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SEMICLASSICAL SPECTRAL ASYMPTOTICS

BY VICTOR IVRII

0. Introduction

The problem of the spectral asymptotics, in particular the problem of asymptotic distribution of eigenvalues is one of the central problems of the spectral theory of partial differential operators. It is also very important for the general theory of partial differential operators. Apart from applications in the quantum mechanics, radiophysics, continuum media mechanics (elasticity, hydrodynamics, theory of shells) etc, there are also applications to the mathematics itself and moreover there are deep though non-obvious links with differential geometry, dynamic systems theory and ergodic theory; even the term "spectral geometry" has arisen. All these circumstances make this topic very attractive for a mathematician.

This problem originated in 1911 when H.Weyl published a paper devoted to eigenvalue asymptotics for the Laplace operator in a bounded domain with a regular boundary. After this article there was published a huge number of papers devoted to the spectral asymptotics and numerous prominent mathematicians were among their authors. The theory was developed in two directions: first of all this theory was extended and there were considered more and more general operators and boundary conditions as well as geometrical domains on which these operators were given; on the other hand the theory was improved and more and more accurate remainder estimates were derived. Namely in the later way the links with differential geometry, dynamic systems theory and ergodic theory appeared. Even the theory of eigenvalue asymptotics for the Laplace (or Laplace-Beltrami) operator has a long, dramatic and yet non-finished history. At a certain moment apart of asymptotics with respect to the spectral parameter there appeared asymptotics with respect to other parameters; the most important among them are (in my opinion) semiclassical asymptotics, i.e. asymptotics with respect to the small parameter h (Planck constant in physics) tending to $+0$. For a long time these asymptotics were in the shadow: most attention was paid to the eigenvalue asymptotics for operators on compact manifolds (with or without a boundary); the results which had been obtained here then were proved again for operators in $\mathbb{R}^d$ such as the Schrödinger operator $-h^2\Delta + V(x)$ with fixed $h > 0$ and with $V(x) \to +\infty$ as $|x| \to \infty$; less attention was paid to semiclassical asymptotics (i.e. asymptotics of eigenvalues less than some fixed level λ as $h \to +0$); moreover the asymptotics of the small negative eigenvalues were considered in the case of fixed h and $V(x)$ decreasing at infinity as $|x|^{2m}$ with $m \in (-1, 0)$; under reasonable conditions in this case the discrete spectrum of an operator has an accumulation point -0 and the essential spectrum coincides with $[0, +\infty)$. The result of the development of the theory described above was that at a certain moment there existed four parallel (though not equally developed) theories and the statements in each of them had to be proved separately. However now this plurality has been finished (at least in my papers) because all the other results are easily derived from the *local semiclassical spectral asymptotics* (*LSSA* in what follows), which are the main object of these lectures All other results are obtained as their applications.

In his papers H.Weyl applied the *variational* method (Dirichlet-Neumann bracketing) invented by himself; later this method was improved in various directions by many mathematicians. Other methods also appeared later and I would like to mention only the *method of a hyperbolic operator* due to B.M.Levitan and Avvakumovič[1)]. All the asymptotics with the most accurate remainder estimates were obtained by this method. It is based on the fact that the fundamental solution to the Cauchy problem (or the initial-boundary value problem) $u(x,x,t)$ for the operator $D_t - A$ is the Schwartz' kernel of the operator $\exp itA$ (where $D_t = -i\partial_t$, etc) and it is connected with the eigenvalue counting function of an operator A by the formula

$$\int u(x,x,t)dx = \int e^{it\lambda}\, d_\lambda N(\lambda); \tag{0.1}$$

in the case of a matrix operator A $u(x,y,t)$ is a matrix-valued function and in the left-hand expression it should be replaced by its trace. Here and below $N(\lambda)$ is the number of eigenvalues of A less than λ (and in this place we consider only operators semi-bounded from below with purely discrete spectra). Then by means of the inverse Fourier transform we can recover $N(\lambda)$ provided we have constructed $u(x,y,t)$ by means of the methods of theory of partial differential operators. However, in fact we are never able (excluding some very special

[1)]This method is a special case of Tauberian methods due to T.Carleman; resolvent method, method of complex power and method of heat equation are other Tauberian methods. The method of the almost spectral projector due to M.Shubin and V.Tulovskii lies between variational and Tauberian methods.

and rare cases when all this machinery is not necessary) to construct $u(x, y, t)$ precisely and for all the values $t \in \mathbb{R}$. Usually (now we assume that A is an elliptic first-order pseudo-differential operator) the fundamental solution is constructed approximately (modulo smooth functions) for t belonging to some interval $[-T, T]$ with $T > 0$. As a consequence we obtain modulo $O(\lambda^{-K})$ with any arbitrarily chosen K an expression for

$$F_{t \to \tau} \chi_T(t) \int u(x, x, t) dx = \int \hat{\chi}_T(\tau - \lambda) d_\lambda N(\lambda) \tag{0.2}$$

where χ is a fixed smooth function supported in $[-1, 1]$, $\chi_T(t) = \chi(\frac{t}{T})$ and a hat as well as $F_{t \to \tau}$ mean the Fourier transform. Then if we know the left-hand expression, using the Tauberian theorem due to Hörmander we are able to recover approximately $N(\lambda)$ by the formula

$$N(\lambda) = \int_{-\infty}^{\lambda} (F_{t \to \tau} \chi_T(t) \sigma)(\tau)) d\tau + O(\lambda^{d-1}) \tag{0.3}$$

where d is the dimension of the domain,

$$\sigma(t) = \int u(x, x, t) dx \tag{0.4}$$

and the explicit construction of $u(x, x, t)$ in this situation yields the formula

$$N(\lambda) = c_0 \lambda^d + O(\lambda^{d-1}) \tag{0.5}$$

with the leading coefficient

$$c_0 = (2\pi)^{-d} \int_{a(x,\xi) < 1} dx d\xi, \tag{0.6}$$

where $a(x, \xi)$ is a principal symbol of A.

We see that the crucial step in this approach is the construction of the fundamental solution. This construction by means of Fourier integral operators[2)] is standard and well-known now, provided we consider a scalar operator for an operator with constant multiplicities of the eigenvalues of the principal symbol and we construct $u(x, y, t)$ at the compact K contained in the interior of our domain X (and T depends on the distance between K and ∂X). If one of these assumptions is violated then the construction is more sophisticated and possible only under some very restrictive conditions. In the presence of a boundary (but only in the case of the constant multiplicities of the eigenvalues of the principal symbol) this construction was realized in certain papers due to R.Seeley, D.Vasil'ev, R.Melrose. However, it is possible to avoid all the troubles by means of another approach suggested by V.Ivrii[4] (see also L.Hörmander [3]) based on the investigation of the propagation of singularities

2)This construction due to L.Hörmander played a very important and stimulating role in the development of Fourier integral operators theory.

for $u(x,y,t)$ and construction of an "approximation" (in a rather exotic sense) for this distribution leading to an approximation in the reasonable sense for $\sigma(t)$ for $|t| \le T$ with appropriate $T > 0$. For h-pseudo-differential operators which are the main subject of this article this approach is essentially more simple and transparent because there is a selected parameter h. We'll discuss this case below. We'll be able to prove in this way the asymptotics (0.3) for an arbitrary self-adjoint m-th order elliptic operator with $m > 0$ and the spectral parameter λ^m now on a compact manifold without or with a boundary (in the former case the boundary conditions are also supposed to be elliptic), scalar or matrix, semi-bounded from below or non-semi-bounded at all (in this case $N(\lambda)$ is replaced by $N^{\pm}(\lambda)$ which is a number of eigenvalues lying between 0 and $\pm\lambda^m$); the formula for c_0 should be changed if it is necessary.

At the same time the two-terms asymptotics

$$N(\lambda) = c_0\lambda^d + c_1\lambda^{d-1} + o(\lambda^{d-1}) \tag{0.7}$$

suggested by H.Weyl (who also gave a formula for c_1) fails to be true unless some additional condition is fulfilled. It is certainly wrong for $d = 1$ and for the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$ of any dimension (this is due to the high multiplicities of its eigenvalues). Moreover, this asymptotics remains wrong in the case when this Laplace-Beltrami operator is perturbed by a potential or even by a symmetric first-order operator with small coefficients; in this case all the eigenvalues of high multiplicities will generate narrow eigenvalue clusters separated by lacunae. On the other hand under some conditions of the global nature the asymptotics (0.7) is valid. For a scalar operator on a compact manifold without a boundary this condition is "*The measure of the {set of all the points of the cotangent bundle periodic with respect to the Hamiltonian flow generated by the principal symbol } equals to 0*"[3]. This condition is more complicated for matrix operators. For a scalar second-order operator on a compact manifold with a boundary one needs to consider only trajectories transversal to the boundary and reflecting according to the geometrical optics law. Though there are some points of the cotangent bundle through which such infinitely long trajectory doesn't pass, but the measure of these dead-end points vanishes and we do not have to take them into account. For higher-order operators as well as for matrix operators the trajectories reflected from the boundary can branch and in this case it is necessary to follow every branch. This makes the situation much more complicated and the following additional condition (which isn't automatically fulfilled now) appears "*the measure of the {set of all the dead-end points} equals to* 0".

