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Astérisque, tome 283 (2002), p. VI-128
<http://www.numdam.org/item?id=AST_2002__283__R3_0>

ANALYTIC THEORY FOR THE QUADRATIC SCATTERING WAVE FRONT SET AND APPLICATION TO THE SCHRÖDINGER EQUATION

Luc Robbiano, Claude Zuily

Abstract. — We consider in this work, the microlocal propagation of analytic singularities for the solutions of the Schrödinger equation with variable coefficients. We introduce, following R. Melrose and J. Wunsch, a $\mathbb{R}^n$ compactification and a cotangent compactification. We define by a FBI transform an analytic wave front set on this cotangent bundle. The main part of this paper is to prove the propagation of microlocal analytic singularities in this wave front set.

Résumé **(Théorie analytique du front d'onde de scattering quadratique et application à l'équation de Schrödinger)**

On examine dans ce travail la propagation des singularités analytiques des solutions de l'équation de Schrödinger à coefficients variables. Nous introduisons, en suivant R. Melrose et J. Wunsch, une compactification de $\mathbb{R}^n$ et une compactification du cotangent. Nous définissons sur ce cotangent un front d'onde analytique par une transformation de FBI. La majeure partie de cet article est consacrée à la preuve de la propagation des singularités analytiques microlocales de ce front d'onde.

CONTENTS

CHAPTER 0

INTRODUCTION

The purpose of this work is to provide a theory for the analytic quadratic scattering wave front set, here denoted ${}^{\mathrm{qsc}}WF_a$, which in the C^∞ case has been introduced by Wunsch [**W1**] after the work of Melrose [**M1**], and to apply it to the propagation of analytic singularities for the linear Schrödinger equation with variable coefficients.

To understand what we are doing here, let us begin by a very simple example. Let us consider the initial value problem for the constant coefficients Schrödinger equation,

$$\text{(0.1)} \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = 0, \quad t > 0, \;\; x \in \mathbb{R}^n \\ u_{|t=0} = u_0 \end{cases} .$$

Taking $u_0 = \delta$ and $u_0 = e^{-i|x|^2}$, it is an easy exercise to see that a data which is a distribution with compact support may give rise to a smooth solution (in x) for every positive t, while an analytic data which oscillates at infinity may produce a singular solution (in x) at some time t. This classical fact, which, roughly speaking, asserts that the smoothness of the solution (in x), for $t > 0$, is under the control of the behavior at infinity of the initial data, is known as "propagation with infinite speed".

It turns out that this fact extends in many directions. It is of microlocal nature, it can be described geometrically and it holds for non trapping Laplacians which are flat perturbation (at infinity) of the constant coefficient case.

These extensions have been the subject of many recent works. See Kapitanski-Safarov [**KS**], Craig-Kappeler-Strauss [**CKS**], Craig [**C**], Shananin [**Sh**], Robbiano-Zuily [**RZ1, RZ2**], Kajitani-Wakabayashi [**KW**], Okaji [**O**], Morimoto-Robbiano-Zuily [**MRZ**]. Related works have been done by Doi [**D1, D2**], Hayashi-Kato [**HK**], Hayashi-Saitoh [**HS**], Kajitani [**K**], Vasy [**V**], Vasy-Zworski [**VZ**] and we refer to the paper [**CKS**] for a more complete bibliography.

In all these works we are handling two informations : behavior at infinity (decay, oscillations...) and smoothness. In a recent paper, Wunsch [**W1**] proposed to embed these two informations in one unique object, which he called the C^∞ quadratic scattering (qsc) wave front set, in which the above phenomena of infinite speed propagation would appear as a propagation of singularities result. Here the word quadratic is used to emphasize that this wave front set takes in account the quadratic oscillations at infinity. Let us note that a scattering wave front set in the C^∞ case was already introduced by Melrose [**M1, M2**] and that related notions have been recently considered by Wunsch-Zworski [**WZ**] (see also Rouleux [**R**]). Moreover, in the same paper Wunsch gave a quite complete description of the propagation of singularities for this C^∞ wave front set which will be described later one.

It is worthwhile to mention that some propagation results have been obtained a long time ago by R. Lascar [**L**] (see also Boutet de Monvel [**B**]). In the C^∞ case, he introduced a parabolic wave front set and he proved its propagation. However this propagation (in x) holds between two points at the same time t ; it is therefore unable to link the "singularities" of the data to those of the solution for positive time.

The work of Wunsch relies on some geometrical point of view of Melrose. It begins by working on a compact manifold M with boundary ∂M, which comes from a (stereographic) compactification of $\mathbb{R}^n$. Roughly speaking this corresponds to set, for large x, $x = \omega/\rho$, where $\rho > 0$ and $\omega \in S^{n-1}$. The boundary ∂M corresponds then to the infinity of $\mathbb{R}^n$. The second step is to define a cotangent bundle. The natural one, coming from the above compactification would be the one where the canonical one form is given by $\alpha = \lambda \frac{d\rho}{\rho^2} + \mu \cdot \frac{dy}{\rho}$ if (ρ, y) are local coordinates near the boundary. However, having in mind that this bundle should hold the singularities of the quadratic oscillatory data, Wunsch introduced the quadratic scattering (qsc) cotangent bundle, ${}^{\mathrm{qsc}}T^*M$ where the canonical one form is given by $\alpha = \lambda \frac{d\rho}{\rho^3} + \mu \cdot \frac{dy}{\rho^2}$. Indeed if $u_0(x) = e^{i\langle Ax, x\rangle}$, where A is an $n \times n$ symmetric real matrix, we have $u_0 = e^{\frac{i}{\rho^2}\langle A\omega, \omega\rangle}$ and the differential of the phase is

$$d\left(\frac{1}{\rho^2}\langle A\omega, \omega\rangle\right) = -2\langle A\omega, \omega\rangle \frac{d\rho}{\rho^3} + \sum_{j=1}^{n} \frac{\partial}{\partial \omega_j}(\langle A\omega, \omega\rangle) \frac{d\omega_j}{\rho^2}.$$

Local coordinates, near the boundary, in this qsc cotangent bundle are given by (ρ, y, λ, μ). Now, since only high frequencies are involved in the occurring of singularities, Melrose suggests to make a radial compactification in the fibers, that is to set, for large $\lambda + |\mu|$,

$$\sigma = \frac{1}{(\lambda^2 + |\mu|^2)^{1/2}}, \quad \overline{\lambda} = \sigma\lambda, \quad \overline{\mu} = \sigma\mu.$$

Then we may define the extended qsc cotangent bundle ${}^{\mathrm{qsc}}\overline{T}^*M$ in which local coordinates, near the boundary of M, are given by $(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu}))$, where $\rho \geqslant 0$, $\sigma \geqslant 0$. Its boundary $\mathcal{C}$ is the union of two faces, ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M = \{(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu})) : \rho = 0\}$ and ${}^{\mathrm{qsc}}S^*M = \{(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu})) : \sigma = 0\}$.

The qsc wave front set is a subset of $\mathcal{C}$. To define it, in the C^∞ case, Wunsch uses Melrose's theory of pseudo-differential operators on manifolds with corners [**M1**]. Here, in the analytic case (but also in the C^∞ or Gevrey cases) we use instead the Sjöstrand machinery of FBI transforms. Our analytic qsc wave front set will be defined through a FBI transform with two scales (h, k), instead of only one scale $\lambda = 1/k$ in the usual case. More precisely we shall set for $u \in L^2(M)$,

$$(0.2) \quad \mathcal{T}u(\alpha, h, k) = \iint e^{ih^{-2}k^{-1}\varphi(\rho/h, y, \alpha, h)} a(\rho/h, y, \alpha, h, k)\chi(\rho/h, y)\overline{u(\rho, y)}\, d\rho dy\,.$$

Here φ is a phase, a a symbol and χ a cut-off function. (See § 2 for the precise definitions of phases, symbols and ${}^{\mathrm{qsc}}WF_a$).

The simplest phase is the following

$$\varphi(s, y, \alpha, h) = (s - \alpha_s)\alpha_\tau + (y - \alpha_y)\cdot\alpha_\eta + ih[(s - \alpha_s)^2 + (y - \alpha_y)^2]\,,$$

where $\alpha = (\alpha_s, \alpha_y, \alpha_\tau, \alpha_\eta) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1}$.

Now, if $u(t, \cdot)$ is a solution of (0.1) and $t_0 > 0$, the ${}^{\mathrm{qsc}}WF_a(u(t_0, \cdot))$ does not propagate ; instead we introduce a uniform qsc analytic wave front set ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0, \cdot))$ which will propagate.

In (0.2), the parameter h is used to describe the behavior at infinity (decay, oscillations...) while k is used to test the analytic smoothness. However near the corner $\{\rho = \sigma = 0\}$ these two informations are mixed. As in the usual case, it is necessary to define such transforms for a large class of phases. Moreover one should be able to change phases, symbols and cut-off functions, in particular, to show the invariance of the ${}^{\mathrm{qsc}}WF_a$; to achieve these invariances, in particular to go from one phase to another, one has to make a careful study of the pseudo-differential operators in the complex domain, then in the real domain and to pass from the first theory to the second by some delicates changes of contours. Here the situation is complicated by the fact that our FBI phases have an imaginary part which goes to zero with h. In the appendix the reader will find a complete Sjöstrand's theory in the case of two scales.

Concerning the propagation theorems we consider a Schrödinger equation with a Laplacian Δ_g with respect to a scattering metric g in the sense of Melrose ; this means that, near the boundary one can write $g = \frac{d\rho^2}{\rho^4} + \frac{h}{\rho^2}$, where h is a metric such that $h|_{\partial M}$ is positive definite. This includes, of course the flat metric for which $h = d\omega^2$, but also the asymptotically flat metrics on $\mathbb{R}^n$. In this setting we try to answer the following question. Let m_0 be a point in $\mathcal{C} = {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M \cup {}^{\mathrm{qsc}}S^*M$, u be a solution of the initial value problem for this Schrödinger equation and $T > 0$. On what condition on u_0 do we have $m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T, \cdot))$? The answer, which depends strongly on the position of m_0 in $\mathcal{C}$, requires a careful study of the flow of the Laplacian on $\mathcal{C}$. This can be found in Wunsch [**W1**] ; however a still more precise description near the corner $\{\rho = \sigma = 0\}$ is needed here. The different statements, according to the position of m_0 in $\mathcal{C}$, will follow from four propagation results : propagation inside ${}^{\mathrm{qsc}}T^*_{\partial M}M$, inside ${}^{\mathrm{qsc}}S^*M$ or along the corner (for the uniform ${}^{\mathrm{qsc}}WF_a$ and fixed t), from the interior to the corner and finally from the boundary at infinity to the corner. To give a flavor of the results obtained, let us describe the case of the first situation. Let $0 \leqslant t_1 < t_2$ and $m_0 \in {}^{\mathrm{qsc}}\overline{T}^*M$. Assume that $\exp tH_\Delta(m_0)$ (the flow of the Laplacian at infinity through m_0) stays, for $t \in [t_1, t_2]$, inside the interior of ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$. Then $\exp t_1 H_\Delta(m_0)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(t_1,\cdot))$ if and only if $\exp t_2 H_\Delta(m_0)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(t_2,\cdot))$. Coming back to the above question, this result can be applied (with $t_1 = 0, t_2 = t$) when $m_0 = (0, y_0, \lambda_0, \mu_0)$ in the following cases : $\mu_0 \neq 0$ or $\mu_0 = 0$, $\lambda_0 > 0$ or $\mu_0 = 0$, $\lambda_0 < 0$, $t < -1/2\lambda_0$, because, in the later case, the flow starting from m_0 reaches the corner after a finite time $t = -1/2\lambda_0$.

A complete description of the other cases can be found in § 4.

Let us now describe the method of proofs. The first idea, which comes from Sjöstrand's work [**Sj**], is that the FBI transform can be used, at the same time, to test the microlocal smoothness and, as a Fourier integral operator, to reduce an operator to a simpler form. Let us be more precise. We look for a family of phases $\varphi = \varphi(\theta; \rho/h, y, \alpha, h)$ and symbols $a = a(\theta; \rho/h, y, \alpha, h, k)$ depending on a parameter θ, such that

$$\left(\frac{\partial}{\partial\theta} + i\Delta_g^*\right)(ae^{ih^{-2}k^{-1}\varphi}) = \mathcal{O}\left(e^{-\varepsilon/hk}\right), \quad \varepsilon > 0, \tag{0.3}$$

where Δ_g^* is the adjoint of the Laplacian Δ_g.

This leads to the eikonal equation for φ,

$$\frac{\partial\varphi}{\partial\theta} + p\left(sh, y, s^2\frac{\partial\varphi}{\partial s}, s\frac{\partial\varphi}{\partial y}\right) = 0 \tag{0.4}$$

and to the transport equations,

$$Xa_j + h^2kQa_{j-1} = b_j, \quad \text{if} \quad a = \sum_{j=0}^{+\infty}(h\sqrt{k})^j a_j, \tag{0.5}$$

where X is a non degenerate real vector field and Q a second order differential operator.

As soon as we have solved these equations, we see that the corresponding FBI transform $\mathcal{T}u(\theta; t, \alpha, h, k)$ satisfies the real transport equation

$$\left(\frac{1}{k}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}\right)\mathcal{T}u(\theta; t, \alpha, h, k) = \mathcal{O}\left(e^{-c/hk}\right), \quad c > 0, \tag{0.6}$$

and the propagation theorems follow easily.

The main point of the paper is therefore to solve (0.4) and (0.5). The resolution of the eikonal equation (0.4) requires the use of the complex symplectic geometry. We make a careful study of the bicharacteristic flow to span a nice complex Lagrangian manifold on which the symbol $q = \theta^* + p(sh, y, \tau s^2, s\eta)$ vanishes. It should be noted that one has to make a global (backward and forward) study of the bicharacteristic system.

Since the transport equations are linear, they are easily solvable step by step. However it is not straightforward that the corresponding symbol $a = \sum (h\sqrt{k})^j a_j$ is an analytic symbol ; the proof of this fact requires the use of a method of "nested neighborhood" as described by Sjöstrand [**Sj**]. In our context these constructions are to be made either globally on $[0, +\infty[$ or until a time T_* at which all the coefficients of X in (0.5) blow-up ; this leads to significant complications.

Finally we would like to thank the referee for its careful reading of the paper, leading to many improvements of the original manuscript.

CHAPTER 1

THE GEOMETRICAL CONTEXT

The content of this section is taken from Melrose [**M1**]. Here smooth will mean analytic and all the objects will be smooth. Let M be a smooth compact manifold with boundary ∂M. A boundary defining function for M is a smooth function ρ on M such that $\rho = 0$ and $d\rho \neq 0$ on ∂M. A scattering metric on M is a smooth metric g such that, for some choice of boundary defining function ρ, we have, in a neighborhood of ∂M

$$g = \frac{d\rho^2}{\rho^4} + \frac{h}{\rho^2}, \tag{1.1}$$

where h is a smooth symmetric bilinear form on T^*M such that $h|_{\partial M}$ is a metric.

This class of metrics has been built to include asymptotically flat metrics on the Euclidian space $\mathbb{R}^n$. Indeed let us consider the upper hemisphere of the unit sphere in $\mathbb{R}^n$,

$$M = S^n_+ = \{(t_0, t') \in \mathbb{R} \times \mathbb{R}^n : t_0 \geqslant 0,\ \ t_0^2 + |t'|^2 = 1\},$$

with boundary $\partial S^n_+ = \{(t_0, t') \in M : t_0 = 0\}$.

The function $\rho(t_0, t') = t_0/(1 - t_0^2)^{1/2}$, defined in a neighborhood of ∂S^n_+ and extended smoothly to S^n_+, is a boundary defining function for S^n_+. Then, a neighborhood of ∂S^n_+ is diffeomorphic to a subset of $[0, +\infty[\times S^{n-1}$ by the map $\Phi : (t_0, t') \mapsto (\rho(t_0, t'), \omega)$ where $\omega = t'/|t'|$. On the other hand, $\mathbb{R}^n$ is diffeomorphic to

$$\overset{\circ}{S}{}^n_+ = \{(t_0, t') \in \mathbb{R} \times \mathbb{R}^n : t_0 > 0, t_0^2 + |t'|^2 = 1\}$$

by the stereographic compactification $SP : \mathbb{R}^n \to \overset{\circ}{S}{}^n_+$, $z \mapsto \big(t_0 = 1/\langle z\rangle, t' = z/\langle z\rangle\big)$, where $\langle z\rangle = (1 + |z|^2)^{1/2}$. Thus, by $\Phi \circ SP$, we can identify $\mathbb{R}^n \setminus \{z : |z| < 1\}$ with a subset of $]0, +\infty[\times S^{n-1}$. It is easy to see that this corresponds to set $\rho = 1/|z|$, $\omega = z/|z|$ for $|z| \geqslant 1$. Since $z = \omega/\rho$, we check easily that, for $|z| \geqslant 1$, we have

$$(\Phi \circ SP)^*(dz^2) = \frac{d\rho^2}{\rho^4} + \frac{d\omega^2}{\rho^2}.$$

In that follows we shall denote by (ρ, y) a system of local coordinates in a neighborhood of the boundary. Then the metric h appearing in (1.1) can be written

$$h = h_{00}(\rho, y)\, d\rho^2 + 2 \sum_{j=1}^{n-1} h_{0j}(\rho, y)\, d\rho\, dy_j + \sum_{i,j=1}^{n-1} h_{ij}(\rho, y)\, dy_i\, dy_j \tag{1.2}$$

where the coefficients are analytic and

$$\text{the matrix } (h_{ij}(0, y))_{1 \leqslant i,j \leqslant n-1} \text{ is positive definite on } \partial M. \tag{1.3}$$

Following Wunsch we shall denote by $\nu_{\mathrm{qsc}}(M)$ (qsc means quadratic scattering) the space of vector fields on M which are, near the boundary, linear combination of $\rho^3 \partial_\rho$ and $\rho^2 \partial_{y_j}$, $1 \leqslant j \leqslant n-1$. Then ${}^{\mathrm{qsc}}TM$ will be the space of smooth section of $\nu_{\mathrm{qsc}}(M)$ and ${}^{\mathrm{qsc}}T^*M$ its dual. The $1-$canonical form on ${}^{\mathrm{qsc}}T^*M$ can be written, in local coordinates near ∂M, as

$$\alpha = \lambda \frac{d\rho}{\rho^3} + \mu \cdot \frac{dy}{\rho^2}. \tag{1.4}$$

Then the current point in ${}^{\mathrm{qsc}}T^*M$ near ∂M will be determined by its coordinates (ρ, y, λ, μ).

We shall set

$$ {}^{\mathrm{qsc}}T^*_{\partial M}M = \{m \in {}^{\mathrm{qsc}}T^*M : \rho = 0\}. \tag{1.5}$$

Now if $\lambda^2 + |\mu|^2$ is very large it will be more convenient to introduce new coordinates by setting

$$\sigma = \frac{1}{(\lambda^2 + |\mu|^2)^{1/2}}, \quad \overline{\lambda} = \sigma\lambda, \quad \overline{\mu} = \sigma\mu, \quad \overline{\lambda}^2 + |\overline{\mu}|^2 = 1. \tag{1.6}$$

This corresponds to make a radial compactification of the fibers variables and we shall denote by ${}^{\mathrm{qsc}}\overline{T}^*M$ the radial compactification of ${}^{\mathrm{qsc}}T^*M$. Then, near $\sigma = 0$ we shall take $(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu}))$ as local coordinates of a point of ${}^{\mathrm{qsc}}\overline{T}^*M$.

It follows that ${}^{\mathrm{qsc}}\overline{T}^*M$ is a manifold with corner and two faces. If we set

$$\begin{cases} {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M = \{m \in {}^{\mathrm{qsc}}\overline{T}^*M : \rho = 0\}, \\ {}^{\mathrm{qsc}}S^*M = \{m \in {}^{\mathrm{qsc}}\overline{T}^*M : \sigma = 0\}, \end{cases} \tag{1.7}$$

then

$$\mathcal{C} = \partial\, {}^{\mathrm{qsc}}\overline{T}^*M = {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M \cup {}^{\mathrm{qsc}}S^*M. \tag{1.8}$$

CHAPTER 2

THE ANALYTIC *QSC* WAVE FRONT SET

It will be defined as a subset of $\mathcal{C}$, through a FBI transform with two parameters. Let us describe what will be the phases and the symbols.

2.1. The FBI phases

Let $M_0 = (X_0, \Xi_0, \alpha^0, h_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} \times [0, +\infty[$, with $\alpha^0 = (\alpha^0_X, \alpha^0_\Xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.1. — We shall say that $\varphi = \varphi(X, \alpha, h)$ is a FBI phase at M_0 if one can find a neighborhood V of (X_0, α^0) in $\mathbb{C}^n \times \mathbb{C}^{2n}$, a neighborhood I_{h_0} of h_0 in $[0, +\infty[$ such that

$$\varphi(X,\alpha,h) = \varphi_2(X,\alpha_\Xi) + \varphi_3(\alpha) + ih\varphi_1(X,\alpha), \quad \alpha = (\alpha_X, \alpha_\Xi), \tag{2.1}$$

where

$$\begin{cases} \varphi_j, \ j = 1,2,3 \text{ are holomorphic functions in } V \\ \text{and } \varphi_2 \text{ is real if } (X,\alpha_\Xi) \in \mathbb{R}^n \times \mathbb{R}^n, \end{cases} \tag{2.2}$$

$$\frac{\partial \varphi}{\partial X}(X_0, \alpha^0, h_0) = \Xi_0, \tag{2.3}$$

$$\begin{cases} \varphi_1(X_0,\alpha^0) = \dfrac{\partial \operatorname{Re}\varphi_1}{\partial X}(X_0,\alpha^0) = 0, \Big(\dfrac{\partial^2 \operatorname{Re}\varphi_1}{\partial X^2}\Big)(X_0,\alpha^0) \text{ is positive} \\ \text{definite and } \Big(\dfrac{\partial^2 \operatorname{Re}\varphi_1}{\partial X\,\partial \alpha_X}\Big)(X_0,\alpha^0) \text{ is invertible}, \end{cases} \tag{2.4}$$

(2.5) i) If $h_0 = 0$, $\big(\frac{\partial^2 \varphi_2}{\partial X \partial \alpha_\Xi}\big)(X_0, \alpha^0_\Xi)$ is invertible,

ii) if $h_0 \neq 0$, the matrices $(\frac{\partial^2 \varphi}{\partial X \partial \alpha_\Xi})(X_0, \alpha^0, h_0)$ and

$$\begin{pmatrix} \frac{\partial^2 \operatorname{Re} \varphi}{\partial X \partial \alpha_\Xi} & \frac{\partial^2 \operatorname{Re} \varphi}{\partial X \partial \alpha_X} \\ \\ \frac{\partial^2 \operatorname{Im} \varphi}{\partial X \partial \alpha_\Xi} & \frac{\partial^2 \operatorname{Im} \varphi}{\partial X \partial \alpha_X} \end{pmatrix} (X_0, \alpha^0, h_0)$$

are invertible.

Examples 2.2

(i) $\varphi(X, \alpha, h) = (X - \alpha_X)\alpha_\Xi + ih(X - \alpha_X)^2$ is a FBI phase at $(X_0, \Xi_0, \alpha^0, h_0)$ if $\alpha^0 = (X_0, \Xi_0)$.

(ii) More generally let $\varphi = (X - \alpha_X)\alpha_\Xi + ih\varphi_1(X, \alpha)$, where φ_1 is holomorphic, real if (X, α) is real and satisfies $\varphi_1(X, \alpha) = \frac{\partial \varphi_1}{\partial X}(X, \alpha) = 0$ if $\alpha_X = X$, $\varphi_1(X, \alpha) \geqslant c|X - \alpha_X|^2$, for (X, α) in a real neighborhood of $(X_0, (X_0, \Xi_0))$. Then φ is a FBI phase at $(X_0, \Xi_0, \alpha^0, h_0)$ if $\alpha^0 = (X_0, \Xi_0)$.

2.2. The analytic symbols

Our symbols will be formally of the following form

$$a(X, \alpha, h, k) = \sum_{j \geqslant 0} a_j(X, \alpha, h, k)(h\sqrt{k})^j \tag{2.6}$$

where the a_j's are holomorphic with respect to (X, α) in a same complex neighborhood of (X_0, α^0), bounded in (h, k) in a same neighborhood of (h_0, σ_0) in $[0, +\infty[\times [0, +\infty[$ and satisfy in these neighborhoods

$$|a_j(X, \alpha, h, k)| \leqslant C^{j+1} j^{j/2}, \quad j \geqslant 0. \tag{2.7}$$

Actually we will take finite sums of such a_j. The symbol a will be called elliptic at $(X_0, \alpha^0, h_0, \sigma_0)$ if $a_0(X_0, \alpha^0, h_0, \sigma_0) \neq 0$.

2.3. The analytic *qsc* wave front set ${}^{\mathrm{qsc}}WF_a$

A point m_0 in $\mathcal{C} = \partial\, {}^{\mathrm{qsc}}\overline{T}^* M$ is given by $m_0 = (\rho_0, y_0, \sigma_0, (\overline{\lambda}_0, \overline{\mu}_0))$ in local coordinates, where $\rho_0 \geqslant 0$, $\sigma_0 \geqslant 0$, $\rho_0 \cdot \sigma_0 = 0$, $y_0 \in \mathbb{R}^{n-1}$ and $\overline{\lambda}_0^2 + |\overline{\mu}_0|^2 = 1$. Let $s_0 > 0$ be given and set $h_0 = \rho_0 / s_0$. We set

$$X_0 = (s_0, y_0) \in \mathbb{R}^n, \quad \Xi_0 = \left(\frac{\overline{\lambda}_0}{s_0^3}, \frac{\overline{\mu}_0}{s_0^2}\right) \in \mathbb{R}^n. \tag{2.8}$$

Definition 2.3. — Let $u \in \mathcal{D}'(M)$ and $m_0 \in \mathcal{C}$. We say that $m_0 \notin {}^{\mathrm{qsc}}WF_a(u)$ if one can find $s_0 > 0$, $\alpha^0 \in \mathbb{R}^{2n}$, a neighborhood V_{α^0} of α^0 in $\mathbb{R}^{2n}$, a FBI phase φ at $(X_0, \Xi_0, \alpha^0, h_0)$, neighborhoods V_{h_0}, V_{σ_0} of h_0, σ_0 in $[0, +\infty[$, positive constants C,

ε_0, an analytic symbol a, elliptic at $(X_0, \alpha^0, h_0, \sigma_0)$, a cut-off $\chi \in C_0^\infty$ equal to one near X_0 such that

$$
(2.9)\quad |\mathcal{T}u(\alpha,h,k)| \doteqdot \Big| \iint e^{ih^{-2}k^{-1}\varphi(\rho/h,y,\alpha,h)}\, a(\rho/h,y,\alpha,h,k)\,\chi(\rho/h,y)\,\overline{u(\rho,y)}\,d\rho\,dy\Big| \leqslant C e^{-\varepsilon_0/hk}
$$

for all α in V_{α^0}, h in $V_{h_0} \setminus 0$, k in $V_{\sigma_0} \setminus 0$.

2.4. The uniform analytic *qsc* wave front set ${}^{\mathrm{qsc}}\widetilde{WF}_a$

Definition 2.4. — Let I be an interval in $\mathbb{R}$, $(u(t;\cdot))_{t\in I}$ be a family of distributions on M and $t_0 \in I$. Let $m_0 \in \mathcal{C}$ (see (1.8)). We shall say that $m_0 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot))$ if one can find s_0, α^0, φ, a, V_{α^0}, V_{h_0}, V_{σ_0}, C, ε_0 as in Definition 2.3 and $\delta_0 > 0$ such that

$$
|\mathcal{T}u(t;\alpha,h,k)| \doteqdot \Big| \iint e^{-ih^{-2}k^{-1}\varphi(\cdots)}\, a(\cdots)\,\chi(\cdots)\,\overline{u(t;\rho,y)}\,d\rho\,dy\Big| \leqslant C e^{-\varepsilon_0/hk}
$$

for all α, h, k respectively in V_{α^0}, $V_{h_0} \setminus 0$, $V_{\sigma_0} \setminus 0$ and all $t \in I$ such that $|t - t_0| \leqslant \delta_0$.

2.5. Invariance

An important result in this theory is the following.

Theorem 2.5. — *The definitions of* ${}^{\mathrm{qsc}}WF_a$ *and* ${}^{\mathrm{qsc}}\widetilde{WF}_a$ *are independent of* s_0, α^0, φ, a, χ *which satisfy the conditions in the Definitions 2.1 and 2.3.*

The proof of this result is given in the Appendix.

2.6. More general phases

Later on we will be lead to handle FBI transform with more general phases than those described in Definition 2.1, which may also depend on a parameter $\nu \in \mathbb{R}^d$. Let $M_0 = (X_0, \Xi_0, \beta^0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n}$.

Definition 2.6. — We shall say that $\psi = \psi(X,\beta,\nu,h)$ is a phase at M_0 if one can find a neighborhood W of $(X_0,\beta^0) \in \mathbb{C}^n \times \mathbb{C}^{2n}$, a set $U \subset \mathbb{R}^d \times]0,+\infty[$, $\varepsilon_0 > 0$, $C_0 > 0$ such that

$$
(2.10)\quad \begin{cases} \psi \text{ is holomorphic in } W, \text{ for all } (\nu,h) \in U \text{ and } \operatorname{Im}\psi(X,\beta,\nu,h) \geqslant 0 \\ \text{if } (X,\beta) \in W_{\mathbb{R}} = W \cap (\mathbb{R}^n \times \mathbb{R}^{2n}) \text{ and } (\nu,h) \in U, \end{cases}
$$

$$
(2.11)\quad |\psi(X,\beta,\nu,h)| + \Big|\frac{\partial\psi}{\partial X}(X,\beta,\nu,h) - \Xi_0\Big| \leqslant \varepsilon_0, \text{ if } (X,\beta) \in W,\ (\nu,h) \in U,
$$

$$
(2.12)\quad \Big|\frac{\partial \operatorname{Im}\psi}{\partial X}(X,\beta,\nu,h)\Big| \leqslant \varepsilon_0 h, \text{ if } (X,\beta) \in W_{\mathbb{R}},\ (\nu,h) \in U,
$$

$$\frac{\partial^2 \operatorname{Im}\psi}{\partial X^2}(X,\beta,\nu,h) \geqslant -\varepsilon_0 h \operatorname{Id}, \ \text{if } (X,\beta)\in W_{\mathbb{R}}, \ (\nu,h)\in U, \tag{2.13}$$
$$|\partial^\alpha \psi(X,\beta,m,h)| \leqslant C_0, \ \text{for } |\alpha|\leqslant 3, \ \text{if } (X,\beta)\in W, \ (\nu,h)\in U.$$

Let us set now $X_0=(s_0,y_0)$ where $s_0>0$ and $y_0\in\mathbb{R}^{n-1}$, $\Xi_0=(\tau_0,\eta_0)\in\mathbb{R}\times\mathbb{R}^{n-1}$, $r_0^2=s_0^6\tau_0^2+s_0^4|\eta_0|^2$.

Theorem 2.7. — *Let ψ be a phase at (X_0,Ξ_0,β^0). Let b be an analytic symbol in a neighborhood of (X_0,β^0). Let us consider the point*

$$m_0=\big(h_0,y_0,k_0/r_0,(s_0^3\tau_0/r_0,s_0^2\eta_0/r_0)\big)\in\mathcal{C}.$$

Then, if $m_0\notin {}^{\mathrm{qsc}}WF_a(u)$, one can find $\chi\in C_0^\infty$, $\chi=1$ in a neighborhood of X_0, positive constants C_0, δ_0, ε_0 such that

$$\Big|\iint e^{ih^{-2}k^{-1}\psi(s,y,\beta,\nu,h)}b(s,y,\beta,\nu,h)\chi(s,y)\overline{u(sh,y)}\,ds\,dy\Big|\leqslant C_0\,e^{-\delta_0/hk},$$

for all (β,ν,h,k) such that $(\nu,h)\in U$ and $|\beta-\beta^0|+|h-h_0|+|k-k_0|<\varepsilon_0$.

Remark 2.8

(1) Two parameters h,k appear in (2.9). The parameter k is used to check the microlocal smoothness of u (in particular at points m_0 where $\rho_0>0$, $\sigma_0=0$) whereas h is used to test the behavior at infinity (decay, oscillations, etc.).

(2) In the case where $m_0=(0,y_0,\sigma_0,(\overline{\lambda}_0,\overline{\mu}_0))$ with $\sigma_0>0$, it is more convenient to use the coordinates $(0,y_0,\lambda_0,\mu_0)$ where $\lambda_0=\overline{\lambda}_0/\sigma_0$, $\mu_0=\overline{\mu}_0/\sigma_0$. Let us set $X_0=(s_0,y_0)$, where $s_0>0$, $\Xi_0=(\lambda_0/s_0^3,\mu_0/s_0^2)$. Let φ be a FBI phase at $(X_0,\Xi_0,\alpha^0,0)$. Assume that on can find positive constants C, δ, ε_h, an analytic symbol a, a cut-off χ equal to one near X_0 such that

$$\Big|\iint e^{ih^{-2}\varphi(s,y,\alpha,h)}a(s,y,\alpha,h)\chi(s,y)\overline{u(hs,y)}\,ds\,dy\Big|\leqslant C\,e^{-\delta/h}, \tag{2.9$'$}$$

for all α in a real neighborhood of α^0 and $h\in\,]0,\varepsilon_h]$. Then $m_0\notin {}^{\mathrm{qsc}}WF_a(u)$. The converse is also true (take $k=\sigma_0$ in (2.9)). In other terms we can ignore the parameter k in (2.9) and fix it to the value σ_0. This fact is proved in the Appendix (Corollary A.16).

(3) If $m_0=(\rho_0,y_0,0,(\overline{\lambda}_0,\overline{\mu}_0))$, we set $X_0=(\rho_0,y_0)$, $\Xi_0=(\overline{\lambda}_0/\rho_0^3,\overline{\mu}_0/\rho_0^2)$ and the fact that $m_0\notin {}^{\mathrm{qsc}}WF_a(u)$ is characterized by the inequality (2.9) where φ is a FBI phase at $(X_0,\Xi_0,\alpha^0,\rho_0)$ and $h=1$, that is we may ignore the parameter h. This shows that ${}^{\mathrm{qsc}}WF_a(u)\cap(S^*M)^0$ coincide with the usual analytic wave front set since then the transformation appearing in (2.9) is a usual FBI transform in the sense of Sjöstrand [**Sj**].

(4) In section 1, we have identified $\mathbb{R}^n\smallsetminus\{z:|z|<1\}$ with a subset of S^n_+, thus with a subset of $]0,+\infty[\,\times S^{n-1}$, which corresponds to set $z=\omega/\rho$. Working in $\mathbb{R}^n$, it is more convenient to use the coordinates (ρ,ω) instead of local coordinates. The one form on ${}^{\mathrm{qsc}}T^*M$ is then equal to $\lambda\frac{d\rho}{\rho^3}+\mu\cdot\frac{d\omega}{\rho^2}$. Here μ has to be taken

in an $(n-1)$–dimensional subspace. Since the forms $\mu\cdot d\omega$ and $(\mu-(\mu\cdot\omega)\omega)d\omega$ coincide (because $\omega\cdot d\omega=0$) it is natural to take μ in $\omega^\perp$. Thus the coordinates of $m_0\in {}^{\mathrm{qsc}}\overline{T}^*M$ will be $(\rho_0,\omega_0,\sigma_0,(\overline{\lambda}_0,\overline{\mu}_0))$, $\overline{\lambda}_0^2+|\overline{\mu}_0|^2=1$, $\overline{\mu}_0\perp\omega_0$. Let us set

$$(2.14)\qquad \begin{cases} X_0=(s_0,\omega_0), \quad \Xi_0=\Big(\dfrac{\overline{\lambda}_0}{s_0^3},\dfrac{\overline{\mu}_0}{s_0^2}\Big)\\ (\Pi_0)=\big\{\alpha=(\alpha_\rho,\alpha_\omega,\alpha_\tau,\alpha_\zeta)\in\mathbb{R}^{2n+2}:\alpha_\omega\cdot\omega_0=1,\,\alpha_\zeta\cdot\omega_0=0\big\}.\end{cases}$$

Claim. — *Let $u\in\mathcal{D}'(\mathbb{R}^n)$. Then $m_0\notin {}^{\mathrm{qsc}}WF_a(u)$ if and only if one can find $s_0>0$, $\alpha^0\in(\Pi_0)$, a FBI phase φ at (X_0,Ξ_0,α^0,h_0), an elliptic symbol a at $(X_0,\alpha^0,h_0,\sigma_0)$ a cut-off χ near X_0, positive constants C, ε_0 such that*

$$(2.15)\quad |\mathcal{T}u(\alpha,h,k)| \doteq \Big|\int_0^{+\infty}\int_{S^{n-1}} e^{ih^{-2}k^{-1}\varphi(\rho/h,\omega,\alpha,h)}a\big(\rho/h,\omega,\alpha,h,k\big)\chi\big(\rho/h,\omega\big)\overline{u(\omega/\rho)}\,\frac{d\rho}{\rho^{n+1}}\,d\omega\Big| \leqslant Ce^{-\varepsilon_0/hk}$$

for α close to α^0 in (Π_0), (h,k) close to (h_0,σ_0) in $[0,+\infty[^2$, where

$$\varphi(s,\omega,\alpha,h)=(s-\alpha_s)\alpha_\tau+(\omega-\alpha_\omega)\cdot\alpha_\zeta+ih[(s-\alpha_s)^2+(\omega-\alpha_\omega)^2].$$

Indeed, in some local coordinates, (2.15) will coincide with (2.9). Let $(\theta_1,\dots,\theta_{n-1})$ be an orthonormal basis of $\omega_0^\perp$. Writing $\alpha_\omega=\omega_0+\sum_{j=1}^{n-1}a_j\theta_j$, $\alpha_\zeta=\sum_{j=1}^{n-1}b_j\theta_j$ we see that $\alpha_\omega\cdot\alpha_\zeta=a\cdot b$; therefore in these coordinates (2.1) is preserved and (2.2) to (2.5) are satisfied.

Examples 2.9

(1) Let u_0 be such that $e^{\delta|x|}u_0\in L^2(\mathbb{R}^n)$ for some $\delta>0$. Then ${}^{\mathrm{qsc}}WF_a(u_0)\cap({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0=\varnothing$. Indeed let $m_0=(0,\omega_0,\lambda_0,\mu_0)$ be a point of $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$. We set $\alpha^0=\big(s_0,\omega_0,\lambda_0/s_0^3,\mu_0/s_0^2\big)$. According to Theorem 2.5 and Remark 2.8 (2), (4), we can take $k=1$, $a=1$ and $\varphi(X,\alpha,h)=(X-\alpha_X)\alpha_\Xi+ih(X-\alpha_X)^2$ (where $X=(\rho/h,\omega))$ in (2.9). In the coordinates (ρ,ω), our assumption on u_0 reads : $u_0\big(\omega/\rho\big)=\rho^{\frac{n}{2}+\frac12}e^{-\delta/\rho}v(\rho,\omega)$ with $v\in L^2(\mathbb{R}_+\times S^{n-1})$. Let χ be a C^∞ cut-off supported in $\{|s-s_0|+|\omega-\omega_0|<\varepsilon\}$ with $\varepsilon\leqslant 1/2s_0$. Then

$$\mathcal{T}u_0(\alpha,h)=\int_0^{+\infty}\int_{S^{n-1}}e^{ih^{-2}\varphi(\rho/h,\omega,\alpha,h)}\chi\big(\rho/h,\omega\big)e^{-\delta/\rho}\overline{v(\rho,\omega)}\,\frac{d\rho}{\rho^{\frac{n}{2}+\frac12}}\,d\omega.$$

On the support of χ we have $\frac12 s_0\leqslant\rho/h\leqslant\frac32 s_0$ so $-\delta/\rho\leqslant-\frac{2\delta}{3s_0}\frac1h$. Since $|e^{ih^{-2}\varphi}|\leqslant1$ we get $|\mathcal{T}u_0(\alpha,h)|\leqslant Ce^{-\varepsilon_0/h}$, for all α near α_0, which means that $m_0\notin {}^{\mathrm{qsc}}WF_a(u_0)$.

(2) Let $u_0(x)=e^{\frac{i}{2}\langle Ax,x\rangle}$, where A is a real $n\times n$ symmetric matrix. Then

$${}^{\mathrm{qsc}}WF_a(u_0)\subset\Lambda_0=\big\{\big(0,\omega_0,-A\omega_0\cdot\omega_0,A\omega_0-(A\omega_0\cdot\omega_0)\omega_0\big),\omega_0\in S^{n-1}\big\}.$$

First of all, since u_0 is analytic, it has no usual analytic wave front set ; by the Remark 2.8, (3), it has no ${}^{\mathrm{qsc}}WF_a$ in $({}^{\mathrm{qsc}}S^*M)^0$. We show now that

$$ {}^{\mathrm{qsc}}WF_a(u_0) \cap ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0 \subset \Lambda_0. $$

Here we may use the transformation (2.15) with

$$ k=1, \quad a=1 \quad \text{and} \quad \varphi = (X-\alpha_X)\alpha_\Xi + ih(X-\alpha_X)^2. $$

Let $m_0 = (0, \omega_0, \lambda_0, \mu_0)$ with $\mu_0 \perp \omega_0$ but

$$ (\lambda_0, \mu_0) \neq (-A\omega_0\cdot\omega_0, A\omega_0 - (A\omega_0\cdot\omega_0)\omega_0). $$

We set $X_0 = (s_0, \omega_0)$, $\Xi_0 = \big(\lambda_0/s_0^3, \mu_0/s_0^2\big)$ and we take $\alpha^0 = (X_0, \Xi_0) = \big(s_0, \omega_0, \lambda_0/s_0^3, \mu_0/s_0^2\big)$. Then we have

$$ \mathcal{T}u_0(\alpha,h) = h^{-n}\int_0^{+\infty}\int_{S^{n-1}} e^{ih^{-2}\theta(s,\omega,\alpha,h)}\chi(s,\omega)\frac{ds}{s^{n+1}}\,d\omega \tag{2.16} $$

where

$$ \begin{cases} \theta(s,\omega,\alpha,h) = \theta_2(s,\omega,\alpha) + ih\theta_1(s,\omega,\alpha) \\ \theta_2(s,\omega,\alpha) = (s-\alpha_s)\alpha_\tau + (\omega-\alpha_\omega)\alpha_\zeta - \dfrac{1}{2}\dfrac{A\omega\cdot\omega}{s^2} \\ \theta_1(s,\omega,\alpha) = (s-\alpha_s)^2 + (\omega-\alpha_\omega)^2 \\ \alpha_\omega\cdot\omega_0 = 1, \ \ \alpha_\zeta\cdot\omega_0 = 0. \end{cases} \tag{2.17} $$

We have

$$ \frac{\partial\theta_2}{\partial s}(s,\omega,\alpha) = \alpha_\tau + \frac{A\omega\cdot\omega}{s^3} \tag{2.18} $$

and if $t\in T_\omega S^{n-1}$ *i.e.* $t\cdot\omega = 0$ we have

$$ t\cdot\frac{\partial\theta_2}{\partial\omega} = t\cdot\alpha_\zeta - \frac{t\cdot A\omega}{s^2} = t\cdot\alpha_\zeta - t\cdot\frac{A\omega-(A\omega\cdot\omega)\omega}{s^2}. \tag{2.18'} $$

Claim. — *One can find $t\in T_{\omega_0}S^{n-1}$, $|t|=1$, $C_0>0$, $\varepsilon>0$ such that for all (s,ω,α) in $\mathbb{R}_+\times S^{n-1}\times(\Pi_0)$ such that $|s-s_0|+|\omega-\omega_0|+|\alpha-\alpha_0|\leqslant\varepsilon$ we have*

$$ \Big|\frac{\partial\theta_2}{\partial s}(s,\omega,\alpha)\Big| + \Big|t\cdot\frac{\partial\theta_2}{\partial\omega}(s,\omega,\alpha)\Big| \geqslant C_0. \tag{2.19} $$

Otherwise for every $t\in T_{\omega_0}S^{n-1}$ one can find sequences (s_j), (ω_j), (α_j) converging to s_0, ω_0, α_0 such that

$$ \Big|\frac{\partial\theta_2}{\partial s}(s_j,\omega_j,\alpha_j)\Big| + \Big|t\cdot\frac{\partial\theta_2}{\partial\omega}(s_j,\omega_j,\alpha_j)\Big| \leqslant \frac{1}{j}, \quad j\geqslant 1. $$

It follows, according to (2.18), (2.18)' that

$$ \frac{\lambda_0}{s_0^3} + \frac{A\omega_0\cdot\omega_0}{s_0^3} = 0 \ \text{and} \ t\cdot\Big(\frac{\mu_0}{s_0^2} - \frac{A\omega_0-(A\omega_0\cdot\omega_0)\omega_0}{s_0^2}\Big) = 0 $$

but this is in contradiction with our choice of (λ_0,μ_0).

Now let us fix α close to α^0 and let us set $X = (s,\omega)$, $\vec{V}(X) = \begin{pmatrix} \frac{\partial\theta_2}{\partial s}(s,\omega,\alpha) \\ \left(t \cdot \frac{\partial\theta_2}{\partial\omega}(s,\omega,\alpha)\right) t \end{pmatrix}$.
We introduce the following contour in $\mathbb{C}^n$,

$$\Sigma = \left\{ Z \in \mathbb{C}^{n+1} : Z = X + i\nu\chi_1(X)\vec{V}(X),\, X = (s,\omega) \in]0,+\infty[\times S^{n-1} \right\} \tag{2.20}$$

where ν is a small positive constant to be chosen and

$$\begin{cases} \chi_1(X) = 1 \text{ if } |X - X_0| \leqslant \varepsilon_1 \\ \chi_1(X) = 0 \text{ if } |X - X_0| \geqslant 2\varepsilon_1 \end{cases}, \quad 0 \leqslant \chi_1 \leqslant 1,$$

ε_1 being such that $\chi(X) = 1$ on the support of χ_1 in (2.16). Since θ given in (2.17) can be extended as a holomorphic function of (s,ω) in $\mathbb{C} \times \mathbb{C}^n$ and since $\chi(X) = 1$ if Σ is not real, we can apply Stokes formula and deduce that

$$\mathcal{T}u_0(\alpha,h) = h^{-n} \iint_{\Sigma} e^{ih^{-2}\theta(Z,\alpha,h)} \chi(Z) \frac{dZ}{Z_1^{n+1}} \cdot \tag{2.21}$$

It follows from Taylor's formula that, for Z in Σ, we have

$$\theta_2(Z,\alpha) = \theta_2(X,\alpha) + i\nu\chi_1(X)\,\|\vec{V}(x)\|^2 + \mathcal{O}(\nu^2\chi_1^2(X)\|\vec{V}(X)\|^2).$$

On the other hand we have

$$\theta_1(Z,\alpha) = (Z - \alpha_X)^2 = \|X - \alpha_X\|^2 + 2i\nu\chi_1(X)(X - \alpha_X)\cdot\vec{V}(X) - \nu^2\chi_1^2(X)\|\vec{V}(X)\|^2.$$

We deduce that for Z in Σ we have

$$\begin{aligned} &\mathrm{Re}(ih^{-2}\theta(Z,\alpha,h)) = \\ &\quad -\nu h^{-2}\chi_1(X)\|\vec{V}(X)\|^2 + h^{-2}\mathcal{O}(\nu^2\chi_1^2(X)\|\vec{V}(X)\|^2) \\ &\quad - h^{-1}\|X - \alpha_X\|^2 + \nu^2 h^{-1}\chi_1^2(X)\|\vec{V}(X)\|^2 = (1) + (2) + (3) + (4). \end{aligned} \tag{2.22}$$

We have

$$\begin{cases} (1) & = -\nu h^{-2}\chi_1(X)\|\vec{V}(X)\|^2 \\ |(2)| & \leqslant C\nu h^{-2}\nu\chi_1(X)\|\vec{V}(X)\|^2 \\ (3) & = -h^{-1}\|X - \alpha_X\|^2 \\ |(4)| & \leqslant C\nu h^{-1}\nu\chi_1(X)\|\vec{V}(X)\|^2. \end{cases} \tag{2.23}$$

Taking ν small, we deduce from (2.22) and (2.23), that

$$\mathrm{Re}(ih^{-2}\theta(Z,\alpha,h)) \leqslant -\frac{1}{2}\nu h^{-2}\chi_1(X)\|\vec{V}(X)\|^2 - h^{-1}\|X - \alpha_X\|^2. \tag{2.24}$$

We fix ν and we write $\Sigma = \Sigma_1 \cup \Sigma_2$ where

$$\begin{cases} \Sigma_1 = \{Z \in \Sigma : \|X - X_0\| \leqslant \varepsilon_1\} \\ \Sigma_2 = \{Z \in \Sigma : \varepsilon_1 < \|X - X_0\|\}. \end{cases}$$

On Σ_1 we have $\chi_1 = 1$ and on Σ_2, $0 \leqslant \chi_1 \leqslant 1$.

On Σ_1 we have by (2.19) and (2.24)

$$\operatorname{Re}(ih^{-2}\theta(Z,\alpha,h)) \leqslant -\frac{1}{4}\nu C_0^2 h^{-2}. \tag{2.25}$$

On the other hand on Σ_2 we have, $\|X-\alpha_X\| \geqslant \|X-X_0\| - \|\alpha_X^0-\alpha_X\| \geqslant \frac{1}{2}\varepsilon_1$, if α is sufficiently close to α^0. It follows from (2.24) that

$$\operatorname{Re}(ih^{-2}\theta(Z,\alpha,h)) \leqslant -\frac{1}{4}\varepsilon_1^2 h^{-1}. \tag{2.26}$$

We deduce from (2.25), (2.26) and (2.21) that

$$|\mathcal{T}u_0(\alpha,h)| \leqslant Ce^{-\varepsilon_0/h}$$

if α and h are sufficiently close to α^0 and 0. It follows that $(0,\omega_0,\lambda_0,\mu_0) \notin {}^{\mathrm{qsc}}WF_a(u_0)$ as claimed.

By the same argument we can prove that u_0 has not ${}^{\mathrm{qsc}}WF_a$ on the corner $\rho = \sigma = 0$. Indeed, in this case we have to estimate

$$\mathcal{T}u_0(\alpha,h,k) = h^{-n}\int_0^{+\infty}\int_{S^{n-1}} e^{ih^{-2}k^{-1}\theta(s,\omega,\alpha,h,k)}\chi(s,\omega)\frac{ds}{s^{n+1}}\,d\omega$$

where

$$\begin{cases} \operatorname{Re}\theta = (s-\alpha_s)\alpha_\tau + (\omega-\alpha_\omega)\alpha_\zeta - \dfrac{k}{2s^2}A\omega\cdot\omega \\ \operatorname{Im}\theta = (s-\alpha_s)^2 + (\omega-\alpha_\omega)^2. \end{cases}$$

Then

$$\frac{\partial}{\partial s}(\operatorname{Re}\theta) = \alpha_\tau + \mathcal{O}(k), \quad t\cdot\frac{\partial}{\partial\omega}(\operatorname{Re}\theta) = t\cdot\alpha_\zeta + \mathcal{O}(k).$$

If α is close to $\alpha^0 = \big(s_0,\omega_0,\overline{\lambda}_0/s_0^3,\overline{\mu}_0/s_0^2\big)$, then $|\alpha_\tau|+|\alpha_\zeta| \neq 0$, since $\overline{\lambda}_0^2 + |\overline{\mu}_0|^2 = 1$. So, if k is small enough we still have (2.19), and the same proof applies.

CHAPTER 3

THE LAPLACIAN AND ITS FLOW

3.1. The Laplacian

The Laplacian on M related to the metric g can be written in any system of coordinates as

$$\Delta_g = \frac{1}{\sqrt{G}} \sum_{j,k=0}^{n-1} D_j \left(\sqrt{G}\, g^{jk} D_k\right), \tag{3.1}$$

where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $G = \det(g_{jk})$, $(g^{jk}) = (g_{jk})^{-1}$.

Since g is a scattering metric, (1.1) and (1.2) show that

$$g_{00} = \frac{1+\rho^2 h_{00}}{\rho^4}, \quad g_{0k} = \frac{h_{0k}}{\rho^2}, \quad g_{jk} = \frac{h_{jk}}{\rho^2}, \quad 1 \leqslant j,k \leqslant n-1.$$

It is easy to see that, for small ρ, we have

$$\begin{cases} G = \left(\dfrac{1}{\rho^2}\right)^{n+1} (H + \mathcal{O}(\rho)), \quad H = \det(h_{jk}(0,y))_{1\leqslant j,k\leqslant n-1} \\ g^{00} = \rho^4 + \mathcal{O}(\rho^6), \quad g^{0k} = \mathcal{O}(\rho^4), \quad 1 \leqslant k \leqslant n-1 \\ g^{jk} = \rho^2 \overline{h}^{jk}(y) + \mathcal{O}(\rho^3), \quad 1 \leqslant j,k \leqslant n-1, \quad \text{where} \\ \qquad (\overline{h}^{jk}) = (h_{jk}(0,y))^{-1}_{1\leqslant j,k\leqslant n-1}. \end{cases} \tag{3.2}$$

Here $\mathcal{O}(\rho^\ell)$ denotes an analytic function on $[0,\varepsilon[\, \times \partial M$ which can be written as $\rho^\ell a(\rho,y)$ with a analytic.

It follows from (3.1) and (3.2) that

$$
(3.3)\quad \begin{cases} \Delta_g = \dfrac{1}{\rho^2}\left[(\rho^3 D_\rho)^2 + \rho^4 \Delta_0 + c(n)\,\rho^5 D_\rho + \rho R\right], \text{where} \\ \Delta_0 = \dfrac{1}{\sqrt{H}} \displaystyle\sum_{j,k=1}^{n-1} D_j\big(\sqrt{H}\,\overline{h}^{jk}(y)\,D_k\big) \ \text{ and} \\ R = \displaystyle\sum_{1\leqslant|\alpha|+\ell\leqslant 2} a_{\alpha\ell}(\rho,y)(\rho^3 D_\rho)^\ell (\rho^2 D_y)^\alpha, \ a_{\alpha\ell}(\rho,y) = \rho^{\sigma(|\alpha|,\ell)}\widetilde{a}_{\alpha\ell}(\rho,y), \\ \sigma(0,2)=1,\ \sigma(1,1)=0,\ \sigma(2,0)=0,\ \sigma(1,0)=2,\ \sigma(0,1)=3. \end{cases}
$$

Let us remark that one can also write

$$
(3.4)\quad \begin{cases} \Delta_g = (\rho^2 D_\rho)^2 + \rho^2 \Delta_0 + c'(n)\,\rho^3 D_\rho + \rho R', \\ R' = \displaystyle\sum_{1\leqslant|\alpha|+\ell\leqslant 2} b_{\alpha\ell}(\rho,y)(\rho^2 D_\rho)^\ell (\rho D_y)^\alpha, \ b_{\alpha\ell}(\rho,y) = \rho^{\theta(|\alpha|,\ell)}\widetilde{b}_{\alpha\ell}(\rho,y), \\ \theta(0,2)=1,\ \theta(1,1)=0,\ \theta(2,0)=0,\ \theta(1,0)=1,\ \theta(0,1)=2. \end{cases}
$$

3.2. The Hamiltonian

In the pseudo-differential calculus of Melrose [**M2**], the principal symbol of Δ_g is a function on ${}^{\mathrm{qsc}}\overline{T}^*M$ which can be written as

$$
(3.5)\quad \begin{cases} \sigma(\Delta_g)(\rho,y,\lambda,\mu) = \dfrac{1}{\rho^2}\,p(\rho,y,\lambda,\mu) \ \text{ where} \\ p(\rho,y,\lambda,\mu) = \lambda^2 + \|\mu\|^2 + \rho\, r(\rho,y,\lambda,\mu) \ \text{ with} \\ \|\mu\|^2 = \displaystyle\sum_{j,k=1}^{n-1} \overline{h}^{jk}(y)\,\mu_j\,\mu_k, \ r(\cdots) = \displaystyle\sum_{|\alpha|+\ell=2} a_{\alpha\ell}(\rho,y)\,\lambda^\ell\,\mu^\alpha, \\ \qquad a_{0\ell}(\rho,y) = \rho\,\widetilde{a}_{0\ell}(\rho,y). \end{cases}
$$

The symplectic two forms on ${}^{\mathrm{qsc}}\overline{T}^*M$ is $\omega = d\alpha$ where α has been defined in (1.4). Therefore

$$
(3.6)\quad \omega = \frac{d\lambda\wedge d\rho}{\rho^3} + \frac{d\mu\wedge dy}{\rho^2} - 2\mu\cdot\frac{d\rho\wedge dy}{\rho^3}.
$$

The Hamiltonian H_Δ of the symbol of Δ_g is then defined by

$$
(3.7)\quad d\Big(\frac{1}{\rho^2}p\Big)(\cdot) = -omega(H_\Delta,\cdot).
$$

An easy computation shows that

$$
(3.8)\quad H_\Delta = \rho\frac{\partial p}{\partial\lambda}\partial_\rho + \frac{\partial p}{\partial\mu}\cdot\partial_y + \Big(2p - 2\mu\cdot\frac{\partial p}{\partial\mu} - \rho\frac{\partial p}{\partial\rho}\Big)\partial_\lambda + \Big(2\mu\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial y}\Big)\cdot\partial_\mu.
$$

Using (3.5) we see that

(3.9)
$$\begin{cases} H_\Delta = X_0 + \widetilde{X} \ \text{ where} \\ X_0 = 2\lambda\rho\partial_\rho + 2(\lambda^2 - \|\mu\|^2)\partial_\lambda + 2\langle\mu, \partial_y\rangle + 4\lambda\mu\cdot\partial_\mu - (\partial_y\|\mu\|^2)\partial_\mu\,, \\ \text{where } \langle a, b\rangle = \sum_{j,k=1}^{n-1} \overline{h}^{jk}(y)\,a_j\,b_k\,,\ \ \|a\|^2 = \langle a, a\rangle\,, \ \text{ and} \\ \widetilde{X} = p_1\rho^2\partial_\rho + p_2\rho\partial_y + q_1\rho\partial_\lambda + q_0\rho\partial_\mu \ \text{ where } p_i \text{ (resp. } q_i) \text{ are} \\ \text{polynomials of degree 1 (resp. 2) in } \lambda,\mu \text{ with analytic coefficients in } (\rho, y). \end{cases}$$

3.3. The flow on $({}^{\mathrm{qsc}}\overline{T}_{\partial M}M)^0$

On this set the flow of the Laplacian will be the flow of X_0 since $\widetilde{X}$ vanishes on this set. Let $m_0 = (0, y_0, \lambda_0, \mu_0) \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$. The flow of X_0 starting from m_0 is given by the equations

$$\begin{cases} \dot\rho(t) = 2\lambda(t)\rho(t)\,, & \rho(0) = 0 \\ \dot y_j(t) = 2\sum_{k=1}^{n-1} \overline{h}^{jk}(y(t))\,\mu_k(t)\,, & y(0) = y_0 \\ \dot\lambda(t) = 2(\lambda^2(t) - \|\mu(t)\|^2)\,, & \lambda(0) = \lambda_0 \\ \dot\mu(t) = 4\lambda(t)\mu(t) - \partial_y\|\mu(t)\|^2\,, & \mu(0) = \mu_0\,. \end{cases} \tag{3.10}$$

This system has a unique maximal solution defined on $[0, T^*[$ (and in $]T_*, 0]$).

