# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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*Cluster ensembles, quantization and the dilogarithm*

# CLUSTER ENSEMBLES, QUANTIZATION
# AND THE DILOGARITHM

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Abstract. – A cluster ensemble is a pair $(\mathcal{X}, \mathcal{A})$ of positive spaces (i.e. varieties equipped with positive atlases), coming with an action of a symmetry group $\Gamma$. The space $\mathcal{A}$ is closely related to the spectrum of a cluster algebra [12]. The two spaces are related by a morphism $p : \mathcal{A} \longrightarrow \mathcal{X}$. The space $\mathcal{A}$ is equipped with a closed 2-form, possibly degenerate, and the space $\mathcal{X}$ has a Poisson structure. The map $p$ is compatible with these structures. The dilogarithm together with its motivic and quantum avatars plays a central role in the cluster ensemble structure. We define a non-commutative $q$-deformation of the $\mathcal{X}$-space. When $q$ is a root of unity the algebra of functions on the $q$-deformed $\mathcal{X}$-space has a large center, which includes the algebra of functions on the original $\mathcal{X}$-space.

The main example is provided by the pair of moduli spaces assigned in [6] to a topological surface $S$ with a finite set of points at the boundary and a split semisimple algebraic group $G$. It is an algebraic-geometric avatar of higher Teichmüller theory on $S$ related to $G$.

We suggest that there exists a duality between the $\mathcal{A}$ and $\mathcal{X}$ spaces. In particular, we conjecture that the tropical points of one of the spaces parametrise a basis in the space of functions on the Langlands dual space. We provide some evidence for the duality conjectures in the finite type case.

Résumé. – Un ensemble amassé est une paire $(\mathcal{X}, \mathcal{A})$ d'espaces positifs (i.e. de variétés munies d'un atlas positif) munis de l'action d'un groupe discret. L'espace $\mathcal{A}$ est relié au spectre d'une algèbre amassée [12]. Les deux espaces sont liés par un morphisme $p : \mathcal{A} \longrightarrow \mathcal{X}$. L'espace $\mathcal{A}$ est muni d'une 2-forme fermée, éventuellement dégénérée, et l'espace $\mathcal{X}$ est muni d'une structure de Poisson. L'application $p$ est compatible avec ces structures. Le dilogarithme avec ses avatars motiviques et quantiques joue un rôle fondamental dans la structure d'un ensemble amassé. Nous définissons une déformation non-commutative de l'espace $\mathcal{X}$. Nous montrons que, dans le cas où le paramètre de la déformation $q$ est une racine de l'unité, l'algèbre déformée a un centre qui contient l'algèbre des fonctions sur l'espace $\mathcal{X}$ originel.

Notre exemple principal est celui de l'espace des modules associé dans [6] à une surface topologique $S$ munie d'un nombre fini de points distingués sur le bord et à un groupe algébrique semi-simple $G$. C'est un avatar algébro-géométrique de la théorie de Teichmüller d'ordre supérieur sur la surface $S$ à valeurs dans $G$.

Nous évoquons l'existence d'une dualité entre les espaces $\mathcal{A}$ et $\mathcal{X}$. Une des manifestations de cette dualité est une conjecture de dualité affirmant que les points tropicaux d'un espace paramètrent une base dans l'espace d'une certaine classe de fonctions sur l'espace Langlands-dual. Nous démontrons cette conjecture dans un certain nombre d'exemples.

## Contents

## 1. Introduction and main definitions with simplest examples

Cluster algebras are a remarkable discovery of S. Fomin and A. Zelevinsky [12]. They are certain commutative algebras defined by a very simple and general data.

We show that a cluster algebra is part of a richer structure, which we call a *cluster ensemble*. A cluster ensemble is a pair $(\mathcal{X}, \mathcal{A})$ of *positive spaces* (which are varieties equipped with

positive atlases), coming with an action of a certain discrete symmetry group $\Gamma$. These two spaces are related by a morphism $p : \mathcal{A} \longrightarrow \mathcal{X}$, which in general, as well as in many interesting examples, is neither injective nor surjective. The space $\mathcal{A}$ has a degenerate symplectic structure, and the space $\mathcal{X}$ has a Poisson structure. The map $p$ relates the Poisson and degenerate symplectic structures in a natural way. Amazingly, the dilogarithm together with its motivic and quantum avatars plays a central role in the cluster ensemble structure. The space $\mathcal{A}$ is closely related to the spectrum of a cluster algebra. On the other hand, in many situations the most interesting part of the structure is the space $\mathcal{X}$.

We define a canonical non-commutative $q$-deformation of the $\mathcal{X}$-space. We show that when $q$ is a root of unity the algebra of functions on the $q$-deformed $\mathcal{X}$-space has a large center, which contains a subalgebra identified with the algebra of functions on the original $\mathcal{X}$-space.

The main example, as well as the main application of this theory so far, is provided by the $(\mathcal{X}, \mathcal{A})$-pair of moduli spaces assigned in [6] to a topological surface $S$ with a finite set of points at the boundary and a semisimple algebraic group $G$. In particular, the $\mathcal{X}$-space in the simplest case when $G = PGL_2$ and $S$ is a disc with $n$ points at the boundary is the moduli space $\mathcal{M}_{0,n}$.

This pair of moduli spaces is an algebraic-geometric avatar of higher Teichmüller theory on $S$ related to $G$. In the case $G = SL_2$ we get the classical Teichmüller theory, as well as its generalization to surfaces with a finite set of points on the boundary. A survey of the Teichmüller theory emphasizing the cluster point of view can be found in [7].

We suggest that there exists a duality between the $\mathcal{A}$ and $\mathcal{X}$ spaces. One of its manifestations is our package of duality conjectures in Section 4. These conjectures assert that the *tropical points* of the $\mathcal{A}/\mathcal{X}$-space parametrise a basis in a certain class of functions on the *Langlands dual* $\mathcal{X}/\mathcal{A}$-space. It can be viewed as a canonical function (the *universal kernel*) on the product of the set of tropical points of one space and the Langlands dual space.

To support these conjectures, we define in Section 5.1 the tropical limit of such a universal kernel in the finite type case. Another piece of evidence is provided by Chapter 12 in [6].

In the rest of the introduction we define cluster $\mathcal{X}$- and $\mathcal{A}$-varieties and describe their key features. Section 1.1 provides background on positive spaces, borrowed from Chapter 4 of [6]. Cluster varieties are defined in Section 1.2. In Section 1.3 we discuss one of the simplest examples: cluster $\mathcal{X}$-variety structures of the moduli space $\mathcal{M}_{0,n+3}$. In Section 1.4 we summarize the main structures of cluster varieties. In Section 1.4 we discuss how they appear in our main example: higher Teichmüller theory.

## 1.1. Positive schemes and positive spaces

A *semifield* is a set $P$ equipped with the operations of addition and multiplication, so that addition is commutative and associative, multiplication makes $P$ an abelian group, and they are compatible in a usual way: $(a + b)c = ac + bc$ for $a, b, c \in P$. A standard example is given by the set $\mathbb{R}_{>0}$ of positive real numbers. Here are more exotic examples. Let $\mathbb{A}$ be one of the sets $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The *tropical semifield* $\mathbb{A}^t$ associated with $\mathbb{A}$ is the set $\mathbb{A}$ with the multiplication $\otimes$ and addition $\oplus$ given by

$$a \otimes b := a + b, \quad a \oplus b := \max(a, b).$$

One more example is given by the semifield $\mathbb{R}_{>0}((\varepsilon))$ of Laurent series in $\varepsilon$ with real coefficients and a positive leading coefficient, equipped with the usual addition and multiplication. There is a homomorphism of semifields $-\deg : \mathbb{R}_{>0}((\varepsilon)) \to \mathbb{Z}^t$, given by $f \longmapsto -\deg(f)$. It explains the origin of the tropical semifield $\mathbb{Z}^t$.

Recall the standard notation $\mathbb{G}_m$ for the multiplicative group. It is an affine algebraic group. The ring of regular functions on $\mathbb{G}_m$ is $\mathbb{Z}[X, X^{-1}]$, and for any field $F$ one has $\mathbb{G}_m(F) = F^*$. A product of multiplicative groups is known as a *split algebraic torus* over $\mathbb{Z}$, or simply a split algebraic torus.

Let $H$ be a split algebraic torus. A rational function $f$ on $H$ is called *positive* if it belongs to the semifield generated, in the field of rational functions on $H$, by the characters of $H$. So it can be written as $f = f_1/f_2$ where $f_1, f_2$ are linear combinations of characters with positive integral coefficients. A *positive rational map* between two split tori $H_1, H_2$ is a rational map $f : H_1 \to H_2$ such that $f^*$ induces a homomorphism of the semifields of positive rational functions. Equivalently, for any character $\chi$ of $H_2$ the composition $\chi \circ f$ is a positive rational function on $H_1$. A composition of positive rational functions is positive. Let Pos be the category whose objects are split algebraic tori and morphisms are positive rational maps. A *positive divisor* in a torus $H$ is a divisor given by an equation $f = 0$, where $f$ is a positive rational function on $H$.

**DEFINITION 1.1.** – *A positive atlas on an irreducible scheme/stack $X$ over $\mathbb{Q}$ is a family of birational isomorphisms*

$$(1) \qquad \psi_\alpha : H_\alpha \longrightarrow X, \quad \alpha \in \mathcal{C}_X,$$

*between split algebraic tori $H_\alpha$ and $X$, parametrised by a non empty set $\mathcal{C}_X$, such that:*

i) *each $\psi_\alpha$ is regular on the complement of a positive divisor in $H_\alpha$;*

ii) *for any $\alpha, \beta \in \mathcal{C}_X$ the map $\psi_{\alpha,\beta} := \psi_\beta^{-1} \circ \psi_\alpha : H_\alpha \longrightarrow H_\beta$ is a positive rational map* [1].

*A positive atlas is called regular if each $\psi_\alpha$ is regular.*

Birational isomorphisms (1) are called positive coordinate systems on $X$. A *positive scheme* is a scheme equipped with a positive atlas. We will need an equivariant version of this definition.

**DEFINITION 1.2.** – *Let $\Gamma$ be a group of automorphisms of $X$. A positive atlas (1) on $X$ is $\Gamma$-equivariant if $\Gamma$ acts on the set $\mathcal{C}_X$, and for every $\gamma \in \Gamma$ there is an isomorphism of algebraic tori $i_\gamma : H_\alpha \xrightarrow{\sim} H_{\gamma(\alpha)}$ making the following diagram commutative:*

$$(2) \qquad
\begin{array}{ccc}
H_\alpha & \xrightarrow{\psi_\alpha} & X \\
\downarrow i_\gamma & & \downarrow \gamma \\
H_{\gamma(\alpha)} & \xrightarrow{\psi_{\gamma(\alpha)}} & X.
\end{array}$$

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[1] A positive atlas covers a non-empty Zariski open subset of $X$, but not necessarily the whole space $X$.

Quite often a collection of positive coordinate systems is the only data we need when working with a positive scheme. Axiomatizing this observation, we arrive at the *category of positive spaces* defined below.

A groupoid is a category where all morphisms are isomorphisms. We assume that the set of morphisms between any two objects is non-empty. The *fundamental group of a groupoid* is the automorphism group of an object of the groupoid. It is well defined up to an inner automorphism.

DEFINITION 1.3. – *Let $\mathcal{G}_{\mathcal{X}}$ be a groupoid. A* positive space *is a functor*

$$(3) \qquad\qquad \psi_{\mathcal{X}} : \mathcal{G}_{\mathcal{X}} \longrightarrow \mathrm{Pos}.$$

The groupoid $\mathcal{G}_{\mathcal{X}}$ is called the *coordinate groupoid* of a positive space. Thus for every object $\alpha$ of $\mathcal{G}_{\mathcal{X}}$ there is an algebraic torus $H_{\alpha}$, called a *coordinate torus* of the positive space $\mathcal{X}$, and for every morphism $f : \alpha \longrightarrow \beta$ in the groupoid there is a positive birational isomorphism $\psi_f : H_{\alpha} \longrightarrow H_{\beta}$.

Let $\psi_1$ and $\psi_2$ be functors from coordinate groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ to the category Pos. A morphism from $\psi_1$ to $\psi_2$ is a pair consisting of a functor $\mu : \mathcal{G}_2 \to \mathcal{G}_1$ and a natural transformation $F : \psi_2 \to \psi_1 \circ \mu$. A morphism is called a *monomial morphism* if for every object $\alpha \in \mathcal{G}_2$ the map $F_{\alpha} : \psi_2(\alpha) \to \psi_1(\mu(\alpha))$ is a homomorphism of algebraic tori.

EXAMPLE 1. – A positive variety $X$ provides a functor (3) as follows. The fundamental group of the coordinate groupoid $\mathcal{G}_{\mathcal{X}}$ is trivial, so it is just a set. Precisely, the objects of $\mathcal{G}_{\mathcal{X}}$ form the set $\mathcal{C}_X$ of coordinate charts of the positive atlas on $X$. The morphisms are given by the subset of $\mathcal{C}_X \times \mathcal{C}_X$ consisting of pairs of charts with nontrivial intersection, with the obvious source and target maps. In particular, the morphisms form the set $\mathcal{C}_X \times \mathcal{C}_X$ if $X$ is irreducible. The functor $\psi_{\mathcal{X}}$ is given by $\psi_{\mathcal{X}}(\alpha) := H_{\alpha}$ and $\psi_{\mathcal{X}}(\alpha \to \beta) := \psi_{\alpha,\beta}$.

EXAMPLE 2. – A $\Gamma$-equivariant positive scheme $X$ provides a positive space $\mathcal{X}$ given by a functor (3). The fundamental group of its coordinate groupoid is isomorphic to $\Gamma$.

Given a split torus $H$ and a semifield $P$, we define the set of $P$-valued points of $H$ as

$$H(P) := X_*(H) \otimes_{\mathbb{Z}} P,$$

where $X_*(H)$ is the group of cocharacters of $H$, and the tensor product is with the abelian group defined by the semifield $P$. A positive birational isomorphism $\psi : H \to H'$ induces a map $\psi_* : H(P) \to H'(P)$.

EXAMPLE 3. – If $\mathbb{A}^t$ is a tropical semifield, then the map $\psi_*$ is given by a piece-wise linear map, the *tropicalization* of the map $\psi$.

An inverse to a positive map may not be positive – the inverse of the map $x' = x+y, y' = y$ is $x = x' - y', y = y'$. If $\psi^{-1}$ is also positive, the map $\psi_*$ is an isomorphism.

Given a positive space $\mathcal{X}$ there is a unique set $\mathcal{X}(P)$ of $P$-points of $\mathcal{X}$. It can be defined as

$$\mathcal{X}(P) = \coprod_{\alpha} H_{\alpha}(P)/(\text{identifications } \psi_{\alpha,\beta*}).$$

For every object $\alpha$ of the coordinate groupoid $\mathcal{G}_\mathcal{X}$ there are functorial (with respect to the maps $\mathcal{X} \to \mathcal{X}'$) isomorphisms

$$\mathcal{X}(P) \cong H_\alpha(P) \cong P^{\dim \mathcal{X}}.$$

Therefore the fundamental group of the coordinate groupoid acts on the set $\mathcal{X}(P)$.

A positive space $\mathcal{X}$ gives rise to a prescheme $\mathcal{X}^*$. It is obtained by gluing the tori $H_\alpha$, where $\alpha$ runs through the objects of $\mathcal{G}_\mathcal{X}$, according to the birational maps $\psi_f$ corresponding to morphisms $f : \alpha \to \beta$. It, however, may not be separable, and thus may not be a scheme. Each torus $H_\alpha$ embeds to $\mathcal{X}^*$ as a Zariski open dense subset $\psi_\alpha(H_\alpha)$.

The ring of regular functions on $\mathcal{X}^*$ is called the ring of *universally Laurent polynomials* for $\mathcal{X}$ and denoted by $\mathbb{L}(\mathcal{X})$. In simple terms, the ring $\mathbb{L}(\mathcal{X})$ consists of all rational functions which are regular at every coordinate torus $H_\alpha$.

It is often useful to take the affine closure of $\mathcal{X}^*$, understood as the spectrum $\mathrm{Spec}(\mathbb{L}(\mathcal{X}))$.

The positive structure on $\mathcal{X}^*$ provides the semifield of all positive rational functions on $\mathcal{X}^*$. Intersecting it with the ring $\mathbb{L}(\mathcal{X})$ we get the semiring $\widetilde{\mathbb{L}}_+(\mathcal{X})$. As the example $1 - x + x^2 = (1 + x^3)/(1 + x)$ shows, a rational function can be positive, while the coefficients of the corresponding Laurent polynomial may be not. So we define a smaller semiring $\mathbb{L}_+(\mathcal{X})$ of *positive universally Laurent polynomials* for $\mathcal{X}$ as follows: an element of $\mathbb{L}_+(\mathcal{X})$ is a rational function on $\mathcal{X}^*$ whose restriction to one (and hence any) of the embedded coordinate tori $\psi_\alpha(H_\alpha)$ is a linear combination of characters of this torus with positive integral coefficients.

## 1.2. Cluster ensembles: definitions

They are defined by a combinatorial data seed – similar [2] to the one used in the definition of cluster algebras [12].

1.2.1. *Seeds and seed tori.* – Recall that a lattice is a free abelian group.

DEFINITION 1.4. – *A seed is a datum* $(\Lambda, (*, *), \{e_i\}, \{d_i\})$, *where*
i) $\Lambda$ *is a lattice;*
ii) $(*, *)$ *is a skewsymmetric $\mathbb{Q}$-valued bilinear form on $\Lambda$;*
iii) $\{e_i\}$ *is a basis of the lattice $\Lambda$, and $I_0$ is a subset of basis vectors, called frozen basis vectors;*
iv) $\{d_i\}$ *are positive integers assigned to the basis vectors, such that*

$$\varepsilon_{ij} := (e_i, e_j)d_j \in \mathbb{Z} \quad unless\ i, j \in I_0 \times I_0.$$

The numbers $\{d_i\}$ are called the *multipliers*. We assume that their greatest common divisor is 1.

*Seeds as quivers.* A seed is a version of the notion of a *quiver*. Precisely, let us assume for simplicity that the set of frozen basis vectors is empty. A quiver corresponding to a seed is a graph whose set of vertices $\{i\}$ is identified with the set of basis vectors $\{e_i\}$; two vertices $i, j$ are connected by $|(e_i, e_j)|$ arrows going from $i$ to $j$ if $(e_i, e_j) > 0$, and from $j$ to $i$ otherwise;

[2] Although different in detail – we do not include the cluster coordinates in the definition of a seed, and give a coordinate free definition.

the $i$-th vertex is marked by $d_i$, see Fig. 1. The (enhanced by multipliers) quivers we get have the following property: all arrows between any two vertices are oriented the same way, and there are no arrows from a vertex to itself. Clearly any enhanced quiver like this corresponds to a unique seed.
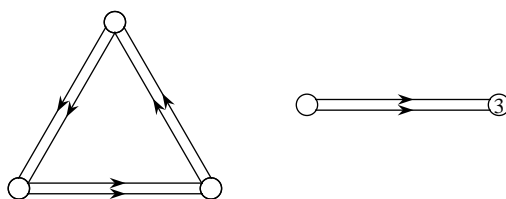


FIGURE 1. Picturing seeds by quivers – we show $d_i$'s only if they differ from 1.

*Lattices and split algebraic tori.* Recall that a lattice $\Lambda$ gives rise to a split algebraic torus

$$\mathcal{X}_\Lambda := \operatorname{Hom}(\Lambda, \mathbb{G}_m).$$

The set of its points with values in a field $F$ is the group $\operatorname{Hom}(\Lambda, F^*)$.

An element $v \in \Lambda$ provides a character $X_v$ of $\mathcal{X}_\Lambda$. Its value on a homomorphism $x \in \mathcal{X}_\Lambda$ is $x(v)$. The assignment $\Lambda \longrightarrow \mathcal{X}_\Lambda$ is a contravariant functor providing an equivalence of categories

the dual to the category of finite rank lattices $\xrightarrow{\sim}$ the category of split algebraic tori.