Let us clarify for the scalar first-order operator on a manifold without boundary a link between asymptotics (0.7) and periodic Hamiltonian trajec-

[3] This condition appeared first in the papers of J.J.Duistermaat and V.Guillemin.

tories. It is well-known that singularities of the solutions of the hyperbolic equations propagate along Hamiltonian trajectories. This fact leads us to a conclusion that the singular support $\sigma(t)$ is contained in the set of all the periods of the Hamiltonian trajectories including $t=0$; in particular $t=0$ is an isolated point of this singular support (this fact remains true in very general situations). Hence if there is no periodic trajectory with the period not exceeding T, we know that on the interval $[-T,T]$ the distribution $\sigma(t)$ is singular only at $t=0$ and hence we know $\sigma(t)$ on this interval modulo a smooth function. The Tauberian theorem permits us to obtain that the remainder in the asymptotics (0.7) doesn't exceed $\frac{C}{T}\lambda^{d-1}+O(\lambda^{d-2})$ with the constant C which doesn't depend on T; however "O" here isn't necessarily uniform with respect to T. Hence if $t=0$ was the unique period (I am aware that it is impossible!) then we would choose T and obtain the remainder estimate $o(\lambda^{d-1})$. In the general (realistic) case one should consider the partition of unity given by two pseudo-differential operators Q_j for every chosen T such that the support of the first operator contains no periodic point with the period not exceeding T and the measure of the support of the symbol of the second operator is less than ϵ with arbitrary chosen $\epsilon>0$ (due to our condition all periodic trajectories with the period not exceeding T form a closed nowhere dense set of measure 0). Applying the Tauberian theorem to every term

$$N_j(\lambda)=\int(Q_je)(x,x,\lambda)dx$$

in $N(\lambda)$ we obtain the remainder estimate $\frac{C}{T}\lambda^{d-1}+O(\lambda^{d-2})$ for $j=1$ and $C\epsilon\lambda^{d-1}+O(\lambda^{d-2})$ for $j=2$ and these estimates imply (0.7) again. Here and in what follows $e(x,y,\lambda)$ is a Schwartz' kernel of the spectral projector. Moreover, under certain more restrictive conditions to the Hamiltonian flow one can improve the remainder estimate in (0.7) to $O(\lambda^{d-1}/\log\lambda)$ or even $O(\lambda^{d-1-\delta})$ with a small exponent $\delta>0$[4].

It has been discovered recently that even in the presence of the periodic trajectories and in the presence of eigenvalue clusters one can have the two-term asymptotics of the form

$$N(\lambda)=c_0\lambda^d+f(\lambda)\lambda^{d-1}+o(\lambda^{d-1}) \tag{0.8}$$

with the explicitly calculable function $f(\lambda)$ which is bounded and oscillating as $\lambda\to+\infty$ with the characteristic "period" of oscillations $\asymp 1$. In particular, this fact enables us to obtain an asymptotic distribution of eigenvalues inside of clusters[5]. Moreover, under some assumptions including an assumption that

[4] See papers of P.Bérard and B.Randoll and more recent papers of A.Volovoy and the author.

[5] See papers of Yu.Safarov and more recent papers of the author.

all the trajectories are periodic one can obtain the asymptotics (0.8) with the remainder estimate $O(\lambda^{d-2})$!

It is well-known that in a large number of cases the Weylian formula fails to be applicable in its standard form. In these cases it is necessary either to remove from the domain of integration some part of the phase space or to divide variables x and ξ in two parts: $x = (x', x'')$, $\xi = (\xi', \xi'')$ and consider only the variables (x', ξ') as Weylian; that means that one should consider the operator in question as a (partial) differential operator with respect to x' with operator-valued coefficients and apply the Weylian procedure to this operator. In a more general case one should divide the phase space in a few parts. One of them should be removed from consideration and in the other parts the "Weylevization" (preceded by a certain transform) should be made only with respect to certain variables.

These remarks do not pretend to be a survey (even an incomplete one). Their goal is only to motivate this article in particular and all my works in general. I would like to recommend to the reader the books of M.S.Birman and M.Z.Solomyak [1] and G.Rozenblyum, M.Z.Solomyak and M.Shubin [11] as the best surveys. One can find accurate references in these books and additional references in the book of D.Robert [10] and in the author' preprints [7.1-7.9].

1. Why One Should Study Local Semiclassical Spectral Asymptotics

Local semiclassical spectral asymptotics (LSSA) are asymptotics of

$$\operatorname{Tr}\psi E(\lambda_1,\lambda_2) = \int \psi(x)\operatorname{tr} e(x,x,\lambda_1,\lambda_2)dx \tag{1.1}$$

as $h \to +0$ where $E(\lambda_1, \lambda_2)$ is a spectral projector of the operator $A = A_h$ depending on a small parameter h and corresponding to the interval $[\lambda_1, \lambda_2)$, $e(x, y, \lambda_1, \lambda_2)$ is its Schwartz kernel, ψ is a C_0^∞-function and Tr and tr mean operator and matrix traces respectively (for scalar operators tr in the right-hand expression is absent). If $\psi = 1$, we obtain $N(\lambda_1, \lambda_2)$. Therefore we hope that taking an appropriate partition of unity we can obtain asymptotics for $N(\lambda_1, \lambda_2)$ starting from LSSA.

Moreover, it is often better to start from microlocal semiclassical spectral asymptotics when ψ is an h-pseudo-differential operator with compactly supported symbol.

Now I would like to present a few well-known results (in a slightly stronger form[6]) and show how they can "improve themselves". We consider only the

[6] The improvement is that the remainder estimates are uniform and that no condition outside the ball $B(0,1)$ is assumed to be fulfilled. This enhancement adds no difficulties in the proofs but is very important for applications.

Schrödinger operator

$$A = -h^2\Delta + V(x) \tag{1.2}$$

in the domain $X \subset \mathbb{R}^d$, where we assume that X contains the unit ball $B(0,1)$ and V is uniformly smooth in this ball:

$$|D^\alpha V| \le c \qquad \forall\alpha : |\alpha| \le K \tag{1.3}$$

where $K = K(d)$ is large enough. We assume that A is self-adjoint in $L^2(X)$, $D(A) \supset C_0^2(B(0,1))$ and in $B(0,1)$ operator A is given by (1.2). One can take $\lambda_1 = -\infty$ now and without any loss of generality one can take $\lambda_2 = 0$.

Theorem 1.1. *(duetoJ.Chazarain). Let A be a self-adjoint operator of the form* (1.2) *and let condition* (1.3) *be fulfilled in $B(0,1) \subset X$. Then for $h \in (0,1]$*

(i) In the general case

$$|e(x,y,-\infty,0)| \le Ch^{-d} \qquad \forall x,y \in B(0,\frac{1}{2}) \tag{1.4}$$

where $C = C(d,c)$;

(ii) If $B(0,1)$ is classically forbidden, i.e. if

$$V \ge \epsilon \qquad \text{in } B(0,1); \tag{1.5}$$

for some $\epsilon > 0$ then

$$|e(x,y,-\infty,0)| \le C'h^s \qquad \forall x,y \in B(0,\frac{1}{2}); \tag{1.6}$$

where s is arbitrary and $C' = C'(d,c,s,\epsilon)$;

(iii) Finally, if 0 isn't a critical value of V, i.e.

$$|V| + |\nabla V| \ge \epsilon_0 \qquad \forall x \in B(0,1) \tag{1.7}$$

then

$$|\int \psi(e(x,x,-\infty,0) - \varkappa(x)h^{-d})dx| \le Ch^{1-d} \tag{1.8}$$

where

$$\varkappa(x) = (2\pi)^{-d}\int_{\{|\xi|^2+V(x)\le 0\}} d\xi = (2\pi)^{-d}\omega_d V_-^{\frac{d}{2}}, \tag{1.9}$$

$V_\pm = \max(\pm V, 0)$, ω_k is a volume of the unit ball in $\mathbb{R}^k$ and we assume that $\psi \in C_0^K(B(0,\frac{1}{2}))$ and

$$|D^\alpha\psi| \le c \qquad \forall\alpha : |\alpha| \le K. \tag{1.10}$$

We don't discuss here (1.8)-type asymptotics without spatial mollification (this asymptotics holds when 0 isn't value of V) and or asymptotics with more accurate remainder estimate (when some condition on the classical dynamic system should be assumed). In this theorem the boundary conditions are not important (because the boundary doesn't intersect $B(0,1)$) and, moreover, the nature of the operator and domain outside $B(0,1)$ isn't important (the self-adjointness is the only assumption of the global nature).

