Approximation of holomorphic functions in Banach spaces admitting a Schauder decomposition

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Abstract. Let *X* be a complex Banach space. Recall that *X* admits a *finite-dimensional Schauder decomposition* if there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of *X*, such that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for every *n*. The finite-dimensional Schauder decomposition is said to be *unconditional* if, for every $x \in X$, the series $x = \sum_{n=1}^{\infty} x_n$, which represents *x*, converges unconditionally, that is, $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers. For short, we say that *X* admits an unconditional F.D.D.

We show that if X admits an unconditional F.D.D. then the following Runge approximation property holds:

(R.A.P.) There is $r \in (0, 1)$ such that, for any $\epsilon > 0$ and any holomorphic function f on the open unit ball of X, there exists a holomorphic function h on X satisfying $|f - h| < \epsilon$ on the open ball of radius r centered at the origin.

Mathematics Subject Classification (2000): 32H02.

1. Introduction

The Runge approximation problem (R.A.P.) of a holomorphic function in a complex Banach space X plays a crucial role in the study of the vanishing of the sheaf cohomology groups $H^q(\Omega, \mathcal{O})$, where $\Omega \subset X$ is an open subset, \mathcal{O} is the sheaf of germs of holomorphic functions on X, and $q \ge 1$. (See for instance [5]). This problem has been studied in particular by L. Lempert [4, 6], B. Josefson [3] and I. Patyi [9]. L. Lempert and B. Josefson show that the Runge approximation property (R.A.P.) holds if X has an unconditional Schauder basis while I. Patyi proves it for ℓ_1 -sum Banach spaces. In this paper, we consider an important class of Banach spaces, namely those which admit an unconditional finite-dimensional Schauder decomposition (F.D.D.). (See [8]). We refer the interested reader to [8, page 51],

The author was partially supported by Swiss NSF Grant 2100-063464.00/1 Received June 27, 2005; accepted in revised form December 27, 2005.

for instance, for examples of Banach spaces which admit an unconditional F.D.D. but which do not have an unconditional Schauder basis.

We have the following theorem.

Theorem 1.1. Let (X, ||||) be a complex Banach space admitting an unconditional F.D.D. Then there exists an equivalent norm $||||_1$ on X such that the following holds. Let $B_1(R) \subset X$ be the ball of radius R > 0 about the origin associated to $||||_1$, and let f be a holomorphic function in $B_1(R)$. Then for any $r \in (0, R)$ and any $\epsilon > 0$, there exists a holomorphic function h in X satisfying $|f - h| < \epsilon$ in $B_1(r)$.

The following corollary is immediate.

Corollary 1.2. *Let X be a complex Banach space admitting an unconditional* F.D.D. *Then the Runge approximation property* (R.A.P.) *holds.*

Notice that among the Banach spaces which do not admit an unconditional F.D.D., there are the spaces C[0, 1] and $L^1(0, 1)$ ([10]). The Runge approximation problem (R.A.P.) is still open, in particular, for these two spaces. It would be interesting to know if the Runge approximation property (R.A.P.) holds for C[0, 1] since it is well known that every separable Banach space is isometric to a subspace of C[0, 1].

ACKNOWLEDGEMENTS. I would like to thank L. Lempert for many helpful discussions and comments on this paper. I am also grateful to the referee for several useful remarks.

2. Preliminaries

Let *X* be a complex Banach space. Recall that a function $f : U \subset X \longrightarrow \mathbb{C}$, where *U* is an open subset of *X*, is *holomorphic* if *f* is continuous on *U*, and $f_{|U \cap X_1|}$ is holomorphic, in the classical sense, as a function of several complex variables, for each finite-dimensional subspace X_1 of *X*. (See [1].)

The following known lemma gives a criterion of compactness in X (See [2], IV.5.4).

