2 x^2,$$

for $a_0, a_1 \in \mathbb{C}$ and $a_2 \in \mathbb{R}$ such that $a_2 < \frac{1}{2T^2}$. In this case the function in (6.4) might be unbounded. So one has to estimate the potential as in Lemma 6.2, separately.

6.2. Non-perturbative accessible potentials. — In this section $\mathcal{O} = \mathbb{R} \setminus \{b\}$, $b \in \mathbb{R}$. We first consider analytic potentials of the form

$$\begin{aligned} V_0 &: \mathcal{D} \to \mathbb{C} \\ z &\mapsto \exp\left(\log(a) - \frac{n}{2}\log\left((z-b)^2\right)\right), \end{aligned} \tag{6.7}$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Note that for $x \in \mathcal{O}$ one has that $V(x) = \frac{a}{|x-b|^n}$.

Lemma 6.5. — *Let V_0 be defined as in (6.7). Then V_0 is analytic on $\mathcal{D}$ and for all $z \in \mathcal{D}$, $z = x + \sqrt{i}y$, $x \in \Theta$, $y \in \mathbb{R}$, and all $0 \le t \le T$ we get that*

$$\left|V_0\left(z+\sqrt{i}B_t\right)\right| = |a| \left|\exp\left(-\frac{n}{2}\log\left((z-b+\sqrt{i}B_t)^2\right)\right| \le |a| \exp\left(-\frac{n}{2}\log\left(\frac{(x-b)^2}{2}\right)\right).$$

For the proof see [**4**].

Theorem 6.6. — *Let $0 < T < \infty$, $\Theta = \mathbb{R}\setminus\{b\}$, V_0 as in Lemma 6.5 and f_m, $m \in \mathbb{N}$, as in (6.1). Then it is possible to define the corresponding Feynman integrand I_{V_0,φ,f_m}, $\varphi : \mathbb{R}\setminus\{b\} \to \mathbb{C}$, Borel measurable, bounded with compact support and I_{V_0,δ_x,f_m}, $x \in \mathbb{R}\setminus\{b\}$, as in Definition 5.1 and Definition 5.2, respectively.*

Proof. — W.l.o.g. we set $a = 1$ and $b = 0$. Then $\Theta = \mathbb{R}\setminus\{0\}$. So let $z \in \mathcal{D}$, $z = x + \sqrt{i}y$, $x \in \Theta$, $y \in \mathbb{R}$, and $0 \le t \le T$. Again we have to check Assumption 3.1 and 4.1. From Lemma 6.5 know that V_0 is analytic on $\mathcal{D}$.

Now we check whether relation (3.1) is true. From Lemma 6.5 we know that

$$\left|\exp\left(-i\int_0^t V_0\left(z+\sqrt{i}B_s\right)ds\right)\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)f_m\left(z+\sqrt{i}B_t\right)\right|$$
$$\le \exp\left(t\exp\left(-\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right)\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|.$$

So with

$k_{z,l} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$

$$u \mapsto \exp\left(T\exp\left(-\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right)c_{m,l}(z)u^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}u^2\right), \tag{6.8}$$

$l \in \mathbb{N}_0$, Lemma 6.1 and Lemma 6.2 we get that

$$E\left[\left|\exp\left(-i\int_0^t V_0\left(z+\sqrt{i}B_s\right)ds\right)\right|\exp\left(\frac{\varepsilon\|B\|^2_{\sup,T}}{2}\right)\left|f_m\left(z+\sqrt{i}B_t\right)\right|\right]$$
$$\le 2\left(\frac{2}{\pi T}\right)^{1/2}\exp\left(T\exp\left(\frac{n}{2}\log\left(\frac{x^2}{2}\right)\right)\right) \tag{6.9}$$
$$\times\int_0^\infty c_{m,l}(z)u^{m+l}\exp\left(\left(\frac{1}{2}+\frac{1}{\sqrt{2\varepsilon}}\right)|z|^2\right)\exp\left(\frac{\varepsilon}{2}u^2\right)e^{-\frac{u^2}{2T}}du =: I(z), \tag{6.10}$$

for all $z \in \mathcal{D}$, $0 < \varepsilon < \frac{1}{4T}$ and $c_{m,l}$ as in Lemma 6.2. Again since $c_{m,l}$ is measurable and locally bounded it follows that the restriction of I to Θ is also measurable and locally bounded. Now we check whether Assumption 4.1 is true. Relation (4.1) follows by Lemma 6.1, Lemma 6.2 and Lemma 6.5. Again with Lemma 6.1, Lemma 6.2 and Lemma 6.5 and the functions $k_{z,0}$, $k_{z,1}$ and $k_{z,2}$ one can show integrability for (4.2) and (4.3), respectively. □

Corollary 6.7. — *In the same way one can also show the existence of the Feynman integrand for potentials of the form*

$$\begin{aligned} V_0 : \mathcal{D} &\to \mathbb{C} \\ z &\mapsto \frac{a}{(z-b)^n}, \end{aligned} \tag{6.11}$$

for $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover one can choose linear combination of the potentials given in (6.2),(6.7) and (6.11).

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