In order to improve this theorem let us reformulate it first for ball $B(\bar{x},\gamma)$ with arbitrary $\gamma > 0$. By dilatation $x_{\text{new}} = \frac{1}{\gamma}(x-\bar{x})$ and multiplication by ρ^{-2} this case can be reduced to the previous one; we consider γ, ρ as additional parameters. After reduction we obtain an operator of the form (1.2) again with $h_{\text{new}} = \frac{h}{\rho\gamma}$ and with $V_{\text{new}} = \rho^{-2}V(\bar{x}+\gamma x)$. Then one should assume that $h_{\text{new}} \in (0,1]$ i.e. that

$$\rho\gamma \geq h > 0 \tag{1.11}$$

instead of the previous condition $h \in (0,1]$ and in order to fulfill conditions (1.3),(1.10) after reduction one should assume that

$$|D^\alpha V| \leq c\rho^2\gamma^{-|\alpha|} \qquad \forall\alpha : |\alpha| \leq K \tag{1.3$'$}$$

and

$$|D^\alpha \psi| \leq c\gamma^{-|\alpha|} \qquad \forall\alpha : |\alpha| \leq K \tag{1.10$'$}$$

where now $\psi \in C_0^K(B(\bar{x},\frac{1}{2}\gamma))$. Taking in account that $e(x,x,-\infty,0)$ is a density i.e. that $e(x,x,-\infty,0)dx$ is invariant in this procedure we obtain

Theorem 1.1$'$. *Let A be a self-adjoint operator of the form* (1.2) *and let in $B(\bar{x},\gamma) \subset X$ condition* (1.3$'$) *be fulfilled. Then for $h \leq \rho\gamma$*

(i) In the general case

$$|e(x,y,-\infty,0)| \leq Ch^{-d}\rho^d \qquad \forall x,y \in B(\bar{x},\frac{1}{2}\gamma) \tag{1.4$'$}$$

where $C = C(d,c)$;

(ii) If $B(\bar{x},\gamma)$ is classically forbidden i.e. if

$$V \geq \epsilon\rho^2 \qquad \text{in } B(\bar{x},\gamma); \tag{1.5$'$}$$

with some $\epsilon > 0$ then

$$|e(x,y,-\infty,0)| \leq C'h^s\rho^{-s}\gamma^{-d-s} \qquad \forall x,y \in B(\bar{x},\frac{1}{2}\gamma); \tag{1.6$'$}$$

where s is arbitrary and $C' = C'(d,c,s,\epsilon)$;

(iii) Finally, if 0 isn't a critical value of V i.e.

$$|V| + |\nabla V|\gamma \geq \epsilon_0\rho^2 \qquad \forall x \in B(\bar{x},\gamma) \tag{1.7$'$}$$

then

$$|\int \psi(e(x,x,-\infty,0) - \varkappa(x)h^{-d})dx| \leq Ch^{1-d}\rho^{1-d}\gamma^{1-d} \tag{1.8$'$}$$

where $\varkappa$ is given by (1.9) *and $\psi \in C_0^K(B(\bar{x},\frac{1}{2}\gamma)$ satisfies* (1.10)$'$.

Let us treat the case $\rho = \gamma = 1$ without condition (1.7). Let us introduce the function

$$\gamma = \epsilon_1(|V| + |\nabla V|^2)^{\frac{1}{2}} + h^{\frac{1}{2}}. \tag{1.12}$$

Then it is easy to check that

$$|\nabla\gamma| \leq \frac{1}{2} \tag{1.13}$$

for small enough constant $\epsilon_1 = \epsilon_1(d,c)$ and that for $\bar{x} \in B(0,\frac{3}{4})$ in $B(\bar{x},\gamma)$ conditions (1.11) and (1.3)′ are fulfilled with $\rho = \gamma = \gamma(\bar{x})$. Moreover, for $\gamma \geq 2h^{\frac{1}{2}}$ condition (1.7)′ is also fulfilled. Let us take a γ-admissible partition of unity $\{\psi_n\}$ in $B(0,\frac{3}{4})$; this means that ψ_n is supported in $B(x_n,\frac{1}{2}\gamma(x_n))$ and satisfies (1.10)′ with $\gamma = \gamma(x_n)$ and that the multiplicity of the covering of $B(0,\frac{3}{4})$ by balls $B(x_n,\gamma(x_n))$ doesn't exceed $C_0 = C_0(d)$. Condition (1.13) implies that this partition exists. Then the contribution of every ball with $\gamma(x_n) \geq h^{\frac{1}{2}}$ to the remainder estimate in (1.8) doesn't exceed

$$Ch^{1-d}\gamma(x_n)^{2d-2} = h^{1-d}\gamma^{d-2}\int_{B(x_n,\gamma(x_n))} dx$$

and therefore the total contribution of all the balls of this type doesn't exceed Ch^{-d} for $d \geq 2$ and $h^{-\frac{1}{2}}$ for $d = 1$. According to (i) $e(x,x,-\infty,0) \leq Ch^{-\frac{d}{2}}$ if $\gamma(x) \asymp h^{\frac{1}{2}}$ and hence the total contribution of this zone to every term of asymptotics (1.8) can be estimated in the same way.

We have therefore proved

Theorem 1.2. *In frames of theorem 1.1 estimate* (1.8) *holds in the general case for* $d \geq 2$. *Moreover, for* $d = 1$ *the left-hand expression of* (1.8) *doesn't exceed* $Ch^{-\frac{1}{2}}$.

Let us treat case $d = 1$ more carefully.

Theorem 1.3. *Let* $d = 1$ *and conditions of theorem 1.1 be fulfilled. Then*

(i) If

$$|V| + |V'| + \cdots + |V^{(n)}| \geq \epsilon_0 \tag{1.14}$$

for some $n \geq 1$ *then the left-hand expression of* (1.8) *doesn't exceed* $C(|\log h| + 1)^{n-1}$ *where now* $K = K(n)$ *in conditions* (1.3), (1.10).

(ii) In the general case the left-hand expression of (1.8) *doesn't exceed* $Ch^{-\delta}$ *with arbitrary* $\delta > 0$ *where now* $K = K(\delta)$ *in conditions* (1.3), (1.10).

Proof. (i) For $n = 1$ this statement has been proved. Assume we have proved it for $n < \bar{n}$ and let us consider the case $n = \bar{n}$. Let us introduce the function

$$\gamma(x) = \epsilon_1 \left(\sum_{0 \le k \le n-1} |V^{(k)}|^{\frac{n}{n-k}} \right)^{\frac{1}{n}} + h^{\frac{2}{n+2}}$$

with small enough $\epsilon_1 = \epsilon_1(n, c, \epsilon_0)$. It is easy to check that (1.13) is fulfilled and that in $B(x,\gamma)$ conditions $(1.3)'$, $(1.10)'$ are fulfilled with $\gamma = \gamma(x)$ and $\rho = \gamma^{\frac{n}{2}}$. Moreover, if $\gamma \ge 2h^{\frac{2}{n+2}}$ then after dilatation and multiplication condition $(1.14)_{n-2}$ is fulfilled; so contribution of this ball doesn't exceed $C(|\log h| + 1)^{n-1}$ (more refined estimate doesn't improve the final answer). Furthermore, it is easy to check that under condition $(1.14)_n$ $\gamma(x) \ge \epsilon_2 \min_l |x - x_{(l)}|$ for appropriate points $x_{(l)}$ with $l = 1, \dots, L \le L_0 = L_0(c, n)$, $\epsilon_1 = \epsilon_1(c, n, \epsilon_0) > 0$ and then the total contribution of these balls (intervals) to remainder estimate in (1.8) doesn't exceed $C(|\log h| + 1)^{n-1}$. Moreover, it is easy to check that the contribution of remaining L intervals doesn't exceed $C(|\log h| + 1)^{n-1}$ either and then the induction step is made. The statement (i) is proved.

(ii) Applying the same arguments as before and using (i) we obtain the remainder estimate $Ch^{-2/(n+2)}(|\log h| + 1)^{n-2}$ with arbitrary n because $(1.14)_n$ isn't assumed to be fulfilled. This yields (ii). □

Remark 1.4. This proof can be extended to a wide class of scalar operators in other dimensions. On the other hand, even in dimension $d = 1$ the propagation of singularities arguments improve the final answer.

2. How LSSA Yield Standard Spectral Asymptotics

Now I would like to discuss three examples which show how LSSA yield asymptotics with respect to the spectral parameter.

(i) Let X be a compact Riemannian manifold without a boundary, Δ the Laplace-Beltrami operator on X. Let us consider the Schrödinger operator $A_h = -h^2\Delta + V(x)$; we know that

$$N^-(A_h) = c_0 h^{-d} + O(h^{1-d})$$

with the Weylian constant c_0 provided either $d \ge 2$ or 0 isn't a critical value of $V(x)$ (otherwise asymptotics with a worse remainder estimate holds). I recall that $N^-(A_h)$ is the number of negative eigenvalues of A_h counting their multiplicities. The Birman-Schwinger principle implies that

$$N^-(A_h) - N^-(A_0) = N(h^{-2})$$

where $N(\lambda)$ is the number of eigenvalues $\mu \in (0, \lambda)$ of the spectral problem $(-\Delta + \mu V(x))u = 0$ and $N^-(A_0) = \lim_{h \to +0} N^-(A_h) < \infty$. These two equalities immediately yield the asymptotics

$$N(\lambda) = c_0 \lambda^{\frac{d}{2}} + O(\lambda^{\frac{d-1}{2}}) \tag{2.1}$$

provided either $d \geq 2$ or 0 isn't a critical value of $V(x)$. Let somebody try to obtain this result directly in the case when $V(x)$ vanishes at some point! This is striking that nobody has yet observed this non-trivial new result (provided 0 isn't critical value of V) which trivially follows from two well-known facts!