The inverse functor assigns to a split algebraic torus $\mathcal{T}$ its lattice of characters $\operatorname{Hom}(\mathcal{T}, \mathbb{G}_m)$.

There is the dual lattice

$$\Lambda^* := \operatorname{Hom}(\Lambda, \mathbb{Z}).$$

An element $a \in \Lambda^*$ gives rise to a cocharacter

$$\varphi_a : \mathbb{G}_m \longrightarrow \operatorname{Hom}(\Lambda, \mathbb{G}_m).$$

On the level of $F$-points, $\varphi_a(f)$ is the homomorphism $v \longmapsto f^{a(v)}$.

The *seed $\mathcal{X}$-torus* is a split algebraic torus $\mathcal{X}_\Lambda := \operatorname{Hom}(\Lambda, \mathbb{G}_m)$. It carries a Poisson structure provided by the form $(*, *)$:

$$\{X_v, X_w\} = (v, w) X_v X_w.$$

The basis $\{e_i\}$ provides *cluster $\mathcal{X}$-coordinates* $\{X_i\}$. They form a basis in the group of characters of $\mathcal{X}_\Lambda$.

The basis $\{e_i\}$ provides a dual basis $\{e_i^*\}$ of the lattice $\Lambda^*$. We need a quasidual basis $\{f_i\}$ given by

(4) $$f_i = d_i^{-1} e_i^*.$$

Let $\Lambda^\circ \subset \Lambda^* \otimes \mathbb{Q}$ be the sublattice spanned by the vectors $f_i$. The *seed $\mathcal{A}$-torus* is a split algebraic torus

$$\mathcal{A}_\Lambda := \operatorname{Hom}(\Lambda^\circ, \mathbb{G}_m).$$

The basis $\{f_i\}$ provides *cluster $\mathcal{A}$-coordinates* $\{A_i\}$.

Let $\mathcal{O}(Y)^*$ be the group of invertible regular functions on a variety $Y$. There is a map

$$d\log \wedge d\log : \mathcal{O}(Y)^* \wedge \mathcal{O}(Y)^* \longrightarrow \Omega^2(Y), \quad f \wedge g \longmapsto d\log(f) \wedge d\log(g).$$

The skew-symmetric bilinear form $(*, *)$, viewed as an element of $\bigwedge^2 \Lambda_{\mathbf{i}}^\circ$, provides an element

$$W \in \mathcal{O}(\mathcal{A}_\Lambda)^* \wedge \mathcal{O}(\mathcal{A}_\Lambda)^*.$$

Applying the map $d\log \wedge d\log$ to $W$ we get a closed 2-form $\Omega$ on the torus $\mathcal{A}_\Lambda$:

$$\Omega := d\log \wedge d\log(W) \in \Omega^2(\mathcal{A}_\Lambda).$$

There is a non-symmetric bilinear form $[*, *]$ on the lattice $\Lambda$, defined by setting

(5)
$$[e_i, e_j] := (e_i, e_j)d_j.$$

There is a natural map of lattices

$$p^* : \Lambda \longrightarrow \Lambda^\circ, \qquad v \longmapsto \sum_j (v, e_j)e_j^* = \sum_j [v, e_j]f_j.$$

It gives rise to a homomorphism of seed tori

(6)
$$p : \mathcal{A}_\Lambda \longrightarrow \mathcal{X}_\Lambda.$$

The following lemma is straightforward.

LEMMA 1.5. – *The fibers of the map $p$ are the leaves of the null-foliation of the 2-form $\Omega$. The subtorus $\mathcal{U}_\Lambda := p(\mathcal{A}_\Lambda)$ is a symplectic leaf of the Poisson structure on $\mathcal{X}_\Lambda$.*

*The symplectic structure on $\mathcal{U}_\Lambda$ induced by the form $\Omega$ on $\mathcal{A}_\Lambda$ coincides with the symplectic structure given by the restriction of the Poisson structure on $\mathcal{X}_\Lambda$.*

Summarising, a seed $\mathbf{i}$ gives rise to seed $\mathcal{X}$- and $\mathcal{A}$-tori. Although they depend only on the lattice $\Lambda$, we denote them by $\mathcal{X}_\mathbf{i}$ and $\mathcal{A}_\mathbf{i}$ to emphasize the cluster coordinates on these tori provided by the seed $\mathbf{i}$.

1.2.2. *Seed mutations.* – Set $[\alpha]_+ = \alpha$ if $\alpha \geq 0$ and $[\alpha]_+ = 0$ otherwise. So $[\alpha]_+ = \max(0, \alpha)$.

Given a seed $\mathbf{i}$ and a non-frozen basis vector $e_k$, we define a new seed $\mathbf{i}'$, called *the seed obtained from $\mathbf{i}$ by mutation in the direction of a non-frozen basis vector $e_k$.* The seed $\mathbf{i}'$ is obtained by changing the basis $\{e_i\}$ – the rest of the datum stays the same. The new basis $\{e_i'\}$ is

(7)
$$e_i' := \begin{cases} e_i + [\varepsilon_{ik}]_+ e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k. \end{cases}$$

We denote by $\mu_{e_k}(\mathbf{i})$, or simply by $\mu_k(\mathbf{i})$, the seed $\mathbf{i}$ mutated in the direction of a basis vector $e_k$. By definition, the frozen/non-frozen basis vectors of the mutated seed are the images of the frozen/non-frozen basis vectors of the original seed.

The basis $\{f_i\}$ in $\Lambda^\circ$ mutates as follows:

(8)
$$f_i' := \begin{cases} -f_k + \sum_{j \in I} [-\varepsilon_{kj}]_+ f_j & \text{if } i = k \\ f_i & \text{if } i \neq k. \end{cases}$$

Therefore, although the definition of the lattice $\Lambda^\circ$ involves a choice of a seed, the lattice does not depend on it.

REMARK. – The basis $\mu_k^2(\{e_i\})$ does not necessarily coincide with $\{e_i\}$. For example, let $\Lambda$ be a rank two lattice with a basis $\{e_1, e_2\}$, and $(e_1, e_2) = 1$. Then

$$\{e_1, e_2\} \xrightarrow{\mu_2} \{e_1 + e_2, -e_2\} \xrightarrow{\mu_2} \{e_1 + e_2, e_2\}.$$

However, although the seeds $\mu_k^2(\mathbf{i})$ and $\mathbf{i}$ are different, they are canonically isomorphic.

### 1.2.3. *Coordinate description*

DEFINITION 1.6. – *A seed* $\mathbf{i}$ *is a quadruple* $(I, I_0, \varepsilon, d)$, *where*

i) $I$ *is a finite set, and* $I_0$ *is a subset of* $I$;

ii) $\varepsilon = \varepsilon_{ij}$ *is a* $\mathbb{Q}$-*valued function on* $I \times I$, *such that* $\varepsilon_{ij} \in \mathbb{Z}$, *unless* $(i, j) \in I_0 \times I_0$;

iii) $d = \{d_i\}$, *where* $i \in I$, *is a set of positive rational numbers, such that the function*

$$\widehat{\varepsilon}_{ij} = \varepsilon_{ij} d_j^{-1} \ \text{ is skew-symmetric: } \ \widehat{\varepsilon}_{ij} = -\widehat{\varepsilon}_{ji}.$$

Definitions 1.4 and 1.6 are equivalent. Indeed, given a seed from Definition 1.6, we set

$$\Lambda := \mathbb{Z}[I], \quad e_i := \{i\}, \ i \in I, \quad (e_i, e_j) := \varepsilon_{ij} \, d_j^{-1}.$$

The non-symmetric bilinear form is the function $\varepsilon$:

$$[e_i, e_j] = \varepsilon_{ij}.$$

The function $\varepsilon$ is called the *exchange function*. The numbers $\{d_i\}$ are the multipliers. The subset $I_0 \subset I$ is the *frozen subset* of $I$, and its elements are the *frozen elements* of $I$. Elements of the set $I$ are often called *vertices*.

The Poisson structure on the torus $\mathcal{X}_{\mathbf{i}}$ looks in coordinates as follows:

$$(9) \qquad \{X_i, X_j\} = \widehat{\varepsilon}_{ij} X_i X_j, \qquad \widehat{\varepsilon}_{ij} := \varepsilon_{ij} d_j^{-1}.$$

The 2-form $\Omega$ on the torus $\mathcal{A}_{\mathbf{i}}$ is

$$(10) \qquad \Omega = \sum_{i,j \in I} \widetilde{\varepsilon}_{ij} d\log A_i \wedge d\log A_j, \qquad \widetilde{\varepsilon}_{ij} := d_i \varepsilon_{ij}.$$

The homomorphism $p$ – see (6) – is given by

$$(11) \qquad p : \mathcal{A}_{\mathbf{i}} \longrightarrow \mathcal{X}_{\mathbf{i}}, \qquad p^* X_i = \prod_{j \in I} A_j^{\varepsilon_{ij}}.$$

Given a seed $\mathbf{i} = (I, I_0, \varepsilon, d)$, every non-frozen element $k \in I - I_0$ provides a mutation in the direction $k$ seed $\mu_k(\mathbf{i}) = \mathbf{i}' = (I', I_0', \varepsilon', d')$: one has $I' := I, I_0' := I_0, d' := d$ and

$$(12) \qquad \varepsilon_{ij}' := \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \quad k \notin \{i, j\} \\ \varepsilon_{ij} + |\varepsilon_{ik}| \cdot \varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \quad k \notin \{i, j\}. \end{cases}$$

This procedure is involutive: the mutation of $\varepsilon_{ij}'$ at the vertex $k$ is the original function $\varepsilon_{ij}$.

This definition of mutations is equivalent to the coordinate free definition thanks to the following lemma.

LEMMA 1.7. – *One has* $\varepsilon_{ij}' = (e_i', e_j')d_j$, *where* $\varepsilon_{ij}'$ *is given by formula* (12).

*Proof.* – Clearly $(e'_i, e'_k) = (e_i + [\varepsilon_{ki}]_+ e_k, -e_k) = -\widehat{\varepsilon}_{ik} = \widehat{\varepsilon}'_{ik}$. Assume that $k \notin \{i, j\}$. Then

$$(e'_i, e'_j) = (e_i + [\varepsilon_{ik}]_+ e_k, e_j + [\varepsilon_{jk}]_+ e_k) = \widehat{\varepsilon}_{ij} + [\varepsilon_{ik}]_+ \widehat{\varepsilon}_{kj} + \widehat{\varepsilon}_{ik}[\varepsilon_{jk}]_+$$
$$= \widehat{\varepsilon}_{ij} + [\varepsilon_{ik}]_+ \widehat{\varepsilon}_{kj} + \varepsilon_{ik}[-\widehat{\varepsilon}_{kj}]_+ = \widehat{\varepsilon}'_{ij}.$$

The lemma is proved. $\qquad\square$

1.2.4. *Cluster transformations.* – This is the heart of the story. A seed mutation $\mu_k$ induces positive rational maps between the corresponding seed $\mathcal{X}$- and $\mathcal{A}$-tori, denoted by the same symbol $\mu_k$. Namely, denote the cluster coordinates related to the seed $\mu_k(\mathbf{i})$ by $X'_i$ and $A'_i$. Then we define

$$(13) \qquad \mu_k^* X'_i := \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

$$(14) \qquad A_k \cdot \mu_k^* A'_k := \prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}; \qquad \mu_k^* A'_i = A_i, \quad i \neq k.$$

Here if just one of the sets $\{j|\varepsilon_{kj} > 0\}$ and $\{j|\varepsilon_{kj} < 0\}$ is empty, the corresponding monomial is 1. If $\varepsilon_{kj} = 0$ for every $j$, the right hand side of the formula is 2, and $\mu_k^* X'_k = X_k^{-2}$, $\mu_k^* X'_i = X_i$ for $i \neq k$.

Seed isomorphisms $\sigma$ obviously induce isomorphisms between the corresponding seed tori, which are denoted by the same symbols $\sigma$:

$$(15) \qquad \sigma^* X'_{\sigma(i)} = X_i, \qquad \sigma^* A'_{\sigma(i)} = A_i.$$

A *seed cluster transformation* is a composition of seed isomorphisms and mutations. It gives rise to a *cluster transformation* of the corresponding seed $\mathcal{X}$- or $\mathcal{A}$-tori. The latter is a rational map obtained by the composition of isomorphisms and mutations corresponding to the seed isomorphisms and mutations forming the seed cluster transformation. Given a semifield $P$, cluster transformations induce isomorphisms between the sets of $P$-points of the corresponding cluster tori.

Two seeds are called *equivalent* if they are related by a cluster transformation. The equivalence class of a seed $\mathbf{i}$ is denoted by $|\mathbf{i}|$.

Mutation formulas (12) and (14) were invented by Fomin and Zelevinsky [12]. Clearly the functions obtained by cluster $\mathcal{A}$- (resp. $\mathcal{X}$-) transformations from the coordinate functions on the initial seed $\mathcal{A}$-torus (resp. seed $\mathcal{X}$-torus) are positive rational functions on this torus. The rational functions $A_i$ obtained this way generate the cluster algebra.

1.2.5. *Cluster modular groupoids.* – A seed cluster transformation $\mathbf{i} \to \mathbf{i}$ is called *trivial*, if the corresponding maps of the seed $\mathcal{A}$-tori as well as of the seed $\mathcal{X}$-tori are the identity maps. [3]

We define the *cluster modular groupoid* $\mathcal{G}_{|\mathbf{i}|}$ as a groupoid whose objects are seeds equivalent to a given seed $\mathbf{i}$, and morphisms are cluster transformations modulo the trivial ones. The fundamental group $\Gamma_{\mathbf{i}}$ of this groupoid (based at $\mathbf{i}$) is called the *cluster modular group*.

---

[3] We conjecture that one of them implies the identity of the other.

1.2.6. *The $\mathcal{A}$- and $\mathcal{X}$- positive spaces.* – We have defined three categories. The first is the groupoid $\mathcal{G}_{|\mathbf{i}|}$. The other two have seed $\mathcal{A}$-/$\mathcal{X}$-tori as objects and cluster transformations of them as morphisms. There are canonical functors from the first category to the second and third. They provide a pair of positive spaces of the same dimension, denoted by $\mathcal{A}_{|\mathbf{i}|}$ and $\mathcal{X}_{|\mathbf{i}|}$, which share a common coordinate groupoid $\mathcal{G}_{|\mathbf{i}|}$. We skip the subscript $|\mathbf{i}|$ whenever possible, writing $\mathcal{X}$ for $\mathcal{X}_{|\mathbf{i}|}$ etc.

1.2.7. *Examples of trivial cluster transformations.* – Given a seed $\mathbf{i}$, denote by $\sigma_{ij}(\mathbf{i})$ a new seed induced by the map of sets $I \to I$ interchanging $i$ and $j$.

PROPOSITION 1.8. – *Let* $h = 2, 3, 4, 6$ *when* $p = 0, 1, 2, 3$ *respectively.* *Then if* $\varepsilon_{ij} = -p\varepsilon_{ji} = -p,$

(16) $$(\sigma_{ij} \circ \mu_i)^{h+2} = a \; trivial \; cluster \; transformation.$$

Relations (16) are affiliated with the rank two Dynkin diagrams, i.e. $A_1 \times A_1, A_2, B_2, G_2$. The number $h = 2, 3, 4, 6$ is the Coxeter number of the diagram. One can present these relations in the form

$$\mu_i \circ \mu_j \circ \mu_i \circ \mu_j \circ \ldots \stackrel{\sim}{=} \sigma_{ij}^{h+2},$$

where the number of mutations on the left equals $h + 2$. Notice that the right hand side is the identity in all but $A_2$ cases. We prove Proposition 1.8 in Section 2.5.

1.2.8. *Special cluster modular groupoid and modular groups.* – *Special trivial seed cluster transformations* are compositions of the one given by (16) and isomorphisms. We do not know any other general procedure to generate trivial cluster transformations.

DEFINITION 1.9. – *Special cluster modular groupoid $\widehat{\mathcal{G}}$ is a connected groupoid whose objects are seeds, and morphisms are cluster transformations modulo the special trivial ones.*

*The fundamental group $\widehat{\Gamma}$ of the groupoid $\widehat{\mathcal{G}}$ is called the* special cluster modular group.

So there is a canonical functor $\widehat{\mathcal{G}} \to \mathcal{G}$ inducing a surjective map $\widehat{\Gamma} \to \Gamma$.

The groupoid $\widehat{\mathcal{G}}$ has a natural geometric interpretation, which justifies Definition 1.9. Namely, thanks to Theorem 2.23 the group $\Gamma$ acts, with finite stabilizers, on a certain manifold with a polyhedral decomposition. So it acts on the dual polyhedral complex $\widehat{M}$, called *the modular complex*. The groupoid $\widehat{\mathcal{G}}$ is identified with the fundamental groupoid of this polyhedral complex, see Theorem 2.30.

1.2.9. *The cluster ensemble.* – We show (Proposition 2.2) that cluster transformations commute with the map $p$ – see (11). So the map $p$ gives rise to a monomial morphism of positive spaces

(17) $$p : \mathcal{A} \longrightarrow \mathcal{X}.$$

DEFINITION 1.10. – *The cluster ensemble related to seed $\mathbf{i}$ is the pair of positive spaces $\mathcal{A}_{|\mathbf{i}|}$ and $\mathcal{X}_{|\mathbf{i}|}$, with common coordinate groupoid $\mathcal{G}_{|\mathbf{i}|}$, related by a (monomial) morphism of positive spaces (17).*

The algebra of regular functions on the $\mathcal{A}$-space is the same thing as the algebra of universal Laurent polynomials $\mathbb{L}(\mathcal{A})$. The Laurent phenomenon theorem [13] implies that the cluster algebra of [12] is a subalgebra of $\mathbb{L}(\mathcal{A})$. The algebra $\mathbb{L}(\mathcal{A})$ is bigger than the cluster algebra in most cases. It coincides with the upper cluster algebra introduced in [1].

Alternatively, one can describe the above families of birational isomorphisms of seed tori by introducing *cluster $\mathcal{X}$- and $\mathcal{A}$-schemes*. By the very definition, a cluster $\mathcal{X}$-scheme is the scheme $\mathcal{X}^*$ related to the positive space $\mathcal{X}$, and similarly the cluster $\mathcal{A}$-scheme. Below we skip the superscript $*$ in the notation for cluster schemes.

Cluster transformations respect both the Poisson structures and the forms $\Omega$. Thus $\mathcal{X}$ is a Poisson space, and there is a 2-form on the space $\mathcal{A}$. (Precisely, $\mathcal{X}$ is understood as a functor from the coordinate groupoid to the appropriate category of Poisson tori.) In particular the manifold $\mathcal{X}(\mathbb{R}_{>0})$ has a $\Gamma$-invariant Poisson structure. We show in Sections 3 and 6 that the Poisson structure on the space $\mathcal{X}$ and the 2-form on the space $\mathcal{A}$ are shadows of more sophisticated structures, namely a non-commutative $q$-deformation of the $\mathcal{X}$-space, and motivic avatars of the form $\Omega$.

1.2.10. *The chiral and Langlands duality for seeds*. – We define the Langlands dual seed by

$$\mathbf{i}^\vee := (I, I_0, \varepsilon_{ij}^\vee, d_i^\vee), \qquad \varepsilon_{ij}^\vee := -\varepsilon_{ji}, \quad d_i^\vee := d_i^{-1}D, \qquad D := \text{l.c.m.}\{d_i\} \,^{(4)}.$$

This procedure is evidently involutive. Here is an alternative description.

   (i) We define the *transposed* seed $\mathbf{i}^t := (I, I_0, \varepsilon_{ij}^t, d_i^t)$, where $\varepsilon_{ij}^t := \varepsilon_{ji}$ and $d_i^t := d_i^{-1}D$.

   (ii) We define the *chiral dual* seed $\mathbf{i}^o := (I, I_0, \varepsilon_{ij}^o, d_i^o)$, where $\varepsilon_{ij}^o := -\varepsilon_{ij}$, and $d_i^o := d_i$.