Lemma 2.1. Let T_n be a uniformly bounded sequence of linear operators in X. If $\lim_n T_n x = x$ for every $x \in X$, then this limit exists uniformly on any compact set. Conversely, if $\lim_n T_n x = x$ uniformly for x in a bounded set K, and if, in addition, $T_n\{x \mid |x| \le 1\}$ is compact for each n, then K is compact.

Assume now that *X* admits an unconditional F.D.D. $\{X_n\}_{n=1}^{\infty}$. Let $x = \sum_{n=1}^{\infty} x_n$, $x_n \in X_n$, be the unique representation of *x*. It is known that for every sequence of complex numbers $\theta = \{\theta_n\}, |\theta_n| \le 1, n \in \mathbb{N}$, the operator M_{θ} defined by

$$M_{\theta} \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \theta_n x_n$$

is a bounded linear operator. The (finite) constant $\sup_{\theta} ||M_{\theta}||$ is called the *unconditional constant* of the decomposition. (See [8] and [10] for details). It is clear that one can always define on X an equivalent norm $||\|_1$ so that the unconditional constant becomes 1. (Take $||x||_1 = \sup_{\theta} ||M_{\theta}x||$). In other words, we have

$$\left\|\sum_{n=1}^{\infty}\theta_n x_n\right\|_1 \le \left\|\sum_{n=1}^{\infty}x_n\right\|_1, \quad |\theta_n| \le 1, \ n \in \mathbb{N}.$$
(2.1)

Using Lemma 2.1 and (2.1), one can prove the following proposition.

Proposition 2.2. Let X be a complex Banach space admitting an unconditional F.D.D. Let $||||_1$ be the equivalent norm for which the unconditional constant is one. Let $B_1(R) \subset X$ be the ball of radius R > 0 about the origin associated to $||||_1$. Then the following holds.

- (1) $M_{\theta}(B_1(R))$ is relatively compact in $B_1(R)$ for any sequence of complex numbers $\theta = \{\theta_n\}, |\theta_n| < 1$, which converges to 0.
- (2) For any compact $K \subset B_1(R)$, there exist a sequence of complex numbers $\theta = \{\theta_n\}, |\theta_n| < 1$ which converges to 0, and a compact $L \subset B_1(R)$ so that $M_{\theta}L = K$.

3. Description of the proof of Theorem 1.1

The proof given below has been inspired by a talk given by Lempert ([7]) on complex Banach spaces admitting an unconditional Schauder basis, and differs from the proofs given in [6] and [3]. Let $|| ||_1$ be the equivalent norm in X given by (2.1) and let $B_1(R) \subset X$ be the ball of radius R > 0 about the origin associated to $|| ||_1$. Let $x \in X$ be given by its unique representation $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for every *n*. As in [6], we restrict our attention to the bounded linear operators $M_{e^{2\pi it}}$, $t = \{t_n\}, t_n \in \mathbb{R}, e^{2\pi i t} = \{e^{2\pi i t_n}\}$. $M_{e^{2\pi i t}}$ is clearly an isometry of X because of (2.1).

For $m \in \mathbb{N}$, let $[\sqrt{m}]$ be the integer part of \sqrt{m} . As in [7], we define

$$T(m) = \left\{ e^{2\pi i t}, \ t = \{t_n\}, \ t_n \in \mathbb{R}, \ t_n = 0 \text{ for } n > \left[\sqrt{m}\right] \right\}$$
(3.1)

and

$$K(m) = \left\{ k = \{k_n\}, \ k_n \in \mathbb{N} \cup \{0\}, \ k_n = 0 \text{ for } n > [\sqrt{m}], \ \sum_{n=1}^{\lceil \sqrt{m} \rceil} k_n \le m \right\} .$$
(3.2)

Definition 3.1. Let g be a holomorphic function on $B_1(r)$, r > 0. We say that g is homogeneous of degree a in x if $g(\lambda x) = \lambda^a g(x)$ for $\lambda \in \mathbb{C}$, $|\lambda| \le 1$. We say that g is homogeneous of degree b in x_N if $g(M_{\theta_\lambda} x) = \lambda^b g(x)$, where $\theta_\lambda = \{\theta_{\lambda n}\}$, $\theta_{\lambda n} = \lambda$ if n = N, $\theta_{\lambda n} = 1$ if $n \ne N$, $\lambda \in \mathbb{C}$, $|\lambda| \le 1$.