(ii) Let us consider the Schrödinger operator $A = -\Delta + V(x)$ in $\mathbb{R}^d$ where Δ is the Laplacian and a real-valued potential $V(x)$ satisfies the following conditions:

$$|D^\alpha V| \leq c\langle x\rangle^{2m-\alpha} \qquad \forall\alpha : |\alpha| \leq K, \tag{2.2}$$

$$V \geq \epsilon_0 |x|^{2m} \qquad \text{for } x : |x| \geq c \tag{2.3}$$

where $\langle x\rangle = (|x|^2+1)^{\frac{1}{2}}$ here and in what follows. Let us take a γ-admissible partition of unity with $\gamma(x) = \frac{1}{4}\langle x\rangle$. If we make a dilatation transforming $B(\bar{x}, \gamma(\bar{x}))$ into $B(0,1)$ and multiply $A-\lambda$ by λ^{-1} we obtain for $|x| \leq C\lambda^{1/2m}$ in $B(0,1)$ the Schrödinger operator with standard restrictions on potential and with $h = \frac{1}{\gamma(\bar{x})\sqrt{\lambda}}$. Let us apply theorems 1.2,1.1 (then one should assume that either $d \geq 2$ or $V-\lambda$ is non-degenerating, i.e.

$$|\nabla V| \geq \epsilon_0 |x|^{2m-1} \qquad \text{for } |x| \geq c); \tag{2.4}$$

then we obtain for spatial means of $e(x,x,\lambda)$ in $B(\bar{x},\gamma(\bar{x}))$ the Weylian asymptotics with the remainder estimate $O(h^{1-d}) = O(\lambda^{\frac{d-1}{2}}\gamma(\bar{x})^{d-1})$; here is no additional factor because $e(x,x,\lambda)$ is a density but not a function. Summing with respect to the partition of unity we obtain remainder estimate $O(\lambda^{(d-1)l})$ with $l = \frac{1}{2m}(1+m)$. If we consider the ball $B(\bar{x},\gamma(\bar{x}))$ with $|x| \geq C\lambda^{\frac{1}{2m}}$ then after dilatation and multiplication of $A-\lambda$ by $\rho^{-2}(\bar{x})$ with $\rho(\bar{x}) = \langle x\rangle^m$ we obtain the Schrödinger operator with standard restrictions on the potential $v(x) \geq \epsilon_1$ and with $h = \frac{1}{\gamma(\bar{x})\rho(\bar{x})}$; then we derive an estimate $|e(x,x,\lambda)| \leq Ch^s\gamma(\bar{x})^{-d}$ with an arbitrary s and hence the contribution of the domain $\{|x| \geq C\lambda^{\frac{1}{2m}}\}$ in the remainder estimate is $O(\lambda^{-s})$ where we decrease s if it is necessary. We obtain also the asymptotics

$$N(\lambda) = \mathcal{N}(\lambda) + O(\lambda^{(d-1)l}) \tag{2.5}$$

with

$$\mathcal{N}(\lambda) = c_0 \int (\lambda - V)_+^{\frac{d}{2}} dx \tag{2.6}$$

as $\lambda \to +\infty$ and $\mathcal{N}(\lambda) \asymp \lambda^{dl}$.

(iii) Let us consider the Schrödinger operator in $\mathbb{R}^d$ with a potential $V(x)$ satisfying (2.2) with $m \in (-1,0)$ and let now $\lambda < 0$ be a small parameter. Then for the same ρ and γ as before the dilatation of $B(\bar{x},\gamma(\bar{x}))$ into $B(0,1)$ and multiplication of $A-\lambda$ by $\rho^{-2}(\bar{x})$ give the Schrödinger operator with the standard restrictions on the potential and with $h = \frac{1}{\gamma(\bar{x})\rho(\bar{x})}$ provided $|x| \leq |\lambda|^{\frac{1}{2m}}$; hence for the spatial mean of $e(x,x,\lambda)$ in $B(\bar{x},\gamma(\bar{x}))$ the Weylian formula holds with the remainder estimate $O(h^{1-d}) = O(\rho(\bar{x})^{d-1}\gamma(\bar{x})^{d-1})$ (provided that either $d \ge 2$ or condition (2.4) is fulfilled). On the other hand, for $|x| \ge C|\lambda|^{\frac{1}{2m}}$ dilatation and multiplication by $|\lambda|^{-1}$ give the Schrödinger operator with the standard restrictions to potential $v(x) \ge \epsilon_1$ and with $h = \frac{1}{\gamma(\bar{x})\sqrt{|\lambda|}}$. Hence the estimate $|e(x,x,\lambda)| \le Ch^s\gamma(\bar{x})^{-d}$ holds again. Summing with respect to the partition of unity we obtain asymptotics (2.5)-(2.6) as $\lambda \to -0$ with the same l as in (i); moreover, $\mathcal{N}(\lambda) \asymp |\lambda|^{dl}$ provided

$$V \le -\epsilon_1|x|^{2m} \qquad \text{for } x : |x| \ge c \tag{2.7}$$

in some non-empty open cone in $\mathbb{R}^d$.

In these three cases for $d=1$ without the non-degeneracy condition the final remainder estimate is slightly worse.

3. How One Can Derive LSSA in the General Case

The only possible (or at least the best) way to derive spectral asymptotics with accurate remainder estimate is the hyperbolic operator method (I don't discuss here special cases when one can find eigenvalues explicitly). There are few implementation of this method for semiclassical spectral asymptotics; for example, for Schrödinger operator one can consider $U(t) = \exp ith^{-1}B$ with either $B = A$ or $B = A^{\frac{1}{m}}$ where $m = 2$ is the order of operator. The second definition leads to the wave equation $h^2D_t^2u = A_hu$ which is hyperbolic in the classical sense and has the useful finite speed of propagation property; however, difficulties arise in the case when A isn't semi-bounded from below and in the case of higher-order operator make this way rather poor. The best way is to consider $U(t) = \exp ith^{-1}A$; the corresponding non-stationary Schrödinger equation

$$hD_tu = A_hu \tag{3.1}$$

isn't hyperbolic in the classical sense but it has all the useful properties of hyperbolic equations (with reasonable modifications). In particular, there is a finite speed of propagation property in the compact domains of the phase space. Thus, as we have mentioned, one should apply methods of partial differential equations and construct $\operatorname{Tr} QU(t)$ at some interval $[-T,T] \ni t$ and then use the formula

$$F_{t\to h^{-1}\tau} \operatorname{Tr} \chi_T(t)QU(t) = T\int \hat{\chi}(\frac{(\tau-\lambda)T}{h})d_\lambda \operatorname{Tr} QE(\lambda) \tag{3.2}$$

and the Tauberian theorem in order to recover asymptotics of $\operatorname{Tr} QE(\tau_1,\tau_2)$; here and below Q is an h-pseudo-differential operator with a compactly supported symbol. For a scalar operator A in the interior of domain (or in similar cases) the construction of QU is well-known for $T = \text{const} > 0$ (or even on longer interval under very restrictive conditions); however, for matrix operators and near the boundary this explicit construction at this interval is either very complicated or impossible. The idea which was suggested eleven years ago by the author in order to avoid this difficulty is very transparent in the semi-classical case: to construct QU for a shorter interval $[-T', T']$ (with T' depending on h) and then to prove that $\operatorname{Tr} QU$ is negligible at $[-T, -T'] \cup [T', T]$.

In order to make the first step we apply the successive approximation method. Let us consider the equation (3.1) with the initial data

$$U|_{t=0} = \delta(x-y). \tag{3.3}$$

Hence

$$(hD_t - A)U^{\pm} = \mp ih\delta(x-y)\delta(t) \tag{3.4}$$

where $U^{\pm} = \theta(\pm t)U$ and θ is Heaviside function; therefore

$$U^{\pm} = \mp ihG^{\pm}\delta(x-y)\delta(t) \tag{3.5}$$

where $G^{\pm}$ is the parametrix of the problem

$$(hD_t - A)v = f, \qquad v|_{\pm t<0} = 0. \tag{3.6}$$

On the other hand, (3.4) yields that

$$(hD_t - \bar{A})U^{\pm} = \mp ih\delta(x-y)\delta(t) + RU^{\pm} \tag{3.7}$$

where $\bar{A} = A(y, hD_x, 0)$ is operator obtained from A by freezing coefficients at y and dropping lower-order terms (in the semi-classical sense) and $R = A - \bar{A}$. Therefore

$$U^{\pm} = \mp ih\bar{G}^{\pm}\delta(x-y)\delta(t) + \bar{G}^{\pm}RU^{\pm}$$

where $\bar{G}$ is the parametrix of problem (3.6) for operator $\bar{A}$. Iterating this equality and using (3.5) once we obtain that

$$U^{\pm} = \mp ih \sum_{n\le N-1} (\bar{G}^{\pm}R)^n\bar{G}^{\pm}\delta(x-y)\delta(t) \mp ih(\bar{G}^{\pm}R)^N G^{\pm}\delta(x-y)\delta(t) \tag{3.8}$$

with arbitrarily large N. Let us notice that

$$R = \bar{R} + R' = \sum_{1\le|\alpha|+k\le M-1} h^k(x-y)^{\alpha}B_{\alpha,k}(y, hD_x) + \sum_{|\alpha|+k=M} B_{\alpha,k}(x, y, hD_x). \tag{3.9}$$

Let us substitute (3.9) for (3.8) and let us move all the factors $(x_j - y_j)$ to the right. If one factor reaches $\delta(x-y)$ the corresponding term vanishes; so the term survives only if this factor is killed on his way. The factor can be killed by commutation either with some h-pseudo-differential operator or with some parametrix. In the first case a factor h arises. In the second case one can apply equality

$$[G^{\pm}, x_j - y_j] = G^{\pm}[A, x_j - y_j]G^{\pm} \tag{3.10}$$

and the similar equality for $\bar{A}$, $\bar{G}$; the first equality is due to identity

$$(hD_t - A)(x_j - y_j)v = -[A, x_j - y_j]v + (x_j - y_j)(hD_t - A)v$$

which yields that

$$(x_j - y_j)v = -G^{\pm}[A, x_j - y_j]v + G^{\pm}(x_j - y_j)(hD_t - A)v$$

provided $v|_{\pm t<0} = 0$; substituting $v = G^{\pm}f$ with $f|_{\pm t<0} = 0$ we obtain (3.10). Thus, in this commutation $(x_j - y_j)$ is replaced by an additional factor h and an additional parametrix appears.