Definitions (i)-(ii) are consistent with mutations. Combining them, we get the Langlands duality on seeds. On the language of lattices. the Langlands duality amounts to replacing the bilinear form $[a, b]$ to the one $-[b, a]$, and changing the multipliers.

Here is a natural realization of the Langlands dual seed. Let $\Lambda^\vee$ be the lattice dual to the lattice $\Lambda^\circ$. So, given a seed $\mathbf{i} = \left(\Lambda, [*, *]_\Lambda, \{e_i\}, d_i\right)$, there is an isomorphicm of lattices

(18)                    $$\delta_\mathbf{i} : \Lambda \longrightarrow \Lambda^\vee, \qquad e_i \longmapsto e_i^\vee := d_i e_i.$$

Let us introduce a bilinear form on $\Lambda^\vee$ by setting

$$[e_i^\vee, e_j^\vee]_{\Lambda^\vee} := -[e_j, e_i]_\Lambda.$$

LEMMA 1.11. – *The map $\delta_\mathbf{i}$ provides an isomorphism of the Langlands dual seed $\mathbf{i}^\vee$ with the seed*

$$\left(\Lambda^\vee, [*, *]_{\Lambda^\vee}, \{e_i^\vee\}, d_i^\vee\right).$$

*This isomorphism is compatible with mutations, i.e. the following diagram is commutative:*

$$
\begin{array}{ccc}
\{e_i\} & \xrightarrow{\delta_\mathbf{i}} & \{e_i^\vee\} \\
\mu_k \downarrow & & \downarrow \mu_k^\vee \\
\{e_i'\} & \xrightarrow{\delta_{\mathbf{i}'}} & \{(e_i')^\vee\}
\end{array}
$$

---

$^{(4)}$ Here $D$ is the least common multiple of the set of positive integers $d_i$.

*Proof.* – The case $i = k$ is obvious. If $i \neq k$, we have $\mu_k(d_i e_i) = (d_i e_i) + d_i[\varepsilon_{ik}]_+ d_k^{-1}(d_k e_k)$. So the lemma follows from the formula

$$(19) \qquad\qquad d_i \varepsilon_{ik} d_k^{-1} = -\varepsilon_{ki}. \qquad\qquad \square$$

### 1.3. An example: the cluster $\mathcal{X}$-variety structure of the moduli space $\mathcal{M}_{0,n+3}$

The moduli space $\mathcal{M}_{0,n+3}$ parametrises configurations of $n + 3$ distinct points $(x_1, \ldots, x_{n+3})$ on $\mathbb{P}^1$ considered modulo the action of $PGL_2$.

EXAMPLE. – The cross-ratio of four points on $\mathbb{P}^1$, normalized by $r^+(\infty, -1, 0, z) = z$, provides an isomorphism

$$r^+ : \mathcal{M}_{0,4} \xrightarrow{\sim} \mathbb{P}^1 - \{0, -1, \infty\}, \qquad r^+(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_1 - x_4)}.$$

The moduli space $\mathcal{M}_{0,n+3}$ has a cluster $\mathcal{X}$-variety atlas [6], which we recall now. It is determined by a cyclic order of the points $(x_1, \ldots, x_{n+3})$. So although the symmetric group $S_{n+3}$ acts by automorphisms of $\mathcal{M}_{0,n+3}$, only its cyclic subgroup $\mathbb{Z}/(n+3)\mathbb{Z}$ will act by automorphisms of the cluster structure.

Let $P_{n+3}$ be a convex polygon with vertices $p_1, \ldots, p_{n+3}$. We assign the points $x_i$ to the vertices $p_i$, so that the order of points $x_i$ is compatible with the clockwise cyclic order of the vertices. The cluster coordinate systems are parametrized by the set $\mathcal{T}_{n+3}$ of complete triangulations of the polygon $P_{n+3}$. Given such a triangulation $T$, the coordinates are assigned to the diagonals of $T$. The coordinate $X_E^T$ corresponding to a triangulation $T$ and its diagonal $E$ is defined as follows. There is a unique rectangle formed by the sides and diagonals of the triangulation, with the diagonal given by $E$. Its vertices provide a cyclic configuration of four points on $\mathbb{P}^1$. We order them starting from a vertex of the $E$, getting a configuration of four points $(x_1, x_2, x_3, x_4)$ on $\mathbb{P}^1$. Then we set

$$X_E^T := r^+(x_1, x_2, x_3, x_4).$$

There are exactly two ways to order the points as above, which differ by a cyclic shift by two.
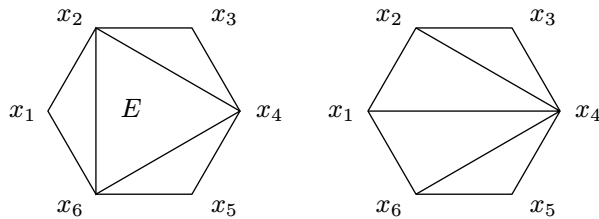


FIGURE 2. The two triangulations of the hexagon are related by the flip at the edge $E$.

Since the cyclic shift by one changes the cross-ratio to its inverse, the rational function $X_E^T$ is well defined. For example the diagonal $E$ on Fig. 2 provides the function $r^+(x_2, x_4, x_6, x_1)$.

We define the cluster seed assigned to a triangulation $T$ as follows. The lattice $\Lambda$ is the free abelian group generated by the diagonals of the triangulation, with a basis given by the diagonals. The bilinear form is given by the adjacency matrix. Namely, two diagonals $E$ and $F$ of the triangulation are called adjacent if they share a vertex, and there are no diagonals

of the triangulation between them. We set $\varepsilon_{EF} = 0$ if $E$ and $F$ are not adjacent. If they are, $\varepsilon_{EF} = 1$ if $E$ is before $F$ according to the clockwise orientation of the diagonals at the vertex $v$ shared by $E$ and $F$, and $\varepsilon_{EF} = -1$ otherwise.

EXAMPLE. – Let us consider a triangulation of $P_{n+3}$ which has the following property: every triangle of the triangulation contains at least one side of the polygon. Then it provides a seed of type $A_n$. For example, a zig-zag triangulation, see Fig. 3, has this property.



FIGURE 3. A zig-zag triangulation of the 11-gon.

One shows that a mutation at a diagonal $E$ corresponds to the flip of the diagonal, see Fig. 2. This means that formula (12) describes the adjacency matrix of the f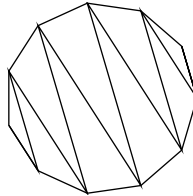lipped triangulation. This way we get a cluster $\mathcal{X}$-variety atlas. One easily sees that the zig-zag triangulation provides a quiver of type $A_n$.

The cyclic order of the points $(x_1, \ldots, x_{n+3})$ provides a connected component $\mathcal{M}^0_{0,n+3}(\mathbb{R})$ in $\mathcal{M}_{0,n+3}(\mathbb{R})$, parametrising configurations of points on $\mathbb{P}^1(\mathbb{R})$ whose cyclic order is compatible with an orientation of $\mathbb{P}^1(\mathbb{R})$. The space of positive points of the cluster $\mathcal{X}$-variety defined above coincides with $\mathcal{M}^0_{0,n+3}(\mathbb{R})$.

REMARK.. – We show in [10] that the Knudsen-Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,n+3}$ can be recovered in a natural way as a cluster compactification of the cluster $\mathcal{X}$-variety of type $A_n$.

## 1.4. Cluster ensemble structures

Below we outline the structures related to a cluster ensemble.

A cluster ensemble gives rise to the following data:

i) A pair of real manifolds $\mathcal{A}(\mathbb{R}_{>0})$ and $\mathcal{X}(\mathbb{R}_{>0})$, provided by the positive structures on $\mathcal{A}$ and $\mathcal{X}$, and a map $p : \mathcal{A}(\mathbb{R}_{>0}) \to \mathcal{X}(\mathbb{R}_{>0})$. For a given seed $\mathbf{i}$, the functions $\log |A_i|$ (resp. $\log |X_i|$) provide diffeomorphisms

$$\alpha_{\mathbf{i}} : \mathcal{A}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathbb{R}^I; \qquad \beta_{\mathbf{i}} : \mathcal{X}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathbb{R}^I.$$

Similarly there are sets of points of $\mathcal{A}$ and $\mathcal{X}$ with values in the tropical semifields $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$.

ii) A modular group $\Gamma$ acts by automorphisms of the whole structure.

iii) A $\Gamma$–invariant Poisson structure $\{*, *\}$ on $\mathcal{X}$. In any $\mathcal{X}$–coordinate system $\{X_i\}$ it is the quadratic Poisson structure given by (9).

iv) A $\Gamma$–invariant 2–form $\Omega$ on $\mathcal{A}$, which in any $\mathcal{A}$–coordinate system $\{A_i\}$ is given by (10). It can be viewed as a presymplectic structure on $\mathcal{A}$.

v) A pair of split algebraic tori of the same dimension, $H_\mathcal{X}$ and $H_\mathcal{A}$. The torus $H_\mathcal{A}$ acts freely on $\mathcal{A}$. The orbits are the fibers of the map $p$, and the leaves of the null foliation for the 2-form $\Omega$. Thus $\mathcal{U} := p(\mathcal{A})$ is a positive symplectic space. Dually, there is a canonical projection $\theta : \mathcal{X} \to H_\mathcal{X}$. Its fibers are the symplectic leaves of the Poisson structure. Moreover $\mathcal{U} = \theta^{-1}(e)$, where $e$ is the unit of $H_\mathcal{X}$. So the natural embedding $i : \mathcal{U} \hookrightarrow \mathcal{X}$ is a Poisson map.

vi) A *quantum space* $\mathcal{X}_q$. It is a non-commutative $q$-deformation of the positive space $\mathcal{X}$, equipped with an involutive antiautomorphism $*$, understood as a functor

$$\psi_q : \widehat{\mathcal{G}} \longrightarrow \mathrm{QPos}^*$$

where $\mathrm{QPos}^*$ is the category of quantum tori with involutive antiautomorphism $*$. Precisely, the category $\mathrm{QPos}^*$ is the opposite category to the category whose objects are quantum tori algebras, and morphisms are positive rational $*$-maps. [5]  For a seed $\mathbf{i}$, the corresponding quantum torus algebra $\psi_q(\mathbf{i})$ is the algebra $\mathbf{T_i}$ generated by the elements $X_i$, $i \in I$, subject to the relations

(20)        $$q^{-\widehat{\varepsilon}_{ij}} X_i X_j = q^{-\widehat{\varepsilon}_{ji}} X_j X_i, \qquad *X_i = X_i, \quad *q = q^{-1}.$$

We denote by $\mathbb{T}_\mathbf{i}$ its non-commutative field of fractions. Given a mutation $\mathbf{i} \to \mathbf{i}'$, the birational map $\psi_q^{\mathbf{i},\mathbf{i}'} : \mathbb{T}_{\mathbf{i}'} \to \mathbb{T}_\mathbf{i}$ is a $q$-deformation of the mutation map (13) from the definition of the positive space $\mathcal{X}$. It is given by the conjugation by the quantum dilogarithm.

There is a canonical projection $\theta_q : \mathcal{X}_q \longrightarrow H_\mathcal{X}$. The inverse images of the characters of $H_\mathcal{X}$ are "quantum Casimirs": they are in the center of $\mathcal{X}_q$ and generate it for generic $q$.

*The quantum space $\mathcal{X}_q$ at roots of unity.* Now suppose that $q^{DN} = 1$, where $D$ is the least common multiple of $d_i$'s. Then, under certain assumption on the exchange function, see Theorem 3.11, there is a *quantum Frobenius map* of quantum spaces

$$\mathbb{F}_N : \mathcal{X}_q \longrightarrow \mathcal{X},$$

which in any cluster coordinate system acts on the cluster coordinates $Y_i$ on $\mathcal{X}$ as $\mathbb{F}_N^* Y_i = X_i^N$. Here $X_i$ the coordinates on $\mathcal{X}_q$. Notice that $\mathbb{F}_N^*$ is a ring homomorphism since $q^N = 1$. So there is a diagram:

$$\mathcal{X}_q \xrightarrow{\theta_q} H_\mathcal{X}$$
$$\mathbb{F}_N \Big\downarrow$$
$$\mathcal{X}$$

The center of the algebra of regular functions $\mathbb{L}(\mathcal{X}_q)$ is generated by the inverse images of the functions on $\mathcal{X}$ and $H_\mathcal{X}$.

vii) *Motivic data.* [6]  There are two levels of understanding:

a) $K_2$–*avatar of* $\Omega$. It is given by a $\Gamma$–invariant class $W \in K_2(\mathcal{A})^\Gamma$.

b) *Motivic dilogarithm class.* For any seed $\mathbf{i}$ the class $W$ can be lifted to an element

$$W_\mathbf{i} = \sum_{i,j} \widetilde{\varepsilon}_{ij} A_i \wedge A_j \in \bigwedge\nolimits^2 \mathbb{Q}(\mathcal{A})^*.$$

---

[5] Quantum torus algebra satisfies Ore's condition, so its non-commutative fraction field and hence rational functions as its elements are defined. The reader may skip positivity from the above definition.

[6] The reader is advised to look at Section 6.2 for the background related to the motivic data.

It has the zero tame symbol at every divisor. It is $H_A$-invariant, and thus is a lift of an element of $\Lambda^2 \mathbb{Q}(\mathcal{U})^*$ by the map $p$. However elements $W_{\mathbf{i}}$ are not equivariant under the action of the cluster modular groupoid, and in particular are not $\Gamma$-invariant. Their behavior under the action of the cluster modular groupoid is described by a class, called *motivic dilogarithm class of cluster ensemble*:

$$\mathbb{W} \in H^2_\Gamma(\mathcal{U}, \mathbb{Q}(2)_\mathcal{M})$$

in the weight two $\Gamma$–equivariant motivic cohomology of the scheme $\mathcal{U}$. We define the weight two motivic cohomology via the dilogarithm complex, also known as the Bloch-Suslin complex.

The simplest way to see the dilogarithm in our story is the following. Mutations act by Poisson automorphisms of $\mathcal{X}(\mathbb{R}_{>0})$. The generating function describing a mutation $\mathbf{i} \to \mathbf{i}'$ at a vertex $k$ is Roger's version of the dilogarithm, applied to the coordinate function $X_k = e^{x_k}$.

In [8], which is the second part of this paper, we introduce one more ingredient of the data which, unlike everything else above, is of analytic nature:

viii) *-*representations*. a) A series of $*$-representations by unbounded operators in Hilbert spaces of the *modular double*

(21)                 $\mathcal{X}_{q,|\mathbf{i}|} \times \mathcal{X}_{q^\vee,|\mathbf{i}^\vee|}, \qquad q = e^{i\pi h},\ q^\vee = e^{i\pi/h}, \quad h \in \mathbb{R}_{\geq 0},$

where $\mathbf{i}^\vee$ is the Langlands dual seed. It is constructed explicitly using the quantum dilogarithm.

b) The modular double of the quantum space $\mathcal{X}_{q,|\mathbf{i}|} \times \mathcal{X}_{q,|\mathbf{i}^\circ|}$, where $\mathbf{i}^\circ$ defines the chiral dual seed, has *canonical* unitary projective representation in the Hilbert space $L_2(\mathcal{A}(\mathbb{R}_{>0}))$.

ix) Finally, in [9], which continues this paper, we define the third ingredient of the cluster ensemble, the *symplectic double*. It is the quasiclassical counterpart of the canonical representation from the part b) of viii). We define its non-commutative $q$-deformation. We construct a series of infinite dimensional unitary projective representations of the special cluster modular groupoid, parametrised by unitary characters of the torus $H_\mathcal{X}(\mathbb{R}_{>0})$. We show that they intertwine the $*$-representations of (21) in certain Schwartz type spaces, and using this prove all claims from viii).

The described cluster quantization is a rather general quantization scheme, which we hope has many applications.

Remarkably the quantization is governed by the motivic avatar of the Weil-Petersson form on the $\mathcal{A}$-space. This and the part b) of vii) show that although the $\mathcal{X}$-space seems to be the primary part of a cluster ensemble – in our basic example it gives the Teichmüller space, while the $\mathcal{A}$-space gives only its decorated unipotent part – one has to study the $\mathcal{X}$ and $\mathcal{A}$ spaces in a package.

### 1.5. Our basic example

Let $G$ be a split semi-simple simply-connected algebraic group over $\mathbb{Q}$. Denote by $G'$ the quotient of $G$ modulo the center. Let $S$ be a hyperbolic surface with non-empty boundary and $m$ distinguished points on the boundary. We defined in [6] a pair of moduli spaces $(\mathcal{X}_{G',S}, \mathcal{A}_{G,S})$, and proved that for $G = SL_m$ it gives rise to a cluster ensemble, leaving the case of general $G$ to the sequel of that paper. We proved that, regardless of the cluster ensemble structure, this pair of moduli spaces for general $G$ has all the described above classical

structures. Here is a more detailed account. The references are made to chapters of [6]. The example discussed in Section 1.3 is the special case when $S$ is a disc with $n+3$ marked points on the boundary, and $G = PGL_2$.

The pair of positive spaces $(\mathcal{X}, \mathcal{A})$ is provided by the pair of positive stacks $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$.

i) The corresponding pair of positive real spaces is the higher Teichmüller space $\mathcal{X}_{G',S}(\mathbb{R}_{>0})$ and its decorated version $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$. The $\mathbb{A}^t$-points of $\mathcal{A}_{SL_2,S}$ give Thurston's laminations (Chapter 12). The space of positive real points of $\mathcal{X}_{PGL_2,S}$ is a version of the classical Teichmüller space on $S$, and the one of $\mathcal{A}_{SL_2,S}$ is Penner's [24] decorated Teichmüller space (Chapter 11).

ii) There is a modular group $\Gamma_{G,S}$ provided by the cluster ensemble structure of the pair $(\mathcal{X}_{G',S}, \mathcal{A}_{G,S})$. The positive spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are $\Gamma_{G,S}$-equivariant positive spaces. The group $\Gamma_{G,S}$ contains as a subgroup the modular group $\Gamma_S$ of the surface $S$. If $G = SL_2$, these two groups coincide. Thus the cluster modular group is a generalization of the classical modular group. Otherwise $\Gamma_{G,S}$ is bigger than $\Gamma_S$. For example, the cluster modular group $\Gamma_{G,S}$ where $G$ is of type $G_2$ and $S$ is a disc with three points on the boundary was calculated in [5]: it is (an infinite quotient of) the braid group of type $G_2$, while its classical counterpart is $\mathbb{Z}/3\mathbb{Z}$.

The quotient $\mathcal{M} := \mathcal{X}(\mathbb{R}_{>0})/\Gamma$ is an analog of the moduli space of complex structures on a surface. We conjecture that the space $\mathcal{M}_{G',S}$ is related to the $W$-algebra for the group $G$ just the same way the classical moduli space (when $G = SL_2$) is related to the Virasoro algebra. We believe that $\mathcal{M}_{G',S}$ is the moduli space of certain objects, $W$-structures, but cannot define them.

iii) There is a canonical projection from the moduli space $\mathcal{X}_{G',S}$ to the moduli space of $G'$-local systems on $S$. The Poisson structure on $\mathcal{X}_{G',S}$ is the inverse image of the standard Poisson structure on the latter by this map.

iv) The form $\Omega$ on the space $\mathcal{A}_{G,S}$ was defined in Chapter 15. For $G = SL_2$ its restriction to the decorated Teichmüller space $\mathcal{A}_{SL_2,S}(\mathbb{R}_{>0})$ is the Weil-Petersson form studied by Penner [24].

v) *The tori $H_\mathcal{A}$ and $H_\mathcal{X}$.* Let $H$ be the Cartan group of $G$, and $H'$ the Cartan group of $G'$. The canonical projection $\theta : \mathcal{X} \longrightarrow H_\mathcal{X}$ and the action of the torus $H_\mathcal{A}$ on the $\mathcal{A}$-space generalize similar structures defined in Chapter 2 for a hyperbolic surface $S$: the canonical projection

$$\mathcal{X}_{G',S} \longrightarrow H'^{\{\text{punctures of } S\}}$$

and the action of $H^{\{\text{punctures of } S\}}$ on the moduli space $\mathcal{A}_{G,S}$.

vi) The results of this paper plus the cluster ensemble structure of the pair $(\mathcal{X}_{PGL_m,S}, \mathcal{A}_{SL_m,S})$ (Chapter 10) provide a quantum space $\mathcal{X}^q_{PGL_m,S}$. For $m = 2$ it is equivalent to the one in [11].

vii) The motivic data for the pair $(\mathcal{X}_{G',S}, \mathcal{A}_{G,S})$ was defined in Chapter 15. It was previously missing even for the classical Teichmüller space. In the case $G = SL_m$ an explicit cocycle representing the class $\mathbb{W}$ is obtained from the explicit cocycle representing the second motivic Chern class of the simplicial classifying space $BSL_m$ defined in [18]. The investigation of this cocycle for $\mathbb{W}$ led us to discovery of the whole picture.

viii) Replacing the Dynkin diagram of the group $G$ by its Langlands dual we get the Langlands dual cluster ensemble. Changing the orientation on $S$ we get the chiral dual cluster ensemble.