Case 1: if $\mu_0 = 0$. — By the first equation we have $\rho(t) = 0$ for $t \in [0, T^*[$ and the last one shows that $\mu(t) = 0$, $t \in [0, T^*[$. Then, by the second equation, $y(t) = y_0$, $t \in [0, T^*[$, and the third one can be written $\dot\lambda(t) = 2\lambda^2(t)$; thus we have $\lambda(t) = \frac{\lambda_0}{1-2\lambda_0 t}$ for $t \in \big[0, 1/2\lambda_0\big[$ if $\lambda_0 > 0$ and for $t \in [0, +\infty[$ if $\lambda_0 < 0$. Moreover if $\lambda_0 > 0$ we have $\lim_{t\to 1/2\lambda_0} \lambda(t) = +\infty$. Summing up, if $\lambda_0 > 0$ we have $T^* = 1/2\lambda_0$ and every integral curve of X_0 starting from $m_0 = (0, y_0, \lambda_0, 0)$ reaches the corner $\rho = \sigma = 0$ at finite time $1/2\lambda_0$. If $\lambda_0 < 0$ then the integral curve is defined for all t in $[0, +\infty[$ and stays in $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$. The same discussion applies to the case $t \in]T_*, 0]$. We introduce the sets

$$\begin{cases} \mathcal{N} = \{m = (\rho, y, \lambda, \mu) : \rho = \mu = 0\} \\ \mathcal{N}^+ = \{m \in \mathcal{N} : \lambda > 0\}\,,\ \ \mathcal{N}^- = \{m \in \mathcal{N} : \lambda < 0\}\,. \end{cases} \tag{3.11}$$

Case 2 : if $\mu_0 \neq 0$. — In that case the solution of (3.10) exists for all time in $\mathbb{R}$ and the integral curve stays in the interior of ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$. Here is a sketch of the proof of these facts (the details are in [W], section 11). If $\mu_0 \neq 0$ then $\mu(t) \neq 0$ for all t in $]T_*, T^*[$. We set $\widehat{\mu}(t) = \mu(t)/\|\mu(t)\|$ and we parametrize the curve by s where

$\dot{s}(t) = 2\|\mu(t)\|$, $s(0) = 0$. The equations (3.10) give

$$(3.12)\quad \begin{cases} \text{(i)} & \dfrac{d\rho}{ds} = \dfrac{\lambda\rho}{\|\mu\|} \qquad \text{(iv)} \quad \dfrac{d\lambda}{ds} = \dfrac{\lambda^2 - \|\mu\|^2}{\|\mu\|} \\ \text{(ii)} & \dfrac{dy_i}{ds} = \sum \overline{h}^{ij}\,\widehat{\mu}_j \qquad \text{(v)} \quad \dfrac{d\widehat{\mu}_\ell}{ds} = -\dfrac{1}{2}\sum_{j,k=1}^{n-1} \dfrac{\partial \overline{h}^{jk}}{\partial y_\ell}\widehat{\mu}_j\,\widehat{\mu}_k\,. \\ \text{(iii)} & \dfrac{d\|\mu\|}{ds} = 2\lambda \end{cases}$$

Then we set $\alpha = \lambda/\|\mu\|$ and we see that $\dot{\alpha} = -(1+\alpha^2)$. The solution of this equation, such that $\alpha(0) = \tan\theta_0$, $\theta_0 \in\,]-\pi/2, \pi/2[$, is $\alpha(s) = \tan(\theta_0 - s)$ where $\theta_0 - s \in\,]-\pi/2, \pi/2[$. It follows that $\lambda(s) = \|\mu(s)\|\tan(\theta_0 - s)$. Using the equation (iii) in (3.12), we get

$$(3.13)\quad \begin{cases} \|\mu(s)\| = A\cos^2(\theta_0 - s) \\ \lambda(s) = \dfrac{A}{2}\sin 2(\theta_0 - s)\,. \end{cases}$$

Then, using (i), we obtain

$$(3.14)\quad \rho(s) = C\cos^2(\theta_0 - s)\,.$$

Since $\alpha(0) = \lambda_0/\|\mu_0\| = \tan\theta_0$, we have $\theta_0 = \operatorname{Arc\,tan} \lambda_0/\|\mu_0\|$. On the other hand, by (3.13), $\lambda_0^2/\|\mu_0\| = A\sin^2\theta_0$. Therefore $\frac{\lambda_0^2}{\|\mu_0\|} + \|\mu_0\| = A$. Moreover, $\dot{s}(t) = 2\|\mu(t)\| = 2A\cos^2(\theta_0 - s(t))$. It follows that $s(t)$ exists for all $t \in \mathbb{R}$. This implies that the solution of (3.12) exists for all $t \in \mathbb{R}$ and (3.13) shows that $|\lambda| + |\mu|$ is bounded so the integral curve stays in the interior of ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$.

3.4. The flow on ${}^{\mathrm{qsc}}S^*M$

When $|\lambda| + |\mu|$ is large, we make the change of variables in the cotangent space, $(\rho, y, \lambda, \mu) \mapsto (\rho, y, \sigma, (\overline{\lambda}, \overline{\mu}))$ where

$$(3.15)\quad \sigma = \frac{1}{[p(\rho, y, \lambda, \mu)]^{1/2}}, \quad \overline{\lambda} = \sigma\lambda, \quad \overline{\mu} = \sigma\mu\,.$$

In these new coordinates the Hamiltonian H_Δ is singular at $\sigma = 0$. However σH_Δ is a smooth vector field and we have

$$(3.16)\quad \sigma H_\Delta = \rho\frac{\partial p}{\partial\lambda}\partial_\rho + \frac{\partial p}{\partial\mu}\partial_y - \Big(\overline{\mu}\frac{\partial p}{\partial\mu} + \rho\frac{\partial p}{\partial\rho}\Big)\partial_{\overline{\lambda}} + \Big(\overline{\mu}\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial y}\Big)\partial_{\overline{\mu}} + \sigma f(\rho, y, \overline{\lambda}, \overline{\mu})\partial_\sigma\,.$$

By definition, the flow of the Laplacian on ${}^{\mathrm{qsc}}S^*M$ will be that of σH_Δ. It is therefore given by the following equations

$$
\tag{3.17}
\begin{cases}
\dot\rho = \rho \dfrac{\partial p}{\partial \lambda}(\rho, y, \overline{\lambda}, \overline{\mu}) & \rho(0) = \rho_0 \\
\dot y = \dfrac{\partial p}{\partial \mu}(\cdots) & y(0) = y_0 \\
\dot{\overline{\lambda}} = -\Big(\overline{\mu}\dfrac{\partial p}{\partial \mu} + \rho\dfrac{\partial p}{\partial \rho}\Big)(\cdots) & \overline{\lambda}(0) = \overline{\lambda}_0 \\
\dot{\overline{\mu}} = \Big(\overline{\mu}\cdot\dfrac{\partial p}{\partial \lambda} - \dfrac{\partial p}{\partial y}\Big)(\cdots) & \overline{\mu}(0) = \overline{\mu}_0 \\
\dot\sigma = \sigma f(\rho, y, \overline{\lambda}, \overline{\mu}) & \sigma(0) = 0.
\end{cases}
$$

The last equation shows that $\sigma(t) = 0$ for all t.

3.5. Behavior of the flow for large time

Definition 3.1. — A maximal integral curve of σH_Δ on ${}^{\mathrm{qsc}}\overline{T}^*M$ will be called non trapped backward (resp. forward) if it is defined for all t in $]-\infty, 0]$ (resp. $[0, +\infty[$) and $\rho(t) \to 0$ as $t \to -\infty$ (resp. $t \to +\infty$).

In (3.11) we have introduced the sets $\mathcal{N}$, $\mathcal{N}^{\pm}$. Here we set

$$
\tag{3.18}
\mathcal{N}^c_{\pm} = \big\{m = (\rho, y, \sigma, (\overline{\lambda}, \overline{\mu})) : \rho = \overline{\mu} = \sigma = 0,\ \ \overline{\lambda} = \pm 1\big\}.
$$

Definition 3.2

(i) Let $m \in {}^{\mathrm{qsc}}\overline{T}^*M$, $m \notin \mathcal{N}$. We shall say that m is non trapped backward (resp. forward) if the integral curve of σH_Δ starting from m is non trapped backward (resp. forward).

(ii) Let $m \in \mathcal{N}$, $m = (0, y_0, \sigma_0, (\pm 1, 0))$. We shall say that m is non trapped backward (resp. forward) if the point $(0, y_0, 0, (\pm 1, 0)) \in \mathcal{N}^c_{\pm}$ does not belong to the closure of any integral curve of σH_Δ trapped backward (resp. forward).

We shall denote by T_- (resp. T_+) the set of points which are trapped backward (resp. forward).

Proposition 3.3. — *Let $m_0 \in {}^{\mathrm{qsc}}S^*M \smallsetminus (\mathcal{N}^c_- \cup T_-)$. Then*

$$
N_{-\infty}(m_0) = \lim_{t \to -\infty} \exp t\sigma H_\Delta(m_0) \in \mathcal{N}^c_+.
$$

(Same result when $-$ and $+$ are exchanged).

Proof. — Here $\exp t\sigma H_\Delta$ denotes the flow of σH_Δ described in (3.17). Let $m_0 = (\rho_0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0))$ be non trapped backward. We have the following cases.

Case 1 : $\rho_0 = \overline{\mu}_0 = 0$. — The first equation in (3.18) shows that $\rho(t) \equiv 0$, $t \in (-\infty, 0]$. According to (3.5) we see that the other equations reduce to

$$\dot{y}_j = 2\sum_{k=1}^{n-1} \overline{h}^{jk}(y)\overline{\mu}_k\,, \quad \dot{\overline{\lambda}} = -2\|\overline{\mu}\|^2, \quad \dot{\overline{\mu}} = 2\overline{\lambda}\overline{\mu} - \partial_y\|\overline{\mu}\|^2.$$

Therefore $\overline{\mu}(t) \equiv 0$, $\overline{\lambda}(t) \equiv 1$, $y(t) \equiv y_0$, $\sigma(t) \equiv 0$, so for all t we have $\exp t\sigma H_\Delta(m_0) = m_0 \in \mathcal{N}_+^c$.

Case 2 : $\rho_0 = 0$, $\overline{\mu}_0 \neq 0$. — Then $\rho(t) \equiv 0$ but $\overline{\mu}(t) \neq 0$ for all t in $(-\infty, 0]$. The above equations show that $\overline{\lambda}$ is strictly decreasing on $(-\infty, 0]$. Since $-1 \leqslant \overline{\lambda}(t) \leqslant 1$, $\overline{\lambda}(t)$ has a limit ℓ when t goes to $-\infty$. It follows that $\|\overline{\mu}(t)\|^2 = 1 - \overline{\lambda}^2(t) \to 1 - \ell^2$ so $\dot{\overline{\lambda}} \to -2(1-\ell^2)$. This implies that $\ell = \mp 1$ so $\|\overline{\mu}(t)\| \to 0$ and $\overline{\lambda}(t) \to \pm 1$. On the other hand we deduce from the above equations that $\frac{d}{dt}\|\overline{\mu}(t)\|^2 = 4\overline{\lambda}(t)\|\overline{\mu}(t)\|^2$, so if $\overline{\lambda}(t) \to -1$ when $t \to -\infty$ we would have $\|\overline{\mu}(t)\|^2 \to +\infty$. Therefore $\overline{\lambda}(t) \to 1$, $\|\overline{\mu}(t)\| \leqslant Ce^{\delta t}$, $\delta > 0$, $t \leqslant 0$. It follows from the equation in y that $\dot{y}_j \in L^1(-\infty, 0)$ so $y_j(t)$ tends to a limit as $t \to -\infty$.

Case 3: $\rho_0 \neq 0$. — In that case, m_0 non trapped backward implies that $\rho(t) \to 0$ as $t \to -\infty$. Moreover, by the first equation in (3.17), we have $\rho(t) \neq 0$ for all t. Now we check easily from (3.17) that $\frac{d}{dt}[p(\rho, y, \overline{\lambda}, \overline{\mu})] = 0$; since, by ellipticity of $p = \lambda^2 + \|\mu\|^2 + \rho r$ we have $c(\lambda^2 + \|\mu\|^2) \leqslant p \leqslant \frac{1}{c}(\lambda^2 + \|\mu\|^2)$, it follows that $\overline{\lambda}$, $\overline{\mu}$, $r(\rho, y, \overline{\lambda}, \overline{\mu})$ and their derivatives are uniformly bounded. We have

$$\dot{\rho} = \rho\frac{\partial p}{\partial \lambda} = 2\overline{\lambda}\rho + \rho^2 r_1(\rho, y, \overline{\lambda}, \overline{\mu}).$$

Then, using the Euler relation, we get

$$\dot{\overline{\lambda}} = \overline{\lambda}\cdot\frac{\partial p}{\partial \lambda} - 2p - \rho\frac{\partial p}{\partial \rho} = \overline{\lambda}\Big(2\overline{\lambda} + \rho\frac{\partial r}{\partial \lambda}\Big) - 2p - \rho\frac{\partial p}{\partial \rho}.$$

Since $p(\rho, y, \overline{\lambda}, \overline{\mu}) = 1$, we obtain $\dot{\overline{\lambda}} = 2\overline{\lambda}^2 - 2 + \rho r_2(\rho, y, \overline{\lambda}, \overline{\mu})$ where r_1 and r_2 are bounded. Let us set $\alpha(t) = (\overline{\lambda}(t) - 1)/\rho(t)$. Then

$$\dot{\alpha} = \frac{\dot{\overline{\lambda}}\rho - (\overline{\lambda}-1)\dot{\rho}}{\rho^2} = \frac{1}{\rho^2}\big[2\rho(\overline{\lambda}-1)(\overline{\lambda}+1) - 2\overline{\lambda}\rho(\overline{\lambda}-1) + \rho^2 r_3\big]$$

$$\dot{\alpha} = \frac{1}{\rho^2}\big[2\rho(\overline{\lambda}-1) + \rho^2 r_3\big] = 2\alpha(t) + f(t)$$

where f is bounded on $(-\infty, 0]$. It follows that α is bounded on $(-\infty, 0]$, so $|\overline{\lambda}(t) - 1| \leqslant M\rho(t)$. Therefore $\overline{\lambda}(t) \to 1$ and $\|\mu(t)\|^2 = 1 - \lambda^2(t) - \rho(t)r(\cdots) \to 0$ so $\lim_{t\to-\infty} \exp t\sigma H_\Delta(m_0) \in \mathcal{N}_+^c$. □

Given $\varepsilon > 0$ we set

$$\Omega_\varepsilon = \big\{(\rho, y, \mu) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} : |\rho| + |\mu| < \varepsilon\big\}.$$

Then one can find $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$ and all (ρ, y, μ) in Ω_ε the problem

$$\begin{cases} p(\rho, y, \lambda, \mu) = 1 \\ \operatorname{Re} \lambda < 0 \end{cases} \tag{3.19}$$

has a unique solution $\lambda = \lambda(\rho, y, \mu)$ which depends holomorphically on the parameters.

By extension we shall say that the point $m = (\rho, y, 0, (\lambda, \mu))$ belongs to Ω_ε if (ρ, y, μ) belongs to Ω_ε and λ is the corresponding unique solution of (3.19).

Lemma 3.4. — *There exists $\varepsilon_0 > 0$ such that for all $m^* = (\rho^*, y^*, 0, (\lambda^*, \mu^*))$ in Ω_{ε_0} we have*

(a) $\exp t\sigma H_\Delta(m^*)$ *exists for all $t \geqslant 0$,*
(b) $\exp t\sigma H_\Delta(m^*)$ *converges, as t goes to infinity, to a point $(0, \underline{y}, 0, (-1, 0)) \in \mathcal{N}_-^c$,*
(c) $\underline{y}$ *depends holomorphically on (ρ^*, y^*, μ^*) in Ω_{ε_0} and*
(d) $\underline{y} = y^* + \rho^* F_1(\rho^*, y^*, \mu^*) + \mu^* F_2(\rho^*, y^*, \mu^*)$.

Proof. — Let us introduce the following subset A of $]0, +\infty[$. We shall say that $T \in A$ if the system (3.17) with data $(\rho^*, y^*, \lambda^*, \mu^*)$ has a solution on $[0, T]$ satisfying

$$\begin{cases} |\rho(t)| \leqslant 2\varepsilon_0 e^{-2t} \\ |y(t) - y^*| \leqslant \varepsilon_0^{1/2} \\ |\lambda(t) + 1| \leqslant 2\varepsilon_0 e^{-4t} \\ |\mu(t)| \leqslant 2\varepsilon_0 e^{-2t}. \end{cases} \tag{3.20}$$

Our purpose is to show that $A =]0, +\infty[$. Let $T^* = \sup A$ and assume that $T^* < +\infty$. Let $T < T^*$. By the first equation of (3.17), our solution on $[0, T]$ satisfies

$$\dot\rho = \rho \frac{\partial p}{\partial \lambda} = 2\lambda\rho + \rho^2 a(\rho, y)\lambda + \rho b(\rho, y)\mu,$$

a, b bounded. Then

$$\dot\rho = -2\rho + 2(\lambda + 1)\rho + a\rho^2\lambda + \rho b\mu = -2\rho + f_1(t).$$

It follows from (3.20) that $|f_1(t)| \leqslant C_1 \varepsilon_0^2 e^{-4t}$. Since

$$\rho(t) = \rho^* e^{-2t} + e^{-2t} \int_0^t e^{2s} f_1(s)\, ds$$

we get

$$|\rho(t)| \leqslant \varepsilon_0 e^{-2t} + \frac{C_1}{2} \varepsilon_0^2 e^{-2t} \leqslant \frac{3}{2} \varepsilon_0 e^{-2t} \tag{3.21}$$

if $C_1 \varepsilon_0 \leqslant 1$.

Now

$$\begin{aligned}\dot\mu &= \mu\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial y}\\ &= -2\mu + 2(\lambda+1)\,\mu + a_1\,\rho^2\lambda\mu + a_2\,\rho\mu - \partial_y\|\mu\|^2 + a_3\,\rho^2\lambda + a_4\,\rho\lambda\mu + a_5\,\rho\mu^2\\ &= -2\mu + f_2(t)\end{aligned}$$

where $|f_2(t)| \leqslant C_2\,\varepsilon_0^2\,e^{-4t}$. Since $|\mu^*| \leqslant \varepsilon_0$ we get easily, as above,

$$|\mu(t)| \leqslant \frac{3}{2}\,\varepsilon_0\,e^{-2t} \tag{3.22}$$

if $C_2\,\varepsilon_0 \leqslant 1$. Let us look to λ. We have, since $p(\rho, y, \lambda, \mu) = 1$,

$$\lambda^2 - 1 = -\|\mu\|^2 - \rho(a\rho\lambda^2 + b\lambda\mu + c\mu^2)\,. \tag{3.23}$$

Now

$$|\lambda^2 - 1| = |\lambda+1|\,|\lambda-1| = |\lambda+1|\,|2-(\lambda+1)| \geqslant |\lambda+1|(2-2\varepsilon_0) \geqslant |\lambda+1|$$

if $\varepsilon_0 \leqslant 1/2$. It follows from (3.23) and (3.20) that

$$|\lambda+1| \leqslant C_3\,\varepsilon_0^2\,e^{-4t} \leqslant \varepsilon_0\,e^{-4t} \tag{3.24}$$

if $C_3\,\varepsilon_0 \leqslant 1$.

Finally, $\dot y_k = \partial p/\partial\mu_k = 2\sum_{j=1}^n \overline{h}^{jk}(y)\,\mu_j + \rho\, a_1\lambda + \rho\, a_k\cdot\mu$. Then

$$|y(t) - y^*| \leqslant C_4\,\varepsilon_0\int_0^t e^{-2s}\,ds \leqslant \frac{C_4}{2}\,\varepsilon_0\,,$$

so

$$|y(t) - y^*| \leqslant \frac{1}{2}\,\varepsilon_0^{1/2} \tag{3.25}$$

if $C_4\,\varepsilon_0^{1/2} \leqslant 1$.

Moreover for t, t' in $[0, T]$, we have

$$|y(t) - y(t')| \leqslant C_4\,\varepsilon_0\Big|\int_t^{t'} e^{-2s}\,ds\Big|\,. \tag{3.26}$$

Now it is easy to see that $(\rho(T), y(T), \lambda(T), \mu(T))$ have a limit as T goes to T^* and these limits satisfy estimates as (3.21), (3.22), (3.24) and (3.25). Applying the Cauchy-Lipschitz theorem, we then see that a solution of (3.17) can be found, which satisfies the estimates (3.20) on $[0, T^*+\delta]$; this contradicts the definition of T^* and proves that $T^* = +\infty$. Thus a) is proved and b) follows from (3.20) and (3.26). Since $\exp t\sigma H_\Delta(m^*)$ depends holomorphically on (ρ^*, y^*, μ^*) in Ω_{ε_0} and since, by (3.20), (3.26) the convergence to $(0, \underline{y}, 0, (-1, 0))$ is uniform, the claim c) is proved. Finally assume in (3.20) that the data $\rho(0) = \rho^*$ and $\mu(0) = \mu^*$ are equal to zero. Then $\rho(t) = \mu(t) = 0$ for all t in $[0, +\infty[$. It follows that $\dot y(t) = 0$ for all t so $y(t) = y(0) = y^*$. This proves d). □

Corollary 3.5. — *Let*

$$m_0 = (0, y_0, 0, (-1,0)), \quad \delta^* > 0 \quad \textit{and} \quad V = \{m^* : d(m^*, N_{+\infty}^{-1}(m_0)) < \delta^*\},$$

where d is Euclidian distance. Let $m^* \in \Omega_{\varepsilon_0}$ be such that $\exp t\sigma H_\Delta(m^*)$ converges, as t goes to $+\infty$, to a point $(0, \underline{y}, 0, (-1, 0))$. Then, if ε_0 is small enough, one can find $\delta > 0$ such that if $|\underline{y} - y_0| < \delta$ we have $m^* \in V$.

Proof. — Let $m^* = (\rho^*, y^*, 0, (\lambda^*, \mu^*))$. By the implicit function theorem, keeping the notations in Lemma 3.4 d), one can find $\lambda_0^* \in \mathbb{C}$ with $\operatorname{Re}\lambda_0^* < 0$ and $y_0^* \in \mathbb{C}^{n-1}$ such that

$$\begin{cases} p(\rho^*, y_0^*, \lambda_0^*, \mu^*) = 1 \\ y_0 = y_0^* + \rho^* F_1(\rho^*, y_0^*, \mu^*) + \mu^* F_2(\rho^*, y_0^*, \mu^*). \end{cases} \tag{3.27}$$

It follows from Lemma 3.4 that $m_0^* = (\rho^*, y_0^*, 0, (\lambda_0^*, \mu^*))$ belongs to Ω_{ε_0} and to $N_{+\infty}^{-1}(m_0)$. Since $\lambda_0^* = G(\rho^*, y_0^*, \mu^*)$, where G is holomorphic in Ω_{ε_0}, we see that $|m^* - m_0^*| \leqslant C|y^* - y_0^*|$. From Lemma 3.4 and (3.27) we deduce that

$$|m^* - m_0^*| \leqslant C|y_0 - \underline{y}| + C'(|\rho^*| + |\mu^*|) \leqslant C|y_0 - \underline{y}| + C'\varepsilon_0 \leqslant C\delta + C'\varepsilon_0 < \delta^*$$

if δ and ε_0 are small enough. It follows that $m^* \in V$. □

Corollary 3.6. — *One can find $\varepsilon_0 > 0$ and a holomorphic function G in the set $\{(\rho^*, \mu^*) : |\rho^*| + |\mu^*| < \varepsilon_0\}$ such that if $m^* \in \Omega_{\varepsilon_0} \cap N_{+\infty}^{-1}(m_0)$, then $\mu^* = G(\rho^*, y^*)$.*

Proof. — This follows from Lemma 3.4 d) and the implicit function theorem if we can show that $F_2(0, y_0, 0)$ is invertible. To compute this term, we may take, in (3.17), $\rho^* = 0$, $y^* = y_0$ and $\mu^* = \mu_\ell^* e_\ell$ where $\mu_\ell^* \in \mathbb{C}$ and $(e_1, \dots, e_{n-1})$ is the canonical basis in $\mathbb{C}^{n-1}$. Then $\rho(t) = 0$ for all t. Let us set $\mu_\ell^* = z \in \mathbb{C}$, $y(t) = y_0 + zY(t)$, $\overline{\mu}(t) = z\eta(t)$. Then from (3.17), we get

$$\dot{y}_k(t) = 2\sum_{j=1}^{n-1} \overline{h}^{jk}(y_0) z\eta_j(t) + \mathcal{O}(|z|^2) = z\dot{Y}_k(t)$$

$$\dot{\overline{\mu}}(t) = -2z\eta(t) + \mathcal{O}(|z|^2) = z\dot{\eta}(t)$$

since, by (3.23), we have $\lambda + 1 = \mathcal{O}(|z|^2)$.

It follows that (Y, η) satisfies the system

$$\begin{cases} \dot{Y}_k(t) = 2\sum_{j=1}^{n-1} \overline{h}^{jk}(y_0)\eta_j(t) + \mathcal{O}(|z|), & Y_k(0) = 0 \\ \dot{\eta}(t) = -2\eta(t) + \mathcal{O}(|z|), & \eta(0) = e_\ell. \end{cases}$$

To compute $F_2(0, y_0, 0)$, we have to solve this system with $z = 0$. We obtain $\eta_j(t) = 0$ if $j \neq \ell$ and $\eta_\ell(t) = e^{-2t}$. We deduce that $\dot{Y}_k(t) = 2\overline{h}^{\ell k}(y_0)e^{-2t}$, which shows that $\lim_{t\to+\infty} Y_k(t) = \overline{h}^{\ell k}(y_0)$. It follows that $F_2(0, y_0, 0) = (\overline{h}^{\ell k}(y_0))_{1\leqslant k,\ell\leqslant n-1}$ which is invertible. □

CHAPTER 4

STATEMENTS OF THE MAIN RESULTS AND REDUCTIONS

We consider in this section for $u_0 \in L^2(\mathbb{R}^n)$, a solution $u(t)$ in the space $C^0([0,+\infty[, L^2(\mathbb{R}^n)) \cap C^1([0,+\infty[, H^{-2}(\mathbb{R}^n))$ of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + i\,\Delta_g u = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{4.1}$$

where Δ_g is the Laplacian with respect to a scattering metric g.

4.1. Main results

Our purpose is to answer the following question : given a point m_0 in $\mathcal{C} = {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M \cup {}^{\mathrm{qsc}}S^*M$ and a time $T > 0$, on what condition on the data u_0 do we have $m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T,\cdot))$?

The point m_0 will be described by its coordinates

(i) if $\sigma_0 = 0$, $m_0 = (\rho_0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0))$, $\overline{\lambda}_0^2 + |\overline{\mu}_0|^2 = 1$

(ii) if $\sigma_0 > 0$, $m_0 = (0, y_0, \lambda_0, \mu_0)$ with $\lambda_0 = \overline{\lambda}_0/\sigma_0$, $\mu_0 = \overline{\mu}_0/\sigma_0$.

We shall consider several different cases.

Case 1. — $\rho_0 = 0$, $\sigma_0 > 0$ and

(1.i) $\mu_0 \neq 0$, $T > 0$, or

(1.ii) $\mu_0 = 0$, $\lambda_0 > 0$, $T > 0$, or

(1.iii) $\mu_0 = 0$, $\lambda_0 < 0$, $T < -1/2\lambda_0$.

Theorem 4.1. — *We have $m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T,\cdot))$ if and only if* $\exp(-TX_0)(m_0) \notin {}^{\mathrm{qsc}}WF_a(u_0)$.

Case 2. — $\rho_0 = 0$, $\sigma_0 > 0$ and

(2.i) $\mu_0 = 0$, $\lambda_0 < 0$, $T = -1/2\lambda_0$.

Let us set $m_1 = \exp\left(\frac{1}{2\lambda_0}X_0\right)(m_0)$. It follows from § 3.3 that $m_1 \in \mathcal{N}_-^c$, that is $m_1 = (0, y_1, 0, (-1, 0))$. We shall denote by $\dot{N}_{+\infty}^{-1}(m_1)$ the set $N_{+\infty}^{-1}(m_1) \setminus \{m_1\}$, that is the set of points in S^*M, different from m_1, which arrive at time $t = +\infty$ at the point m_1 by the flow of σH_Δ.

Theorem 4.2

Assume that one can find a neighborhood U of $m_1 = \exp\big(\frac{1}{2\lambda_0}X_0\big)(m_0)$ such that $\dot N^{-1}_{+\infty}(m_1) \cap U$ does not intersect ${}^{\mathrm{qsc}}\widetilde{WF}_a(u_0)$. Then $m_0 \notin {}^{\mathrm{qsc}}WF_a\big(u(-1/2\lambda_0, \cdot)\big)$.

Case 3. — $\rho_0 = 0$, $\sigma_0 > 0$ and

(3.i) $\mu_0 = 0$, $\lambda_0 < 0$ and $T > -1/2\lambda_0$.

As before let us set $m_1 = \exp\big(\frac{1}{2\lambda_0}X_0\big)(m_0) \in \mathcal{N}^c_-$. If m_1 is not backward trapped then, by Definition 3.2, all the points of $N^{-1}_{+\infty}(m_1)$ (that is the points arriving at time $+\infty$ at m_1 by the flow of σH_Δ) are not backward trapped ; therefore the set $N_{-\infty}(N^{-1}_{+\infty}(m_1))$ is well defined. We shall set

$$\mathrm{scat}(m_1) = N_{-\infty}(N^{-1}_{+\infty}(m_1)) \subset \mathcal{N}^c_+.$$

Theorem 4.3

Let $m_1 = \exp\big(\frac{1}{2\lambda_0}X_0\big)(m_0) \in \mathcal{N}^c_-$. Assume that m_1 is not backward trapped and that $\exp\big[-\big(T + \frac{1}{2\lambda_0}\big)\big](\mathrm{scat}(m_1)) \cap {}^{\mathrm{qsc}}WF_a(u_0) = \varnothing$. *Then $m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T, \cdot))$.*

Case 4. — $\sigma_0 = 0$ and (4.i) $\rho_0 > 0$, $T > 0$, or (4.ii) $\rho_0 = 0$, $m_0 \notin \mathcal{N}^c_-$, $T > 0$.

Theorem 4.4

Assume that m_0 is not backward trapped (then $N_{-\infty}(m_0) \in \mathcal{N}^c_+$). Assume that $\exp(-TX_0)(N_{-\infty}(m_0)) \notin {}^{\mathrm{qsc}}WF_a(u_0)$. *Then $m_0 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T, \cdot))$.*

Case 5. — $\sigma_0 > 0$ and

(5.i) $\rho_0 = 0$, $m_1 = \exp(-TX_0)(m_0) \in \mathcal{N}^c_-$, $T > 0$.

Theorem 4.5

Assume that $m_1 \notin {}^{\mathrm{qsc}}WF_a(u_0)$. Then $m_0 = \exp(TX_0)(m_1) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T, \cdot))$.

Remark 4.6. — Theorem 4.4 contains the so called "smoothing effect". Using Examples 2.7 (1) and (2), we can recover results which, in this context, are analogue to those of [**RZ1**] and [**RZ2**].

The results described above will follow from several other ones which we state now.

4.2. Propagation inside ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$

Theorem 4.6

Let $0 \leqslant \theta_* < \theta^*$ and $m \in {}^{\mathrm{qsc}}\overline{T}^*M$. Assume that $\exp(\theta X_0)(m) \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$ *for $\theta \in [\theta_*, \theta^*]$. Then*

$$\exp(\theta_* X_0)(m) \notin {}^{\mathrm{qsc}}WF_a(u(\theta_*, \cdot)) \Longleftrightarrow \exp(\theta^* X_0)(m) \notin {}^{\mathrm{qsc}}WF_a(u(\theta^*, \cdot)).$$

4.3. Propagation of the uniform wave front set in $({}^{\mathrm{qsc}}S^*M)^0$ or on the corner

Theorem 4.7. — *Let $t_0 > 0$ be fixed. Let $0 \leqslant \theta_* < \theta^*$ and $m \in {}^{\mathrm{qsc}}\overline{T}^*M$. Assume that $\exp(\theta\sigma H_\Delta)(m) \in ({}^{\mathrm{qsc}}S^*M)^0$ (resp. ${}^{\mathrm{qsc}}S^*M \cap {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$) for $\theta \in [\theta_*, \theta^*]$. Then*

$$\exp(\theta_*\sigma H_\Delta)(m) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot)) \iff \exp(\theta^*\sigma H_\Delta)(m) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot)).$$

4.4. Propagation from the interior to the corner

Theorem 4.8. — *Let $m \in \mathcal{N}^c_-$ and $t_0 \geqslant 0$. Assume that one can find a neighborhood U of m in ${}^{\mathrm{qsc}}S^*M$ such that $\dot{N}^{-1}_{+\infty}(m)$ does not intersect ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot))$ in U. Then $m \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot))$.*

4.5. Propagation from the boundary at infinity to the corner

Theorem 4.9. — *Let $m \in \mathcal{N}^c_+$. Assume that $\exp(-TX_0)(m) \notin {}^{\mathrm{qsc}}WF_a(u_0)$. Then $m \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$.*

4.6. Proofs of Theorems 4.1 to 4.5

Let us now show how Theorems 4.6 to 4.9 imply the main results.

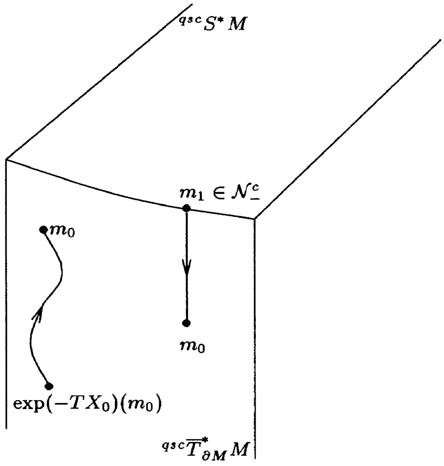
A. Proof of Theorem 4.1. — According to the description of the flow on $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$ in § 3.3, we see that in the cases (1.i), (1.ii) and (1.iii) the bicharacteristic stays, for $\theta \in [0,T]$, inside $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$. Thus Theorem 4.1 follows from Theorem 4.6 taking $\theta_* = 0$, $\theta^* = T$.

B. Proof of Theorem 4.2. — Let $m_1 = \exp\big(\frac{1}{2\lambda_0}X_0\big)(m_0) \in \mathcal{N}^c_-$ (since $\lambda_0 < 0$). It follows from Theorem 4.8 (with $t_0 = 0$) that $m_1 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u_0)$. Then one can find $\varepsilon \in \,]0, -1/2\lambda_0[$ such that $\exp(\varepsilon X_0)(m_1) \notin {}^{\mathrm{qsc}}WF_a(u(\varepsilon,\cdot))$. Applying Theorem 4.6 with $\theta_* = \varepsilon$, $\theta^* = -1/2\lambda_0$ we get $\exp\big(-\frac{1}{2\lambda_0}X_0\big)(m_1) = m_0 \notin {}^{\mathrm{qsc}}WF_a\big(u(-1/2\lambda_0,\cdot)\big)$.

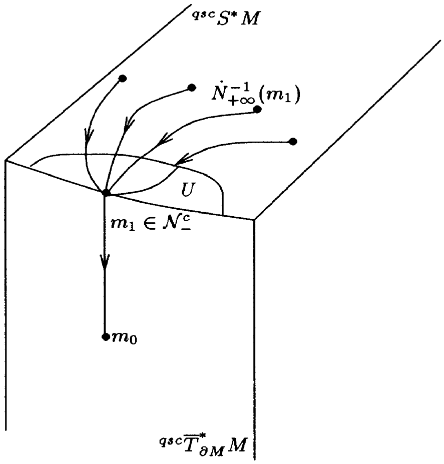
C. Proof of Theorem 4.3. — Assume that

$$\exp\big[-\big(T+\tfrac{1}{2\lambda_0}\big)X_0\big](\mathrm{scat}(m_1)) \cap {}^{\mathrm{qsc}}WF_a(u_0) = \varnothing.$$

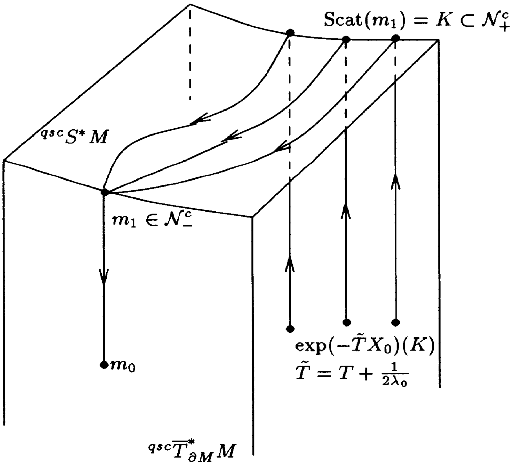
Let $m \in \mathrm{scat}(m_1)$. Then $m \in \mathcal{N}^c_+$. We apply Theorem 4.9 with $T + \frac{1}{2\lambda_0}$ instead of T. It follows that $m \notin {}^{\mathrm{qsc}}\widetilde{WF}_a\big(u\big(T+\frac{1}{2\lambda_0},\cdot\big)\big)$. Then a small neighborhood of m in ${}^{\mathrm{qsc}}S^*M$ does not intersect this set. We apply Theorem 4.7. We deduce that all the bicharacteristic issued from m does not intersect ${}^{\mathrm{qsc}}\widetilde{WF}_a\big(u\big(T+\frac{1}{2\lambda_0},\cdot\big)\big)$. Using this argument for all points in $\mathrm{scat}(m_1)$, we see that one can find a neighborhood U of m_1 in S^*M such that $N^{-1}_{+\infty}(m_1)$ does not intersect ${}^{\mathrm{qsc}}\widetilde{WF}_a\big(u\big(T+\frac{1}{2\lambda_0},\cdot\big)\big)$ in U. It



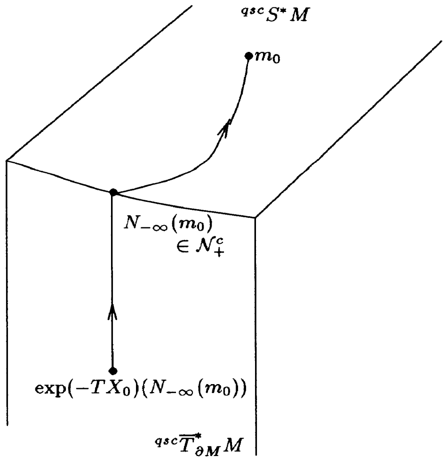
Theorem 4.1 **Theorem 4.5**



Theorem 4.2



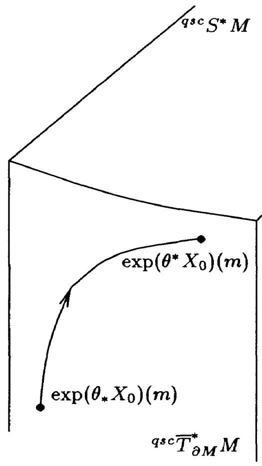
Theorem 4.3



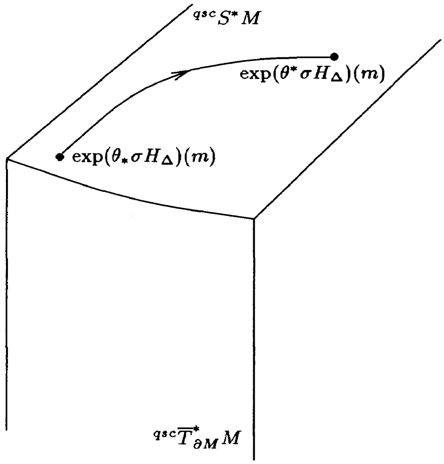
Theorem 4.4

follows from Theorem 4.8, with $t_0 = T + \frac{1}{2\lambda_0} > 0$, that $m_1 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a\big(u(T + \frac{1}{2\lambda_0}, \cdot)\big)$. Let us introduce $m_2 = \exp(-TX_0)(m_0)$. Then $m_1 = \exp\big((T + \frac{1}{2\lambda_0})\, X_0\big)(m_2) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a\big(u(T + \frac{1}{2\lambda_0}, \cdot)\big)$. Then one can find $\varepsilon > 0$ such that

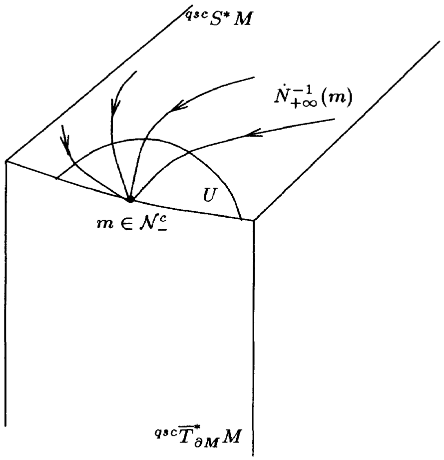
$$
\begin{aligned}
&\exp(\varepsilon X_0)(m_1) \\
&\quad = \exp\big((T + \tfrac{1}{2\lambda_0} + \varepsilon)\, X_0\big)(m_2) \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0 \cap \big({}^{\mathrm{qsc}}WF_a\big(u(T + \tfrac{1}{2\lambda_0} + \varepsilon, \cdot)\big)\big)^c.
\end{aligned}
$$



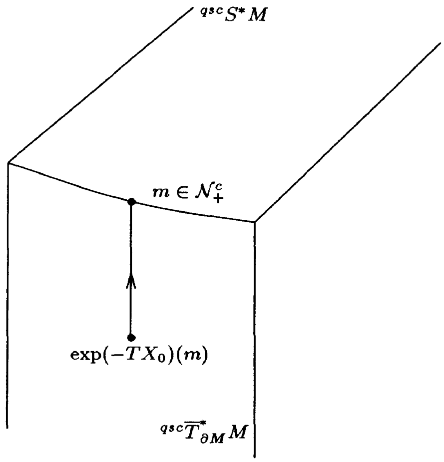
Theorem 4.6



Theorem 4.7



Theorem 4.8



Theorem 4.9

Applying Theorem 4.6 with $\theta_* = T + \frac{1}{2\lambda_0} + \varepsilon < \theta^* = T$ we see that $\exp(TX_0)(m_2) = m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T,\cdot))$.

D. Proof of Theorem 4.4. — Let $m_1 = N_{-\infty}(m_0) \in \mathcal{N}^c_+$. Since

$$\exp(-TX_0)(m_1) \notin {}^{\mathrm{qsc}}WF_a(u_0)$$

we have, by Theorem 4.9, $m_1 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$. If $m_0 \in {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$ then, by Theorem 4.7, all the bicharacteristic starting at m_0 is contained in ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$.

Since this set is closed it would follow that

$$N_{-\infty}(m_0) = m_1 = \lim_{t \to -\infty} \exp(tX_0)(m_0) \in {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$$

which is a contradiction. So $m_0 \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(T,\cdot))$.

E. Proof of Theorem 4.5. — The complementary of ${}^{\mathrm{qsc}}\widetilde{WF}_a$ is an open set; then, there exists $\varepsilon > 0$ such that $m_2 = \exp(\varepsilon X_0)(m_1) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(\varepsilon,\cdot))$. We can now apply Theorem 4.1 with $T - \varepsilon$ to obtain Theorem 4.5.

CHAPTER 5

PROOF OF THEOREM 4.6

In $({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$ we can, according to the Remark 2.8 (2), forget the parameter k in the FBI transform and use (2.9)'. Using (3.4) we see that the adjoint Δ_g^* of our Laplacian can be written as

$$(5.1)\qquad \begin{cases} \Delta_g^* = (\rho^2 D_\rho)^2 + \rho^2 \Delta_0 + c(n)\,\rho^3 D_\rho + d(n)\,\rho^2 + \rho R \ \text{ where} \\ R = \displaystyle\sum_{0\leqslant|\alpha|+\ell\leqslant 2} b_{\alpha\ell}(\rho,y)(\rho^2 D_\rho)^\ell (\rho D_y)^\alpha\,,\ b_{0\ell}(\rho,y) = \rho^{3-\ell}\widetilde{b}_{0,\ell}(\rho,y)\,. \end{cases}$$

Let $(\theta_0; s_0, y_0, \alpha_0) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^{2n}$ be a fixed point. Roughly speaking our goal is to find phases $\varphi = \varphi\big(\theta; \rho/h, y, \alpha, h\big)$, symbols $a = a\big(\theta; \rho/h, y, \alpha, h\big)$ depending smoothly on all variables in a real neighborhood of $(\theta_0; s_0, y_0, \alpha_0, 0)$ such that, at least formally, we have

$$(5.2)\qquad \Big(\frac{\partial}{\partial\theta} + i\,\Delta_g^*\Big)(a\,e^{ih^{-2}\varphi}) = \mathcal{O}(e^{-\varepsilon/h}),\quad \varepsilon > 0\,.$$

We shall seek for φ and a on the following form

$$(5.3)\qquad \varphi = \varphi_2\Big(\theta; \frac{\rho}{h}, y, \alpha\Big) + ih\varphi_1\Big(\theta; \frac{\rho}{h}, y, \alpha\Big)$$

$$(5.4)\qquad a = \sum_{j\geqslant 0} h^j\, a_j\Big(\theta; \frac{\rho}{h}, y, \alpha, h\Big)\,.$$

An easy computation shows that, working with the variable $s = \rho/h$, we have

$$(5.5)\qquad \Big(\frac{\partial}{\partial\theta} + i\,\Delta_g^*\Big)(a\,e^{ih^{-2}\varphi}) = h^{-2}\,e^{ih^{-2}\varphi}(I + II + III + IV)\,.$$

$$(5.6)\qquad \begin{cases} I = i\Big(\dfrac{\partial\varphi_2}{\partial\theta} + s^4\Big(\dfrac{\partial\varphi_2}{\partial s}\Big)^2 + s^4\Big\|\dfrac{\partial\varphi_2}{\partial y}\Big\|^2\Big)\,a, \quad \text{where} \\ \|Y\|^2 = \displaystyle\sum_{j,k=1}^{n-1} \overline{h}^{jk}(y)\,Y_j\,Y_k = \langle Y, Y\rangle\,. \end{cases}$$

$$(5.7)\qquad \begin{cases} II = -h\Big(\mathcal{L}\varphi_1 + i F_0\Big(s, y, \dfrac{\partial\varphi_2}{\partial s}, \dfrac{\partial\varphi_2}{\partial y}\Big)\Big) a, \ \text{where} \\ \mathcal{L} = \dfrac{\partial}{\partial\theta} + 2s^4 \dfrac{\partial\varphi_2}{\partial s}\dfrac{\partial}{\partial s} + 2s^2\Big\langle \dfrac{\partial\varphi_2}{\partial y}, \dfrac{\partial}{\partial y}\Big\rangle \ \text{and} \\ F_0 \ \text{is real if } (s,y) \text{ is real and is a polynomial in } \dfrac{\partial\varphi_2}{\partial s}, \dfrac{\partial\varphi_2}{\partial y}. \end{cases}$$

$$(5.8)\qquad III = h^2\big(\mathcal{L}a + F_1\big(s, y, (\partial^\alpha\varphi_\ell)_{|\alpha|\leqslant 2, \ell=1,2}\big)\, a\big).$$

$$(5.9)\qquad IV = \sum_{j=1}^{2} h^{2+j} X_j\big(sh, s, y, (\partial^\alpha\varphi_\ell)_{|\alpha|\leqslant 2, \ell=1,2}; \partial_s, \partial_y\big)\, a.$$

Here F_1 is analytic in (s,y), polynomial in $(\partial^\alpha\varphi_\ell)$, $|\alpha| \leqslant 2$, $\ell = 1,2$ and X_j is a homogeneous differential operator of order j whose coefficients are finite sums of terms of the form $b(sh,y)c(s,y)(\partial^\alpha\varphi_1)^{\ell_1}(\partial^\beta\varphi_2)^{\ell_2}$ where $|\alpha| \leqslant 2$, $|\beta| \leqslant 2$, $\ell_1 + \ell_2 \leqslant 2$ and c, b are smooth.

5.1. The first phase equation

Our purpose here is to find φ_2 such that the term I in (5.6) vanishes. We shall solve, for (θ, s, y) real, the Cauchy problem

$$(5.10)\qquad \begin{cases} \dfrac{\partial\varphi_2}{\partial\theta} + s^4\Big(\dfrac{\partial\varphi_2}{\partial s}\Big)^2 + s^2\Big\|\dfrac{\partial\varphi_2}{\partial y}\Big\|^2 = 0 \\ \varphi_2|_{\theta=\theta_0} = (s-\alpha_s)\alpha_\tau + (y-\alpha_y)\cdot\alpha_\eta \end{cases}$$

where $\alpha = (\alpha_s, \alpha_y, \alpha_\tau, \alpha_\eta) \in \mathbb{R}^{2n}$ is a parameter close to α_0. If we set

$$(5.11)\qquad \varphi_2(\theta; s, y, \alpha) = \widetilde{\varphi}_2(\theta; s, y, \alpha_\tau, \alpha_\eta) - \alpha_s\alpha_\tau - \alpha_y\alpha_\eta$$

then (5.10) is equivalent to

$$(5.12)\qquad \begin{cases} \dfrac{\partial\widetilde{\varphi}_2}{\partial\theta} + s^4\Big(\dfrac{\partial\widetilde{\varphi}_2}{\partial s}\Big)^2 + s^2\Big\|\dfrac{\partial\widetilde{\varphi}_2}{\partial y}\Big\|^2 = 0 \\ \widetilde{\varphi}_2|_{\theta=\theta_0} = s\alpha_\tau + y\cdot\alpha_\eta \end{cases}.$$

Let us consider the symbol

$$(5.13)\qquad \ell(s, y, \tau, \eta, \theta^*) = \theta^* + q(s, y, \tau, \eta), \quad q(\cdots) = s^4\tau^2 + s^2\|\eta\|^2.$$

The equation in (5.12) is equivalent to

$$\ell\Big(s, y, \frac{\partial\widetilde{\varphi}_2}{\partial s}, \frac{\partial\widetilde{\varphi}_2}{\partial y}, \frac{\partial\widetilde{\varphi}_2}{\partial\theta}\Big) = 0.$$

The bicharacteristic of ℓ starting from $(\theta_0, \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta)$ is described by the equations

$$(5.14)\quad \begin{cases} \dot\theta(t) = 1 & \theta(0) = \theta_0 \\ \dot s(t) = \dfrac{\partial q}{\partial \tau}(s(t), y(t), \tau(t), \eta(t)) & s(0) = \widetilde{s} \\ \dot y(t) = \dfrac{\partial q}{\partial \eta}(\cdots) & y(0) = \widetilde{y} \\ \dot\theta^*(t) = 0 & \theta^*(0) = -q(\widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta}) \\ \dot\tau(t) = -\dfrac{\partial q}{\partial s}(s(t), y(t), \tau(t), \eta(t)) & \tau(0) = \alpha_\tau \\ \dot\eta(t) = -\dfrac{\partial q}{\partial y}(\cdots) & \eta(0) = \alpha_\eta . \end{cases}$$

We have $\theta(t) = t + \theta_0$, $\theta^*(t) = \theta^*(0)$ and the system in (s, y, τ, η) has, for small $|t|$, a unique solution

$$\big(s(t; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta),\ y(t; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta),\ \tau(t, \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta),\ \eta(t; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta)\big).$$

Let us consider, for fixed $(\alpha_\tau, \alpha_\eta)$, the set

$$(5.15)\quad \begin{aligned} \Lambda = \Big\{ &\big(\theta, s(\theta-\theta_0; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta),\ y(\theta-\theta_0; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta), \theta^*(0), \\ &\tau(\theta-\theta_0; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta),\ \eta(\theta-\theta_0; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta)\big),\ (\theta, \widetilde{s}, \widetilde{y}) \text{ close to } (\theta_0, s_0, y_0) \Big\}. \end{aligned}$$

Then Λ is a Lagrangian submanifold and, since ℓ is constant on the bicharacteristics, we have

$$(5.16)\quad \ell|_\Lambda = 0.$$

Now the map $(\theta, \widetilde{s}, \widetilde{y}) \mapsto (\theta, s(\theta-\theta_0; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta), y(\theta-\theta_0; \cdots))$ has a Jacobian with determinant equal to one at $\theta = \theta_0$. It follows that the projection on the basis $\Pi : \Lambda \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$, is a local diffeomorphism. Therefore one can find a real function $\widetilde{\varphi}_2(\theta; s, y, \alpha_\tau, \alpha_\eta)$ in a real neighborhood of (θ_0, s_0, y_0) such that

$$(5.17)\quad \Lambda = \Big\{ \Big(\theta, s, y, \frac{\partial\widetilde{\varphi}_2}{\partial\theta}, \frac{\partial\widetilde{\varphi}_2}{\partial s}, \frac{\partial\widetilde{\varphi}_2}{\partial y}\Big),\ (\theta, s, y) \text{ close to } (\theta_0, s_0, y_0) \Big\}.$$

Then (5.16), (5.17) show that $\widetilde{\varphi}_2$ solves (5.12). Let us note that

$$(5.18)\quad \begin{cases} \dfrac{\partial\widetilde{\varphi}_2}{\partial s}(\theta; s, y, \alpha_\tau, \alpha_\eta) = \tau\big(\theta-\theta_0; \kappa_1(\theta-\theta_0; s, y, \alpha_\tau, \alpha_\eta), \\ \qquad\qquad\qquad\qquad \kappa_2(\theta-\theta_0; s, y, \alpha_\tau, \alpha_\eta), \alpha_\tau, \alpha_\eta\big) \\ \dfrac{\partial\widetilde{\varphi}_2}{\partial y}(\cdots) = \eta\big(\theta-\theta_0; K_1(\cdots), \kappa_2(\cdots), \alpha_\tau, \alpha_\eta\big) \end{cases}$$

where

$$\text{(5.19)}\qquad \begin{cases} \theta=\theta \\ s(\theta-\theta_0;\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta)=s \\ y(\theta-\theta_0;\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta)=y \end{cases} \iff \begin{cases} \theta=\theta \\ \widetilde{s}=\kappa_1(\theta-\theta_0;s,y,\alpha_\tau,\alpha_\eta). \\ \widetilde{y}=\kappa_2(\theta-\theta_0;\cdots) \end{cases}$$

Now the solution $\widetilde{\varphi}_2$ in (5.18) is determined up to a constant. We shall take the constant such that

$$\text{(5.20)}\qquad \widetilde{\varphi}_2(\theta_0;s_0,y_0,\alpha_\tau,\alpha_\eta)=\alpha_\tau\, s_0+\alpha_\eta\cdot y_0\,.$$

This determines $\widetilde{\varphi}_2$ uniquely. Now we write

$$\widetilde{\varphi}_2(\theta_0;s,y,\alpha_\tau,\alpha_\eta)=\widetilde{\varphi}_2(\theta_0;s_0,y_0,\alpha_\tau,\alpha_\eta)+\int_0^1\Big[(s-s_0)\frac{\partial\widetilde{\varphi}_2}{\partial s}+(y-y_0)\cdot\frac{\partial\widetilde{\varphi}_2}{\partial y}\Big]\big(\theta_0;ts+(1-t)s_0,ty+(1-t)y_0,\alpha_\tau,\alpha_\eta\big)\,dt.$$

It follows from (5.20), (5.18), (5.14) that

$$\widetilde{\varphi}_2(\theta_0;s,y,\alpha_\tau,\alpha_\eta)=\alpha_\tau\, s_0+\alpha_\eta\cdot s_0+(s-s_0)\alpha_\tau+(y-y_0)\cdot\alpha_\eta=\alpha_\tau\, s+\alpha_y\cdot\eta\,.$$

This proves that $\widetilde{\varphi}_2$ satisfies also the initial condition in (5.12). Let us note that φ_2 defined in (5.11) satisfies then (5.10).

5.2. The second phase equation

Our purpose here is to find φ_1 such that the term II in (5.7) vanishes. More precisely, we shall solve, for (θ,s,y) real, the Cauchy problem

$$\text{(5.21)}\qquad \begin{cases} \mathcal{L}\varphi_1=\dfrac{\partial\varphi_1}{\partial\theta}+2s^4\dfrac{\partial\varphi_2}{\partial s}\dfrac{\partial\varphi_1}{\partial s}+2s^2\Big\langle\dfrac{\partial\varphi_2}{\partial y},\dfrac{\partial\varphi_1}{\partial y}\Big\rangle=-iF_0\Big(s,y,\dfrac{\partial\varphi_2}{\partial s},\dfrac{\partial\varphi_2}{\partial y}\Big) \\ \varphi_1|_{\theta=\theta_0}=(s-\alpha_s)^2+(y-\alpha_y)^2 \end{cases}$$

where F_0 is real.

Since $\mathcal{L}$ is a real vector field with smooth coefficients, the problem (5.21) has a unique solution $\varphi_1=\varphi_1(\theta;s,y,\alpha)$ near (θ_0,s_0,y_0) which is a smooth function of its arguments. Now, since F_0 is real, we have

$$\text{(5.22)}\qquad \begin{cases} \mathcal{L}(\operatorname{Re}\varphi_1)=0, & \mathcal{L}(\operatorname{Im}\varphi_1)=-F_0(s,y,\partial\varphi_2) \\ \operatorname{Re}\varphi_1|_{\theta=\theta_0}=(s-\alpha_s)^2+(y-\alpha_y)^2, & \operatorname{Im}\varphi_1|_{\theta=\theta_0}=0. \end{cases}$$

Notation 5.1. — Let $\alpha_0=(s_0,y_0,\alpha_\tau^0,\alpha_\eta^0)$. We shall denote by $s(t;\alpha_0)$, $y(t;\alpha_0)$, etc. the solution of (5.14) with data $(\theta_0,s_0,y_0,\alpha_\tau^0,\alpha_\eta^0)$.

Lemma 5.2. — *Let us set*

$$A=\big(\theta;s(\theta-\theta_0;\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta),y(\theta-\theta_0;\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta),\alpha\big),$$

$$\frac{\partial}{\partial X}=\frac{\partial}{\partial s}\ \textit{or}\ \frac{\partial}{\partial y},\quad \frac{\partial}{\partial\alpha_X}=\frac{\partial}{\partial\alpha_s}\ \textit{or}\ \frac{\partial}{\partial\alpha_y}.$$

Then

(i) $\operatorname{Re}\varphi_1(A) = (\widetilde{s}-\alpha_s)^2 + (\widetilde{y}-\alpha_y)^2$.

(ii) $\dfrac{\partial}{\partial X}(\operatorname{Re}\varphi_1)(A) = \dfrac{\partial}{\partial \alpha_X}(\operatorname{Re}\varphi_1)(A) = 0,\ \dfrac{\partial^2}{\partial X^2}(\operatorname{Re}\varphi_1)(A) \gg 0$ *if* $\alpha_s = \widetilde{s},\ \alpha_y = \widetilde{y}$.

Proof. — If, instead of working in the (θ, s, y) coordinates, we take the $(\widetilde{\theta}, \widetilde{s}, \widetilde{y})$ coordinates given by (5.19) (where $\widetilde{\theta} = \theta$) we see, using (5.14) and (5.18) for φ_2, that $\mathcal{L} = \partial/\partial\widetilde{\theta}$. Thus, setting $F(\widetilde{\theta}, \widetilde{s}, \widetilde{y}, \alpha) = \operatorname{Re}\varphi_1(A)$, we get $\partial F/\partial\widetilde{\theta} = 0$; therefore $F(\widetilde{\theta}; \widetilde{s}, \widetilde{y}, \alpha) = F(\theta_0; \widetilde{s}, \widetilde{y}, \alpha)$. Since for $\widetilde{\theta} = \theta_0$, $A = (\theta_0; \widetilde{s}, \widetilde{y}, \alpha)$ and since $\operatorname{Re}\varphi_1$ satisfies (5.22) we get (i). Then (ii) follows easily from (i). □

Let us note that (i) implies, in particular, that

$$\operatorname{Re}\varphi_1\big(\theta; s(\theta-\theta_0;\alpha_0), y(\theta-\theta_0;\alpha_0), \alpha_0\big) = 0.$$

5.3. The link between the bicharacteristics of q and the flow of X_0

Let q and X_0 be defined in (5.13) and (3.9). Then we have the following lemma.

