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Compensated compactness and one-dimensional elastodynamics

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1. - Introduction and Statement of Results

One of the standard methods of solving nonlinear equations is to replace the equation with another one involving a regularizing parameter ϵ, show that this equation has a solution and then pass to the limit $\epsilon \downarrow 0$. One of the difficulties is then to show that one obtains a solution in the limit. Usually it is not too hard to show that the approximating solutions converge weakly in some sense, but if the equation is nonlinear, this is not of much use because if $u_\epsilon \to u$ weakly then $f(u_\epsilon)$ need not converge weakly toward $f(u)$. If one knew that $u_\epsilon \to u$ in some stronger sense this would not be a problem in most cases. Thus the issue is one of lack of compactness.

In the context of certain conservation laws, in particular the equations of one-dimensional elastodynamics of the form

$$\begin{aligned} w_t &= v_x, \\ v_t &= \sigma(w)_x, \end{aligned} \tag{1}$$

there is successful theory of how to, under certain assumptions, compensate for this lack of compactness with other reasonable assumptions that can be shown to hold true in many cases. Luc Tartar invented the method which first appeared in [11] (see also [12]) using a crucial lemma due to Murat [7]. DiPerna [2] was the first to find out how to use the method for systems (as in (1)) and in [10] a more abstract question related to Tartar's method was considered. Later developments can be found in [5] and [6]. The lecture notes [3] are useful as a general reference.

The purpose of this paper is to show that this theory works for conservation laws of the form (1) under weaker hypotheses on σ (concerning the zeroes of the second derivative) than those used in e.g. [2] and [10]. Moreover, these

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assumptions are not too restrictive in terms of what one usually has to assume of the nonlinearity in order to obtain the apriori bound, see e.g. [1], although there is cleary still room for improvements. Note that these results are applicable to equations that are perturbations of (1) as well, see [8].

Recall that a continuously differentiable function $\eta : \mathbb{R}^2 \to \mathbb{R}$ is an entropy for the conservation law (1) with corresponding entropy flux q if

$$\frac{\partial}{\partial t}\eta(w(x,t),v(x,t)) + \frac{\partial}{\partial x}q(w(x,t),v(x,t)) = 0,$$

for every smooth solution (w,v) of (1). Such pairs (η,q) are obtained as solutions of the system of equations

$$\begin{aligned} q_w(w,v) &= -\sigma'(w)\eta_v(w,v), \\ q_v(w,v) &= -\eta_w(w,v), \end{aligned} \qquad w,\ v \in \mathbb{R}. \tag{2}$$

THEOREM. *Assume that*

(i) $\sigma \in C^1(\mathbb{R};\mathbb{R})$ *is such that* $\inf_{u\in\mathbb{R}}\sigma'(u) > 0$ *and there exists a point* $w_\diamond \in \mathbb{R}\cup\{\pm\infty\}$ *such that* σ' *is nonincreasing on* $(-\infty, w_\diamond)$ *and nondecreasing on* $(w_\diamond,\infty)$.

(ii) *If the point* $w_\diamond$ *in* (i) *cannot be chosen to be* $\pm\infty$, *then it is unique,* $\sigma \in C^2(\mathbb{R};\mathbb{R})$, *and* $(w - w_\diamond)\sigma''(w) > 0$ *on the interval* $(w_\diamond, w_\diamond+\delta)$ *or on the interval* $(w_\diamond-\delta, w_\diamond)$ *for some* $\delta > 0$.

(iii) w_ϵ, $v_\epsilon \in L^\infty(\mathbb{R}\times(0,\infty);\mathbb{R})$ *for each* $\epsilon \in (0,1)$ *and*

$$\sup_{\epsilon\in(0,1)} (\|w_\epsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))} + \|v_\epsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))}) < \infty.$$

(iv) *For every continuously differentiable entropy-entropy flux pair* (η,q) *of* (1) *and for each bounded open set* $U \subset \mathbb{R}\times(0,\infty)$ *the functions* $\left\{\frac{\partial}{\partial t}\eta(w_\epsilon,v_\epsilon) + \frac{\partial}{\partial x}q(w_\epsilon,v_\epsilon)\right\}_{\epsilon\in(0,1)}$ *belong to a compact set in* $H^{-1}(U;\mathbb{R})$.

Then there exist functions w *and* $v \in L^\infty(\mathbb{R}\times(0,\infty);\mathbb{R})$ *and some sequence* $\{\epsilon_j\}_{j=1}^\infty$ *such that* $w_{\epsilon_j} \to w$, $\sigma(w_{\epsilon_j}) \to \sigma(w)$, *and* $v_{\epsilon_j} \to v$ *weakly-∗ in* $L^\infty(\mathbb{R}\times(0,\infty);\mathbb{R})$ *as* $\epsilon_j \downarrow 0$. *Moreover, if* σ' *is not constant on any open interval, then* $w_{\epsilon_j} \to w$, *and* $v_{\epsilon_j} \to v$ *pointwise in* $\mathbb{R}\times(0,\infty)$.