The classical Teichmüller space was quantized, independently, in [22] and in [11]: the Poisson manifold $\mathcal{X}_{PGL_2,S}(\mathbb{R}_{>0})$ was quantized in [11], and its symplectic leaf $\mathcal{U}_{PGL_2,S}(\mathbb{R}_{>0})$ in [22].

The principal embedding $SL_2 \hookrightarrow G$, defined up to a conjugation, leads to natural embeddings

$$\mathcal{X}_{PGL_2,S} \hookrightarrow \mathcal{X}_{G,S}, \quad \mathcal{A}_{SL_2,S} \hookrightarrow \mathcal{A}_{G,S}$$

and their counterparts for the Teichmüller, lamination and moduli spaces. However since the cluster modular group $\Gamma_{G,S}$ is bigger than the modular group $\Gamma_S$, it is hard to expect natural $\Gamma_{G,S}$-equivariant projections like $\mathcal{X}_{G,S} \longrightarrow \mathcal{X}_{PGL_2,S}$. We do not know whether $W$-structures on $S$ can be defined as a complex structure plus some extra data on $S$.

The elements $p^*(X_i)$ of the cluster algebra were considered by Gekhtman, Shapiro and Vainshtein [16] who studied various Poisson structures on a cluster algebra. The form $\Omega$ and the connection between Penner's decorated Teichmüller spaces to cluster algebras were independently discovered in [17]. The relation of the form $\Omega$ to the Poisson structures is discussed there.

After the first version of this paper appeared in the ArXiv (math/0311245), Berenstein and Zelevinsky released a paper [2], where they defined and studied $q$-deformations of cluster algebras. In general there is a family of such $q$-deformations, matching the Poisson structures on cluster algebras defined in [16]. The cluster modular group does not preserve individual $q$-deformations. However if $\det \varepsilon_{ij} \neq 0$, the $q$-deformation of cluster algebra is unique, and thus $\Gamma$-invariant.

## 1.6. The structure of the paper

Cluster ensembles are studied in Section 2. We discuss the cluster nature of the Teichmüller theory on a punctured torus, as well as the cluster structure of the pair of universal Teichmüller spaces. In the latter case the set $I$ is the set of edges of the Farey triangulation of the hyperbolic plane, and the modular group is the Thompson group.

In Section 3 we define the non-commutative $\mathcal{X}$-space and establish its properties.

In Section 4 we present our duality conjectures.

In Section 5 we furnish some evidence for the duality conjectures in the finite type case: We define, in the finite type case, a canonical pairing between the tropical points of dual cluster varieties and one of the two canonical maps. Our definitions do not depend on the Classification Theorem, and do not use root systems etc. As a byproduct, we show that the space of real tropical points of a finite type cluster $\mathcal{X}$-variety has a canonical decomposition into cones. It is dual to the generalised associahedra defined in [15].

In Section 6 we introduce motivic structures related to a cluster ensemble. They are defined using the dilogarithmic motivic complex, and play a key role in our understanding of cluster ensembles.

In [8] we started a program of quantization of cluster ensembles using the quantum dilogarithm intertwiners. It is a quantum version of the motivic data from Section 6. It was completed in [9].

## 2. Cluster ensembles and their properties

### 2.1. Cluster ensembles revisited

2.1.1. *Cluster transformations of cluster seed tori.* – A seed mutation $\mu_k : \mathbf{i} \to \mathbf{i}'$ provides birational isomorphisms

$$(22) \qquad \mu_k^x : \mathcal{X}_\mathbf{i} \longrightarrow \mathcal{X}_{\mathbf{i}'} \quad \text{and} \quad \mu_k^a : \mathcal{A}_\mathbf{i} \longrightarrow \mathcal{A}_{\mathbf{i}'}$$

acting on cluster coordinates by formulas (13) and (14), respectively. Seed isomorphisms provide isomorphism of tori, see (15). So a seed cluster transformation $\mathbf{c} : \mathbf{i} \to \mathbf{i}'$ gives rise to birational isomorphisms

$$(23) \qquad \mathbf{c}^a : \mathcal{A}_\mathbf{i} \longrightarrow \mathcal{A}_{\mathbf{i}'}, \qquad \mathbf{c}^x : \mathcal{X}_\mathbf{i} \longrightarrow \mathcal{X}_{\mathbf{i}'}.$$

Recall the coordinate groupoid $\mathcal{G}$. The following lemma results from the very definition.

LEMMA 2.1. – *There are well defined functors*

$$\psi_\mathcal{A} : \mathcal{G} \longrightarrow \text{Pos}, \quad \psi_\mathcal{A}(\mathbf{i}) := \mathcal{A}_\mathbf{i}, \quad \psi_\mathcal{A}(\mu_k) := \mu_k^a,$$

$$\psi_\mathcal{X} : \mathcal{G} \longrightarrow \text{Pos}, \quad \psi_\mathcal{X}(\mathbf{i}) := \mathcal{X}_\mathbf{i}, \quad \psi_\mathcal{X}(\mu_k) := \mu_k^x.$$

Let $\mathcal{A}$ and $\mathcal{X}$ be the positive spaces defined by the functors from Lemma 2.1. These spaces are related as follows. Given a seed $\mathbf{i}$, the map $p$ looks in coordinates as follows:

$$(24) \qquad p_\mathbf{i} : \mathcal{A}_\mathbf{i} \longrightarrow \mathcal{X}_\mathbf{i}, \qquad p_\mathbf{i}^* X_i := \prod_{j \in I} A_j^{\varepsilon_{ij}}.$$

PROPOSITION 2.2. – *The maps of the seed tori* (24) *give rise to a map of positive spaces* $p : \mathcal{A} \longrightarrow \mathcal{X}$.

We will give a proof after a discussion of decomposition of mutations.

2.1.2. *Decomposition of mutations.* – The seed tori $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{A}_{\mathbf{i}'}$ (respectively $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{i}'}$) are canonically identified with the torus $\mathcal{A}_{\Lambda}$ (respectively $\mathcal{X}_{\Lambda}$). Therefore there are tautological isomorphisms

$$\mu_k' : \mathcal{A}_{\mathbf{i}} \xrightarrow{\sim} \mathcal{A}_{\mathbf{i}'}, \qquad \mu_k' : \mathcal{X}_{\mathbf{i}} \xrightarrow{\sim} \mathcal{X}_{\mathbf{i}'}.$$

These isomorphisms, however, do not respect the cluster coordinates on these tori. Therefore there are two ways to write the mutation transformations:

(i) Using the cluster coordinates assigned to the seeds $\mathbf{i}$ and $\mathbf{i}'$, or

(ii) Using the cluster coordinates assigned to the seeds $\mathbf{i}$ only.

Equivalently, in the approach (ii) we present mutation birational isomorphisms (22) as compositions

$$\mu_k^a = \mu_k' \circ \mu_k^\sharp, \qquad \mu_k^\sharp : \mathcal{A}_{\mathbf{i}} \longrightarrow \mathcal{A}_{\mathbf{i}}, \quad \mu_k' : \mathcal{A}_{\mathbf{i}} \xrightarrow{\sim} \mathcal{A}_{\mathbf{i}}',$$

$$\mu_k^x = \mu_k' \circ \mu_k^\sharp, \qquad \mu_k^\sharp : \mathcal{X}_{\mathbf{i}} \longrightarrow \mathcal{X}_{\mathbf{i}}, \quad \mu_k' : \mathcal{X}_{\mathbf{i}} \xrightarrow{\sim} \mathcal{X}_{\mathbf{i}}',$$

and then look at the birational isomorphisms $\mu_k^\sharp$ only.

We usually use the approach (i). In particular formulas (13) and (14) were written this way. However the approach (ii) leads to simpler formulas, which are easier to deal with, especially for the $\mathcal{X}$-space. What is more important, the conceptual meaning of the map $\mu_k^\sharp$ in the $\mathcal{X}$-case becomes crystal clear when we go to the $q$-deformed spaces, see Section 4.

Below we work out these formulas, i.e. compute mutation birational automorphisms $\mu_k^\sharp$ in the cluster coordinates assigned to the seed $\mathbf{i}$.

It is handy to employ the following notation:

$$\mathbb{A}_k^+ := \prod_{i | \varepsilon_{ki} > 0} A_i^{\varepsilon_{ki}}, \qquad \mathbb{A}_k^- := \prod_{i | \varepsilon_{ki} < 0} A_i^{-\varepsilon_{ki}}.$$

Then

$$(25) \qquad \frac{\mathbb{A}_k^+}{\mathbb{A}_k^-} = \prod_j A_j^{\varepsilon_{kj}} = p^* X_k.$$

PROPOSITION 2.3. – *Given a seed $\mathbf{i}$, the birational automorphism $\mu_k^\sharp$ of the seed $\mathcal{A}$-torus (respectively $\mathcal{X}$-torus) acts on the cluster coordinates $\{A_i\}$ (respectively $\{X_i\}$) related to the seed $\mathbf{i}$ as follows:*

$$(26) \qquad A_i \longmapsto A_i^\sharp := A_i(1 + p^* X_k)^{-\delta_{ik}} = \begin{cases} A_i & \text{if } i \neq k, \\ A_k(1 + \mathbb{A}_k^+/\mathbb{A}_k^-)^{-1} & \text{if } i = k, \end{cases}$$

$$(27) \qquad X_i \longmapsto X_i^\sharp := X_i(1 + X_k)^{-\varepsilon_{ik}}.$$

*Proof.* – Let $\{A_i'\}$ be the cluster coordinates in the function field of $\mathcal{A}_{\Lambda}$ assigned to the mutated seed $\mathbf{i}'$. They are related to the cluster coordinates $\{A_i'\}$ assigned to the seed $\mathbf{i}$ as follows:

$$(28) \qquad A_i' \longmapsto \begin{cases} A_i & \text{if } i \neq k, \\ \mathbb{A}_k^-/A_k & \text{if } i = k. \end{cases}$$

These formulas describe the action of the tautological mutation isomorphism $\mu_k'$ on the cluster coordinates. They reflect the action of the seed mutation on the quasidual basis $\{f_i\}$, see (8).

Then the transformation $\mu_k' \circ \mu_k^\sharp$ acts on the coordinates $A_i'$ as follows:

$$(\mu_k^\sharp)^* \circ (\mu_k')^* : A_k' \longmapsto \mathbb{A}_k^- / A_k = \mathbb{A}_k^- (1 + \frac{\mathbb{A}_k^+}{\mathbb{A}_k^-}) A_k^{-1} = \frac{\mathbb{A}_k^+ + \mathbb{A}_k^-}{A_k}.$$

This coincides with the action of the mutation $\mu_k$ on the coordinate $A_k'$. This proves the $\mathcal{A}$-part of the proposition.

Similarly, the tautological mutation isomorphism $\mu_k'$ acts on the coordinates by

$$(29) \qquad X_i' \longmapsto \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(X_k)^{[\varepsilon_{ik}]_+} & \text{if } i \neq k. \end{cases}$$

This reflects the action of the seed mutation on the basis $\{e_i\}$.

Therefore the transformation $\mu_k' \circ \mu_k^\sharp$ acts on the coordinates $X_i'$ as follows:

$$X_k' \longmapsto X_k^{-1} \longmapsto X_k^{-1},$$

and, if $i \neq k$,

$$X_i' \longmapsto X_i(X_k)^{[\varepsilon_{ik}]_+} \longmapsto X_i(1 + X_k)^{-\varepsilon_{ik}}(X_k)^{[\varepsilon_{ik}]_+} = X_i(1 + X_k^{-\operatorname{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}}.$$

This coincides with the action of the mutation $\mu_k$ on the coordinate $X_i'$. The proposition is proved. □

2.1.3. *Proof of Proposition 2.2.* – It is equivalent to the following statement [7]: For each mutation $\mu_k$ of the seed **i** there is a commutative diagram

$$(30) \qquad \begin{array}{ccc} \mathcal{A}_\Lambda & \xrightarrow{\mu_k^\sharp} & \mathcal{A}_\Lambda \\ p \downarrow & & \downarrow p \\ \mathcal{X}_\Lambda & \xrightarrow{\mu_k^\sharp} & \mathcal{X}_\Lambda. \end{array}$$

Let us prove this statement. Going up and to the left we get

$$X_i \longmapsto \prod_{j \in I} A_j^{\varepsilon_{ij}} \longmapsto \prod_{j \in I} A_j^{\varepsilon_{ij}} \cdot (A_k^\sharp / A_k)^{\varepsilon_{ik}}.$$

Going to the left and up we get the same:

$$X_i \longmapsto X_i(1 + X_k)^{-\varepsilon_{ik}} \longmapsto \prod_{j \in I} A_j^{\varepsilon_{ij}}(1 + p^* X_k)^{-\varepsilon_{ik}}.$$

DEFINITION 2.4. – *The space $\mathcal{U}$ is the image of the space $\mathcal{A}$ under the map $p$.*

We leave to the reader to check that the space $\mathcal{U}$ is indeed a positive space.

COROLLARY 2.5. – *Assume that* $\det \varepsilon_{ij} \neq 0$. *Let* $\mathbf{c} : \mathbf{i} \to \mathbf{i}$ *be a seed cluster transformation. It gives rise to cluster transformations* $\mathbf{c}^a$ *and* $\mathbf{c}^x$ *of the $\mathcal{A}$ and $\mathcal{X}$ spaces. Then* $\mathbf{c}^a = \operatorname{Id}$ *implies* $\mathbf{c}^x = \operatorname{Id}$.

*Proof.* – Assume that $\det \varepsilon_{ij} \neq 0$. Then the map of algebras $p^* : \mathbb{Z}[\mathcal{X}_\mathbf{i}] \longrightarrow \mathbb{Z}[\mathcal{A}_\mathbf{i}]$ is an injection, and commutes with the cluster transformations thanks to Proposition 2.2. This implies the claim. □

[7] As pointed out by a referee, relation between $p^* X_i$ and $p^* X_i'$ is equivalent to Lemma 1.2 in [16].

2.1.4. *A Poisson structure on the $\mathcal{X}$-space*

LEMMA 2.6. – *Cluster transformations preserve the Poisson structure on the seed $\mathcal{X}$-tori. Therefore the space $\mathcal{X}$ has a Poisson structure.*

*Proof.* – This can easily be checked directly, and also follows from a similar but stronger statement about the $q$-deformed cluster $\mathcal{X}$-varieties proved, independently of the lemma, in Lemma 3.3. □

*A Poisson structure on the real tropical $\mathcal{X}$-space.* Given a seed, we define a Poisson bracket $\{*, *\}$ on the space $\mathcal{X}(\mathbb{R}^t)$ by $\{x_i, x_j\} := \widehat{\varepsilon}_{ij}$. Since mutations are given by piecewise linear transformations, it makes sense to ask whether it is invariant under mutations – the invariance of the Poisson structure should be understood on the domain of differentiability. It is easy to check that this Poisson bracket does not depend on the choice of the seed.

## 2.2. The $\mathcal{X}$-space is fibered over the torus $H_\mathcal{X}$

Consider the left kernel of the form (5):

$$(31) \qquad \mathrm{Ker}_L[*, *] := \{l \in \Lambda \mid [l, v] = 0 \quad \text{for every } v \in \Lambda\} .$$

Given a seed $\mathbf{i}$, there is an isomorphism

$$\mathrm{Ker}_L[*, *] = \left\{ \{\alpha_i\} \in \mathbb{Z}^I \mid \sum_{i \in I} \alpha_i \varepsilon_{ij} = 0 \quad \text{for every } j \in I \right\}.$$

Denote by $H_\mathcal{X}$ the split torus with the group of characters $\mathrm{Ker}_L[*, *]$. The tautological inclusion $\mathrm{Ker}_L[*, *] \hookrightarrow \Lambda$ provides a surjective homomorphism

$$\theta : \mathcal{X}_\Lambda \to H_\mathcal{X}.$$

Denote by $\chi_\alpha$ the character of the torus $H_\mathcal{X}$ corresponding to $\alpha \in \mathrm{Ker}_L[*, *]$. In the cluster coordinates assigned to a seed $\mathbf{i}$ we have $\theta^* \chi_\alpha = \prod_{i \in I} X_i^{\alpha_i}$.

LEMMA 2.7. – *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{X}_\Lambda & \xrightarrow{\mu_k^x} & \mathcal{X}_\Lambda \\ \theta \downarrow & & \downarrow \theta \\ H_\mathcal{X} & \xrightarrow{\sim} & H_\mathcal{X}. \end{array}$$

*Proof.* – Follows from the quantum version, see Lemma 3.10. □

Let us interpret the torus $H_\mathcal{X}$ as a tautological positive space, i.e. as a functor

$$(32) \qquad \theta : \mathcal{G} \longrightarrow \text{ the category of split algebraic tori,}$$

sending objects to the torus $H_\mathcal{X}$, and morphisms to the identity map. Then Lemma 2.7 implies

COROLLARY 2.8. – *There is a unique map of positive spaces $\theta : \mathcal{X} \longrightarrow H_\mathcal{X}$ such that for any seed*

$$\theta^*(\chi_\alpha) := \prod_{i \in I} X_i^{\alpha_i}.$$

Let $e$ be the unit of $H_{\mathcal{X}}$. Thanks to Lemma 2.7 the fibers $\theta^{-1}(e)$ of the maps $\theta : \mathcal{X}_{\Lambda} \longrightarrow H_{\mathcal{X}}$ are glued into a positive space. It is the space $\mathcal{U}$ from Definition 2.4.

Similarly, for a general $h$, the fibers $\theta^{-1}(h)$ can be glued into an object generalizing positive space. We are not going to develop this point of view, observing only that the $\mathbb{R}_{>0}$-points of the fibers make sense as manifolds.

PROPOSITION 2.9. – a) *The fibers of the map $\theta$ are the symplectic leaves of the Poisson space $\mathcal{X}$.*

b) *In particular the fibers of the map $\theta : \mathcal{X}(\mathbb{R}_{>0}) \longrightarrow H_{\mathcal{X}}(\mathbb{R}_{>0})$ are the symplectic leaves of the Poisson manifold $\mathcal{X}(\mathbb{R}_{>0})$.*

REMARK. – Here in a) by the fibers we mean the collection of varieties $\theta^{-1}(h) \subset \mathcal{X}_{\mathbf{i}}$ and birational isomorphisms between them provided by Lemma 2.7. The claim is that they are symplectic leaves in the tori $\mathcal{X}_{\mathbf{i}}$, and the gluing maps respect the symplectic structure.