Lemma 5.3. — *Let* $m_0 = (0, y_0, \lambda_0, \mu_0) \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$ *and* $\alpha_0 = \big(s_0, y_0, \lambda_0/s_0^3, \mu_0/s_0^2\big)$. *Let* $\big(s(\theta-\theta_0;\alpha_0), y(\theta-\theta_0;\alpha_0), \tau(\theta-\theta_0;\alpha_0), \eta(\theta-\theta_0;\alpha_0)\big)$ *be the bicharacteristic of* q *(defined in (5.14)) starting from* α_0. *Let us set*

(5.23) $\lambda(\theta-\theta_0) = [s(\theta-\theta_0;\alpha_0)]^3\tau(\theta-\theta_0;\alpha_0),\ \mu(\theta-\theta_0) = [s(\theta-\theta_0;\alpha_0)]^2\eta(\theta-\theta_0;\alpha_0)$.

Then $\big(0, y(\theta-\theta_0;\alpha_0), \lambda(\theta-\theta_0), \mu(\theta-\theta_0)\big) = \exp[(\theta-\theta_0)X_0](m_0)$.

Proof. — It is an easy computation using (5.14) and (3.10). □

5.4. φ is a FBI phase

Lemma 5.4. — *Let* $m_0 = (0, y_0, \lambda_0, \mu_0) \in ({}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M)^0$ *and* $\alpha_0 = \big(s_0, y_0, \lambda_0/s_0^3, \mu_0/s_0^2\big)$. *Let us set, according to Notation 5.1 and (5.23),*

$$X(\theta) = \big(s(\theta-\theta_0;\alpha_0), y(\theta-\theta_0;\alpha_0)\big),\quad \Xi(\theta) = \Big(\frac{\lambda(\theta-\theta_0)}{s^3(\theta-\theta_0;\alpha_0)}, \frac{\mu(\theta-\theta_0)}{s^2(\theta-\theta_0;\alpha_0)}\Big).$$

Let us set $\varphi(\theta; s, y, \alpha, h) = \varphi_2(\theta; s, y, \alpha) + ih\varphi_1(\theta; s, y, \alpha)$, *where* φ_2 *and* φ_1 *are the solutions of (5.10) and (5.21). Then, for small* $|\theta-\theta_0|$, $\varphi(\theta,\cdot)$ *is a FBI phase at* $(X(\theta), \Xi(\theta), \alpha_0, 0)$ *(see Definition 2.1).*

Proof. — Let us set $X = (s, y)$, $\alpha = (\alpha_X, \alpha_\Xi)$, $\alpha_X = (\alpha_s, \alpha_y)$, $\alpha_\Xi = (\alpha_\tau, \alpha_\eta)$. By (5.10) and (5.11) we can write

$$\varphi_2(\theta; X, \alpha) = \varphi_2(\theta_0; X, \alpha) + \psi_2(\theta; X, \alpha_\Xi) = (X-\alpha_X)\alpha_\Xi + \psi_2(\theta; X, \alpha_\Xi).$$

Now, we deduce from (5.18) and (5.23) that

$$\frac{\partial\varphi_2}{\partial X}(\theta; X(\theta), \alpha_0) = \big(\tau(\theta-\theta_0;\alpha_0), \eta(\theta-\theta_0;\alpha_0)\big) = \Xi(\theta).$$

It follows from Lemma 5.2 that $\frac{\partial \operatorname{Re} \varphi_1}{\partial X}(\theta; X(\theta), \alpha_0) = 0$. On the other hand (5.21) shows that $\operatorname{Re} \varphi_1|_{\theta=\theta_0} = (X - \alpha_X)^2$. It follows that

$$\frac{\partial^2 \operatorname{Re} \varphi_1}{\partial X^2}(\theta_0; X, \alpha) = -\frac{\partial^2 \operatorname{Re} \varphi_1}{\partial X \, \partial \alpha_X}(\theta_0; X, \alpha) = 2 \operatorname{Id},$$

for all (X, α).

Therefore for (θ, α) close to (θ_0, α_0), we get

$$\Big(\frac{\partial^2 \operatorname{Re} \varphi_1}{\partial X^2}(\theta; X(\theta), \alpha)\Big) \gg 0$$

$$\det \frac{\partial^2 \operatorname{Re} \varphi_1}{\partial X \, \partial \alpha_X}(\theta; X(\theta), \alpha) \neq 0.$$

Finally, since $\varphi_2(\theta_0; X, \alpha) = (X - \alpha_X)\alpha_\Xi$, we get $\frac{\partial^2 \varphi_2}{\partial X \, \partial \alpha_\Xi}(\theta_0; X, \alpha) = \operatorname{Id}$, which implies that, for θ close to θ_0, $\det \frac{\partial^2 \varphi_2}{\partial X \, \partial \alpha_\Xi}(\theta; X(\theta), \alpha) \neq 0$. This proves our claim. □

5.5. The transport equations

Here we look for a symbol a such that the terms III and IV in (5.8), (5.9) vanish. We shall take a of the form

$$\text{(5.24)} \qquad \begin{cases} a(\theta; s, y, \alpha, h) = \sum_{j \geqslant 0} a_j(\theta; s, y, \alpha) h^j, \quad \text{with} \\ |a_j(\theta; s, y, \alpha)| \leqslant M^{j+1} j^{j/2}. \end{cases}$$

Setting $h^2 = \lambda^{-1}$ we see that $a = \Sigma a_j \lambda^{-j/2}$. Compared with the symbols used in Sjöstrand **[Sj]**, these symbols are non classical. However, we follow essentially **[Sj]**. We shall work in the coordinates $(\widetilde{\theta}, \widetilde{s}, \widetilde{y})$ of Lemma 5.2 where $\mathcal{L} = \partial/\partial\widetilde{\theta}$ and we skip the $\sim$ for convenience. Coming back to (5.8), (5.9) and setting $\underline{a} = a - 1$, we have to solve the Cauchy problem

$$\text{(5.25)} \qquad \begin{cases} \Big(\frac{\partial}{\partial \theta} + c(\theta, s, y, \alpha)\Big)\underline{a} + h^{-2}(h^3 X_1 + h^4 X_2)\underline{a} = b \\ \underline{a}|_{\theta=\theta_0} = 0 \end{cases}$$

where X_j, $j = 1, 2$, are homogeneous differential operators of order j with smooth coefficients in $(s, y, \theta, \alpha, h)$ in a neighborhood of $(s_0, y_0, \theta_0, \alpha_0, 0)$ and b is a symbol. Setting $a_1 = \exp\big(\int_{\theta_0}^{\theta} c(\sigma, s, y, \alpha)\, d\sigma\big)\, \underline{a}$, we are lead to solve (5.25) with $c = 0$. Here α is fixed, so we skip it in that follows.

With $r > 0$ small enough and $0 \leqslant t < r$, we set

$$\text{(5.26)} \qquad \Omega_t = \big\{(\theta, s, y) : |\theta - \theta_0| + |s - s_0| + |y - y_0| < r - t\big\}.$$

Given $\rho > 0$, we shall say that $a \in \mathcal{A}_\rho$, if $a = \sum_{j \geqslant 0} a_j(\theta, s, y) h^j$ with

$$\text{(5.27)} \qquad \sup_{\Omega_t} |a_j| \leqslant f_j(a) j^{j/2} t^{-j/2}, \quad 0 < t < r,$$

(where $f_j(a)$ is the best constant for which (5.27) holds) and

$$\sum_{j=0}^{+\infty} f_j(a)\,\rho^j = \|a\|_\rho < +\infty. \tag{5.28}$$

Let us set

$$\partial_\theta^{-1} f(\theta,s,y,h) = (\theta-\theta_0)\int_0^1 f(\sigma\theta+(1-\sigma)\theta_0,s,y,h)\,d\sigma. \tag{5.29}$$

Then the problem (5.25) (with $c=0$) is equivalent to

$$(\mathrm{Id}+B)\,a = d, \quad \text{where } B = h^{-2}\partial_\theta^{-1}(h^3X_1+h^4X_2). \tag{5.30}$$

We want to show that one can find $\rho>0$ such that $\|B\|_{\mathcal{L}(\mathcal{A}_\rho,\mathcal{A}_\rho)} \leqslant c_0 < 1$, which will imply that $I+B$ is invertible.

Lemma 5.5. — *Let $\mathcal{A}'_\rho$ be the subspace of $\mathcal{A}_\rho$ of symbols of the form $a=\sum_{j\geqslant 3} a_j\,h^j$. One can find a positive constant C_0 such that for any $\rho>0$ and a in $\mathcal{A}'_\rho$ we have*

$$\|h^{-2}\partial_\theta^{-1}a\|_\rho \leqslant \frac{C_0}{\rho^2}\,\|a\|_\rho.$$

Proof. — We have

$$h^{-2}\partial_\theta^{-1}a = h^{-2}\sum_{j\geqslant 3} h^j\,\partial_\theta^{-1}a_j = \sum_{j\geqslant 1} h^j\,\partial_\theta^{-1}a_{j+2} = \sum_{j\geqslant 1} h^j\,b_j$$

where

$$b_j(\theta,s,y) = (\theta-\theta_0)\int_0^1 a_{j+2}(\sigma\theta+(1-\sigma)\theta_0,s,y)\,d\sigma.$$

Now, if $(\theta,s,y)\in\Omega_t$ then

$$\begin{aligned}|\sigma\theta+(1-\sigma)\theta_0-\theta_0| &+ |s-s_0|+|y-y_0|\\ &= |\theta-\theta_0|+|s-s_0|+|y-y_0|+(\sigma-1)|\theta-\theta_0|\\ &\leqslant r-t-(1-\sigma)|\theta-\theta_0|\ ;\end{aligned}$$

so $(\sigma\theta+(1-\sigma)\theta_0,s,y)\in\Omega_{t+(1-\sigma)|\theta-\theta_0|}$. Therefore

$$|b_j(s,\theta,y)| \leqslant |\theta-\theta_0|\int_0^1 f_{j+2}(a)(j+2)^{\frac{j}{2}+1}(t+(1-\sigma)|\theta-\theta_0|)^{-\frac{j}{2}-1}\,d\sigma.$$

So

$$|b_j(s,\theta,y)| \leqslant f_{j+2}(a)(j+2)^{\frac{j}{2}+1}\,\frac{2}{j}\,t^{-j/2}.$$

Now, for $j\geqslant 1$, we have

$$\frac{(j+2)^{j/2}}{j^{j/2}}\cdot\frac{j+2}{j} = \Big(1+\frac{1}{j/2}\Big)^{j/2}\Big(1+\frac{2}{j}\Big) \leqslant 3e.$$

Therefore, for (θ,s,y) in Ω_t, we have

$$|b_j(\theta,s,y)| < 6e\,f_{j+2}(a)\,j^{j/2}\,t^{-j/2}.$$

This shows that

$$f_j(b) \leqslant 6e\, f_{j+2}(a). \tag{5.31}$$

It follows that

$$\|h^{-2}\partial_\theta^{-1} a\|_\rho = \|b\|_\rho = \sum_{j\geqslant 1} f_j(b)\,\rho^j \leqslant 6e \sum_{j\geqslant 1} f_{j+2}(a)\,\rho^j \leqslant \frac{6e}{\rho^2}\,\|a\|_\rho. \qquad \square$$

Lemma 5.6. — *The operator $h^3 X_1 + h^4 X_2$ maps $\mathcal{A}_\rho$ to $\mathcal{A}'_\rho$ and there exists a positive constant C_1 such that for all ρ in $]0,1[$ and all a in $\mathcal{A}_\rho$ we have*

$$\|(h^3 X_1 + h^4 X_2)\,a\|_\rho \leqslant C_1\,\rho^3\,\|a\|_\rho.$$

Proof. — Since X_ℓ is an homogeneous differential operator of order ℓ ($\ell = 1, 2$), the Cauchy formula shows that for $t' < t$

$$\sup_{\Omega_t} |X_\ell f| \leqslant C(t-t')^{-\ell} \sup_{\Omega_{t'}} |f|. \tag{5.32}$$

Now $h^{2+\ell} X_\ell a = \sum_{j\geqslant 0} h^{2+\ell+j} X_\ell a_j = \sum_{j\geqslant 2+\ell} h^j X_\ell a_{j-2-\ell}$. The use of (5.32) shows that

$$\begin{aligned}\sup_{\Omega_t} |X_\ell a_{j-2-\ell}| &\leqslant C(t-t')^{-\ell} \sup_{\Omega_{t'}} |a_{j-2-\ell}| \\ &\leqslant C(t-t')^{-\ell} f_{j-2-\ell}(a)(j-2-\ell)^{\frac12(j-2-\ell)}\, t'^{-\frac12(j-2-\ell)}.\end{aligned}$$

Let us take $t' = \frac{j-2-\ell}{j} t$. Then

$$\begin{aligned}&\sup_{\Omega_t} |X_\ell a_{j-2-\ell}| \\ &\leqslant C t^{-\ell} \frac{(\ell+2)^{-\ell}}{j^{-\ell}} (j-2-\ell)^{\frac12(j-2-\ell)} \frac{(j-2-\ell)^{-\frac12(j-2-\ell)}}{j^{-\frac12(j-2-\ell)}} t^{-\frac12(j-2-\ell)} f_{j-2-\ell}(a).\end{aligned}$$

The right hand side of this inequality can be written as

$$C t^{-j/2} t^{1-\ell/2} j^{j/2} \cdot \frac{1}{(\ell+2)^\ell} \frac{j^\ell}{j^{1+\ell/2}} f_{j-2-\ell}(a).$$

Since $1 \leqslant \ell \leqslant 2$, we have $\frac{1}{(\ell+2)^\ell} \cdot \frac{j^\ell}{j^{1+\ell/2}} \leqslant 1$ and $t^{1-\ell/2} \leqslant 1$ ($t < r \leqslant 1$), so we get

$$\sup_{\Omega_t} |X_\ell a_{j-2-\ell}| \leqslant C f_{j-2-\ell}(a)\, j^{j/2} t^{-j/2}.$$

It follows that $f_j(h^{2+\ell} X_\ell a) \leqslant C f_{j-2-\ell}(a)$, so

$$\|h^{2+\ell} X_\ell a\|_\rho = \sum_{j\geqslant \ell+2} f_j(h^{2+\ell} X_\ell a)\,\rho^j \leqslant C \sum_{j\geqslant \ell+2} f_{j-2-\ell}(a)\,\rho^j \leqslant C\rho^{\ell+2} \sum_{j\geqslant 0} f_j(a)\,\rho^j$$

which proves the lemma. $\square$

Using the Lemmas 5.5 and 5.6, we deduce

$$\|Ba\|_\rho \leqslant C_0 C_1 \rho \|a\|_\rho, \text{ for all } a \in \mathcal{A}_\rho \text{ and } \rho \in]0,1[.$$

Taking ρ small enough we get our conclusion.

5.6. Proof of Theorem 4.6 (continued)

Let us set

$$A = \{\theta \in [\theta_*, \theta^*] : \exp(\theta X_0)(m) \notin {}^{\mathrm{qsc}}WF_a(u(\theta, \cdot))\}. \tag{5.33}$$

If we show that A is open and closed in $[\theta_*, \theta^*]$ then the claim in Theorem 4.6 will follow.

(i) *A is open*

Let $\theta_0 \in A$, that is $m_0 = \exp(\theta_0 X_0)(m) \notin {}^{\mathrm{qsc}}WF_a(u(\theta_0, \cdot))$. We set $m_0 = (0, y_0, \lambda_0, \mu_0)$. Since the definition of ${}^{\mathrm{qsc}}WF_a$ is independent of the phase and the symbol, we may take $a \equiv 1$ and $\varphi = \varphi^0 = \varphi_2^0 + ih\varphi_1^0$ where

$$\begin{cases} \varphi_2^0(s, y, \alpha) = (s - \alpha_s)\alpha_\tau + (y - \alpha_y) \cdot \alpha_\eta \\ \varphi_1^0(s, y, \alpha) = (s - \alpha_s)^2 + (y - \alpha_y)^2, \end{cases} \tag{5.34}$$

and $\alpha_0 = \big(s_0, y_0, \lambda_0/s_0^3, \mu_0/s_0^2\big)$. Then one can find a cut-off function $\chi(s, y)$ equal to one in a neighborhood of (s_0, y_0), a neighborhood V_{α_0} of α_0, strictly positive constants C, c_0, ε_0 such that for all (α, h) in $V_{\alpha_0} \times]0, c_0[$,

$$|\mathcal{T}_0 u(\theta_0; \alpha, h)| \leqslant C e^{-\varepsilon_0/h} \tag{5.35}$$

where

$$\mathcal{T}_0 u(\theta_0; \alpha, h) = \iint e^{ih^{-2}\varphi^0(\rho/h, y, \alpha, h)} \chi\Big(\frac{\rho}{h}, y\Big) \overline{u(\theta_0; \rho, y)}\, d\rho\, dy. \tag{5.36}$$

Let, for $|\theta - \theta_0|$ small enough, $\varphi(\theta; s, y, \alpha, h) = \varphi_2(\theta; s, y, \alpha) + ih\varphi_1(\theta; s, y, \alpha)$ be the phase given by the Lemma 5.4 which, for $\theta = \theta_0$, is equal to φ^0 given in (5.34). Let $a(\theta; s, y, \alpha, h)$ the analytic symbol constructed in § 5.5, which is equal to one for $\theta = \theta_0$. Let $\chi(\theta; s, y)$ be a cut-off which is equal to one in a neighborhood of $X(\theta) = (s(\theta - \theta_0; \alpha_0), y(\theta - \theta_0; \alpha_0))$. Let us set

(5.37)

$$\mathcal{T}u(\theta; t, \alpha, h) = \iint e^{ih^{-2}\varphi(\theta; \rho/h, y, \alpha, h)} a\Big(\theta; \frac{\rho}{h}, y, \alpha, h\Big) \chi\Big(\theta; \frac{\rho}{h}, y\Big) \overline{u(t; \rho, y)}\, d\rho\, dy.$$

It follows from (5.35), (5.36) that

$$|\mathcal{T}u(\theta_0; \theta_0, \alpha, h)| \leqslant C e^{-\varepsilon_0/h}, \quad \forall \alpha \in V_{\alpha_0}, \ \forall h \in]0, c_0[. \tag{5.38}$$

Then we have

Lemma 5.7. — *One can find two smooth functions U and V, $\varepsilon_1 > 0$, $c_1 > 0$ and neighborhoods V_{θ_0}, V_{α_0} such that*

$$\mathcal{T}u(\theta;t,\alpha,h) = U(\theta - t;\alpha,h) + V(\theta;t,\alpha,h) \tag{5.39}$$

$$|V(\theta;t,\alpha,h)| \leqslant C_1 e^{-\varepsilon_1/h}, \quad \forall (\theta,t,\alpha,h) \in V_{\theta_0} \times [0,T] \times V_{\alpha_0} \times]0,c_1[. \tag{5.40}$$

Let us assume this lemma proved ; then, for $|\theta - \theta_0|$ small enough, we can write

$$\mathcal{T}u(\theta;\theta,\alpha,h) = \mathcal{T}u(\theta_0;\theta_0,\alpha,h) + W(\theta,\theta_0,\alpha,h)$$

where W satisfies the estimate (5.40) with a larger C_1. It follows from (5.38) that $\mathcal{T}u(\theta;\theta,\alpha,h)$ satisfies also the same kind of estimate. Therefore by the Definition 2.3, we have $(0, y(\theta-\theta_0), \lambda(\theta-\theta_0), \mu(\theta-\theta_0)) = \exp((\theta-\theta_0)X_0)(m_0) \notin {}^{\mathrm{qsc}}WF_a(u(\theta;\cdot))$ when $|\theta - \theta_0|$ is small enough. Since $m_0 = \exp(\theta_0 X_0)(m)$ it follows that $\theta \in A$ if $\theta \in V_{\theta_0}$, which proves that A is open.

Proof of Lemma 5.7. — First of all we have $|e^{ih^{-2}\varphi}| = e^{-h^{-1}\operatorname{Re}\varphi_1(\theta;\rho/h,y,\alpha)} \leqslant 1$ by Lemma 5.2 (i). Now we take the symbol $a = \sum_{j\leqslant\delta/h^2} a_j(\theta;\rho/h,y,\alpha)\,h^j$, where a_j satisfies (5.24) and δ is small enough. Then by (5.5) and the choice of φ and a we get

$$\Big|\Big(\frac{\partial}{\partial\theta} + i\Delta_g^*\Big)(a e^{ih^{-2}\varphi})\Big| \leqslant M\Big(M\frac{\sqrt{\delta}}{h}h\Big)^{\delta/h^2} = M e^{\frac{\delta}{h^2}\operatorname{Log}(M\sqrt{\delta})} \leqslant e^{-\delta_0/h^2} \tag{5.41}$$

with $\delta_0 > 0$ if $M\sqrt{\delta} < 1$.

It follows from (5.37) that we have,

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathcal{T}u(\theta;t,\alpha,h) = &-\iint i\Delta_g^*(a e^{ih^{-2}\varphi})\chi\Big(\theta;\frac{\rho}{h},y\Big)\overline{u(t;\rho,y)}\,d\rho\,dy\\ &+\iint e^{ih^{-2}\varphi}a\frac{\partial\chi}{\partial\theta}(\cdots)\overline{u(t,\rho,y)}\,d\rho\,dy.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathcal{T}u(\theta;t,\alpha,h) = &\iint e^{ih^{-2}\varphi(\cdots)}a(\cdots)\Big[\frac{\partial}{\partial\theta} - i\Delta_g,\chi\Big]\overline{u(t;\rho,y)}\,d\rho\,dy\\ &+\iint e^{ih^{-2}\varphi}a(\cdots)\chi(\cdots)\overline{i\Delta_g u(t;\rho,y)}\,d\rho\,dy.\end{aligned}$$

Since $i\Delta_g u = -\partial u/\partial t$ we get

$$\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial\theta}\Big)\mathcal{T}u(\theta;t,\alpha,h) = \iint e^{ih^{-2}\varphi(\cdots)}a(\cdots)\Big[\frac{\partial}{\partial\theta}-i\Delta_g,\chi\Big]\overline{u(t;\rho,y)}\,d\rho\,dy. \tag{5.42}$$

We prove now that the integral in the right hand side of the above inequality satisfies an estimate like (5.40). Indeed, on the support of $\big[\frac{\partial}{\partial\theta} - i\Delta_g,\chi\big]$, we have by definition of χ,

$$\varepsilon_1 \leqslant \|X - X(\theta)\| \leqslant 2\varepsilon_1 \tag{5.43}$$

where $X = (s, y)$ and $X(\theta) = (s(\theta - \theta_0; \alpha_0), y(\theta - \theta_0; \alpha_0))$. Moreover

$$\operatorname{Re}\varphi_1(\theta; X, \alpha) = \operatorname{Re}\varphi_1(\theta; X(\theta), \alpha) + \frac{\partial \operatorname{Re}\varphi_1}{\partial X}(\theta; X(\theta), \alpha)(X - X(\theta)) + \frac{1}{2}\frac{\partial^2 \operatorname{Re}\varphi_1}{\partial X^2}(\theta; X(\theta), \alpha)(X - X(\theta))^2 + o(\|X - X(\theta)\|^2).$$

It follows from Lemma 5.2 (ii), that

$$\operatorname{Re}\varphi_1(\theta; X, \alpha) \geqslant c_0 \|X - X(\theta)\|^2. \tag{5.44}$$

Since $|e^{ih^{-2}\varphi}| = e^{-h \operatorname{Re}\varphi_1}$, our claim follows from (5.43), (5.44). Then (5.42) implies

$$\begin{cases} (\partial_t + \partial_\theta)\mathcal{T}u(\theta, t; \alpha, h) = V(\theta, t, \alpha, h) \text{ where} \\ |V(\theta, t, \alpha, h)| \leqslant Ce^{-\varepsilon/h}, \quad (\theta, t, \alpha, h) \in V_{\theta_0} \times [0, T] \times V_{\alpha_0} \times]0, c_0[\end{cases}$$

from which Lemma 5.7 follows. □

(ii) *A is closed*

Let $\theta_1 \in \overline{A}$. For every $\varepsilon > 0$ there exists $\theta_0 \in A$ such that $|\theta_0 - \theta_1| \leqslant \varepsilon$. We take ε so small that θ_1 belongs to the neighborhood of θ_0 where $\varphi(\theta; \cdots)$ $a(\theta; \cdots)$ have been constructed. Then, as above we can write

$$\mathcal{T}u(\theta_1; \theta_1, \alpha, h) = \mathcal{T}u(\theta_0; \theta_0, \alpha, h) + V(\theta_0, \theta_1, \alpha, h)$$

where $V = \mathcal{O}(e^{-\varepsilon/h})$. Since $\theta_0 \in A$, we have $\mathcal{T}u(\theta_0; \theta_0, \alpha, h) = \mathcal{O}(e^{-\delta/h})$ so the above equality shows that $\exp[(\theta_1 - \theta_0)X_0](m_0) = \exp(\theta, X_0)(m)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(\theta_1, \cdot))$ thus $\theta_1 \in A$ and A is closed. The proof of Theorem 4.6 is complete. □

CHAPTER 6

PROOF OF THEOREM 4.7

6.1. Propagation in $({}^{\mathrm{qsc}}S^*M)^0$

In this set, ${}^{\mathrm{qsc}}\widetilde{WF}_a$ coincide with the locally uniform analytic wave front set introduced in [**RZ1**], Definition 1.1. Moreover Theorem 4.7 is of local nature, thus independent of the asymptotic behavior of the metric. Therefore Theorem 4.7 in $({}^{\mathrm{qsc}}S^*M)^0$ will follow from Theorem 6.1 in [**RZ1**] as soon as we show that the flow of σH_Δ described by (3.17) coincide with the bicharacteristic flow of the Laplacian Δ_g described in (3.4).

The principal symbol of Δ_g is equal to $p(\rho, y, \rho^2\tau, \rho\eta)$ where $p(\rho, y, \lambda, \mu) = \lambda^2 + \|\mu\|^2 + \rho r(\rho, y, \lambda, \mu)$ (see (3.4), (3.5)). Therefore the bicharacteristics of Δ_g are described by the equations

$$
(6.1) \qquad \begin{cases} \dot\rho = \rho^2 \dfrac{\partial p}{\partial \lambda}(\rho, y, \rho^2\tau, \rho\eta) \\ \dot y = \rho \dfrac{\partial p}{\partial \mu}(\cdots) \\ \dot\tau = \Big[-\dfrac{\partial p}{\partial \rho} - 2\rho\tau \dfrac{\partial p}{\partial \lambda} - \eta \dfrac{\partial p}{\partial \mu}\Big](\rho, y, \rho^2\tau, \rho\eta) \\ \dot\eta = -\dfrac{\partial p}{\partial y}(\cdots). \end{cases}
$$

Then we have the following result.

Lemma 6.1. — *Let* $m_0 = (\rho_0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0)) \in ({}^{\mathrm{qsc}}S^*M)^0$, $\rho_0 > 0$. *We set* $\tau_0 = \overline{\lambda}_0/\rho_0^3$, $\eta_0 = \overline{\mu}_0/\rho_0^2$. *Let* $(\rho(t), y(t), \tau(t), \eta(t))$ *be the bicharacteristic of* Δ_g *starting at* $(\rho_0, y_0, \tau_0, \eta_0)$. *Then* $\rho(t) \neq 0$ *for all* t. *Let* $\chi(t)$ *be the solution of the problem* $\dot{\chi}(t) = \rho_0/\rho(\chi(t))$, $\chi(0) = 0$. *Then*

$$(\overline{\rho}(t) = \rho(\chi(t)), \overline{y}(t) = y(\chi(t)), \overline{\lambda}(t) = \rho_0(\tau\rho^2)(\chi(t)), \overline{\mu}(t) = \rho_0(\rho\eta)(\chi(t)))$$

is the flow of σH_Δ *described in (3.17).*

The proof of this lemma is a straightforward computation.

6.2. Propagation on the corner

Let $m = (0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0))$ be a point of the corner. We take $s_0 > 0$ and we set $\alpha_0 = \big(s_0, y_0, \overline{\lambda}_0/s_0^3, \overline{\mu}_0/s_0^2\big)$. Let $\theta_0 \in \mathbb{R}$. Here we look for a phase φ and a symbol a such that for some $\varepsilon > 0$,

$$A = \Big(\frac{1}{k}\frac{\partial}{\partial\theta} + i\Delta_g^*\Big)\Big[e^{ih^{-2}k^{-1}\varphi(\theta;\rho/h,y,\alpha,h)}\, a\Big(\theta; \frac{\rho}{h}, y, \alpha, h, k\Big)\Big] = \mathcal{O}(e^{-\varepsilon/hk}) \tag{6.2}$$

for $\big(\theta, \rho/h, y, \alpha\big)$ in a complex neighborhood of $(\theta_0, s_0, y_0, \alpha_0)$ and (h, k) in a neighborhood of $(0,0)$ in $]0, +\infty[\, \times\,]0, +\infty[$.

Setting $s = \rho/h$ we see easily that

$$A = e^{ih^{-2}k^{-1}\varphi}(I + II) \tag{6.3}$$

where

$$I = h^{-2}k^{-2}\Big(\frac{\partial\varphi}{\partial\theta} + p\Big(sh, y, s^2\frac{\partial\varphi}{\partial s}, s\frac{\partial\varphi}{\partial y}\Big)\Big)a \tag{6.4}$$

$$\begin{cases} II = \dfrac{1}{k}\left(\mathcal{L}a + i(\Delta_g^*\varphi)a + ih^2k\Delta_g^* a\right) \text{ with} \\ \mathcal{L} = \dfrac{\partial}{\partial\theta} + s^2\dfrac{\partial p}{\partial\lambda}\Big(sh, y, s^2\dfrac{\partial\varphi}{\partial s}, s\dfrac{\partial\varphi}{\partial y}\Big)\dfrac{\partial}{\partial s} + s\dfrac{\partial p}{\partial\mu}(\cdots)\dfrac{\partial}{\partial y}. \end{cases} \tag{6.5}$$

6.2.1. Resolution of the phase equation

Proposition 6.2. — *There exists a holomorphic function* $\varphi = \varphi(\theta; s, y, \alpha, h)$ *in a complex neighborhood of* $(\theta_0, s_0, y_0, \alpha_0)$ *depending smoothly on* h *such that*

$$\begin{cases} \dfrac{\partial\varphi}{\partial\theta} + p\Big(sh, y, s^2\dfrac{\partial\varphi}{\partial s}, s\dfrac{\partial\varphi}{\partial y}\Big) = 0 \\ \varphi|_{\theta=\theta_0} = (X - \alpha_X)\alpha_\Xi + ih(X - \alpha_X)^2 \end{cases}$$

where $X = (s, y)$, $\alpha = (\alpha_X, \alpha_\Xi)$, $\alpha_X = (\alpha_s, \alpha_y)$, $\alpha_\Xi = (\alpha_\tau, \alpha_\eta)$.

Proof. — We introduce the symbol $q = \theta^* + p(sh, y, s^2\tau, s\eta)$ and for fixed α, h we consider the bicharacteristic system of q which is given by the equations

$$
(6.6)\quad
\begin{cases}
\dot\theta(t) = 1 & \theta(0) = \theta_0 \\
\dot s(t) = s^2 \dfrac{\partial p}{\partial \lambda}(sh, y, s^2\tau, s\eta) & s(0) = \widetilde{s} \\
\dot y(t) = s\,\dfrac{\partial p}{\partial \mu}(\cdots) & y(0) = \widetilde{y} \\
\dot\theta^*(t) = 0 & \theta^*(0) = -p(\widetilde{s}h, \widetilde{y}, \widetilde{s}^2\widetilde{\tau}, \widetilde{s}\widetilde{\eta}) \\
\dot\tau(t) = -\Big[h\dfrac{\partial p}{\partial \rho} + 2s\tau\dfrac{\partial p}{\partial \lambda} + \eta\dfrac{\partial p}{\partial \mu}\Big](\cdots) & \tau(0) = \widetilde{\tau} = \alpha_\tau + 2ih(\widetilde{s} - \alpha_s) \\
\dot\eta(t) = -\dfrac{\partial p}{\partial y}(\cdots) & \eta(0) = \widetilde{\eta} = \alpha_\eta + 2ih(\widetilde{y} - \alpha_y).
\end{cases}
$$

Here t is complex and $(\widetilde{s}, \widetilde{y})$ are taken in a neighborhood of (s_0, y_0) in $\mathbb{C} \times \mathbb{C}^{n-1}$. By the Cauchy-Lipschitz theorem this system has, for small t, a unique holomorphic solution which depends holomorphically on the initial data. Since $\theta(t) = t + \theta_0$ we can set $t + \theta_0 = \theta$ and we consider

$$
(6.7)\quad \Lambda = \Big\{\big(\theta; s(\theta - \theta_0; \widetilde{s}, \widetilde{y}, \alpha, h), y(\theta - \theta_0; \cdots), \theta^*(0), \tau(\theta - \theta_0; \widetilde{s}, \widetilde{y}, \alpha, h), \eta(\theta - \theta_0; \cdots)\big),\ (\widetilde{s}, \widetilde{y}) \text{ near } (s_0, y_0)\Big\}.
$$

Then Λ is a Lagrangian manifold on which q vanishes. Moreover we see from the equations (6.6) that the projection π from Λ to the basis is a local diffeomorphism. Therefore one can find $\varphi = \varphi(\theta; s, y, \alpha, h)$ such that

$$
(6.8)\quad \Lambda = \Big\{\Big(\theta, s, y; \frac{\partial \varphi}{\partial \theta}(\theta; s, y, \alpha, h), \frac{\partial \varphi}{\partial s}(\cdots), \frac{\partial \varphi}{\partial y}(\cdots)\Big),\ (\theta, s, y) \text{ in a neighborhood of } (\theta_0, s_0, y_0)\Big\}.
$$

Since q vanishes on Λ, the function φ solves the equation in Proposition 6.2. However one can add to φ any constant without changing Λ. We shall take the constant such that

$$
(6.9)\quad \varphi(\theta_0; s_0, y_0, \alpha, h) = (s_0 - \alpha_s)\alpha_\tau + (y_0 - \alpha_y)\cdot\alpha_\eta + ih[(s_0 - \alpha_s)^2 + (y_0 - \alpha_y)^2].
$$

Let us show then that φ satisfies also the initial condition. We can write

$$
\varphi(\theta_0; s, y, \alpha, h) = \varphi(\theta_0; s_0, y_0, \alpha, h) + \int_0^1 \Big[(s - s_0)\frac{\partial \varphi}{\partial s}(\theta_0; ts + (1-t)s_0, ty + (1-t)y_0, \alpha, h) + (y - y_0)\cdot\frac{\partial \varphi}{\partial y}(\theta_0; \cdots)\Big]\,dt.
$$

It follows from (6.8), (6.7), (6.6) that

$$\varphi(\theta_0; s, y, \alpha, h) = \varphi(\theta_0; s_0, y_0, \alpha, h) + \int_0^1 \Big[(s - s_0)\big(\alpha_\tau + 2ih(ts + (1-t)s_0 - \alpha_s)\big) \\ +(y - y_0)\cdot\big(\alpha_\eta + 2ih(ty + (1-t)y_0 - \alpha_y)\big)\Big]\, dt$$

$$\varphi(\theta_0; s, y, \alpha, h) = \varphi(\theta_0; s_0, y_0, \alpha, h) + (s - s_0)\alpha_\tau + (y - y_0)\cdot\alpha_\eta \\ +2ih\Big[\frac{1}{2}(s - s_0)^2 + (s - s_0)(s_0 - \alpha_s) + \frac{1}{2}(y - y_0)^2 + (y - y_0)\cdot(y_0 - \alpha_y)\Big].$$

Using (6.9), we deduce that

$$\varphi(\theta_0; s, y, \alpha, h) = (s - \alpha_s)\alpha_\tau + (y - \alpha_y)\cdot\alpha_\eta + ih\big[(s - \alpha_s)^2 + (y - \alpha_y)^2\big]$$

which is the initial condition in Proposition 6.2. □

6.2.2. $\varphi(\theta, \cdot)$ is a phase. — Let us show now that $\varphi(\theta; s, y, \alpha, h)$ is a phase in the sense of Definition 2.6 at $(X(\theta), \Xi(\theta), \alpha, h_0 = 0)$ (independent of ν) where

$$X(\theta) = \big(s(\theta - \theta_0; s_0, y_0, \alpha_0, 0), y(\theta - \theta_0; \cdots)\big),$$
$$\Xi(\theta) = \big(\tau(\theta - \theta_0; s_0, y_0, \alpha_0, 0),\ \eta(\theta - \theta_0; \cdots)\big).$$

We set

$$\varphi(\theta; s, y, \alpha, h) = \psi_2(\theta; s, y, \alpha) + ih\psi_1(s, y, \alpha) + h^2\psi_0(s, y, \alpha, h)$$

and

$$F(h) = p\Big(sh, y, s^2\frac{\partial\varphi}{\partial s}, s\frac{\partial\varphi}{\partial y}\Big).$$

Then writing $F(h) = F(0) + hF'(0) + h^2G(h)$ and using Proposition 6.2, we see that ψ_2 satisfies the equation

$$\frac{\partial\psi_2}{\partial\theta} + s^4\Big(\frac{\partial\psi_2}{\partial s}\Big)^2 + s^2\Big\|\frac{\partial\psi_2}{\partial y}\Big\|^2 = 0.$$

Thus ψ_2 is real if (s, y, α) are real. Moreover by (6.7), (6.8),

$$\frac{\partial\psi_2}{\partial X}(\theta, X(\theta), \alpha_0) = \frac{\partial\varphi}{\partial X}\big(\theta; s(\theta - \theta_0; \cdots), y(\theta - \theta_0, \cdots), \alpha_0, 0\big) = \Xi(\theta).$$

On the other hand ψ_1 satisfies the equation

$$\begin{cases} \mathcal{L}\psi_1 = is\dfrac{\partial p}{\partial\rho}\Big(0, y; s^2\dfrac{\partial\psi_2}{\partial s}, s\dfrac{\partial\psi_2}{\partial y}\Big) \\ \psi_1|_{\theta=\theta_0} = (X - \alpha_X)^2. \end{cases}$$

It follows that $\mathcal{L}\,\mathrm{Re}\,\psi_1 = 0$ and $\mathrm{Re}\,\psi_1|_{\theta=\theta_0} = (X - \alpha_X)^2$. Working in the coordinates $(\theta, \widetilde{s}, \widetilde{y})$ as in Lemma 5.2, the vector field $\mathcal{L}$ becomes $\partial/\partial\theta$. It follows that

$$\mathrm{Re}\,\psi_1\big(\theta; s(\theta - \theta_0; \widetilde{s}, \widetilde{y}, \alpha, 0), y(\theta - \theta_0; \cdots), \alpha\big) = (\widetilde{s} - \alpha_s)^2 + (\widetilde{y} - \alpha_y)^2$$

which shows that $\mathrm{Re}\,\psi_1 \geqslant 0$ if (s, y, α) are real. Finally,

$$\mathrm{Re}\,\psi_1(X(\theta), \alpha_0) = (s_0 - s_0)^2 + (y_0 - y_0)^2 = 0.$$

6.2.3. Resolution of the transport equation. — We look for a symbol a of the form

$$a(\theta; s, y, \alpha, h, k) = \sum_{j\geqslant 0} a_j(\theta; s, y, \alpha, h)(h\sqrt{k})^j$$

where the a_j's satisfy the following estimates

$$|a_j(\theta; s, y, \alpha, h)| \leqslant M^{j+1} j^{j/2}.$$

If, instead of working in the (θ, s, y) variables we shift to the new variables $(\theta, \widetilde{s}, \widetilde{y})$, where $s(\theta - \theta_0; \widetilde{s}, \widetilde{y}, \alpha, h) = s$ and $y(\theta - \theta_0; \widetilde{s}, \widetilde{y}, \alpha, h) = y$, the operator $\mathcal{L}$ becomes $\partial/\partial\theta$. Therefore solving the equation $II = 0$ in (6.5) is equivalent to solve

$$\begin{cases} \Big[\dfrac{\partial}{\partial\theta} + c(\theta; \widetilde{s}, \widetilde{y}, \alpha)\Big] b + (h\sqrt{k})^{-2}\big((h\sqrt{k})^4 P_2\big) b = 0 \\ b|_{\theta=\theta_0} = 1 \end{cases}$$

where P_2 is a second order differential operator. Then the same argument as used in [**Sj**] or in the proof of Theorem 4.6 ensures the existence of such a symbol.

Now, before giving the proof of Theorem 4.7, we must link the flow of σH_Δ with the bicharacteristic of p described in (6.1).

Proposition 6.3. — *Let $m_0 = (0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0))$ be a point in the corner. Let $s_0 > 0$ and set $\tau_0 = \overline{\lambda}_0/s_0^3$, $\eta_0 = \overline{\mu}_0/s_0^2$. Let $(s(t), y(t), \tau(t), \eta(t))$ be the bicharacteristic of the symbol $p(0, y, s^2\tau, s\eta)$ issued from $(s_0, y_0, \tau_0, \eta_0)$. Then $s(t) \neq 0$ for all t. Let $\chi(t)$ be the solution of the problem $\dot{\chi}(t) = s_0/s(\chi(t))$, $\chi(0) = 0$. Then*

$$(\overline{\rho}(t) = 0, \overline{y}(t) = y(\chi(t)), \overline{\lambda}(t) = s_0(\tau s^2)(\chi(t)), \overline{\mu}(t) = s_0(s\eta)(\chi(t))$$

is the flow of σH_Δ (described in (3.17)) through m_0.

Proof. — This is a straightforward computation. □

6.2.4. Proof of Theorem 4.7. — Let us introduce the set

$$A = \big\{\theta \in [\theta_*, \theta^*] : \exp(\theta\sigma H_\Delta)(m) \notin {}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0, \cdot))\big\}. \tag{6.10}$$

If we show that A is open and closed in $[\theta_*, \theta^*]$ we are done. A is open because ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0, \cdot))$ is closed. It remains to prove that A is closed. Let (θ_n) be a sequence in A which converges to some $\theta_0 \in \mathbb{R}$. Let us set $\exp(\theta_0\sigma H_\Delta)(m) = m_0 = (0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0))$. Let V_{θ_0} be an open neighborhood of θ_0 in $\mathbb{R}$ in which the phase $\varphi(\theta; s, y, \alpha, h)$ given by Proposition 6.2 and the symbol solving the transport equations exist. Let γ be the solution of the problem $\dot{\gamma}(t) = s_0/s(\gamma(t))$, $\gamma(0) = 0$ introduced in Proposition 6.3. Then one can find θ_n such that $\widetilde{\theta}_0 = \theta_0 + \gamma(\theta_n - \theta_0) \in V_{\theta_0}$ and we fix it. Now let us set, for θ in V_{θ_0},

(6.11)

$$\mathcal{T}u(\theta; t, \alpha, h, k) = \iint e^{ih^{-2}k^{-1}\varphi(\theta; X_h, \alpha, h)} a(\theta; X_h, \alpha, h, k)\chi(\theta; X_h) u(t; \rho, y)\, dy\, d\rho$$

where $X_h = (\rho/h, y)$ and $\chi(\theta;\cdot)$ is a cut-off function localizing at

$$X(\theta) = \big(s(\theta-\theta_0; s_0, y_0, \alpha_0),\, y(\theta-\theta_0; s_0, y_0, \alpha_0)\big),$$

$s_0 > 0$, $\alpha_0 = \big(s_0, y_0, \overline{\lambda}_0/s_0^3, \overline{\mu}_0/s_0^2\big)$.

Lemma 6.4. — *One can find two smooth functions U, V, complex neighborhoods W_{θ_0}, W_{α_0} of θ_0, α_0, positive constants C, ε_0, δ_0, δ_1 such that*

$$\text{(6.12)}\qquad \begin{cases} \mathcal{T}u(\theta; t, \alpha, h, k) = U(k\theta - t; \alpha, h, k) + V(\theta; t, \alpha, h, k) \\ |V(\theta; t, \alpha, h, k)| \leqslant Ce^{-\varepsilon_0/hk} \end{cases}$$

for all $(\theta; t, \alpha, h, k)$ in $W_{\theta_0} \times]t_0 - \delta_0, t_0 + \delta_0[\times W_{\alpha_0} \times]0, \delta_1[\times]0, \delta_1[$.

Proof. — It is very similar to that of Lemma 5.7 so we only sketch it. It follows from (6.3), (6.4), (6.5), Proposition 6.2 and from the construction of the symbol that (6.2) is true. It follows that $\mathcal{T}u$ satisfies

$$\Big(\frac{1}{k}\frac{\partial}{\partial\theta} + \partial_t\Big)\mathcal{T}u(\theta; t, \alpha, h, k)$$
$$= \iint e^{ih^{-2}k^{-1}\varphi(\cdots)} a(\cdots)\Big[\frac{1}{k}\frac{\partial}{\partial\theta} - i\Delta_g, \chi\Big](\cdots)\overline{u(t;\rho,y)}\,d\rho\,dy + V_1$$

where V_1 is a smooth function satisfying the estimate in (6.12). Then we use the properties of the phase φ on the support of $\big[\frac{1}{k}\frac{\partial}{\partial\theta} - i\Delta_g, \chi\big]$ to achieve the proof. □

It follows from Lemma 6.4 that

$$\text{(6.13)}\qquad \mathcal{T}u(\theta_0; t, \alpha, h, k) = \mathcal{T}u\big(\widetilde{\theta}_0; t - k(\theta_0 - \widetilde{\theta}_0), \alpha, h, k\big) + V_2$$

where V_2 satisfies the estimate in (6.12).

Let us check at what point does $\mathcal{T}u(\widetilde{\theta}_0;\cdots)$ microlocalize. By (6.7) and (6.8) we have, with $X = (s, y)$,

$$\frac{\partial\varphi}{\partial X}\big(\widetilde{\theta}_0; s(\widetilde{\theta}_0 - \theta_0; s_0, y_0, \alpha_0, 0), y(\widetilde{\theta}_0 - \theta_0; \cdots), \alpha_0, 0\big)$$
$$= \big(\tau(\widetilde{\theta}_0 - \theta_0; s_0, y_0, \alpha_0, 0), \eta(\widetilde{\theta} - \theta_0; \cdots)\big).$$

Since $\widetilde{\theta}_0 - \theta_0 = \gamma(\theta_n - \theta_0)$, setting $\overline{s} = s \circ \gamma$ and using Proposition 6.3 we see that

$$\frac{\partial\varphi}{\partial X}\big(\widetilde{\theta}_0; \overline{s}(\theta_n - \theta_0), \overline{y}(\theta_n - \theta_0), \alpha_0, 0\big) = \frac{\overline{s}(\theta_n - \theta_0)}{s_0}\Big(\frac{\overline{\lambda}(\theta_n - \theta_0)}{\overline{s}^3(\theta_n - \theta_0)}, \frac{\overline{\mu}(\theta_n - \theta_0)}{\overline{s}^2(\theta_n - \theta_0)}\Big).$$

Since $\theta_n \in A$ and $\widetilde{\varphi} = \frac{s(\theta_n - \theta_0)}{s_0}\varphi$ is a phase in the sense of Definition 2.6, it follows from Definition 2.4 that $\exp((\theta_n - \theta_0)\sigma H_\Delta)(m_0) = \exp(\theta_n \sigma H_\Delta)(m)$ does not belong to ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot))$. Therefore, taking k small enough, the right hand side of (6.13) is bounded by $e^{-\varepsilon_0/hk}$ uniformly for t in $]t_0 - \delta, t_0 + \delta[$. Therefore the left hand side has the same bound, which proves that $\theta_0 \in A$, so A is closed.

CHAPTER 7

PROOF OF THEOREM 4.8

Let $m_0 = (0, y_0, 0, (-1, 0)) \in \mathcal{N}_-^c$. We take $s_0 > 0$ and we set $\alpha_0 = (s_0, y_0, -1/s_0^3, 0)$. The scheme of the proof is the same as in Theorem 4.7. We look for a phase φ and a symbol a such that (6.2) holds.

In this case we have to study in particular the flow starting from a real point $(\widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta)$ on the interval $]-T^*, 0]$ where $-T^*$ looks like $1/2\alpha_\tau \widetilde{s}^3$. The problem then is that the solution $s(\theta)$ blows up at $\theta = -T^*$. This forces us to stay slightly far from $-T^*$ at a distance KH, see Theorem 7.1 below, where K is a large constant and $H = h + |\alpha_\eta|$. Then we will have to control (with respect to K) precisely all the quantities which may blow up at $-T^*$. This is a kind of renormalization. In the case of the flat Laplacian it is easy to see that

$$s(\theta; \widetilde{s}, \widetilde{y}, \alpha_\tau, 0) = \frac{\widetilde{s}}{1 - 2\alpha_\tau \widetilde{s}^3 \theta}.$$

For fixed $(\widetilde{y}, \alpha_\tau)$, the map $(\theta, \widetilde{s}) \mapsto (\theta, s(\theta; \widetilde{s}, \widetilde{y}, \alpha_\tau, 0))$ is a diffeomorphism from O_1 to O_2 (see fig. 1, 2).

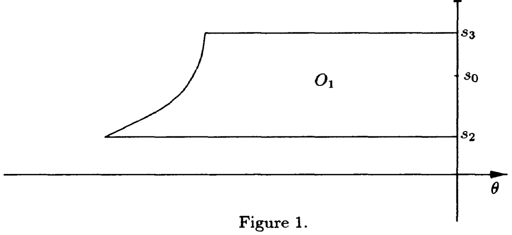


Figure 1.

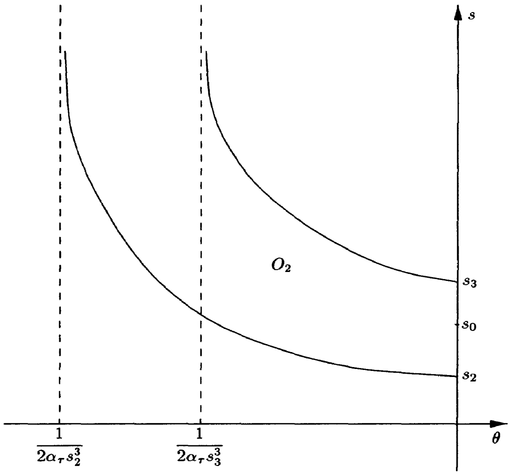


Figure 2.

7.1. The phase equation

Let us set

$$(7.1)\qquad \begin{cases} I_\alpha = \{\alpha \in \mathbb{R}^{2n} : |\alpha - \alpha_0| < \varepsilon_\alpha\}, \\ I_h = \{h \in \mathbb{R} : 0 < h < \varepsilon_h\}, \\ H = h + |\alpha_\eta|, \text{ if } (\alpha, h) \in I_\alpha \times I_h. \end{cases}$$

Theorem 7.1. — *There exist positive constants ε_α, ε_h, ε_s, K, K' such that for (α, h) in $I_\alpha \times I_h$, if we set*

$$D_1 = \{(\theta_1, \widetilde{s}_1) \in \mathbb{R}_- \times \mathbb{R}_+ : |\widetilde{s}_1 - s_0| < \varepsilon_s, Q_1 \doteqdot 1 - 2\alpha_\tau \widetilde{s}_1^3 \theta_1 > KH\}$$

and

$$E = \bigcup_{(\theta_1, \widetilde{s}_1) \in D_1} \Big\{(\theta, s, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1} : |\theta - \theta_1| < \frac{1}{K'} Q_1, \Big|s - \frac{\widetilde{s}_1}{Q_1}\Big| < \frac{1}{K'} \frac{1}{Q_1}, |y - y_1| < \frac{1}{K'}\Big\},$$

(where $y_1 \in \mathbb{C}^n$ is a certain point depending on $(\theta_1, \widetilde{s}_1, y_0, \alpha, h)$ defined in (7.16)), one can find a function $\varphi = \varphi(\theta; s, y, \alpha, h)$ holomorphic in (θ, s, y) in E depending smoothly on (α, h) in $I_\alpha \times I_h$ such that

$$(7.2)\qquad \begin{cases} \dfrac{\partial \varphi}{\partial \theta} + p\Big(hs, y, s^2 \dfrac{\partial \varphi}{\partial s}, s \dfrac{\partial \varphi}{\partial y}\Big) = 0 \\ \varphi|_{\theta=0} = (s - \alpha_s)\alpha_\tau + (y - \alpha_y)\alpha_\eta + ih\big[(s - \alpha_s)^2 + (y - \alpha_y)^2\big]. \end{cases}$$

Proof. — Let $q = \theta^* + p(hs, y, \tau s^2, s\eta)$ and let us consider the bicharacteristic system for q,

$$(7.3)\qquad \begin{cases} \dot\theta(t) = 1 & \theta(0) = 0 \\ \dot s(t) = s^2 \dfrac{\partial p}{\partial \lambda}(hs, y, \tau s^2, s\eta) & s(0) = \widetilde{s} \\ \dot y(t) = s \dfrac{\partial p}{\partial \mu}(\cdots) & y(0) = \widetilde{y} \\ \dot\theta^*(t) = 0 & \theta^*(0) = -p(h\widetilde{s}, \widetilde{y}, \widetilde{\tau}\widetilde{s}^2, \widetilde{s}\widetilde{\eta}) \\ \dot\tau(t) = -\Big[h \dfrac{\partial p}{\partial \rho} + 2s\tau \dfrac{\partial p}{\partial \lambda} + \eta \dfrac{\partial p}{\partial \mu}\Big](\cdots) & \tau(0) = \widetilde{\tau} \\ \dot\eta(t) = -\dfrac{\partial p}{\partial y}(\cdots) & \eta(0) = \widetilde{\eta}. \end{cases}$$

Then obviously $\theta(t) = t$, for all t and $\theta^*(t) = \theta^*(0)$. Therefore we shall take θ instead of t as parameter on the bicharacteristic.

Proposition 7.2. — *There exist positive constants ε_h, ε_s, ε_y, δ_0, K, M, such that for all (α, h) in $I_\alpha \times I_h$ all $(\theta_1, \widetilde{s}_1)$ in D_1 and all $(\widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$ such that*

$$(7.3)' \quad |\widetilde{s} - s_0| \leqslant \varepsilon_s, \quad |\widetilde{s} - \widetilde{s}_1| < \delta_0 Q_1, \quad |\widetilde{y} - y_0| < \varepsilon_y, \quad |\widetilde{\tau} - \alpha_\tau| < \frac{1}{K} Q_1, \quad |\widetilde{\eta}| < \frac{1}{K} Q_1,$$

the system (7.3) has an unique solution defined for $\operatorname{Re}\theta \in \left[\theta_1 - \frac{1}{K} Q_1, 0\right]$, $|\operatorname{Im}\theta| < \frac{1}{K} Q_1$, *which satisfies*

$$|s(\theta)| \leqslant \frac{2s_0}{1 - 2\alpha_\tau \widetilde{s}_1^3 \operatorname{Re}\theta}, \quad |y(\theta) - \widetilde{y}| \leqslant \frac{M}{K}$$

$$|(\tau s^2)(\theta) - \widetilde{\tau}\widetilde{s}^2| \leqslant \frac{1}{K}, \quad |\eta(\theta)| < \frac{2}{K} Q_1.$$

Moreover

$$\frac{1}{s(\theta)} = \frac{1}{\widetilde{s}} - 2\widetilde{\tau}\widetilde{s}^2\theta + F \quad \textit{where } |F| \leqslant \frac{C}{K^2} Q_1.$$

On the other hand the solution $(s(\theta; \widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta}, h)$, $y(\theta; \cdots)$, $\tau(\theta; \cdots)$, $\eta(\theta; \cdots))$ is holomorphic with respect to $(\theta; \widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$ in the set,

$$\Delta = \bigcup_{(\theta, \widetilde{s}_1) \in D_1} \Big\{ |\widetilde{s} - s_1| < \delta_0 Q_1, |\widetilde{y} - y_0| < \varepsilon_y, |\widetilde{\tau} - \alpha_\tau| < \frac{1}{K} Q_1, |\widetilde{\eta}| < \frac{1}{K} Q_1,$$
$$\operatorname{Re}\theta \in \Big[\theta_1 - \frac{1}{K} Q_1, 0\Big], |\operatorname{Im}\theta| < \frac{1}{K} Q_1 \Big\}.$$

Proof. — We begin with the case where θ is real. The existence of a small $T > 0$ for which (7.3) has a solution on $[-T, 0]$ satisfying the estimates in the proposition follows from the Cauchy-Lipschitz theorem. Let $-T_*(\widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$ be the maximal time for which this solution exists and satisfies the estimates.

Case 1. — For any data $(\widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$, we have $-T_*(\cdots) \leqslant \theta_1 - \frac{1}{K} Q_1$. Then the proposition is proved.

Case 2. — Assume there is a data $(\widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$ for which one has $\theta_1 - \frac{1}{K} Q_1 < -T_*(\cdots)$ and let $T > 0$ be such that $-T_*(\cdots) < -T$. Then on $[-T, 0]$ we have a solution of (7.3) which satisfies the above estimates. It follows that $|y(\theta)|$, $|\tau(\theta) s^2(\theta)|$, $|\eta(\theta)|$ are bounded by constants depending only on (s_0, y_0).

For any integer $p \geqslant 2$ and any θ in $[-T, 0]$, we have

$$(7.4) \qquad \int_\theta^0 |s(\sigma)|^p \, d\sigma \leqslant \frac{C(s_0, y_0)}{(1 - 2\alpha_\tau \widetilde{s}_1^3 \theta)^{p-1}}.$$

Since $\theta > \theta_1 - \frac{1}{K} Q_1$ it follows that

$$1 - 2\alpha_\tau \widetilde{s}_1^3 \theta > 1 - 2\alpha_\tau \widetilde{s}_1^3 \theta_1 + \frac{2\alpha_\tau \widetilde{s}_1^3}{K} Q_1 \geqslant \Big(1 - \frac{C}{K}\Big) Q_1 \geqslant \frac{1}{2} Q_1$$

if K is large. Therefore

$$(7.5) \qquad |s(\theta)| \leqslant \frac{4s_0}{Q_1}, \quad \int_\theta^0 |s(\sigma)|^p \, d\sigma \leqslant \frac{C}{Q_1^{p-1}}, \quad \text{if } p \geqslant 2.$$

On the other hand, since $H = h + |\alpha_\eta|$ and $Q_1 > KH$, we have

$$h|s(\theta)| \leqslant \frac{4s_0}{K}, \quad |s(\theta)| \cdot |\eta(\theta)| \leqslant \frac{8s_0}{K}. \tag{7.6}$$

Now it follows from (7.3) that

$$|\eta(\theta)| \leqslant |\widetilde{\eta}| + \int_\theta^0 \left|\frac{\partial p}{\partial y}\right| d\sigma$$

and it is easy to see that

$$\left|\frac{\partial p}{\partial y}\right| \leqslant C\big(s^2|\eta|^2 + s^2h^2(\tau s^2) + sh(\tau s^2)\,s|\eta| + shs^2|\eta|^2\big),$$

where C depends only on a bound of the coefficients of p.