The Duhamel formula yields that the operator norms of $G^{\pm}$ and $\bar{G}^{\pm}$ in $L^2([-T,T],\mathbb{R}^d)$ don't exceed $C\frac{T}{h}$. Let us note that in the original expansion every parametrix was accompanied either by h or by $(x_j - y_j)$ factors. This yields that under the condition

$$T \le h^{\frac{1}{2}+\delta} \tag{3.11}$$

with arbitrarily small $\delta > 0$ for $M = M(d,s,\delta)$ and $N = N(d,s,\delta)$ the remainder term (with $n = N$) in (3.8) is negligible (i.e., less than h^s) and the equality (3.8) remains true modulo negligible terms if one replaces R by $\bar{R}$[7)].

Now only operators with symbols not depending on x remain and the Fourier transform on x and Fourier-Laplace transform on t provide us with the final answer:

$$F_{t\to h^{-1}\tau} \operatorname{Tr} QU^{\pm} = \pm \int F(\tau, y, \xi, h)dyd\xi \tag{3.12}$$

for $\mp\tau > 0$ and

$$F_{t\to h^{-1}\tau} \operatorname{Tr} QU = \int \mathcal{F}(\tau, y, \xi, h)dyd\xi \tag{3.13}$$

for $\tau \in \mathbb{R}$. Here F is a sum of the terms of the type

$$\operatorname{tr}(\tau - a(y,\xi))^{-1}b_1(y,\xi)(\tau - a(y,\xi))^{-1}\dots$$
$$b_{r-1}(y,\xi)(\tau - a(y,\xi))^{-1}b_r(y,\xi)h^{-d+n}$$

(a is the principal symbol of A) and $\mathcal{F}(\tau,.,.,.) = F(\tau - i0,.,.,.) - F(\tau + i0,.,.,.)$. Thus at rather short time interval $\operatorname{Tr} QU$ is constructed modulo negligible term. Multiplying $\operatorname{Tr} QU$ by $\varphi(hD_t/L)\bar{\chi}_T(t)$ with $\bar{\chi} \in C_0^K([-1,1])$, $\bar{\chi} = 1$ on $[-\frac{1}{2},\frac{1}{2}]$,

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7) In fact this deduction works only under some restrictions on A. It is sufficient to assume that its symbol is compactly supported; otherwise the appropriate cutoff should be done.

$\bar{\chi}_T(t) = \bar{\chi}(t/T)$, $\varphi \in C_0^K(\mathbb{R})$, $LT \geq h^{1-\delta}$ and setting $t=0$ we obtain complete asymptotics of the spectral mean

$$\int \phi(\frac{\tau}{L}) \operatorname{Tr} Q d_\tau E(\tau) \sim \sum_n h^{-d+n} \int \phi(\frac{\tau}{L}) \varkappa_k'(\tau) d\tau \tag{3.14}$$

provided $L \geq h^{\frac{1}{2}-\delta}$.

In order to derive asymptotics without mollifications additional arguments linked with propagation of singularities should be applied.

4. Propagation of Singularities

In fact, the results of this type (namely, locally finite speed of propagation) were used in order to justify the construction in the previous section. However here more refined results are necessary.

Let us assume first that A is a scalar operator. Let $\mathcal{V}$ be a small neighborhood of the point $(\bar{x}, \bar{\xi})$ such that $a(\bar{x}, \bar{\xi}) = 0$; if $|a(\bar{x}, \bar{\xi})| \geq \epsilon > 0$ then standard elliptic arguments yield that $F_{t \to h^{-1}\tau} Q_x U = O(h^s)$ for $|\tau| \leq \epsilon_1 = \epsilon_1(d, c, \epsilon) > 0$ provided the symbol of Q is supported in $\mathcal{V}$. Let us assume first that

$$|\nabla_\xi a(\bar{x}, \bar{\xi})| \geq c^{-1}; \tag{4.1}$$

without loss of generality one can assume that $\partial_{\xi_1} a(\bar{x}, \bar{\xi}) \geq \epsilon_0$; otherwise one can reach it by change of co-ordinates. Then singularities in the neighborhood of $(\bar{x}, \bar{\xi})$ propagate with velocity disjoint from 0 in the x_1-direction (one can obtain this from the classical Hamiltonian system) and since at $t=0$ all the singularities of U lie on $\{x = y, \xi = -\eta\}$ then for $0 \leq \pm t \leq T_0$ all the singularities of QU lie in $\{\mp(x_1 - y_1) \geq \pm \epsilon_0 t\}$; therefore there is no singularity of $QU|_{x=y}$ in $[-T_0, T_0] \setminus 0$ where $T_0 = T_0(d, c, \epsilon_0) > 0$ is small enough. Then $F_{t \to h^{-1}\tau} \chi_T' QU|_{x=y} = O(h^s)$ and $F_{t \to h^{-1}\tau} \chi_T' \operatorname{Tr} QU = O(h^s)$ for $\chi \in C_0^K([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $0 < T \leq T_0$; this estimate is uniform for T disjoint from 0. However, this result can be improved by means of dilatation method and the derived estimates are uniform for $h^{1-\delta} \leq T \leq T_0$ with arbitrarily small exponent $\delta > 0$ (this restriction is due to uncertainty principle). Therefore under the above condition the construction of the previous section provides us with $F_{t \to h^{-1}\tau} \bar{\chi}_T \operatorname{Tr} QU$ for $T = T_0$ (because intervals $[-T_0, -h^{1-\delta}]$ (or $[h^{1-\delta}, T_0]$) and $[-h^{\frac{1}{2}+\delta}, h^{\frac{1}{2}+\delta}]$ overlap for small $\delta > 0$).

This and the arguments at the end of the previous section yield immediately the complete asymptotics for spectral means with mollification parameter $L \geq h^{1-\delta}$.

The asymptotics of $\operatorname{Tr} QE(\tau_1, \tau_2)$ without mollification and with remainder estimate $O(h^{1-d})$ follows from the construction of section 3 (extended to interval $[-T_0, T_0]$ now) via the Tauberian theorem (see below). Moreover, one can replace condition (4.1) by condition

$$|\nabla_{x,\xi} a(\bar{x}, \bar{\xi})| \geq \epsilon_0. \tag{4.2}$$

Actually, if this condition is fulfilled one can always reach (4.1) by means of symplectic change of phase co-ordinates (and there is always implementation by unitary Fourier integral operator preserving trace but not restriction to the diagonal[8)]). Moreover, referring to above elliptic arguments one can replace (4.2) by

$$|a| + |\nabla_{x,\xi} a| \geq \epsilon_0 \qquad \text{in } \mathcal{V}. \tag{4.3}$$

Then we obtain the asymptotics of the above type for $|\tau_i| \leq \epsilon_1$ with a small enough constant $\epsilon_1 > 0$.[9)] Finally, the condition that $\mathcal{V}$ is small isn't necessary: one always can use an appropriate partition of unity.

This construction is done rigorously in [Ivrii 7.2]. There is also generalization to matrix operators and condition (4.3) is replaced by *microhyperbolicity* condition

$$\langle (\mathcal{T} a)(x, \xi) v, v \rangle \geq \epsilon_0 |v|^2 - c|a(x, \xi) v|^2 \qquad \forall v \tag{4.4}$$

for appropriate $\mathcal{T} \in T_{x,\xi}\mathcal{V}$ depending on (x, ξ) and such that $|\mathcal{T}| \leq 1$. Moreover this construction can be done near the boundary but with more sophisticated *microhyperbolicity* condition involving also the boundary operators [Ivrii 7.3]. One of the main statements obtained in [Ivrii 7.2] is the following

Theorem 4.1. *Let A be a self-adjoint h-pseudo-differential operator in $X = \mathbb{R}^d$ and let Q be a (fixed) h-pseudo-differential operator with the symbol supported in $\Omega \subset \{|x - \bar{x}| \leq c,\ |x - \bar{\xi}| \leq c\}$. Let at every point $(x, \xi) \in \Omega$ the microhyperbolicity condition (4.4) be fulfilled with $\mathcal{T} \in T_{(x,\xi)}T^*X$, $|\mathcal{T}| \leq 1$. Then the estimate*

$$|\operatorname{Tr} QE(\tau_1, \tau_2) - \varkappa_0 h^{-d}| \leq Ch^{1-d} \quad \forall \tau_1, \tau_2 \in [-\epsilon_1, \epsilon_1] \tag{4.5}$$

holds with

$$\varkappa_0 = (2\pi)^{-d} \int \operatorname{tr} q^0(x, \xi) \mathcal{E}(x, \xi, \tau_1, \tau_2) dx d\xi \tag{4.6}$$

where q^0 is the principal symbol of Q, $\mathcal{E}$ the spectral projector of $a(x, \xi)$ and $C = C(d, c, c')$, $\epsilon_1 = \epsilon_1(d, c, \epsilon_0)$; here c' is a constant in the routine smoothness conditions to symbol of Q.

5. Tauberian Theorem

The following Tauberian theorem (a variant of Tauberian theorem due

[8)] Therefore we cannot replace (4.1) by (4.2) in the case when we are interested in asymptotics without spatial mollification.