In [2] and [10] it is, instead of (ii), assumed that σ is twice continuously differentiable and σ'' vanishes at one point at most, in which case (ii) must clearly be satisfied.

In [4] it is shown how one can construct a function σ which satisfies (i) (with $w_\diamond = 0$) and is such that $\sigma \in C^2((-\infty,0)\cup(0,\infty);\mathbb{R})$ and $w\sigma''(w) > 0$ when $w \neq 0$, but $\lim_{w\downarrow 0}\sigma''(w) \neq \lim_{w\uparrow 0}\sigma''(w)$, so that (iii) and (iv) do not imply the second conclusion of the theorem.

The boundedness assumption (iii) could easily be replaced by the assumption that w_ϵ, $v_\epsilon \in L^\infty_{\text{loc}}(\mathbb{R} \times (0,\infty);\mathbb{R})$ for each $\epsilon \in (0,1)$ and that one has $\sup_{\epsilon\in(0,1)}(\|w_\epsilon\|_{L^\infty(U)} + \|v_\epsilon\|_{L^\infty(U)}) < \infty$ for every bounded open set $U \subset \mathbb{R}\times(0,\infty)$, if a corresponding small change is made in the statement of the conclusion.

One sees from the proof below that if σ' is strictly decreasing on $(-\infty, w_\diamond)$ and strictly increasing on $(w_\diamond, \infty)$, then one can find a subsequence so that w_{ϵ_j} and v_{ϵ_j} converge pointwise almost everywhere. Moreover, if one can take δ in (ii) to be infinite, then it is not necessary to assume that there is only one point $w_\diamond$ for which (i) holds.

When proving that the functions w_ϵ and v_ϵ are bounded and when deciding which of perhaps many weak solutions of (1) is the physically relevant one, it is in most cases of crucial importance that the entropies one studies are convex. In this paper, the entropies are not necessarily convex.

2. - Proof

It follows immediately from (iii) that there exist a sequence $\{\epsilon_j\}$ and functions w and $v \in L^\infty(\mathbb{R}\times(0,\infty);\mathbb{R})$ so that $w_{\epsilon_j} \to w$ and $v_{\epsilon_j} \to v$ weakly-$*$ in $L^\infty(\mathbb{R}\times(0,\infty);\mathbb{R})$ as $\epsilon_j \downarrow 0$.

By the Div-Curl Lemma (see [3, p. 53] or [7]) we conclude that if μ denotes the Young measure at some point (x,t) (see [3, p. 15]) associated with some subsequence $(w_{\epsilon_j}, v_{\epsilon_j})$ then it follows (for almost every (x,t)) that

$$\begin{aligned}
&\int_\Omega (\eta(w,v)q^*(w,v) - q(w,v)\eta^*(w,v))\mu(dwdv) \\
&\qquad = \int_\Omega \eta(w,v)\mu(dwdv) \int_\Omega q^*(w,v)\mu(dwdv) \\
&\qquad - \int_\Omega q(w,v)\mu(dwdv) \int_\Omega \eta^*(w,v)\mu(dwdv),
\end{aligned} \tag{3}$$

where (η, q) and (η^*, q^*) are two continuously differentiable entropy-entropy flux pairs, see [3, p. 60].

Suppose for the moment that we have already shown that it follows from (3) that σ' is constant on an interval containing the set $\{w \in \mathbb{R} \,|\, (w,v) \in \text{supp}(\mu)\}$.

We know that at the points where the support of the Young measure μ consists of a single point the functions w_{ϵ_j} and v_{ϵ_j} converge pointwise, see [3, p. 16]. For the other points we deduce by a similar argument that the distance from $w_{\epsilon_j}(x,t)$ to an interval where σ' is constant tends to 0. Because the sequence $\{w_{\epsilon_j}\}$ converges weakly in $L^2(E;\mathbb{R})$ for every measurable bounded subset E of $\mathbb{R}\times(0,\infty)$ as well, there are convex combinations of this sequences that converge in the norm of $L^2(E;\mathbb{R})$ (see, [9, Thm. 3.13]) and hence some subsequence of these convex combinations that converges pointwise. But if for some point (x,t) the distance from the point $w_{\epsilon_j}(x,t)$ to an interval where σ' is constant tends to 0, then

$$\lim_{k\to\infty}\sum_{j=k}^{\infty} c_{k,j}\sigma(w_{\epsilon_j}(x,t)) - \sigma\left(\sum_{j=k}^{\infty} c_{k,j}w_{\epsilon_j}(x,t)\right) = 0,$$

where $\sum_{j=k}^{\infty} c_{k,j} = 1$ and thus we can conclude that the weak-$*$ limit of $\sigma(w_{\epsilon_j})$ must equal $\sigma(w)$, which, of course, is the crucial point in the statement of the theorem.