Using the fact that τs^2 is bounded, the estimate $|\eta(\theta)| \leqslant \frac{2}{K}Q_1$ and (7.6), we see that

$$\left|\frac{\partial p}{\partial y}\right| \leqslant \frac{C_1}{K^2}Q_1^2 s^2 + \frac{C_2}{K}Q_1 s^2|\eta|.$$

It follows from (7.5) that

$$|\eta(\theta)| \leqslant |\widetilde{\eta}| + \frac{C_3}{K^2}Q_1 + \frac{C_2}{K}Q_1\int_\theta^0 s^2(\sigma)\,|\eta(\sigma)|\,d\sigma.$$

We can use Gronwall's inequality and (7.5) to get

$$|\eta(\theta)| \leqslant \frac{1}{K}\Big(1 + \frac{C_3}{K}\Big)Q_1 \exp\frac{C_4}{K}$$

since $|\widetilde{\eta}| \leqslant \frac{1}{K}Q_1$. Taking K so large that $\big(1 + \frac{C_3}{K}\big)e^{C_4/K} \leqslant 3/2$, we deduce that

$$|\eta(\theta)| \leqslant \frac{3}{2}\frac{1}{K}Q_1, \quad \theta \in [-T, 0]. \tag{7.7}$$

Let us now estimate τs^2. From (7.3), we get

$$\frac{d}{d\theta}(\tau s^2) = \dot{\tau}s^2 + 2\tau s\dot{s} = -hs^2\frac{\partial p}{\partial \rho}(\cdots) - s^2\eta\cdot\frac{\partial p}{\partial \mu}(\cdots).$$

Since τs^2 is bounded, it follows from (3.5) that

$$\left|\frac{\partial p}{\partial \rho}(\cdots)\right| + \left|\frac{\partial p}{\partial \mu}(\cdots)\right| \leqslant C(h + |\eta(\theta)|)\,s(\theta).$$

Using (7.7) and the fact that $Q_1 > KH \geqslant Kh$ we see that the right hand side is bounded by $\frac{C}{K^2}Q_1^2 s^3$. It follows that

$$|(\tau s^2)(\theta) - \widetilde{\tau}\widetilde{s}^2| \leqslant \frac{C}{K^2}Q_1^2\int_\theta^0 s^3(\sigma)\,d\sigma$$

which implies, using (7.5), that

$$|(\tau s^2)(\theta) - \widetilde{\tau}\widetilde{s}^2| \leqslant \frac{C'}{K^2} \leqslant \frac{1}{2K}, \quad \theta \in [-T, 0]. \tag{7.8}$$

Now $\dot{y} = s\frac{\partial p}{\partial \mu}$; since $\left|s\frac{\partial p}{\partial \mu}(\cdots)\right| \leqslant \frac{C}{K}Q_1 s^2$, where C depends only on the data (α_0, the coefficient of p ...), using (7.5) we get,

$$(7.9) \qquad |y(\theta) - \widetilde{y}| \leqslant \frac{C'}{K} \leqslant \frac{1}{2}\frac{M}{K},$$

if $M \geqslant 2C'$.

Finally, we consider $s(\theta)$. We have $\dot{s}(\theta) = s^2\frac{\partial p}{\partial \lambda}(sh, y, \tau s^2, s\eta)$ from which we deduce that

$$\frac{\dot{s}(\theta)}{s^2(\theta)} = \big(2\tau s^2 + a(sh,y)\,h^2\,s^2(\tau s^2) + h\,s^2\,b(sh,y)\cdot\eta)\big)(\theta).$$

It follows that

$$-\frac{1}{\widetilde{s}}+\frac{1}{s(\theta)} = -2\theta\widetilde{\tau}\widetilde{s}^2-2\int_\theta^0\int_\sigma^0 \frac{d}{d\theta}\underbrace{(\tau s^2)(x)\,dx\,d\sigma}_{(1)} + \int_\theta^0 \underbrace{[ah^2\,s^2(\tau s^2) + h\,s^2 b\cdot\eta](\sigma)\,d\sigma}_{(2)}.$$

Using (7.5) we see that

$$|(2)| \leqslant \frac{CQ_1}{K^2}.$$

We have seen in the proof of (7.8) that

$$\left|\frac{d}{d\theta}(\tau s^2)\right| \leqslant \frac{CQ_1^2}{K^2}s^3,$$

so using (7.5) twice, we see that

$$|(1)| \leqslant \frac{CQ_1}{K^2}.$$

Therefore we can write

$$(7.10) \qquad \begin{cases} \dfrac{1}{s(\theta)} = \dfrac{1}{\widetilde{s}} - 2\widetilde{\tau}\widetilde{s}^2\theta + F(\theta) \\ |F(\theta)| \leqslant \dfrac{CQ_1}{K^2}. \end{cases}$$

It follows that we can write

$$(7.10)' \qquad s(\theta) = \frac{\widetilde{s}}{1 - 2\widetilde{\tau}\widetilde{s}^3\theta + \widetilde{s}F(\theta)}.$$

Now we have

$$|1 - 2\widetilde{\tau}\widetilde{s}^3\theta + \widetilde{s}F| \geqslant 1 - 2\alpha_\tau\widetilde{s}_1^3\theta - C\Big(\frac{1}{K} + \delta_0\Big)Q_1$$

$$1 - 2\alpha_\tau\widetilde{s}_1^3\theta \geqslant \Big(1 - \frac{C}{K}\Big)Q_1.$$

So we can write

$$|s(\theta)| \leqslant \frac{|\widetilde{s}|(1-\frac{C}{K})}{1 - C(\frac{2}{K} + \delta_0)}\cdot\frac{1}{1 - 2\alpha_\tau\widetilde{s}_1^3\theta}.$$

Since $|\widetilde{s}| \leqslant s_0 + \varepsilon_s$, taking ε_s, δ_0, small and K large so that

$$\frac{(s_0+\varepsilon_s)(1-\frac{C}{K})}{1 - C(\frac{2}{K}+\delta_0)} \leqslant \frac{3}{2}s_0$$

we get

$$|s(\theta)| \leqslant \frac{\frac{3}{2}s_0}{1-2\alpha_\tau \widetilde{s}_1^3\theta}, \quad \theta\in[-T,0]. \tag{7.11}$$

Now the estimates (7.7) to (7.11) are true for any $T < T_*$, so letting T go to T_* we conclude that they are true up to $\theta = -T_*$. Then we consider the system (7.3) with data $s(-T_*)$, $y(-T_*)$, $\tau(-T_*)$, $\eta(-T_*)$ and we solve it on $[-T_*-\delta, -T_*]$ by the Cauchy-Lipschitz theorem. Matching this solution with the previous one, we obtain a solution of (7.3) on $[-T_*-\delta, 0]$ which satisfies the estimates in Proposition 7.2, getting a contradiction. This proves the Proposition 7.2 in the case where θ is real.

Let us consider the case of complex θ. We recall the following well known result. Let $(\theta_0, X_0)\in\mathbb{R}\times\mathbb{C}^N$ and

$$\Omega = \big\{(\theta, X)\in\mathbb{C}\times\mathbb{C}^N : |\theta-\theta_0|<a,\ |X-X_0|<b\big\}.$$

Let $F:\Omega\to\mathbb{C}^N$ be a holomorphic function such that $\sup_\Omega\|F(X)\| = M < +\infty$. Then the Cauchy problem

$$\begin{cases} \dot X(\theta) = F(\theta, X(\theta)) \\ X(\theta_0) = X_0 \end{cases}$$

has a unique holomorphic solution defined in $\{\theta\in\mathbb{C} : |\theta-\theta_0|<\rho\}$ where

$$\rho < a\Big(1-\exp\Big(\frac{-b}{(N+1)aM}\Big)\Big). \tag{7.12}$$

Let us fix $(\theta_1, \widetilde{s}_1)$ in D_1 and let us take $\theta_0\in\big[\theta_1-\frac{1}{K}Q_1, 0\big]$. We introduce

$$s_1(\theta) = \frac{s(\theta)}{s(\theta_0)},\ y_1(\theta)=y(\theta),\quad \tau_1(\theta)=\tau(\theta)s(\theta_0)^2,\quad \eta_1(\theta)=\eta(\theta)s(\theta_0),$$

and we consider the system satisfied by

$$X(\theta) = \big(s_1(\theta), y_1(\theta), \tau_1(\theta), \eta_1(\theta)\big)$$

which is derived from (7.3). It can be written as $\dot X(\theta) = F(X(\theta))$. Let us introduce

$$\Omega = \big\{(s_1, y_1, \tau_1, \eta_1)\in\mathbb{C}^{2n} : |s_1-1| + |y_1-y(\theta_0)| + |\tau_1-\tau(\theta_0)s(\theta_0)^2| \\ + |\eta_1 - s(\theta_0)\eta_1(\theta_0)| < \delta\big\},$$

where δ depends on the domain on which the coefficients of p extend holomorphically ; then, using the estimates (7.5) to (7.11) for real θ_0, we see that $\sup_\Omega\|F(X)\| \leqslant C_0(1+s(\theta_0)) \leqslant \frac{C_0(1+4s_0)}{Q_1} = M$, by (7.5) since $Q_1\leqslant 1$. We take $a=\frac{2}{K}Q_1$ where K is so large that $\frac{\delta}{(2n+1)aM} = \frac{\delta K}{2(2n+1)C_0(1+4s_0)} \geqslant \operatorname{Log} 2$.

It follows that $\exp\big(-\frac{\delta}{(2n+1)aM}\big)\leqslant\frac{1}{2}$; so our system has a holomorphic solution in the set $\big\{|\theta-\theta_0|<\frac{a}{2}\big\} = \big\{|\theta-\theta_0|<\frac{1}{K}Q_1\big\}$. Matching these solution for $\theta_0\in\big[\theta_1-\frac{Q_1}{K}, 0\big]$, we obtain a solution of (7.3) for $\operatorname{Re}\theta\in\big[\theta_1-\frac{1}{K}Q_1, 0\big]$, $|\operatorname{Im}\theta|<\frac{1}{K}Q_1$ as claimed. The proof of Proposition 4.2 is complete. □

Corollary 7.3. — *There exist constants C_j, $j = 1, \dots, 8$, depending only on the data, such that*

$$\frac{C_1}{Q_1^2} \leqslant \left|\frac{\partial s(\theta)}{\partial \widetilde{s}}\right| \leqslant \frac{C_2}{Q_1^2}, \quad \left|\frac{\partial s(\theta)}{\partial \widetilde{y}}\right| \leqslant \frac{C_3}{K^2 Q_1}, \quad \left|\frac{\partial s(\theta)}{\partial \widetilde{\tau}}\right| \leqslant \frac{C_4}{Q_1^2}, \quad \left|\frac{\partial s(\theta)}{\partial \widetilde{\eta}}\right| \leqslant \frac{C_5}{K Q_1^2}$$

$$\frac{\partial y(\theta)}{\partial \widetilde{y}} = \mathrm{Id} + \mathcal{O}\Big(\frac{1}{K}\Big), \quad \left|\frac{\partial y(\theta)}{\partial \widetilde{s}}\right| \leqslant \frac{C_6}{Q_1}, \quad \left|\frac{\partial y(\theta)}{\partial \widetilde{\tau}}\right| \leqslant \frac{C_7}{Q_1}, \quad \left|\frac{\partial y(\theta)}{\partial \widetilde{\eta}}\right| \leqslant \frac{C_8}{Q_1}.$$

Proof. — For the estimates on $s(\theta)$ use (7.10), (7.10)′. We obtain

$$\frac{\partial s(\theta)}{\partial \widetilde{s}} = \frac{1 + 4\widetilde{\tau}\widetilde{s}^3\theta - \widetilde{s}^2 \frac{\partial F(\theta)}{\partial \widetilde{s}}}{(1 - 2\widetilde{\tau}\widetilde{s}^3\theta + \widetilde{s}F(\theta))^2}.$$

From (7.10) and the Cauchy formula we have $\partial F(\theta)/\partial \widetilde{s} = \mathcal{O}(1/K^2)$. From the lines after (7.10)′ we get

$$|1 - 2\widetilde{\tau}\widetilde{s}^3\theta + \widetilde{s}F(\theta)| \geqslant \Big(1 - \frac{C}{K} - C\delta_0\Big) Q_1 \geqslant \frac{1}{2} Q_1,$$

if $1/K$ and δ_0 are small enough.

Moreover, since $\widetilde{\tau}\widetilde{s}^3 = -1 + \mathcal{O}(Q_1/K)$, $\operatorname{Re}\theta \leqslant 0$, $|\operatorname{Im}\theta| \leqslant Q_1/K$, we get,

$$1 + 4\widetilde{\tau}\widetilde{s}^3\theta - \widetilde{s}^2 \frac{\partial F(\theta)}{\partial \widetilde{s}} = 1 + 4|\operatorname{Re}\theta| + \mathcal{O}(1/K).$$

Then the estimates on $\partial s(\theta)/\partial \widetilde{s}$ follow. For the estimates on $y(\theta)$, we use the equality $y(\theta) = \widetilde{y} + g(\theta)$, $g = \mathcal{O}(1/K)$ and the Cauchy formula. □

Let us remark that, in Proposition 7.2, we can take

$$\widetilde{\tau} = \alpha_\tau + 2ih(\widetilde{s} - \alpha_s), \quad \widetilde{\eta} = \alpha_\eta + 2ih(\widetilde{y} - \alpha_y).$$

Indeed this follows from the estimate

$$|\widetilde{\tau} - \alpha_\tau| = 2h|\widetilde{s} - \alpha_s| \leqslant 2H(|\widetilde{s} - s_0| + |s_0 - \alpha_s|) \leqslant 4\varepsilon H \leqslant H \leqslant \frac{1}{K} Q_1,$$

if $\varepsilon \leqslant 1/4$ and from the analogue for $|\widetilde{\eta} - \alpha_\eta|$.

So we introduce the following notation

$$\underline{f}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = f\big(\theta; \widetilde{s}, \widetilde{y}, \alpha_\tau + 2ih(\widetilde{s} - \alpha_s), \alpha_\eta + 2ih(\widetilde{y} - \alpha_y), h\big) \tag{7.13}$$

which will be used for $f = s, y, \tau, \eta$. The function $\underline{f}$ is then defined in the set

$$\underline{\Delta} = \bigcup_{(\theta_1, \widetilde{s}_1) \in D_1} \Big\{(\theta, \widetilde{s}, \widetilde{y}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1} : |\widetilde{s} - \widetilde{s}_1| < \delta_0 Q_1, |\widetilde{y} - y_0| < \varepsilon_y, \operatorname{Re}\theta \in \Big[\theta_1 - \frac{1}{K} Q_1, 0\Big], |\operatorname{Im}\theta| < \frac{1}{K} Q_1\Big\}. \tag{7.14}$$

We introduce now, for fixed α, h, the set

$$\Lambda = \Big\{\big(\theta, \underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), \underline{y}(\theta; \cdots), \theta^*(0), \underline{\tau}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), \underline{\eta}(\theta; \cdots)\big), (\theta, \widetilde{s}, \widetilde{y}) \in \underline{\Delta}\Big\}. \tag{7.15}$$

Let us consider the set

$$(7.16)\; E = \bigcup_{(\theta_1,\widetilde{s}_1)\in D_1} \Big\{(\theta,s,y)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^{n-1} : |\theta-\theta_1| < \frac{1}{K'}Q_1,$$
$$|y-\underline{y}(\theta_1;\widetilde{s}_1,y_0,\alpha,h)| < \frac{1}{K'},\ \Big|s-\frac{\widetilde{s}_1}{1-2\alpha_\tau\widetilde{s}_1^3\theta_1}\Big| < \frac{1}{K'}\cdot\frac{1}{Q_1}\Big\}.$$

Then we have

Proposition 7.4. — *Let Λ be defined by (7.15) and π be the projection on the basis. Then if K' is large enough, we have $E\subset\pi(\Lambda)$ and, if we set $\Lambda_1=\pi^{-1}(E)$ then the map $\pi:\Lambda_1\to E$ is bijective.*

Proof

Claim. — *For any (θ,s,y) in E one can find $(\theta,\widetilde{s},\widetilde{y})$ in $\underline{\Delta}$ such that*

$$(7.17)\qquad \begin{cases} \theta &= \theta \\ \underline{s}(\theta;\widetilde{s},\widetilde{y},\alpha,h) &= s \\ \underline{y}(\theta;\widetilde{s},\widetilde{y},\alpha,h) &= y. \end{cases}$$

Here α and h are fixed in I_α, I_h.

We set $\widetilde{s}=\widetilde{s}_1+\widetilde{t}$ and $s=\frac{\widetilde{s}_1}{1-2\alpha_\tau\widetilde{s}_1^3\theta_1}+t=\frac{\widetilde{s}_1}{Q_1}+t$. Let us recall that, according to (7.10), we have

$$\underline{s}(\theta;\widetilde{s},\widetilde{y},\alpha,h)=\frac{\widetilde{s}}{1-2\widetilde{\tau}\widetilde{s}^3\theta+\widetilde{s}F(\theta)}$$

where $\widetilde{\tau}=\alpha_\tau+ih(\widetilde{s}-\alpha_s)$ and $|F|\leqslant\frac{C}{K^2}Q_1$.

Now, if $|\theta-\theta_1|<\frac{1}{K'}Q_1$, $|\widetilde{t}|\leqslant\delta_0Q_1$ and $|\widetilde{s}-\alpha_s|\leqslant\varepsilon_s$, we have

$$1-2\widetilde{\tau}\widetilde{s}^3\theta+\widetilde{s}F(\theta)=Q_1-6\alpha_\tau\widetilde{s}_1^2\widetilde{t}\theta_1+G(\theta;\widetilde{t},\widetilde{y},\alpha,h)$$

where $|G|\leqslant C\big(\frac{1}{K'}+\delta_0^2+\frac{1}{K^2}+\frac{\varepsilon_s+\varepsilon_\alpha}{K}\big)Q_1$.

It follows that the equation $\underline{s}(\theta;\cdots)=s$ is equivalent to

$$\widetilde{s}_1Q_1+\widetilde{t}Q_1=Q_1\widetilde{s}_1-6\alpha_\tau\widetilde{s}_1^3\widetilde{t}\theta_1+tQ_1(Q_1-6\alpha_\tau\widetilde{s}_1^2\widetilde{t}\theta_1)+(\widetilde{s}_1+tQ_1)G.$$

Now, since $Q_1+6\alpha_\tau\widetilde{s}_1^3\theta_1=1+4\alpha_\tau\widetilde{s}_1^3\theta_1=c_1\geqslant1$, this equation is equivalent to

$$(7.18)\qquad \widetilde{t}=\frac{1}{c_1}tQ_1(Q_1-6\alpha_\tau\widetilde{s}_1^2\widetilde{t}\theta_1)+\frac{1}{c_1}(\widetilde{s}_1+tQ_1)G(\theta;\widetilde{t},\widetilde{y},\alpha,h)=H(\widetilde{t},\widetilde{y}).$$

On the other hand, forgetting α and h which are fixed, we can write

$$\underline{y}(\theta;\widetilde{s},\widetilde{y})=y(\theta;\widetilde{s},y_0)+(\widetilde{y}-y_0)\frac{\partial\underline{y}}{\partial\widetilde{y}}(\theta;\widetilde{s},y_0)+\mathcal{O}(|\widetilde{y}-y_0|^2).$$

Since $\partial\underline{y}/\partial\widetilde{y}(\theta;\widetilde{s},y_0)=1+\mathcal{O}(1/K)$, we see that the equation $\underline{y}(\theta;\widetilde{s},\widetilde{y})=y$ is equivalent to

$$(7.19)\qquad \widetilde{y}-y_0=a(\theta;\widetilde{s})(y-\underline{y}(\theta;\widetilde{s},y_0))+\mathcal{O}(|\widetilde{y}-y_0|^2).$$

Now

$$\underline{y}(\theta;\widetilde{s},y_0)=\underline{y}(\theta_1;\widetilde{s}_1,y_0)+(\theta-\theta_1)\frac{\partial\underline{y}}{\partial\theta}(\theta^*;\widetilde{s}^*,y_0)+\widetilde{t}\frac{\partial\underline{y}}{\partial\widetilde{s}}(\cdots),$$

and we have

$$\Big|\frac{\partial\underline{y}}{\partial\theta}(\theta^*;\widetilde{s}^*,y_0)\Big|\leqslant C,\quad \frac{\partial\underline{y}}{\partial\widetilde{s}}=\mathcal{O}\Big(\frac{1}{K}\Big),\quad |\theta-\theta_1|<\frac{1}{K'}.$$

It follows that (7.19) is equivalent to

$$\widetilde{y}-y_0=a(\theta;\widetilde{s})(y-\underline{y}(\theta_1;\widetilde{s}_1,y_0))+\mathcal{O}\Big(\frac{1}{K'}\Big)+\widetilde{t}\,\mathcal{O}\Big(\frac{1}{K}\Big)+\mathcal{O}(|\widetilde{y}-y_0|^2). \tag{7.20}$$

Setting $Y=\big(\widetilde{t}/(\widetilde{y}-y_0)\big)$, we see, according to (7.18) and (7.20), that (7.17) can be written as $Y=\Phi(Y)$.

Taking $|t|\leqslant\frac{1}{K'}Q_1$, $|y-\underline{y}(\theta_1;\widetilde{s}_1,y_0)|\leqslant 1/K'$ and setting

$$B=\big\{(\widetilde{t},\widetilde{y}):|\widetilde{t}|\leqslant\delta_0Q_1,\ |\widetilde{y}-y_0|<\varepsilon_y\big\}$$

we see that, if δ_0 is small enough and $K'\gg K$, then Φ maps B into itself and satisfies $|\Phi(Y)-\Phi(Y')|\leqslant\delta|Y-Y'|$ with $\delta<1$. Thus the first part of Proposition 7.4 follows from the fixed point theorem.

Let $\Lambda_1=\pi^{-1}(E)$; we must show that $\pi:\Lambda_1\to E$ is injective. We recall that

$$\frac{1}{\underline{s}(\theta)}=\frac{1}{\widetilde{s}}-2\widetilde{\tau}\widetilde{s}^2\theta+F(\theta),\quad \underline{y}(\theta)=\widetilde{y}+G(\theta)$$

where $|\frac{\partial F}{\partial\widetilde{s}}|+|\frac{\partial F}{\partial\widetilde{y}}|+|\frac{\partial G}{\partial\widetilde{s}}|+|\frac{\partial G}{\partial\widetilde{y}}|=\mathcal{O}(\frac{1}{K})$, and $\widetilde{\tau}=\alpha_\tau+2ih(\widetilde{s}-\alpha_s)$. Forgetting α,h, which are fixed, assume that

$$\underline{s}(\theta;\widetilde{s},\widetilde{y})=\underline{s}(\theta;\widetilde{s}',\widetilde{y}'),\quad \underline{y}(\theta;\widetilde{s},\widetilde{y})=\underline{y}(\theta;\widetilde{s}',\widetilde{y}').$$

It follows, from the above formulas that

$$(\widetilde{s}'-\widetilde{s})\big(1+2\alpha_\tau\theta\widetilde{s}\widetilde{s}'(\widetilde{s}+\widetilde{s}')\big)+4ih\theta(\widetilde{s}'-\widetilde{s})f(\widetilde{s},\widetilde{s}',\alpha)=\widetilde{s}\widetilde{s}'\big[\widetilde{s}F(\theta;\widetilde{s},\widetilde{y})-\widetilde{s}'F(\theta;\widetilde{s}',\widetilde{y}')\big]$$

$$\widetilde{y}-\widetilde{y}'=G(\theta;\widetilde{s}',\widetilde{y}')-G(\theta;\widetilde{s},\widetilde{y})$$

where $f(\widetilde{s},\widetilde{s}',\alpha)=\mathcal{O}(1)$. Since $|1+2\alpha_\tau\theta\widetilde{s}\widetilde{s}'(\widetilde{s}+\widetilde{s}')|$ is bounded below the above equations lead to the estimates

$$\begin{cases}|\widetilde{s}-\widetilde{s}'|\leqslant C\Big(h+\dfrac{1}{K}\Big)\big(|\widetilde{s}-\widetilde{s}'|+|\widetilde{y}-\widetilde{y}'|\big),\\ |\widetilde{y}-\widetilde{y}'|\leqslant\dfrac{C}{K}\big(|\widetilde{s}-\widetilde{s}'|+|\widetilde{y}-\widetilde{y}'|\big).\end{cases}$$

Taking h and $1/K$ small enough we see that this implies $\widetilde{s}=\widetilde{s}'$ and $\widetilde{y}=\widetilde{y}'$. □

Proposition 7.5. — *For all λ in Λ_1, the map $d\pi:T_\lambda\Lambda_1\to T_{\pi(\lambda)}E$ is surjective.*

Proof. — Let G be the map

$$(\theta,\widetilde{s},\widetilde{y})\longmapsto(\theta,\underline{s}(\theta;\widetilde{s},\widetilde{y},\alpha,h),\underline{y}(\theta;\cdots),\theta^*(0),\underline{\tau}(\theta;\widetilde{s},\widetilde{y},\alpha,h),\underline{\eta}(\theta;\cdots))$$

from the set

$$\underline{\Delta}_1 = \bigcup_{(\theta_1,\widetilde{s}_1)\in D_1} \Big\{(\theta,\widetilde{s},\widetilde{y}) : |\widetilde{s}-s_1| < \delta Q_1,\, |\widetilde{y}-y_0| < \delta,$$

$$\operatorname{Re}\theta \in [\theta_1 - \delta Q_1, 0],\, |\operatorname{Im}\theta| < \delta Q_1\Big\}$$

to Λ_1. If we show that $d(\pi\circ G)$ is surjective, then we are done. Now it is easy to see that $d(\pi\circ G)$ is surjective if and only if the determinant of the matrix $A = \begin{pmatrix} \frac{\partial \underline{s}(\theta)}{\partial \widetilde{s}} & \frac{\partial \underline{s}(\theta)}{\partial \widetilde{y}} \\ \frac{\partial \underline{y}(\theta)}{\partial \widetilde{s}} & \frac{\partial \underline{y}(\theta)}{\partial \widetilde{y}} \end{pmatrix}$ is different from zero. This will follow from Corollary 7.3. Indeed we have

$$\Big|\frac{\partial \underline{s}}{\partial \widetilde{s}}\Big| \geqslant \frac{C}{Q_1^2}, \quad \frac{\partial \underline{s}}{\partial \widetilde{y}} = \mathcal{O}\Big(\frac{1}{K^2 Q_1}\Big), \quad \frac{\partial \underline{y}}{\partial \widetilde{s}} = \mathcal{O}\Big(\frac{1}{Q_1}\Big), \quad \frac{\partial \underline{y}}{\partial \widetilde{y}} = \operatorname{Id} + \mathcal{O}\Big(\frac{1}{K}\Big).$$

This implies that

$$|\det A| \geqslant \frac{C}{Q_1^2}. \qquad \square$$

Proof of Theorem 7.1. — It follows from Propositions 7.4 and 7.5 that one can find a smooth function $\varphi = \varphi(\theta; s, y, \alpha, h)$ which, for fixed α, h, is defined on the set E (see (7.16)) such that

$$\Lambda_1 = \Big\{\Big(\theta, s, y, \frac{\partial\varphi}{\partial\theta}(\theta; s, y, \alpha, h), \frac{\partial\varphi}{\partial s}(\theta;\cdots), \frac{\partial\varphi}{\partial y}(\theta;\cdots)\Big), (\theta, s, y) \in E\Big\}.$$

Since $\Lambda_1 \subset \Lambda$ and the symbol $q^* = \theta^* + p(hs, y, \tau s^2, s\eta)$ vanishes on Λ, we have solved the first equation in (7.2). Obviously φ is defined up to a constant and we can choose it such that $\varphi(0; s_0, y_0, \alpha, h) = (s_0 - \alpha_s)\alpha_\tau + (y_0 - \alpha_y)\alpha_\eta + ih[(s_0-\alpha_s)^2 + (y_0 - \alpha_y)^2]$. Then we write

$$\varphi(0; s, y, \alpha, h) = \varphi(0; s_0, y_0, \alpha, h) + \int_0^1 \Big[\frac{\partial\varphi}{\partial s}(0; ts + (1-t)s_0, ty + (1-t)y_0, \alpha, h)\cdot$$

$$\cdot(s - s_0) + \frac{\partial\varphi}{\partial y}(0; ts + (1-t)s_0, ty + (1-t)y_0, \alpha, h)\cdot(y - y_0)\Big]\,dt.$$

Now

$$\frac{\partial\varphi}{\partial s}(0;\cdots) = \underline{\tau}(0; \widetilde{s}, \widetilde{y}, \alpha, h) = \alpha_\tau + 2ih(\widetilde{s} - \alpha_s)$$

where $\underline{s}(0; \widetilde{s}, \widetilde{y}, \alpha, h) = ts + (1-t)s_0 = \widetilde{s}$. Using these relations and the same one for $\partial\varphi/\partial y$, we find that φ satisfies also the initial condition in (7.2). $\square$

Proposition 7.6. — *Let (α, h) be fixed in $I_\alpha \times I_h$. Then the phase given in Theorem 7.1 satisfies, for $(\theta, \widetilde{s}, \widetilde{y})$ in $\underline{\Delta}$*

$$\varphi\big(\theta; \underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), \underline{y}(\theta;\cdots), \alpha, h\big) = (\widetilde{s} - \alpha_s)\alpha_\tau + (\widetilde{y} - \alpha_y)\cdot\alpha_\eta$$
$$+ ih\big[(\widetilde{s}-\alpha_s)^2 + (\widetilde{y} - \alpha_y)^2\big] + \theta p(h\widetilde{s}, \widetilde{y}, \widetilde{\tau}\widetilde{s}^2, \widetilde{s}\widetilde{\eta})$$

where $\widetilde{\tau} = \alpha_\tau + ih(\widetilde{s} - \alpha_s)$ and $\widetilde{\eta} = \alpha_\eta + ih(\widetilde{y} - \alpha_y)$.

Proof. — Let us write $f(\theta)$ instead of $\underline{f}(\theta;\widetilde{s},\widetilde{y},\alpha,h)$. Then

$$(1)=\frac{d}{d\theta}[\varphi(\theta;s(\theta),y(\theta),\alpha,h)]=\Big(\frac{\partial\varphi}{\partial\theta}+\dot s\,\frac{\partial\varphi}{\partial s}+\dot y\,\frac{\partial\varphi}{\partial y}\Big)(\theta;s(\theta),y(\theta),\alpha,h).$$

Using (7.3) and the definition of φ, we get

$$\dot s\,\frac{\partial\varphi}{\partial s}+\dot y\,\frac{\partial\varphi}{\partial y}=\tau s^2\,\frac{\partial p}{\partial\lambda}+s\eta\cdot\frac{\partial p}{\partial\mu}=2p(hs,y,\tau s^2,s\eta).$$

Since p is constant on the bicharacteristics and

$$\frac{\partial\varphi}{\partial\theta}=\theta^*(0)=-p(h\widetilde{s},\widetilde{y},\widetilde{\tau}\widetilde{s}^2,\widetilde{\eta}\widetilde{s}),$$

where $\widetilde{\tau}=\alpha_\tau+ih(\widetilde{s}-\alpha_s)$, $\widetilde{\eta}=\cdots$, we get $(1)=p(h\widetilde{s},\widetilde{y},\widetilde{\tau}\widetilde{s}^2,\widetilde{s}\widetilde{\eta})$. On the other hand we have

$$\varphi(0;\widetilde{s},\widetilde{y},\alpha,h)=(\widetilde{s}-\alpha_s)\alpha_\tau+(\widetilde{y}-\alpha_y)\cdot\alpha_\eta+ih[(\widetilde{s}-\alpha_s)^2+(\widetilde{y}-\alpha_y)^2],$$

which proves our claim. □

7.2. Link between the flow of σH_Δ and the bicharacteristics

Proposition 7.7. — *Let $(\theta,\widetilde{s},\widetilde{y})$, $\theta<0$, and (α,h) be fixed real points in $\underline{\Delta}$ and $I_\alpha\times I_h$.*

(i) *We set $R^2=p(h\widetilde{s},\widetilde{y},\alpha_\tau\widetilde{s}^2,\widetilde{s}\alpha_\eta)>0$. Then the problem*

$$\begin{cases}\dot\chi(t)=\dfrac{1}{Rs(\chi(t);\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h)}\\ \chi(0)=\theta\end{cases}\tag{7.21}$$

has a unique solution defined on $[0,T^*]$ with $\chi(T^*)=0$.

(ii) *If we set, for $t\in[0,T^*]$,*

$$\rho(t)=hs(\chi(t);\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h),\qquad y(t)=y(\chi(t);\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h)$$
$$\overline{\lambda}(t)=\frac{1}{R}(\tau s^2)(\chi(t);\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h),\qquad \overline{\mu}(t)=\frac{1}{R}(s\eta)(\chi(t);\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h)$$

then $\big(\rho(t),y(t),0,(\overline{\lambda}(t),\overline{\mu}(t))\big)=\exp t\sigma H_\Delta\big(\rho(0),y(0),0,(\overline{\lambda}(0),\overline{\mu}(0))\big)$.

Proof

(i) Let us introduce the following set

$$A=\big\{T>0:\text{(7.21) has a solution on }[0,T]\text{ with }(\chi(t),\widetilde{s},\widetilde{y})\in\underline{\Delta}\big\}.$$

Then A is an interval which is non empty, by the Cauchy-Lipschitz theorem. Let $T^*=\sup A$. Then, on $[0,T^*[$ one has $\chi(t)\leqslant 0$ (by the definition of $\underline{\Delta}$). Since $s(\theta;\widetilde{s},\widetilde{y},\cdots)>0$ and $R>0$ we have $\dot\chi>0$ so $\lim_{t\to T^*}\chi(t)=\ell\leqslant 0$ exists. By (7.21) we have then

$$\lim_{t\to T^*}\dot\chi(t)=\frac{1}{Rs(\ell;\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta,h)}>0.$$

Therefore $T^* < +\infty$. We can extend χ to $[0,T^*]$ by setting $\chi(T^*) = \ell$. If $\ell < 0$, the equation in (7.21) with $\chi(T^*) = \ell$ would have a solution on $[T^*, T^* + \varepsilon]$ with $(\chi(t), \widetilde{s}, \widetilde{y}) \in \underline{\Delta}$ with contradicts the maximality of T^* ; so $\chi(T^*) = 0$.

(ii) This claim follows from (7.3) and (3.17) by a simple computation. □

7.3. The transport equation

As before we look for an analytic symbol a such that

$$\Big(\frac{1}{k}\,\partial_\theta + i\,\Delta_g^*\Big)(a\,e^{ih^{-2}k^{-1}\varphi}) = \mathcal{O}(e^{-\delta/hk}), \quad \delta > 0. \tag{7.22}$$

Working in the $(\theta, \widetilde{s}, \widetilde{y})$ coordinates instead of (θ, s, y) we are lead to solve the transport equation

$$\begin{cases} \Big(\dfrac{\partial}{\partial\theta} + c(\theta; \widetilde{s}, \widetilde{y}, \alpha) + h^2 k Q\Big)\, a = b \\ a|_{\theta=0} = 0 \end{cases} \tag{7.23}$$

where Q is of second order and is a linear combination with bounded coefficients of $\partial_{\widetilde{s}}^2$, $s^2(\theta)\,\partial_{\widetilde{y}}^2$, $s(\theta)\,\partial_{\widetilde{s}}\,\partial_{\widetilde{y}}$, $s(\theta)\,\partial_{\widetilde{s}}$, $s(\theta)\,\partial_{\widetilde{y}}$. To see this, we first note that

$$\partial_{\widetilde{s}} = \frac{\partial s(\theta)}{\partial \widetilde{s}}\,\partial_s + \frac{\partial y(\theta)}{\partial \widetilde{s}} \cdot \partial_y$$
$$\partial_{\widetilde{y}_j} = \frac{\partial s(\theta)}{\partial \widetilde{y}_j}\,\partial_s + \frac{\partial y(\theta)}{\partial \widetilde{y}_j} \cdot \partial_y\,.$$

Now, it follows from (7.10)′ that

$$\frac{\partial s(\theta)}{\partial \widetilde{s}} = a(\theta; \widetilde{s}, \widetilde{y})\, s^2(\theta)$$

where $a(\theta) \neq 0$,

$$\frac{\partial s(\theta)}{\partial \widetilde{y}_j} = b_j(\theta)\, s^2(\theta)$$

where $b_j(\theta) = \mathcal{O}\big(Q_1/K\big)$ (because $\partial F/\partial\widetilde{y} = \mathcal{O}\big(Q_1/K\big)$ by (7.10) and the Cauchy formula),

$$\frac{\partial y_k(\theta)}{\partial \widetilde{s}} = \mathcal{O}\Big(\frac{1}{K}\Big), \quad \frac{\partial y_k(\theta)}{\partial \widetilde{y}_j} = \delta_{jK} + \mathcal{O}\Big(\frac{1}{K}\Big).$$

Moreover, from the line after (7.10)′, we have $|s(\theta)| \leqslant C/Q_1$. Inverting the system above and using these informations, we see that

$$s^2\,\partial_s = \alpha(\theta; \widetilde{s}, \widetilde{y})\,\partial_{\widetilde{s}} + \beta(\theta; \widetilde{s}, \widetilde{y}) \cdot \partial_{\widetilde{y}}$$
$$\partial_y = \gamma(\theta; \widetilde{s}, \widetilde{y})\,\partial_{\widetilde{s}} + \delta(\theta; \widetilde{s}, \widetilde{y}) \cdot \partial_{\widetilde{y}}$$

where α, β, δ are bounded and $\gamma(\theta) = \mathcal{O}\big(Q_1/K\big)$. Since $|s(\theta)| \leqslant C/Q_1$, it follows that the coefficient of $\partial_{\widetilde{s}}$ coming from $s\,\partial_y$ is bounded. Then our claim follows from the fact that, in the coordinates (s, y) the operator Q is given by Δ_g^* which is described in (5.1). We shall set $\lambda^{-1} = h^2 k$ and take $a = \sum_{j \geqslant 0} \lambda^{-j}\, a_j$.

We shall define our symbol in a subset of $\underline{\Delta}$ introduced in (7.14). First of all by the usual trick we can assume that $c = 0$ in (7.23). Formally, the system (7.23) can be solved in $\underline{\Delta}$ since it is a linear problem ; our goal will be, therefore, to show that this formal resolution leads to an analytic symbol. The equation (7.23) is equivalent to

$$\begin{cases} (I+B)a = b, \quad \text{where} \\ B = \partial_\theta^{-1}\lambda^{-1}Q \quad \text{and} \quad \partial_\theta^{-1}a(\theta;\widetilde{s},\widetilde{y}) = \theta \displaystyle\int_0^1 a(\mu\theta;\widetilde{s},\widetilde{y})d\mu. \end{cases} \tag{7.24}$$

According to (7.14) let $(\theta_1, \widetilde{s}_1) \in D_1$. We introduce for $t > 0$, $0 < t' < \varepsilon_y$,

$$\Omega_{tt'} = \left\{(\theta,\widetilde{s},\widetilde{y}) : 0 > \theta > \theta_1 - \frac{Q_1}{K} + t, |\widetilde{s}-\widetilde{s}_1| < \delta_0 t, |\widetilde{y} - y_0| < \varepsilon_y - t'\right\}$$

where $Q_1 = 1 - 2\alpha_\tau \widetilde{s}_1^3\theta_1$ and δ_0 is small enough.

Claim 1. — *$\Omega_{tt'} \subset \underline{\Delta}$ (see (7.14)).*

Let $(\theta,\widetilde{s},\widetilde{y}) \in \Omega_{tt'}$; let us take $\theta_1' = \theta_1 + t$, $\widetilde{s}_1 = s_1$ and let us show that $(\theta_1', \widetilde{s}_1) \in D_1$. To see this, we write

$$Q_1' = 1 - 2\alpha_\tau \widetilde{s}_1^3(\theta_1 + t) = Q_1 + 2|\alpha_\tau|\widetilde{s}_1^3 t > KH.$$

This shows that $Q_1' \geqslant Q_1$, $Q_1' > KH$, $Q_1' \geqslant 2|\alpha_\tau|\widetilde{s}_1^3 t$. Thus $(\theta_1', \widetilde{s}_1) \in D_1$. Now

$$0 > \theta > \theta_1 - \frac{Q_1}{K} + t \geqslant \theta_1' - \frac{Q_1'}{K}, \quad |\widetilde{s} - \widetilde{s}_1| < \delta_0 t \leqslant \frac{\delta_0}{2|\alpha_\tau|\widetilde{s}_1^3}Q_1' \quad \text{and} \quad |\widetilde{y} - y_0| < \varepsilon_y.$$

It follows that $(\theta,\widetilde{s},\widetilde{y}) \in \underline{\Delta}$.

Claim 2. — *If $(\theta,\widetilde{s},\widetilde{y}) \in \Omega_{tt'}$ then, for $\mu \in (0,1)$, $(\mu\theta,\widetilde{s},\widetilde{y}) \in \Omega_{t_\mu t'}$ where $t_\mu = I - J\mu$, $I = \left|\theta_1 - \frac{Q_1}{K}\right|$, $J = \left|\theta_1 - \frac{Q_1}{K}\right| - t$. The expression of t_μ follows easily from the definition of $\Omega_{tt'}$ and $t_\mu \geqslant t$, since $t < \left|\theta_1 - \frac{Q_1}{K}\right|$.*

Let us remark that $J = I - t$ and $0 \geqslant \theta > -J$.

Now, given $\rho > 0$, we shall say that $a \in \mathcal{A}_\rho$ if $a = \sum_{j\geqslant 0} \lambda^{-j}a_j$ with

$$\sup_{\Omega_{tt'}} |a_j| \leqslant f_j(a) j^j t^{-j} t'^{-2j} \tag{7.25}$$

where $f_j(a)$ is the best constant for which such an estimate holds and

$$\sum_{j=0}^{+\infty} f_j(a)\rho^j = \|a\|_\rho < +\infty. \tag{7.26}$$

Claim 3. — *One can find a positive constant C such that for all $\rho > 0$ and $a \in \mathcal{A}_\rho$*

$$\|Ba\|_\rho \leqslant C\rho\|a\|_\rho. \tag{7.27}$$

Proof. — We shall prove (7.27) when $B = \partial_\theta^{-1}\lambda^{-1}s^2(\theta)\Delta_{\widetilde{y}}$ and $B = \partial_\theta^{-1}\lambda^{-1}\partial_{\widetilde{s}}^2$, the other terms are easier to handle. In the first case we have

$$Ba = \sum_{j\geqslant 0}\lambda^{-j-1}\,\partial_\theta^{-1}s^2(\theta)\,\Delta_{\widetilde{y}}\,a_j$$
$$= \sum_{j\geqslant 1}\lambda^{-j}\,\theta\int_0^1 s^2(\mu\theta,\cdots)\,\Delta_{\widetilde{y}}\,a_{j-1}(\mu\theta,\cdots)\,d\mu = \sum_{j\geqslant 1} b_j\,.$$

Then

$$\sup_{\Omega_{tt'}}|b_j| \leqslant |\theta|\int_0^1 \sup_{\Omega_{t_\mu t'}}|s^2\,\Delta_{\widetilde{y}}\,a_{j-1}|\,d\mu, \quad j\geqslant 1.$$

Now, in $\Omega_{t_\mu t'}$, we have $1 - 2\alpha_\tau\theta(\mathrm{Re}\,\widetilde{s})^3 > 2|\alpha_\tau|(\mathrm{Re}\,\widetilde{s})^3\,t_\mu \geqslant C_0\,t_\mu$. It follows from (7.10) that $\sup_{\Omega_{t_\mu t'}}|s^2| \leqslant C_1\,t_\mu^{-2}$.

Let $t_0' < t'$; then by the Cauchy formula we have

$$\sup_{\Omega_{t_\mu t'}}|\Delta_{\widetilde{y}}\,a_{j-1}| \leqslant C_2(t'-t_0)^{-2}\sup_{\Omega_{t_\mu t_0'}}|a_{j-1}|\,.$$

Using (7.25) we get

$$\sup_{\Omega_{tt'}}|b_j| \leqslant C_3|\theta|\,f_{j-1}(a)(j-1)^{j-1}(t'-t_0')^{-2}\,t_0'^{-2j+2}\int_0^1 t_\mu^{-2}\,t_\mu^{-j+1}\,d\mu\,.$$

Since $t_\mu = I - J\mu$, we have

$$\int_0^1 t_\mu^{-j-1}\,d\mu = \frac{(I-J)^{-j}}{jJ} - \frac{I^{-j}}{jJ} \leqslant \frac{t^{-j}}{jJ},$$

since $I - J = t$ and $I \geqslant 0$.

Let us take $t_0' = \sqrt{j/(j+1)}\,t'$, for $j\geqslant 0$. Then, $t' - t_0' \geqslant t'/2(j+1)$, so we get

$$\sup_{\Omega_{tt'}}|b_j| \leqslant C_3|\theta|\,f_{j-1}(a)(j-1)^{j-1}\,4(j+1)^2\Big(\frac{j+1}{j}\Big)^{j-1}\,t'^{-2j}\,\frac{1}{jJ}\,t^{-j}\,.$$

Since

$$\frac{|\theta|}{J} \leqslant 1 \quad \text{and}(j-1)^{j-1}(j+1)^2\Big(\frac{j+1}{j}\Big)^{j-1}\frac{1}{j} \leqslant C_0\,j^j,$$

where C_0 is an absolute constant we obtain

$$\sup_{\Omega_{tt'}}|b_j| \leqslant C_4\,f_{j-1}(a)\,j^j\,t'^{-2j}\,t^{-j}\,.$$

It follows that $f_j(Ba) \leqslant C_4\,f_{j-1}(a)$, for $j\geqslant 1$, which implies (7.27). In the case where $B = \partial_\theta^{-1}\lambda^{-1}\partial_{\widetilde{s}}^2$, we take $t_0 < t_\mu$. Then

$$\sup_{\Omega_{t_\mu t'}}|\partial_{\widetilde{s}}^2\,a_{j-1}| < C_5(t_\mu - t_0)^{-2}\sup_{\Omega_{t_0 t'}}|a_{j-1}|\,.$$

If we take $t_0 = \frac{j}{j+1}\,t_\mu$ then $t_\mu - t_0 = \frac{t_\mu}{j+1}$. It follows that

$$\sup_{\Omega_{tt'}}|b_j| \leqslant C_6\,|\theta|\,f_{j-1}(a)(j-1)^{j-1}(j+1)^2\,t'^{-2j+2}\int_0^1 t_\mu^{-2}\,t_\mu^{-j+1}\,d\mu\,.$$

We obtain the same estimate as before (since $t'^{-2j+2} \leqslant \varepsilon_y^2 t'^{-2j}$) and we conclude in the same way. □

It follows from (7.27) that $(I + B)$ is invertible in $\mathcal{A}_\rho$, if ρ is small enough. Now we can take $t' = \frac{1}{2}\varepsilon_y$ and $t = Q_1/K$ in the definition of $\Omega_{tt'}$. Moreover we take $\lambda^{-1} = h^2 k$. Since $Q_1 > KH$ we have $t^{-1} \leqslant H^{-1}$; therefore the a_j's satisfy the estimates $|a_j| \leqslant M^j j^j H^{-j}$ on $\Omega_{tt'}$. The analytic symbols that we shall handle will be on the form

$$a = \sum_{j<\delta H/h^2 k} (h^2 k)^j a_j$$

where δ is a small positive constant. Then the size of the first term which has been neglected is as follows : if $j_0 \sim \delta H/h^2 k$ then,

$$|(h^2k)^{j_0} a_{j_0}| \leqslant \Big(h^2 k M j_0 \frac{1}{H}\Big)^{j_0} \leqslant (\delta M)^{j_0} \leqslant e^{-\gamma H/h^2 k} \leqslant e^{-\gamma/hk}$$

since $H \geqslant h$, where $\gamma > 0$ if $\delta M < 1$.

Summing up we have obtained an analytic symbol in a set which is slightly smaller than $\underline{\Delta}$ but has the same form. For convenience we shall still call it $\underline{\Delta}$. Then, if we denote by Φ the map

$$(\theta, \widetilde{s}, \widetilde{y}) \longmapsto \big(\theta; \underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), \underline{y}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h)\big)$$

then the symbol $a(\theta; s, y, \alpha, h)$ is well defined in $E_1 = \Phi(\underline{\Delta}) \subset E$.

7.4. End of the proof of Theorem 4.8

We introduce a cut-off function $\chi = \chi(h, s, y)$ supported in $E_1 = \Phi(\underline{\Delta})$ such that, with $Q = 1 - 2\alpha_\tau (\operatorname{Re} \widetilde{s})^3 \theta$,

$$\begin{aligned} \chi \circ \Phi = 1 \quad &\text{if} \quad Q \geqslant 2KH \quad \text{and } |\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y| \leqslant \frac{\varepsilon_0}{C_1 K} \\ \chi \circ \Phi = 0 \quad &\text{if} \quad Q \leqslant \frac{1}{2}KH \quad \text{or } |\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y| \geqslant \frac{\varepsilon_0}{C_2 K} \end{aligned}$$

where ε_0 is the constant appearing in the Definition B in the Appendix and $C_2 < C_1$ are constants (independent of K) which depend only on the data. Then the support of a derivative of χ is contained in the image by Φ of the set $W_1 \cup W_2$, where

$$\text{(7.28)} \quad \begin{cases} W_1 = \Big\{(\theta, \widetilde{s}, \widetilde{y}) : \dfrac{\varepsilon_0}{C_1 K} \leqslant |\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y| \leqslant \dfrac{\varepsilon_0}{C_2 K}, Q \geqslant \dfrac{1}{2}KH\Big\}, \\ W_2 = \Big\{(\theta, \widetilde{s}, \widetilde{y}) : \dfrac{1}{2}KH \leqslant Q \leqslant 2KH, |\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y| \leqslant \dfrac{\varepsilon_0}{C_2 K}\Big\}. \end{cases}$$

Let φ, a be the phase and the amplitude which satisfy (7.22) and u be a solution of our Schrödinger equation. We set

(7.29)

$$\mathcal{T}u(\theta; t, \alpha, h, k) = \int e^{ih^{-2}k^{-1}\varphi(\theta;\rho/h,y,h)} a\Big(\theta; \frac{\rho}{h}, y, \alpha, h\Big) \chi\Big(\theta; \frac{\rho}{h}, y\Big) \overline{u}(t, \rho, y)\, d\rho\, dy.$$

Using (7.22) and the equation satisfied by u, we see easily that

$$(\partial_\theta - k\partial_t)\mathcal{T}u(\theta;t,\alpha,h,k) = F(\theta;t,\alpha,h,k) + G(\theta;t,\alpha,h,k)$$

where

$$|G(\theta;t,\alpha,h,k)| \leqslant Ce^{-\delta/hk}, \quad \delta > 0 \tag{7.30}$$

for θ in $\operatorname{supp}\chi$, $|t-t_0| \leqslant \varepsilon$, $\alpha \in I_\alpha$, $h \in I_h$, $k > 0$ small and F is a finite sum of terms of the form

$$\int e^{ih^{-2}k^{-1}\varphi(\theta;\rho/h,y,\alpha,h)}a\Big(\theta;\frac{\rho}{h},y,\alpha,h,k\Big)\widetilde{\chi}\Big(\theta;\frac{\rho}{h},y\Big)\overline{v}(t,\rho,y)\,d\rho\,dy \tag{7.31}$$

where $\widetilde{\chi}$ is a derivative of χ of order $\geqslant 1$ and v is a derivative of u of order $\leqslant 1$. Then,

$$\operatorname{supp}\widetilde{\chi} \subset \operatorname{supp}\chi' \subset W_1 \cup W_2\,. \tag{7.32}$$

It follows that

$$\frac{d}{d\theta}\big(\mathcal{T}u(\theta;t-k\theta,\alpha,h,k)\big) = F(\theta;t-k\theta,\alpha,h,k) + \mathcal{O}(e^{-\delta/hk}).$$

We take $\theta_0 < 0$ such that $\chi(\theta_0;\rho/h,y) \equiv 0$ and we integrate both sides of the above equality from $\theta = \theta_0$ to $\theta = 0$. Since $\varphi(0;\rho/h,y,\alpha,h)$ is a FBI phase at $(s_0,y_0,(-1/s_0^3,0),\alpha_0,0)$, Theorem 4.8 will be proved if we can show that

$$\begin{aligned} I = \int_{\theta_0}^{0}\int e^{ih^{-2}k^{-1}\varphi(\theta;\rho/h,y,\alpha,h)}a\Big(\theta;\frac{\rho}{h},\cdots\Big)\widetilde{\chi}\Big(\theta;\frac{\rho}{h},y\Big)\overline{v}(t-k\theta;\rho,y)\,d\rho\,dy\,d\theta \\ = \mathcal{O}(e^{-\delta/hk}), \quad \delta > 0, \end{aligned} \tag{7.33}$$

uniformly in t when $|t-t_0| \leqslant \varepsilon$ and for $\alpha \in I_\alpha$, $h \in I_h$, $k > 0$ small. Since $\operatorname{supp}\widetilde{\chi} \subset W_1 \cup W_2$ we divide the proof of (7.33) into two cases.

Case 1. — We consider the part of the integral where $(\theta,\rho/h,y) \in W_2$ (this is the hard case). We set

$$\begin{cases} \widetilde{h} = \dfrac{h}{H}, \ \ \widetilde{k} = Hk, \ \ s = \dfrac{\rho}{\widetilde{h}} \\ \psi(\theta;s,y,\alpha,h) = \dfrac{1}{H}\varphi\Big(\theta;\dfrac{s}{H},y,\alpha,h\Big). \end{cases} \tag{7.34}$$

We obtain

$$I_1 = \widetilde{h}\iint_{W_2} e^{i\widetilde{h}^{-2}\widetilde{k}^{-1}\psi(\theta;s,y,\alpha,h)}a\Big(\theta;\frac{s}{H},y,\alpha,h,k\Big)\widetilde{\chi}\Big(\theta;\frac{s}{H},y\Big)\overline{v}(t-k\theta;\widetilde{h}s,y)\,ds\,dy\,d\theta.$$

If $(\theta,s/H,y) \in W_2$ then one can find $(\theta_1,\widetilde{s}_1) \in D_1$ (see Theorem 7.1) such that

$$|\theta-\theta_1| \leqslant \frac{1}{K'}Q_1, \quad \Big|\frac{s}{H} - \frac{\widetilde{s}_1}{Q_1}\Big| \leqslant \frac{1}{K'}\cdot\frac{1}{Q_1}$$

where $Q_1 = 1 - 2\alpha_\tau\widetilde{s}_1^3\theta_1$ (see (7.14)). Let $(\theta,\widetilde{s},\widetilde{y}) \in \underline{\Delta}$ satisfying $\Phi(\theta,\widetilde{s},\widetilde{y}) = (\theta,s/H,y)$. Then $Q_1 = 1 - 2\alpha_\tau(\operatorname{Re}\widetilde{s})^3\operatorname{Re}\theta + \mathcal{O}(\frac{1}{K}+\delta_0)Q_1$. Therefore

$$\frac{1}{2}KH + \mathcal{O}\Big(\frac{1}{K}+\delta_0\Big)Q_1 \leqslant Q_1 \leqslant 2KH + \mathcal{O}\Big(\frac{1}{K}+\delta_0\Big)Q_1$$

so, if $\frac{1}{K} + \delta_0$ is small enough,

$$\frac{1}{4} KH \leqslant Q_1 \leqslant 3KH. \tag{7.35}$$

It follows that

$$|s| \leqslant \left|s - \frac{\widetilde{s}_1}{Q_1} H\right| + \left|\widetilde{s}_1 \frac{H}{Q_1}\right| \leqslant \frac{H}{K'Q_1} + |\widetilde{s}_1| \frac{H}{Q_1} \leqslant \left(\frac{1}{K'} + |\widetilde{s}_1|\right) \frac{3}{K}.$$

This shows that $|s|$ is small if K is large enough. Our goal is to apply Theorem A.14 in the Appendix. Therefore we have to show that ψ is a phase satisfying the conditions of Definition A.4 in the Appendix.

Let us check that $\operatorname{Im}\psi \geqslant 0$, when the variables are real. Since φ satisfies (7.2) and p is a real quadratic form in (λ, μ), we see easily that $\operatorname{Im}\varphi$ is the solution of a linear vector field which is transverse to the hypersurface $\theta = 0$. Since $\operatorname{Im}\varphi|_{\theta=0} \geqslant 0$, the positivity propagates as long as φ exists.

The second point is to check condition 3) and, first of all to find the point ξ_0.

Let us recall that, according to Lemma 3.4 and Corollary 3.6, if $(\rho, y, 0, (\lambda, \mu))$ is a point such that $\rho + |\mu| \leqslant \varepsilon_0$, $(\rho, y) \neq (0, y_0)$ which satisfies $y_0 = y + \rho F_1(\rho, y, \mu) + \mu F_2(\rho, y, \mu)$ and λ is the unique negative solution of $p(\rho, y, \lambda, \mu) = 1$, then $(\rho, y, 0, (\lambda, \mu))$ belongs to $\dot{N}^{-1}_{+\infty}(m_0)$. Moreover $\mu = \mu(\rho, y), \lambda = \lambda(\rho, y)$.

We fix $(\widetilde{h}_1, s_1, y_1)$ in $[0,1[\times \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and we consider the following neighborhood of this point

$$V_r = \left\{(\widetilde{h}, s, y) \in [0,1[\times \mathbb{C} \times \mathbb{C}^{n-1} : |\widetilde{h} - \widetilde{h}_1| + \left|\frac{1}{s^2} - \frac{1}{s_1^2}\right| + |y - y_1| < r\right\} \tag{7.36}$$

where r is to be chosen. We also assume that H= $h + |\alpha_\eta| < r$. Then we set

$$\begin{cases} \lambda_1 = \lambda(\widetilde{h}_1 s_1, y_1), \quad \mu_1 = \mu(\widetilde{h}_1 s_1, y_1) \text{ and} \\ \xi_0 = \left(\frac{\lambda_1}{s_1^3}, \frac{\mu_1}{s_1^2}\right). \end{cases} \tag{7.37}$$

Proposition 7.8. — *If r is small enough, we have for all $(\widetilde{h}, s, y)$ in V_r,*

$$\left|\frac{\partial\psi}{\partial s}(\theta; s, y) - \frac{\lambda_1}{s_1^3}\right| + \left|\frac{\partial\psi}{\partial y}(\theta; s, y) - \frac{\mu_1}{s_1^2}\right| \leqslant \varepsilon_0 \tag{7.38}$$

where ε_0 is the constant appearing in Definition A.4 in the Appendix.

Proof. — From the definition of ψ we have, (see (7.13) and (7.15)),

$$\frac{\partial\psi}{\partial s}(\theta, s, y) = \frac{1}{H^2} \underline{\tau}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), \quad \frac{\partial\psi}{\partial y}(\theta; s, y) = \frac{1}{H} \underline{\eta}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h)$$

where $\Phi(\theta; \widetilde{s}, \widetilde{y}) = (\theta, s/H, y)$ that is $\underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = s/H$ and $\underline{y}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = y$.

Lemma 7.9. — *Let $(\widetilde{s}', \widetilde{y}')$ be the solution, which is real, of*

$$\begin{cases} s(\theta; \widetilde{s}', \widetilde{y}', \alpha_\tau, \alpha_\eta, h) = \dfrac{s}{H} (= \underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h)), \ |s| \sim \dfrac{c}{K}, \\ y(\theta; \widetilde{s}', \widetilde{y}', \alpha_\tau, \alpha_\eta, h) = y (= \underline{y}(\cdots)). \end{cases}$$

Then

$$(i) \qquad \begin{cases} |\widetilde{s} - \widetilde{s}'| \leqslant Ch(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|) \\ |\widetilde{y} - \widetilde{y}'| \leqslant C\dfrac{\widetilde{h}}{K}(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|) \end{cases}$$

$$(ii) \qquad |\underline{g}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) - g(\theta; \widetilde{s}', \widetilde{y}', \alpha_\tau, \alpha_\eta, h)| \leqslant C\widetilde{h}(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|),$$

where $g = \frac{1}{H^2}\tau$ *or* $\frac{1}{H}\eta$, $\underline{g} = \frac{1}{H^2}\underline{\tau}$ *or* $\frac{1}{H}\underline{\eta}$.

Proof. — By the proof of Proposition 7.4 (see (7.17)) for fixed θ the solution $(\widetilde{s}', \widetilde{y}')$ (resp. $(\widetilde{s}, \widetilde{y})$) exists and $(\theta, \widetilde{s}', \widetilde{y}')$ (resp. $(\theta, \widetilde{s}, \widetilde{y})$) belongs to $\underline{\Delta}$ (see (7.14)). This means that $|\widetilde{s}' - \widetilde{s}_1| < \delta_0 Q_1$, $|\widetilde{y}' - y_0| < \varepsilon_y$, $|\widetilde{s} - \widetilde{s}_1| < \delta_0 Q_1$, $|\widetilde{y} - y_0| < \varepsilon_y$. It follows that $|\widetilde{s} - \widetilde{s}'| < 2\delta_0 Q_1$ and $|\widetilde{y} - \widetilde{y}'| < 2\varepsilon_y$.