[9)] The condition (4.1) can be changed in a similar way.

to Hörmander) and its modification linked with Fourier transform plays the central role in the proof of our results:

Theorem 5.1. *Let $\nu(\tau)$ be a monotone non-decreasing function such that*

$$|\nu(\tau)| \le M'(|\tau|+1)^p \qquad \forall \tau \in \mathbb{R}. \tag{5.1}$$

Let $\chi \in C_0^K([-1,1])$ be a fixed function equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$.

(i) Let us assume that

$$|\int \widehat{\chi_T}(\tau-\tau')d_{\tau'}\nu(\tau')| \le M'h^s \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.2}$$

with $T \ge h^{1-\delta}$, $\delta > 0$. Then

$$|\nu(\tau)-\nu(0)| \le C'Mh^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}] \tag{5.3}$$

where $q = q(\delta,p)$, $K = K(s,\delta,p)$, $C' = C'(s,\delta,p,\epsilon,\chi)$;

(ii) Let us assume that

$$\int \widehat{\chi_T}(\tau-\tau')d_{\tau'}\nu(\tau) = \vartheta(\tau) \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.4}$$

with $h^{-p} \ge T \ge h^{1-\delta}$. Let $\phi \in C_0^K([-c,c])$ be a fixed function. Then

$$|\int (\nu(\tau)-\nu(\tau')-\Theta(\tau,\tau'))\phi(\tau')d\tau'| \le CMT^{-1}+C'M'h^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}] \tag{5.5}$$

where

$$\Theta(\tau,\tau') = h^{-1}\int_{\tau'}^{\tau} \vartheta(\tau'')d\tau'', \tag{5.6}$$

$C = C(\phi,\chi,\epsilon)$, $M = \sup_{[-\epsilon,\epsilon]}|\vartheta(\tau)|$ and K, q, C' are the same exponents and constants as before. Moreover, if

$$|\vartheta(\tau)| \le M_0 + M|\tau| \qquad \forall \tau \in [-\epsilon,\epsilon] \tag{5.7}$$

then

$$|\int (\nu(0)-\nu(\tau')-\Theta(0,\tau'))\phi(\tau')d\tau'| \le CM_0T^{-1}+CMhT^{-2}+C'M'h^{s-q} \qquad \forall \tau \in [-\frac{\epsilon}{2},\frac{\epsilon}{2}]. \tag{5.8}$$

Thus, one can see easily that in order to obtain a good remainder estimate in spectral asymptotics one needs to construct $\operatorname{Tr} QU(t)$ for large T and estimate $F_{t\to h^{-1}\tau}\chi_T(\tau)\operatorname{Tr} QU(t)$ in an appropriate way (the condition that Q is non-negative definite operator provides that $\nu(\tau) = \operatorname{Tr} QE(0,\tau)$ is monotone non-decreasing function; one can easily extend the final remainder estimate to an arbitrary h-pseudo-differential operator Q). In the proof of theorem 4.1 the microhyperbolicity condition is used twice: in order to go from $T = h^{\frac{1}{2}+\delta}$ to $T = \text{const} > 0$ and in order to estimate $F_{t\to h^{-1}\tau}\chi_T(\tau)\operatorname{Tr} QU(t)$ when it has been calculated by the method of successive approximations.

6. How to Improve Remainder Estimate in the Case of Non-Periodic Trajectories

The above Tauberian theorem yields that in order to improve the remainder estimate in the asymptotics one should increase T in our analysis. We treat only scalar operators in this and in the following subsections; certain generalizations can be found in [Ivrii 7.2,7.3]. In this section we consider the easiest case when all the singularities of $\operatorname{Tr} QU(t)$ in$[-T,T]$ lie in fact in $[-T',T']$ with $T' = h^{1-\delta}$ where T is either a large constant or even temperately large parameter; in view of section 4 one should prove that intervals $[-T,-T_0]$ and $[T_0,T]$ contain no singularity, where $T_0 > 0$ is an arbitrarily small constant (if T is a large parameter then one should prove that $\operatorname{Tr} QU(t)$ is uniformly negligible at these intervals). The properties of the trace yield that $\operatorname{Tr} QU(t)$ can be replaced by $\operatorname{Tr} QU(t)Q'$ in this analysis. Moreover, taking an adjoint operator we obtain $\operatorname{Tr} Q_1U(-t)Q'_1$ with $Q_1 = Q'^*$ and $Q'_1 = Q^*$. Therefore it is enough to consider only one of the intervals $[-T,-T_0]$ and $[T_0,T]$, This is quite different from the analysis of resonances.

It is well-known that in the scalar case singularities propagate along classical Hamiltonian trajectories. This means that all the trajectories of the length $\le T$ starting from supp Q' in the positive (or negative) direction don't meet the boundary and don't leave the zone where the coefficients of the operator are regular. If for $T_0 \le t \le T$ they are disjoint from supp Q then interval $[T_0,T]$ (or $[-T_0,T]$ respectively) contains no singularity of $QU(t)Q'$ and both intervals contain no singularity of its trace; in the latter case Q and Q' are supported in the neighborhood of the same point. Here and below a support of h-pseudo-differential operator means a support of its symbol. If we refer to classical results then "disjoint" means that distance is greater than some positive constant. However, it is possible to obtain the same result for "disjoint" meaning that the distance is greater than $\bar\gamma = h^{\frac{1}{2}-\delta}$ with an arbitrarily small exponent $\delta > 0$. Moreover , we can treat the case when T is a large parameter; in this case one should assume that $T \le h^{-\sigma}$, $|J| \le h^{-\sigma}$ and $|D^\alpha a_\beta| \le h^{-\sigma}$ along trajectories where a_β are coefficients of operator, J means a Jacobi matrix of the Hamiltonian flow and $\sigma = \sigma(d,\delta,s) > 0$ is a small enough exponent.

The easiest proof uses the Heisenberg' representation. Let us introduce $Q_t = U(-t)QU(t)$. To prove that $QU(t)Q'$ is negligible it is sufficient to prove thatQ_tQ' is negligible. Moreover,

$$D_tQ_t = -h^{-1}[A,Q_t] \tag{6.1}$$

and

$$Q_0 = Q. \tag{6.2}$$

Let us assume for a moment that Q_t is a h-pseudo-differential operator. Then Cauchy problem (6.1)-(6.2) yields a sequence of Cauchy problems for different terms of the symbol. We can prove under our hypothesis that this sequence of problems has a solution belonging to the appropriate symbol class. Then the quantization of this symbol $\tilde{Q}_t$ satisfies Cauchy problem (6.1)-(6.2) modulo a negligible operator. It is easy to show that $Q_t - \tilde{Q}_t$ is also negligible. Here the conjecture that Q_t was a pseudo-differential operator was used only in order to pass from the operator to its symbol but we can write the Cauchy problem for a symbol formally, and justify this conjecture a posteriori when we prove that the solution is an admissible symbol and therefore operator $\tilde{Q}_t$ is an appropriate operator.

Moreover, even the case when the trajectory meets the boundary can be treated under certain hypothesis (see [Ivrii 7.3]). Let us assume $\operatorname{diam}\operatorname{supp} Q \le \bar{\gamma} = h^{\frac{1}{2}-\delta}$ [10] Then the arguments of the previous sections yield the estimate

$$\Bigl|\int \phi(\tau)(\operatorname{Tr} QE(\tau,0) - \Theta(\tau,0))d\tau\Bigr| \le \\ Ch^{1-d}T^{-1}\int_{\Sigma_0\cap\mathcal{V}} d\mu_0 + Ch^{1-d+\sigma'}\bar{\gamma}^{2d-1} + C'h^s \tag{6.3}$$

where $\Theta(\tau',\tau)$ is given by (5.6) with $\vartheta(\tau) = F_{t\to h^{-1}\tau}\chi_T(t)\operatorname{Tr} QU(t)$, $\Sigma_\tau = \{(x,\xi) : a(x,\xi) = \tau\}$ is an energy surface in the phase space, $\mu_\tau = dxd\xi : da|_{\Sigma_\tau}$ is a natural density on Σ_τ, $\mathcal{V}$ is a $\bar{\gamma}$-neighborhood of $\operatorname{supp} Q$, $\sigma' = \sigma'(d,\delta) > 0$ is a small enough exponent and under weak conditions C' depends on T but under more restrictive conditions C' doesn't depend on T.

Let us take partition of unity of diameter $\bar{\gamma}$ in the neighborhood of $\operatorname{supp} Q_0$ where Q_0 is a h-pseudo-differential operator supported in a fixed ball in the phase space. Applying estimate (6.3) we obtain the estimate

$$\Bigl|\int \phi(\tau)(\operatorname{Tr} QE(\tau,0) - h^{-d}\varkappa_0(\tau,0) - h^{1-d}\varkappa_1(\tau,0))d\tau\Bigr| \le \\ C\sum_{0\le i\le n} h^{1-d}T_i^{-1}\int_{\Lambda_i\cap\mathcal{V}} d\mu_0 + Ch^{1-d+\sigma'} + C'h^s \tag{6.4}$$

where Λ_i $(i = 1,\dots,n)$ are closed subsets of Σ_0 on which appropriate conditions discussed above including the condition of non-periodicity

$$\operatorname{dist}((x,\xi),\Phi_t(x,\xi)) \ge \bar{\gamma} \tag{6.5}$$

[10] In our analysis x and ξ are of equal right and we should remember about uncertainty principle $\Delta x \cdot \Delta\xi \ge h^{1-\delta}$.

are fulfilled with T_i instead of T and $\Lambda_0 = \Sigma_0 \setminus (\Lambda_1 \cup \dots \cup \Lambda_n)$, $T_0 = 1$. Of course, one can take Λ_i depending on h and under appropriate conditions the right-hand expression of (6.4) is $o(h^{1-d})$ or even better (up to $O(h^{1-d+\sigma'})$).