In the remaining parts of the proof, in view of the above argument, we only have to show that it follows from (3) that σ' is constant on an interval containing the set $\{w \in \mathbb{R}\,|(w,v) \in \operatorname{supp}(\mu)\}$, or, in particular, that the support of μ consists of a single point. In this argument we may and shall without loss of generality assume that we have $w_\diamond = 0$ unless $|w_\diamond| = \infty$.

Although the entropy-entropy flux pairs that we shall consider and the first steps of the proof are almost standard ones, we shall for completeness and notational convenience describe them in fairly close detail. The entropies are of the form

$$\eta_{\alpha,\beta,k}(w,v) = \mathrm{e}^{\alpha k\int_0^w \psi_{\alpha k}(s)\mathrm{d}s+\beta k v}, \quad w,\ v \in \mathbb{R},$$

and the corresponding entropy fluxes are

$$q_{\alpha,\beta,k} = -\alpha\beta\psi_{\alpha k}(w)\eta_{\alpha,\beta,k}(w,v), \quad w,\ v \in \mathbb{R}, \tag{4}$$

where $k > 0$ and $\alpha,\ \beta \in \{-1,1\}$. In order for (2) to be satisfied the function ψ_κ must be a solution of the differential equation

$$\psi_\kappa'(w) + \kappa\psi_\kappa^2(w) = \kappa\sigma'(w). \tag{5}$$

(Note that with this formulation of the entropy-entropy flux pairs the expansions usually employed are in way hidden in the relation (5)). We require that ψ_κ satisfies the initial condition

$$\psi_\kappa(-M-1) = \sqrt{\sigma'(-M-1)}, \text{ if } \kappa > 0,$$

$$\psi_\kappa(M+1) = \sqrt{\sigma'(M+1)}, \text{ if } \kappa < 0,$$

where M is some sufficiently large number so that the support of μ is contained in $[-M,M]\times[-M,M]$. With this initial condition (5) has a global solution.

Next, we investigate how ψ_κ behaves when $|\kappa| \to \infty$ and we assume that σ is twice continuously differentiable. Let

$$\varphi_\kappa = \sqrt{\sigma'(\bullet)} - \psi_\kappa. \tag{6}$$

It is clear that φ_κ is a solution of the equation

$$\varphi_\kappa'(w) = \frac{\sigma''(w)}{2\sqrt{\sigma'(w)}} - \kappa\left(\sqrt{\sigma(w)} + \psi_\kappa(w)\right)\varphi_\kappa(w),$$

and if we solve this equation we obtain the following expressions for φ_κ:

$$\varphi_\kappa(w) = \begin{cases} \displaystyle\int_{-M-1}^{w} e^{-\kappa\int_t^w (\sqrt{\sigma'(s)}+\psi_\kappa(s))\mathrm{d}s} \frac{\sigma''(t)}{2\sqrt{\sigma'(t)}}\,\mathrm{d}t, & \kappa > 0, \\ \displaystyle\int_{w}^{M+1} e^{\kappa\int_w^t (\sqrt{\sigma'(s)}+\psi_\kappa(s))\mathrm{d}s} \frac{\sigma''(t)}{2\sqrt{\sigma'(t)}}\,\mathrm{d}t, & \kappa < 0. \end{cases} \tag{7}$$

Is is straightforward to see that if $\psi_\kappa(w) \geq \inf_{s\in\mathbb{R}} \sqrt{\sigma'(s)}$ does not hold for some $w \in [-M, M]$, then we get a contradiction from (5). Therefore it follows from (7) that

$$\varphi_\kappa \to 0 \text{ uniformly on } [-M, M] \text{ as } |\kappa| \to \infty. \tag{8}$$

We claim that it is possible to conclude from (7) that in addition

$$\kappa\varphi_\kappa \to \frac{\sigma''(\bullet)}{4\sigma'(\bullet)} \text{ uniformly on } [-M, M] \text{ as } |\kappa| \to \infty. \tag{9}$$

To see this, consider for example the case $\kappa > 0$ and note that

$$e^{-\kappa\int_t^w (\sqrt{\sigma'(s)}+\psi_\kappa(s))\mathrm{d}s} \leq e^{-2\kappa \inf_{s\in\mathbb{R}} \sqrt{\sigma'(s)}(w-t)},$$

when $-M - 1 \leq t \leq w \leq M$. Thus it follows from (7) that

$$\lim_{\kappa\to\infty} \sup_{w\in[-M,M]} \left| \kappa\varphi_\kappa(w) - \kappa \int_{-M-1}^{w} e^{-\kappa\int_t^w (\sqrt{\sigma'(s)}+\psi_\kappa(s))\mathrm{d}s} \frac{\sigma''(w)}{2\sqrt{\sigma'(w)}}\,\mathrm{d}t \right| = 0.$$