Now for t in $[0, 1]$ let us set

$$M_t = t(\widetilde{s}, \widetilde{y}) + (1-t)(\widetilde{s}', \widetilde{y}') = (\widetilde{s}_t, \widetilde{y}_t).$$

With the notations of Proposition 7.2 we have

$$1 - 2\alpha_\tau \widetilde{s}_t^3 \theta + \widetilde{s}_t F(\theta; M_t, \alpha, h) = 1 - 2\alpha_\tau \widetilde{s}_1^3 \theta + \mathcal{O}\Big(|\widetilde{s} - \widetilde{s}_1| + |\widetilde{s}' - \widetilde{s}_1| + \frac{Q_1}{K_2}\Big).$$

It follows then that we have

$$\big|1 - 2\alpha_\tau \widetilde{s}_t^3 \theta + \widetilde{s}_t F(\theta; M_t, \alpha, h)\big| \geqslant C_1 Q_1.$$

Let us set

$$a = \frac{\partial s}{\partial \widetilde{s}}(\theta; \widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta}, h), \ u_j = \frac{\partial s(\theta)}{\partial \widetilde{y}_j}, \ v_j = \frac{\partial y_j(\theta)}{\partial \widetilde{s}}, \quad 1 \leqslant j \leqslant n,$$

$$D = \Big(\frac{\partial y_j(\theta)}{\partial \widetilde{y}_k}\Big)_{1 \leqslant j,k \leqslant n-1}, \quad U = (u_j), \ V = (v_j).$$

Then Corollary 7.3 shows that,

$$\frac{C_1}{Q_1^2} \leqslant |a| \leqslant \frac{C_2}{Q_1^2}, \ |u_j| \leqslant \frac{C}{K^2 Q_1}, \ |v_j| \leqslant \frac{C}{Q_1}, \ D = \mathrm{Id} + \mathcal{O}\Big(\frac{1}{K}\Big).$$

Moreover

$$\frac{\partial s(\theta)}{\partial \widetilde{\tau}} = \mathcal{O}\Big(\frac{1}{Q_1^2}\Big), \ \frac{\partial s(\theta)}{\partial \widetilde{\eta}} = \mathcal{O}\Big(\frac{1}{K Q_1^2}\Big), \ \frac{\partial y(\theta)}{\partial \widetilde{\tau}} = \mathcal{O}\Big(\frac{1}{Q_1}\Big) \text{ and } \frac{\partial y(\theta)}{\partial \widetilde{\eta}} = \mathcal{O}\Big(\frac{1}{Q_1}\Big).$$

Using the formula (7.10)′ we obtain

$$\begin{aligned}
&\Big|\frac{\partial^2 s}{\partial \widetilde{s}^2}(\theta, M_t, \alpha, h)\Big| \leqslant \frac{C}{Q_1^3}, \qquad &&\Big|\frac{\partial^2 s}{\partial \widetilde{s}\partial \widetilde{y}}(\theta, M_t, \alpha, h)\Big| \leqslant \frac{C}{KQ_1^2}\\
&\Big|\frac{\partial^2 s}{\partial \widetilde{y}^2}(\theta, M_t, \alpha, h)\Big| \leqslant \frac{C}{KQ_1}, &&\Big|\frac{\partial^2 y}{\partial \widetilde{s}^2}\Big| \leqslant \frac{C}{KQ_1^2}\\
&\Big|\frac{\partial^2 y}{\partial \widetilde{s}\partial \widetilde{y}}(\theta, M_t, \alpha, h)\Big| \leqslant \frac{C}{KQ_1}, &&\Big|\frac{\partial^2 y}{\partial \widetilde{y}^2}\Big| \leqslant \frac{C}{K}.
\end{aligned}$$

By the Taylor formula with integral remainder we get

$$\begin{aligned}
\begin{pmatrix}\underline{s}\\ \underline{y}\end{pmatrix}(\theta, \widetilde{s}, \widetilde{y}, \alpha, h) &= \begin{pmatrix}s\\ y\end{pmatrix}(\theta, \widetilde{s}, \widetilde{y}, \alpha_\tau + 2ih(\widetilde{s}-\alpha_s), \alpha_\eta + 2ih(\widetilde{y}-\alpha_y), \alpha, h)\\
&= \begin{pmatrix}s\\ y\end{pmatrix}(\theta, \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta, h) + \begin{pmatrix}Y_1\\ Y'\end{pmatrix}
\end{aligned}$$

where

$$|Y_1| \leqslant \frac{Ch}{Q_1^2}\big(|\widetilde{s}-\alpha_s| + |\widetilde{y}-\alpha_y|\big), \quad |Y'| \leqslant \frac{Ch}{Q_1}\big(|\widetilde{s}-\alpha_s| + |\widetilde{y}-\alpha_y|\big).$$

Let us set $B = |\widetilde{s}-\widetilde{s}'| + Q_1|\widetilde{y}-\widetilde{y}'|$. Then we have $B \leqslant C(\delta_0 + \varepsilon_y)Q_1$.

Now, using the Taylor formula up to the second order and the above estimates on the second derivatives of s, y, we get

$$\begin{pmatrix}s\\ y\end{pmatrix}(\theta, \widetilde{s}, \widetilde{y}, \alpha, h) - \begin{pmatrix}s\\ y\end{pmatrix}(\theta, \widetilde{s}', \widetilde{y}', \alpha, h) = \begin{pmatrix}a & U\\ V & D\end{pmatrix}\begin{pmatrix}\widetilde{s}-\widetilde{s}'\\ \widetilde{y}-\widetilde{y}'\end{pmatrix} + \begin{pmatrix}\mathcal{O}\big(B^2/Q_1^3\big)\\ \mathcal{O}\big(B^2/Q_1^2\big)\end{pmatrix}.$$

We deduce from the above computations and the hypotheses in the Lemma 7.9 that,

$$\begin{aligned}
\begin{pmatrix}0\\ 0\end{pmatrix} &= \begin{pmatrix}\underline{s}\\ \underline{y}\end{pmatrix}(\theta, \widetilde{s}, \widetilde{y}, \alpha, h) - \begin{pmatrix}s\\ y\end{pmatrix}(\theta, \widetilde{s}', \widetilde{y}', \alpha, h)\\
&= \begin{pmatrix}a & U\\ V & D\end{pmatrix}\begin{pmatrix}\widetilde{s}-\widetilde{s}'\\ \widetilde{y}-\widetilde{y}'\end{pmatrix} + \begin{pmatrix}\mathcal{O}\big(B^2/Q_1^3\big)\\ \mathcal{O}\big(B^2/Q_1^2\big)\end{pmatrix} + \begin{pmatrix}Y_1\\ Y'\end{pmatrix}.
\end{aligned}$$

Let us set $M = (m_{jk}) = \big(\frac{1}{a}v_j u_k\big)$; then $|m_{jk}| \leqslant C/K^2$. It follows that $D - M$ is invertible and $(D-M)^{-1} = I + \mathcal{O}(1/K)$. Then we have

$$\begin{cases}
\widetilde{y}-\widetilde{y}' = (D-M)^{-1}\Big(-\dfrac{1}{a}Y_1 V + Y'\Big) + \mathcal{O}\Big(\dfrac{B^2}{Q_1^2}\Big)\\[2ex]
\widetilde{s}-\widetilde{s}' = \dfrac{1}{a}Y_1 - \dfrac{1}{a}U\cdot(\widetilde{y}-\widetilde{y}') + \mathcal{O}\Big(\dfrac{B^2}{Q_1}\Big).
\end{cases}$$

This implies that

$$B \leqslant Ch\big(|\widetilde{s}-\alpha_s| + |\widetilde{y}-\alpha_y|\big) + C\frac{B^2}{Q_1}.$$

Now $B/Q_1 \leqslant C(\delta_0 + \varepsilon_y)$; it follows that $B \leqslant C'h(|\widetilde{s}-\alpha_s| + |\widetilde{y}-\alpha_y|)$ since $\delta_0 + \varepsilon_y$ is small enough, which proves the first part of the lemma, since $Q_1 \geqslant KH$ and $\widetilde{h} = h/H$.

Since $s(\theta) = s/H = \underline{s}(\theta)$, with $|s| \sim C/K$, we have

$$\left|\frac{1}{H^2}\underline{\tau}(\theta) - \frac{1}{H^2}\tau(\theta)\right| = \frac{1}{s^2}|\underline{\tau}\underline{s}^2(\theta) - \tau s^2(\theta)| = (1).$$

Now, by (7.8), $(\underline{\tau}\underline{s}^2)(\theta) = \widetilde{\tau}\widetilde{s}^2 + \underline{f}$ and $(\tau\widetilde{s}^2)(\theta) = \alpha_\tau \widetilde{s}'^2 + f$ where $|f| + |\underline{f}| \leqslant C/K^2$. It follows that

$$(1) \leqslant CK^2 h(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|) + \frac{1}{s^2}|\underline{f}(\theta) - f(\theta)|.$$

On the other hand,

$$\frac{\partial f}{\partial \widetilde{s}} = \mathcal{O}\Big(\frac{1}{K^2 Q_1}\Big), \quad \frac{\partial f}{\partial \widetilde{y}} = \mathcal{O}\Big(\frac{1}{K^2}\Big), \quad \frac{\partial f}{\partial \widetilde{\tau}} = \mathcal{O}\Big(\frac{1}{KQ_1}\Big) \quad \text{and} \quad \frac{\partial f}{\partial \widetilde{\eta}} = \mathcal{O}\Big(\frac{1}{KQ_1}\Big).$$

Then using Taylor's formula and the part (i) of the lemma we obtain

$$(1) \leqslant C\widetilde{h}(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|),$$

if $K^2 H$ is bounded, and (ii) follows. The same argument applies to $\frac{1}{H}\eta$. □

Let us now prove (7.38). Using the notations of Lemma 7.9, we write

$$\frac{\partial\psi}{\partial s} - \frac{\lambda_1}{s_1^3} = \underbrace{\frac{1}{H^2}\underline{\tau}(\theta) - \frac{1}{H^2}\tau(\theta)}_{(1)} + \underbrace{\Big(\frac{1}{s^2} - \frac{1}{s_1^2}\Big)\tau s^2}_{(2)} + \underbrace{\frac{1}{s_1^2}\Big(\tau s^2 - \frac{\lambda_1}{s_1}\Big)}_{(3)}.$$

The term (1) is bounded by $C\widetilde{h}\delta_2\varepsilon_0 \leqslant \frac{1}{3}\varepsilon_0$, by Lemma 7.9. Since τs^2 is bounded, we have $|(2)| \leqslant Cr \leqslant \frac{1}{3}\varepsilon_0$. Let us look to (3).

In that follows we shall write $(\widetilde{s}, \widetilde{y})$ instead of $(\widetilde{s}', \widetilde{y}')$. Let us set

$$\text{(7.39)} \qquad \begin{cases} \rho^* = hs(\theta; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta, h) \\ y^* = y(\theta; \cdots) \\ \lambda^* = \dfrac{1}{R}(\tau s^2)(\theta; \cdots) \\ \mu^* = \dfrac{1}{R}(s\eta)(\theta; \cdots) \end{cases}$$

where $R = p(h\widetilde{s}, \widetilde{y}, \alpha_\tau \widetilde{s}^2, \alpha_\eta \widetilde{s}) \neq 0$.

Let us set $m^* = (\rho^*, y^*, 0, (\lambda^*, \mu^*))$. It follows from Proposition 7.7 that one can find $T^* > 0$ such that

$$\exp T^*\sigma H_\Delta(m^*) = \Big(h\widetilde{s}, \widetilde{y}, 0, \Big(\frac{1}{R}\alpha_\tau \widetilde{s}^2, \frac{1}{R}\widetilde{s}\alpha_\eta\Big)\Big).$$

Therefore, if $h + |\alpha_\eta|$ is small enough, Lemma 3.4 shows that

$$\lim_{t\to+\infty} \exp t\sigma H_\Delta(m^*) = \lim_{t\to+\infty} \exp t\sigma H_\Delta(h\widetilde{s}, \widetilde{y}, \cdots) = (0, \underline{y}, 0, (-1, 0))$$

where $|\underline{y} - \widetilde{y}| \leqslant C_1 H$.

Then Corollary 3.5 shows that $d(m^*, N_{+\infty}^{-1}(m_0)) \leqslant C_2 H$. This means that one can find $(\rho_2, y_2, 0, (\lambda_2, \mu_2)) \in N_{+\infty}^{-1}(m_0)$ such that

$$|\rho^* - \rho_2| + |y^* - y_2| + |\lambda^* - \lambda_2| + |\mu^* - \mu_2| \leqslant C_2 H.$$

It follows that

$$\begin{aligned}|\lambda^* - \lambda(\rho^*, y^*)| + |\mu^* - \mu(\rho^*, y^*)| \\ \leqslant |\lambda^* - \lambda_2| + |\lambda_2 - \lambda(\rho^*, y^*)| + |\mu^* - \mu_2| + |\mu_2 - \mu(\rho^*, y^*)| \\ \leqslant 2C_2 H + C_3(|\rho_2 - \rho^*| + |y_2 - y_2^*|) \leqslant C_4 H\end{aligned}$$

because $\lambda_2 = \lambda(\rho_2, y_2)$, $\mu_2 = \mu(\rho_2, y_2)$ where λ and μ are holomorphic functions.

Therefore we have

$$\begin{aligned}|\lambda^* - \lambda_1| + |\mu^* - \mu_1| \leqslant |\lambda^* - \lambda(\rho^*, y^*)| \\ + |\mu^* - \mu(\rho^*, y^*)| + |\lambda(\rho^*, y^*) - \lambda(\rho_1, y_1)| + |\mu(\rho^*, y^*) - \mu(\rho_1, y_1)| \\ \leqslant CH + C(|\rho^* - \rho_1| + |y^* - y_1|) \leqslant CH.\end{aligned}$$

Now $R^2 = \widetilde{s}^4 \alpha_\tau^2 + \widetilde{s}^2 \alpha_\eta^2 + O(h) = 1/s_1^2 + \mathcal{O}(H)$. Then

$$\Big|R\lambda^* - \frac{\lambda_1}{s_1}\Big| + \Big|R\mu^* - \frac{\mu_1}{s_1}\Big| \leqslant CH$$

which, according to (7.39) proves (7.38) and completes the proof of Proposition 7.8. □

Let us now check condition 4 in the Definition A.4 in the Appendix.

Lemma 7.10. — *One can find a positive constant C such that for real (θ, s, y),*

$$\Big|\operatorname{Im} \frac{\partial \psi}{\partial x}(\theta, s, y, \alpha, h)\Big| \leqslant \varepsilon_0 \widetilde{h}, \quad x = s \text{ or } y.$$

Proof. — We use Lemma 7.9. The observation is that, if s/H and y are real, then $(\theta, \widetilde{s}', \widetilde{y}')$ is real ; this implies that $\tau(\theta; \widetilde{s}', \widetilde{y}', \alpha_\tau, \alpha_\eta, h)$ is real. Now

$$\frac{\partial \psi}{\partial s}(\theta, s, y, \alpha, h) = \frac{1}{H^2}\underline{\tau}(\theta) = \frac{1}{H^2}\tau(\theta) + \frac{1}{H^2}(\underline{\tau}(\theta) - \tau(\theta)).$$

So $\big|\operatorname{Im} \partial\psi/\partial s\big| \leqslant \frac{1}{H^2}|\underline{\tau}(\theta) - \tau(\theta)| \leqslant C\widetilde{h}\delta_2\varepsilon_0 \leqslant \varepsilon_0$, by Lemma 7.9, since $\widetilde{h} \leqslant 1$, taking $C\delta_2 \leqslant 1$. The same argument works for $\operatorname{Im} \partial\psi/\partial y$. □

The condition 5) in Definition A.4 follows from the holomorphy of ψ, so we are left with condition 6) which is

(7.40) $$(\operatorname{Im} \psi'')(\theta, s, y) \geqslant -\varepsilon_0 \widetilde{h}, \quad \text{for real } (\theta, s, y).$$

Let us recall that we have set

$$\begin{cases} \underline{s}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = \dfrac{s}{H} \\ \underline{y}(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = y. \end{cases}$$

Then, if we set

$$a = H\frac{\partial \underline{s}}{\partial \widetilde{s}}, \quad b_j = \frac{\partial \underline{y}_j}{\partial \widetilde{s}}, \quad \gamma_k = H\frac{\partial \underline{s}}{\partial \widetilde{y}_k}, \quad D = \Big(\frac{\partial \underline{y}_j}{\partial \widetilde{y}_k}\Big),$$

we have

$$\frac{C'}{K^2H}\leqslant |a|\leqslant \frac{C}{K^2H},\quad |b_j|\leqslant \frac{C}{K^2H},\quad |\gamma_k|\leqslant \frac{C}{K^3},\quad D=I+\mathcal{O}\Big(\frac{1}{K}\Big).$$

Let us recall (see (7.10)) that $1/\underline{s}(\theta)=1/\widetilde{s}-2\widetilde{\tau}\widetilde{s}^2\theta+F$, where $\widetilde{\tau}=\alpha_\tau+2ih(\widetilde{s}-\alpha_s)$ and $F=\mathcal{O}\big(Q_1/K^2\big)$. It follows that

$$\frac{\partial\underline{s}(\theta)}{\partial\widetilde{s}}=\frac{1+4\widetilde{\tau}\widetilde{s}^3\theta+4ih\widetilde{s}^4\theta-\widetilde{s}^2\frac{\partial F}{\partial\widetilde{s}}}{(1-2\widetilde{\tau}\widetilde{s}^3\theta+\widetilde{s}F)^2}=\frac{U}{V}.$$

From the Cauchy formula, integrating on a ball $|\widetilde{s}-\zeta|=\delta Q_1$, we see that

$$\Big|\frac{\partial F(\theta)}{\partial\widetilde{s}}\Big|\leqslant C\frac{Q_1}{K^2}\cdot\frac{1}{\delta Q_1}=\mathcal{O}\Big(\frac{1}{K^2}\Big)^2.$$

Now, since $4\widetilde{\tau}\widetilde{s}^3\theta$ is close to $4\alpha_\tau s_0^3\operatorname{Re}\theta$ which is non negative, we deduce that $|U|$ is bounded above and below by strictly positive constants. On the other hand we can write $2\widetilde{\tau}\widetilde{s}^3\theta+\widetilde{s}F=Q_1+\mathcal{O}\big(\frac{1}{K}+\delta_0\big)Q_1+\mathcal{O}(h)$. Since $h\leqslant H<Q_1/K$ (see (7.1)) we deduce that $|V|$ is bounded above and below by CQ_1^2. Since, by (7.35), Q_1 is equivalent to KH we deduce that $H\,\partial\underline{s}(\theta)/\partial\widetilde{s}$ is uniformly equivalent to $H/K^2H^2=1/K^2H$, which is our first claim. For the estimate on b_j, we use the fact that $\underline{y}_j(\theta)=\widetilde{y}_j+G_j(\theta)$, where $G_j=\mathcal{O}\big(1/K\big)$. By the argument used above we get,

$$\Big|\frac{\partial G_j}{\partial\widetilde{s}}\Big|\leqslant\frac{C}{K}\cdot\frac{1}{\delta Q_1}=\frac{C}{\delta}\frac{1}{K^2H}.$$

The other estimates follow also from the expressions of $\underline{s}(\theta)$, $\underline{y}(\theta)$ and the Cauchy formula on a ball $|\widetilde{y}_j-\zeta|=\varepsilon$. It follows that

$$\frac{b_j}{a}=\mathcal{O}(1),\quad\frac{\gamma_k}{a}=\mathcal{O}\Big(\frac{H}{K}\Big),\quad m_{jk}=\frac{b_j\gamma_k}{a}=\mathcal{O}\Big(\frac{1}{K^3}\Big).$$

We set $M=(m_{jk})$, then $(D-M)^{-1}=\operatorname{Id}+\mathcal{O}\big(1/K\big)$ and

$$\text{(7.41)}\qquad\begin{cases}\dfrac{\partial}{\partial s}=\dfrac{1}{a}\Big(1+\dfrac{b}{a}(D-M)^{-1}\gamma\Big)\dfrac{\partial}{\partial\widetilde{s}}-\dfrac{b}{a}(D-M)^{-1}\dfrac{\partial}{\partial\widetilde{y}}\\[2mm]\dfrac{\partial}{\partial y}=(D-M)^{-1}\Big[\dfrac{\partial}{\partial\widetilde{y}}-\dfrac{\gamma}{a}\dfrac{\partial}{\partial\widetilde{s}}\Big].\end{cases}$$

Let us recall that we have

$$\frac{\partial\psi}{\partial y}=\frac{1}{H}\underline{\eta}(\theta;\widetilde{s},\widetilde{y},\alpha,h)$$

and, by (7.3),

$$\underline{\dot\eta}(t)=-\frac{\partial p}{\partial y}(h\underline{s}(t),\underline{y}(t),\underline{\tau}\,\underline{s}^2(t),\underline{s}\underline{\eta}(t))$$

with $\underline{\eta}(0)=\widetilde{\eta}=\alpha_\eta+2ih(\widetilde{y}-\alpha_y)$. It follows that, with $\widetilde{h}=h/H$,

$$\frac{\partial^2\psi}{\partial y^2}=(D-M)^{-1}\Big(2i\widetilde{h}\operatorname{Id}+\frac{1}{H}\int_\theta^0\Big(\frac{\partial}{\partial\widetilde{y}}-\frac{\gamma}{a}\frac{\partial}{\partial\widetilde{s}}\Big)\Big[\frac{\partial p}{\partial y}(\cdots)\Big]d\sigma\Big).$$

We claim that, with $A = |\operatorname{Re}\widetilde{s} - \alpha_s| + |\operatorname{Re}\widetilde{y} - \alpha_y|$, we have

$$\operatorname{Im} \frac{\partial^2\psi}{\partial y^2} = 2\widetilde{h}\,\mathrm{Id} + \frac{\widetilde{h}}{K}\mathcal{O}\Big(\frac{1}{K} + A\Big). \tag{7.42}$$

This will be achieved if we prove that

$$\operatorname{Im}\int_\theta^0 \Big(\frac{\partial}{\partial\widetilde{y}} - \frac{\gamma}{a}\frac{\partial}{\partial\widetilde{s}}\Big)\Big[\frac{\partial p}{\partial y}(\cdots)\Big]\,d\sigma = \mathcal{O}\Big(\frac{h}{K^2} + \frac{hA}{K}\Big) \tag{7.43}$$

since $(D - M)^{-1} = \mathrm{Id} + \mathcal{O}(1/K)$.

Now $p = \lambda^2 + \sum h^{jk}(y)\,\mu_j\,\mu_k + \rho r$, where $r = \rho^2 r_0(\rho, y)\lambda^2 + \rho r_1(\rho, y)\lambda\mu$ and it will be clear from the method that the term ρr can be handled in the same manner. Therefore let us assume that $p = \lambda^2 + f(y)\mu^2$. Then $\partial p/\partial y = f_1(y)\mu^2$ and

$$\begin{aligned}(7.43)'\quad \Big(\frac{\partial}{\partial\widetilde{y}} - \frac{\gamma}{a}\frac{\partial}{\partial\widetilde{s}}\Big)\Big[\frac{\partial p}{\partial y}(h\underline{s}(\sigma), \underline{y}(\sigma), \underline{\tau}\,\underline{s}^2, \underline{s}\,\underline{\eta})\Big] &= \underline{s}^2\,\underline{\eta}^2\cdot f_2(\underline{y})\cdot\Big(\frac{\partial \underline{y}}{\partial\widetilde{y}} - \frac{\gamma}{a}\frac{\partial \underline{y}}{\partial\widetilde{s}}\Big)\\ +2f_1(\underline{y})\,\underline{\eta}^2\,\underline{s}\Big(\frac{\partial \underline{s}}{\partial\widetilde{y}} - \frac{\gamma}{a}\frac{\partial \underline{s}}{\partial\widetilde{s}}\Big) + 2f_1(\underline{y})\,\underline{s}^2\,\underline{\eta}\Big(\frac{\partial \underline{\eta}}{\partial\widetilde{y}} - \frac{\gamma}{a}\frac{\partial \underline{\eta}}{\partial\widetilde{s}}\Big) &= (1) + (2) + (3)\end{aligned}$$

where f_j, j$= 1, 2$ are smooth function which are real if y is real.

We recall that, if $f = s, y, \tau, \eta$, we have

$$\underline{f}(\sigma; \widetilde{s}, \widetilde{y}, \alpha, h) = f\big(\sigma; \widetilde{s}, \widetilde{y}, \alpha_\tau + 2ih(\widetilde{s} - \alpha_s), \alpha_\eta + 2ih(\widetilde{y} - \alpha_y), h\big).$$

Lemma 7.11. — *With the notations of (7.28) and Lemma 7.9, let $(\theta, \widetilde{s}, \widetilde{y}) \in W_2$. Then, for $\sigma \in [\theta, 0]$ we have the following estimates.*

$$\left\{\begin{aligned}
&|\underline{s}(\sigma) - s(\Sigma)| \leqslant ChAu_0^2,\quad |\underline{y}(\sigma) - y(\Sigma)| \leqslant C\frac{hA}{KH},\\
&|\underline{\eta}(\sigma) - \eta(\Sigma)| \leqslant ChA,\quad |\underline{s}(\sigma)| + |s(\Sigma)| \leqslant Cu_0,\\
&\Big|\frac{\partial}{\partial\widetilde{s}}(\underline{s})(\sigma) - \frac{\partial s}{\partial\widetilde{s}}(\Sigma)\Big| \leqslant Chu_0^2\Big(1 + \frac{A}{KH}\Big),\\
&\Big|\frac{\partial}{\partial\widetilde{y}}(\underline{s})(\sigma) - \frac{\partial s}{\partial\widetilde{y}}(\Sigma)\Big| \leqslant C\frac{h}{K}u_0^2,\\
&\Big|\frac{\partial}{\partial\widetilde{s}}(\underline{y})(\sigma) - \frac{\partial y}{\partial\widetilde{s}}(\Sigma)\Big| \leqslant C\Big(\frac{h}{K^2H} + \frac{hA}{K^2H^2}\Big),\\
&\Big|\frac{\partial\underline{y}(\sigma)}{\partial\widetilde{y}} - \frac{\partial y}{\partial\widetilde{y}}(\Sigma)\Big| \leqslant C\frac{h}{KH},\\
&\Big|\frac{\partial}{\partial\widetilde{s}}(\underline{\eta})(\sigma) - \frac{\partial\eta}{\partial\widetilde{s}}(\Sigma)\Big| \leqslant C\Big(h + \frac{hA}{KH}\Big),\\
&\Big|\frac{\partial}{\partial\widetilde{y}}(\underline{\eta})(\sigma) - \frac{\partial\eta}{\partial\widetilde{y}}(\Sigma)\Big| \leqslant Ch.
\end{aligned}\right. \tag{7.44}$$

where $\Sigma = (\sigma; \operatorname{Re}\widetilde{s}, \operatorname{Re}\widetilde{y}, \alpha_\tau, \alpha_\eta, h)$, $u_0(\sigma) = 1/(1 - 2\alpha_\tau(\operatorname{Re}\widetilde{s})^3\sigma)$ *and*

$$A = |\operatorname{Re}\widetilde{s} - \alpha_s| + |\operatorname{Re}\widetilde{y} - \alpha_y|.$$

Proof. — Since $-2\alpha_\tau(\operatorname{Re}\widetilde{s})^3$ is positive, we have in W_2 (see (7.28)),

$$1 - 2\alpha_\tau(\operatorname{Re}\widetilde{s})^3\sigma \geqslant 1 - 2\alpha_\tau(\operatorname{Re}\widetilde{s})^3\theta \geqslant Q_1 \geqslant \frac{1}{2}KH.$$

Thus $u_0(\sigma) \leqslant 2/KH$.

Moreover for any z, ζ such that $z = \operatorname{Re}\widetilde{s} + \mathcal{O}(H)$, $\zeta = \operatorname{Re}\widetilde{y} + \mathcal{O}(H)$ we have

$$1 - 2\widetilde{\tau}z^3\sigma + zF(\sigma; z, \zeta, \dots) = 1 - 2\alpha_\tau(\operatorname{Re}\widetilde{s})^3\sigma + \mathcal{O}(H)$$

since $\widetilde{\tau} = \alpha_\tau + 2ih(z - \alpha_s)$ and $F = \mathcal{O}(H)$. It follows that

$$\left|1 - 2\widetilde{\tau}z^3\sigma + zF(\sigma; z, \zeta, \dots)\right| \geqslant \frac{1}{2}u_0(\sigma)$$

if K is large enough.

Let us recall the rule of differentiation. By the Cauchy formula applied in the set (7.3)$'$, each time we differentiate a holomorphic function ψ with respect to $\widetilde{s}$ (resp. $\widetilde{y}, \widetilde{\tau}, \widetilde{\eta}$) we loose a factor which is $\mathcal{O}(1/Q_1)$ (resp. $\mathcal{O}(1)$, $\mathcal{O}(K/Q_1)$, $\mathcal{O}(K/Q_1)$) with respect to ψ. Note that $Q_1 \sim KH$ by (7.35). Recall now that,

$$s(\sigma; \widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta}, h) = \frac{\widetilde{s}}{1 - 2\widetilde{\tau}\widetilde{s}^3\sigma + \widetilde{s}F(\sigma)},$$

where $F = \mathcal{O}(H/K)$ (see (7.10)). By the above rule, we get

$$\begin{aligned}
&\frac{\partial F}{\partial \widetilde{s}} = \mathcal{O}\Big(\frac{1}{K^2}\Big), \quad \frac{\partial F}{\partial \widetilde{y}} = \mathcal{O}\Big(\frac{H}{K}\Big), \quad \frac{\partial F}{\partial x} = \mathcal{O}\Big(\frac{1}{K}\Big) \text{ if } x = \widetilde{\tau} \text{ or } \widetilde{\eta},\\
&\frac{\partial^2 F}{\partial \widetilde{s}^2} = \mathcal{O}\Big(\frac{1}{K^3H}\Big), \quad \frac{\partial^2 F}{\partial \widetilde{s}\partial \widetilde{y}} = \mathcal{O}\Big(\frac{1}{K^2}\Big), \quad \frac{\partial^2 F}{\partial \widetilde{s}\partial x} = \mathcal{O}\Big(\frac{1}{KH}\Big),\\
&\frac{\partial^2 F}{\partial \widetilde{y}^2} = \mathcal{O}\Big(\frac{H}{K}\Big), \quad \frac{\partial^2 F}{\partial \widetilde{y}\partial x} = \mathcal{O}\Big(\frac{1}{K}\Big), \quad \text{etc.}
\end{aligned}$$

Using the explicit expression of $\underline{s}(\sigma)$ given above we obtain the following estimates.

$$\begin{aligned}
&\frac{\partial s(\sigma)}{\partial \widetilde{s}} = \mathcal{O}(u_0^2), \quad \frac{\partial s(\sigma)}{\partial \widetilde{y}} = \mathcal{O}\Big(\frac{H}{K}u_0^2\Big), \quad \frac{\partial s(\sigma)}{\partial \widetilde{\tau}} = \mathcal{O}(u_0^2), \quad \frac{\partial s(\sigma)}{\partial \widetilde{\eta}} = \mathcal{O}\Big(\frac{1}{K}u_0^2\Big),\\
&\frac{\partial^2 s(\sigma)}{\partial \widetilde{s}^2} = \mathcal{O}\Big(\frac{1}{KH}u_0^2\Big), \quad \frac{\partial^2 s(\sigma)}{\partial \widetilde{s}\partial \widetilde{y}} = \mathcal{O}\Big(\frac{1}{K^2}u_0^2\Big), \quad \frac{\partial^2 s(\sigma)}{\partial \widetilde{s}\partial x} = \mathcal{O}\Big(\frac{1}{KH}u_0^2\Big),\\
&\frac{\partial^2 s(\sigma)}{\partial \widetilde{y}^2} = \mathcal{O}\Big(\frac{H}{K}u_0^2\Big), \quad \frac{\partial^2 s(\sigma)}{\partial \widetilde{y}\partial x} = \mathcal{O}\Big(\frac{1}{K}u_0^2\Big),
\end{aligned}$$

where we have used the estimate $u_0(\sigma) \leqslant 1/KH$.

Finally let us remark that, according to Lemma 7.9 we have,

$$|\operatorname{Im}\widetilde{s}| \leqslant ChA, \quad |\operatorname{Im}\widetilde{y}| \leqslant \frac{ChA}{HK}, \quad h\big(|\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y|\big) \leqslant ChA.$$

Let us now prove the estimates on s in (7.44)

$$\begin{aligned}
\underline{s}(\sigma) - s(\Sigma) = s(\sigma; \widetilde{s}, \widetilde{y}, \alpha_\tau + 2ih(\widetilde{s} - \alpha_s), \alpha_\eta + 2ih(\widetilde{y} - \alpha_y), h)&\\
- s(\sigma; \operatorname{Re}\widetilde{s}, \operatorname{Re}\widetilde{y}, \alpha_\tau, \alpha_\eta, h)&.
\end{aligned}$$

Thus, the first estimate in (7.44) follows from the Taylor formula and the above bounds on the derivatives of s.

Consider now $(*) = \frac{\partial}{\partial\widetilde{s}}(\underline{s})(\sigma) - \frac{\partial s}{\partial\widetilde{s}}(\Sigma)$. We have

$$(*) = \frac{\partial s}{\partial\widetilde{s}}(\sigma;\widetilde{s},\widetilde{y},\alpha_\tau + 2ih(\widetilde{s}-\alpha_s),\dots) - \frac{\partial s}{\partial\widetilde{s}}(\sigma;\operatorname{Re}\widetilde{s},\operatorname{Re}\widetilde{y},\dots) + 2ih\frac{\partial s}{\partial\widetilde{\tau}}(\sigma;\widetilde{s},\widetilde{y},\alpha_\tau + 2ih(\widetilde{s}-\alpha_s),\dots).$$

The last term in the right hand side is bounded by hu_0^2. Then

$$\operatorname{Im}\widetilde{s}\cdot\frac{\partial^2 s}{\partial\widetilde{s}^2} = \mathcal{O}\Big(\frac{hA}{KH}u_0^2\Big),\quad \operatorname{Im}\widetilde{y}\frac{\partial^2 s}{\partial\widetilde{s}\partial\widetilde{y}} = \mathcal{O}\Big(\frac{hA}{KH}\cdot\frac{1}{K^2}u_0^2\Big),$$
$$2ih(x^*-\alpha_s)\frac{\partial^2 s}{\partial\widetilde{s}\partial x} = \mathcal{O}\Big(hA\cdot\frac{1}{KH}u_0^2\Big),\quad x=\widetilde{\tau}\text{ or }\widetilde{\eta},\quad x^*=\widetilde{s}\text{ or }\widetilde{y}.$$

This proves the claimed bound for $(*)$.

Let us consider now $(**) = \frac{\partial}{\partial\widetilde{y}}(\underline{s})(\sigma) - \frac{\partial s}{\partial\widetilde{y}}(\Sigma)$. We have the term $2ih\,\partial s/\partial\widetilde{\eta}$ which is $\mathcal{O}(hu_0^2)$. Moreover

$$\operatorname{Im}\widetilde{s}\frac{\partial^2 s}{\partial\widetilde{s}\partial\widetilde{y}} = \mathcal{O}\Big(hA\frac{1}{K^2}u_0^2\Big),\quad \operatorname{Im}\widetilde{y}\frac{\partial^2 s}{\partial\widetilde{y}^2} = \mathcal{O}\Big(\frac{hA}{KH}\frac{H}{K}u_0^2\Big) = \mathcal{O}\Big(\frac{hA}{K^2}u_0^2\Big),$$
$$h(\widetilde{s}-\alpha_s)\frac{\partial^2 s}{\partial\widetilde{y}\partial\widetilde{\tau}} = \mathcal{O}\Big(hA\frac{1}{K}u_0^2\Big).$$

This proves the bound for $(**)$.

For the bounds concerning $y(\sigma)$ and $\eta(\sigma)$, we use the fact that we have,

$$y(\sigma) = \widetilde{y} + \mathcal{O}\Big(\frac{1}{K}\Big),\quad \eta(\sigma) = \widetilde{\eta} + \mathcal{O}(H).$$

Details are left to the reader. □

It follows from these estimates that (see (7.43)′)

$$(1) = \text{ real term } + \mathcal{O}\Big(hHAu_0^2 + \frac{hH}{K}u_0^2\Big).$$

Indeed we have, with $\Sigma = (\sigma;\operatorname{Re}\widetilde{s},\operatorname{Re}\widetilde{y},\alpha_\tau,\alpha_\eta,h)$

$$(1) = \underline{s}^2(\sigma)\underline{\eta}^2(\sigma)f_2(\underline{y}(\sigma))\Big(\frac{\partial\underline{y}(\sigma)}{\partial\widetilde{y}} - \frac{\gamma(\sigma)}{a(\sigma)}\frac{\partial\underline{y}(\sigma)}{\partial\widetilde{s}}\Big).$$

So we can write

$$(1) = s^2(\Sigma)\eta^2(\Sigma)f_2(y(\Sigma))\Big(\frac{\partial y}{\partial\widetilde{y}}(\Sigma) - \frac{\gamma(\Sigma)}{a(\Sigma)}\frac{\partial y}{\partial\widetilde{s}}(\Sigma)\Big) + R$$

where the first term in the right hand side is real. Moreover R is a finite sum of R_j, $j=1,\dots,6$, which we consider now. We have,

$$R_1 = (\underline{s}^2(\sigma) - s^2(\Sigma))\underline{\eta}^2(\sigma)f_2(\underline{y}(\sigma))\Big(\frac{\partial\underline{y}(\sigma)}{\partial\widetilde{y}} - \frac{\gamma(\sigma)}{a(\sigma)}\frac{\partial\underline{y}(\sigma)}{\partial\widetilde{s}}\Big).$$

By Lemma 7.11 we have $|\underline{s}^2(\sigma)-s^2(\Sigma)| \leqslant ChAu_0^3$; moreover $\underline{\eta}^2(\sigma)\, f_2(y(\sigma))$ is $\mathcal{O}(H^2)$,

$$\frac{\partial y}{\partial \widetilde{\underline{y}}} = O(1), \quad \frac{\gamma}{a} = \mathcal{O}\Big(\frac{H}{K}\Big) \text{ (see after (7.40)) and } \frac{\partial y}{\partial \widetilde{\underline{s}}} = \mathcal{O}\Big(\frac{1}{KH}\Big).$$

It follows that

$$R_1 = \mathcal{O}(hH^2Au_0^3) = \mathcal{O}\Big(\frac{hHA}{K}u_0^2\Big)$$

since $u_0 = \mathcal{O}(1/KH)$.

$$R_2 = s^2(\Sigma)(\underline{\eta}^2(\sigma) - \eta^2(\Sigma))\, f_2(\cdots)\Big(\frac{\partial y}{\partial \widetilde{\underline{y}}} - \frac{\gamma}{a}\frac{\partial y}{\partial \widetilde{\underline{s}}}\Big).$$

We have $|s^2(\Sigma)| \leqslant Cu_0^2$, $|\underline{\eta}^2(\sigma) - \eta^2(\Sigma)| \leqslant ChHA$. Therefore we get, $R_2 = \mathcal{O}(hHAu_0^2)$. Now $R_3 = s^2(\Sigma)\eta^2(\Sigma)[f_2(\underline{y}(\sigma)) - f_2(y(\Sigma))(\cdots)]$. We have

$$|f_2(\underline{y}(\sigma)) - f_2(y(\Sigma))| \leqslant C\,|\underline{y}(\sigma) - y(\Sigma)| \leqslant C\frac{hA}{KH}.$$

It follows that

$$R_3 = \mathcal{O}\Big(u_0^2H^2\frac{hA}{KH}\Big) = \mathcal{O}\Big(\frac{hHA}{K}u_0^2\Big).$$

Now

$$R_4 = s^2(\Sigma)\eta^2(\Sigma)\, f_2(y(\Sigma))\Big(\frac{\partial y}{\partial \widetilde{\underline{y}}}(\sigma) - \frac{\partial y}{\partial \widetilde{y}}(\Sigma)\Big) = \mathcal{O}\Big(u_0^2H^2\frac{h}{KH}\Big) = \mathcal{O}\Big(\frac{hH}{K}u_0^2\Big).$$

Then

$$R_5 = s^2(\Sigma)\eta^2(\Sigma)\, f_2(y(\Sigma))\Big(\frac{\gamma}{a}(\sigma) - \frac{\gamma}{a}(\Sigma)\Big)\frac{\partial y}{\partial \widetilde{\underline{s}}}(\sigma).$$

Recall that $a = H\,\partial s/\partial\widetilde{s}$ and $\gamma = H\,\partial s/\partial\widetilde{y}$. It follows that

$$\frac{\gamma}{a}(\sigma) - \frac{\gamma}{a}(\Sigma) = \frac{1}{\frac{\partial s}{\partial \widetilde{s}}(\sigma)\cdot\frac{\partial s}{\partial \widetilde{s}}(\Sigma)}\Big[\frac{\partial s}{\partial \widetilde{y}}(\sigma)\Big(\frac{\partial s}{\partial \widetilde{s}}(\Sigma) - \frac{\partial s}{\partial \widetilde{s}}(\sigma)\Big) + \frac{\partial s}{\partial \widetilde{s}}(\sigma)\Big(\frac{\partial s}{\partial \widetilde{y}}(\sigma) - \frac{\partial s}{\partial \widetilde{y}}(\Sigma)\Big)\Big].$$

The denominator is bounded below by Cu_0^4 and from Lemma 7.11, the numerator can be estimated by

$$C\Big(\frac{H}{K}u_0^2\cdot\Big(hu_0^2 + \frac{hA}{KH}u_0^2\Big) + u_0^2\cdot\frac{h}{K}u_0^2\Big) \leqslant C'\frac{h}{K}u_0^4.$$

It follows that

$$R_5 = \mathcal{O}\Big(H^2u_0^2\frac{h}{K}\Big)\frac{1}{KH} = \mathcal{O}\Big(\frac{hH}{K^2}u_0^2\Big).$$

Finally

$$R_6 = s^2(\Sigma)\eta^2(\Sigma)\, f_2(y(\Sigma))\,\frac{\gamma}{a}(\Sigma)\Big(\frac{\partial \underline{y}(\sigma)}{\partial \widetilde{s}} - \frac{\partial y}{\partial \widetilde{s}}(\Sigma)\Big)$$

can be estimated by

$$u_0^2H^2\cdot\frac{H}{K}\Big(\frac{h}{K^2H} + \frac{hA}{K^2H^2}\Big) = \mathcal{O}\Big(\frac{hH^2}{K^3}u_0^2 + \frac{hHA}{K^3}u_0^2\Big).$$

Summing up we get

$$R = \sum_{j=1}^{6} R_j = \mathcal{O}\Big(hHAu_0^2 + \frac{hH}{K}u_0^2\Big).$$

Since
$$\Big|\int_\theta^0 u_0^p(\sigma)\,d\sigma\Big| \leqslant \frac{C}{(KH)^{p-1}} \text{ if } p \geqslant 2,$$
we obtain
$$\operatorname{Im}\int_\theta^0 (1)\,d\sigma = \mathcal{O}\Big(\frac{hA}{K}+\frac{h}{K^2}\Big). \tag{7.45}$$
The other terms can be handled in the same manner, using the above estimates. This proves (7.42).

Our next claim is
$$\operatorname{Im}\frac{\partial^2\psi}{\partial s\partial y_j} = -2\widetilde{h}\operatorname{Re}\frac{b_j}{a} + \widetilde{h}\mathcal{O}\Big(\frac{1}{K}+A\Big). \tag{7.46}$$
Since
$$\frac{\partial\psi}{\partial y_j} = \frac{1}{H}(\alpha_\eta^j + 2ih(\widetilde{y}_j - \alpha_y^j)) + \frac{1}{H}\int_\theta^0 \frac{\partial p}{\partial y_j}(h\underline{s}(\sigma), \underline{y}(\sigma), \dots)\,d\sigma$$
and, by (7.41),
$$\frac{\partial}{\partial s} = -\frac{b}{a}\cdot\frac{\partial}{\partial\widetilde{y}} + \mathcal{O}\Big(\frac{1}{K}\Big)\cdot\frac{\partial}{\partial\widetilde{y}} + \frac{1}{a}\Big(1+\frac{b}{a}(D-M)^{-1}\gamma\Big)\frac{\partial}{\partial\widetilde{s}},$$
we can write,
$$\frac{\partial^2\psi}{\partial s\partial y_j} = -2i\widetilde{h}\frac{b_j}{a}\Big(1+\mathcal{O}\Big(\frac{1}{K}\Big)\Big) + \frac{1}{H}\frac{\partial}{\partial s}\Big(\int_\theta^0 \frac{\partial p}{\partial y_j}(h\underline{s}(\sigma), \underline{y}(\sigma), \dots)\,d\sigma\Big).$$
Thus (7.46) will follow from,
$$\operatorname{Im}\int_\theta^0 \Big[\frac{1}{a}\Big(1+\frac{b}{a}(D-M)^{-1}\gamma\Big)\frac{\partial}{\partial\widetilde{s}} - \frac{b}{a}(D-M)^{-1}\frac{\partial}{\partial\widetilde{y}}\Big]\Big(\frac{\partial p}{\partial y}(h\underline{s}(\sigma),\cdots)\Big)\,d\sigma = hO\Big(\frac{1}{K}+A\Big). \tag{7.47}$$
Since $b/a = O(1)$, the estimate of
$$\operatorname{Im}\int_\theta^0 \frac{b}{a}(D-M)^{-1}\frac{\partial}{\partial\widetilde{y}}\Big(\frac{\partial p}{\partial y}\Big)\,d\sigma$$
has been obtained in the proof of the preceding case. On the other hand we have
$$\frac{b}{a}(D-M)^{-1}\gamma = \mathcal{O}\Big(\frac{1}{K^3}\Big).$$
Therefore the main term remaining is
$$\operatorname{Im}\int_\theta^0 \frac{1}{a}\frac{\partial}{\partial\widetilde{s}}\Big(\frac{\partial p}{\partial y}\Big)\,d\sigma.$$
As before, we will assume that $p = \lambda^2 + f(y)\mu^2$, which implies that $\partial p/\partial y = f_1(y)\mu^2$. So we are left with the estimate of
$$\begin{aligned} I &= \operatorname{Im}\Big[\int_\theta^0 \frac{1}{a}f_2(\underline{y})\frac{\partial\underline{y}}{\partial\widetilde{s}}\underline{s}^2\underline{\eta}^2\,d\sigma + \int_\theta^0 \frac{2}{a}f_1(\underline{y})\underline{s}\cdot\frac{\partial\underline{s}}{\partial\widetilde{s}}\underline{\eta}^2\,d\sigma + \int_\theta^0 \frac{2}{a}f_1(\underline{y})\underline{s}^2\underline{\eta}\cdot\frac{\partial\underline{\eta}}{\partial\widetilde{s}}\,d\sigma\Big] \\ &= \operatorname{Im}\big((1)+(2)+(3)\big). \end{aligned}$$

Let us look to the first term. By (7.44) we have $|f_2(\underline{y}) - f_2(y)| < ChA/KH$. It follows that

$$\begin{aligned}
\Big|\Big(\frac{1}{a} - \frac{1}{a(\mathrm{real})}\Big) f_2(\underline{y}) \frac{\partial \underline{\underline{y}}}{\partial \widetilde{s}} \underline{s}^2 \underline{\eta}^2\Big| &\leqslant CKhA \cdot \frac{1}{K^2H} H^2 u_0^2 \leqslant ChHAu_0^2, \\
\Big|\frac{1}{a}(f_2(\underline{y}) - f_2(y)) \frac{\partial \underline{\underline{y}}}{\partial \widetilde{s}} \underline{s}^2 \underline{\eta}^2\Big| &\leqslant CK^2H \frac{h}{K^2H} A \frac{1}{K^2H} H^2 u_0^2 = \frac{ChAH}{K^2} u_0^2, \\
\Big|\frac{1}{a} f_2(y) \Big(\frac{\partial \underline{\underline{y}}}{\partial \widetilde{s}} - \frac{\partial \underline{y}}{\partial \widetilde{s}}\Big) \underline{s}^2 \underline{\eta}^2\Big| &\leqslant CK^2H \frac{hA}{K^2H^2} H^2 u_0^2 = ChAHu_0^2, \\
\Big|\frac{1}{a} f_2(y) \frac{\partial \underline{y}}{\partial \widetilde{s}} (\underline{s}^2 - s^2) \underline{\eta}^2\Big| &\leqslant CK^2H \frac{1}{K^2H} hAH^2 u_0^3 = ChAH^2 u_0^3, \\
\Big|\frac{1}{a} f_2(y) \frac{\partial y}{\partial \widetilde{s}} s^2 (\underline{\eta}^2 - \eta^2)\Big| &\leqslant CK^2H \frac{1}{K^2H} ChHAu_0^2 = ChAH^2 u_0^2.
\end{aligned}$$

It follows that

$$(1) = \text{ real term } + \Big(\mathcal{O} \int_\theta^0 hHAu_0^2(\sigma)\, d\sigma\Big).$$

Therefore

$$\operatorname{Im}(1) = \mathcal{O}\Big(\frac{hA}{K}\Big).$$

The same estimates and the same method apply to the term (2) and (3). Then (7.46) follows.

The last step in the proof of (7.40) is the following claim

$$\operatorname{Im} \frac{\partial^2 \psi}{\partial s^2} = 2\Big(\operatorname{Re} \frac{b}{a}\Big)^2 \widetilde{h} + \mathcal{O}(K\widetilde{h}A). \tag{7.48}$$

To prove this we shall use (7.34) and Proposition 7.6 which give

$$\psi = \frac{1}{H}\Big((\widetilde{s} - \alpha_s)\alpha_\tau + (\widetilde{y} - \alpha_y)\alpha_\eta + ih\big((\widetilde{s} - \alpha_s)^2 + (\widetilde{y} - \alpha_y)^2\big) + \theta p(h\widetilde{s}, \widetilde{y}, \widetilde{\tau}\widetilde{s}^2, \widetilde{s}\widetilde{\eta})\Big)$$

where $\widetilde{\tau} = \alpha_\tau + 2ih(\widetilde{s} - \alpha_s)$, $\widetilde{\eta} = \alpha_\eta + 2ih(\widetilde{y} - \alpha_y)$.

Let us recall that, by (7.41), $\frac{\partial}{\partial s} = P \frac{\partial}{\partial \widetilde{s}} + Q \frac{\partial}{\partial \widetilde{y}}$, where

$$P = \frac{1}{a} + \frac{b}{a}(D - M)^{-1}\gamma, \quad Q = -\frac{b}{a} \cdot (D - M)^{-1}.$$

Let us also assume, for simplicity that $p = \lambda^2 + \sum \overline{h}^{jk}(y)\mu_j\mu_k$. Then

$$\begin{aligned}
\frac{\partial \psi}{\partial s} = \frac{1}{H} P\Big(\alpha_\tau + 2ih(\widetilde{s} - \alpha_s) + (4ih\widetilde{\tau}\widetilde{s}^4 + 4\widetilde{\tau}^2\widetilde{s}^3 + \widetilde{s} f_1(\widetilde{y})\widetilde{\eta}^2)\theta\Big) \\
+ \frac{1}{H} Q\Big(\alpha_\eta + 2ih(\widetilde{y} - \alpha_y) + \big(\widetilde{s}^2 f_2(\widetilde{y})\widetilde{\eta}^2 + \widetilde{s}^2 ih f_3(\widetilde{y})\widetilde{\eta}\big)\theta\Big).
\end{aligned}$$

We write

$$\frac{\partial \psi}{\partial s} = \frac{1}{H}(PU + QV). \tag{7.49}$$

Now we have

$$P = \frac{1}{a}(1 + b(D - M)^{-1}\gamma) = \frac{1}{a}\Big(1 + \mathcal{O}\Big(\frac{H}{K}\Big)\Big)$$

and

$$a = H\frac{\partial \underline{s}}{\partial \widetilde{s}} = \frac{1 + 4ih\widetilde{s}^4\theta + 4\widetilde{\tau}\widetilde{s}^3\theta - \partial F/\partial\widetilde{s}}{\widetilde{s}^2}\frac{s^2}{H},$$

since $\underline{s}(\theta) = s/H$. On the other hand,

$$U = \widetilde{\tau}(1 + 4ih\widetilde{s}^4\theta + 4\widetilde{\tau}\widetilde{s}^3\theta) + \mathcal{O}(H^2).$$

Since $\partial F/\partial\widetilde{s} = \mathcal{O}(1/K^2)$ we get

$$\frac{1}{H}PU = \frac{1}{s^2}(\widetilde{\tau}\widetilde{s}^2 + R), \quad R = \mathcal{O}\Big(H^2 + \frac{1}{K^2}\Big).$$

Moreover, using the Taylor and Cauchy formulas, we see that if R is a holomorphic function, depending on $(\theta, \widetilde{s}, \widetilde{y}, \widetilde{\tau}, \widetilde{\eta})$, which is real on the real and bounded by L, then $|\operatorname{Im} R| \leqslant CL\frac{h}{H}A$.

We can now begin to estimate the second derivative of ψ. We have

$$\frac{\partial}{\partial s}\Big(\frac{1}{H}Pu\Big) = -\frac{2}{s^3}\underbrace{(\widetilde{\tau}\widetilde{s}^2 + R)}_{(1)} + \frac{1}{s^2}\underbrace{\Big((2ih\widetilde{s}^2 + 2\widetilde{\tau}\widetilde{s})P + P\frac{\partial R}{\partial\widetilde{s}} + Q\frac{\partial R}{\partial\widetilde{y}}\Big)}_{(2)}.$$

It follows that

$$\operatorname{Im}(1) = \mathcal{O}\Big(K^3hA + K^3\Big(H^2 + \frac{1}{K^2}\Big)\frac{h}{H}A\Big) = \mathcal{O}(K^3h\cdot\widetilde{h}A + K\widetilde{h}A).$$

On the other hand

$$P\frac{\partial R}{\partial\widetilde{s}} + Q\frac{\partial R}{\partial\widetilde{y}} = \mathcal{O}\Big(KH^2 + \frac{1}{K}\Big)$$

and $\operatorname{Im} P = \mathcal{O}(KhA)$. Therefore we get

$$\operatorname{Im}(2) = \mathcal{O}(K^3H\widetilde{h}A + K\widetilde{h}A).$$

Then

$$\operatorname{Im}\frac{\partial}{\partial s}\Big(\frac{1}{H}PU\Big) = \mathcal{O}(K^3H\widetilde{h}A + K\widetilde{h}A). \tag{7.50}$$

Let us look now to the term

$$\frac{\partial}{\partial s}\Big(\frac{1}{H}QV\Big) = P\frac{\partial}{\partial\widetilde{s}}\Big(\frac{1}{H}QV\Big) + Q\frac{\partial}{\partial\widetilde{y}}\Big(\frac{1}{H}QV\Big) = (1) + (2).$$

We have $QV = \mathcal{O}(H)$ so

$$\frac{\partial}{\partial\widetilde{s}}QV = \mathcal{O}\Big(\frac{1}{K}\Big) \quad \text{and} \quad \operatorname{Im} QV = \mathcal{O}\Big(\frac{1}{K}\widetilde{h}A\Big).$$

Since $P = \mathcal{O}(K^2H)$ and $\operatorname{Im} P = \mathcal{O}(KhA)$ we get

$$\operatorname{Im}(1) = \mathcal{O}(KH\widetilde{h}A).$$

On the other hand,

$$(2) = \frac{1}{H}\frac{\partial Q}{\partial\widetilde{y}}V + \frac{1}{H}Q^2\frac{\partial V}{\partial\widetilde{y}} = (3) + (4).$$

The same argument shows that (3) $= \mathcal{O}(1)$ so $\operatorname{Im}(3) = \mathcal{O}(\widetilde{h}A)$. Now it is easy to see that $\operatorname{Im}(4) = 2\big(\operatorname{Re} b/a\big)^2\widetilde{h} + \mathcal{O}(\widetilde{h}A)$. Then

$$\operatorname{Im}\frac{\partial}{\partial s}\Big(\frac{1}{H}QV\Big) = 2\Big(\operatorname{Re}\frac{b}{a}\Big)^2\widetilde{h} + \mathcal{O}(\widetilde{h}A). \tag{7.51}$$

It follows from (7.49), (7.50) and (7.51) that if $K^2H \leqslant 1$, we have

$$\operatorname{Im}\frac{\partial^2\psi}{\partial s^2} = 2\Big(\operatorname{Re}\frac{b}{a}\Big)^2\widetilde{h} + \mathcal{O}(K\widetilde{h}A) \tag{7.52}$$

which proves (7.48).

Now, since by (7.28) $A \leqslant \varepsilon_0/C_2K$, where C_2 is large, taking $1/K \ll \varepsilon_0$, we deduce from (7.42), (7.46) and (7.52) that (7.40) is satisfied.

Thus we may apply the Theorem A.14 in the Appendix to conclude that the part of the integral (7.33) where $(\theta, p/h, y) \in W_2$ is $\mathcal{O}(e^{-\delta/hk})$, with $\delta > 0$.

Case 2. — Let us look now to the part of the integral where $(\theta, \widetilde{s}, \widetilde{y})$ belongs to W_1 that is

$$Q = 1 - 2\alpha_\tau \operatorname{Re}\widetilde{s}^3\theta \geqslant \frac{1}{2}KH, \quad \frac{\varepsilon_0}{C_1K} \leqslant |\widetilde{s} - \alpha_s| + |\widetilde{y} - \alpha_y| \leqslant \frac{\varepsilon_0}{C_2K}.$$

In this integral, (θ, s, y) is real. It follows from Lemma 7.9 ($(\widetilde{s}', \widetilde{y}', \alpha_s, \alpha_y)$ being real) that

$$\begin{cases} |\operatorname{Im}\widetilde{s}| \leqslant Ch(|\operatorname{Re}\widetilde{s} - \alpha_s| + |\operatorname{Re}\widetilde{y} - \alpha_y|) \\ |\operatorname{Im}\widetilde{y}| \leqslant \dfrac{C}{K}\widetilde{h}(|\operatorname{Re}\widetilde{s} - \alpha_s| + |\operatorname{Re}\widetilde{y} - \alpha_y|). \end{cases} \tag{7.53}$$

Let us set

$$A^2 = |\operatorname{Re}\widetilde{s} - \alpha_s|^2 + |\operatorname{Re}\widetilde{y} - \alpha_y|^2. \tag{7.54}$$

Then in W_1 we have

$$A \geqslant \frac{\varepsilon_0}{2C_1K}. \tag{7.55}$$

Lemma 7.12. — *In W_1 we have*

$$\operatorname{Im}\varphi \geqslant \frac{1}{2}h(|\operatorname{Re}\widetilde{s} - \alpha_s|^2 + |\operatorname{Re}\widetilde{y} - \alpha_y|^2).$$

Proof. — From Proposition 7.6, we have

$$\varphi(\theta; \underline{s}(\theta), \underline{y}(\theta), \alpha, h) = \underbrace{(\widetilde{s} - \alpha_s)\alpha_\tau + (\widetilde{y} - \alpha_y)\alpha_\eta}_{(1)} + ih\underbrace{\big[(\widetilde{s} - \alpha_s)^2 + (\widetilde{y} - \alpha_y)^2\big]}_{(2)} + \theta p\underbrace{(h\widetilde{s}, \widetilde{y}, \widetilde{\tau}\widetilde{s}^2, \widetilde{s}\widetilde{\eta})}_{(3)}.$$

We have

$$\operatorname{Im}(2) = hA^2 - h(\operatorname{Im}\widetilde{s})^2 - h(\operatorname{Im}\widetilde{y})^2.$$

It follows from (7.53) that

$$\text{Im}(2) \geqslant hA^2 - \mathcal{O}\Big(\frac{1}{K^2}hA^2 + h^2A^2\Big). \tag{7.56}$$

We have also

$$\text{Im}(1) = \text{Im}\,\widetilde{s}\alpha_\tau + \text{Im}\,\widetilde{y}\cdot\alpha_\eta. \tag{7.57}$$

On the other hand,

$$p(h\widetilde{s}, \widetilde{y}, \widetilde{\tau}\widetilde{s}^2, \widetilde{s}\widetilde{\eta}) = \widetilde{\tau}^2\widetilde{s}^4 + \widetilde{s}^2\widetilde{\eta}^2 + h\widetilde{s}\big(h\widetilde{s}a(h\widetilde{s},\widetilde{y})(\widetilde{\tau}\widetilde{s}^2)^2 + b(\cdots)\widetilde{\tau}\widetilde{s}^2\widetilde{s}\widetilde{\eta} + c(\cdots)\widetilde{s}^2\widetilde{\eta}^2\big),$$

where $\widetilde{\tau} = \alpha_\tau + 2ih\widetilde{s} - \alpha_s)$, $\widetilde{\eta} = \alpha_\eta + 2ih(\widetilde{y} - \alpha_y)$.