7. How to Improve the Remainder Estimate in the Case of Periodic Trajectories

Let us consider the case when all the trajectories are periodic (or at least there is a domain Ω in the phase space such that all the trajectories starting in Ω remain there and are periodic). As before we consider the scalar case and we assume that

$$|\nabla a| \geq \epsilon_0 \qquad \text{in } \Omega. \tag{7.1}$$

It is well-known that under these conditions generic period is a function of the energy level (exceptional subperiodic trajectories are possible):

$$T(x,\xi) = T(a(x,\xi)) \qquad \forall (x,\xi) \in \Omega.$$

Replacement $A \to A_1 = f(A)$ yields $T(x,\xi) \to T_1(x,\xi) = T(x,\xi)/f'(a(x,\xi))$ where prime means the derivative here; taking $f'(\tau) = T(\tau)$ we obtain $T_1 \equiv 1$ (at least for $(x,\xi) \in \Omega$). On the other hand, spectral projectors of A and $f(A)$ are linked obviously. Therefore without loss of generality one can assume that

$$\Phi_t(\Omega) = \Omega, \qquad \Phi_1(x,\xi) = (x,\xi) \qquad \forall (x,\xi) \in \Omega. \tag{7.2}$$

It is well-known [3,10] that in this case

$$e^{ih^{-1}A}Q \equiv e^{iB}Q$$

provided Q is compactly supported in Ω. Here B is an h-pseudo-differential operator with the principal symbol

$$b(x,\xi) = T(x,\xi)^{-1} \int_0^{T(x,\xi)} a^s(\Phi_t(x,\xi))dt + \alpha$$

(with $T \equiv 1$ here but this formula is invariant under the above replacement) where a^s is the subprincipal symbol and $\alpha = \alpha_1 h^{-1} + \alpha_2$ is the Maslov' constant: $\alpha_1 = \text{action}/T$ and $4\alpha_2/\pi$ is the Maslov' index of closed trajectory ($\alpha, \alpha_1, \alpha_2$ don't depend on trajectory in our case). Without loss of generality one can assume that $\alpha = 0$. In fact, $A \to A + \mu$ yields $B \to B + h^{-1}\mu$ for constant μ.

In order to understand the role of B let us assume first that $B = 0$. Moreover, let us assume that $e^{ih^{-1}A} = I$ (from the heuristic point of view these assumptions are almost equivalent); then

$$e^{ith^{-1}A} = I \tag{7.3}$$

for $t \in \mathbb{Z}$ and therefore $e^{ith^{-1}A} = e^{it'h^{-1}A}$ for $t \in \mathbb{R}$ where t' is the fractional part of t. Therefore in order to construct $\operatorname{Tr} QU(t)$ on $\mathbb{R}$ it is sufficient to construct it on the interval $[-\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ with arbitrarily small $\epsilon > 0$ and then apply the partition of unity $1=\sum_{n\in\mathbb{Z}}\chi(t-n)$ on $\mathbb{R}$ with appropriate $\chi\in C_0^K([-\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon])$. However , this doesn't lead to better remainder estimates in semiclassical spectral asymptotics because one can estimate $|F_{t\to h^{-1}\tau}\bar{\chi}_T\operatorname{Tr}QU|$ only by CTh^{1-d} for large T and increasing T we gain nothing (see the Tauberian theorem). It is reasonable: equality (7.3) yields that $\operatorname{Spec}(A)\subset\mathbb{Z}$ and therefore eigenvalues of A are highly degenerated (with multiplicities $\asymp h^{1-d}$). In this case we can obtain complete asymptotics *inside* spectral gaps.

Let us consider a more general case and let us assume that

$$e^{ih^{-1}A}Q\equiv e^{i\eta B}Q \tag{7.4}$$

for all h-pseudo-differential operators Q supported in Ω. Here B is an h-pseudo-differential operator and $\eta\in(h^n,h^{\delta-1})$ is an additional parameter. A small parameter η can appear because first terms of "original" B vanish and large parameter η can appear because one perturbs A by $\eta hA'$.

One can then easily prove that $e^{inh^{-1}A}Q\equiv e^{in\eta B}Q$ for $n\in\mathbb{Z}$, $|n|\le\epsilon/h\eta$ and Q compactly supported in Ω where $\epsilon>0$ depends on $\operatorname{dist}(\operatorname{supp}Q,\partial\Omega)$ and that

$$e^{ith^{-1}A}Q\equiv e^{it'h^{-1}A}e^{it''\eta h^{-1}B}Q\qquad\forall t:|t|\le\frac{\epsilon}{h\eta} \tag{7.5}$$

for same Q and ϵ as before, where $t''=[t]h\eta$ and $t'=\{t\}$. Hence one can study a long-time propagation of singularities in this case: singularities propagate along Hamiltonian trajectories of a drifting with velocity $h\eta$ along Hamiltonian trajectories of b. The uncertainty principle yields that one can notice this drift only for $|h\eta t|\ge h^{1-\delta'}$. Let us assume that

$$|\nabla_\Sigma b|\ge\epsilon_0\qquad\text{at }\Sigma_\tau \tag{7.6}$$

where ∇_Σ means differential along Σ_τ. Then the periodicity of trajectories is destroyed for $|n|\ge h^{-\delta'}\eta^{-1}$. If $\eta\ge h^{-\delta}$ then the periodicity of trajectories is destroyed after one turn and if we assume that

(7.7) There is no subperiodic trajectory of a

then the singularity of $\operatorname{Tr}QU(t)$ located in a neighborhood of 0 is the only singularity in the interval $[-T,T]$ with $T=\epsilon/h\eta$. The Tauberian theorem yields then the standard semiclassical spectral asymptotics with the remainder estimate $Ch^{1-d}\eta^{-1}$.

Let us assume that $\eta\le h^{-\delta}$. Then (under condition (7.7)) singularities of $\operatorname{Tr}QU(t)$ in $[-T,T]$ are located in $[-T',T']$ where $T=\epsilon/h\eta$ and $T'=1/h^\delta\eta$. Then

$$|F_{t\to h^{-1}\tau}\chi_T\operatorname{Tr}QU|\le CT'h^{1-d} \tag{7.8}$$

and the Tauberian theorem yields the semiclassical spectral asymptotics with the remainder estimate $O(h^{2-d-\delta})$ and with additional term $h^{1-d}F(\tau,\tau/h)$ due to singularities of $\operatorname{Tr} QU(t)$ located in $[-T',T']$ and different from 0. Namely,

$$F(t,z)=(2\pi)^{-d}\int_{\Sigma_\tau}\Upsilon(z-\eta b)d\mu_\tau \tag{7.9}$$

where $\Upsilon(z)$ is 2π-periodic function on $\mathbb{R}$ equal to $\pi-z$ at $[0,2\pi)$. Moreover, under conditions (7.6) and (7.7) more accurate calculations yield the estimate (7.8) with $T'=1/\eta+1$ and we obtain the remainder estimate $O(h^{2-d})$. Thus the estimate

$$|\operatorname{Tr} QE(\tau',\tau)-\varkappa_0(\tau',\tau)h^{-d}\varkappa_1(\tau',\tau)h^{1-d}-F(\tau,\tau/h)h^{1-d}|\le Ch^{2-d}(1+\eta) \tag{7.10}$$

holds.

In this asymptotics the non-Weylian term $F(\tau,\tau/h)h^{1-d}$ is of the same order as the second Weylian term $\varkappa_1 h^{1-d}$. However, at intervals of the length $\asymp h$ oscillations of the non-Weylian term are of the same order as the oscillations of the principal term (at least in the situation described below).

The detailed analysis and generalizations can be found in [Ivrii 1.2]. In particular, conditions (7.6) and (7.7) are weakened there. Moreover, there is proved that under conditions (7.1),(7.2),(7.4) $\operatorname{Tr} QE(\tau',\tau)$ is negligible when τ' and τ belong to the same gap in the semiclassical approximation to spectrum. These gaps are defined (for $\eta\le\epsilon$ only) by the condition

$$|b(x,\xi,h)\eta-ih^{-1}\tau-2\pi m|\ge\epsilon\eta \qquad \forall m\in\mathbb{Z}. \tag{7.11}$$

In the case of τ and τ' belonging to different gaps complete asymptotics (with oscillating non-Weylian terms) is derived.