*Proof.* – Follows immediately from Lemma 1.5.                                   □

## 2.3. The torus $H_{\mathcal{A}}$ acts on the $\mathcal{A}$-space

Consider the right kernel of the form $[*, *]$:

(33) $$\mathrm{Ker}_R[*, *] := \{l \in \Lambda \mid [v, l] = 0 \quad \text{for any } v \in \Lambda\}.$$

Given a seed $\mathbf{i}$, there is an isomorphism

$$\mathrm{Ker}_R[*, *] = \left\{\{\beta_j\} \in \mathbb{Z}^I \stackrel{\sim}{=} \Lambda \mid \sum_{j \in I} \varepsilon_{ij}\beta_j = 0 \quad \text{for any } i \in I\right\}.$$

Recall the map $\delta_{\mathbf{i}}$, see (18). Thanks to Lemma 1.11, the lattice

(34) $$\mathrm{K}_{\mathcal{A}} := \delta_{\mathbf{i}}(\mathrm{Ker}_R[*, *]) \subset \Lambda^{\vee}$$

does not depend on the choice of $\mathbf{i}$. It is, of course, isomorphic to the lattice $\mathrm{Ker}_R[*, *]$.

Let $H_{\mathcal{A}}$ be the torus with the group of cocharacters (34). Observe that the lattice $\Lambda^{\vee}$ is the group of cocharacters of the torus $\mathcal{A}_{\Lambda}$. Thus the homomorphism

$$\mathrm{K}_{\mathcal{A}} \times \Lambda^{\vee} \longrightarrow \Lambda^{\vee}$$

given by action of the lattice $\mathrm{K}_{\mathcal{A}}$ on the lattice $\Lambda^{\vee}$ gives rise to a homomorphism of tori

(35) $$H_{\mathcal{A}} \times \mathcal{A}_{\Lambda} \to \mathcal{A}_{\Lambda}, \quad \chi_{\beta}(t) \times (a_1, \ldots, a_n) \longmapsto (t^{\beta_1}a_1, \ldots, t^{\beta_n}a_n),$$

where $\chi_{\beta} : \mathbb{G}_m \to H_{\mathcal{A}}$ is the cocharacter assigned to $\beta$.

LEMMA 2.10. – a) *The maps (35) glue into an action of the torus $H_{\mathcal{A}}$ on the $\mathcal{A}$-space.*

b) *The projection $p : \mathcal{A} \longrightarrow \mathcal{X}$ is the factorization by the action of the torus $H_{\mathcal{A}}$. Moreover, there is an "exact sequence":*

$$\mathcal{A} \xrightarrow{p} \mathcal{X} \xrightarrow{\theta} H_{\mathcal{X}} \longrightarrow 1, \qquad \mathrm{Im}\, p = \theta^{-1}(1).$$

*Proof.* – a) Taking into account the decomposition of the mutations, the claim amounts to the commutativity of the following diagram:

$$
\begin{array}{ccc}
H_{\mathcal{A}} \times \mathcal{A}_{\Lambda} & \longrightarrow & \mathcal{A}_{\Lambda} \\
\mathrm{Id} \downarrow \mu_k^{\sharp} & & \downarrow \mu_k^{\sharp} \\
H_{\mathcal{A}} \times \mathcal{A}_{\Lambda} & \longrightarrow & \mathcal{A}_{\Lambda}.
\end{array}
$$

The coordinate $A_k$ under the composition of the right and top arrows transforms as follows:

$$
A_i \longmapsto A_i(1 + p^* X_k)^{-\delta_{ik}} \longmapsto t^{\beta_i} A_i(1 + p^* X_k)^{-\delta_{ik}}.
$$

The other composition gives the same. Indeed, since $\sum_j \varepsilon_{kj} \beta_j = 0$, the transformation $A_j \longmapsto t^{\beta_j} A_j$ does not change expression (25). So the two ways to compute the transformation of the coordinate $A_k$ in the diagram lead to the same result.

b) Clear. The lemma is proved. $\qquad\square$

LEMMA 2.11. – *There is a canonical isomorphism $H_{\mathcal{A}} \otimes \mathbb{Q} = H_{\mathcal{X}} \otimes \mathbb{Q}$.*

*Proof.* – This just means that $(\mathrm{Ker}_R[*, *] \otimes \mathbb{Q})^{\vee} = \mathrm{Ker}_L[*, *] \otimes \mathbb{Q}$. $\qquad\square$

REMARK. – Observe that $\mathrm{Ker}_R[*, *] = \mathrm{Ker}_L[*, *]^t$. There is a canonical isomorphism

$$
(36) \qquad\qquad X_*(H_{\mathcal{A}}) = X^*(H_{\mathcal{X}^{\vee}}).
$$

Indeed, both abelian groups are identified with $\mathrm{Ker}_R[*, *]$. It plays an essential role in Section 4.

LEMMA 2.12. – *There are canonical group homomorphisms*

$$
\Gamma \longrightarrow \mathrm{Aut}\left(\mathrm{Ker}_L[*, *]\right), \qquad \Gamma \longrightarrow \mathrm{Aut}\left(\mathrm{Ker}_R[*, *]\right).
$$

*Proof.* – Clear from the very definition. $\qquad\square$

## 2.4. Cluster modular groups revisited

*The simplicial complex $\mathbb{S}$ [12].* – Let **i** be a seed, $n := |I|$, $m := |I - I_0|$. Let $S$ be an $(n-1)$-dimensional simplex, equipped with a bijection (decoration)

$$
(37) \qquad\qquad \{\text{Codimension one faces of } S\} \xrightarrow{\sim} I.
$$

We call it a $I$-decorated, or simply decorated simplex.

Take a decorated simplex, and glue to it $m$ other decorated simplices as follows. To each codimension one face of the initial simplex decorated by an element $k \in I - I_0$ we glue a new decorated simplex along its codimension one face decorated by the same $k$. Then to each of the remaining codimension one faces decorated by the elements of $I - I_0$ we glue new decorated simplices, and so on, repeating this construction infinitely many times. We get a simplicial complex $\mathbb{S}$. Let **S** be the set of all its simplices.

We connect two elements of **S** by an edge if the corresponding simplices share a common codimension one face. We get an $m$-valent tree with the set of vertices **S**. Its edges are decorated by the elements of the set $I - I_0$. We denote it by Tr.

LEMMA 2.13. – *There are canonical bijections:*

{*Seeds equivalent to a seed* $\mathbf{i}$} $\leftrightarrow$ {*The set* $\mathbf{S}$ *of simplices of the simplicial complex* $\mathbb{S}$},

{*Compositions of seed mutations*} $\leftrightarrow$ {*Paths on the tree* Tr}.

*Proof.* – The seed $\mathbf{i}$ is assigned to the original simplex $S$. Given any other simplex $S'$ of the simplicial complex $\mathbb{S}$, there is a unique path on the tree Tr connecting $S$ with $S'$. It gives rise to a sequence of mutations parametrised by the edges of the path, so that the edge decorated by $k$ gives rise to the mutation in the direction $k$. Mutating the seed $\mathbf{i}$ by this sequence of mutations, we get the seed assigned to the simplex $S'$. The lemma follows. $\qquad\square$

REMARK. – Cluster $\mathcal{A}$-coordinates are assigned to vertices of the simplicial complex $\mathbb{S}$. Cluster $\mathcal{X}$-coordinates are assigned to cooriented faces of $\mathbb{S}$. Changing coorientation amounts to inversion of the corresponding cluster $\mathcal{X}$-coordinate. Mutations are parametrized by codimension one faces of $\mathbb{S}$.

2.4.1. *Another look at the cluster modular groups.* – Let $\mathbf{F}(\mathbb{S})$ be the set of all pairs $(S, F)$ where $S$ is a simplex of $\mathbb{S}$, and $F$ is a codimension one face of $S$. Pairs of faces belonging to the same simplex are parametrized by the fibered product

$$(38) \qquad\qquad\qquad\qquad \mathbf{F}(\mathbb{S}) \times_{\mathbf{S}} \mathbf{F}(\mathbb{S}).$$

The collection of exchange functions $\varepsilon_{ij}$ can be viewed as a single function $\mathcal{E}$ on the set (38).

Let Aut$(\mathbb{S})$ be the automorphism group of the simplicial complex $\mathbb{S}$. It contains the subgroup Aut$_0(\mathbb{S})$ respecting the decorations (37). The group Aut$(\mathbb{S})$ is a semidirect product:

$$0 \longrightarrow \mathrm{Aut}_0(\mathbb{S}) \longrightarrow \mathrm{Aut}(\mathbb{S}) \longrightarrow \mathrm{Per} \longrightarrow 0.$$

Here Per is the group of automorphisms of the pair $(I, I_0)$. Given $\mathbf{i} \in \mathbf{S}$, the group Per is realized as a subgroup Aut$(\mathbb{S})$ permuting the faces of $S_{\mathbf{i}}$.

The group Aut$(\mathbb{S})$ acts on the set (38), and hence on the set of exchange functions $\mathcal{E}$.

DEFINITION 2.14. – *Let $\mathcal{E}$ be the exchange function corresponding to a seed* $\mathbf{i}$.
*The group $D$ is the subgroup of* Aut$(\mathbb{S})$ *preserving $\mathcal{E}$:*

$$(39) \qquad\qquad\qquad D := \{\gamma \in \mathrm{Aut}(\mathbb{S}) \quad | \quad \gamma^*(\mathcal{E}) = \mathcal{E}\};$$

*The group $\Delta$ is the subgroup of $D$ preserving the cluster $\mathcal{A}$- and $\mathcal{X}$-coordinates:*

$$(40) \qquad\qquad\qquad \Delta := \{\gamma \in D \quad | \quad \gamma^* A_j = A_j, \quad \gamma^* X_j = X_j\};$$

*The cluster complex $C$ is the quotient of $\mathbb{S}$ by the action of the group $\Delta$:*

$$C := \mathbb{S}/\Delta.$$

Clearly $\Delta$ is a normal subgroup of $D$.

LEMMA 2.15. – *The quotient group $\Gamma := D/\Delta$ is the cluster modular group.*

*Proof.* – For any two simplices $S_{\mathbf{i}}$ and $S_{\mathbf{i}'}$ of $\mathbb{S}_{I,I_0}$ there exists a unique element of the group Aut$_0(\mathbb{S}_{I,I_0})$ transforming $S_{\mathbf{i}}$ to $S_{\mathbf{i}'}$. So given an element $d \in D$, there is a cluster transformation $\mathbf{c}_d : \mathbf{i} \to \mathbf{i}'$. Then by (39) there is a seed isomorphism $\sigma_d : \mathbf{i}' \to \mathbf{i}$. Thanks to (40) the cluster transformation $\sigma_d \circ \mathbf{c}_d : \mathbf{i} \to \mathbf{i}$ is trivial. $\qquad\square$

**Variants.** – In Definition 2.14 and Lemma 2.15 we looked how the $\mathcal{A}$- and $\mathcal{X}$-coordinates behave under cluster transformations. There are similar definitions using either $\mathcal{A}$-coordinates, or $\mathcal{X}$-coordinates. This way we get the groups $\Delta_?, \Gamma_?$, and the simplicial complexes $C_?$, where ? stands, respectively, for $\mathcal{A}$ and $\mathcal{X}$. The cluster complex $C_\mathcal{A}$ was defined in [FZII]. Corollary 2.5 immediately implies
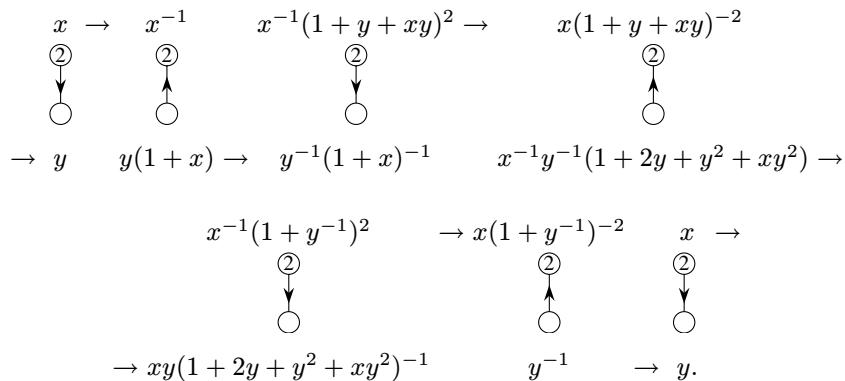
LEMMA 2.16. – *Assume that* $\det \varepsilon_{ij} \neq 0$. *Then* $\Delta_\mathcal{A} = \Delta \subset \Delta_\mathcal{X}$, $C_\mathcal{A} = C$, *and* $\Gamma_\mathcal{A} = \Gamma$.

### 2.5. Example: Cluster transformations for $\mathcal{X}$-varieties of types $A_1 \times A_1, A_2, B_2, G_2$.

The results of Section 2.5 play a crucial role in Section 5.2. We start by an elaboration of the cluster transformations for cluster $\mathcal{X}$-variety of type $B_2$. Its main goal is to tell the reader how we picture mutations, quivers etc. The obtained formulas, however, do not seem very illuminating.

We show that the situation changes dramatically when we go to the tropicalised cluster transformations. Notice that they contain just the same information as the original cluster transformations. The advantage of the tropicalised formulas stems from the fact that they are piecewise linear transformations, and thus can be perceived geometrically. We demonstrate this idea by working out tropicalisations of cluster transformations for every finite type cluster $\mathcal{X}$-variety of rank two, uncovering an interesting geometry standing behind. We calculate cluster modular groups in these cases.

2.5.1. *Quivers and cluster transformations in the $B_2$ case.* – We picture a seed by a quiver with two vertices. The cluster $\mathcal{X}$-coordinates assigned to a seed are the functions written near the corresponding vertices. Every two neighboring seeds are related by a horizontal arrow, associated with one of the vertices of the left quiver. It shows a mutation in the direction of that vertex. The exchange function $\varepsilon$ is determined as follows. Denote by $b$ and $t$ the bottom and top vertices. Then for the very left quiver $\varepsilon_{bt} = -2$, $\varepsilon_{tb} = 1$. For the next one, $\varepsilon_{bt} = 2$, $\varepsilon_{tb} = -1$, and so on. This sequence of mutations has period 6. Similar calculations can be done for the seeds of types $A_1 \times A_1, A_2, G_2$.

$$x \;\to\quad x^{-1} \qquad x^{-1}(1+y+xy)^2 \;\to\qquad x(1+y+xy)^{-2}$$



$$\to \; y \qquad y(1+x) \;\to\quad y^{-1}(1+x)^{-1} \qquad x^{-1}y^{-1}(1+2y+y^2+xy^2) \;\to$$

$$x^{-1}(1+y^{-1})^2 \qquad \to x(1+y^{-1})^{-2} \qquad x \;\to$$



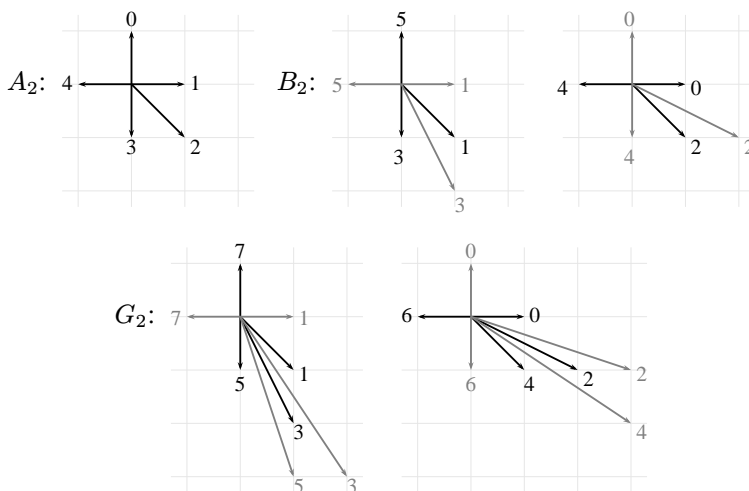$$\to \; xy(1+2y+y^2+xy^2)^{-1} \qquad y^{-1} \qquad \to \; y.$$

FIGURE 4. Orbits of mutations.

2.5.2. *Geometry of the tropicalised cluster transformations in the finite type rank two case.* –
Take a finite type rank two seed $\mathbf{i}$, with $I = \{1, 2\}$.

*Case $A_2$.* In this case all seeds are isomorphic. Consider a cluster transformation

$$\mu := \sigma_{1,2} \circ \mu_1 : \mathbf{i} \longrightarrow \sigma_{1,2}(\mathbf{i}) \sim \mathbf{i}.$$

Let $P$ be the tropicalisation of the cluster $\mathcal{X}$-torus corresponding to the seed $\mathbf{i}$. It is a plane
with coordinates $(x, y)$. The cluster transformation $\mu$ induces a map

$$\mu^t : P \longrightarrow P, \quad x \longmapsto y + \max(0, x), \quad y \longmapsto -x.$$

The plane $P$ is decomposed into 5 sectors as shown on Fig. 4. Three of them are coordi-
nate quadrants. The other two are obtained by subdividing the remaining quadrant into
two sectors. We order the sectors clockwise cyclically, starting from the positive quadrant
$\{(x, y) \mid x, y \geq 0\}$.

LEMMA 2.17. – *The map $\mu^t$ moves the $i$-th sector to the $(i + 1)$-st. Its restriction to any
sector is linear. The sectors are the largest domains in $P$ on which any power of the map $\mu^t$ is
linear.*

*Proof.* – The vectors $(0, 1), (1, 0), (1, -1), (0, -1), (-1, 0)$ are the primitive integral vec-
tors generating the boundary arrows of the domains. The map $\mu^t$ moves them cyclically
clockwise. The lemma follows easily from this.                                    □

COROLLARY 2.18. – *One has $\mu^5 = $ Id. The element $\mu$ generates the modular group of the
cluster $\mathcal{X}$-variety of type $A_2$, and identifies it with $\mathbb{Z}/5\mathbb{Z}$.*

*Cases $A_1 \times A_1$.* This is the simplest case. We have

$$\mu^t : P \longrightarrow P, \quad x \longmapsto y, \quad y \longmapsto -x.$$

There are four sectors in this case, given by the coordinate quadrangles.

Lemma 2.19. – *The map $\mu^t$ moves the $i$-th sector to the $(i+1)$-st. Its restriction to any sector is linear. The sectors are the largest domains in $P$ on which any power of the map $\mu^t$ is linear.*

*One has $\mu^4 = \mathrm{Id}$. The element $\mu$ generates the modular group of the cluster $\mathcal{X}$-variety of type $A_1 \times A_1$, and identifies it with $\mathbb{Z}/4\mathbb{Z}$.*

*Cases $B_2$ and $G_2$.* In these cases there are two non-isomorphic seeds, denoted $\mathbf{i}_-$ and $\mathbf{i}_+$. Consider cluster transformations

$$\mu_- := \sigma_{1,2} \circ \mu_1 : \mathbf{i}_- \longrightarrow \sigma_{1,2}(\mathbf{i}_+), \qquad \mu_+ := \sigma_{1,2} \circ \mu_1 : \mathbf{i}_+ \longrightarrow \sigma_{1,2}(\mathbf{i}_-).$$

There are two tropical planes $P_-$ and $P_+$ with coordinates $(x,y)$, which are tropicalisations of the cluster $\mathcal{X}$-tori corresponding to the seeds $\mathbf{i}_-$ and $\mathbf{i}_+$. The cluster transformations $\mu_\pm$ induce the maps

$$\mu_-^t : P_- \longrightarrow P_+, \qquad x \longmapsto y + \max(0,x), \qquad y \longmapsto -x,$$
$$\mu_+^t : P_+ \longrightarrow P_-, \qquad x \longmapsto y + c\max(0,x), \qquad y \longmapsto -x.$$

Here $c = 2,3$ for the Dynkin diagrams $B_2, G_2$ respectively. The maps $\mu_\pm^t$ have the following geometric description. Each of the planes $P_\pm$ is decomposed into a union of $h+2$ sectors. These sectors include all coordinate quadrants but the bottom right one. The remaining sectors subdivide that quadrant as shown on Fig. 4. The boundaries of these sectors are arrows whose directing vectors are:

$$B_2 : \qquad P_- : (1,-1), (1,-2); \qquad P_+ : (2,-1), (1,-1).$$

$$G_2 : \qquad P_- : (1,-1), (2,-3)(1,-2), (1,-3); \qquad P_+ : (3,-1), (2,-1), (3,-2), (1,-1).$$

The remaining four directing vectors are $(0,1), (1,0), (0,-1), (-1,0)$ in both cases. Let us order the sectors clockwise cyclically, starting from the positive coordinate quadrant.