The proof of the claim now follows from the fact that

$$\kappa \int_{-M-1}^{w} e^{-\kappa\int_t^w (\sqrt{\sigma'(s)}+\psi_\kappa(s))\mathrm{d}s} = \kappa \int_{-M-1}^{w} e^{-2\kappa\sqrt{\sigma'(w)}(w-t)}\mathrm{d}t$$

$$+ \kappa \int_{-M-1}^{w} e^{-2\kappa\sqrt{\sigma'(w)}(w-t)}(e^{\kappa\int_t^w (2\sqrt{\sigma'(s)}-2\sqrt{\sigma'(w)}-\varphi_\kappa(s))\mathrm{d}s} - 1)\mathrm{d}t,$$

and when $\kappa \to \infty$ the first term converges to $1/\left(2\sqrt{\sigma'(w)}\right)$ and the second to 0 (by (8) and the continuity of $\sqrt{\sigma'(\bullet)}$), both uniformly for $w \in [-M, M]$.

If σ is not twice continuously differentiable, then the distribution derivative of σ' is a continuous measure with locally bounded variation and it is easy to see that (8) holds, and although (9) need not hold we have at least

$$\lim_{|\kappa|\to\infty} \kappa \int_{w_1}^{w_2} \varphi_\kappa(s)\mathrm{d}s = \frac{1}{4}(\ln(\sigma'(w_2)) - \ln(\sigma'(w_1))), \quad w_1,\ w_2 \in [-M, M], \tag{10}$$

and that will be sufficient for our purposes in this case.

Finally we let

$$\phi_\kappa = \varphi_\kappa - \frac{\sigma''(\bullet)}{4\kappa\sigma'(\bullet)}.$$

A simple calculation shows that

$$\phi_\kappa = \frac{\varphi_\kappa(\bullet)^2}{2\sqrt{\sigma'(\bullet)}} - \frac{\varphi_\kappa'(\bullet)}{2\kappa\sqrt{\sigma'(\bullet)}}.$$

If we integrate both sides of this equality over (w_1, w_2), do an integration by parts, and use the definition of ϕ_κ once more, then we conclude that

$$\int_{w_1}^{w_2} \phi_\kappa(s)\mathrm{d}s = -\int_{w_1}^{w_2} \frac{\varphi_\kappa(s)^2}{2\sqrt{\sigma'(s)}}\mathrm{d}s - \frac{\varphi_\kappa(w_2)}{2\kappa\sqrt{\sigma'(w_2)}} + \frac{\varphi_\kappa(w_1)}{2\kappa\sqrt{\sigma'(w_1)}} + \int_{w_1}^{w_2} \frac{\varphi_\kappa(s)\phi_\kappa(s)}{\sqrt{\sigma'(s)}}\mathrm{d}s.$$

Thus it follows from (9) and the definition of ϕ_κ that

$$\begin{aligned}\lim_{|\kappa|\to\infty} \kappa^2 \int_{w_1}^{w_2} \phi_\kappa(s)\mathrm{d}s = &-\int_{w_1}^{w_2} \frac{\sigma''(s)^2}{32\sigma'(s)^{5/2}}\mathrm{d}s \\ &- \frac{\sigma''(w_2)}{8\sigma'(w_2)^{3/2}} + \frac{\sigma''(w_1)}{8\sigma'(w_1)^{3/2}}, \quad w_1,\ w_2 \in [-M, M].\end{aligned} \tag{11}$$

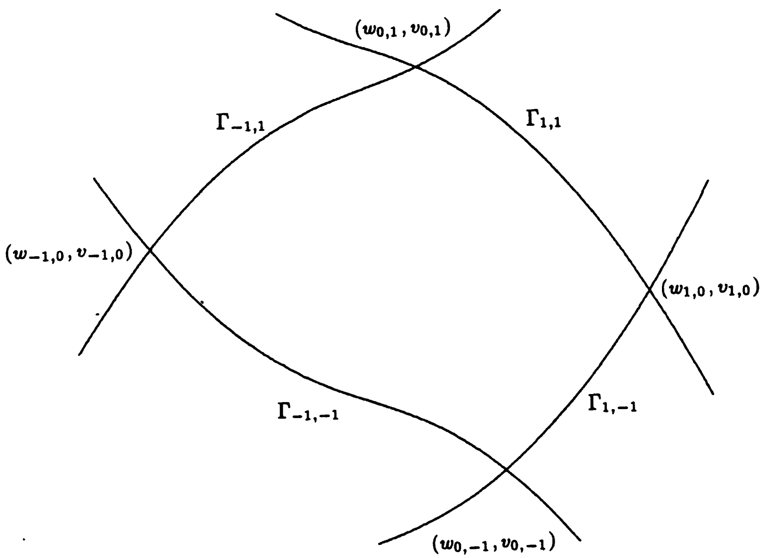
This completes our investigation of the function ψ_κ.