It is easy to see that

$$\begin{cases} \text{Im}\,\widetilde{\tau}^2\widetilde{s}^4 = 4\alpha_\tau^2\,\text{Re}\,\widetilde{s}^3\,\text{Im}\,\widetilde{s} + 4h\alpha_\tau\,\text{Re}\,\widetilde{s}^4(\text{Re}\,\widetilde{s} - \alpha_s) + \mathcal{O}(h^2A) \\ \text{Im}\,\widetilde{s}^2\widetilde{\eta}^2 = \mathcal{O}(hHA) \\ \text{Im}\,h\widetilde{s}(h\widetilde{s}a(\cdots)(\widetilde{\tau}\widetilde{s}^2)^2 + \cdots) = \mathcal{O}(h^2A) = \mathcal{O}(hHA). \end{cases} \tag{7.58}$$

It follows from (7.56) to (7.58) that

$$\begin{aligned} \text{Im}\,\varphi \geqslant \text{Im}\,\widetilde{y}\cdot\alpha_\eta + \alpha_\tau\Big[(1 + 4\theta\alpha_\tau\,\text{Re}\,\widetilde{s}^3)\,\text{Im}\,\widetilde{s} + 4\theta h\,\text{Re}\,\widetilde{s}(\text{Re}\,\widetilde{s} - \alpha_s)\Big] \\ + hA^2 + \mathcal{O}\Big(\frac{1}{K^2}hA^2 + h^2A^2 + hHA\Big). \end{aligned} \tag{7.59}$$

On the other hand (7.10) shows that

$$\frac{H}{s} = \frac{1}{\underline{s}(\theta)} = \frac{1}{\widetilde{s}} - 2\widetilde{\tau}\widetilde{s}^2\theta + F, \quad |F| \leqslant C\frac{H}{K},$$

and, since H/s is real, we get

$$-\frac{\text{Im}\,\widetilde{s}}{|\widetilde{s}|^2} - 2\,\text{Im}\,\widetilde{\tau}\widetilde{s}^2\theta + \text{Im}\,F = 0.$$

Since $|\widetilde{s}|^2 = \text{Re}\,\widetilde{s}^2 + \mathcal{O}(h^2A^2)$, we obtain

$$\text{Im}\,\widetilde{s}(1 + 4\alpha_\tau\,\text{Re}\,\widetilde{s}^3\theta) + 4\theta h\,\text{Re}\,\widetilde{s}^4(\text{Re}\,\widetilde{s} - \alpha_s) = \text{Re}\,\widetilde{s}^2\,\text{Im}\,F + \mathcal{O}(h^2A).$$

Then (7.59) implies that

$$\text{Im}\,\varphi \geqslant hA^2 + \text{Im}\,\widetilde{y}\cdot\alpha_\eta + \alpha_\tau\,\text{Re}\,\widetilde{s}^2\,\text{Im}\,F + \mathcal{O}\Big(\frac{1}{K^2}hA^2 + hHA\Big).$$

Then, Lemma 7.12 will follow from the following lemma.

Lemma 7.13. — *We have*

$$\text{Im}\,\widetilde{y}\cdot\alpha_\eta + \alpha_\tau\,\text{Re}\,\widetilde{s}^2\,\text{Im}\,F = \mathcal{O}\Big(\frac{1}{K^2}hA + hHA + \frac{1}{K}hA^2\Big).$$

Indeed, since $A \geqslant \varepsilon_0/2C_1K$, we have $\mathcal{O}\big(\frac{1}{K^2}hA + hHA + \frac{1}{K}hA^2\big) \leqslant \frac{1}{2}hA^2$ which implies that $\text{Im}\,\varphi \geqslant \frac{1}{2}hA^2$, as claimed. □

Proof. — Let us recall that $p = \lambda^2 + \sum \overline{h}^{jk} \mu_j \mu_k + \rho r$, with

$$r = a\rho\lambda^2 + b\lambda\mu + c\mu^2 \quad \text{and} \quad \dot{y} = s\frac{\partial p}{\partial \mu}(hs, y, \tau s^2, s\eta) = 2s^2\langle\eta\rangle + hs^2\frac{\partial r}{\partial \mu},$$

where $\langle\eta\rangle = \big(\sum_k \overline{h}^{jk}\eta_\kappa\big)_{j=1,\dots,n}$. Then

$$\underline{y}(\theta) = \widetilde{y} - 2\int_\theta^0 \underline{s}^2(\sigma)\langle\underline{\eta}(\sigma\rangle\, d\sigma - \int_\theta^0 h\underline{s}^2(\sigma)\frac{\partial r}{\partial \mu}(h\underline{s}(\sigma)\cdots)\, d\sigma = y \in \mathbb{R}^{n-1}.$$

Denoting by $s(\theta), y(\theta)$ the real functions such that

$$s(\theta; \widetilde{s}', \widetilde{y}', \alpha_\tau, \alpha_\eta) = \frac{s}{H}, \quad y(\theta, \cdots) = y$$

(see Lemma 7.9) we can write

$$\text{(7.60)}\ \operatorname{Im}\widetilde{y} = 2\operatorname{Im}\int_\theta^0 \left[\underline{s}^2(\sigma)\langle\underline{\eta}(\sigma)\rangle - s^2(\sigma)\langle\eta(\sigma)\rangle\right] d\sigma$$
$$+\operatorname{Im}\int_\theta^0 h\Big[\underline{s}^2(\sigma)\frac{\partial r}{\partial \mu}(h\underline{s}(\sigma)\cdots) - s^2(\sigma)\frac{\partial r}{\partial \mu}(hs(\sigma)\cdots)\Big]\, d\sigma = I + II.$$

We have

$$I = 2\operatorname{Im}\int_\theta^0 \Big[s^2(\sigma)\big(\langle\underline{\eta}(\sigma)\rangle - \langle\eta(\sigma)\rangle\big) + \langle\underline{\eta}(\sigma)\rangle(\underline{s}^2(\sigma) - s^2(\sigma))\Big]\, d\sigma.$$

Let us introduce the following function

$$u_0(\theta) = \frac{1}{1 - 2\operatorname{Re}\widetilde{s}^3\alpha_\tau\theta} \geqslant 0. \tag{7.61}$$

Then, using the Lemma 7.9 and the estimates on $\partial s(\theta)/\partial\widetilde{s}$, $\partial s(\theta)/\partial\widetilde{y}\cdots$ (see Corollary 7.3) we obtain, $|\underline{s}(\theta) - s(\theta)| \leqslant ChAu_0^2$. Moreover we have $|\underline{s}(\theta) + s(\theta)| \leqslant Cu_0$ and $|\langle\underline{\eta}(\sigma)\rangle| \leqslant CH$. It follows that

$$\Big|\int_\theta^0 \langle\underline{\eta}(\sigma)\rangle(\underline{s}^2(\sigma) - s^2(\sigma))\, d\sigma\Big| \leqslant CHhA\int_\theta^0 u_0^3(\sigma)\, d\sigma \leqslant \frac{CHhA}{K^2H^2} = \frac{ChA}{K^2H}. \tag{7.62}$$

Now we have $\underline{\eta}(\sigma) = \widetilde{\eta} + \underline{G}(\sigma)$, $\eta(\theta) = \alpha_\eta + G(\sigma)$, where G and $\underline{G}$ are bounded by CH/K. Let us write for convenience $\langle\underline{\eta}\rangle = f(\underline{y})\underline{\eta}$. Then

$$\langle\underline{\eta}\rangle - \langle\eta\rangle = f(y)(\underline{\eta}(\sigma) - \eta(\sigma)) + \underline{\eta}(\sigma)(f(\underline{y}) - f(y))$$
$$\langle\underline{\eta}\rangle - \langle\eta\rangle = 2ihf(y(\sigma))(\widetilde{y} - \alpha_y) + f(y(\sigma))(\underline{G}(\sigma) - G(\sigma)) + \underline{\eta}(\sigma)\big(f(\underline{y}(\sigma)) - f(y(\sigma))\big).$$

Since $|\underline{y}(\sigma) - y(\sigma)| \leqslant ChA/KH$, we see easily that

$$|f(y(\sigma))(\underline{G}(\sigma) - G(\sigma))| + |\underline{\eta}(\sigma)(f(\underline{y}(\sigma)) - f(y(\sigma))| \leqslant \frac{ChA}{K}.$$

It follows that

$$2\operatorname{Im}\int_\theta^0 \Big[s^2(\sigma)\big(\langle\underline{\eta}(\sigma)\rangle - \langle\eta(\sigma)\rangle\big)\Big]\, d\sigma = 4h\langle\operatorname{Re}\widetilde{y} - \alpha_y\rangle\int_\theta^0 s^2(\sigma)\, d\sigma + \mathcal{O}\Big(\frac{hA}{K^2H}\Big). \tag{7.63}$$

We can apply exactly the same technique to the term II and obtain

$$II = \mathcal{O}\Big(\frac{hA}{K^2H}\Big). \tag{7.64}$$

Then, using (7.60), (7.62), (7.63), (7.64) we obtain

$$\operatorname{Im}\widetilde{y}\cdot\alpha_\eta = 4h\sum_{j,k}\overline{h}^{jk}(\operatorname{Re}\widetilde{y})(\operatorname{Re}\widetilde{y}_j - \alpha_y^j)\alpha_\eta^k\int_\theta^0 s^2(\sigma)\,d\sigma + \mathcal{O}\Big(\frac{hA}{K^2}\Big). \tag{7.65}$$

Let us look now to the term $\operatorname{Im} F$. According to the computations made before (7.10), we have

$$\begin{aligned}F = 4\int_\theta^0\int_\sigma^0 \underline{s}^3(t)\sum_{jk}h^{jk}(\underline{y}(t))\underline{\eta}_j(t)\underline{\eta}_k(t)\,dt\,d\sigma\\ +2\int_\theta^0\int_\sigma^0\Big(h\underline{s}^3\underline{\eta}\frac{\partial r}{\partial\mu}+h\underline{s}^2\frac{\partial r}{\partial\rho}\Big)(t)\,dt\,d\sigma+\int_\theta^0 h\underline{s}(\sigma)\frac{\partial r}{\partial\lambda}\,d\sigma = (I)+(II)+(III).\end{aligned} \tag{7.66}$$

Let us look to the term (I). Since $|\underline{s}^3(t)-s^3(t)| \leqslant C|\underline{s}(t)-s(t)|u_0^2 \leqslant C' hAu_0^4$, we can replace $\underline{s}^3(t)$ by $s^3(t)$ modulo an error which is $\mathcal{O}(\frac{hA}{K^2})$. Then we write $\underline{\eta}(t) = \widetilde{\eta}+\underline{G}(t)$, $\eta(t)=\alpha_\eta+G(t)$ where $\underline{G}$ and G are bounded by $C\frac{H}{K}$. It follows as before that

$$\operatorname{Im}(I) = 16h\int_\theta^0\int_\sigma^0 s^3(t)\,dt\,d\sigma\sum_{j,k}\overline{h}^{jk}(\operatorname{Re}\widetilde{y})\alpha_\eta^k(\operatorname{Re}\widetilde{y}_j-\alpha_y^j) + \mathcal{O}\Big(\frac{hA}{K^2}\Big).$$

The same computation can be applied to the terms (II) and (III) and we find finally

$$\operatorname{Im} F = 16h\sum_{j,k}\overline{h}^{jk}(\operatorname{Re}\widetilde{y})\alpha_\eta^k(\operatorname{Re}\widetilde{y}_j-\alpha_\eta^j)\int_\theta^0\int_\sigma^0 s^3(t)\,dt\,d\sigma + \mathcal{O}\Big(\frac{hA}{K^2}+\frac{hA^2}{K}\Big). \tag{7.67}$$

Now we see easily, using (7.10), that, with u_0 defined in (7.61),

$$\begin{cases}|s^2(\sigma)-(\operatorname{Re}\widetilde{s})^2u_0^2(\sigma)| \leqslant C\dfrac{H}{K}u_0^3(\sigma),\\ |s^3(t)-(\operatorname{Re}\widetilde{s})^3u_0^3(t)| \leqslant C\dfrac{H}{K}u_0^4(t).\end{cases} \tag{7.68}$$

Using (7.4) we see that we can replace, in (7.65) and (7.67) s by $(\operatorname{Re}\widetilde{s})u_0$ modulo an error which is $\mathcal{O}(\frac{1}{K^2}hA)$. On the other hand we have

$$\int_\sigma^0((\operatorname{Re}\widetilde{s})u_0(t))^3\,dt = \frac{1}{4\alpha_\tau}(1-u_0^2(t)).$$

Then we get

$$\operatorname{Im} F = -\frac{4h}{\alpha_\tau}\sum_{jk}\overline{h}^{jk}(\operatorname{Re}\widetilde{y})\alpha_\eta^k(\operatorname{Re}\widetilde{y}_j-\alpha_\eta^j)\int_\theta^0 u_0^2(t)\,dt + \mathcal{O}\Big(\frac{hA}{K^2}+hHA+\frac{hA^2}{K}\Big).$$

Using (7.65) and (7.68) we conclude that

$$\operatorname{Re}\widetilde{s}^2\alpha_\tau\operatorname{Im} F = -\operatorname{Im}\widetilde{y}\cdot\alpha_\eta + \mathcal{O}\Big(\frac{hA}{K^2}+hHA+\frac{hA^2}{K}\Big)$$

which is the claim in Lemma 7.13. □

End of the proof of Theorem 4.8. — By the Lemma 7.11, the part of the integral, in (7.35), lying in W_1 is bounded by $Ce^{-\delta/hk}$. This proves (7.35) and completes the proof of Theorem 4.8. □

CHAPTER 8

PROOF OF THEOREM 4.9

We consider the case $m_0 \in \mathcal{N}_+^c$, $m_0 = (0, y_0, 0, (1,0))$. We proceed as in the proof of Theorem 4.8, § 7 ; we look for a phase φ and a symbol a satisfying the phase and transport equations.

8.1. Resolution of the phase equation

Let d be a strictly positive integer. We denote by $\mathcal{P}_d$ the set of polynomials of the following form

$$r(s, y, h, \lambda, \mu) = hb(s, y, h)\lambda^d + \sum_{\substack{|\alpha|+j \leqslant d \\ \alpha \neq 0}} b_{\alpha j}(s, y, h)\mu^\alpha \lambda^j \tag{8.1}$$

where b and $b_{\alpha j}$ extend to holomorphic functions near (s_0, y_0) and are smooth in h on $[0, +\infty[$. Then we have

(i) Let $r \in \mathcal{P}_d$; then for all $K_0 > 0$ one can find $C(K_0) > 0$ such that for all (s, y, h, λ, μ) satisfying $|s - s_0| + |h| + |y - y_0| + |\lambda| + |\mu| \leqslant K_0$ one has

$$|r(s, y, h, \lambda, \mu)| \leqslant C(K_0)(h + |\mu|). \tag{8.2}$$

(ii) If $r \in \mathcal{P}_d$, $\partial r/\partial s \in \mathcal{P}_d$, $\partial r/\partial y \in \mathcal{P}_d$ and $\partial r/\partial \lambda \in \mathcal{P}_{d-1}$ if $d \geqslant 2$.

Recall that the symbol of Δ_g is $p(\rho, y, \lambda, \mu) = \lambda^2 + \|\mu\|^2 + \rho r$ so

$$p(sh, y, s^2\tau, s\eta) = s^4\tau^2 + s^2\|\eta\|^2 + hs^2\widetilde{r}(s, y, h, s^2\tau, \eta), \quad \widetilde{r} \in \mathcal{P}_2. \tag{8.3}$$

Proposition 8.1. — *Let $\alpha_0 = \big(s_0, y_0, 1/s_0^3, 0\big)$. There exist positive constants ε_θ, ε_s, ε_y, ε_α, ε_h and for h in $]0, \varepsilon_h[$ a holomorphic function $\varphi = \varphi(\theta; s, y, \alpha, h)$ in the set*

$$E = \Big\{(\theta, s, y, \alpha) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{2n} : \operatorname{Re}\theta \in (-\infty, \varepsilon_\theta],$$
$$|\operatorname{Im}\theta| < \varepsilon_\theta,\ \Big|s - \frac{s_0}{1 + 2|\theta|}\Big| < \frac{\varepsilon_s}{1 + |\theta|},\ |y - y_0| < \varepsilon_y,\ |\alpha - \alpha_0| < \varepsilon_\alpha\Big\}$$

such that
(8.4)

$$\begin{cases} \dfrac{\partial\varphi}{\partial\theta} + p\Big(sh, y, s^2\dfrac{\partial\varphi}{\partial s}, s\dfrac{\partial\varphi}{\partial y}\Big) = 0 \text{ in } E \\ \varphi|_{\theta=0} = (s-\alpha_s)\alpha_\tau + (y-\alpha_y)\cdot\alpha_\eta - A(s-s_0)^2 + ih[(s-\alpha_s)^2 + (y-\alpha_y)^2] \end{cases}$$

where $A = (1+\delta)s_0^{-4}$, δ *small.*

Proof. — We introduce the symbol

$$q = \theta^* + p(sh, y, s^2\tau, s\eta) \tag{8.5}$$

and we study the bicharacteristic system of q when the parameter on the curve is real. For $(\widetilde{s}, \widetilde{y}, \alpha, h)$ in $\mathbb{C}\times\mathbb{C}^{n-1}\times\mathbb{C}^{2n}\times\,]0,+\infty[$ such that

$$|\widetilde{s}-s_0| < \varepsilon_s^0, \quad |\widetilde{y}-y_0| \leqslant \varepsilon_y^0, \quad \Big|\alpha_\tau - \frac{1}{s_0^3}\Big| + |\alpha_\eta| \leqslant \varepsilon_\alpha^0, \quad 0 < h < \varepsilon_h^0,$$

we consider the system
(8.6)

$$\begin{cases} \dot\theta(t) = 1, & \theta(0) = 0 \\ \dot s(t) = 2\tau s^4 + hs^4 r_1(s, y, h, s^2\tau, \eta), & s(0) = \widetilde{s} \\ \dot y(t) = 2s^2\langle\eta\rangle + hs^2\big[a(hs,y)(\tau s^2) + sb(hs,y)\cdot\eta\big], & y(0) = \widetilde{y} \\ \dot\theta^*(t) = 0, & \theta^*(0) = -p(\widetilde{s}h, \widetilde{y}, \cdots) \\ \dot\tau(t) = -\big[4s^3\tau^2 + 2s\|\eta\|^2 + shr_2(s,y,h,s^2\tau,\eta)\big], & \\ \qquad\qquad \tau(0) = \widetilde{\tau} = \alpha_\tau - 2A(\widetilde{s}-s_0) + 2ih(\widetilde{s}-\alpha_s) & \\ \dot\eta(t) = -s^2\partial_y\|\eta\|^2 + s^2hr_3(s,y,h,s^2\tau,\eta), \quad \eta(0) = \widetilde{\eta} = \alpha_\eta + 2ih(\widetilde{y}-\alpha_y), & \end{cases}$$

where $r_1 \in \mathcal{P}_1$, $r_2, r_3 \in \mathcal{P}_2$, a, b (and their derivatives) are uniformly bounded and

$$\langle\eta\rangle = \Big(\sum_{i=1}^{n-1} \overline{h}^{ij}(y)\eta_j\Big), \quad \partial_y\|\eta\|^2 = \sum_{j,k=1}^{n} \frac{\partial\overline{h}^{jk}}{\partial y}(y)\eta_j\eta_k.$$

Lemma 8.2. — *The system (8.6) has, for* ε_s^0, ε_y^0, ε_α^0, ε_h^0 *small enough, a unique global solution on* $(-\infty, 0]$ *which is holomorphic with respect to* $(\widetilde{s}, \widetilde{y}, \alpha)$.

Proof. — First of all we have $\theta(t) = t$ and $\theta^*(t) = \theta^*(0)$ which are globally defined on $(-\infty, 0]$. Then we introduce the following subset I of $[0, +\infty[$: $T \in I$ iff the problem (8.6) has a unique solution on $[-T, 0]$, which is holomorphic with respect $(\widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta)$

and satisfies

$$
(8.7)\qquad \begin{cases} \text{(i)} & \dfrac{1}{10}\,\dfrac{s_0}{1+2|t|} \leqslant |s(t)| \leqslant 10\,\dfrac{s_0}{1+2|t|} \\ \text{(ii)} & |y(t)-\widetilde{y}| \leqslant (|\alpha_\eta|+h)^{1/2} \\ \text{(iii)} & |\eta(t)-\widetilde{\eta}| \leqslant |\alpha_\eta|+h \\ \text{(iv)} & |\tau(t)\,s^2(t)-\widetilde{\tau}\,\widetilde{s}^2| \leqslant |\alpha_\eta|+h \ \text{ where } \widetilde{\tau}=\alpha_\tau-2A(\widetilde{s}-s_0). \end{cases}
$$

(We assume $|\alpha_\eta|+h\leqslant \varepsilon_\alpha^0+\varepsilon_h^0\leqslant 1$).

The set I is of course an interval which is non empty. Indeed, the Cauchy-Lipschitz theorem shows that (5.51) has a unique solution on $[-T,0]$ for some small $T>0$ which is holomorphic with respect to the data. This solution satisfies $|s(t)-\widetilde{s}|\leqslant C|t|$ so (i) will be satisfied if T and ε_{s_0} are small enough (with respect to s_0) ; now, according to the equation satisfied by η we have $\eta(t)\equiv 0$ if α_η and h are equal to zero ; since η is smooth with respect to α_η and h, we will have $|\eta(t)|\leqslant C(|\alpha_\eta|+h)$; then using the equation satisfied by η and (8.2) we get

$$
|\eta(t)-\widetilde{\eta}| \leqslant \int_t^0 |\dot{\eta}(\sigma)|d\sigma \leqslant C(|\alpha_\eta|+h)^2 \leqslant |\alpha_\eta|+h
$$

if ε_α^0 and ε_h^0 are small enough. On the other hand we can write

$$
|y(t)-\widetilde{y}| \leqslant \int_t^0 |\dot{y}(\sigma)|d\sigma \leqslant C_1(|\alpha_\eta|+h)+C_2h \leqslant (|\alpha_\eta|+h)^{1/2}.
$$

Finally,

$$
(8.8)\qquad \frac{d}{dt}(\tau s^2) = \dot{\tau}s^2+2\tau s\dot{s} = -2s^3\|\eta\|^2+s^3hr(s,y,h,s^2\tau,\eta),
$$

with $r\in\mathcal{P}_2$. Therefore, using (8.2) we get

$$
|\tau(t)s^2(t)-\widetilde{\tau}\widetilde{s}^2| \leqslant C_1(|\alpha_\eta|+h)^2+C_2h(h+|\alpha_\eta|+h) \leqslant |\alpha_\eta|+h.
$$

Let us set $T^*=\sup I$. If $T^*=+\infty$ our lemma is proved ; so assume $T^*<+\infty$ and let $T_0\in I$, $T_0<T^*$. On $[-T_0,0]$ we have a solution which satisfies (8.7). By (8.8) we have,

$$
|\tau(t)s^2(t)-\widetilde{\tau}\widetilde{s}^2| \leqslant 2\int_t^0 |s(\sigma)|^3\|\eta(\sigma)\|^2\,d\sigma + h\int_t^0 |s^3(\sigma)|\,|r(s(\sigma),y(\sigma),\cdots)|\,d\sigma.
$$

It follows from (8.7) and (8.2) that one can find a constant C_1 depending only on the data such that

$$
|\tau(t)s^2(t)-\widetilde{\tau}\widetilde{s}^2| \leqslant C_1(|\alpha_\eta|+h)^2\int_{-\infty}^0 \frac{d\sigma}{(1+|\sigma|)^3},
$$

therefore

$$
(8.9)\qquad |\tau(t)s^2(t)-\widetilde{\tau}\widetilde{s}^2| \leqslant \frac{1}{2}(|\alpha_\eta|+h),
$$

if ε_α and ε_h are small enough with respect to the data.

Now we use the second equation of (8.6). We get

$$\frac{\dot{s}(t)}{s^2(t)} = 2\tau s^2 + s^2 h r_1(s, y, h, s^2\tau, \eta), \quad r_1 \in \mathcal{P}_1. \tag{8.10}$$

Let us set

$$f(t) = (\tau s^2)(t) - \widetilde{\tau}\widetilde{s}^2. \tag{8.11}$$

Integrating (8.10) between t and zero we get

$$\frac{1}{s(t)} - \frac{1}{\widetilde{s}} = -2t\widetilde{\tau}\widetilde{s}^2 + 2\int_t^0 f(\sigma)\,d\sigma + h\int_t^0 s^2(\sigma) r_1(s(\sigma), y(\sigma), \cdots)\,d\sigma. \tag{8.12}$$

Then we write

$$\int_t^0 f(\sigma)d\sigma = [\sigma f(\sigma)]_t^0 - \int_t^0 \sigma f'(\sigma)d\sigma = -tf(t) - \int_t^0 \sigma f'(\sigma)d\sigma. \tag{8.13}$$

It follows from (8.8) and (8.11)

$$f'(\sigma) = -2s^3(\sigma)\|\eta(\sigma)\|^2 + hs^3(\sigma)r(s(\sigma), y(\sigma), \cdots), \quad r \in \mathcal{P}_2$$

so,

$$|\sigma f'(\sigma)| \leqslant C_1 \frac{(|\alpha_\eta| + h)^2}{(1+|\sigma|)^2}. \tag{8.14}$$

Therefore, using (8.13), (8.14), (8.15) and (8.9) we get

$$\begin{cases} \dfrac{1}{s(t)} - \dfrac{1}{\widetilde{s}} = 2|t|\widetilde{\tau}\widetilde{s}^2 + 2|t|f(t) + g(t) \\ 2|f(t)| \leqslant |\alpha_\eta| + h \\ |g(t)| \leqslant C_2(|\alpha_\eta| + h)^2. \end{cases}$$

It follows that

$$\begin{cases} s(t) = \dfrac{\widetilde{s}}{1 + g_1(t) + 2|t|(\widetilde{\tau}\widetilde{s}^3 + f_1(t))} \\ |f_1(t)| + |g_1(t)| \leqslant C_3(|\alpha_\eta| + h). \end{cases} \tag{8.15}$$

Now

$$\begin{aligned} \widetilde{\tau}\widetilde{s}^3 &= (\alpha_\tau - 2A(\widetilde{s} - s_0))\widetilde{s}^3 \\ &= \Big(\frac{1}{s_0^3} + \Big(\alpha_\tau - \frac{1}{s_0^3}\Big) - 2A(\widetilde{s} - s_0)\Big)(s_0^3 + \widetilde{s}^3 - s_0^3) = 1 + \mathcal{O}(\varepsilon_s + \varepsilon_\alpha), \end{aligned}$$

where $\mathcal{O}(\varepsilon)$ stands for a quantity bounded by $C\varepsilon$ where C depends only on the data. It follows from (8.15) that

$$1 + g_1(t) + 2|t|(\widetilde{\tau}\widetilde{s}^3 + f_1(t)) = 1 + 2|t| + \mathcal{O}(\varepsilon_\alpha + \varepsilon_s + \varepsilon_h) + |t|\mathcal{O}(\varepsilon_\alpha + \varepsilon_h).$$

Therefore, if ε_α, ε_s, ε_h are small enough we will have

$$\frac{1}{2}\frac{s_0}{1+2|t|} \leqslant |s(t)| \leqslant 2\frac{s_0}{1+2|t|}, \quad t \in [-T_0, 0]. \tag{8.16}$$

Next, using (8.6) we get

$$|y(t)-\widetilde{y}| \leqslant 2\int_t^0 |s^2(\sigma)\langle\eta(\sigma)\rangle|\,d\sigma + h\int_t^0 |s^2(\sigma)|\,|\ell_1(\cdots)+\ell_2(\cdots)|\,d\sigma$$
$$\leqslant C_4(|\alpha_\eta|+h)\int_{-\infty}^0 \frac{d\sigma}{(1+|\sigma|)^2} + C_5\,h\int_{-\infty}^0 \frac{d\sigma}{(1+|\sigma|)^2}$$

so

$$|y(t)-\widetilde{y}| \leqslant \frac{1}{2}(|\alpha_\eta|+h)^{1/2}, \quad t\in[-T_0,0]. \tag{8.17}$$

Finally

$$|\eta(t)-\alpha_\eta|$$
$$\leqslant \int_t^0 |s^2(\sigma)|\sum_{i,j}\Big|\frac{\partial \overline{h}^{ij}}{\partial y}y(\sigma)\Big|\,|\eta_i(\sigma)|\,|\eta_j(\sigma)|\,d\sigma + \int_t^0 h|s^2(\sigma)|\,|r_3(s(\sigma),\cdots)|\,d\sigma$$
$$\leqslant \big[C_6(|\alpha_\eta|+h)^2 + C_7 h(|\alpha_\eta|+h)\big]\int_{-\infty}^0 \frac{d\sigma}{(1+|\sigma|)^2}$$

so

$$|\eta(t)-\alpha_\eta| \leqslant \frac{1}{2}(|\alpha_\eta|+h). \tag{8.18}$$

It follows from (8.6) and (8.7) that $\dot{s}$, $\dot{y}$, $\dot{\tau}$, $\dot{\eta}$ are integrable on $(-T^*,0]$ therefore $s(t)$, $y(t)$, $\tau(t)$, $\eta(t)$ have limits $s(-T^*)$, $y(-T^*)$, $\tau(-T^*)$, $\eta(-T^*)$ as $t\to -T^*$. Moreover these limits satisfy the estimates (8.9), (8.16), (8.17), (8.18). Then we solve the system (8.6) with data $s(-T^*)$, $y(-T^*)$, $\tau(-T^*)$, $\eta(-T^*)$ on $t=-T^*$; by the Cauchy Lipschitz theorem, we find a solution on $[-T^*-\delta,-T^*]$ close to the data ; matching this solution with the previous one, we get a solution on $[-T^*-\delta,0]$ which will satisfy the estimates (8.7). This contradicts the definition of T^* and proves that $T^*=+\infty$.

We show now that we can complexify the time t and obtain a solution of (8.6) in the set $\operatorname{Re} t\in(-\infty,-\varepsilon]$, $|\operatorname{Im} t|\leqslant\varepsilon$. The equations (8.6) show that we can take θ as a new variable on the bicharacteristic.

Lemma 8.3. — *The system (8.6) in $(s(\theta),y(\theta),\tau(\theta),\eta(\theta))$ has, for small ε_s^0, ε_y^0, ε_α^0, ε_h^0, ε_θ^0 a unique holomorphic solution for $\operatorname{Re}\theta\in(-\infty,-\varepsilon_\theta]$, $|\operatorname{Im}\theta|<\varepsilon_\theta$, which is holomorphic with respect to the data $(\widetilde{s},\widetilde{y},\alpha_\tau,\alpha_\eta)$.*

Proof. — Let us recall the following well known result. Let $(\theta_0,X_0)\in\mathbb{C}\times\mathbb{C}^N$ and $Q=\{(\theta,X)\in\mathbb{C}\times\mathbb{C}^N : |\theta-\theta_0|<a, |X-X_0|<b\}$. Let $F:Q\to\mathbb{C}^N$ be a holomorphic function such that $\sup_Q|F|=M<+\infty$. Then the Cauchy problem

$$\begin{cases}\dot{X}(\theta)=F(\theta,X(\theta))\\ X(\theta_0)=X_0\end{cases} \tag{8.19}$$

has a unique solution, holomorphic in $\{\theta \in \mathbb{C} : |\theta - \theta_0| < \rho\}$ where

$$\rho < a\Big(1 - \exp\Big(\frac{-b}{(N+1)aM}\Big)\Big). \tag{8.20}$$

We apply this result to the system (8.6). We take $\theta_0 \in [0, +\infty[$ and we call $X_0 = (s(\theta_0), y(\theta_0), \tau(\theta_0), \eta(\theta_0))$, the value at $\theta = \theta_0$ of the solution found in Lemma 8.2. Here $N = 2n$. We take b small depending on (s_0, y_0) ; then M also depends only on (s_0, y_0) ; finally we take $b/a = (2n+1)M\,Ln2$. It follows that the system (8.6) with data X_0 at $\theta = \theta_0$ has a unique holomorphic solution in $\{|\theta \in \mathbb{C}^n : |\theta - \theta_0| < \rho\}$ where ρ depends only on (s_0, y_0) but is independent of θ_0. Therefore moving θ_0 from 0 to $+\infty$, we get a solution of (8.6) in a fixed small complex neighborhood of $[0, +\infty[$. We can check that this solution satisfies the estimates (8.7) on this set.

Proof of Proposition 8.1. — We introduce for ε_θ^0, ε_s^0, ε_α^0, ε_y^0, ε_h^0 small enough the sets

$$\begin{aligned}\Lambda = \Big\{&(\theta, s(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), y(\theta; \cdots), \theta^*(0), \tau(\theta; \cdots), \eta(\theta; \cdots))\\ &\operatorname{Re}\theta \in\,]-\infty, \varepsilon_\theta^0[, |\operatorname{Im}\theta| < \varepsilon_\theta^0, |\widetilde{s} - s_0| < \varepsilon_s^0, |\widetilde{y} - y_0| < \varepsilon_y^0\Big\}\end{aligned} \tag{8.21}$$

where $\alpha_\tau, \alpha_\eta, h$ are fixed such that $|\alpha_\tau - 1/s_0^3| + |\alpha_\eta| < \varepsilon_\alpha^0$, $h \in\,]0, \varepsilon_h^0[$. We also introduce the set

$$\begin{aligned}E = \Big\{&(\theta, z, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}, \operatorname{Re}\theta \in\,]-\infty, \varepsilon_\theta^0[,\\ &|\operatorname{Im}\theta| < \varepsilon_\theta^0, \Big|z - \frac{s_0}{1+2|\theta|}\Big| < \frac{\varepsilon_z}{1+|\theta|}, |y - y_0| < \varepsilon_y\Big\}.\end{aligned} \tag{8.22}$$

Let $\pi : \Lambda \to \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$ be the projection on the basis.

Lemma 8.4. — *If ε_s^0, ε_y^0, ε_α^0 are small (depending on the data (s_0, y_0)) one can find $\varepsilon_z > 0$, $\varepsilon_y > 0$ such that the map $\pi : \Lambda \to E$ is bijective.*

Proof. — We fix (α, h). For fixed θ and (θ, z, y) in E we must find $\widetilde{s}, \widetilde{y}$ such that $|\widetilde{s} - s_0| < \varepsilon_s^0$, $|\widetilde{y} - y_0| < \varepsilon_y^0$ and

$$\begin{cases} s(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = z \\ y(\theta; \widetilde{s}, \widetilde{y}, \alpha, h) = y. \end{cases} \tag{8.23}$$

It follows from (8.15) and (8.7) (ii) that this system is equivalent to

$$\begin{cases} \widetilde{s} = z\big(1 - 2\theta(\alpha_\tau - 2A(\widetilde{s} - s_0))\widetilde{s}^3 + g_1(\theta) + 2|\theta|\, f_1(\theta)\big) \\ \widetilde{y} = y + g_2(\theta; \widetilde{s}, \widetilde{y}, \alpha_\tau, \alpha_\eta, h). \end{cases} \tag{8.24}$$

To solve (8.24) we use the fixed point theorem. For (θ, α, h) fixed and $(\theta, z, y) \in E$ let us consider the map from $\mathbb{C} \times \mathbb{C}^{n-1}$ in itself

$$F(\widetilde{s}, \widetilde{y}) = \begin{cases} z\big(1 - 2\theta(\alpha_\tau - 2A(\widetilde{s} - s_0))\widetilde{s}^3 + g_1(\theta) + 2|\theta|\, f_1(\theta)\big) \\ y + g_2(\theta, \cdots). \end{cases} \tag{8.25}$$

We shall show that F maps the set

$$B = \{(\widetilde{s},\widetilde{y}) \in \mathbb{C}\times\mathbb{C}^{n-1} : |\widetilde{s}-s_0| < \varepsilon_s,\, |\widetilde{y}-y_0| < \varepsilon_y\}$$

in itself. Let us denote by $F_1(\widetilde{s},\widetilde{y})$ (resp. F_2) the first (resp. the second) component of F. We have, from (8.7),

$$(8.26)\quad |F_2(\widetilde{s},\widetilde{y}) - y_0| \leqslant |y-y_0| + |g_2(\theta,\cdots)| \leqslant \widetilde{\varepsilon}_y + (|\alpha_\eta|+h)^{1/2} \leqslant \widetilde{\varepsilon}_y + (\varepsilon_\alpha+\varepsilon_h)^{1/2}.$$

Now if (θ,z,y) is in E, we have

$$z = \frac{s_0}{1+2|\theta|} + \frac{\mathcal{O}(\varepsilon_z)}{1+|\theta|}\ ;$$

moreover

$$\alpha_\tau = \frac{1}{s_0^3} + \mathcal{O}(\varepsilon_\alpha),\quad \widetilde{s}^3 = s_0^3 + 3s_0^2(\widetilde{s}-s_0) + \mathcal{O}((\widetilde{s}-s_0)^2),\quad |\widetilde{s}-s_0| < \varepsilon_s.$$

Let us skip the $\sim$. We have, from (8.25)

$$(1) = F_1(s,y) - s_0 = \Big(\frac{s_0}{1-2\theta} + \frac{\mathcal{O}(\varepsilon_z)}{1+|\theta|}\Big)\Big[\Big(1-2\theta\Big(\frac{1}{s_0^3} - 2A(s-s_0) + \mathcal{O}(\varepsilon_\alpha)\Big)$$
$$(s_0^3 + 3s_0^2(s-s_0) + \mathcal{O}(|s-s_0|^2)\Big) + g_1(\theta) + 2|\theta|f_1(\theta)\Big] - s_0.$$

Using the fact that $A = (1+\delta_A)s_0^{-4}$ we get

$$(1) = \frac{-2\theta}{1-2\theta}(s-s_0)(1-2\delta_A) + \mathcal{O}(\varepsilon_z+\varepsilon_\alpha+\varepsilon_h+\varepsilon_s^2).$$

It follows from (8.26) and the fact that $\left|\frac{-2\theta}{1-2\theta}\right| \leqslant 1$ if $\operatorname{Re}\theta \leqslant 0$, that

$$|F(s,y) - (s_0,y_0)| \leqslant \widetilde{\varepsilon}_y + (\varepsilon_\alpha+\varepsilon_h)^{1/2} + (1-2\delta_A)\varepsilon_s + \mathcal{O}(\varepsilon_z+\varepsilon_\alpha+\varepsilon_h+\varepsilon_s^2).$$

If we take $\delta_A \in \left]0,\frac{1}{2}\right[$ and $\widetilde{\varepsilon}_y + (\varepsilon_\alpha+\varepsilon_h)^{1/2} + \mathcal{O}(\varepsilon_z+\varepsilon_\alpha+\varepsilon_h+\varepsilon_s^2) \leqslant \delta_A\varepsilon_s$, then $F(s,y)\in B$.

We show now that $F: B\to B$ satisfies

$$(8.27)\qquad |F(s,y) - F(s',y')| \leqslant k(|s-s'|+|y-y'|),\ (s,y),\ (s',y')\in B,\ \ k<1.$$

Since g_1, g_2, f_1 are smooth in s,y and satisfy (8.15), we have,

$$(|z||\theta||f_1(\theta)| + |z|)|g_j(\theta;s,y,\cdots) - g_j(\theta;s',y',\cdots)| \leqslant C(\varepsilon_\alpha+\varepsilon_h)^{1/2}(|y-y'|+|s-s'|).$$

Let us estimate

$$\begin{aligned} I &= -2\theta z\big[(\alpha_\tau - 2A(s-s_0))s^3 - (\alpha_\tau - 2A(s'-s_0))s'^3\big] \\ &= -2\theta z(\alpha_\tau + 2As_0)(s^3 - s'^3) + 4A\theta z(s^4 - s'^4). \end{aligned}$$

We have

$$\alpha_\tau = \frac{1}{s_0^3} + \mathcal{O}(\varepsilon_\alpha),\quad s^3 - s'^3 = (s-s')(3s_0^2 + \mathcal{O}(s-s_0))$$

and

$$s^4 - s'^4 = (s-s')(4s_0^2 + \mathcal{O}(s-s_0)), \quad z = \frac{s_0}{1-2\theta} + \mathcal{O}(\varepsilon_z)\frac{1}{1+|\theta|}.$$

Then

$$I = -2\theta\Big(\frac{s_0}{1-2\theta} + \frac{\mathcal{O}(\varepsilon_z)}{1+|\theta|}\Big)\Big(\frac{1}{s_0^3} + 2As_0 + \mathcal{O}(\varepsilon_\alpha)\Big)(s-s')(3s_0^2 + \mathcal{O}(s-s_0))$$
$$+ 4A\theta\Big(\frac{s_0}{1-2\theta} + \frac{\mathcal{O}(\varepsilon_z)}{1+|\theta|}\Big)(s-s')(4s_0^3 + \mathcal{O}(s-s_0)).$$

$$I = \frac{-2\theta}{1-2\theta}s_0\Big(\frac{1}{s_0^3} + 2As_0\Big)3s_0^2(s-s') + 4A\theta\frac{4s_0^4}{1-2\theta}(s-s')$$
$$+ |s-s'|\mathcal{O}(\varepsilon_z + \varepsilon_\alpha + \varepsilon_s).$$

$$I = \Big(\frac{-6\theta}{1-2\theta} - \frac{12\theta}{1-2\theta}As_0^4 + \frac{16\theta}{1-2\theta}As_0^4\Big)(s-s') + |s-s'|\mathcal{O}(\varepsilon_z + \varepsilon_\alpha + \varepsilon_s).$$

Since $As_0^4 = 1 - 2\delta_A$ we get

$$|I| \leqslant \Big(\frac{2|\theta|}{1+2|\theta|}(1-2\delta_A) + \mathcal{O}(\varepsilon_z + \varepsilon_\alpha + \varepsilon_h)\Big)|s-s'|.$$

Taking $\varepsilon_z, \varepsilon_\alpha, \varepsilon_h$ and $\delta \in \left]0, \frac{1}{2}\right[$ we get (8.27). The proof of Lemma 8.4 is complete. ☐

Lemma 8.5. — *The map $d\pi : T_\lambda\Lambda \to T_{\pi(\lambda)}E$ is surjective for all λ in Λ.*

Proof. — Let G be the map (for fixed α, h)

$$(\theta, \widetilde{s}, \widetilde{y}) \longmapsto (\theta, s(\theta; \widetilde{s}, \widetilde{y}, \alpha, h), y(\theta; \widetilde{s}, \widetilde{y}, \cdots), \theta^*(\theta), \tau(\theta; \cdots), \eta(\theta, \cdots))$$

from the set $\{\operatorname{Re}\theta \in (-\infty, \varepsilon_\theta[, |\operatorname{Im}\theta| < \varepsilon_\theta, |\widetilde{s} - s_0| < \varepsilon_s, |\widetilde{y} - y| < \varepsilon_y\}$ to Λ. If $d(\pi \circ G)$ is surjective then $d\pi$ is also surjective. Now $d(\pi \circ G)$ is surjective if and only if $\det\begin{pmatrix} \frac{\partial s(\theta)}{\partial \widetilde{s}} & \frac{\partial s(\theta)}{\partial \widetilde{y}} \\ \frac{\partial y(\theta)}{\partial \widetilde{s}} & \frac{\partial y(\theta)}{\partial \widetilde{y}} \end{pmatrix}$ is non zero. According to (8.7) (ii), this will be the case if $|\partial s(\theta)/\partial\widetilde{s}| \geqslant c_0 > 0$. By (8.15) we have $s(\theta) = \widetilde{s}/D$ where

$$D = 1 - 2\theta(\alpha_\tau - 2A(\widetilde{s} - s_0))\widetilde{s}^3 + g_1(\theta) + 2\theta f_1(\theta).$$

Then

$$D^2\frac{\partial s(\theta)}{\partial \widetilde{s}} = 1 - 2\theta\Big(\frac{1}{s_0^3} + \mathcal{O}(\varepsilon_\alpha) + \mathcal{O}(\varepsilon_s)\Big)(s_0^3 + \mathcal{O}(\varepsilon_s) + (1+|\theta|)\mathcal{O}((\varepsilon_\alpha + \varepsilon_h)^{1/2})$$
$$- s_0(-2\theta)\Big(\frac{1}{s_0^3}\cdot 3s_0^2 - 2As_0^3\Big) + (1+|\theta|)\mathcal{O}(\varepsilon_s + \varepsilon_\alpha + (\varepsilon_\alpha + \varepsilon_h)^{1/2}),$$

$$D^2\frac{\partial s}{\partial \widetilde{s}} = 1 - 2\theta - 4As_0^4 + 6\theta + (1+|\theta|)\mathcal{O}(\varepsilon_s + (\varepsilon_\alpha + (\varepsilon_\alpha + \varepsilon_h)^{1/2})$$
$$= 1 - 4\delta_A\theta + (1+|\theta|)\mathcal{O}(\varepsilon_s + (\varepsilon_\alpha + \varepsilon_h)^{1/2}).$$

Since $\operatorname{Re}\theta \in]-\infty, \varepsilon_\theta]$ we will have

$$\Big|D^2\frac{\partial s}{\partial\theta}\Big| \geqslant c_0(1+|\theta|) - \mathcal{O}(\varepsilon_s + (\varepsilon_\alpha + \varepsilon_h)^{1/2})(1+|\theta|).$$

Corollary 8.6. — *There exists* $\varphi = \varphi(\theta; s, y, \alpha, h)$ *defined on* E, *holomorphic with respect to* (θ, s, y, α), *smooth in* h *such that*

$$\Lambda = \Big\{\Big(\theta, s, y, \frac{\partial\varphi}{\partial\theta}(\theta; s, y, \alpha, h), \frac{\partial\varphi}{\partial s}(\theta; \cdots), \frac{\partial\varphi}{\partial y}(\theta; \cdots)\Big), (\theta, s, y) \in E\Big\}.$$

Then Proposition 8.1 follows from Corollary 8.6 since q (defined in (8.5)) vanishes on Λ.

8.2. Resolution of the transport equation

As before, working in the coordinates $(\theta, \widetilde{s}, \widetilde{y})$ we are led to solve the problem

$$\begin{cases} \Big(\dfrac{\partial}{\partial\theta} + c(\theta, \widetilde{s}, \widetilde{y})\Big) a + i h^2 k P_2 a = 0 \\ a|_{\theta=0} = 1 \end{cases} \tag{8.28}$$

where c is equal to $i\Delta_g^*\varphi$ in the new coordinates. The solution should exist in the set $\{\operatorname{Re}\theta \in (-\infty, 0], |\operatorname{Im}\theta| \leqslant \varepsilon_\theta, |\widetilde{s} - s_0| \leqslant \varepsilon_s, |\widetilde{y} - y_0| \leqslant \varepsilon_y\}$. Using the properties of φ it is not difficult to see that

$$|c(\theta; \widetilde{s}, \widetilde{y})| \leqslant \frac{C}{1 + |\theta|}, \quad \operatorname{Re}\theta \in (-\infty, 0], \quad |\operatorname{Im}\theta| \leqslant \varepsilon_\theta. \tag{8.29}$$

Therefore we are in the same situation as in [**RZ1**] (4.16) and the same construction can be made showing that (8.28) can be solved in a space of symbols. We refer to [**RZ1**] for the details.

8.3. Proof of Theorem 4.9

Let $m_0 = (0, y_0, 0, (1, 0)) \in \mathcal{N}_+^c$. Our assumption is that

$$\exp(-TX_0)(m_0) = \Big(\rho = 0, y_0, \lambda_0 = \frac{1}{2T}, \mu_0 = 0\Big)$$

does not belong to ${}^{\mathrm{qsc}}WF_a(u_0)$.

Let us introduce the continuous family of FBI transform

$$\begin{aligned} &\mathcal{T}u(\theta; t, \alpha, h, k) \\ &\quad = \iint e^{ih^{-2}k^{-1}\varphi(\theta; \frac{\rho}{h}, y, \alpha, h)} a\Big(\theta; \frac{\rho}{h}, y, \alpha, h, k\Big) \chi\Big(\theta; \frac{\rho}{h}, y\Big) \overline{u(t; \rho, y)} d\rho dy \end{aligned} \tag{8.30}$$

where φ and a have been constructed in § 8.1, 8.2 and χ is a cut-off function equal to one when

$$\Big|\frac{\rho}{h} - \frac{s_0}{1 + 2|\theta|}\Big| \leqslant \frac{1}{2}\frac{\varepsilon_s}{1 + |\theta|}, \quad |y - y_0| \leqslant \frac{1}{2}\varepsilon_y.$$

As in the proof of Lemma 6.4, we see that

$$\Big(\frac{1}{k}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}\Big)\mathcal{T}u(\theta; t, \cdots) = -i \iint e^{ih^{-2}k^{-1}\varphi(\theta; \cdots)} a(\theta; \cdots)[\Delta_g, \chi](\cdots)\overline{u(t, \rho, y)}\, d\rho dy.$$

Using the properties of φ on the support of $[\Delta_g,\chi]$ we deduce, as in Lemma 6.4 that

$$\mathcal{T}u(\theta;t,\alpha,h,k)=U(k\theta-t;\alpha,h,k)+V(\theta;t,\alpha,h,k) \tag{8.31}$$

where U,V are continuous in

$$\mathcal{O}=\big\{(\theta,t,\alpha,h,k):\theta\in(-\infty,0],\ |t-T|\leqslant\delta_0,\ |\alpha-\alpha_0|\leqslant\varepsilon_\alpha,\ h\in[0,\varepsilon_h],\ k\in[0,\varepsilon_k]\big\}$$

and

$$|V(\theta;t,\alpha,h,k)|\leqslant Ce^{-\varepsilon_0/hk}\ \text{ in }\ \mathcal{O}\cap\{h>0,k>0\}. \tag{8.32}$$

It follows from (8.31) and (8.32) that

$$\mathcal{T}u(0;t,\alpha,h,k)=\mathcal{T}u\Big(-\frac{t}{k};0,\alpha,h,k\Big)+V_1(\theta;t,\alpha,h,k) \tag{8.33}$$

where V_1 satisfies (8.32).

Now the phase which appears in the FBI transform (8.30) can be written, according to § 6.2.2, as

$$\varphi\Big(\theta;\frac{\rho}{h},y,\alpha,h\Big)=\varphi_2\Big(\theta;\frac{\rho}{h},y,\alpha\Big)+ih\varphi_1\Big(\theta;\frac{\rho}{h},y,\alpha,h\Big).$$

It follows that

$$\begin{aligned}(1)&=ih^{-2}k^{-1}\varphi\Big(-\frac{t}{k};\frac{\rho}{h},y,\alpha,h\Big)\\&=i(hk)^{-2}\Big[k\varphi_2\Big(-\frac{t}{k};k\cdot\frac{\rho}{hk},y,\alpha\Big)+i(hk)\varphi_1\Big(-\frac{t}{k};k\cdot\frac{\rho}{hk},y,\alpha,h\Big)\Big].\end{aligned}$$

Therefore if we set

$$\begin{cases}H=hk,\quad \nu=(t,h,k),\\ \psi_2(s,y,\alpha,\nu)=k\varphi_2\Big(-\dfrac{t}{k};ks,y,\alpha\Big),\\ \psi_1(s,y,\alpha,\nu)=\varphi_1\Big(-\dfrac{t}{k};ks,y,\alpha,h\Big),\end{cases} \tag{8.34}$$

then

$$(1)=iH^{-2}\Big[\psi_2\Big(\frac{\rho}{H},y,\alpha,\nu\Big)+iH\psi_1\Big(\frac{\rho}{H},y,\alpha,\nu\Big)\Big]=iH^{-2}\psi\Big(\frac{\rho}{H},y,\alpha,\nu,H\Big). \tag{8.35}$$

Lemma 8.7. — *Let*

$$\widetilde{s}_0=\frac{s_0}{2T},\quad X_0=(\widetilde{s}_0,y_0),\quad \Xi_0=\Big(\frac{1/2T}{\widetilde{s}_0^3},0\Big).$$

Then, when $\nu=(t,h,k)$ tends to $\nu_0=(T,0,0)$, $\psi(\rho/H,y,\alpha,\nu)$ tends (uniformly in $\rho/H,y,\alpha$) to $\psi(\rho/H,y,\alpha,\nu_0)$ and ψ is a phase at $(X_0,\Xi_0,\alpha_0,0,\nu_0)$.

Proof. — Let first h go to zero. Since the phase φ is smooth in h up to $h=0$, $\varphi(-t/k,ks,y,\alpha,h)$ tends to $\varphi(-t/k,ks,y,\alpha,0)$. Let $(\widetilde{s},\widetilde{y},\alpha)$ be given and let us denote by $s(\theta)$, $y(\theta)$, $\tau(\theta)$, $\eta(\theta)$ the solution of (8.6) given by Lemma 8.2. We claim that

$$(8.36)\quad \varphi_2(\theta; s(\theta), y(\theta), \alpha) = [\widetilde{s}^4(\alpha_\tau - 2A(\widetilde{s}-s_0))^2 + \widetilde{s}^2\|\alpha_\eta\|^2]\theta + (\widetilde{s}-\alpha_s)\alpha_\tau + (\widetilde{y}-\alpha_y)\alpha_\eta - A(\widetilde{s}-s_0)^2.$$

Indeed we have

$$\frac{d}{d\theta}[\varphi(\theta; s(\theta), y(\theta), \alpha, 0)] = \Big(\frac{\partial\varphi}{\partial\theta} + \dot{s}(\theta)\frac{\partial\varphi}{\partial s} + \dot{y}(\theta)\frac{\partial\varphi}{\partial y}\Big)(\theta; s(\theta), y(\theta), \alpha) = (1).$$

Now $\dot{s}(\theta) = 2(\tau s^4)(\theta) = 2s^4(\theta)\cdot\partial\varphi/\partial s$, $\dot{y} = 2s^2(\theta)\langle\partial\varphi/\partial y\rangle$. It follows that

$$(1) = \frac{\partial\varphi}{\partial\theta} + 2p\Big(0, y(\theta), \Big(s^2\frac{\partial\varphi}{\partial s}\Big)(\theta;\cdots), \Big(s\frac{\partial\varphi}{\partial y}\Big)(\theta;\cdots)\Big) = -\frac{\partial\varphi}{\partial\theta}(\theta;\cdots)$$

by (8.4). We deduce from (8.6), (8.21) and Corollary 8.6 that

$$-\frac{\partial\varphi}{\partial\theta}(\theta; s(\theta), y(\theta), \alpha, 0) = -\theta^*(\theta) = -\theta^*(0) = p(0, \widetilde{y}, \widetilde{s}^2\widetilde{\tau}, \widetilde{s}\widetilde{\eta}) = \widetilde{s}^4\widetilde{\tau}^2 + \widetilde{s}^2\|\alpha_\eta\|^2$$

where $\widetilde{\tau} = \alpha_\tau - 2A(\widetilde{s}-s_0)$. Then (8.36) follows using (8.4) and the fact that $\varphi = \varphi_2$ if $h = 0$. It follows that

$$(8.37)\quad \begin{cases} k\varphi_2\Big(-\frac{t}{k}; s\Big(-\frac{t}{k}\Big), y\Big(-\frac{t}{k}\Big), \alpha\Big) = -Ut + kV \ \text{ where} \\ U = \widetilde{s}^4(\alpha_\tau - 2A(\widetilde{s}-s_0))^2 + \widetilde{s}^2\|\alpha_\eta\|^2, \\ V = (\widetilde{s}-\alpha_s)\alpha_\tau + (\widetilde{y}-\alpha_y)\cdot\alpha_\eta - A(\widetilde{s}-s_0)^2. \end{cases}$$

Now let (ks, y) be given. The system

$$(8.38)\quad \begin{cases} s\Big(-\frac{t}{k}; \widetilde{s}, \widetilde{y}, \alpha\Big) = ks \\ y\Big(-\frac{t}{k}; \widetilde{s}, \widetilde{y}, \alpha\Big) = y \end{cases}$$

is equivalent, according to (8.15), to

$$\frac{\widetilde{s}}{k(1+g_1(t)) + 2t(\widetilde{\tau}\widetilde{s}^3 + f_1(t))} = s, \quad \widetilde{y} + \mathcal{O}(|\alpha_\eta|^{1/2}) = y.$$

We know from (8.23) that this system has a unique solution which is moreover continuous in $k \in [0, +\infty[$. It follows from (8.37), (8.38) that $k\varphi_2\big(-t/k; ks, y, \alpha\big)$ has a limit when $t \to T$ and $k \to 0$. Let us now look to φ_1. We have seen in § 6.2.2 that

$$(8.39)\quad \operatorname{Re}\varphi_1\Big(-\frac{t}{k}; s\Big(-\frac{t}{k}; \widetilde{s}, \widetilde{y}, \alpha\Big), y\Big(-\frac{t}{k};\cdots\Big), \alpha, 0\Big) = (\widetilde{s}-\alpha_s)^2 + (\widetilde{y}-\alpha_y)^2$$

so the same argument as before works. Concerning the imaginary part of φ_1 ; according to § 6.2.2, we have

$$\frac{\partial}{\partial\theta}[\operatorname{Im}\varphi_1(\theta; s(\theta), y(\theta), \alpha)] = \mathcal{L}_1(\operatorname{Im}\psi)(\theta; s(\theta),\dots) = s(\theta)\frac{\partial p}{\partial\rho}\Big(0, y, s^2\frac{\partial\varphi_2}{\partial s}, s\frac{\partial\varphi_2}{\partial y}\Big) = \mathcal{O}(s^2(\theta)).$$

Since $s^2(\theta)$ is integrable on $(-\infty, 0]$ it follows that

$$\operatorname{Im}\varphi_1\Big(-\frac{t}{k}; s\Big(-\frac{t}{k}; \widetilde{s}, \widetilde{y}, \alpha\Big), y\Big(-\frac{t}{k}; \cdots\Big), \alpha\Big)$$

has also a limit when $k \to 0$, $t \to T$. Let us denote by

$$\psi_2(\frac{\rho}{H}, y, \alpha, \nu_0) + iH\psi_1(\frac{\rho}{h}, y, \alpha, \nu_0)$$

this limit. Let

$$\widetilde{s}_0 = \frac{s_0}{2T}, \quad X_0 = (\widetilde{s}_0, y_0), \quad \Xi_0 = (\frac{1/2T}{\widetilde{s}_0^3}, 0).$$

It remains to show that ψ is a phase at $(X_0, \Xi_0, \alpha_0, h_0 = 0, \nu_0)$ in the sense of Definition 2.6. Conditions (2.10), (2.11), (2.12) are easy satisfied. Let us look to (2.13). We have

$$\frac{\partial\psi_2}{\partial s}\Big(\frac{s_0}{2T}, y_0, \alpha_0, \nu_0\Big) = \lim_{k\to 0} k^2 \frac{\partial\varphi_2}{\partial s}\Big(-\frac{T}{k}; k\frac{s_0}{2T}, y_0, \alpha_0\Big).$$

Now

$$\begin{aligned}
s\Big(-\frac{T}{k}; s_0, y_0, \alpha_0\Big) &= \frac{ks_0}{k+2T} = \frac{ks_0}{2T} + O(k),\\
y\Big(-\frac{T}{k}; s_0, y_0, \alpha_0\Big) &= y_0\\
\frac{\partial\varphi_2}{\partial s}\Big(-\frac{T}{k}; s\Big(-\frac{T}{k}; s_0, y_0, \alpha_0\Big), y\Big(-\frac{T}{k}, \cdots\Big), \alpha_0\Big) &= \tau\Big(-\frac{T}{k}; s_0, y_0, \alpha_0\Big)\\
&= \frac{\tau_0 s_0^2}{s^2(-T/k; s_0, y_0, \alpha_0)}\\
&= \frac{(1+2T/k)^2}{s_0^3}.
\end{aligned}$$

Therefore

$$k^2\frac{\partial\varphi_2}{\partial s}\Big(-\frac{T}{k}, k\frac{s_0}{2T}, y_0, \alpha_0\Big) = \frac{(k+2T)^2}{s_0^3} + O(k) \longrightarrow \frac{(2T)^2}{s_0^3} = \frac{1/2T}{\widetilde{s}_0^3}.$$

It follows that

$$\frac{\partial\psi_2}{\partial s}\Big(\frac{s_0}{2T}, y_0, \alpha_0, \nu_0\Big) = \frac{1/2T}{\widetilde{s}_0^3}.$$

Moreover

$$\frac{\partial\psi}{\partial y}\Big(\frac{s_0}{2T}, y_0, \alpha_0, \nu_0\Big) = \lim_{k\to 0}\frac{\partial\varphi_2}{\partial y}\Big(-\frac{T}{k}, \frac{ks_0}{2T}, y_0, \alpha_0\Big) = 0.$$

Finally

$$\operatorname{Re}\psi_1\Big(\frac{s_0}{2T}, y_0, \alpha_0, \nu_0\Big) = \lim_{k\to 0}\operatorname{Re}\varphi_1\Big(-\frac{T}{k}, k\frac{s_0}{2T}, y_0, \alpha_0\Big)$$

and

$$\begin{aligned}
\operatorname{Re}\varphi_1\big(-\tfrac{T}{k}, k\tfrac{s_0}{2T}, y_0, \alpha_0\big) &= \operatorname{Re}\varphi_1\big(-\tfrac{T}{k}, s\big(-\tfrac{T}{k}; s_0, y_0, \alpha_0\big), y\big(-\tfrac{T}{k}; \cdots\big), \alpha_0\big) + O(k)\\
&= (s_0 - s_0)^2 + (y_0 - y_0)^2 + O(k) \longrightarrow 0.
\end{aligned}$$

□

We can now give the final argument of the proof of Theorem 4.9. It follows from Lemma 8.7, Theorem 2.7 and the fact that $(0, y_0, 1/2T, 0)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(0,\cdot))$ that

$$\Big|\mathcal{T}u\Big(-\frac{t}{k}; 0, \alpha, h, k\Big)\Big| \leqslant C e^{-\varepsilon/H}, \quad \varepsilon_1 > 0, \ H = hk,$$

for all α in V_{α_0}, $0 < h < \varepsilon_h$, $0 < k < \varepsilon_k$ and $|t - T| \leqslant \delta$ (since $\nu = (T, h, k) \in V_{\nu_0}$). We use (8.33) to show that

$$|\mathcal{T}u(0; t, \alpha, h, k)| \leqslant C' e^{-\varepsilon_2/hk}, \quad \varepsilon_0 > 0$$

for the same value of the parameters. Since the phase of the later FBI transform is, by Proposition 8.1,

$$\varphi(0; s, y, \alpha, h) = (s - \alpha_s)\alpha_\tau + (y - \alpha_y)\alpha_\eta - A(s - s_0)^2 + ih[(s - \alpha_s)^2 + (y - \alpha_y)^2]$$

which is a FBI phase in the sense of Definition 2.1, we deduce from Definition 2.4 that the point m_0 does not belong to ${}^{\mathrm{qsc}}\widetilde{WF}(u(T,\cdot))$ which is our claim. □

APPENDIX

We develop here the Sjöstrand theory of FBI transform in the case of two scales. This will allow us to define the qsc analytic wave front set. The main difficulty will be to prove the invariance of this notion under the change of phase, amplitude and cut-off functions.