Lemma 2.20. – *The map $\mu_-^t$ moves the $i$-th sector on $P_-$ to the $(i+1)$-st sector on $P_+$. The map $\mu_+^t$ moves the $i$-th sector on $P_+$ to the $(i+1)$-st sector on $P_-$. They are linear maps on the sectors. The sectors are the largest domains in $P_\pm$ on which any composition of the map $\mu_\pm^t$ is linear.*

*Proof.* – In the $B_2$ case we get two sequences of vectors, shown by black and grey on Fig. 4:

(41) $\qquad (0,1) \xrightarrow{\mu_-} (1,0) \xrightarrow{\mu_+} (1,-1) \xrightarrow{\mu_-} (1,-1) \xrightarrow{\mu_+} (0,-1) \xrightarrow{\mu_-} (-1,0) \xrightarrow{\mu_+} (0,1).$

(42) $\qquad (1,0) \xrightarrow{\mu_-} (2,-1) \xrightarrow{\mu_+} (1,-2) \xrightarrow{\mu_-} (0,-1) \xrightarrow{\mu_+} (-1,0) \xrightarrow{\mu_-} (0,1) \xrightarrow{\mu_+} (1,0).$

In the $G_2$ case we also get two sequences of vectors, shown by black and grey on Fig. 4:

$(0,1) \xrightarrow{\mu_-} (1,0) \xrightarrow{\mu_+} (1,-1) \xrightarrow{\mu_-} (2,-1) \xrightarrow{\mu_+} (1,-2) \xrightarrow{\mu_-} (1,-1) \xrightarrow{\mu_+} (0,-1) \xrightarrow{\mu_-} (-1,0) \xrightarrow{\mu_+} (0,1).$

$(1,0) \xrightarrow{\mu_-} (3,-1) \xrightarrow{\mu_+} (2,-3) \xrightarrow{\mu_-} (3,-2) \xrightarrow{\mu_+} (1,-3) \xrightarrow{\mu_-} (0,-1) \xrightarrow{\mu_+} (-1,0) \xrightarrow{\mu_-} (0,1) \xrightarrow{\mu_+} (1,0).$

One checks that the cluster transformations $\mu_\pm$ are linear on each of the sectors. $\qquad \square$

COROLLARY 2.21. – *One has $(\mu_+\mu_-)^3 = \mathrm{Id}$ in the case $B_2$, and $(\mu_+\mu_-)^4 = \mathrm{Id}$ in the case $G_2$.*

*The modular group of the cluster $\mathcal{X}$-variety of type $B_2$ is $\mathbb{Z}/3\mathbb{Z}$. Its generator is $\mu_+\mu_-$.*

*The modular group of the cluster $\mathcal{X}$-variety of type $G_2$ is $\mathbb{Z}/4\mathbb{Z}$. Its generator is $\mu_+\mu_-$.*

2.5.3. *Proof of Proposition 1.8.* – Since $\det\varepsilon_{ij} \neq 0$, the $\mathcal{A}$-coordinates in the rank two cases have the same period. This settles the proposition for the rank two case. The claim in general for the $\mathcal{A}$-coordinates as well as the exchange functions $\varepsilon_{ij}$ was proved in [12]. The claim for the $\mathcal{X}$-coordinates can be reduced to it via the following trick. One can find a set $I'$ containing $I$ and a skew-symmetrisable exchange function $\varepsilon'_{ij}$ on $I' \times I'$ extending $\varepsilon_{ij}$ on $I \times I$ such that $\det\varepsilon'_{ij} \neq 0$. Then the claim follows from Corollary 2.5, since the composition of the $\mathcal{A}$-mutations assigned to the standard $(h+2)$-gon is trivial for any seed.

## 2.6. Modular complexes, modular orbifolds and special modular groups

A simplex of the simplicial complex $C_?$ is of *finite type* if the set of all simplices of $C_?$ containing it is finite. Here "?" stands for $\mathcal{A}$, $\mathcal{X}$, or no label at all.

DEFINITION 2.22. – *The reduced cluster complex $C_?^*$ is the union of finite type simplices of $C_?$.*

The reduced cluster complex is not a simplicial complex: certain faces of its simplices may not belong to it. But it has a topological realization.

THEOREM 2.23. – *Topological realizations of the reduced cluster complexes $C^*$ and $C_{\mathcal{A}}^*$ are homeomorphic to manifolds.*

*Proof.* – We give a proof for the cluster complex $C^*$ – the case of $C_{\mathcal{A}}^*$ similar, and a bit simpler.

A simplicial complex is of *finite type* if it has a finite number of simplices. According to the Classification Theorem [14] cluster algebras of finite type, i.e. the ones with the cluster complexes of finite type, are classified by the Dynkin diagrams of type $A, B, \ldots, G_2$. The cluster complex of type $A_n$ is a Stasheff polytope. The cluster complexes corresponding to other finite type cluster algebras are the *generalized associahedra*, or generalized Stasheff polytopes [15].

We need the following crucial lemma.

LEMMA 2.24. – *Let $S'_{\mathbf{i}}$ be a simplex of finite type in the cluster complex $C$. Then the set of all simplices containing $S'_{\mathbf{i}}$ is naturally identified with the set of all faces of a generalized Stasheff polytope, so that the codimension $i$ simplices correspond to the $i$-dimensional faces.*

*Proof.* – Let $I_{\mathbf{i}}$ be the set of vertices of a top dimensional simplex $S_{\mathbf{i}}$ of the simplicial complex $C$. The exchange function $\varepsilon$ is a function on $I_{\mathbf{i}} \times I_{\mathbf{i}}$. Let $S'_{\mathbf{i}}$ be a finite type simplex contained in the simplex $S_{\mathbf{i}}$. Let $I'_{\mathbf{i}} \subset I_{\mathbf{i}}$ be the subset of the vertices of $S'_{\mathbf{i}}$. The set of top dimensional simplices of $C$ containing $S'_{\mathbf{i}}$ is obtained from the simplex $S_{\mathbf{i}}$ as follows. Consider the seed defined by the function $\varepsilon'$ with the frozen variables parametrized by the subset $I'_{\mathbf{i}}$. Recall that this means that we do mutations only at the vertices of $I_{\mathbf{i}} - I'_{\mathbf{i}}$. Since the simplex $S'_{\mathbf{i}}$ is finite type, it gives rise to a finite type cluster algebra. Indeed, since by the very definition

$\Delta$ is a subgroup of $\Delta_{\mathcal{A}}$, the simplicial complex $C$ has more simplices than $C_{\mathcal{A}}$. So if $C$ is of finite type, $C_{\mathcal{A}}$ is also of finite type. Therefore the matrix $\varepsilon_{ij}$ is non-degenerate by the Classification Theorem. So by Corollary 2.5 in our case, $C_{\mathcal{A}} = C$. The corresponding cluster complex is the generalized Stasheff polytope corresponding to the Cartan matrix assigned to the exchange function $\varepsilon'$ on the set $(I_{\mathbf{i}} - I'_{\mathbf{i}})^2$. It is a convex polytope [4]. This proves the lemma. $\qquad\square$

Let us deduce the theorem from this lemma. Consider a convex polyhedron $P$. Take the dual decomposition of its boundary, and connect each of the obtained polyhedrons with a point inside of $P$ by straight lines. We get a conical decomposition of $P$. Let us apply this construction to the generalized Stasheff polytope. Then the product of the interior part of the simplex $S'_{\mathbf{i}}$ and the defined above conical decomposition of the generalized Stasheff polytope corresponding to the exchange function $\varepsilon'$ on $(I_{\mathbf{i}} - I'_{\mathbf{i}})^2$ gives the link of the interior part of the simplex $S'_{\mathbf{i}}$. In particular a neighborhood of any interior point of $S'_{\mathbf{i}}$ is topologically a ball. The theorem is proved. $\qquad\square$

CONJECTURE 2.25. – *The simplicial complex $C_{\mathcal{X}}$ is of finite type if and only if $C$ is of finite type.*

Proposition 1.8 implies that Conjecture 2.25 is valid if $|I - I_0| = 2$.

*The cluster modular complex.* Suppose that we have a decomposition of a manifold on simplices, although some faces of certain simplices may not belong to the manifold. Then the dual polyhedral decomposition of the manifold is a polyhedral complex. Its topological realization is homotopy equivalent to the manifold.

Thanks to Theorem 2.23 the reduced cluster complex $C^*$ is homeomorphic to a manifold. Therefore the dual polyhedral complex for $C^*$ is a polyhedral complex whose topological realization is homotopy equivalent to a manifold. This motivates the following two definitions.

DEFINITION 2.26. – *The cluster modular complex $\widehat{M}$ is the dual polyhedral complex for the reduced cluster complex $C^*$.*

*The modular orbifold.* The cluster modular group $\Gamma$ acts on $C^*$, and hence on $\widehat{M}$. The stabilizers of points are finite groups.

DEFINITION 2.27. – *The cluster modular orbifold $M$ is the orbifold $\widehat{M}/\Gamma$.*

The fundamental groupoid of a polyhedral complex $P$ is a groupoid whose objects are vertices of $P$, and morphisms are homotopy classes of paths between the vertices.

THEOREM 2.28. – *The special modular groupoid $\widehat{\mathcal{G}}$ is the fundamental groupoid of the cluster modular orbifold $M$. The special modular group $\widehat{\Gamma}$ is the fundamental group of the orbifold $M$, centered at a vertex of $M$*

*Proof.* – The morphisms in the fundamental groupoid of a polyhedral complex $P$ can be described by generators and relations as follows. The generators are given by the edges of $P$. The relations correspond to the two dimensional cells of $P$. So to prove the theorem, we describe the 2-skeleton of the polyhedral complex $\widehat{M}$. We start from a reformulation of Lemma 2.24:

COROLLARY 2.29. – *Any cell of the polyhedral complex $\widehat{M}$ is isomorphic to the generalized Stasheff polytope corresponding to a Dynkin diagram from the Cartan-Killing classification.*

This implies the following description of the 2-skeleton of the polyhedral complex $\widehat{M}$. The 1-skeleton of $\widehat{M}$ is the quotient $\mathrm{Tr}/\Delta$. Let us describe the 2-cells of $\widehat{M}$. As was discussed in Section 2.5, if $\varepsilon_{ij}$ is one of the following matrices

$$(43) \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$$

then performing mutations at the vertices $i, j, i, j, i, \ldots$ we get an $(h + 2)$-gon, where $h$ is the Coxeter number of the Dynkin diagram of type $A_1 \times A_1, A_2, B_2, G_2$ respectively, i.e. $h = 2, 3, 4, 6$. These $(h + 2)$-gons are called the *standard $(h + 2)$-gons* in $\widehat{M}$. We get

COROLLARY 2.30. – *Any 2-cell of the polyhedral complex $\widehat{M}$ is a standard $(h + 2)$-gon, $h = 2, 3, 4, 6$.*

Therefore the edges and 2-cells of the polyhedral complex $\widehat{M}$ match the generators and the relations of the special modular groupoid from Definition 1.9. The theorem is proved.  □

REMARK. – The vertices of the modular orbifold are parametrized by the functions on the set $I \times I$ up to permutation equivalence obtained by mutations of an initial exchange function $\varepsilon_{ij}$. So the number of cells is infinite if and only if the absolute value of the cluster function $\mathcal{E}$ is unbounded.

HYPOTHESIS 2.31. – *The reduced cluster complex $C^*$ (or, equivalently, the modular complex $\widehat{M}$) is simply connected.*

Hypothesis 2.31 is equivalent to the one that the canonical epimorphism $\widehat{\Gamma} \to \Gamma$ is an isomorphism. So in the cases when it is satisfied we have a transparent description of the modular group $\Gamma$: all relations come from the standard $(h + 2)$-gons.

A cluster ensemble is of *finite type* if the cluster complex $C$ is of finite type. Lemma 2.16 implies that the classification of finite type cluster ensembles is the same as the one for cluster algebras. Hypothesis 2.31 is valid for cluster ensembles of finite type if and only if $|I| > 2$. Indeed, in the finite type case the cluster complex $C$ is the boundary of a generalized Stasheff polytope, which is a convex polytope [4]. So the topological realization of $C$ is homeomorphic to a sphere.

REMARK. – Just the same way as we prove Theorem 2.23, Conjecture 2.25 implies that the topological realization of the reduced cluster complex $C_{\mathcal{X}}^*$ is homeomorphic to a manifold.

## 2.7. Cluster nature of the classical Teichmüller space

2.7.1. *Cluster data for the Teichmüller space of an oriented hyperbolic surface $S$ with punctures*. – It was defined in Chapter 11 of [6]. Consider a trivalent tree $T$ embedded into $S$, homotopy equivalent to $S$. Let $\Lambda$ be the lattice generated by the edges of $T$. It has a basis given by the edges. The skew-symmetric matrix $\varepsilon_{EF}$, where $E$ and $F$ run through the set of edges of the tree $T$, is defined as follows. Each edge $E$ of $T$ determines two flags, defined as pairs $(v, E)$ where $v$ is a vertex of an edge $E$. Given two flags $(v, E)$ and $(v, F)$ sharing the same vertex, we define $\delta_{v,E,F} \in \{-1, 1\}$ as follows: $\delta_{v,E,F} = +1$ (respectively $\delta_{v,E,F} = -1$) if the edge $F$ goes right after (respectively right before) the edge $E$ according to the orientation of the surface, see Fig. 5. For each pair $(E, F)$ of the edges of $T$, consider the set $v(E, F)$ of their common vertices. It has at most two elements. We set

$$(44) \qquad\qquad \varepsilon_{EF} := \sum_{v \in v(E,F)} \delta_{v,E,F} \in \{\pm 2, \pm 1, 0\}.$$



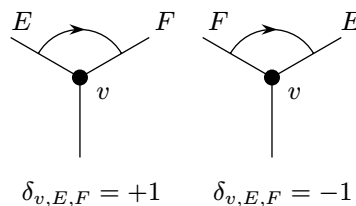$$\delta_{v,E,F} = +1 \qquad \delta_{v,E,F} = -1$$

FIGURE 5. The function $\delta_{E,F}$.

We defined in Chapter 3 of [6] a polyhedral complex $\mathbb{G}_S$, called the *modular complex*. Its vertices are parametrized by the isotopy classes of trivalent graphs on $S$ which are homotopy equivalent to $S$. Its dimension $k$ faces correspond to the isotopy classes of graphs $G$ on $S$, homotopy equivalent to $S$, such that valency $\text{val}(v)$ of each vertex $v$ of $G$ is $\geq 3$, and $k = \sum_v (\text{val}(v) - 3)$.

The modular complex $\mathbb{G}_S$ can be identified with the cluster modular complex for the exchange function (44). Since $\mathbb{G}_S$ is known to be contractible, Hypothesis 2.31 is valid in this case. This suggests that Hypothesis 2.31 may be valid in a large class of examples. The faces of the modular complex $\mathbb{G}_S$ are the Stasheff polytopes or their products: this illustrates Theorem 2.23. Since $\mathbb{G}_S$ is contractible, the two versions $T$ and $\widehat{T}$ of the cluster modular group are isomorphic, and identified with the modular group of $S$.

2.7.2. *Cluster data for the Teichmüller space of the punctured torus $S$*. – There is a unique up to isomorphism trivalent ribbon graph corresponding to a punctured torus. It is shown on Fig. 6 embedded in the punctured torus: the puncture is at the identified vertices of the square. Let us number its edges by $\{1, 2, 3\}$. The general recipe in the case of the punctured

torus leads to an exchange function $\varepsilon_{ij}$ given by the skew-symmetric matrix

$$(45) \qquad \varepsilon_{ij} = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Its quiver is shown on the left of Fig. 1. Mutations change the sign of the function $\varepsilon_{ij}$: $\varepsilon'_{ij} = -\varepsilon_{ij}$.
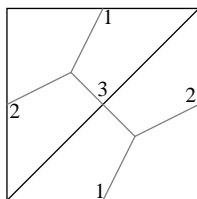


FIGURE 6. The trivalent ribbon graph corresponding to a punctured torus.

It is well known that in this case the modular group is $PSL_2(\mathbb{Z})$. Consider the classical modular triangulation of the upper half plane, obtained by reflections of the geodesic triangle with vertices at $0, 1, \infty$. If we mark the vertices of one of the triangles by elements of the set $\{1, 2, 3\}$ then there is a unique way to mark the vertices of the modular triangulation by the elements of the same set so that the vertices of each triangle get distinct marks, see Fig. 7. The set of vertices of the modular triangulation is the set of the cusps, identified with $P^1(\mathbb{Q})$. The set of the edges of the modular triangulation inherits a decoration by the elements of the set $\{1, 2, 3\}$ such that a vertex of each of the modular triangles and the side opposite to this vertex are labeled by the same element.

The modular triangulation is the simplicial complex $\mathbb{S}$. The dual graph of this triangulation without the vertices is a trivalent tree. Its edges inherit labels by the elements of the set $\{1, 2, 3\}$. So it is an $\{1, 2, 3\}$-decorated tree. It is the polyhedral complex $\mathbb{G}_S$ for the punctured torus $S$: an embedded graph as on Fig. 6 corresponds to a vertex of this tree. A flip at an edge of this graph corresponds to the flip at the corresponding edge of the tree.
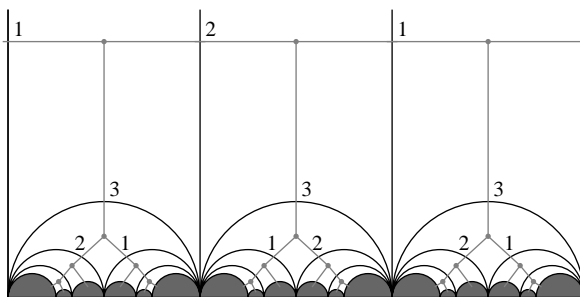


FIGURE 7. The modular triangulation of the upper half plane, and the dual tree.

LEMMA 2.32. – *For the cluster ensemble related to the Teichmüller space on the punctured torus there are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{S}) = PGL_2(\mathbb{Z}), \quad D = PSL_2(\mathbb{Z}), \quad \Delta = \{e\}, \quad \Gamma = PSL_2(\mathbb{Z}), \quad C - C^* = P^1(\mathbb{Q}).$$

*Proof.* – Recall that $PGL_2(\mathbb{R})$ acts on $\mathbb{C} - \mathbb{R}$. This action commutes with the complex conjugation $c : z \rightarrow \bar{z}$ acting on $\mathbb{C} - \mathbb{R}$. Thus $PGL_2(\mathbb{R})$ acts on the quotient $(\mathbb{C} - \mathbb{R})/c$, which is identified with the upper half plane. The subgroup $PGL_2(\mathbb{Z})$ preserves the modular picture. The canonical homomorphism $p : \mathrm{Aut}(\mathbb{S}) \rightarrow \mathrm{Perm}(I)$ is the projection $PGL_2(\mathbb{Z}) \rightarrow S_3$ provided by the action on the set $\{1, 2, 3\}$. The subgroup $PSL_2(\mathbb{Z})$ of $PGL_2(\mathbb{Z})$ preserves the function $\varepsilon_{ij}$. So it is the cluster subgroup $D$. The cluster subgroup $\Delta$ is trivial in this case. This agrees with the fact that the matrix $\varepsilon_{ij}$ has no principal $2 \times 2$ submatrix from the list (43), and thus the dual polyhedral complex is reduced to a tree. So the modular group is $PSL_2(\mathbb{Z})$. The modular triangulation of the upper half plane coincides with the topological realization of the reduced cluster complex $C^*$. One has $C - C^* = P^1(\mathbb{Q})$. The lemma is proved.                                                    □

Below we explain the cluster nature of the universal Teichmüller space, from which the case of a hyperbolic surface $S$ can be obtained by taking the $\pi_1(S)$-invariants.