Let $\Omega = \mathrm{supp}(\mu)$, $\alpha,\ \beta \in \{-1, 1\}$, and let

$$\Gamma_{\alpha,\beta} = \left\{(w,v) \,\middle|\, \alpha\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + \beta v = \sup_{(w',v')\in\Omega}\left(\alpha\int_0^{w'}\sqrt{\sigma'(s)}\mathrm{d}s + \beta v'\right)\right\}, \tag{12}$$

(see the picture below). The functions $\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s \pm v$ are the so-called Riemann invariants and one could take them as independent variables instead of w and v.

We denote by $(w_{\alpha,0}, v_{\alpha,0})$ the point where the curves $\Gamma_{\alpha,\beta}$ and $\Gamma_{\alpha,-\beta}$ intersect and by $(w_{0,\beta}, v_{0,\beta})$ the point where the curves $\Gamma_{\alpha,\beta}$ and $\Gamma_{-\alpha,\beta}$ intersect. (If $\sigma' \equiv 1$ and $\Omega = \{(w,v) \mid |w| + |v| \le 1\}$, then $(w_{\alpha,0}, v_{\alpha,0})$ is actually the point $(\alpha, 0)$ and $(w_{0,\beta}, v_{0,\beta})$ is the point $(0, \beta)$).



Let $\mu_{\alpha,\beta}$ be a probability measure that one obtains as the weak limit (of some subsequence) of the sequence of probability measures

$$E \mapsto \frac{\int_E \eta_{\alpha,\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)}{\int_\Omega \eta_{\alpha,\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)},$$

as $k \to \infty$. It is quite easy to see from (6), (8), and (12) that the support $\Omega_{\alpha,\beta}$ of $\mu_{\alpha,\beta}$ is contained in $\Gamma_{\alpha,\beta}$.

Let us now choose $\eta^* = \eta_{\alpha,\beta,k}$ and $q^* = q_{\alpha,\beta,k}$ in (3) where α, $\beta \in \{-1, 1\}$ and $k > 0$. If we divide both sides of (3) by $\int_\Omega \eta_{\alpha,\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)$ and let

$k \to \infty$, then we obtain from (4), (8), and the definition of $\mu_{\alpha,\beta}$ that

$$\int_{\Omega_{\alpha,\beta}} \left(\alpha\beta\sqrt{\sigma'(w)}\eta(w,v) + q(w,v)\right)\mu_{\alpha,\beta}(\mathrm{d}w,\mathrm{d}v)$$

$$(13) \qquad = \alpha\beta \int_{\Omega} \eta(w,v)\mu(\mathrm{d}w\mathrm{d}v) \int_{\Omega_{\alpha,\beta}} \sqrt{\sigma'(w)}\mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v) + \int_{\Omega} q(w,v)\mu(\mathrm{d}w\mathrm{d}v).$$

Next we take $\eta = \eta_{-\alpha,-\beta,k}$ and $q = q_{-\alpha,-\beta,k}$, divide both sides of equation (13) by $\int_{\Omega} \eta_{-\alpha,-\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)$ and let $k \to \infty$. The conclusion we shall get is that

$$(14) \qquad \int_{\Omega_{\alpha,\beta}} \sqrt{\sigma'(w)}\mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v) = \int_{\Omega_{-\alpha,-\beta}} \sqrt{\sigma'(w)}\mu_{-\alpha,-\beta}(\mathrm{d}w\mathrm{d}v), \quad \alpha,\ \beta \in \{-1,1\}.$$

If $\Gamma_{\alpha,\beta} \neq \Gamma_{-\alpha,-\beta}$, then (14) follows from the fact that in this case we have

$$\lim_{k\to\infty} \frac{\int_{\Omega_{\alpha,\beta}} \left(\alpha\beta\sqrt{\sigma'(w)}\eta_{-\alpha,-\beta,k}(w,v) + q_{-\alpha,-\beta,k}(w,v)\right)\mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v)}{\int_{\Omega} \eta_{-\alpha,-\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)} = 0.$$

If on the other hand $\Gamma_{\alpha,\beta} = \Gamma_{-\alpha,-\beta}$, then if follows that the function $(w,v) \mapsto \alpha \int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + \beta v$ is constant on Ω and if we combine this result with (10) and with the fact that $\eta_{\alpha,\beta,k}(w,v) = \mathrm{e}^{\alpha \int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + \beta v - \alpha k \int_0^w \varphi_{\alpha k}(s)\mathrm{d}s}$, then we see that $\mu_{\alpha,\beta} = \mu_{-\alpha,-\beta}$ and this gives, of course, (14).