A.1. The phases

Definition A.1. — Let $m_0 = (x_0, \xi_0, \alpha^0, h_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} \times [0, +\infty[$. We shall say that $\varphi = \varphi(x, \alpha, h)$ is an FBI phase at m_0 if one can find a neighborhood V of (x_0, α^0) in $\mathbb{C}^n \times \mathbb{C}^{2n}$, a neighborhood I_{h_0} of h_0 in $[0, +\infty[$ such that, in $V \times I_{h_0}$,

$$\varphi(x, \alpha, h) = \varphi_2(x, \alpha_\xi) + \varphi_3(\alpha) + ih\varphi_1(x, \alpha), \quad \alpha = (\alpha_x, \alpha_\xi),$$

where

(1) φ_j, $j = 1, 2, 3$, are holomorphic in V,

(2) φ_2 is real when $(x, \alpha_\xi) \in (\mathbb{R}^n \times \mathbb{R}^n) \cap V$, φ_3 is real when α is real,

(3) $\dfrac{\partial \varphi}{\partial x}(x_0, \alpha^0, h_0) = \xi_0$,

(4) $\varphi_1(x_0, \alpha^0) = 0$, $\dfrac{\partial \operatorname{Re} \varphi_1}{\partial x}(x_0, \alpha^0) = 0$, $\Big(\dfrac{\partial^2 \operatorname{Re} \varphi_1}{\partial x^2}(x_0, \alpha^0)\Big)$ is positive definite, $\Big(\dfrac{\partial^2 \operatorname{Re} \varphi_1}{\partial x \partial \alpha_x}(x_0, \alpha^0)\Big)$ is invertible,

(5)

(a) if $h_0 = 0$, the matrix $\Big(\dfrac{\partial^2 \varphi_2}{\partial x \partial \alpha_\xi}(x_0, \alpha_\varepsilon^0)\Big)$ is invertible,

(b) if $h_0 \neq 0$, the matrices $\Big(\dfrac{\partial^2 \varphi}{\partial x \partial \alpha_\xi}\Big)(x_0, \alpha^0, h_0)$ and

$$\begin{pmatrix} \frac{\partial^2 \operatorname{Re} \varphi}{\partial x \partial \alpha_\xi} & \frac{\partial^2 \operatorname{Re} \varphi}{\partial x \partial \alpha_x} \\ \frac{\partial^2 \operatorname{Im} \varphi}{\partial x \partial \alpha_\xi} & \frac{\partial^2 \operatorname{Im} \varphi}{\partial x \partial \alpha_x} \end{pmatrix} (x_0, \alpha^0, h_0)$$

are invertible.

The simplest example of such a phase is given by

$$\varphi(x,\alpha,h) = (x-\alpha_x)\alpha_\xi + ih(x-\alpha_x)^2$$

with $\alpha^0 = (x_0, \xi_0)$.

Now, if f is a complex function defined on the complex domain, we define

$$(A.1) \qquad \begin{cases} f^r(z) = \dfrac{1}{2}(f(z) + \overline{f(\overline{z})}) \\ f^i(z) = \dfrac{1}{2i}(f(z) - \overline{f(\overline{z})}). \end{cases}$$

Definition A.2. — With the notations of Definition A.1, we shall say that φ is a precised FBI phase at m_0 if it is an FBI phase at m_0 and moreover, $(x,\alpha) \in V$ and $\partial\varphi_1^r/\partial x(x,\alpha) = 0$ imply $\varphi_1^r(x,\alpha) = 0$.

Then we have the following result.

Proposition A.3. — *Let φ be an FBI phase at $m_0 = (x_0, \xi_0, \alpha^0, h_0)$. Then one can find a precised FBI phase $\widetilde{\varphi}$ at m_0 such that*

$$\widetilde{\varphi}(x,\alpha,h) = \varphi(x,\alpha,h) + g(\alpha,h)$$

with $g(\alpha^0, h_0) = 0$. Moreover if the inequality (2.9) (defining ${}^{\text{qsc}}WF_a(u)$) is true with φ, it is also true, with other constants, with $\widetilde{\varphi}$.

Proof. — Using the hypothesis 4) in Definition A.1 and the implicit function theorem, we see that there exists a holomorphic function $x(\alpha)$ such that $\partial\varphi_1^r/\partial x(x(\alpha),\alpha) = 0$, with $x(\alpha^0) = x_0$ and $x(\alpha)$ is real if α is real. Let us set

$$\widetilde{\varphi}(x,\alpha,h) = \varphi(x,\alpha,h) - ih\varphi_1^r(x(\alpha),\alpha).$$

Since $\widetilde{\varphi} = \varphi_2(x,\alpha_\xi) + \varphi_3(\alpha) + ih(\varphi_1(x,\alpha) - \varphi_1^r(x(\alpha),\alpha))$, we see that $\widetilde{\varphi}$ satisfies the hypotheses 1) to 5) in Definition A.1. Since $\partial\widetilde{\varphi}_1^r/\partial x = \partial\varphi_1^r/\partial x$, the solution of $\partial\widetilde{\varphi}_1^r/\partial x(x,\alpha) = 0$ is also $x(\alpha)$ and $\widetilde{\varphi}_1^r(x(\alpha),\alpha) = 0$.

We introduce now a weaker notion of phase. The reason for that is that, in a propagation process, even if we begin at the initial time with an FBI phase, after a while the phase could only be a phase in the following sense.

Definition A.4. — Let $m_0 = (x_0, \xi_0, \beta^0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n}$. We shall say that $\psi = \psi(x,\beta,m,h)$, defined for (x,β) in a neighborhood W of (x_0,β^0) in $\mathbb{C}^n \times \mathbb{C}^{2n}$ and for the parameters (m,h) in a set $U \subset \mathbb{R}^N \times \mathbb{R}^+$, is a phase at m_0 if there exist positive constants ε_0, C_0 such that

(1) ψ is holomorphic in W for any (m,h) in U,
(2) $\operatorname{Im}\psi(x,\beta,m,h) \geqslant 0$ if $(x,\beta) \in W_{\mathbb{R}} = W \cap \mathbb{R}^n \times \mathbb{R}^{2n}$ and $(m,h) \in U$,
(3) $|\psi(x,\beta,m,h)| + \left|\frac{\partial\psi}{\partial x}(x,\beta,m,h) - \xi_0\right| \leqslant \varepsilon_0$, for all (x,β) in W and (m,h) in U,
(4) $\left|\frac{\partial \operatorname{Im}\psi}{\partial x}(x,\beta,m,h)\right| \leqslant \varepsilon_0 h$, for all $(x,\beta) \in W_{\mathbb{R}}$, and $(m,h) \in U$,
(5) $|\partial^\alpha\psi(x,\beta,m,h)| \leqslant C_0$ for $|\alpha| \leqslant 3$, $(x,\beta) \in W$, $(m,h) \in U$,
(6) $\operatorname{Im}\frac{\partial^2\psi}{\partial x^2}(x,\beta,m,h) \geqslant -\varepsilon_0 h \operatorname{Id}$ if $(x,\beta) \in W_{\mathbb{R}}$ and $(m,h) \in U$.

For the purpose of the theory, we introduce now the phases of pseudo-differential operators.

Definition A.5. — Let $m_0 = (x_0, \xi_0, \alpha^0, h_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} \times [0, +\infty[$. We shall say that

$$\varphi = \varphi(x, y, \alpha, h) = \varphi_2(x, y, \alpha_\xi) + ih\varphi_1(x, y, \alpha)$$

is a pseudo-differential phase, near m_0 if

(1) φ_j, $j = 1, 2$, are holomorphic on a neighborhood V of $\rho_0 = (x_0, x_0, \alpha^0)$ in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{2n}$.

(2) φ_2 is real if $(x, y, \alpha_\xi) \in V_{\mathbb{R}} = V \cap (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n})$.

(3) $\varphi_2(x, x, \alpha_\xi) = \varphi_1^i(x, x, \alpha) = 0$.

(4) $\dfrac{\partial \varphi_1^r}{\partial \alpha_x}(x, x, \alpha) = 0$ implies $\varphi_1^r(x, x, \alpha) = 0$.

(5) $\dfrac{\partial \varphi_1}{\partial \alpha_x}(\rho_0) = 0$ and $\Big(\dfrac{\partial^2 \varphi_1^r}{\partial \alpha_x^2}(\rho_0)\Big)$ is positive definite. We shall denote by $\alpha_x(x, \alpha_\xi)$ the solution of $\dfrac{\partial \varphi_1^r}{\partial \alpha_x}(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi) = 0$ with $\alpha_x(x_0, \alpha_\xi^0) = \alpha_x^0$, $\alpha^0 = (\alpha_x^0, \alpha_\xi^0)$.

(6) $\dfrac{\partial \varphi}{\partial x}(x_0, x_0, \alpha^0, h) = -\dfrac{\partial \varphi}{\partial y}(x_0, x_0, \alpha^0, h) = \xi_0$, for all h in a neighborhood of h_0. Moreover the matrices

$$\frac{\partial^2 \varphi_1^r}{\partial x^2}(\rho_0) + 2\frac{\partial^2 \varphi_1^r}{\partial x \partial y}(\rho_0) + \frac{\partial^2 \varphi_1^r}{\partial y^2}(\rho_0)$$
$$\frac{\partial^2 \varphi_1^r}{\partial x \partial \alpha_x}(\rho_0) + \frac{\partial^2 \varphi_1^r}{\partial y \partial \alpha_x}(\rho_0)$$

are invertible.

(7) One can find $C > 0$ such that for every (x, y, α) in $V_{\mathbb{R}}$

$$\varphi_1^r(x, y, \alpha) \geqslant C\big[(\alpha_x - \alpha_x(x, \alpha_\xi))^2 + (\alpha_x - \alpha_x(y, \alpha_\xi))^2\big].$$

(8) If $h_0 = 0$, the matrix $\dfrac{\partial^2 \varphi_2}{\partial y \partial \alpha_\xi}(x_0, x_0, \alpha_\xi^0)$ is invertible. If $h_0 \neq 0$, the matrices

$$\Big(\frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi}\Big)(\rho_0) \text{ and } M = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi}(\rho_0) & \frac{\partial^2 \varphi}{\partial y \partial \alpha_x}(\rho_0) \\ \frac{\partial^2 \varphi}{\partial \alpha_x \partial \alpha_\xi}(\rho_0) & \frac{\partial^2 \varphi}{\partial \alpha_x^2}(\rho_0) \end{pmatrix}$$

are invertible.

Then we have,

Proposition A.6. — *Let $m_0 = (x_0, \xi_0, \alpha^0, h_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} \times [0, +\infty[$. Let $\widetilde{\varphi}$ be a precised FBI phase at m_0. Then*

$$\varphi(x, y, \alpha, h) = \widetilde{\varphi}(x, \alpha, h) - \widetilde{\varphi}_2(y, \alpha_\xi) - \widetilde{\varphi}_3(\alpha) + ih\widetilde{\varphi}_1^r(y, \alpha) + h\widetilde{\varphi}_1^i(y, \alpha)$$

is a pseudo-differential phase at m_0.

Proof. — Let us remark that if (x,y,α) is real then

$$\varphi(x,y,\alpha,h) = \widetilde{\varphi}(x,\alpha,h) - \overline{\widetilde{\varphi}(y,\alpha,h)}.$$

We have $\varphi_2(x,y,\alpha_\xi) = \widetilde{\varphi}_2(x,\alpha_\xi) - \widetilde{\varphi}_2(y,\alpha_\xi)$ and $\varphi_1(x,y,\alpha) = \widetilde{\varphi}_1(x,\alpha) + \widetilde{\varphi}_1^r(y,\alpha) - i\widetilde{\varphi}_1^i(y,\alpha)$. Then, the conditions 1), 2), 3) in Definition A.5, are trivially satisfied. Condition 4) is also easily satisfied since $\widetilde{\varphi}$ is precised. Let us check 5). We set $\psi = \widetilde{\varphi}_1^r$. Let $x(\alpha)$ be the local solution of the problem

$$\frac{\partial\psi}{\partial x}(x(\alpha),\alpha) = 0, \quad x(\alpha^0) = x_0. \tag{A.2}$$

Since $\widetilde{\varphi}$ is precised, we have $\psi(x(\alpha),\alpha) = 0$. Differentiating with respect to α_x we obtain,

$$\frac{\partial\psi}{\partial x}(x(\alpha),\alpha)\frac{\partial x}{\partial\alpha_x}(\alpha) + \frac{\partial\psi}{\partial\alpha_x}(x(\alpha),\alpha) = 0.$$

Therefore

$$\frac{\partial\psi}{\partial\alpha_x}(x(\alpha),\alpha) = 0, \quad x(\alpha^0) = x_0. \tag{A.3}$$

Now,

$$\frac{\partial\varphi_1}{\partial\alpha_x}(\rho_0) = 2\frac{\partial\widetilde{\varphi}_1^r}{\partial\alpha_x}(x_0,\alpha^0) = 0,$$

which is the first part of condition 5). If we differentiate (A.2) and (A.3) with respect to α_x we get,

$$\frac{\partial x}{\partial\alpha_x} = -\Big(\frac{\partial^2\psi}{\partial x^2}\Big)^{-1}\frac{\partial^2\psi}{\partial x\partial\alpha_x} \tag{A.4}$$

$$\frac{\partial^2\psi}{\partial\alpha_x^2} = -\frac{\partial^2\psi}{\partial\alpha_x\partial x}\cdot\frac{\partial x}{\partial\alpha_x} = \Big(\frac{\partial^2\psi}{\partial\alpha_x\partial x}\Big)\Big(\frac{\partial^2\psi}{\partial x^2}\Big)^{-1}\Big(\frac{\partial^2\psi}{\partial x\partial\alpha_x}\Big) \gg 0, \tag{A.5}$$

by condition 4) in Definition A.1.

Since

$$\frac{\partial^2\varphi_1^r}{\partial\alpha_x^2}(\rho_0) = 2\frac{\partial^2\psi}{\partial\alpha_x^2}(x_0,\alpha^0),$$

the second part of 5) follows.

Let us check now condition 7) since condition 6) follows easily from condition 3), 4) in Definition A.1. We deduce from (A.5) that we can find $\alpha_x(x,\alpha_\xi)$ such that, with $\psi = \widetilde{\varphi}_1^r$,

$$\frac{\partial\psi}{\partial\alpha_x}(x,\alpha_x(x,\alpha_\xi),\alpha_\xi) = 0, \quad \alpha_x(x_0,\alpha_\xi^0) = \alpha_x^0. \tag{A.6}$$

By (A.4), the map $\alpha_x \mapsto x(\alpha)$ is, for any α_ξ, a local diffeomorphism. The inverse map $x^{-1}(x,\alpha_\xi)$ satisfies, by (A.2), $\partial\psi/\partial\alpha_x(x,x^{-1}(x,\alpha_\xi),\alpha_\xi) = 0$. By uniqueness in (A.2) we obtain $x^{-1}(x,\alpha_\xi) = \alpha_x(x,\alpha_\xi)$. Then

$$\psi(x,\alpha_x(x,\alpha_\xi),\alpha_\xi) = \psi(x(\alpha_x(x,\alpha_\xi),\alpha_\xi),\alpha_x(x,\alpha_\xi),\alpha_\xi) = 0. \tag{A.7}$$

It follows from Taylor's formula that

$$\begin{aligned}\psi(x,\alpha) &= \psi(x,\alpha_x(x,\alpha_\xi),\alpha_\xi) + \frac{\partial\psi}{\partial x}(x,\alpha_x(x,\alpha_\xi),\alpha_\xi)(\alpha_x - \alpha_x(x,\alpha_\xi)) \\ &\qquad\qquad + M(x,\alpha)(\alpha_x - \alpha_x(x,\alpha_\xi))^2 \\ &= M(x,\alpha)(\alpha_x - \alpha_x(x,\alpha_\xi))^2 .\end{aligned}$$

By (A.5) and condition 4) in Definition A.1 we have $M(x_0,\alpha^0) \gg 0$. Therefore, with $\psi = \widetilde{\varphi}_1^r$, we have

$$\psi(x,\alpha) \geqslant C|\alpha_x - \alpha_x(x,\alpha_\xi)|^2, \text{ if } (x,\alpha) \text{ is real.} \tag{A.8}$$

Then condition 7) follows from (A.8) since $\varphi_1^r(x,y,\alpha) = \widetilde{\varphi}_1^r(x,\alpha) + \widetilde{\varphi}_1^r(y,\alpha)$.

To check condition 8) when $h_0 \neq 0$, we differentiate (A.2) and (A.3) with respect to α_ξ and α_x. We get, with $\psi = \widetilde{\varphi}_1^r = \widetilde{\varphi}_1^r(y,\alpha)$

$$\begin{cases} \dfrac{\partial^2\psi}{\partial\alpha_x\partial\alpha_\xi} = \Big(\dfrac{\partial^2\psi}{\partial\alpha_x\partial y}\Big)\Big(\dfrac{\partial^2\psi}{\partial y^2}\Big)^{-1}\Big(\dfrac{\partial^2\psi}{\partial y\partial\alpha_\xi}\Big) \\ \dfrac{\partial^2\psi}{\partial\alpha_x^2} = \Big(\dfrac{\partial^2\psi}{\partial\alpha_x\partial y}\Big)\Big(\dfrac{\partial^2\psi}{\partial y^2}\Big)^{-1}\Big(\dfrac{\partial^2\psi}{\partial y\partial\alpha_x}\Big). \end{cases} \tag{A.9}$$

Let us set $A = \big(\frac{\partial^2\psi}{\partial\alpha_x\partial y}\big)\big(\frac{\partial^2\psi}{\partial y^2}\big)^{-1}$. By condition 4) of Definition A.1 this is an invertible matrix at (x_0,α^0,h_0). We set also

$$B = \Big(\frac{\partial^2\psi}{\partial y\partial\alpha_\xi}\Big), \quad C = \Big(\frac{\partial^2\psi}{\partial y\partial\alpha_x}\Big), \quad D = \Big(\frac{\partial^2\widetilde{\varphi}^r}{\partial y\partial\alpha_\xi}\Big), \quad E = \Big(\frac{\partial^2\widetilde{\varphi}^r}{\partial y\partial\alpha_x}\Big).$$

Then the matrix M occurring in condition 8) can be written at ρ_0 as

$$M = \begin{pmatrix} D + ihB & E + ihC \\ 2ihAB & 2ihAC \end{pmatrix}.$$

Now, condition 5) of Definition A.1 ensures that the matrix $\begin{pmatrix} D & E \\ B & C\end{pmatrix}$ is invertible. Since A is invertible it follows that M is uniformly invertible when $h \geqslant h_1 > 0$. The invertibility of $\big(\frac{\partial^2\varphi}{\partial y\partial\alpha_\xi}\big)(\rho_0)$ follows from that of $\big(\frac{\partial^2\widetilde{\varphi}}{\partial y\partial\alpha_\xi}\big)$. □

The case $h_0 = 0$ in 8) is easier since $\frac{\partial^2\varphi_2}{\partial y\partial\alpha_\xi}(x_0,x_0,\alpha_\xi^0) = -\frac{\partial^2\widetilde{\varphi}_2}{\partial y\partial\alpha_\xi}(x_0,\alpha^0)$ is invertible by condition 5), Definition A.1.

Remark A.7. — For a general pseudo-differential phase, we still have the correspondence between $x(\alpha)$ and $\alpha_x(x,\alpha_\xi)$. Indeed, by conditions 5), 6) in Definition A.5 we can solve the problems

$$\begin{aligned} &\frac{\partial\varphi_1^r}{\partial\alpha_x}(x,x,\alpha_x(x,\alpha_\xi),\alpha_\xi) = 0, && \alpha_x(x_0,\alpha_\xi^0) = \alpha_x^0 \\ &\Big(\frac{\partial\varphi_1^r}{\partial x} + \frac{\partial\varphi_1^r}{\partial y}\Big)(x(\alpha),x(\alpha),\alpha) = 0, && x(\alpha^0) = x_0 . \end{aligned}$$

We have $\varphi_1^r(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi) = 0$ by condition 4) so

$$\Big(\frac{\partial \varphi_1^r}{\partial x} + \frac{\partial \varphi_1^r}{\partial y}\Big)(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi) = 0.$$

The map $x \mapsto \alpha_x(x, \alpha_\xi)$ is, by condition 6), a local diffeomorphism so we deduce, as above, that $\alpha_x(x, \alpha_\xi)$ and $x(\alpha)$ are inverse of each other.

A.2. Good contours

Pseudo-differential operators in the complex domain will lead to integrals along some contours. In this section we define these objects which will be called "good contours". Let W an open subset of $\mathbb{R}^{2k} \times \mathbb{R}^{2n}$ and V a subset of $\mathbb{R}^N \times]0, +\infty[$. Let $f(x, y, z, h) = f_2(x, y, h) + h f_1(x, y, z, h)$ be a real function defined for (y, z) in W and (x, h) in V.

We shall assume that

$$(A.10) \qquad \exists\, C > 0 : |\partial^\alpha_{(y,z)} f_j| \leqslant C,\ j = 1, 2,\ \forall\, (y, z) \in W,\ \forall\, (x, h) \in V,\ \forall\, |\alpha| \leqslant 3.$$

$$(A.11) \qquad \begin{cases} \text{For any } (x,h) \text{ in } V,\ f \text{ has a unique critical point in } (y,z) \\ \text{(denoted } (y(x,h), z(x,h))) \text{ in } W. \end{cases}$$

$(A.12)$

$$\begin{cases} \text{The matrix } \Big(\dfrac{\partial^2 f}{\partial (y,z)^2}\Big)(x, y(x,h), z(x,h), h) \text{ has signature } (n+k, n+k), \\ \forall\, h_1 > 0,\ \exists\, C_{h_1} > 0 : \forall\, (x,h) \text{ with } h \geqslant h_1 \text{ we have} \\ \qquad \Big\| \Big[\dfrac{\partial^2 f}{\partial (y,z)^2}(x, y(x,h), z(x,h), h)\Big]^{-1} \Big\| \leqslant C_{h_1}. \end{cases}$$

$(A.13)$

$$\begin{cases} \exists\, h_2 > 0,\ C_0 > 0 : \forall\, (x,h) \in V,\ h \in]0, h_2],\ \dfrac{\partial^2 f}{\partial y^2}(x, y(x,h), z(x,h), h) \\ \text{has signature } (k,k),\ \dfrac{\partial^2 f_1}{\partial z^2}(x, y(x,h), z(x,h), h) \text{ has signature } (n,n) \text{ and} \\ \qquad \Big\| \Big[\dfrac{\partial^2 f}{\partial y^2}(x, y(x,h), z(x,h), h)\Big]^{-1} \Big\| \leqslant C_0 \\ \qquad \Big\| \Big[\dfrac{\partial^2 f_1}{\partial z^2}(x, y(x,h), z(x,h), h)\Big]^{-1} \Big\| \leqslant C_0. \end{cases}$$

Let us remark that (A.13) implies (A.12) for small h.

Definition A.8. — Let f be satisfying (A.10) to (A.13). Let

$$\Gamma_{x,h} : (\widetilde{Y}, \widetilde{Z}) \longmapsto (y(x, \widetilde{Y}, \widetilde{Z}, h), z(x, \widetilde{Y}, \widetilde{Z}, h))$$

be a map from a neighborhood of $(0,0)$ in $\mathbb{R}^k \times \mathbb{R}^n$ to $W \subset \mathbb{R}^{2k} \times \mathbb{R}^{2n}$, such that $y(x,0,0,h) = y(x,h)$, $z(x,0,0,h) = z(x,h)$. We shall say that $\Gamma_{x,h}$ is a good contour for f if there exists a positive constant C_0 such that, for every (x,h) in V,

$$(A.14)\quad f(x,y,z,h) - f(x,y(x,h),z(x,h),h) \leqslant -C_0\big[|y-y(x,h)|^2 + h|z-z(x,h)|^2\big]$$

on the contour $\Gamma_{x,h}$ (that means for $(y,z) = (y(x,\widetilde{Y},\widetilde{Z},h), z(x,\widetilde{Y},\widetilde{Z},h)))$. We assume moreover that $y(x,\widetilde{Y},\widetilde{Z},h) = y_1(x,\widetilde{Y},h) + hy_2(x,\widetilde{Y},\widetilde{Z},h)$ and that for all (x,h) in V,
$(A.15)$

$$\begin{cases} |\partial^\alpha_{(Y,Z)} y_j(\cdots)| + |\partial^\alpha_{(Y,Z)} z(x,\widetilde{Y},\widetilde{Z},h)| \leqslant C_0, \quad |\alpha| \leqslant 2 \\ \Big|D_{Y,Z}y(x,0,0,h)\begin{pmatrix}Y\\Z\end{pmatrix}\Big|^2 + h\Big|D_{Y,Z}z(x,0,0,h)\begin{pmatrix}Y\\Z\end{pmatrix}\Big|^2 \geqslant \dfrac{1}{C_0}(|Y|^2 + h|Z|^2). \end{cases}$$

Proposition A.9. — *Let $\Gamma_{x,h,0}$ and $\Gamma_{x,h,1}$ be two good contours for f. Then, there exist for $s \in [0,1]$ a smooth family $\widetilde{\Gamma}_{x,h,s}$ a good contours and $\delta > 0$ such that for every (x,h) in V,*

$$\big(\Gamma_{x,h,0} \smallsetminus \widetilde{\Gamma}_{x,h,0}\big) \cup \big(\Gamma_{x,h,1} \smallsetminus \widetilde{\Gamma}_{x,h,1}\big) \cup \{\partial\widetilde{\Gamma}_{x,h,s} : s \in [0,1]\} \subset \big\{(y,z) : f(x,y,z,h) \leqslant -\delta h + f(x,y(x,h),z(x,h),h)\big\}.$$

Proof. — To prove this result, we first write f in a set of Morse coordinates. This leads us to check that the change of coordinates is well defined in a fixed neighborhood of the critical point, that means independent of $(x,h) \in V$ and that the constants are also uniform.

Lemma A.10. — *Let A_0 be a $2n \times 2n$ matrix which is real, symmetric and has signature (n,n). Then there exists a matrix Q_0 such that $A_0 = {}^tQ_0 D Q_0$ with $D = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and, for all symmetric matrix A, such that $\|A - A_0\| \leqslant 1/2\|A_0^{-1}\|$, one can find $Q = Q(A)$ such that*

(i) $A = {}^tQDQ, \quad Q(A_0) = Q_0,$
(ii) $\|Q(A) - Q(B)\| \leqslant \|A_0^{-1}\|^{1/2}\|A - B\|$, *when* $\|A_0 - B\| \leqslant 1/2\|A_0^{-1}\|$,
(iii) $\|Q(A)^{-1}\| \leqslant 2\|A_0^{-1}\|\,\|Q(A)\|$,
(iv) $\|Q(A)\| \leqslant \|A_0\|^{1/2} + 1/2\|A_0^{-1}\|^{1/2}$.

Here $\|\cdot\|$ is the matrix norm related to the Euclidian norm in $\mathbb{R}^{2n}$.

Proof. — We write $A_0 = {}^tO\Lambda O$, where O is orthogonal and Λ diagonal ; then we write $\Lambda = {}^tK_0DK_0$ where K_0 is the diagonal matrix which entries are the square roots of the absolute values of the eigenvalues of A_0. We set $Q_0 = K_0O$. Then $A_0 = {}^tQ_0DQ_0$ and $\|Q_0\|^2 = \|A_0\|$, $\|Q_0^{-1}\|^2 = \|A_0^{-1}\|$. Now we set $Q = Q_0 + R$; then R must satisfy

$$A - A_0 = {}^tQ_0DR + {}^tRDQ_0 + {}^tRDR.$$

To solve this equation we define, by induction, R_j, K_j such that

$$(A.16)\qquad \begin{cases} K_0 = 0, \;\; K_{j+1} = A - A_0 - {}^t R_j D R_j, & j \geqslant 0 \\ R_j = \dfrac{1}{2} D\, {}^t Q_0^{-1} K_j, & j \geqslant 0. \end{cases}$$

It is easily proved that

$$\|K_{j+1} - K_j\| \leqslant \frac{1}{4}\big(\|K_j\| + \|K_{j-1}\|\big)\, \|A_0^{-1}\|\, \|K_j - K_{j-1}\|, \quad j \geqslant 1.$$

We deduce, by induction, from this inequality that, for $j \geqslant 1$,

$$\begin{cases} \|K_k\| \leqslant \dfrac{1}{\|A_0^{-1}\|}, & 0 \leqslant k \leqslant j \\ \|K_j - K_{j-1}\| \leqslant \dfrac{1}{2^{j-1}} \|A - A_0\|. \end{cases}$$

It follows that $K_j \to K_\infty$, $R_j \to R_\infty$ and R_∞ solves our initial equation. Now, if we denote by K_j', R_j' the solution of (A.16) with B instead of A, we have,

$$\|K_j' - K_j\| \leqslant \|A - B\| + \frac{1}{2} \|K_{j-1}' - K_{j-1}\|,$$

which implies that

$$\|K_j' - K_j\| \leqslant 2\|A - B\| - \frac{1}{2^{j-1}} \|A - B\|.$$

Then

$$\|Q(A) - Q(B)\| = \|R_\infty - R_\infty'\| \leqslant \frac{1}{2} \|Q_0^{-1}\|\, \|K_\infty - K_\infty'\| \leqslant \|Q_0^{-1}\| \cdot \|A - B\|.$$

Finally

$$\|Q\| \leqslant \|Q_0\| + \|R\| \leqslant \|A_0\|^{1/2} + \frac{1}{2} \|Q_0^{-1}\| \frac{1}{\|A_0^{-1}\|} \leqslant \|A_0\|^{1/2} + \frac{1}{2\|A_0^{-1}\|^{1/2}}$$

and

$$\|Q^{-1}\| = \|A^{-1}\, {}^t Q D\| \leqslant \|A^{-1}\| \cdot \|Q\| \leqslant 2\|A_0^{-1}\| \cdot \|Q\|. \qquad \square$$

Proof of Proposition A.9. — We shall consider the case where h is small ; the case h large follows the same lines and y, z play the same role. We write

$$(A.17)\quad f(x,y,z,h) = f(x, y(x,h), z(x,h), h) + \big({}^t(y - y(x,h)), {}^t(z - z(x,h))\big) \begin{pmatrix} A_1 & hB \\ h\,{}^t B & hA_2 \end{pmatrix} \begin{pmatrix} y - y(x,h) \\ z - z(x,h) \end{pmatrix},$$

where A_1, B, A_2 depend on (x,y,z,h) and satisfy the estimates

$$\left\| A_1 - \frac{\partial^2 f}{\partial y^2}(m_{x,h}) \right\| \leqslant C_1\big(|y - y(x,h)| + h|z - z(x,h)|\big),$$

$$\left\| B - \frac{\partial^2 f}{\partial y \partial z}(m_{x,h}) \right\| \leqslant C_1\big(|y - y(x,h)| + |z - z(x,h)|\big),$$

$$\left\|A_2 - \frac{\partial^2 f}{\partial z^2}(m_{x,h})\right\| \leqslant C_1\big(|y - y(x,h)| + |z - z(x,h)|\big),$$

where $m_{x,h} = (x, y(x,h), z(x,h), h)$ and C_1 is a constant which depends only on the constant C in (A.10).

We wish to apply the Lemma A.10 to A_1, so we need that

$$C_1\big(|y - y(x,h)| + h|z - z(x,h)|\big) \leqslant \frac{1}{2\left\|\left(\frac{\partial^2 f}{\partial y^2}(m_{x,h})\right)^{-1}\right\|}.$$

It follows from (A.13) that this will be achieved if

$$|y - y(x,h)| + h|z - z(x,h)| \leqslant \frac{1}{2C_0C_1}. \tag{A.18}$$

Under this condition, the Lemma A.10 implies that one can find $Q_1 = Q_1(x,y,z,h)$ such that

$$A_1 = {}^tQ_1 D_1 Q_1 \text{ where } D_1 = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}.$$

Since $A_1 = A_1'(x,y,h) + hA_1''(x,y,z,h)$, it follows from Lemma A.10, (ii), that $Q_1 = Q_1'(x,y,h) + hQ_1''(x,y,z,h)$, with $Q_1' = Q(A_1')$. Moreover $\|Q_1\|$ and $\|Q_0^{-1}\|$ are uniformly bounded by constants which depend only on C_0, C in (A.10), (A.13).

It follows that we have

$$\begin{pmatrix} A_1 & hB \\ h\,{}^tB & hA_2 \end{pmatrix} = \begin{pmatrix} {}^tQ_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_1 & h\,{}^tQ_1^{-1}B \\ h\,{}^tBQ_1^{-1} & hA_2 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}.$$

Let us set ${}^tQ_1^{-1}B = B_1$ and let us look for Q_2 such that

$$\begin{pmatrix} I & 0 \\ h\,{}^tQ_2 & I \end{pmatrix} \begin{pmatrix} D_1 & hB_1 \\ h\,{}^tB_1 & hA_2 \end{pmatrix} \begin{pmatrix} I & hQ_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & hA_3 \end{pmatrix}.$$

This will be achieved if $D_1Q_2 + B_1 = 0$ and we find

$$A_3 = A_2 + h({}^tB_1Q_2 + {}^tQ_2B_1 + {}^tQ_2D_1Q_2).$$

Then $Q_2 = -D_1^{-1}B_1$ and $\|Q_2\|$ is uniformly bounded. Moreover if h is small enough, A_3 will satisfy the hypothesis of Lemma A.10 if

$$|y - y(x,h)| + |z - z(x,h)| \leqslant C_2,$$

where C_2 depends only on C_0, C, C_1 in (A.10), (A.13). It follows that one can find Q_3 such that $A_3 = {}^tQ_3D_2Q_3$ with $D_2 = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Then

$$\begin{pmatrix} A_1 & hB \\ h\,{}^tB & hA_2 \end{pmatrix} = \begin{pmatrix} Q_1 & 0 \\ h\,{}^tQ_2 & Q_3 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & hD_2 \end{pmatrix} \begin{pmatrix} Q_1 & hQ_2 \\ 0 & Q_3 \end{pmatrix}.$$

Now we introduce the coordinates

$$Y = Q_1(x,y,z,h)(y - y(x,h)) + hQ_2(x,y,z,h)(z - z(x,h))$$
$$Z = Q_3(x,y,z,h)(z - z(x,h)).$$

This is a change of coordinates from U to $\widetilde{U}$, where U contains a fixed ball with center $(y(x,h), z(x,h))$ and $\widetilde{U}$ contains a fixed ball with center $(0,0)$. Moreover, there exists a uniform constant C_3 such that

$$\frac{1}{C_3}|z - z(x,h)| \leqslant |Z| \leqslant C_3\,|z - z(x,h)|$$

$$\frac{1}{C_3}\big[|y - y(x,h)| + h|z - z(x,h)|\big] \leqslant |Y| + h|Z| \leqslant C_3\big[|y - y(x,h)| + h|z - z(x,h)|\big].$$

Now, if we write f is the coordinates (X, Z) we get

$$f(x,Y,Z,h) = f(x, y(x,h), z(x,h), h) + Y_1^2 - Y_2^2 + h(Z_1^2 - Z_2^2)$$

where $Y = (Y_1, Y_2) \in \mathbb{R}^k \times \mathbb{R}^k$, $Z = (Z_1, Z_2) \in \mathbb{R}^n \times \mathbb{R}^n$. So a good contour for f must satisfy

$$|Y_1|^2 - |Y_2|^2 + h(|Z_1|^2 - |Z_2|^2) \leqslant -C\big(|Y_1|^2 + |Y_2|^2 + h|Z_1|^2 + h|Z_2|^2\big).$$

Therefore, on such contour we have

$$(A.19) \qquad |Y_1|^2 + h|Z_1|^2 \leqslant \delta(|Y_2|^2 + h|Z_2|^2), \quad 0 < \delta < 1.$$

The contour, in the coordinates (Y, Z) satisfies the conditions (A.15), since we have seen that $Q_1 = Q_1'(x,y,h) + hQ_1''(x,y,z,h)$. Let us denote by $(\widetilde{Y}, \widetilde{Z})$ the parameters on the contour and

$$Y(x,\widetilde{Y},\widetilde{Z},h) = Y^1(x,\widetilde{Y},h) + hY^2(x,\widetilde{Y},\widetilde{Z},h).$$

It follows from (A.19), using a Taylor expansion of (Y, Z), that there exists a constant C_4, depending only on fixed constants, such that

$$(A.20) \qquad \Big|\frac{\partial Y_1^1}{\partial \widetilde{Y}}\widetilde{Y}\Big|^2 + h\Big|\frac{\partial Z_1}{\partial \widetilde{Z}}\widetilde{Z}\Big|^2 \leqslant \delta\Big(\Big|\frac{\partial Y_2^1}{\partial \widetilde{Y}}\widetilde{Y}\Big|^2 + h\Big|\frac{\partial Z_2}{\partial \widetilde{Z}}\widetilde{Z}\Big|\Big) + C_3\big(|\widetilde{Y}|^3 + h|\widetilde{Z}|^3 + h^{1/2}|\widetilde{Y}|^2 + h^{3/2}|\widetilde{Z}|^2\big).$$

Therefore, if

$$\frac{\partial Y_2^1}{\partial \widetilde{Y}}\widetilde{Y} = 0, \qquad \frac{\partial Z_2}{\partial \widetilde{Z}}\widetilde{Z} = 0,$$

it follows from (A.15) that

$$C_3(\cdots) + \Big|\frac{\partial Y_1^1}{\partial \widetilde{Y}}\widetilde{Y}\Big|^2 + h\Big|\frac{\partial Z_1}{\partial \widetilde{Z}}\widetilde{Z}\Big|^2 \geqslant \frac{1}{C_0}\big(|\widetilde{Y}|^2 + h|\widetilde{Z}|^2\big).$$

Using (A.20) we see that this implies $\widetilde{Y} = \widetilde{Z} = 0$. Thus the map $\begin{pmatrix} \frac{\partial Y_1^1}{\partial \widetilde{Y}} & 0 \\ 0 & \frac{\partial Z_2}{\partial \widetilde{Z}} \end{pmatrix}$: $\mathbb{R}^{k+n} \longrightarrow \mathbb{R}^{k+n}$ is bijective.

It follows that we can solve the system in $(\widetilde{Y}, \widetilde{Z})$

$$\begin{cases} Y_2 = Y_2^1(x,\widetilde{Y},h) + hY_2^2(x,\widetilde{Y},\widetilde{Z},h) \\ Z_2 = Z_2(x,\widetilde{Y},\widetilde{Z},h) \end{cases}$$

if h and $|Y_2| + |Z_2|$ are small enough.

Therefore any good contour can be parametrized by (Y_2, Z_2). If we have two good contours parametrized in the Morse coordinates by (Y_2, Z_2), that means that we have $\Gamma_{x,h,j} = (Y_1^j(x, Y_2, Z_2, h), Z_1^j(x, Y_2, Z_2, h))$, $j = 0, 1$, then

$$\widetilde{\Gamma}_{x,h,s} = \big(sY_1^1(x, Y_2, Z_2, h) + (1-s)Y_1^0(x, Y_2, Z_2, h), s Z_1^1(\cdots) + (1-s) Z_1^0(\cdots)\big)$$

is a good contour, since it satisfies (A.19) and it is the family that we looked for. □

A.3. Pseudo-differential operators in the complex domain

We follow here Sjöstrand [**Sj**]. The parameter λ will be replaced by $h^{-2}k^{-1}$ and the weight of the spaces H_φ will depend on some parameters (including h and k).

Let W be a neighborhood of a point $x_0 \in \mathbb{C}^N$. Let $V \subset \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+$ be the set of parameters m, h, k. Let $\varphi = \varphi(x; m, h, k)$ be a real function which is C^∞ with respect to x in W and satisfies $\sum_{|\alpha| \leqslant 2} \sup_{V \times W} |\partial_x^\alpha \varphi| \leqslant C$. We shall say that $u = u(x; m, h, k)$ belongs to H_φ if

(i) for any (m, h, k) in V, $x \mapsto u(x; m, h, k)$ is holomorphic in W,

(ii) there exist $C > 0$, $M > 0$ such that for any (m, h, k) in V and x in W

$$|u(x; m, h, k)| \leqslant C(hk)^{-M} e^{h^{-2}k^{-1}\varphi(x;m,h,k)}.$$

To any (m, h, k) in V we associate a function $a = a(x, y, \xi; m, h, k)$ holomorphic with respect to (x, y, ξ) in a neighborhood $\widetilde{W}$ of $\big(x_0, x_0, \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0; m, h, k)\big)$ and uniformly bounded. It will be called "analytic symbol". We consider now

$$\Gamma_h(x_0) = \Big\{(y, \xi) \in \mathbb{C}^N \times \mathbb{C}^N : |x_0 - y| < r, \xi = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0; m, h, k) + iR\overline{(x_0 - y)}\Big\}.$$

Here R is large enough but r is so small that $\Gamma_h(x_0)$ is contained in the set where a is holomorphic.

Now, for $u \in H_\varphi$, we set

$$(A.22)\ Au(x; m, h, k) = \Big(\frac{h^{-2}k^{-1}}{2\pi}\Big)^N \iint_{\Gamma_h(x_0)} e^{ih^{-2}k^{-1}(x-y)\cdot\xi} a(x, y, \xi; m, h, k) u(y; m, h, k)\, dy\, d\xi.$$

Then Au is holomorphic with respect to x near x_0 and modulo a term which is uniformly bounded by $e^{-\delta h^{-2}k^{-1}}$, $\delta > 0$, we can integrate, in (A.22), on $\Gamma_h(x)$ instead of $\Gamma_h(x_0)$. Moreover one can see that $Au \in H_\varphi$.

To invert the elliptic symbols, we have to modify slightly the argument of Sjöstrand.

We shall say that $a(x, \xi, \lambda; m, h, k) = \sum_{j \geqslant 0} \lambda^{-j} a_j(x, \xi; m, h, k)$ is a formal analytic symbol if one can find a neighborhood of (x_0, ξ_0), a set V containing the parameters (m, h, k) and $C_0 > 0$ such that

$$(A.23) \qquad |a_j(x, \xi; m, h, k)| \leqslant C_0^{j+1} j^{j/2}, \quad \forall (x, \xi) \in W, \ \forall (m, h, k) \in V.$$

We shall set

$$A = \sum_{j\geqslant 0} \lambda^{-j} \frac{\lambda^{-2|\alpha|}}{\alpha!} \partial_\xi^\alpha a_j D^\alpha = \sum_{k\geqslant 0} \lambda^{-k} A_k(x,\xi;m,h,k,D) = \sum_{k\geqslant 0} \lambda^{-k} A_k .$$

Let $t_0 > 0$ and, for $t \in]0,t_0]$, Ω_t be an open subset of $\mathbb{C}^N$ such that

i) $\Omega_s \subset \Omega_t$, if $s < t$,

ii) $\exists\, \delta > 0 : \forall\, s < t,\ \forall\, x \in \Omega_s,\ B(x,\delta(t-s)) \subset \Omega_t \subset \Omega_{t_0} \subset W$.

Let $\mathcal{H}(\Omega_t)$ be the space of holomorphic functions on Ω_t, endowed with the sup norm and $E_{s,t} = \mathcal{L}(\mathcal{H}(\Omega_t), \mathcal{H}(\Omega_s))$ be the space of bounded linear operator with the corresponding norm $\|\cdot\|_{s,t}$. Then

1) $A_k \in E_{s,t}$

2) $\|A_k\|_{s,t} \leqslant C_1^{k+1} k^{k/2} (t-s)^{-k}$, if $s < t$.

We set $f_k = \sup_{0<s<t\leqslant t_0} \frac{(t-s)^k}{k^{k/2}} \|A_k\|_{s,t}$ and $\|a\|_\rho = \sum_{k\geqslant 0} \rho^k f_k$. Then a is a formal analytic symbol iff one can find $\rho_0 > 0$ such that $\|a\|_{\rho_0} < +\infty$.

To a formal symbol a, we can associate an operator $\mathrm{Op}(a)$ obtained by the formula (A.22), where a has been replaced by $\sum_{|j|\leqslant \frac{1}{C_2} h^{-2}k^{-1}} (h^2 k)^j a_j(x,\xi;m,h,k)$, with C_2 large enough.

Conversely we can associate to the operator defined by (A.22), a formal analytic symbol given by

$$\sigma_A = \sum \frac{1}{\alpha!} \frac{1}{(i\lambda^2)^{|\alpha|}} (\partial_\xi^\alpha \partial_y^\alpha a)(x,x,\xi;m,h,k), \quad \lambda^2 = h^{-2}k^{-1}.$$

The formula (4.4) and the Lemma 4.1 in **[Sj]** show that if $u \in H_\varphi$,

$$\exists\, C > 0,\ \exists\, \varepsilon > 0 : |(\mathrm{Op}(\sigma_A) - A)u| \leqslant C e^{h^{-2}k^{-1}(\varphi-\varepsilon)} .$$

On the other hand if we define, on the set of formal symbols, the composition by

$$a\#b = \sum_\alpha \frac{1}{\alpha!} \frac{1}{(i\lambda^2)^{|\alpha|}} \frac{\partial^\alpha a}{\partial\xi^\alpha} \frac{\partial^\alpha b}{\partial x^\alpha}, \quad \lambda^2 = h^{-2}k^{-1}$$

the Theorem 4.2 in **[Sj]** shows that if $u \in H_\varphi$ one can find $C > 0$, $\varepsilon > 0$ such that

$$\big|[\mathrm{Op}(a\#b) - \mathrm{Op}(a) \circ \mathrm{Op}(b)]u\big| \leqslant C e^{h^{-2}k^{-1}(\varphi-\varepsilon)} .$$

The Lemmas 1.3 and 1.4 in **[Sj]** still hold and we can invert the elliptic formal symbols *i.e.* those for which $|a_0(x,\xi;m,h,k)| \geqslant C > 0$ for all (x,ξ) in W and (m,h,k) in V.

If the operator A given by (A.22) is elliptic, which means that,

$$\exists\, C > 0 : \forall\, (x,y,\xi) \in \widetilde{W},\ \forall\, (m,h,k) \in V, \quad |a(x,y,\xi;m,h,k)| \geqslant C$$

then its associate formal symbol σ_A is elliptic and one can find a formal symbol b such that

$$\mathrm{Id} \equiv \mathrm{Op}(\sigma_A \# b) \equiv \mathrm{Op}(\sigma_A) \circ \mathrm{Op}(b) \equiv A \circ \mathrm{Op}(b) \ \text{ in } \ H_\varphi .$$

The equality $\equiv$ in H_φ means that the difference applied to $u \in H_\varphi$ is bounded by $C e^{h^{-2}k^{-1}(\varphi-\varepsilon)}$.

We shall use the Remark 4.3 in **[Sj]** which we recall. Let $\psi = \psi(x, y, \xi; m, h, k)$ be holomorphic near (x_0, x_0, ξ_0) uniformly bounded which satisfies

$$(A.24)\qquad \begin{cases} \psi|_{x=y} = 0, \\ M = \Big(\dfrac{\partial^2\psi}{\partial x \partial \xi}\Big) \text{ is invertible and } \|M^{-1}\| \text{ is uniformly bounded.} \end{cases}$$

Let us set

$$Au(x; m, h, k) = \Big(\frac{h^{-2}k^{-1}}{2\pi}\Big)^N \iint_\Gamma e^{ih^{-2}k^{-1}\psi(x,y,\xi;\cdots)} a(x, y, \xi; m, h, k)\, u(y; m, h, k)\, dy\, d\xi$$

where a is an analytic symbol near (x_0, y_0, ξ_0) and Γ a contour, which will be described below, such that A will be an operator on the complex domain. Thanks to (A.24), we can write $\psi(x, y, \xi; \cdots) = (x-y) \cdot f(x, y, \xi; m, h, k)$ and the map $\xi \mapsto f(x, y, \xi; m, h, k)$ is a local diffeomorphism on a neighborhood which is independent of (x, y, m, h, k). Let us denote by g the inverse map $\xi = g(x, y, \theta; m, h, k)$ and let $\widetilde{a}$ be an analytic symbol. We set

$$\widetilde{A}u(x; \cdots) = \Big(\frac{h^{-2}k^{-1}}{2\pi}\Big)^N \iint_{\widetilde{\Gamma}} e^{ih^{-2}k^{-1}(x-y)\cdot\theta}\, \widetilde{a}(x, y, \theta; m, h, k)\, u(y; m, h, k)\, dy\, d\theta$$

where

$$\widetilde{\Gamma} = \Big\{(y, \theta) : |x - y| < r\,,\, \theta = \frac{\partial\psi}{\partial x}(x_0, x_0, \xi_0; \cdots) + i\,R\overline{(x-y)}\Big\}.$$

Then $\widetilde{A}$ is an operator on H_φ if $\big|\frac{\partial\psi}{\partial x}(x_0, x_0, \xi_0; \cdots) - \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0; \cdots)\big|$ is small enough.

Now, if in the integral defining Au we took $\Gamma = g(x, \widetilde{\Gamma}; m, h, k)$ and if in $\widetilde{A}$ we took $\widetilde{a}(x, y, \theta; \cdots) = a(x, y, g(x, y, \theta; \cdots), \cdots)\,\mathrm{Jac}(g(x, y, \theta; \cdots))$ then $A = \widetilde{A}$ in H_φ. Moreover a is elliptic iff $\widetilde{a}$ is elliptic. □

We would like now to define an operator on the complex domain using a pseudo-differential phase $\varphi = \varphi(x, y, \alpha, h)$ whose definition is given in Definition A.5. Let $a = a(x, y, \alpha; h, k)$ be an analytic symbol. Here the parameters are (h, k). Formally this operator will be given by

$$(A.25)\qquad Au(x; \alpha, h, k) = \Big(\frac{h^{-2}k^{-1}}{2\pi}\Big)^n \Big(\frac{h^{-1}k^{-1}}{2\pi}\Big)^{n/2} \iint_\Gamma e^{ih^{-2}k^{-1}\varphi(x,y,\alpha;h)}\, a(x, y, \alpha; h, k)\, u(y; h, k)\, dy\, d\alpha.$$

Here φ and a are holomorphic near (x_0, x_0, α^0) and u is holomorphic near x_0.

Let us describe the contour Γ. Let $\varphi = \varphi_2(x, y, \alpha_\xi) + ih\varphi_1(x, y, \alpha)$. Let $\alpha_x(x, y, \alpha_\xi)$ be the solution of $\frac{\partial\varphi_1}{\partial\alpha_x}(x, y, \alpha_x(x, y, \alpha_\xi), \alpha_\xi) = 0$ with $\alpha_x(x_0, x_0, \alpha_\xi^0) = \alpha_x^0$. We have $\alpha_x(x, x, \alpha_\xi) = \alpha_x(x, \alpha_\xi)$, with the notation of Definition A.5, 5). Let Γ_{α_x} be the contour given by $\alpha_x = \alpha_x(x, y, \alpha_\xi) + t$, where $t \in \mathbb{R}^n$, $|t| \leqslant \delta$, and let us set

$$b(x, y, \alpha_\xi, h, k) = \Big(\frac{h^{-1}k^{-1}}{2\pi}\Big)^{n/2} e^{h^{-1}k^{-1}\varphi_1(x,y,\alpha_x(x,y,\alpha_\xi),\alpha_\xi)} \int_{\Gamma_{\alpha_x}} e^{-h^{-1}k^{-1}\varphi_1(x,y,\alpha)} \cdot a(x, y, \alpha, h, k)\, d\alpha_x\,.$$

Since $\left(\frac{\partial^2\varphi_1^r}{\partial\alpha_x^2}(x_0,x_0,\alpha^0)\right) \gg 0$ (Definition A.5, 5)), we obtain easily, from the Taylor formula that b is an analytic symbol near (x_0,x_0,α_ξ^0). Moreover if a is elliptic then b is elliptic if hk is small enough (here α_ξ plays the role of ξ). Let us remark that if we change δ in the definition of the contour Γ_{α_x} then we obtain, in b, an error which is $\mathcal{O}(e^{-\varepsilon h^{-1}k^{-1}})$. This error was not negligible in the case of Sjöstrand [**Sj**]. However it has no consequence here according to our definition of ${}^{\mathrm{qsc}}WF_a$.

Now we want to give a meaning to

$$(A.26)\ Au(x;\alpha,h,k) = \Big(\frac{h^{-2}k^{-1}}{2\pi}\Big)^n \iint_{\Gamma'} e^{ih^{-2}k^{-1}[\varphi_2(x,y,\alpha_\xi)+ih\varphi_1(x,y,\alpha_x(x,y,\alpha_\xi),\alpha_\xi)]} b(x,y,\alpha_\xi;h,k)\,u(y;h,k)\,dy\,d\alpha_\xi\,.$$

Let us show now that the phase

$$\psi(x,y,\alpha_\xi,h) = \varphi_2(x,y,\alpha_\xi) + ih\varphi_1(x,y,\alpha_x(x,y,\alpha_\xi),\alpha_\xi)$$

satisfies the condition (A.24). First of all, conditions 3) and 4) in Definition A.5 show that $\psi = 0$ if $x = y$. Assume now that h is small. Then

$$\frac{\partial^2\psi}{\partial x\partial\alpha_\xi} = \frac{\partial^2\varphi_2}{\partial x\partial\alpha_\xi} + O(h)$$

and since $\varphi_2(x,x,\alpha_\xi) \equiv 0$, we have

$$\frac{\partial^2\varphi_2}{\partial x\partial\alpha_\xi} = -\frac{\partial^2\varphi_2}{\partial y\partial\alpha_\xi},$$

so the second condition on ψ follows from 8), Definition A.5. When $h \geqslant \delta > 0$, we use instead conditions 5) and 8).

Now we have

$$\begin{aligned}\frac{\partial\psi}{\partial x}(x_0,x_0,\alpha_\xi^0) &= \frac{\partial\varphi_2}{\partial x}(x_0,x_0,\alpha_\xi^0) + ih\frac{\partial\varphi_1}{\partial x}(x_0,x_0,\alpha^0)\\ &\qquad + ih\frac{\partial\varphi_1}{\partial\alpha_x}(x_0,x_0,\alpha^0)\cdot\frac{\partial\alpha_x}{\partial x}(x_0,x_0,\alpha_\xi^0)\\ &= \frac{\partial\varphi}{\partial x}(x_0,x_0,\alpha^0,h) = \xi_0,\end{aligned}$$

by condition 6) Definition A.5. By the discussion made after (A.24), if we set,

$$\widetilde{\Gamma} = \big\{(y,\theta) : |x-y| < r,\, \theta = \xi_0 + \overline{iR(x-y)}\big\}$$

then $\Gamma' = g(x,y,\widetilde{\Gamma})$ is a good contour, and A in (A.26) is well defined on H_χ as soon as $\left|\frac{2}{i}\frac{\partial\chi}{\partial x} - \xi_0\right|$ is small enough.

Thus we have obtained a contour Γ in (A.25) where $\alpha_x \in \Gamma_{\alpha_x}$, $(y,\alpha_\xi) \in \Gamma'$ and we show now that this contour is a good contour for $f = \mathrm{Re}(i\varphi)$. We shall use the results of § 2. Our function f is here a function of (x,y,z,h) where y stands for (y,α_ξ) and $z = \alpha_x$. With these notation we have $f(x,y,z,h) = f_2(x,y) + hf_1(x,y,z)$, where $f_2 = \mathrm{Re}(i\varphi_2)$ and $f_1 = -\mathrm{Re}\,\varphi_1 = -\varphi_1^r$.