## 2.8. Cluster nature of universal Teichmüller spaces and the Thompson group

DEFINITION 2.33. – *The universal Teichmüller space $\mathcal{X}^+$ is the space of $PGL_2(\mathbb{R})$-orbits on the set of maps*

$$(46) \qquad\qquad\qquad\qquad \beta : \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\mathbb{R})$$

*respecting the natural cyclic order of both sets.*

A generalization to an arbitrary split simple Lie group with trivial center $G$ is in [6]. The name and relationship with Teichmüller spaces $\mathcal{T}_S^+$ for surfaces $S$ with punctures is explained below.

Let $\mathbb{S}^1(\mathbb{R})$ be the set of all rays in $\mathbb{R}^2 - \{0, 0\}$. There is a $2 : 1$ cover $\mathbb{S}^1(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$. Let $s$ be the antipodal involution. It is the unique non-trivial automorphism of this covering. Let $\mathbb{S}^1(\mathbb{Q})$ be the set of its rational points, given by the rays with rational slopes.

DEFINITION 2.34. – *Consider the set of all maps*

$$(47) \qquad \alpha : \mathbb{S}^1(\mathbb{Q}) \longrightarrow \mathbb{R}^2 - \{0, 0\} \quad \textit{satisfying the condition} \quad \alpha(s(p)) = -\alpha(p)$$

*such that composing $\alpha$ with the projection $\mathbb{R}^2 - \{0, 0\} \longrightarrow \mathbb{S}^1(\mathbb{R})$ we get a map $\bar{\alpha} : \mathbb{S}^1(\mathbb{Q}) \longrightarrow \mathbb{S}^1(\mathbb{R})$ respecting the natural cyclic order of both sets. The universal decorated Teichmüller space $\mathcal{A}^+$ is the quotient of this set by the natural action of the group $SL_2(\mathbb{R})$ on it.*

*The Thompson group $\mathbb{T}$.* It is the group of all piecewise $PSL_2(\mathbb{Z})$ automorphisms of $\mathbb{P}^1(\mathbb{Q})$: for every $g \in \mathbb{T}$, there exists a decomposition of $\mathbb{P}^1(\mathbb{Q})$ into the union of a finite number of segments, which may overlap only at the ends, such that the restriction of $g$ to each segment is given by an element of $PSL_2(\mathbb{Z})$.

Consider the Farey triangulation $T$ of the hyperbolic plane $\mathcal{H}$ shown on Fig. 7 and 8. Let $T$ be the dual trivalent tree. We have canonical identifications

$$(48) \qquad \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty = \{\text{vertices of the Farey triangulation}\}.$$

Let

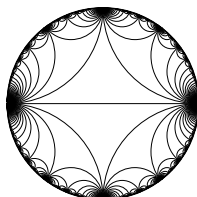$$I_{\mathcal{F}} := \{\text{edges of the Farey triangulation}\}.$$



FIGURE 8. The Farey ideal triangulation of the hyperbolic disc.

It is identified with the edges of the dual tree $T$. Therefore applying (44) to the latter, we get a skew-symmetric function $\varepsilon_{ij} : I_{\mathcal{F}} \times I_{\mathcal{F}} \to \{0, \pm 1\}$. Observe that although $I_{\mathcal{F}}$ is an infinite set, for any $i \in I_{\mathcal{F}}$ the function $I_{\mathcal{F}} \to \mathbb{Z}$, $j \longmapsto \varepsilon_{ij}$ has a finite support. It is easy to see that in such situation all the constructions above work. So we get the corresponding seed and the cluster ensemble, called *Farey cluster ensemble*.

THEOREM 2.35. – *Let $(\mathcal{X}_{\mathcal{F}}, \mathcal{A}_{\mathcal{F}})$ be the Farey cluster ensemble. Then*

i) $\mathcal{X}_{\mathcal{F}}(\mathbb{R}_{>0})$ *is identified with the universal Teichmüller space $\mathcal{X}^+$.*

ii) $\mathcal{A}_{\mathcal{F}}(\mathbb{R}_{>0})$ *is identified with the universal decorated Teichmüller space $\mathcal{A}^+$.*

iii) *The modular group $\Gamma$ of the Farey cluster ensemble is the Thompson group. It is isomorphic to the group $\widehat{\Gamma}$.*

*Proof.* – The proof of the parts i) and ii) is very similar to the proofs in the finite genus case given in Chapter 11 of [6]. Let us outline the proof of i). Let us identify once and for all the sets $\mathbb{P}^1(\mathbb{Q})$ in (48) and (46).

LEMMA 2.36. – *There is a canonical isomorphism $\varphi : \mathcal{X}^+ \xrightarrow{\sim} \mathbb{R}^I_{>0}$.*

*Proof.* – It assigns to a map $\beta$ a function $\varphi(\beta)$ on $I$ defined as follows. Let $E$ be an edge of the Farey triangulation. Denote by $v_1, v_2, v_3, v_4$ the vertices of the 4-gon obtained by taking the union of the two triangles sharing $E$. We assume that the vertices follow an orientation of the circle, and the vertex $v_1$ does not belong to the edge $E$. Recall the cross-ratio $r^+$ normalized by $r^+(\infty, -1, 0, x) = x$. Then

$$\varphi(\beta)(E) := r^+(\beta(v_1), \beta(v_2), \beta(v_3), \beta(v_4)).$$

To prove that every positive valued function on the set $I$ is realized we glue one by one triangles of the triangulation to the initial one in the hyperbolic disc so that the cross-ratio corresponding to each edge $E$ by the above formula is the value of the given function at $E$. The lemma is proved. $\square$

It remains to check that flips at the edges of the Farey triangulation are given by the same formulas as the corresponding mutations in the cluster ensemble. This is a straightforward check, left to the reader.

ii) Let us define a map $\mathcal{A}^+ \to \mathbb{R}^I_{>0}$. Take an edge $E$ of the Farey triangulation. Let $p_1(E)$, $p_2(E)$ be the endpoints of the edge $E$, considered as the points of $\mathbb{P}^1(\mathbb{Q})$. The map (47) assigns to them vectors $v_1(E)$, $v_2(E)$, each well defined up to a sign. The coordinate corresponding to the edge $E$ is the absolute value of the area of the parallelogram in $\mathbb{R}^2$ generated by these vectors. One checks that the exchange relation follows from the Plücker relation.

iii) Here is another way to look at the Thompson group. The Farey triangulation has a distinguished oriented edge, connecting $0$ and $\infty$. The Thompson group contains the following elements, called flips at the edges: Given an edge $E$ of the Farey triangulation $T$, we do a flip at an edge $E$ obtaining a new triangulation $T'$ with a distinguished oriented edge. This edge is the old one if $E$ is not the distinguished oriented edge, and it is the flip of the distinguished oriented edge otherwise. Observe that the ends of the trivalent trees dual to the triangulations $T$ and $T'$ are identified, each of them with $P^1(\mathbb{Q})$. On the other hand, there exists a unique isomorphism of the plane trees $T$ and $T'$ which identifies their distinguished oriented edges. It provides a map of the ends of these trees, and hence an automorphism of $P^1(\mathbb{Q})$, which is easily seen to be piece-wise linear. The Thompson group is generated by flips at the edges ([21]). It remains to check that the relations in the Thompson group corresponds to the standard pentagons in the cluster complex. Indeed, these pentagons are exactly the pentagons of the Farey triangulation, it is well known [3] that they give rise to relations in the Thompson group, and all relations are obtained this way.

Let us prove that $\widehat{\Gamma} = \Gamma$. First, $\widehat{\Gamma} = \Gamma$ is true for the cluster ensemble related to a triangulation of an $n$-gon. Indeed, in this case the cluster modular complex is nothing else than the Stasheff polytope, which is simply connected if $n > 4$. This implies that the same is true for the modular triangulation. The theorem is proved. $\qquad\square$

The universal decorated Teichmüller space $\mathcal{A}^+$ is isomorphic to the one defined by Penner [24].

*Relation with the Teichmüller spaces of surfaces.* Given a torsion free subgroup $\pi \subset PSL_2(\mathbb{Z})$, set $S_\pi := \mathcal{H}/\pi$. The Teichmüller space $\mathcal{X}^+_{S_\pi}$ is embedded into $\mathcal{X}^+$ as the subspace of $\pi$-invariants:

$$\mathcal{X}^+_{S_\pi} = (\mathcal{X}^+)^\pi.$$

Let us define this isomorphism. The Teichmüller space $\mathcal{X}^+_S$ of a surface with punctures $S$ has canonical coordinates corresponding to an ideal triangulation of $S$. We have a natural triangulation on $S_\pi$, the image of the Farey triangulation under the projection $\pi_\pi : \mathcal{H} \to S_\pi$. So $\mathcal{X}^+_{S_\pi}$ is identified with the $\mathbb{R}_{>0}$-valued functions on $I_{\mathcal{F}}/\pi$, i.e. with $\pi$-invariant $\mathbb{R}_{>0}$-valued functions on $I_{\mathcal{F}}$.

### 3. A non-commutative $q$-deformation of the $\mathcal{X}$-space

#### 3.1. Heisenberg groups and quantum tori

Let $\Lambda$ be a lattice equipped with a skew-symmetric bilinear form $(*, *) : \Lambda \times \Lambda \to \mathbb{Z}$. We associate to this datum a Heisenberg group $\mathcal{H}_\Lambda$. It is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_\Lambda \longrightarrow \Lambda \longrightarrow 0.$$

The composition law is given by the rule

$$\{v_1, n_1\} \circ \{v_2, n_2\} = \{v_1 + v_2, n_1 + n_2 + (v_1, v_2)\}, \qquad v_i \in \Lambda, n_i \in \mathbb{Z}.$$

DEFINITION 3.1. – *Let $\Lambda$ be a lattice equipped with a skew-symmetric bilinear form $(*, *) : \Lambda \times \Lambda \to \mathbb{Z}$. The corresponding quantum torus algebra $\mathbf{T}_\Lambda$ is the group ring of the Heisenberg group $\mathcal{H}_\Lambda$.*

Let $q$ be the element of the group algebra corresponding to the central element $(0, 1) \in \mathcal{H}_\Lambda$. Denote by $X_v$ the element of the group algebra corresponding to the element $(v, 0) \in \mathcal{H}_\Lambda$. Then

$$q^{-(v_1, v_2)} X_{v_1} X_{v_2} = X_{v_1 + v_2}.$$

In particular the left hand side is symmetric in $v_1, v_2$. There is an involutive antiautomorphism

$$* : \mathbf{T}_\Lambda \longrightarrow \mathbf{T}_\Lambda, \qquad *(X_v) = X_v, \ *(q) = q^{-1}.$$

Choose a basis $\{e_i\}$ of the lattice $\Lambda$. Set $X_i := X_{e_i}$. Then the algebra $\mathbf{T}_\Lambda$ is identified with the algebra of non-commutative polynomials in $\{X_i\}$ over the ring $\mathbb{Z}[q, q^{-1}]$ subject to the relations

$$(49) \qquad q^{-\widehat{\varepsilon}_{ij}} X_i X_j = q^{-\widehat{\varepsilon}_{ji}} X_j X_i, \qquad \widehat{\varepsilon}_{ij} := (e_i, e_j).$$

Let us choose an order $e_1, \ldots, e_n$ of the basis of $\Lambda$. Then given a vector $v = \sum_{i=1}^n a_i e_i$ of $\Lambda$, one has

$$(50) \qquad X_v = q^{-\sum_{i<j} a_i a_j \widehat{\varepsilon}_{ij}} \prod_{i=1}^n X_i^{a_i}.$$

In particular the right hand side does not depend on the choice of the order.

The above construction gives rise to a functor from the category of lattices with skew-symmetric forms to the category of non-commutative algebras with an involutive antiautomorphism $*$.

There is a version of this construction where $q$ is a complex number with absolute value 1 and $*$ is a semilinear antiautomorphism preserving the generators $X_i$. There is a specialization homomorphism of $*$–algebras sending the formal variable $q$ to its value.

3.1.1. *Center of the quantum torus algebra at roots of unity*. – Let us start with an example. Let $\Lambda$ be a lattice spanned by elements $e_1, e_2$ with $(e_1, e_2) = 1$. Then the quantum torus algebra $\mathbf{T}_\Lambda$ is isomorphic to the algebra of non-commutative Laurent polynomials satisfying the relation $qX_1X_2 = q^{-1}X_2X_1$. Suppose now that $q$ is an $N$-th root of unity. Due to the $q$-binomial formula [23] we have

$$(X_1 + X_2)^N = X_1^N + X_2^N.$$

Obviously $X_i^N$ are central elements. One easily proves that the center of the algebra $\mathbf{T}_\Lambda$ is the algebra of Laurent polynomials in $X_1^N, X_2^N$.

One can reformulate this using the geometric language as follows. Let $Y_1, Y_2$ be the coordinate functions on the two dimensional torus $\mathbb{G}_m \times \mathbb{G}_m$. Then there is a natural homomorphism of algebras

$$\mathbb{F}_N^* : \mathcal{O}(\mathbb{G}_m \times \mathbb{G}_m) \hookrightarrow \mathbf{T}_\Lambda, \qquad Y_i \longmapsto X_i^N.$$

Its image is the center of $\mathbf{T}_\Lambda$. We may think about this map as of the *quantum Frobenius map*

$$\mathbb{F}_N : \operatorname{Spec}(\mathbf{T}_\Lambda) \longrightarrow \mathbb{G}_m \times \mathbb{G}_m.$$

Here is a generalization of this example. Let $\Lambda_0$ be the kernel of the form $(*, *)$. Then $\mathbf{T}_{\Lambda_0}$ is in the center of $\mathbf{T}_\Lambda$. For generic $q$ the center coincides with $\mathbf{T}_{\Lambda_0}$. Geometrically, the center is generated by the preimages of the characters under the Casimir map

$$\theta_q : \operatorname{Spec}(\mathbf{T}_\Lambda) \longrightarrow \operatorname{Spec}(\mathbf{T}_{\Lambda_0}) = \operatorname{Hom}(\Lambda_0, \mathbb{G}_m).$$

In the $q^N = 1$ case we have in addition to this the Frobenius map

$$\mathbb{F}_N : \operatorname{Spec}(\mathbf{T}_\Lambda) \longrightarrow \operatorname{Hom}(\Lambda, \mathbb{G}_m).$$

The center in this case is generated by the preimages of the characters under these maps.

Below we show how to glue quantum tori, so that at roots of unity the quantum Frobenius map will be preserved.

## 3.2. The quantum dilogarithm

Consider the following formal power series, a version of the inverse of the *Pochhammer symbol*:

(51)
$$\mathbf{\Psi}_q(x) := \frac{1}{(-qx; q^2)_\infty} = \prod_{a=1}^\infty (1 + q^{2a-1}x)^{-1} = \frac{1}{(1 + qx)(1 + q^3x)(1 + q^5x)(1 + q^7x)\dots}.$$

It is a *q-analog of the gamma function*. It is characterized, up to a constant, by a difference relation

(52) $\quad \mathbf{\Psi}_q(q^2 x) = (1 + qx)\mathbf{\Psi}_q(x), \quad$ or, equivalently, $\quad \mathbf{\Psi}_q(q^{-2}x) = (1 + q^{-1}x)^{-1}\mathbf{\Psi}_q(x).$

It is also called the *q-exponential*. The name is justified by the power series expansion

$$\mathbf{\Psi}_q(x) = \sum_{n=0}^\infty \frac{q^{-\frac{n(n-1)}{2}} x^n}{(q - q^{-1})(q^2 - q^{-2})\dots(q^n - q^{-n})}.$$

It is easily checked by using the difference relation. There is a power series expansion of the inverse of $\mathbf{\Psi}_q(x)$:

$$\mathbf{\Psi}_q(x)^{-1} = \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(1 - q^2)(1 - q^4) \ldots (1 - q^{2n})}.$$

The difference relation immediately implies the following property of the $q$-exponential power series:

(53) $$\mathbf{\Psi}_{q^{-1}}(x) = \mathbf{\Psi}_q(x)^{-1}.$$

Indeed, both parts of the equation satisfy the equivalent difference relations (52).

Formal power series (51) are also known by the name the *quantum dilogarithm power series*. To justify the name, recall a version of the classical dilogarithm function:

$$\mathrm{L}_2(x) := \int_0^x \log(1 + t) \frac{dt}{t} = -\mathrm{Li}_2(-x).$$

It has a $q$-deformation, called the *$q$-dilogarithm power series*, given by

$$\mathrm{L}_2(x; q) := \sum_{n=1}^{\infty} \frac{x^n}{n(q^n - q^{-n})}.$$

One has the identity

$$\log \mathbf{\Psi}_q(x) = \mathrm{L}_2(x; q).$$

It is proved easily by using the difference relations (52) characterizing $\mathbf{\Psi}_q(x)$.

The precise relation with the classical dilogarithm is the following. If $|q| < 1$ the power series $\mathbf{\Psi}_q(x)^{-1}$ converge, providing an analytic function in $x \in \mathbb{C}$. If, in addition to this, $|x| < 1$, the $q$-dilogarithm power series also converge. There are asymptotic expansions when $q \to 1^-$:

(54) $$\mathrm{L}_2(x; q) \sim \frac{\mathrm{L}_2(x)}{\log q^2}, \qquad \mathbf{\Psi}_q(x) \sim \exp\left(\frac{\mathrm{L}_2(x)}{\log q^2}\right).$$

### 3.3. The quantum space $\mathcal{X}_q$

According to Definition 1.4, a seed $\mathbf{i}$ includes a lattice $\Lambda$ with a skew-symmetric bilinear form $(*, *)$, and thus determines a quantum torus $*$-algebra $\mathbf{T}_\Lambda$, denoted $\mathbf{T}_\mathbf{i}$. Using the basis $\{e_i\}$ it is described by generators and relations, see (49). Denote by $\mathbb{T}_\mathbf{i}$ the non-commutative fraction field of $\mathbf{T}_\mathbf{i}$.

The quantum mutation map $\mu_k^q$ is an isomorphism of skew fields

$$\mu_k^q : \mathbb{T}_{\mathbf{i}'} \longrightarrow \mathbb{T}_\mathbf{i}.$$

The simplest way to define it employs the following fact: The algebras $\mathbf{T}_\mathbf{i}$ for the seeds $\mathbf{i}$ related by seed cluster transformations are canonically isomorphic, since each of them is identified with the algebra $\mathbf{T}_\Lambda$,

DEFINITION 3.2. – *The mutation homomorphism $\mu_k^q : \mathbb{T}_{\mathbf{i}'} \longrightarrow \mathbb{T}_\mathbf{i}$ is the conjugation by $\mathbf{\Psi}_{q_k}(X_k)$, where $X_k = X_{e_k}$ is a basis element for the seed $\mathbf{i}$:*

$$\mu_k^q := \mathrm{Ad}_{\mathbf{\Psi}_{q_k}(X_k)}, \qquad q_k := q^{1/d_k}.$$

In other words, the map $\mu_k^q$ is defined as the unique map making the following diagram commutative. Here the vertical maps are the canonical isomorphisms.

$$
\begin{array}{ccc}
\mathbb{T}_{\mathbf{i}'} & \xrightarrow{\mu_k^q} & \mathbb{T}_{\mathbf{i}} \\
\sim\downarrow & & \downarrow\sim \\
\mathbb{T}_\Lambda & \xrightarrow{\mathrm{Ad}_{\Psi_{q_k}(X_k)}} & \mathbb{T}_\Lambda.
\end{array}
$$

Although $\Psi_{q_k}(X_k)$ is not a rational function, we show in Lemma 3.4 that $\mu_k^q$ is a rational map.