Let *f* be a holomorphic function on $B_1(R)$. For $k \in K(m)$ and $e^{2\pi i s} \in T(m)$, one can define the following holomorphic functions on $B_1(R)$

$$f_m(x) = \int_{S^1} f(e^{2\pi i t} x) e^{-2\pi i m t} dt, \qquad (3.3)$$

$$f^{k}(x) = \int_{T(m)} f(M_{e^{2\pi i s}}x)e^{-2\pi i k.s} ds, \qquad (3.4)$$

where dt (respectively ds) is the normalized Haar measure on the unit circle S^1 (respectively on T(m)).

Remark 3.2. f^k is homogeneous of degree k_n in x_n , for $n \leq \lfloor \sqrt{m} \rfloor$ while f_m is homogeneous of degree m in x.

Consider now the formal series associated to f

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k .$$
(3.5)

Definition 3.3. The formal series associated to f given by (3.5) is called the Josefson series.

It is clear that when dim $X < \infty$, the Josefson series converges to f, uniformly on compact sets of $B_1(R)$. As in [7], we set for $k \in K(m)$ and for any sequence of complex numbers $\theta = \{\theta_n\}, 0 \le \theta_n < 1$, which converges to 0,

$$\theta^{(k)} = \prod_{n \le \lfloor \sqrt{m} \rfloor} \theta_n^{k_n} \left(\max_{n > \lfloor \sqrt{m} \rfloor} \theta_n \right)^{m - |k|}, \tag{3.6}$$

where $|k| = k_1 + \dots + k_{[\sqrt{m}]}$.

Proposition 3.4. Let f be a holomorphic function on $B_1(R)$. Then the following inequality holds

$$\overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\theta^{(k)} | (f_m)^k |_{B_1(R)} \right)^{\frac{1}{m}} < 1$$
(3.7)

for any sequence of complex numbers $\theta = \{\theta_n\}, 0 \le \theta_n < 1$, which converges to 0.

Proof. By Proposition 2.2, $M_{\theta}(B_1(R))$ is relatively compact in $B_1(R)$ for any sequence of complex numbers $\theta = \{\theta_n\}, 0 \le \theta_n < 1$, which converges to 0. Therefore f is bounded on $M_{\theta}(B_1(R))$. Using Remark 3.2 and the fact that $|(f_m)^k|$ on $M_{\theta}(B_1(R))$ is bounded by |f| on $M_{\theta}(B_1(R))$, we easily conclude that

$$\overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\theta^{(k)} | (f_m)^k |_{B_1(R)} \right)^{\frac{1}{m}} \le 1.$$
(3.8)

Replacing θ by $\lambda\theta$, for some good choice of a real number $\lambda > 1$, we see that the inequality (3.8) holds as a strict inequality. This achieves the proof of Proposition 3.4.

We have the following lemma whose proof is left to the reader.

Lemma 3.5. Let K(m) be given by (3.2), and let $\lambda \in \mathbb{R}$, with $\lambda < 1$. Then the series given by

$$\sum_{m=0}^{\infty} |K(m)||\lambda^m|$$

is convergent.

Proposition 3.6. Let h_m be a holomorphic function on $B_1(R)$ homogeneous of degree *m* in *x*. Suppose that for any sequence of complex numbers $\theta = \{\theta_n\}, 0 \le \theta_n < 1$, which converges to 0, and for any set *L* relatively compact in $B_1(R)$, the following holds

$$\overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\theta^{(k)} | (h_m)^k |_L \right)^{\frac{1}{m}} < 1.$$
(3.9)

Then

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (h_m)^k \tag{3.10}$$

converges, uniformly on compact sets of $B_1(R)$.