We may assume, making a translation, that (ignoring x which is fixed)

$$f_2(0) = f_1(0,0) = 0, \quad \frac{\partial^2 f_1}{\partial z^2}(0,0) \text{ invertible.}$$

Let $z(y)$ be the solution of

$$\frac{\partial f_1}{\partial z}(y, z(y)) = 0, \quad z(0) = 0.$$

Then if $z(y) + t$ is a good contour for φ_1, which means that on the contour we have

$$f_1(y, z(y) + t) - f_1(y, z(y)) \leqslant -C\,|t|^2$$

and if we have a good contour in y for $f_2(y) + h\,f_1(y, z(y))$ which reads

$$f_2(y) + h\,f_1(y, z(y)) \leqslant -C\,|y|^2,$$

on the contour, then the contour in (y, z), $(y, z = z(y) + t)$ is a good contour for $f_2(y) + h\,f_1(y, z)$ since

$$\begin{aligned} f_2(y) + h\,f_1(y, z(y) + t) &= f_2(y) + h\,f_1(y, z(y)) + h(f_1(y, z(y) + t) - f_1(y, z(y)) \\ &\leqslant -C\,|y|^2 - C\,h\,|t|^2 \end{aligned}$$

on the contour and conditions (A.15) are satisfied.

A.4. Pseudo-differential operators in the real domain

Let $m_0 = (x_0, \xi_0, \alpha^0, h_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} \times [0, +\infty[$ and φ a pseudo-differential phase near m_0 (Definition A.5). Let V be a neighborhood of α^0 in $\mathbb{R}^{2n}$. We set, following Sjöstrand,

$$\nabla\overline{V} = \{(x, y, \alpha) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2n} : x = y,\, \alpha_x = \alpha_x(x, \alpha_\xi), \alpha \in \overline{V}\}$$

where

$$\frac{\partial \varphi_1^r}{\partial x}(x, y, \alpha_x(x, y, \alpha_\xi), \alpha_\xi) = 0$$

and $\alpha_x(x, \alpha_\xi) = \alpha_x(x, x, \alpha_\xi)$. Let a be an analytic symbol. Then we set, for x real,
(A.27)

$$A^V u(x; h, k) = \iint_{\alpha \in V} e^{ih^{-2}k^{-1}\varphi(x,y,\alpha,h)}\, a(x, y, \alpha; h, k)\, \chi(x, y, \alpha)\, u(y; h, k)\, dy\, d\alpha.$$

Here χ is a cut-off function which localizes in the set where φ satisfies the conditions of Definition A.5, $\chi = 1$ near $\nabla\overline{V}$ and a is an analytic symbol.

Here is an important result in this theory which will be used later on.

Theorem A.11. — *Let ψ be a phase in the sense of Definition A.4, b an analytic symbol, φ a pseudo-differential phase, a an analytic symbol and let A^V be defined by (A.27). Then one can find $\varepsilon_1 > 0$ (depending only on C_0, in Definition A.4, and φ)*

such that if $\varepsilon_0 < \varepsilon_1$ *then there exist* $\delta > 0$, $C > 0$ *such that for all* (x,β) *in* W *and all* (m,h) *in* U, *we have*

$$A^V\big(e^{ih^{-2}k^{-1}\psi(\cdot,\beta,m,h)}b(\cdot,\beta,m,h,k)\big) = e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)}c(x,\beta,m,h,k) + d$$

where $|d| \leqslant Ce^{-\delta h^{-1}k^{-1}}$, *c is an analytic symbol and*

$$c = e^{-ih^{-2}k^{-1}\psi(x,\beta,m,h)}A\big(e^{ih^{-2}k^{-1}\psi(\cdot,\beta,m,h)}b(\cdot,\beta,m,h,k)\big)$$

where, in the last expression, A acts in the complex domain as an operator on $H_{-\operatorname{Im}\psi}$, *modulo error terms bounded by* $Ce^{-\delta h^{-1}k^{-1}}$.

Proof. — The first step is to study the phase $\theta = \varphi + \psi$, which occurs in the expression of $A^V(e^{ih^{-2}k^{-1}\psi}b)$.

Lemma A.12. — *Let* ψ *be a phase in the sense of Definition A.4. Let* φ *be a pseudo-differential phase (Definition A.5). We set*

$$\theta(x,y,\alpha,\beta,m,h) = \varphi(x,y,\alpha,h) + \psi(y,\beta,m,h).$$

Then for all (x,β) *in* W, *all* (m,h) *in* U *there exist* $y(x,\beta,m,h)$, $\alpha(x,\beta,m,h)$ *such that*

$$\frac{\partial\theta}{\partial y}(x,y(x,\beta,m,h),\alpha(x,\beta,m,h),m,h) = \frac{\partial\theta}{\partial\alpha_x}(\cdots) = \frac{\partial\theta}{\partial\alpha_\xi}(\cdots) = 0.$$

Moreover (y,α) *satisfies the following properties*

(i) $y(x,\beta,m,h) = x$.

(ii) $\alpha_x(x,\beta,m,h) = \alpha_x(x,\alpha_\xi(x,\beta,m,h))$ *where* α_x *is the real on the real.*

(iii) *There exist* $\varepsilon_1 > 0$, $C > 0$ *such that, for* $0 < \varepsilon_0 < \varepsilon_1$,

$$|\operatorname{Im}\alpha_\xi(x,\beta,m,h)| \leqslant C\varepsilon_0 h, \quad \text{for } (x,\beta) \in W \cap \mathbb{R}^{3n},$$

ε_1 *and* C *depend only on* C_0 *(Definition A.4) and* φ.

Proof. — Let us note that in ii) the function α_x in the right hand side is the function which appears in 5) Definition A.5, that is (thanks to 3))

$$\frac{\partial\varphi_1}{\partial\alpha_x}(x,x,\alpha_x(x,\alpha_\xi),\alpha_\xi) = 0.$$

Moreover we have

$$\frac{\partial\varphi_2}{\partial\alpha_\xi}(x,x,\alpha_\xi) = 0, \quad \frac{\partial\varphi_1^i}{\partial\alpha_\xi}(x,x,\alpha) = 0$$

and thanks to 4) Definition A.5, differentiating with respect to α_ξ, we get

$$\frac{\partial\varphi_1^r}{\partial\alpha_\xi}(x,x,\alpha_x(x,\alpha_\xi),\alpha_\xi) = 0.$$

It follows that $(y(x,\alpha_\xi) = x, \alpha_x(x,\alpha_\xi))$ is a solution of

$$\frac{\partial\varphi}{\partial\alpha_x}(\cdots) = \frac{\partial\varphi}{\partial\alpha_\xi}(\cdots) = 0, \quad y(x_0,\alpha_\xi^0) = x_0, \quad \alpha_x(x_0,\alpha_\xi^0) = \alpha_x^0.$$

It remains to solve

$$\frac{\partial \theta}{\partial y}(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi, \beta, m, h) = 0,$$

with respect to α_ξ. Let us denote by $\alpha_\xi^1(x, \beta, m, h)$ the solution of

$$\frac{\partial \varphi_2}{\partial y}(x, x, \alpha_\xi) + \frac{\partial \psi^r}{\partial y}(x, \beta, m, h) = 0 \tag{A.28}$$

for α_ξ in a neighborhood of α_ε^0. This equation can be solved since $\left(\frac{\partial^2 \varphi_2}{\partial y \partial \alpha_\xi}\right)$ is invertible.

We note that α_ξ^1 is real if (x, β) is real. Now, let us denote by $\alpha_\xi(x, \beta, m, h)$ the solution in α_ξ (near α_ξ^0) of the equation,

$$\frac{\partial \varphi}{\partial y}(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi, h) + \frac{\partial \psi}{\partial y}(x, \beta, m, h) = 0. \tag{A.29}$$

One can solve (A.29) if the matrix

$$\frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi} - ih\Big(\frac{\partial^2 \varphi_1}{\partial y \partial \alpha_x}\Big) \cdot \Big(\frac{\partial^2 \varphi_1}{\partial \alpha_x^2}\Big)^{-1} \cdot \Big(\frac{\partial^2 \varphi_1}{\partial \alpha_x \partial \alpha_\xi}\Big)$$

is invertible, which is implied by the condition 8) of Definition A.5, since $\begin{pmatrix} A & B \\ D & C \end{pmatrix}$ is invertible iff $A - BC^{-1}D$ is invertible.

Since $\alpha_\xi^1(x, \beta, m, h)$ is real for (x, β) real, we have

$$|\operatorname{Im} \alpha_\xi| = |\operatorname{Im}(\alpha_\xi - \alpha_\xi^1)| \leqslant |\alpha_\xi - \alpha_\xi^1|.$$

To prove that $|\alpha_\xi^1 - \alpha_\xi| \leqslant C\varepsilon_0 h$, we apply the following result.

Lemma A.13. — *Let F, G be C^2 function from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let X_0, η_0 in $\mathbb{R}^N$ and assume that $F(X_0) = \eta_0$, $G(X_0) = 0$. Let us assume that the matrices*

$$\frac{\partial F}{\partial X}(X_0) \quad \textit{and} \quad \frac{\partial (F + hG)}{\partial X}(X_0)$$

are invertible.

Let $X(\eta)$ be the solution of $F(X(\eta)) = \eta$. Let $Y(\eta, h)$ be the solution of

$$(F + hG)(Y(\eta, h)) = \eta,$$

for η close to η_0. Then

$$|Y(\eta, h) - X(\eta)| \leqslant Ch|\eta - \eta_0|.$$

First of all, Lemma A.13 implies the claim iii) in Lemma A.12 since it follows from 4), Definition A.4 that $|\frac{\partial \psi^r}{\partial y} - \frac{\partial \psi}{\partial y}| \leqslant \varepsilon_0 h$. Moreover let us note that we can solve (A.28), (A.29) with a right hand side η, keeping the conclusion of Lemma A.12 if $|\eta| \leqslant \varepsilon_0$, with ε_0 small.

Proof of Lemma A.13. — We have

$$\eta = F(X(\eta)) = F(Y(\eta)) + \frac{\partial F}{\partial X}(Y(\eta))(X(\eta) - Y(\eta)) + O(|X(\eta) - Y(\eta)|^2).$$

Moreover

$$\frac{\partial F}{\partial X}(Y(\eta)) = \frac{\partial F}{\partial X}(X_0) + O(|\eta - \eta_0|) \quad \text{and} \quad F(Y(\eta)) = \eta - hG(Y(\eta)).$$

Since $|G(Y(\eta))| \leqslant C|\eta - \eta_0|$ we get

$$|X(\eta) - Y(\eta)| \leqslant C(h|\eta - \eta_0| + |X(\eta) - Y(\eta)|^2)$$

and the lemma follows. □

Proof of Theorem A.11. — Recall that $\theta = \varphi + \psi = \theta_2 + ih\theta_1$, where $\theta_2 = \varphi_2 + \psi$, $\theta_1 = \varphi_1$. We show first that $f = -\operatorname{Im}\theta$ satisfies the conditions (A.10) to (A.13). We have $f = f_2 + hf_1$ with $f_2 = -\operatorname{Im}\theta_2$, $f_1 = -\operatorname{Re}\theta_1$. The correspondence between the variables in f and θ is the following : the variable y (resp. z) in f is the variable (y, α_ξ) (resp. α_x) in θ. The condition (A.10) is obviously satisfied and (A.11) has been proved in Lemma A.12. Since θ is holomorphic in (y, α) we are reduced to prove that some matrices of second derivatives of θ are invertible with uniformly bounded inverses since the conditions on the signature will follow from the holomorphy. Let us begin by (A.13), which is the case of small h. We have

$$\frac{\partial^2\theta_1}{\partial\alpha_x^2} = \frac{\partial^2\varphi_1}{\partial\alpha_x^2}$$

and the later is uniformly invertible by conditions 3), 5) in Definition A.5 (since they are taken at the point $(x, x, \alpha_x(x, \alpha_\xi), \alpha_\xi, h)$). Now we have

$$\frac{\partial^2\theta_2}{\partial(y, \alpha_\xi)^2} = \begin{pmatrix} \frac{\partial^2\theta_2}{\partial y^2} & \frac{\partial^2\varphi_2}{\partial y\partial\alpha_\xi} \\ \frac{\partial^2\varphi_2}{\partial\alpha_\xi\partial y} & \frac{\partial^2\varphi_2}{\partial\alpha_\xi^2} \end{pmatrix}.$$

We have

$$\frac{\partial^2\varphi_2}{\partial\alpha_\xi^2}(x, x, \alpha_\xi) = 0$$

(condition 3)) and $\frac{\partial^2\varphi_2}{\partial y\partial\alpha_\xi}$ is invertible (condition 8)). If h is small enough, it follows that $\frac{\partial^2\theta}{\partial y\partial\alpha_\xi}$ is invertible with a uniformly bounded inverse.

Let us consider now the case h large. We have

$$\frac{\partial^2\theta}{\partial(y, \alpha)^2} = \begin{pmatrix} \frac{\partial^2\theta}{\partial y^2} & \frac{\partial^2\varphi}{\partial y\partial\alpha_\xi} & \frac{\partial^2\varphi}{\partial y\partial\alpha_x} \\ \frac{\partial^2\varphi}{\partial\alpha_\xi\partial y} & \frac{\partial^2\varphi}{\partial\alpha_\xi^2} & \frac{\partial^2\varphi}{\partial\alpha_\xi\partial\alpha_x} \\ \frac{\partial^2\varphi}{\partial\alpha_x\partial y} & \frac{\partial^2\varphi}{\partial\alpha_x\partial\alpha_\xi} & \frac{\partial^2\varphi}{\partial\alpha_x^2} \end{pmatrix}.$$

At the point $(x, x, \alpha_x(x,\alpha_\xi), \alpha_\xi)$ we have $\frac{\partial^2\varphi_2}{\partial\alpha_\xi^2} = 0$. Then

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial\alpha_\xi^2} & \frac{\partial^2\varphi}{\partial\alpha_\xi\partial\alpha_x} \\ \frac{\partial^2\varphi}{\partial\alpha_x\partial\alpha_\xi} & \frac{\partial^2\varphi}{\partial\alpha_x^2} \end{pmatrix} = ih \begin{pmatrix} \frac{\partial^2\varphi_1}{\partial\alpha_\xi^2} & \frac{\partial^2\varphi_1}{\partial\alpha_\xi\partial\alpha_x} \\ \frac{\partial^2\varphi_1}{\partial\alpha_x\partial\alpha_\xi} & \frac{\partial^2\varphi_1}{\partial\alpha_x^2} \end{pmatrix}.$$

Since

$$\frac{\partial\varphi_1}{\partial\alpha_x} = \frac{\partial\varphi_1}{\partial\alpha_\xi} = \frac{\partial\varphi_2}{\partial\alpha_\xi} = 0$$

at the critical point, we can prove (as in (A.9)) that

$$\frac{\partial^2\varphi}{\partial\alpha_\xi^2} = \frac{\partial^2\varphi_1}{\partial\alpha_\xi\partial\alpha_x}\Big(\frac{\partial^2\varphi_1}{\partial\alpha_x^2}\Big)^{-1}\frac{\partial^2\varphi_1}{\partial\alpha_x\partial\alpha_\xi}.$$

Thus we can write

$$\frac{\partial^2\theta}{\partial(y,\alpha)^2} = \begin{pmatrix} a & B & C \\ {}^tB & ihDE^{-1\,t}D & ihD \\ {}^tC & ih{}^tD & ihE \end{pmatrix} = M.$$

Now, by condition 8), the matrix $\begin{pmatrix} B & C \\ ih{}^tD & ihE \end{pmatrix}$ is invertible. This is equivalent to $(B - CE^{-1\,t}D)$ invertible. Combining the second and the third "line", we see that M is invertible if $({}^tB - DE^{-1\,t}C)$ is invertible, which is the case, since E is symmetric. We are going now to change the contour of integration in the integral giving $A^V(e^{ih^{-2}k^{-1}\psi}b)$, in order to integrate on a good contour. Then proposition will follow, since, by Proposition A.9, we can then change this good contour to the good contour $(\Gamma_{\alpha_x}, \Gamma')$ given after (A.25), (A.26).

Let $\chi_0(x,y,\alpha) \in C_0^\infty$ be a cut-off function with $\operatorname{supp}\chi_0 \subset \{\chi = 1\}$ (where χ appears in the right hand side of (A.27)), $\chi_0 = 1$ in a neighborhood of $\nabla\overline{V}$ (see the beginning of § 4) and $\chi_0 \geqslant 0$.

For $s \in [0,1]$, we set

$$\Gamma_s : y = \widetilde{y} + is\delta\chi_0(x,\widetilde{y},\alpha)\overline{\frac{\partial\theta}{\partial y}(x,\widetilde{y},\alpha,\beta,m,h)}, \text{ where } \widetilde{y} \in \mathbb{R}^n.$$

For each $\alpha \in V$, the contour $\Gamma_0 = \mathbb{R}^n$ is modified in a set where $\chi = 1$. Therefore in $A^V(\cdots)$ (see (A.27)) by holomorphy, we can integrate on Γ_1 instead of Γ_0.

Let now $\chi_2 \in C_0^\infty$, $\operatorname{supp}\chi_2 \subset \{\chi_0 = 1\}$, $\chi_2 = 1$ on a neighborhood of $\nabla\overline{V}$ and $\chi_2 \geqslant 0$.

Let $\chi_1 \in C_0^\infty$, $\operatorname{supp}\chi_1 \subset \{\chi_2 = 1\}$, $\chi_1 = 1$ near $\nabla\overline{V}$, $\chi_1 \geqslant 0$. For $s \in [0,1]$ and $\widetilde{\alpha} \in V$ we set,

$$(A.30)\qquad \begin{cases} \underline{y} = \widetilde{y}, & \widetilde{y} \in \mathbb{R}^n, \\ \underline{\alpha}_\xi = \widetilde{\alpha}_\xi + s\chi_2(x;\widetilde{y},\widetilde{\alpha})(\alpha_\xi(x,\beta,m,h) - \alpha_\xi^0), & \widetilde{\alpha}_\xi \in \mathbb{R}^n, \\ \underline{\alpha}_x = \widetilde{\alpha}_x + s\chi_2(x,\widetilde{y},\widetilde{\alpha})(\alpha_x(x,\beta,m,h) - \alpha_x^0), & \widetilde{\alpha}_x \in \mathbb{R}^n, \end{cases}$$

and we define

$$\widetilde{\Gamma}_s \begin{cases} y = \underline{y} + i\delta\chi_0(x,\widetilde{y},\widetilde{\alpha})\,\overline{\dfrac{\partial\theta}{\partial y}(x,\underline{y},\underline{\alpha},\beta,m,h)}\,, \\ \alpha_\xi = \underline{\alpha_\xi} + is\delta\chi_1(x,\widetilde{y},\widetilde{\alpha})\,\overline{\dfrac{\partial\theta}{\partial\alpha_\xi}(x,\underline{y},\underline{\alpha},\beta,m,h)}\,, \\ \alpha_x = \underline{\alpha_x}\,. \end{cases}$$

We have $\widetilde{\Gamma}_0 = \Gamma_1$. Let us compute θ on $\widetilde{\Gamma}_s$. We have

$$\theta(x,,y,\alpha,m,h) = \theta(x,\underline{y},\underline{\alpha},\beta,m,h) + i\delta\chi_0(x,\widetilde{y},\widetilde{\alpha})\Big|\frac{\partial\theta}{\partial y}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2$$
$$+is\delta\chi_1(x,\widetilde{y},\widetilde{\alpha})\Big|\frac{\partial\theta}{\partial\alpha_\xi}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2 + O\Big(\delta^2\chi_0^2\Big|\frac{\partial\theta}{\partial y}\Big|^2 + s^2\delta^2\chi_1^2\Big|\frac{\partial\theta}{\partial\alpha_\xi}\Big|^2\Big).$$

If δ is small enough we get

$$(A.31)\ \operatorname{Im}\theta(x,y,\alpha,\beta,m,h) \geqslant \operatorname{Im}\theta(x,\underline{y},\underline{\alpha},\beta,m,h) + \frac{\delta}{2}\chi_0(x,\widetilde{y},\widetilde{\alpha})\Big|\frac{\partial\theta}{\partial y}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2$$
$$+\frac{\delta}{2}s\chi_1(x,\widetilde{y},\widetilde{\alpha})\Big|\frac{\partial\theta}{\partial\alpha_\xi}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2.$$

We show now that we can restrict the contour to the set where $\chi_1 = 1$. By Lemma A.12 we have $|\operatorname{Im}\alpha_\xi(x,\beta,m,h)| \leqslant C_\alpha h\varepsilon_0$. Therefore,

$$\theta(x,\underline{y},\underline{\alpha},\beta,m,h) = \theta(x,\underline{y},\operatorname{Re}\underline{\alpha},\beta,m,h) + \mathcal{O}(\varepsilon_0 h)\,,$$

(where $\mathcal{O}$ means uniformly bounded by a constant depending only on C_α and φ). It follows that

$$\operatorname{Im}\theta(x,\underline{y},\underline{\alpha},\beta,m,h) \geqslant h\operatorname{Re}\varphi_1(x,\underline{y},\operatorname{Re}\underline{\alpha},\beta,m,h) + \mathcal{O}(\varepsilon_0 h)$$

since $\operatorname{Im}\psi \geqslant 0$ on the real; then, by (A.30) and condition 7) in Definition A.5, we get on the contour

$$\operatorname{Im}\theta \geqslant Ch\big[(\operatorname{Re}\underline{\alpha_x} - \alpha_x(x,\operatorname{Re}\underline{\alpha_\xi}))^2 + (\operatorname{Re}\underline{\alpha_x} - \alpha_x(\underline{y},\operatorname{Re}\underline{\alpha_\xi}))^2\big]$$
$$+\frac{\delta}{2}\chi_0(x,\widetilde{y},\widetilde{\alpha})\Big|\frac{\partial\theta}{\partial y}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2 + \mathcal{O}(\varepsilon_0 h).$$

Now since $\chi_1 = 1$ on a neighborhood of $\nabla\overline{V}$ we see that if $(x,\widetilde{y},\widetilde{\alpha}) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{2n}$ belongs to the set $\{\chi_1 < 1\}$ we have $|\widetilde{\alpha}_x - \alpha_x(x,\widetilde{\alpha}_\xi)| + |\widetilde{\alpha}_x - \alpha_x(y,\widetilde{\alpha}_\xi)| \geqslant \delta > 0$ with a uniform δ. For this we use that, in the integral (A.27), α is bounded and that, by Remark A.7, $\alpha_x(x,\widetilde{\alpha}_\xi) = \beta_x \Leftrightarrow x = x(\beta_x,\widetilde{\alpha}_\xi)$. Now

$$|\operatorname{Re}\underline{\alpha_x} - \alpha_x(x;\operatorname{Re}\underline{\alpha_\xi})| + |\operatorname{Re}\underline{\alpha_x} - \alpha_x(\underline{y},\operatorname{Re}\underline{\alpha_\xi})|$$
$$\geqslant |\widetilde{\alpha}_x - \alpha_x(x,\widetilde{\alpha}_\xi)| + |\widetilde{\alpha}_x - \alpha_x(y,\widetilde{\alpha}_\xi)| + \mathcal{O}(|\alpha_x(x,\beta,m,h) - \alpha_x^0| + |\alpha_\xi(x,\beta,m,h) - \alpha_\xi^0|$$
$$\geqslant \delta + \mathcal{O}(|x-x_0|) \geqslant \frac{\delta}{2}$$

if $|x - x_0|$ is small enough. It follows that $\operatorname{Im}\theta \geqslant C_1 h$ on this set. This gives an error term which is $\mathcal{O}(e^{-C_1 h^{-1}k^{-1}})$. In the set $\{\chi_1 = 1\}$, on the boundary of the contour $\widetilde{\Gamma}_s$ we have $\widetilde{\alpha} \in \partial V$ so, $|\underline{\alpha} - \alpha^0| \geqslant \delta_0 > 0$. Moreover $\chi_0(x, \widetilde{y}, \widetilde{\alpha}) = 1$. Now, since $\widetilde{\alpha}$ and α_0 are real it follows from (A.30) that,

$$\operatorname{Im}\underline{\alpha} = s\chi_2(x; \widetilde{y}, \widetilde{\alpha}) \operatorname{Im}[\alpha(x, \beta, m, h)] = \mathcal{O}(h),$$

by the Lemma A.12, iii) and ii). Then,

$$\operatorname{Im}\theta \geqslant Ch\big[|\underline{\alpha}_x - \alpha_x(x, \underline{\alpha}_\xi)|^2 + |\alpha_x - \alpha_x(y, \underline{\alpha}_\xi)|^2\big] + \frac{\delta}{2}\Big|\frac{\partial\theta}{\partial y}(x, \underline{y}, \cdots)\Big|^2 + \mathcal{O}(\varepsilon_0 h + h^2).$$

Claim. — *Let $|a| + |b| + |c| \leqslant d_1$. Then, the problem in $(\underline{y}, \underline{\alpha})$,*

$$\begin{cases} \alpha_x(x, \underline{\alpha}_\xi) = \underline{\alpha}_x + a \\ \alpha_x(\underline{y}, \underline{\alpha}_\xi) = \underline{\alpha}_x + b \\ \dfrac{\partial\theta}{\partial y}(x, \underline{y}, \underline{\alpha}, \beta, m, h) = c \end{cases}$$

has a unique solution $(\underline{y}, \underline{\alpha})$ such that,

$$|\underline{\alpha} - \alpha^0| + |\underline{y} - y_0| \leqslant Cd_1$$

with a positive C which is independent of (β, m, h) and d_1. Assume this claim true. Then if d_1 is such that $Cd_1 = \frac{1}{2}\delta_0$ we get

$$|\alpha_x(x, \underline{\alpha}_\xi) - \underline{\alpha}_x|^2 + |\alpha_x(\underline{y}, \underline{\alpha}_x) - \underline{\alpha}_x|^2 + \frac{\delta}{2}\Big|\frac{\partial\theta}{\partial y}(x, \underline{y}, \underline{\alpha}, \beta, m, h)\Big|^2 \geqslant C_1\delta_0^2$$

where C_1 depends only on C and δ. It follows then that $\operatorname{Im}\theta \geqslant C_2 h$ on this set which gives an error term which is $\mathcal{O}(e^{-C_2 h^{-1}k^{-1}})$.

Therefore we can shift the contour $\Gamma_1 = \widetilde{\Gamma}_0$ to $\widetilde{\Gamma}_1 \cap \{\chi_1 = 1\}$.

Proof of the claim. — The map

$$F : (\underline{y}, \underline{\alpha}) \longmapsto \Big(\underline{\alpha}_x - \alpha_x(x, \underline{\alpha}_\xi), \underline{\alpha}_x - \alpha_x(\underline{y}, \underline{\alpha}_\xi), \frac{\partial\theta}{\partial y}(x, \underline{y}, \underline{\alpha}, \beta, m, h)\Big)$$

is a local diffeomorphism.

Indeed we first note that,

$$\frac{\partial\alpha_x}{\partial\alpha_\xi} = -\Big(\frac{\partial^2\varphi_1^r}{\partial\alpha_x^2}\Big)^{-1}\frac{\partial^2\varphi_1^r}{\partial\alpha_x\partial\alpha_\xi},$$

$$\frac{\partial\alpha_x}{\partial y} = -\Big(\frac{\partial^2\varphi_1^r}{\partial\alpha_x^2}\Big)^{-1}\Big(\frac{\partial^2\varphi_1^r}{\partial x\partial\alpha_x} + \frac{\partial^2\varphi_1^r}{\partial y\partial\alpha_x}\Big),$$

and

$$\frac{\partial\theta}{\partial\alpha} = \frac{\partial\varphi}{\partial\alpha}.$$

It follows that the differential of F at (x_0, α^0) is (uniformly in (β, m, h)) invertible if the matrix

$$\begin{pmatrix} 0 & \frac{\partial^2 \varphi_1^r}{\partial \alpha_x^2} & \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial \alpha_\xi} \\ \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial x} + \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial y} & \frac{\partial^2 \varphi_1^r}{\partial \alpha_x^2} & \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial \alpha_\xi} \\ \frac{\partial^2 \varphi}{\partial y^2} & \frac{\partial^2 \varphi}{\partial y \partial \alpha_x} & \frac{\partial^1 \varphi}{\partial y \partial \alpha_\xi} \end{pmatrix} (x_0, x_0, \alpha^0, \beta, m, h)$$

is (uniformly) invertible.

This is equivalent to say that the matrix

$$M = \begin{pmatrix} \frac{\partial^2 \varphi_1^r}{\partial \alpha_x^2} & \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial \alpha_\xi} \\ \frac{\partial^2 \varphi}{\partial y \partial \alpha_x} & \frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi} \end{pmatrix} (x_0, x_0, \alpha^0, h)$$

is (uniformly) invertible, because $\left(\frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial x} + \frac{\partial^2 \varphi_1^r}{\partial \alpha_x \partial y}\right)$ is invertible (see Definition A.5 6)).

Now, if $h_0 = 0$, since $\frac{\partial^2 \varphi_1^r}{\partial \alpha_x^2}$ and $\frac{\partial^2 \varphi_1^r}{\partial y \partial \alpha_\xi}$ are invertible and $\frac{\partial^2 \varphi}{\partial y \partial \alpha_x} = O(h)$ we obtain that M is (uniformly) invertible if h is small enough. If $h_0 \neq 0$, since φ_2 does not depend on α_x and $\varphi_1^i(x, x, \alpha) = 0$ we get $\frac{\partial^2 \varphi_1^i}{\partial \alpha^2}(x_0, x_0, \alpha^0) = 0$. Then,

$$M = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial \alpha_x^2} & \frac{\partial^2 \varphi}{\partial \alpha_x \partial \alpha_\xi} \\ \frac{\partial^2 \varphi}{\partial y \partial \alpha_x} & \frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi} \end{pmatrix}$$

which is (uniformly) invertible by Definitions A.5, 8). It remains to show that the later is a good contour for $-\operatorname{Im}\theta$. For this we are going to use (A.31) with $\chi_0(x, \widetilde{y}, \widetilde{\alpha}) = \chi_1(x, \widetilde{y}, \widetilde{\alpha}) = 1$ since $\operatorname{supp}\chi_1 \subset \{\chi_0 = 1\}$. According to (A.30) with $s = 1$, $\chi_2 = 1$ and to the fact (Lemma A.12) that $(x, x, \alpha(x, \beta, m, h))$ is critical point for θ, we set

$$(A.32) \quad \theta(x, \underline{y}, \underline{\alpha}, m, h) = \theta(x, x, \alpha(x, \beta, m, h), m, h) + \frac{1}{2} D^2\theta(x, x, \alpha(x, \beta, m, h), m, h) \cdot X^2 + \mathcal{O}(E^3)$$

where

$$(A.33) \quad \begin{cases} D = (\partial_y, \partial_\alpha), \quad X = (\widetilde{y} - x, \widetilde{\alpha} - \alpha^0) \\ E^j = |\widetilde{y} - x|^j + |\widetilde{\alpha}_\xi - \alpha_\xi^0|^j + h\,|\alpha_x - \alpha_x^0|^j, \quad j \geqslant 0. \end{cases}$$

Since, by Lemma A.12, $\operatorname{Im}\alpha(x, \beta, m, h) = \mathcal{O}(\varepsilon_0 h)$, we can replace in $D^2\theta$, in the above formula, $\alpha(x, \beta, m, h)$ by $\operatorname{Re}\alpha(x, \beta, m, h)$ modulo an error which is $\mathcal{O}(\varepsilon_0 h \|X\|^2)$. Now $\theta = \varphi_2 + ih\varphi_1 + \psi$. Since φ_2 is real on the real, we have

$$(A.34) \qquad \operatorname{Im}\left[D^2\varphi_2(x, x, \operatorname{Re}\alpha(x, \beta, m, h))X^2\right] = 0.$$

To take care of $\operatorname{Re} D^2\varphi_1$ we recall that (Lemma A.12),

$$\alpha_x(x, \beta, m, h) = \alpha_x(x, \alpha_\xi(x, \beta, m, h))$$

so

$$\operatorname{Re}\alpha_x(x, \beta, m, h) = \alpha_x(x, \operatorname{Re}\alpha_\xi(x, \beta, m, h)) + \mathcal{O}(\varepsilon_0 h).$$

Therefore we can replace in $D^2\varphi_1$, $\operatorname{Re}\alpha_x(x,\beta,m,h)$ by $\alpha_x(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h))$ modulo errors which are $\mathcal{O}(\varepsilon_0 hE^2)$. So we are left with

$$(1) = h\operatorname{Re} D^2\varphi_1(x,x,\alpha_x(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h)),\operatorname{Re}\alpha_\xi(x,\beta,m,h))X^2\,.$$

Now, by conditions 4) and 7), Definition A.5, we have

$$\operatorname{Re}\varphi_1(x,x,\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots)),\operatorname{Re}\alpha_\xi(\cdots))=0\,,$$
$$\operatorname{Re}\varphi_1(x,y,\alpha)\geqslant 0 \text{ if } (x,y,\alpha) \text{ is real.}$$

Then, that $\operatorname{Re} D\varphi_1(x,x,\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots)),\operatorname{Re}\alpha_\xi(\cdots))=0$. It follows from condition 7), Definition A.5 and Taylor's formula that,

$$\begin{aligned}&h\operatorname{Re}\varphi_1(x,\widetilde{y},\widetilde{\alpha}_x-\alpha_x^0+\alpha_x(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h)),\widetilde{\alpha}_\xi-\alpha_\xi^0+\operatorname{Re}\alpha_\xi(x,\beta,m,h))\\&\quad=(1)+\mathcal{O}(E^3)\geqslant Ch(\widetilde{\alpha}_x-\alpha_x^0+\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots))-\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots)+\widetilde{\alpha}_\xi-\alpha_\xi^0))^2.\end{aligned}$$

It follows that

$$\begin{aligned}(A.35)\ h\operatorname{Re}D^2\varphi_1(x,x,\alpha(x,\beta,m,h)X^2&\geqslant Ch(\widetilde{\alpha}_x-\alpha_x^0+\alpha_x(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h))\\&\quad-\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots)+\widetilde{\alpha}_\xi-\alpha_\xi^0))^2+\mathcal{O}(\varepsilon_0hE^2+E^3).\end{aligned}$$

Now by condition 6), Definition A.4, we have

$$(A.36)\qquad \operatorname{Im}D^2\psi(x,\beta,m,h)X^2\geqslant -C\varepsilon_0h\|X\|^2.$$

We deduce from (A.31) to (A.36) that

$$\begin{aligned}(A.37)\ &\operatorname{Im}\theta(x,y,\alpha,\beta,m,h)-\operatorname{Im}\theta(x,x,\alpha(x,\beta,m,h),m,h)\\&\geqslant Ch(\widetilde{\alpha}_x-\alpha_x^0+\alpha_x(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h))-\alpha_x(x,\operatorname{Re}\alpha_\xi(\cdots)+\widetilde{\alpha}_\xi-\alpha_\xi^0))^2\\&\quad+\frac{\delta}{2}\Big|\frac{\partial\theta}{\partial y}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2+\frac{\delta}{2}\Big|\frac{\partial\theta}{\partial\alpha_\xi}(x,\underline{y},\underline{\alpha},\beta,m,h)\Big|^2+\mathcal{O}(E^3+\varepsilon_0hE^2)\end{aligned}$$

where $X=(\widetilde{y}-x,\widetilde{\alpha}-\alpha^0)$.

Let us set $\rho^*=(x,x,\alpha(x,\beta,m,h),m,h)$. Recall that this is a critical point for θ, (Lemma A.12). Then, (A.37) implies

$$\begin{aligned}(A.38)\ \ \operatorname{Im}\theta(x,y,\alpha,\beta,m,h)&\geqslant\frac{\delta}{2}\bigg(\Big|\frac{\partial^2\theta}{\partial y^2}(\rho^*)(\widetilde{y}-x)+\frac{\partial^2\theta}{\partial y\partial\alpha}(\rho^*)(\widetilde{\alpha}-\alpha^0)\Big|^2\\&\qquad+\Big|\frac{\partial^2\theta}{\partial\alpha_\xi\partial y}(\rho^*)(\widetilde{y}-x)+\frac{\partial^2\theta}{\partial\alpha_\xi\partial\alpha}(\rho^*)(\widetilde{\alpha}-\alpha^0)\Big|^2\bigg)\\&\quad+Ch\Big|\widetilde{\alpha}_x-\alpha_x^0-\frac{\partial\alpha_x}{\partial\alpha_\xi}(x,\operatorname{Re}\alpha_\xi(x,\beta,m,h))(\widetilde{\alpha}_\xi-\alpha_\xi^0)\Big|^2+\mathcal{O}(E^3+\varepsilon_0hE^2).\end{aligned}$$

Now, the sum of squares, in the right hand side of (A.38), is equal to $\|M(\rho^*)Y\|^2$ where M is the matrix

$$M=\begin{pmatrix}\frac{\partial^2\theta}{\partial y^2}&\frac{\partial^2\theta}{\partial y\partial\alpha_\xi}&\frac{\partial^2\theta}{\partial y\partial\alpha_x}\\\frac{\partial^2\theta}{\partial\alpha_\xi\partial y}&\frac{\partial^2\theta}{\partial\alpha_\xi^2}&\frac{\partial^2\theta}{\partial\alpha_\xi\partial\alpha_x}\\0&-\sqrt{Ch}\,\frac{\partial\alpha_x}{\partial\alpha_\xi}&\sqrt{Ch}\operatorname{Id}\end{pmatrix}$$

and $Y = (\widetilde{y} - x, \widetilde{\alpha}_\xi - \alpha^0_\xi, \widetilde{\alpha}_x - \alpha^0_x)$. Now we have seen that

$$\frac{\partial \alpha_x}{\partial \alpha_\xi} = -\Big(\frac{\partial^2 \varphi_1}{\partial \alpha_x^2}\Big)^{-1} \cdot \frac{\partial^2 \varphi_1}{\partial \alpha_x \partial \alpha_\xi}$$

and (see the beginning of the proof of Theorem A.11)

$$\frac{\partial^2 \theta}{\partial \alpha_\xi^2}(\rho^*) = \frac{\partial^2 \varphi}{\partial \alpha_\xi^2}(\rho^*) = ih \frac{\partial^2 \varphi_1}{\partial \alpha_\xi \partial \alpha_x}\Big(\frac{\partial^2 \varphi_1}{\partial \alpha_x^2}\Big)^{-1} \frac{\partial^2 \varphi_1}{\partial \alpha_x \partial \alpha_\xi}.$$

It follows that M can be written as

$$M = \begin{pmatrix} A & B & C \\ {}^tB & ih{}^tDE^{-1}D & ih{}^tD \\ 0 & \sqrt{Ch}E^{-1}D & \sqrt{Ch}\,\mathrm{Id} \end{pmatrix}.$$

Moreover, since ψ does not depend on α, it follows from condition 8), Definition A.5 that, when h is large, the matrix $\begin{pmatrix} B & C \\ D & E \end{pmatrix}$ is invertible at ρ^*. Since B is also invertible we see from the second and third "line" of M that M is also uniformly invertible. When h is small we write,

$$\frac{\partial^2 \theta}{\partial \alpha_\xi \partial \alpha}(\rho^*)(\widetilde{\alpha} - \alpha^0) = ih \frac{\partial^2 \varphi_1}{\partial \alpha_\xi^2}(\rho^*)(\widetilde{\alpha}_\xi - \alpha^0_\xi) + ih \frac{\partial^2 \varphi_1}{\partial \alpha_\xi \partial \alpha_x}(\rho^*)(\widetilde{\alpha}_x - \alpha^0_x)$$

since $\frac{\partial^2 \varphi_2}{\partial \alpha_\xi^2}(\rho^*) = 0$. Then

$$\frac{\partial^2 \theta}{\partial \alpha_\xi \partial \alpha}(\rho^*)(\widetilde{\alpha} - \alpha_0) = ih \frac{\partial^2 \varphi_1}{\partial \alpha_\xi \partial \alpha_x}(\rho^*)\Big(- \frac{\partial \alpha_x}{\partial \alpha_\xi}(x, \alpha(x, \beta, m, h))(\widetilde{\alpha}_\xi - \alpha^0_\xi) + \widetilde{\alpha}_x - \alpha^0_x\Big).$$

By condition 8), Definition A.5, $\frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi}(\rho^*)$ is uniformly invertible since ρ^* is close to (x_0, x_0, α^0). It follows that

$$(A.39) \quad |\widetilde{y} - x|^2 \leqslant C\Big|\frac{\partial^2 \varphi}{\partial \alpha_\xi \partial y}(\rho^*)(\widetilde{y} - x)\Big|^2 \leqslant C\Big|\frac{\partial^2 \varphi}{\partial \alpha_\xi \partial y}(\rho^*)(\widetilde{y} - x) + \frac{\partial^2 \theta}{\partial \alpha_\xi \partial \alpha}(\rho^*)(\widetilde{\alpha} - \alpha^0)\Big|^2 + Ch^2\Big|\widetilde{\alpha}_x - \alpha^0_x - \frac{\partial \alpha_x}{\partial \alpha_\xi}(x, \alpha(x, \beta, m, h))(\widetilde{\alpha}_\xi - \alpha^0_\xi)\Big|^2.$$

By the same way

$$(A.40) \quad |\widetilde{\alpha}_\xi - \alpha^0_\xi|^2 \leqslant C\Big|\frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi}(\rho^*)(\widetilde{\alpha}_\xi - \alpha^0_\xi)\Big|^2 \leqslant C\Big|\frac{\partial^2 \varphi}{\partial y \partial \alpha_\xi}(\rho^*)(\widetilde{\alpha}_\xi - \alpha^0_\xi) + \frac{\partial^2 \theta}{\partial y^2}(\rho^*)(\widetilde{y} - x) + \frac{\partial^2 \varphi}{\partial y \partial \alpha_x}(\rho^*)(\widetilde{\alpha}_x - \alpha^0_x)\Big|^2 + C|\widetilde{y} - x|^2 + Ch^2|\widetilde{\alpha}_x - \alpha^0_x|^2.$$

$$(A.41) \quad h|\alpha_x - \alpha^0_x|^2 \leqslant h\Big|\alpha_x - \alpha^0_x - \frac{\partial \alpha_x}{\partial \alpha_\xi}(x, \alpha_\xi(\cdots))(\widetilde{\alpha}_\xi - \alpha^0_\xi)\Big|^2 + Ch|\alpha_\xi - \alpha^0_\xi|^2.$$

Using (A.38) to (A.41) we obtain, if ε_0 is small enough

$$\operatorname{Im}\theta(x,y,\alpha,\beta,m,h) \geqslant C\big(|\widetilde{y}-x|^2 + |\widetilde{\alpha}_\xi - \alpha^0_\xi|^2 + h\,|\widetilde{\alpha}_x - \alpha^0_x|\big)\,.$$

According to Definition A.8, this proves that the contour $\widetilde{\Gamma}_1$ is good for $-\operatorname{Im}\theta$.

Theorem A.14. — *Let ψ be a phase, in the sense of Definition A.4, at $(x_0,\xi_0,\beta^0)\in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$. Let b be an analytic symbol in a neighborhood of (x_0,β^0). We set $x_0=(s_0,y_0)$, where $s_0>0$, $\xi_0=(\tau_0,\eta_0)$, $r_0^2=s_0^6\tau_0^2+s_0^4|\eta_0|^2$, $\zeta_0=\frac{1}{r_0}(s_0^3\tau_0,s_0^2\eta_0)$, $n_0=\big(h_0,y_0,k_0/r_0,\zeta_0\big)$ and $x=(s,y)$. Then, if $n_0\notin {}^{\mathrm{qsc}}WF_a(u)$, one can find $\chi\in C_0^\infty$, $\chi=1$ in a neighborhood of x_0, positive constants C_1, δ_1, ρ_1*

$$(A.42)\qquad \Big|\int e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)}\,b(x,\beta,m,h)\,\chi(x)\,\overline{u(sh,y)}\,ds\,dy\Big| \leqslant C_1\,e^{-\delta_1/hk}\,,$$

for all (β,h,m,k) such that $(m,h)\in U$, $|\beta-\beta^0|+|h-h_0|+|k-k_0|<\rho_1$.

Proof. — Our assumption implies that there exists a precised FBI phase φ (by Proposition A.3, Definition A.2) at (x_0,Ξ_0,α^0,h_0), where $\Xi_0=\big(\tau_0/r_0,\eta_0/r_0\big)$, an analytic elliptic symbol a, a cut-off function $\chi_0\in C_0^\infty$ equal to one in a neighborhood of x_0 and positive constants C_0, δ_0, ρ_0 such that

$$(A.43)\qquad \Big|\iint e^{ih^{-2}k^{-1}\varphi(x,\alpha,h)}\,a(x,\alpha,h,k)\,\chi_0(x)\,\overline{u(hs,y)}\,ds\,dy\Big| \leqslant C_0\,e^{-\delta_0/hk}\,,$$

for all, α,h,k such that $|\alpha-\alpha^0|+|h-h_0|+|k-k_0|<\rho$, $h>0$, $k>0$. Let $\widetilde{\varphi}=\widetilde{\varphi}(x,z,\alpha,h)$ be the pseudo-differential phase (Definition A.5) constructed in Proposition A.6. We set, formally,

$$(A.44)\qquad Av(x,h,k)=\iint e^{ih^{-2}k^{-1}\widetilde{\varphi}(x,z,\alpha,h)}\,a(x,\alpha,h)\,v(z,h,k)\,dz\,d\alpha$$

which can be realized either as an operator in the complex domain or as an operator in the real domain.

Since A is elliptic, there exists an analytic symbol $c(z,\beta,h,k)$ such that

$$(A.45)\qquad A\big(e^{ih^{-2}k^{-1}\psi(\cdot,\beta,m,h)}\,c(\cdot,\beta,m,h,k)\big)=b(x,\beta,m,h,k)\,e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)}$$

where A acts on $H_{-\operatorname{Im}\psi}$. Indeed, if B is the inverse of A in $H_{-\operatorname{Im}\psi}$, we have $e^{-ih^{-2}k^{-1}\psi}B(e^{ih^{-2}k^{-1}\psi}b)=c$, modulo errors which are $\mathcal{O}(e^{-\delta/hk})$. Let V be a neighborhood of α_0. It follows from (A.45) and Theorem A.11 that

$$(A.46)\ A^V\big(e^{ih^{-2}k^{-1}\psi(\cdot,\beta,m,h)}\,c(\cdot,\beta,m,h,k)\big)=b(x,\beta,m,h,k)\,e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)} + \mathcal{O}(e^{-\delta/hk}).$$

Let us recall that the function χ occurring in the expression of A_V in (A.27) is such that, for some $r_0>0$,

$$\chi(x,z,\alpha)=1 \quad\text{if}\quad |x-z|+|\alpha_x-\alpha_x(x,\alpha_\xi)|\leqslant r_0\,,$$
$$\chi(x,z,\alpha)=0 \quad\text{if}\quad |x-z|+|\alpha_x-\alpha_x(x,\alpha_\xi)|\geqslant 2r_0\,.$$

Let $\chi_1 = \chi_1(z) \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\begin{aligned} \chi_1(z) &= 1 \quad \text{if} \quad |z - x_0| \leqslant r_0 \\ &= 0 \quad \text{if} \quad |z - x_0| \geqslant 2r_0 . \end{aligned}$$

Let us set

$$(1) = |\alpha_x - \alpha_x(x, \alpha_\xi)| + |\alpha_x - \alpha_x(z, \alpha_\xi)| .$$

Since the map, $x \mapsto \alpha_x(x, \alpha_\xi)$ is invertible for all α_ξ near α_ξ^0, we can find $C > 0$ such that for all $r_1 > 0$ small enough

$$|\alpha_x(x, \alpha_\xi) - \alpha_x(z, \alpha_\xi)| \leqslant r_1 \Longrightarrow |x - z| \leqslant C r_1 .$$

We claim that if $(1) \leqslant r_0/(1+2C)$ then $\chi_1(z) = \chi(x, z, \alpha) = 1$. Indeed, it follows from this inequality that $|\alpha_x(x,\alpha_\xi) - \alpha_x(z, \alpha_\xi)| \leqslant 2r_0/(1+2C)$ so $|x - z| \leqslant 2Cr_0/(1+2C)$ therefore

$$|x - z| + |\alpha_x - \alpha_x(x, \alpha_\xi)| \leqslant \frac{2Cr_0}{1+2C} + \frac{r_0}{1+2C} = r_0$$

which implies that $\chi(x, z, \alpha) = 1$. Moreover

$$|z - x_0| \leqslant |z - x| + |x - x_0| \leqslant \frac{2Cr_0}{1+2C} + \frac{r_0}{1+2C}, \quad \text{since } |x - x_0| \leqslant \frac{r_0}{1+2C} .$$

It follows that $\chi_1(z) = 1$.

Summing up we have proved that, on the support of $\chi(x, z, \alpha) - \chi_1(z)$ we have $(1) \geqslant r_0/(1+2C)$. We deduce from Definition A.5, 7) that $|e^{ih^{-2}k^{-1}\varphi}| \leqslant e^{-C_1/hk}$ which proves that in the definition of A^V we can replace $\chi(x, z, \alpha)$ by $\chi_1(z)$ (*modulo* controlled errors) if $|x - x_0|$ is small enough. Then

$$A^V v(x, h, k) = \iint_{\alpha \in V} e^{ih^{-2}k^{-1}\widetilde{\varphi}(x,z,\alpha,h)} a(x, \alpha, h, k) \chi_1(z) v(z, h, k)\, dz\, d\alpha + \mathcal{O}(e^{-\delta/hk}),$$

(where the error term is bounded by $\sup |v|$).

Let us write (A.46), replacing χ by χ_1. We have

$$\begin{aligned} \text{(A.47)} \quad & b(x, \beta, m, h, k) e^{ih^{-2}k^{-1}\psi(x,\beta,m,h,k)} \\ &= \int_{\alpha \in V} e^{ih^{-2}k^{-1}\varphi(x,\alpha,h)} a(x, \alpha, h, k) f(\alpha, \beta, m, h, k)\, d\alpha + \mathcal{O}(e^{-\delta/hk}), \end{aligned}$$

where

$$f(\alpha, \beta, m, h, k) = \int \chi_1(z) e^{ih^{-2}k^{-1}[-\varphi_2(z,\alpha_\xi) + ih\varphi_1(z,\alpha) + \psi(z,\beta,m,h)]} c(z, \beta, m, h, k)\, dz .$$

It follows from (A.8) that $\operatorname{Re}\varphi_1(z, \alpha) \geqslant C|\alpha_x - \alpha_x(z, \alpha_\xi)|^2$ and, since $\operatorname{Im}\psi \geqslant 0$ we have $|f| \leqslant C_N (hk)^{-N}$ for some $N \in \mathbb{N}$. Then, using (A.47) we can write, with $x = (s, y)$,

$$\begin{aligned} (A.48) \quad & \iint e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)} b(x, \beta, m, h, k) \chi(x) \overline{u(sh, y)}\, ds\, dy \\ &= \int_{\alpha \in V} f(\alpha, \beta, m, h, k) \cdot \Big(\iint e^{ih^{-2}k^{-1}\widetilde{\varphi}(x,\alpha,h)} a(x, \alpha, h, k) \chi(x) \overline{u(hs, y)}\, ds\, dy \Big)\, d\alpha . \end{aligned}$$

This proves Theorem A.14. □

Corollary A.15. — *The definition of* ${}^{\mathrm{qsc}}WF_a$ *is invariant under a change of phase satisfying Definition A.1.*

Proof. — We have seen in Proposition A.3 that we may assume that the phase is precised. We can change s_0 in the definition of ${}^{\mathrm{qsc}}WF_a$. Indeed, let φ be a FBI phase at $(x_0, \xi_0, \alpha^0, h_0)$, $x_0 = (s_0, y_0)$, $\xi_0 = (\tau_0, \eta_0)$; let us set $\widetilde{\varphi}(s, y, \alpha, h) = \gamma^{-2}\varphi(s/\gamma, y, \alpha, \gamma h)$, $\gamma > 0$. Then $\widetilde{\varphi}$ is an FBI phase at $(\gamma s_0, y_0, \tau_0/\gamma^3, \eta_0/\gamma^2, \alpha^0, \gamma h_0)$. We see that the change of (h, s) to $(\gamma h, s/\gamma)$ in the definition of ${}^{\mathrm{qsc}}WF_a$ gives rise to the phase $\widetilde{\varphi}$ in the integral. The analytic symbol is changed but stays elliptic. Now let us take two precised FBI phase at the same point $x_0 = (s_0, y_0)$. Then we see easily that they satisfy both the Definition A.4 (for instance, condition 2) in Definition A.4 follows from (A.8)), so we may apply the Theorem A.14. □

Corollary A.16. — *Let* $\widehat{\varphi}$ *be a FBI phase at* $(x_0, \xi_0, \alpha^0, 0)$. *Let us assume that one can find positive constants* C, δ, h_1, *an analytic symbol* a *elliptic at* (x_0, α^0, h_0), *a cut-off* $\chi \in C_0^\infty$, *equal to one near* $x_0 = (s_0, y_0)$ *such that*

$$\Big| \int e^{ih^{-2}\widehat{\varphi}(s,y,\alpha,h)} a(s, y, \alpha, h) \chi(s, y) \overline{u(sh, y)} \, ds\, dy \Big| \leqslant C e^{-\delta/h},$$

for all α *in a neighborhood of* α^0 *and* $h \in]0, h_1[$. *Let us set*

$$r_0^2 = s_0^6 \tau_0^2 + s_0^4 |\eta_0|^2 > 0, \quad \overline{\tau}_0 = \frac{s_0^3 \tau_0}{r_0}, \quad \overline{\eta}_0 = \frac{s_0^2 \eta_0}{r_0},$$

where $\xi_0 = (\tau_0, \eta_0)$. *Then* $n_0 = \big(0, y_0, 1/r_0, (\overline{\tau}_0, \overline{\eta}_0)\big) \notin {}^{\mathrm{qsc}}WF_a(u)$. *In the coordinates* (λ, μ) *this reads* $(0, y_0, s_0^3 \tau_0, s_0^2 \eta_0) \notin {}^{\mathrm{qsc}}WF_a(u)$.

Proof. — We may assume that $\widehat{\varphi}$ is precised. Let us set $\widetilde{\varphi} = \frac{1}{r_0}\widehat{\varphi}$ when $r_0 \neq 0$. Then $\widetilde{\varphi}$ is a precised FBI phase at $\big(s_0, y_0, \tau_0/r_0, \eta_0/r_0, \alpha^0, 0\big)$. Let us associate to $\widetilde{\varphi}$, a pseudodifferential phase φ by the formula in Proposition A.6. Finally let us set $\psi(x, \beta, m, h) = \frac{1}{m}\widetilde{\varphi}(x, \beta, h)$. Then ψ is a phase in the sense of Definition A.4 at the point $\big(x_0, \frac{1}{r_0}\xi_0, \alpha^0\big)$, $\xi_0 = (\tau_0, \eta_0)$, $x_0 = (s_0, y_0)$ and $U = B(r_0^{-1}, \delta) \times]0, \delta)$, $(m, h) \in U$. Let us set

$$A^V v(x, \alpha, h, k) = \iint_{\alpha \in V} e^{ih^{-2}k^{-1}\varphi(x,z,\alpha,h)} a(x, \alpha, h) \chi_1(x, z, \alpha) v(z, h, k) \, dz\, d\alpha.$$

We can apply Theorem A.11 as in the proof of Theorem A.14. We get

$$A^V\big(e^{ih^{-2}k^{-1}\psi(\cdot,\beta,m,h)} c(\cdot, \beta, m, h, k)\big) = a(x, \beta, h) e^{ih^{-2}k^{-1}\psi(x,\beta,m,h)} + \mathcal{O}(e^{-\delta/hk}).$$

In this formula we fix $k = k_0 = r_0^{-1}$ and we obtain, with $x = (s, y)$

$$\iint e^{ih^{-2}m^{-1}\widetilde{\varphi}(x,\beta,h)} a(x, \beta, h) \overline{u(hs, y)} \, ds\, dy$$

$$= \int_{\alpha \in V} f(\alpha, \beta, m, h) \Big(\int e^{ih^{-2}\widehat{\varphi}(x,\alpha,h)} \chi(x) a(x, \alpha, h) \overline{u(hs, y)} \, ds\, dy \Big) d\alpha + \mathcal{O}(e^{-\delta/h}),$$

with $f = \mathcal{O}(h^{-N})$.

Then the result follows if we consider m as the parameter k in the definition of ${}^{\rm qsc}WF_a$. $\square$

Corollary A.17. — *Let $m_0 = (\rho_0, y_0, \tau_0, \eta_0)$ with $\rho_0 > 0$. Then $m_0 \notin WF_a(u)$ (the usual analytic wave front set) if and only if $(\rho_0, y_0, 0, (\overline{\lambda}_0, \overline{\mu}_0)) \notin {}^{\rm qsc}WF_a(u)$, where*

$$\overline{\lambda}_0 = \frac{\rho_0 \tau_0}{\sqrt{\rho_0^2 \tau_0^2 + |\eta_0|^2}}, \quad \overline{\mu}_0 = \frac{\eta_0}{\sqrt{\rho_0^2 \tau_0^2 + |\eta_0|^2}}.$$

Proof. — We note that, in the definition of WF_a and ${}^{\rm qsc}WF_a$ we did not take the same coordinates on T^*M. The statement of this corollary takes this difference in account. Indeed, we have

$$\tau\, d\rho + \eta \cdot dy = (\rho^3 \tau) \frac{d\rho}{\rho^3} + (\rho^2 \eta) \cdot \frac{dy}{\rho^2}.$$

If $(\rho_0, y_0, 0, (\overline{\lambda}_0, \mu_0)) \notin {}^{\rm qsc}WF_a(u)$ then if we set $h = h_0$ and $k^{-1} = \lambda$, in our transformation $\mathcal{T}$, we recover a FBI transform in the sense of Sjöstrand. Then we have $(\rho_0, y_0, \tau_0, \eta_0) \notin WF_a(u)$ with $\tau_0 = \overline{\lambda}_0/\rho_0^3$, $\eta_0 = \overline{\mu}_0/\rho_0^2$; our claim follow since WF_a is conical.

Conversely, let us assume that $(\rho_0, y_0, \tau_0, \eta_0) \notin WF_a(u)$. Let us set, with $x = (\rho, y)$, $\widetilde{\varphi} = (x - \alpha_x) \cdot \alpha_\xi + i(x - \alpha_x)^2$. Then one can find positive constants C, δ, λ_0 and a cut-off $\chi \in C_0^\infty$, $\chi(\rho_0, y_0) = 1$ such that

$$\Big| \int e^{i\lambda \widetilde{\varphi}(x,\alpha)} \chi(x) \overline{u(x)}\, dx \Big| \leqslant C e^{-\lambda\delta}$$

for all α in a neighborhood of $(\rho_0, y_0, \tau_0, \eta_0)$ and $\lambda \geqslant \lambda_0$.

Let us set $\varphi(x, \alpha, h) = (x - \alpha_x)\alpha_\xi + ih(x - \alpha_x)^2$ and let us associate to φ a pseudo-differential phase by Proposition A.6. Finally let us set

$$\psi(s, y, \beta, m, h) = m^{-2}\Big[\Big(\frac{s}{m} - \beta_s\Big)\beta_\tau + (y - \beta_y) \cdot \beta_\eta\Big] + \frac{i}{m}\Big[\Big(\frac{s}{m} - \beta_s\Big)^2 + (y - \beta_y)^2\Big].$$

Then ψ satisfies the conditions in Definition A.4 for all ε_0 if

$$|s - \rho_0| + |y - y_0| + |\beta_s - \rho_0| + |\beta_y - y_0| + |\beta_\tau - \tau_0| + |\beta_\eta - \eta_0| + |m - 1|$$

is small enough. It follows from Theorem A.11 that we have the formula (A.48). Let us make $h = 1$ in this formula. The right hand side is $\mathcal{O}(e^{-\delta/k})$. Now if m plays the role of h in the left hand side, we recover the expression $\mathcal{T}u$; this proves our claim. $\square$

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