From (13) and (14) we furthermore deduce that

$$(15) \qquad \int_{\Omega_{\alpha,\beta}} \left(\alpha\beta\sqrt{\sigma'(w)}\eta(w,v) + q(w,v)\right)\mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v) = \int_{\Omega_{-\alpha,-\beta}} \left(\alpha\beta\sqrt{\sigma'(w)}\eta(w,v) + q(w,v)\right)\mu_{-\alpha,-\beta}(\mathrm{d}w\mathrm{d}v).$$

Next we show that

$$(16) \qquad (w_{\alpha,0}, v_{\alpha,0}) \in \Omega_{\alpha,\beta} \cap \Omega_{\alpha,-\beta} \text{ and } (w_{0,\beta}, v_{0,\beta}) \in \Omega_{\alpha,\beta} \cap \Omega_{-\alpha,\beta}.$$

Since all cases are similar it suffices to prove that $(w_{\alpha,0}, v_{\alpha,0}) \in \Omega_{\alpha,\beta}$. Suppose that this is not the case. Then we take $\eta = \eta_{\alpha,-\beta,k}$ and $q = q_{\alpha,-\beta,k}$ in (13), divide both sides by $\int\limits_{\Omega} \eta_{\alpha,-\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)$ and let $k \to \infty$. Because we assumed that the support of $\mu_{\alpha,\beta}$ is bounded away from the curve $\Gamma_{\alpha,-\beta}$ we conclude that

$$\frac{\int\limits_{\Omega_{\alpha,\beta}} \left(\alpha\beta\sqrt{\sigma'(w)}\eta_{\alpha,-\beta,k}(w,v) + q_{\alpha,-\beta,k}(w,v)\right) \mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v)}{\int\limits_{\Omega} \eta_{\alpha,-\beta,k}(w,v)\mu(\mathrm{d}w\mathrm{d}v)} \to 0 \text{ as } k \to \infty.$$

Thus we get

$$0 = \alpha\beta \int\limits_{\Omega_{\alpha,\beta}} \sqrt{\sigma'(w)}\mu_{\alpha,\beta}(\mathrm{d}w\mathrm{d}v) + \alpha\beta \int\limits_{\Omega_{\alpha,-\beta}} \sqrt{\sigma'(w)}\mu_{\alpha,-\beta}(\mathrm{d}w\mathrm{d}v),$$

and since $\sqrt{\sigma'}$ is strictly positive, we have a contradiction.

Let $\beta^* \in \{-1, 1\}$ be such that $w_{0,\beta^*} \geq w_{0,-\beta^*}$. Then $w \geq w'$ for all $(w,v) \in \Omega_{1,\beta^*}$ and $(w',v') \in \Omega_{-1,-\beta^*}$. Therefore it follows from (14) (with $\alpha = 1$ and $\beta = \beta^*$) and (16) that if σ' is monotone on $[w_{-1,0}, w_{1,0}]$, then it must be a constant.

Thus it remains to consider the case where σ' has its minimum point 0 inside the interval $[w_{-1,0}, w_{1,0}]$ and we shall derive a contradiction from the assumption that Ω is not a single point.

First we consider the case where $w_{0,\beta^*} > 0$ or $w_{0,-\beta^*} < 0$. Suppose that the first inequality holds. Because the inflection point 0 of σ is unique by (ii) we know that $\sigma'(w_{0,\beta^*}) > \sigma'(0)$.

Denote by ρ_+ the value of $\int\limits_0^w \sqrt{\sigma'(s)}\mathrm{d}s + \beta^* v$ on Γ_{1,β^*} and by ρ_- the value on $\Gamma_{-1,-\beta^*}$. It follows from our choice of β^* that we must have $\rho_+ > \rho_-$ unless Ω consists of exactly one point. Note that on the sets Ω_{1,β^*} and $\Omega_{-1,-\beta^*}$ we have $\beta^* v = \rho_\pm - \int\limits_0^w \sqrt{\sigma'(s)}\mathrm{d}s$. Thus we can, when integrating over these sets, replace $\mu_{\pm 1,\pm\beta^*}$ by measures $\nu_\pm$ supported on $[w_{0,\beta^*}, w_{1,0}]$ and $[w_{-1,0}, w_{0,-\beta^*}]$, respectively.