Proof. Let $K \subset B_1(R)$ be a compact set. Using Proposition 2.2, we can find $\theta = \{\theta_n\}$, with $\theta_n = \epsilon$, $0 \le \epsilon < 1$, for $n > \lfloor \sqrt{m} \rfloor$, and a compact set $L \subset B_1(R)$, such that $K \subset M_{\theta}L$. Using Remark 3.2 and the assumption, we conclude that there exists $\lambda < 1$ such that for *m* large enough and $k \in K(m)$,

$$|(h_m)^k|_K < \lambda^m \,. \tag{3.11}$$

Using (3.11) and Lemma 3.5, we get that

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} |(h_m)^k|$$
 (3.12)

converges uniformally on *K*. This achieves the proof of Proposition 3.6.

Proposition 3.7. Let h_m be a holomorphic function on $B_1(R)$ homogeneous of degree *m* in *x*. Suppose that for any sequence of complex numbers $\theta = \{\theta_n\}, 0 \le \theta_n < 1$, which converges to 0, the following holds

$$\overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\theta^{(k)} | (h_m)^k |_{B_1(R)} \right)^{\frac{1}{m}} < 1.$$
(3.13)

Suppose also that there exists t > 1 such that

$$|(h_m)^k|_{B_1(R)} = \begin{cases} 0, \ or \\ \ge t^m \end{cases}$$
(3.14)

for $k \in K(m)$. Then

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (h_m)^k \tag{3.15}$$

is a holomorphic function on X.

Proof. Since h_m is a homogeneous holomorphic function, it implies that h_m extends holomorphically to X. Using Proposition 3.6, we see that it is enough to show that (3.9) holds for any set L relatively compact in X.

For *L* relatively compact in *X*, we choose $\delta > 0$ such that $\delta L \subset B_1(R)$.

Let $\alpha > 1$. Then there exits $\Theta = \{\Theta_n\}, \ 0 \le \Theta_n < 1$, which converges to 0, such that

$$\overline{\lim}_{m \to \infty} \sup_{k \in K(m)} \left(\theta^{(k)} | (h_m)^k |_L \right)^{\frac{1}{m}} \le$$
(3.16)

$$\delta^{-1} \overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\Theta^{(k)} | (h_m)^k |_{B_1(R)} \right)^{\frac{\alpha+1}{m}} \overline{\lim_{m \to \infty}} \sup_{\substack{k \in K(m), \\ |(h_m)^k |_{B_1(R)} \neq 0}} \left(|((h_m)^k)^{-1} t^m |_{B_1(R)} \right)^{\frac{\alpha}{m}} t^{-\alpha}.$$

Using Proposition 3.4 and the assumption, we conclude that

$$\overline{\lim_{m \to \infty}} \sup_{k \in K(m)} \left(\theta^{(k)} | (h_m)^k |_L \right)^{\frac{1}{m}} < \delta^{-1} t^{-\alpha} .$$
(3.17)

Using (3.17) for α large enough, we obtain (3.9). This achieves the proof of Proposition 3.7.

Proof of Theorem 1.1. Let f be a holomorphic function in $B_1(R)$. Since X admits an unconditional F.D.D., f and $\sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k$ agree on a dense subset of $B_1(R)$. By Proposition 3.4 and Proposition 3.6, it is then clear that $f = \sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k$. For t > 1 and $M \in \mathbb{N}$, we put

$$h_{t,M} = \sum_{m>M} \sum_{\substack{k \in K(m), \\ |(f_m)^k|_{B_1(R) \ge t^m}}} (f_m)^k + \sum_{m=0}^M \sum_{k \in K(m)} (f_m)^k.$$
(3.18)

By Proposition 3.7, $h_{t,M}$ is holomorphic on *X*. One can check that for $r \in (0, R)$, and $\epsilon > 0$, there exist t > 1 and *M* such that $|f - h_{t,M}| < \epsilon$ in $B_1(r)$. This achieves the proof of Theorem 1.1.

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