8. Eigenvalue Estimates and Asymptotics for Spectral Problems with Singularities

Singularity means the non-smoothness of the coefficients and (or) boundary, unboundedness (exit to infinity) of the domain X or of the classically allowed zone $\{x, V(x)\le\tau\}$ for the Schrödinger operator etc. For a sake of simplicity we consider only Schrödinger operator in dimension $d\ge 3$. It is well-known that in this case (under Dirichlet boundary condition)

$$N^-(A)\le c_0h^{-d}\int V_-^{\frac{d}{2}}dx. \tag{8.1}$$

This is the Lieb-Cwickel-Rozenblyum estimate and many other estimates of this are known for the Schrödinger operator and more general operators. Under certain conditions it is possible to combine this estimate and local semiclassical spectral asymptotics. Namely, let us consider the Schrödinger operator in $\mathbb{R}^d$

or in a domain $X \subset \mathbb{R}^d$ (in this case we refer to LSSA near boundary), $d \ge 2$. Let us assume that in X functions γ and ρ are given such that

$$\gamma > 0,\ \rho > 0,\ |\nabla \gamma| \le 1 \tag{8.2}$$

and in subdomain $X' = \{x \in X, \rho\gamma \ge h\}$ the following conditions are fulfilled:

$$y \in X', x \in B(y, \gamma(y)) \implies c^{-1} \le \frac{\rho(y)}{\rho(x)} \le c,\ |\nabla \rho| \le c\rho\gamma^{-1}, \tag{8.3}$$

$$|D^\alpha V| \le c\rho^2 \gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K \tag{8.4}$$

and

$$\begin{aligned} X \cap B(y, \gamma(y)) = \{x_k = \phi_k(x_{\hat{k}}) \cap B(y, \gamma(y)),\\ |D^\alpha \phi_k| \le c\gamma^{1-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \le K \end{aligned} \tag{8.5}$$

for some $k = k(y)$ where $x_{\hat{k}} = (x_1, \dots, x_{k-1}, \dots, x_{k+1}, \dots, x_d)$. Furthermore, let us assume that

(8.6) For $y \in X'$ on $\partial X \cap B(y, \gamma(y))$ either the Dirichlet or the Neumann condition is satisfied. Then

$$N^-(A) = \int \psi' e(x, x, -\infty, 0) dx + \int \psi'' e(x, x, -\infty, 0) dx$$

where $\psi' + \psi'' = 1$, ψ' and ψ'' are supported in (the closures of) $\{x \in X, \rho\gamma \ge \frac{5}{4}h\}$ and $\{x \in X, \rho\gamma \le \frac{6}{4}h\}$ respectively and

$$|D^\alpha \psi'| \le c\gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K.$$

Moreover, LSSA and dilatation-multiplication procedure yield estimates

$$\mathcal{N}^- - CR_1 \le \int \psi' e(x, x, -\infty, 0) dx \le \mathcal{N}^- + CR_1 + C' R_2$$

where

$$\mathcal{N}^- = (2\pi)^{-d} \omega_d \int \psi' V_-^{\frac{d}{2}} dx \tag{8.7}$$

is the Weylian approximation and

$$R_1 = h^{1-d} \int_{X' \cap \{V \le \epsilon\rho^2\}} \rho^{d-1} \gamma^{-1} dx, \tag{8.8}$$

$$R_2 = h^s \int_{X'} \rho^{-s} \gamma^{-d-s} dx, \tag{8.9}$$

$\epsilon > 0$ is arbitrary and C, C' depend on ϵ. Therefore

$$\mathcal{N}^- - CR_1 \le N^- \le \mathcal{N}^- + CR_1 + CR_2 + \int \psi'' e(x, x, -\infty, 0) dx$$

and the lower estimate is derived. In order to derive an upper estimate one should derive an upper estimate for $\int \psi'' e(x,x,-\infty,0)dx$. Let us consider operator A'' in the domain $X'' = \{x \in X, \rho\gamma \le 2h\}$, coinciding in $\{\rho\gamma \le 7h/4\}$ with A (taking in account boundary conditions). Let us assume that A'' is a self-adjoint operator in $L^2(X'')$ and that the estimate

$$N^-(A'' - tJ) \le R''(t+1)^n \qquad \forall t \ge 0 \tag{8.10}$$

holds for some n and some function $J > 0$ coinciding with ρ^2 in $X' \cap X''$. Then it is possible to prove the following estimate:

$$\int \psi'' e(x,x,-\infty,0)dx \le C''R'' + C'R_2$$

where C'' depends on n and C', C'' doen't depend on the choice of J. Therefore the final estimates are

$$\mathcal{N}^- - CR_1 \le N^- \le \mathcal{N}^- + CR_1 + CR_2 + C''R''. \tag{8.11}$$

Moreover, if $d \ge 3$ and Dirichlet boundary condition is given on $\partial X \cap \{\rho\gamma \le 2h\}$ then one can take $A'' = A$ in X'' with the Diriclet boundary condition on $\partial X''$ and estimate (8.1) yields for appropriate J that

$$R'' = C\int_{X''} V_-^{\frac{d}{2}} dx \tag{8.12}$$

There are some useful modifications. Moreover, one can apply LSSA with more accurate remainder estimates (under restrictions to Hamiltonian trajectories). Finally, other improvements also can be done.

The same idea of splitting works even for operators non semi-bounded from below (for example, for the Dirac operator). In this case it is very useful to reduce the original problem to some modified problems via the Birman-Schwinger principle. This principle is very useful for semi-bounded operators too.

The upper and lower estimates for a number of negative eigenvalues or eigenvalues lying in an interval are very useful. Let us take $h = 1$ (however, in the deduction $h_{\mathrm{eff}} \ll 1$ in some balls). Let us consider operator depending on other parameter(s). Then estimates derived here yield asymptotics with respect to new parameters. Some examples of this type were treated in as in section 2. A number of more sophisticated examples can be found in [Ivrii 5,6,7.7]

9. Generalizations. Non-Weylian Asymptotics

I list here some situations when the results of section 8 are not applicable and where non-Weylian spectral asymptotics arise.

1. Schrödinger and Dirac operators with the strong magnetic field. Let us start from local asymptotics. There are two parameters now: $h \ll 1$ and a coupling parameter $\mu \gg 1$. We use the same ideas as before: the hyperbolic operator method including the construction of $\operatorname{Tr} QU$ at short time interval by successive approximations and then extension to larger interval based on certain version of microhyperbolicity. Depending on the problem at hand, the microlocal canonical form of operators in question is used in both parts of analysis in the second one only.Thus the successive approximations method is applied either to the original operator or to the reduced one. For example, if the magnetic intensity is constant, $d = 2,3$ and $|\nabla V| \geq \epsilon_0$ the first (second) approach works for $1 \leq \mu \leq h^{\delta-1}$ ($h^{-\delta} \leq \mu$ respectively). The asymptotics derived by this method are different and contain many terms which one can calculate only "in principle". However, a comparison of these two asymptotics in zone where both of them hold provides much simpler and more effective answer.

When local semiclassical spectral asymptotics are derived we generalize them by dilatation-multiplication method. Then in order to attack global problems we use partition of unity, treat the singular zones and derive eigenvalue estimates. Finally, we consider operators depending on parameter(s) and derive eigenvalue asymptotics with respect to these parameter(s).

Operators in domains with thick cusps. Spectral asymptotics (with accurate remainder estimates) for operators in domains with thin cusps are due to results of section 8. However, if the cusp is thick, the remainder estimate is not so good or we even may fail to derive asymptotics at all. In this case we change the co-ordinates and transform our cusp to the cylinder. Then we treat the reduced operator as d'-dimensional one with operator-valued coefficients where d' is dimension of the cusp (usually $d' = 1$). We can obtain local semiclassical spectral asymptotics for such operator by methods described above. Moreover, we can use this approach either only in order to extend the time interval (so successive approximations method is applied to original operator) or from the beginning of our analysis. The first (second) approach is useful in the part of cusp near to (far from respectively) origin and we can split these asymptotics.

The same ideas work for the Schrödinger operator in $\mathbb{R}^d$ when V fails to tend to $+\infty$ along some directions. If the "canyons" in the $d+1$-dimensional graph of function $V(x)$ are narrow then results of section 8 yield the desired answer. Otherwise the similar operators with respect to part of variables in the auxiliary Hilbert space should be treated etc.

The same ideas work also in the case when the operator degenerates on a symplectic manifold.

3. Riesz means. In the framework of section 1 the asymptotics of spectral Riesz means can be treated and the remainder estimate $O(h^{1+\vartheta-d})$ can be obtained in local semiclassical spectral asymptotics where ϑ is the order of the Riesz mean. Moreover, under appropriate condition to Hamiltonian flow this remainder estimate can be improved. However, if we apply the approach of section 8 in order to treat the case when V has singularities then we may or may not be able to recover the remainder estimate obtained in the smooth case. It depends on d, ϑ and order of singularity. For example, for Coulomb-like singularity and $\vartheta = 1$ (the most interesting case from the physical point of view) the remainder estimate is $O(h^{2-d})$ for $d \ge 5$ but we obtain $O(h^{-2}\log h)$ for $d = 4$ and $O(h^{-2})$ for $d = 2, 3$. However, under appropriate assumptions we can obtain asymptotics with remainder estimate $O(h^{2-d})$ or even better for $d = 2, 3, 4$. The main idea here is to treat the operator in question as a perturbation of the Schrödinger operator with homogeneous potential and estimate the difference between $\int e_\vartheta(x, x, -\infty, 0)\psi(x/r)dx$ for perturbed and unperturbed operators where subscript ν means that ϑ-th order Riesz means is calculated, $\psi \in C_0^K(\mathbb{R}^d)$ is a fixed function equal 1 near 0 and r is an appropriate parameter. This estimate is based on equality

$$\operatorname{Tr}(E_\vartheta(\tau; A_1) - E_\vartheta(\tau; A_0)) = -\vartheta \int_0^1 \operatorname{Tr} E_{\nu-1}(\tau; A_t) B dt$$

where $A_t = A_0 + Bt$ and $\vartheta \ge 1$; for $0 < \vartheta < 1$ some interpolation arguments are used. On the other hand, for

$$\int (\psi(x) - \psi(\frac{x}{r})) e_\vartheta(x, x, -\infty, 0) dx$$

we apply the local semiclassical spectral asymptotics approach (possibly with remainder estimates improved by very accurate treatment of propagation of singularities near origin). The details and generalizations can be found in [Ivrii 7.9, Ivrii& Sigal 8.1,8.2].

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