In (15) we take $\alpha = 1$, $\beta = \beta^*$, $\eta = \eta_{1,\beta^*,k}$, and $q = q_{1,\beta^*,k}$ where $k > 0$ and use the notation presented above. By (4) and (6) we then have

$$\int_{[w_{0,\beta^*},w_{1,0}]} \varphi_k(w) e^{k\rho_+ + k\int_0^w (\psi_k(s)-\sqrt{\sigma'(s)})\mathrm{d}s} \nu_+(\mathrm{d}w)$$
$$(17) \qquad = \int_{[w_{-1,0},w_{0,-\beta^*}]} \varphi_k(w) e^{k\rho_- + k\int_0^w (\psi_k(s)-\sqrt{\sigma'(s)})\mathrm{d}s} \nu_-(\mathrm{d}w).$$

Let G be the support of σ''. From the fact that $\sigma'(w_{0,\beta^*}) > \sigma'(0)$ it follows that we must have $G \cap (0, w_{0,\beta^*}) \neq \emptyset$. Therefore we conclude from (7) that for sufficiently large values of k we have

$$\varphi_k(w) \geq c_1 e^{-2k\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s}, \quad w \in [w_{0,\beta^*}, w_{1,0}],$$

for some positive constant c_1. On the other hand we have by (8) and (10) that

$$\varphi_k(w) e^{k\rho_- + k\int_0^w (\psi_k(s)-\sqrt{\sigma'(s)})\mathrm{d}s} \leq c_2 e^{k\rho_-}, \quad w \in [w_{-1,0}, w_{0,-\beta^*}],$$

where c_2 is some positive constant. Inserting these two inequalities into (17) we get

$$(18) \qquad c_3 \int_{[w_{0,\beta^*},w_{1,0}]} e^{-2k\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + k\rho_+} \nu_+(\mathrm{d}w) \leq c_2 e^{k\rho_-},$$

where $c_3 \leq c_1 e^{-k\int_0^w \varphi_k(s)\mathrm{d}s}$. Moreover, it is easy to check that because $-\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + \beta^* v$ is constant on the curve Γ_{-1,β^*} connecting the points $(w_{-1,0}, v_{-1,0})$ and $(w_{0,\beta^*}, v_{0,\beta^*})$ we have

$$(19) \qquad \int_0^{w_{0,\beta^*}} \sqrt{\sigma'(s)}\mathrm{d}s - \int_0^{w_{-1,0}} \sqrt{\sigma'(s)}\mathrm{d}s = \frac{1}{2}(\rho_+ - \rho_-).$$

Because $w_{-1,0} < 0$ we have $\int_0^{w_{-1,0}} \sqrt{\sigma'(s)}\mathrm{d}s < 0$ and therefore it follows from (16) and (19) that (18) gives a contradiction when $k \to \infty$ and the proof is completed in this case.

The final case that we have to consider is the one where $w_{0,\beta^*} = w_{0,-\beta^*} = 0$. In order to simplify the notation we define $w_+ = w_{1,0}$ and $w_- = w_{-1,0}$.

Let us first show that the supports of ν_+ and ν_- cannot intersect the set $\{w|\sigma''(w)\neq 0\}$. To do this we multiply both sides of (17) by $k\mathrm{e}^{-k\rho_+}$ and using the fact that $\rho_+ > \rho_-$ we get by (9) when $k \to \infty$ that

$$\int\limits_{[0,w_+]} \frac{\sigma''(w)}{4\sigma'(w)} \left(\frac{\sigma'(0)}{\sigma'(w)}\right)^{\frac{1}{4}} \nu_+(\mathrm{d}w) = 0.$$

Since σ'' is nonnegative on the support of ν_+ it follows that this support cannot intersect the set where σ'' is strictly positive. Similarly one shows that the support of ν_- cannot intersect the set where σ'' is strictly negative.

Let us, without loss of generality, assume that $\sigma''(w) > 0$ if $0 < w < \delta$, where δ is some positive number. When we combine this assumption with (16) and with the result derived above, we conclude that

$$\nu_+(\{0\}) > 0 \text{ and } \nu_+((0,\delta)) = 0. \tag{20}$$

In (15) we now take $\alpha = 1$, $\beta = 1$, $\eta = \eta_{-1,1,k}$, and $q = q_{-1,1,k}$ and $k > 0$. By (4) and the fact that $v = \rho_\pm - \int_0^w \sqrt{\sigma'(s)}\mathrm{d}s$ on $\Omega_{1,1}$ and $\Omega_{-1,-1}$, respectively, we have

$$\int\limits_{[0,w_+]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \mathrm{e}^{k\rho_+ - k\int_0^w (\psi_{-k}(s) + \sqrt{\sigma'(s)})\mathrm{d}s} \nu_+(\mathrm{d}w)$$

$$= \int\limits_{[w_-,0]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \mathrm{e}^{k\rho_- - k\int_0^w (w_{-k}(s) + \sqrt{\sigma'(s)})\mathrm{d}s} \nu_-(\mathrm{d}w),$$

and, since by (19), $\int_{w_-}^0 \sqrt{\sigma'(s)}\mathrm{d}s = (\rho_+ - \rho_-)/2$, we get

$$\int\limits_{[0,w_+]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \mathrm{e}^{k\rho_+ - 2k\int_0^w \sqrt{\sigma'(s)}\mathrm{d}s + k\int_0^w \varphi_{-k}(s)\mathrm{d}s} \nu_+(\mathrm{d}w)$$

$$= \int\limits_{[w_-,0]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \tag{21}$$

$$\times\, \mathrm{e}^{k\rho_+ - 2k\int_{w_-}^w \sqrt{\sigma'(s)}\mathrm{d}s + k\int_0^w \varphi_{-k}(s)\mathrm{d}s} \nu_-(\mathrm{d}w).$$

Therefore it follows from (20) and (21) when we let $k \to \infty$ that ν_- must have a point mass at w_-. More precisely, we see from (9) that we must have

$$\sqrt{\sigma'(0)}\nu_+(\{0\}) = \sqrt{\sigma'(w_-)}\left(\frac{\sigma'(0)}{\sigma'(w_-)}\right)^{\frac{1}{4}}\nu_-(\{w_-\}). \tag{22}$$

Because the support of ν_- cannot intersect the set where $\sigma''(w)\neq 0$ we have $\sigma''(w_-) = 0 = \sigma''(0)$ and it follows from (9) that

$$\begin{aligned} \psi_{-k}(w_-) &= \sqrt{\sigma'(w_-)} + o\left(\frac{1}{k}\right) \\ \psi_{-k}(0) &= \sqrt{\sigma'(0)} + o\left(\frac{1}{k}\right) \end{aligned} \quad \text{as } k \to \infty, \tag{23}$$

and we get from (11) and the definition of ϕ_k that

$$\begin{aligned} k\int_0^{w_-} \varphi_{-k}(s)\mathrm{d}s &= \frac{1}{4}(\ln(\sigma'(0)) - \ln(\sigma'(w_-))) \\ &\quad + \frac{1}{k}\int_{w_-}^{0} \frac{\sigma''(s)^2}{32\sigma'(s)^{5/2}}\mathrm{d}s + o\left(\frac{1}{k}\right) \quad \text{as } k \to \infty. \end{aligned} \tag{24}$$

Since

$$\int_{(w_-,0]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \mathrm{e}^{k\rho_+ - 2k\int_{w_-}^{w}\sqrt{\sigma'(s)}\mathrm{d}s + k\int_0^w \varphi_{-k}(s)\mathrm{d}s}\,\nu_-(\mathrm{d}w) \geq 0,$$

we see by (the first part of) (23) and (24) that

$$\begin{aligned} &\int_{[w_-,0]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) \mathrm{e}^{k\rho_+ - 2k\int_{w_-}^{w}\sqrt{\sigma'(s)}\mathrm{d}s + k\int_0^w \varphi_{-k}(s)\mathrm{d}s}\,\nu_-(\mathrm{d}w) \\ &\quad \geq \left(2\sqrt{\sigma'(w_-)} + o\left(\frac{1}{k}\right)\right)\left(\frac{\sigma'(0)}{\sigma'(w_-)}\right)^{\frac{1}{4}} \mathrm{e}^{k\rho_+ + \gamma/k}\nu_-(\{w_-\}), \end{aligned} \tag{25}$$

where

$$\gamma = \int_{w_-}^{0} \frac{\sigma''(s)^2}{32\sigma'(s)^{5/2}}\mathrm{d}s > 0$$

because $w_\diamond = 0$ was unique. From the second part of (20) we see that there is some positive constant c_4 so that

$$\int_{(0,w_+]} \left(\sqrt{\sigma'(w)} + \psi_{-k}(w)\right) e^{k\rho_+ - 2k\int_0^w \sqrt{\sigma'(s)}ds + k\int_0^w \varphi_{-k}(s)ds} \nu_+(dw) \leq c_4 e^{k\rho_+ - 2k\int_0^\delta \sqrt{\sigma'(s)}ds}. \tag{26}$$

If we insert (25) and (26) in (21) we conclude with the aid of (the second part of) (23) that

$$\left(2\sqrt{\sigma'(w_-)} + o\left(\frac{1}{k}\right)\right)\left(\frac{\sigma'(0)}{\sigma(w_-)}\right)^{\frac{1}{4}} e^{k\rho_+ + \gamma/k}\nu_-(\{w_-\})$$
$$\leq \left(2\sqrt{\sigma'(0)} + o\left(\frac{1}{k}\right)\right) e^{k\rho_+}\nu_+(\{0\}) + c_4 e^{k\rho_+ 2k\int_0^\delta \sqrt{\sigma'(s)}ds}.$$

If we now let $k \to \infty$ we get a contradiction from (22) because $\gamma > 0$. This completes the proof. □

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