LEMMA 3.3. – *The map $\mu_k^q$ is a homomorphism of $*$-algebras.*

*Proof.* – The map $\mu_k^q$ is given by the conjugation. So is a homomorphism of algebras. It commutes with the involution $*$ thanks to (53). Indeed, since $*X_i = X_i$, we have

$$
*\left(\Psi_q(X_k)X_i\Psi_q(X_k)^{-1}\right) = \Psi_{q^{-1}}(X_k)^{-1}X_i\Psi_{q^{-1}}(X_k) \stackrel{(53)}{=} \Psi_q(X_k)X_i\Psi_q(X_k)^{-1}. \quad \square
$$

3.3.1. *Decomposition of quantum mutations.* – Although the algebras $\mathbf{T}_{\mathbf{i}'}$ and $\mathbf{T}_{\mathbf{i}}$ are canonically isomorphic, they are equipped with different sets of the generators – the cluster coordinates – $\{X_{e_i'}\}$ and $\{X_{e_i}\}$. Let us write the mutation map in the cluster coordinates. Then we have

$$
\mu_k^q = \mu_k^\sharp \circ \mu_k', \qquad \mu_k^\sharp := \mathrm{Ad}_{\Psi_{q_k}(X_k)} : \mathbf{T}_{\mathbf{i}} \longrightarrow \mathbf{T}_{\mathbf{i}},
$$

where $\mu_k'$ is a map which tells how the coordinates related to the basis $\{e_i'\}$ are related to the ones related to the basis $\{e_i\}$. It is given in the cluster coordinates as follows:

$$
(55) \qquad \mu_k' : \mathbf{T}_{\mathbf{i}'} \longrightarrow \mathbf{T}_{\mathbf{i}}, \quad X_{e_i'} \longmapsto X_{e_i'} = X_{e_i + [\varepsilon_{ik}]_+ e_k} = q^{-\widehat{\varepsilon}_{ik}[\varepsilon_{ik}]_+} X_{e_i} X_{e_k}^{[\varepsilon_{ik}]_+}.
$$

So although it is the identity map after the canonical identification of algebras $\mathbf{T}_{\mathbf{i}'}$ and $\mathbf{T}_{\mathbf{i}}$, it looks as a non-trivial map when written in the cluster coordinates.

3.3.2. *An explicit computation of the automorphism $\mu_k^\sharp$*

LEMMA 3.4. – *The automorphism $\mu_k^\sharp$ is given on the generators by the formulas*

$$
(56) \quad X_i \longmapsto X_i^\sharp := \begin{cases} X_i(1 + q_k X_k)(1 + q_k^3 X_k)\dots(1 + q_k^{2|\varepsilon_{ik}|-1} X_k) & \text{if } \varepsilon_{ik} \leq 0, \\ X_i\left((1 + q_k^{-1} X_k)(1 + q_k^{-3} X_k)\dots(1 + q_k^{1-2|\varepsilon_{ik}|} X_k)\right)^{-1} & \text{if } \varepsilon_{ik} \geq 0. \end{cases}
$$

*Proof.* – For any formal power series $\varphi(x)$ the relation $q^{-\widehat{\varepsilon}_{ki}} X_k X_i = q^{-\widehat{\varepsilon}_{ik}} X_i X_k$ implies

$$
(57) \qquad\qquad\qquad \varphi(X_k)X_i = X_i\varphi(q^{-2\widehat{\varepsilon}_{ik}} X_k).
$$

The difference equation (52) implies that the formula (56) can be rewritten as

$$
(58) \qquad\qquad\qquad X_i^\sharp = X_i \cdot \Psi_{q_k}(q_k^{-2\varepsilon_{ik}} X_k)\Psi_{q_k}(X_k)^{-1}.
$$

Using (57) and $q_k^{-2\varepsilon_{ik}} = q^{-2\widehat{\varepsilon}_{ik}}$, we get

$$
\Psi_{q_k}(X_k)X_i\Psi_{q_k}(X_k)^{-1} = X_i\Psi_{q_k}(q^{-2\widehat{\varepsilon}_{ik}} X_k)\Psi_{q_k}(X_k)^{-1} \stackrel{(58)}{=} X_i^\sharp.
$$

The lemma is proved. $\qquad\square$

Let $a \geq 0$ be an integer and

$$G_a(q; X) := \Psi_q(q^{2a}X)\Psi_q(X)^{-1} = \begin{cases} \prod_{i=1}^{a}(1 + q^{2i-1}X) & a > 0, \\ 1 & a = 0. \end{cases}$$

Let $\mu_k := \mathbf{i} \to \mathbf{i}'$ be a mutation. The following lemma follows easily from Lemma 3.4.

LEMMA 3.5. – *The quantum mutation homomorphism*

$$\mu_k^q : \mathbb{T}_{\mathbf{i}'} \longrightarrow \mathbb{T}_{\mathbf{i}}$$

*is given in the cluster coordinates by the formula*

$$\mu_k^q : X_i' \longmapsto \begin{cases} X_i F_{ik}(q; X_k) & \text{if } k \neq i \\ X_i^{-1} & \text{if } k = i, \end{cases} \qquad F_{ik}(q; X) = \begin{cases} G_{|\varepsilon_{ik}|}(q_k; X) & \text{if } \varepsilon_{ik} \leq 0 \\ G_{|\varepsilon_{ik}|}(q_k; X^{-1})^{-1} & \text{if } \varepsilon_{ik} \geq 0. \end{cases}$$

In particular, it implies

COROLLARY 3.6. – *Setting $q = 1$ we recover the $\mathcal{X}$-mutation formulae.*

*The Poisson structure on $\mathcal{X}$.* The quasiclassical limit of the non commutative space $\mathcal{X}_q$ is described by a Poisson structure on the $\mathcal{X}$-space. This Poisson structure in any cluster coordinate system $\{X_i\}$ is given by the formula $\{X_i, X_j\} = 2\widehat{\varepsilon}_{ij} X_i X_j$. Lemma 3.3 implies that it is independent of the choice of coordinate system.

LEMMA 3.7. – *We have $(\mu_k^q)^2 = \text{Id}$ for quantum mutations.*

*Proof.* – Suppose that $\varepsilon_{ik} = a > 0$. Then, using (57) and difference equation (52), we have

$$\text{Ad}_{\Psi_{q_k}(X_k^{-1})}\text{Ad}_{\Psi_{q_k}(X_k)}X_i = X_i G_a(q_k; X_k^{-1})G_a(q_k^{-1}, X_k) = q_k^{a^2} X_i X_k^{-a}.$$

Being composed with (55), this gives the identity map. The other case is reduced to this one. The lemma is proved. $\qquad\square$

PROPOSITION 3.8. – *The collection of the quantum tori $\mathbf{T_i}$ and the quantum mutation maps $\mu_k^q$ provide a functor $\widehat{\mathcal{G}} \longrightarrow \text{QPos}^*$.*

The quantum space $\mathcal{X}_q$ is understood as this functor.

REMARK. – We expect to have a functor $\mathcal{G} \longrightarrow \text{QPos}^*$. The problem is that we do not know the relations in the groupoid $\mathcal{G}$ explicitly.

*Proof.* – We have to check that the composition of maps corresponding to the boundary of any standard $(h + 2)$-gon equals to the identity. It can be checked by a calculation. We present its crucial step as Lemma 3.9 below. The proposition is proved. $\qquad\square$

REMARK. – Another proof follows from Lemma 2.22 in [9] plus the trick used in the proof of Proposition 1.8: one embeds the seed $\mathbf{i}$ in a bigger seed $\mathbf{i}'$ with $\det \varepsilon'_{ij} \neq 0$, and observes that cluster transformations (16) remain trivial on the classical level after extension of the seed, so by Lemma 2.22 in [9] they are trivial on the quantum level.

3.3.3. *Examples of quantum relations.* – Consider a sequence of mutations at the vertices $i, j, i, j, i, \ldots$. We picture it by a polygon, whose vertices match the mutations. The seeds are the sides of the polygon, and the $\mathcal{X}$-coordinates for a given seed $\mathbf{i}$ are assigned to the flags (a vertex of the side, the side). The $\mathcal{X}$-coordinates assigned to the flags sharing a vertex are opposite to each other. Below we calculate the sequence of the $\mathcal{X}$-coordinates assigned to the flags oriented the same way, clockwise. They determine the set of all $\mathcal{X}$-coordinates

The $\mathcal{X}$-coordinates on the set of all clockwise oriented flags are obtained from the initial $\mathcal{X}$-coordinates $x_1, x_2$ by the following inductive procedure.

*Classical case.* – Let $F$ be a field, $x_1, x_2 \in F^*$ and $b, c$ are non-negative integers. Consider the recursion

$$(59) \qquad x_{m-1} x_{m+1} = \begin{cases} (1 + x_m)^b & m: \text{ even,} \\ (1 + x_m)^c & m: \text{ odd.} \end{cases}$$

According to Chapters 2 and 6 of [FZ1], this sequence is periodic if and only if the Cartan matrix $\begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ or its transpose is of finite type, i.e. $b = c = 0$ or $1 \le |bc| \le 3$. Therefore, up to a shift $x_i \longmapsto x_{i+1}$, there are only four periodic sequences, corresponding to the root systems $A_1 \times A_1, A_2, B_2, G_2$. The period is $h + 2$, where $h$ is the Coxeter number of the root system.

*Quantum case.* – Let $(\varepsilon_{ij}) = \begin{pmatrix} 0 & -1 \\ c & 0 \end{pmatrix}$. So the commutation relations are $q^{-2c} X_i X_{i+1} = q^{2c} X_{i+1} X_i$. We have:

$$(60) \qquad X_{m-1} X_{m+1} = \begin{cases} (1 + q^c X_m) & m: \text{ even} \\ (1 + q X_m) \cdot (1 + q^3 X_m) \cdots (1 + q^{2c-1} X_m) & m: \text{ odd} \end{cases}$$

Let $h$ be the Coxeter number for the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -c & 2 \end{pmatrix}$ of finite type, i.e. $c = 0, 1, 2, 3$. So $h = 2$ for $c = 0$; $h = 3$ for $c = 1$; $h = 4$ for $c = 2$; and $h = 6$ for $c = 3$.

LEMMA 3.9. – *For any integer $m$ one has $X_{m+h+2} = X_m$.*

*Proof.* – Compute the elements $X_m$:
*Type $A_1 \times A_1$.* Then $b = c = 0, h = 2$ and $X_3 = X_1^{-1}, \quad X_4 = X_2^{-1}$.
*Type $A_2$.* Then $b = c = 1, h = 3$ and

$$X_3 = X_1^{-1}(1 + q X_2), \quad X_4 = (X_1 X_2)^{-1}\Big(X_1 + q(1 + q X_2)\Big), \quad X_5 = X_2^{-1}(1 + q^{-1} X_1), \quad X_6 = X_1.$$

*Type $B_2$.* Then $b = 1, c = 2, h = 4$ and

$$X_1 X_3 = 1 + q^2 X_2, \qquad X_1^2 X_2 X_4 = \Big(X_1 + q(1 + q^6 X_2)\Big)\Big(X_1 + q^3(1 + q^2 X_2)\Big),$$

$$X_1 X_2 X_5 = q^2\Big((1 + q^{-1} X_1)(1 + q^{-3} X_1) + q^2 X_2\Big), \quad X_2 X_6 = (1 + q^{-1} X_1)(1 + q^{-3} X_1), \quad X_7 = X_1.$$

*Type $G_2$.* Then $b = 1, c = 3, h = 6$ and

$$X_1 X_3 = 1 + q^3 X_2, \qquad X_1^3 X_2 X_4 = \left( X_1 + q(1 + q^{15} X_2) \right)\left( X_1 + q^3(1 + q^9 X_2) \right)\left( X_1 + q^5(1 + q^3 X_2) \right),$$

$$X_1^2 X_2 X_5 = (1 + q^{-1} X_1)^3 + (q^3 X_2)^2 + (1 + q^6)q^3 X_2 + 3q^{-1} X_1 q^3 X_2, \dots,$$

$$X_2 X_8 = (1 + q^{-1} X_1)(1 + q^{-3} X_1)(1 + q^{-5} X_1), \quad X_9 = X_1. \qquad \square$$

REMARK. – This way we get just the half of all $\mathcal{X}$-coordinates. They are related to the ones in the example as follows: we get only the $\mathcal{X}$-coordinates assigned to the mutating vertices before the mutations; the initial coordinates are $x_1 := y^{-1}, x_2 := x$.

## 3.4. The quantum Frobenius map

3.4.1. *Center of the quantum space when q is not a root of unity.* – Let $\alpha \in \mathrm{Ker}_L[*, *]$. Then the element $X_\alpha$ is in the center of the quantum torus $\mathrm{T}_{\mathbf{i}}^q$. The torus $H_\mathcal{X}$ can be treated as a commutative quantum positive space, see (32). Recall the character $\chi_\alpha$ of the torus $H_\mathcal{X}$ corresponding to $\alpha$.

LEMMA 3.10. – *There exists a unique map of quantum positive spaces $\theta_q : \mathcal{X}_q \to H_\mathcal{X}$ such that for any seed we have $\theta_q^* \chi_\alpha = X_\alpha$, where $\alpha \in \mathrm{Ker}_L[*, *]$.*

*Proof.* – Since the element $X_\alpha$ of the quantum torus $\mathrm{T}_\Lambda$ corresponding to a vector $\alpha \in \mathrm{Ker}_L[*, *]$ lies in the center, conjugation by $\Psi_q(X_k)$ acts on them as the identity. The lemma follows. $\qquad \square$

3.4.2. *Center of the quantum space when q is a root of unity.* – When $q$ is a root of unity, the quantum space has a much larger center, which we are going to describe now. Let $\widehat{\varepsilon}_{ij} \in \mathbb{Z}$. For a seed $\mathbf{i}$, denote by $\{Y_i\}$ (respectively $\{X_i\}$) the corresponding cluster coordinates on $\mathcal{X}$ (respectively on $\mathcal{X}_q$). If $q^N = 1$ then $X_i^N$ are in the center of the quantum torus algebra $\mathrm{T}_{\mathbf{i}}$.

THEOREM 3.11. – *Let $q = \zeta_N$ be a primitive $N$-th root of unity. Let us assume that $q^{d_k}$ is a primitive $N$-th root of unity, and for every seed $\mathbf{i}$ the corresponding function $\varepsilon_{ij}$ satisfies $(2\varepsilon_{ij}, N) = 1$. Then there exists a map of positive spaces, called the quantum Frobenius map,*

$$\mathbb{F}_N : \mathcal{X}_q \longrightarrow \mathcal{X} \quad \text{such that} \quad \mathbb{F}_N^*(Y_i) := X_i^N$$

*in any cluster coordinate system.*

*Proof.* – To check that the quantum Frobenius map commutes with a mutation $\mu_k : \mathbf{i} \to \mathbf{i}'$, it is sufficient to check that it commutes with the conjugation by $\Psi_q(X_k)$. Here we consider the generic $q$, and only after the conjugation specializes $q$ to a root of unity.

Let us assume that $\varepsilon_{ik} = -a \le 0$. Then we have to show that, specializing $q_k = 1$ in

$$\Psi_{q_k}(q_k^{2aN} X_k)\Psi_q(X_k)^{-1},$$

we get $(1 + X_k^N)^a$. The statement is equivalent to the identity

(61) $$\prod_{b=0}^{N-1} G_a(q_k; q_k^{2ba} X_k) = (1 + X_k^N)^a.$$

Notice that $q_k$ is a primitive $N$-th root of unity and, since $(2a, N) = 1$, the set $\{-2ab\}$, when $b \in \{1, \ldots, N-1\}$, consists of all residues modulo $N$ except zero. Thus each factor of the product

$$G_a(q_k; X_k) = \prod_{i=1}^{a}(1 + q_k^{2i-1}X_k)$$

contributes $(1 + X_k^N)$ thanks to the formula $\prod_{c=0}^{N-1}(1 + q_k^{2i-1+c}Z) = 1 + Z^N$.

The argument in the case $\varepsilon_{ik} = a > 0$ is similar. In fact it can be reduced to the previous case using $(\mu_k^q)^2 = \mathrm{Id}$ and $\varepsilon'_{ik} = -\varepsilon_{ik}$. The theorem is proved.                    □

EXAMPLES. – a) Let $\widehat{S}$ be a marked hyperbolic surface. Then the pair of moduli spaces $(\mathcal{X}_{PGL_2,\widehat{S}}, \mathcal{A}_{SL_2,\widehat{S}})$ has a cluster ensemble structure with $\varepsilon_{ij} \in \{0, \pm 1, \pm 2\}$ ([6], Chapter 10; [7]). Thus it satisfies the assumptions of the theorem for any odd $N$.

b) A cluster ensemble of finite type satisfies the assumptions of the theorem for any odd $N$ in all cases except $G_2$, where the condition is $(N, 6) = 1$.

REMARK. – Sometimes it makes sense to restrict the functor defining the space $\mathcal{X}_q$ to a subgroupoid $\widehat{\mathcal{G}}'$ of $\widehat{\mathcal{G}}$, restricting therefore the set of values of the exchange function. For example for the pair of moduli spaces $(\mathcal{X}_{PGL_m,S}, \mathcal{A}_{SL_m,S})$ one may consider only those mutations which were introduced in Chapter 10 of [6] to decompose flips. Then the restricted cluster function takes values in $\{0, \pm 1, \pm 2\}$, $d_k = 1$, and the fundamental group of the restricted groupoid $\widehat{\mathcal{G}}'$ contains the classical modular group of $S$. So the quantum Frobenius map in this case commutes with the action of the classical modular group. If the modular group $\Gamma$ is finitely generated, we can always restrict to a subgroupoid $\widehat{\mathcal{G}}'$ of $\widehat{\mathcal{G}}$ which has the same fundamental group and a bounded set of values $|\varepsilon_{ij}|$.

## 4. Duality and canonical pairings: conjectures

Below we denote by $\mathcal{A}$ and $\mathcal{A}^\vee$ the positive spaces $\mathcal{A}_{|\mathbf{i}|}$ and $\mathcal{A}_{|\mathbf{i}^\vee|}$, and similarly for the $\mathcal{X}$-spaces.

In this section we show how to extend to cluster ensembles the philosophy of duality between the $\mathcal{X}$ and $\mathcal{A}$ positive spaces developed in [6] in the context of the two moduli spaces related to a split semisimple group $G$ and a surface $S$. We suggest that there exist several types of closely related canonical pairings/maps between the positive spaces $\mathcal{X}$ and $\mathcal{A}^\vee$. An example provided by the cluster ensemble related to the classical Teichmüller theory was elaborated in Chapter 12 of [6]. It was extended to the pair of Teichmüller spaces related to a surface $S$ with $m > 0$ distinguished points on the boundary in [7]. In particular, when $S$ is a disc with $m$ marked points on the boundary, we cover the case of the cluster ensemble of finite type $A_m$. The canonical map $\mathbb{I}_{\mathcal{X}}$ for cluster ensembles of an arbitrary finite type is defined in Section 4.6. Other examples can be obtained using the work [25] on the rank two finite and affine cluster algebras.

Our main conjectures are Conjecture 4.1 and its quantum version, Conjecture 4.8.

Conjecture 4.3 is a variation on the theme of Conjecture 4.1. We show that, under some assumptions, the latter can be deduced from the former.

4.0.3. *Background*. – Let $L$ be a set. Denote by $\mathbb{Z}_+\{L\}$ the abelian semigroup generated by $L$. Its elements are expressions $\sum_i n_i\{l_i\}$ where $n_i \geq 0$, the sum is finite, and $\{l_i\}$ is the generator corresponding to $l_i \in L$. Similarly $\mathbb{Z}\{L\}$ is the abelian group generated by $L$.

Let $\mathcal{X}$ be a positive space. Observe that the ring of regular functions on a split torus $H$ is the ring of Laurent polynomials in characters of $H$. Recall (Chapter 1.1 and [6], Section 4.3) that a *universally Laurent polynomial* on $\mathcal{X}$ is a regular function on one of the coordinate tori $H_\alpha$ defining $\mathcal{X}$ whose restriction to any other coordinate torus $H_\beta$ is a regular function there. $\mathbb{L}(\mathcal{X})$ denotes the ring of all universally Laurent polynomials, and $\mathbb{L}_+(\mathcal{X})$ the semiring of *universally positive Laurent polynomials* obtained by imposing the positivity condition on coefficients of universally Laurent polynomials. Let $\mathbf{E}(\mathcal{X})$ be the set of extremal elements, that is universally positive Laurent polynomials which cannot be decomposed into a sum of two non zero universally positive Laurent polynomials with positive coefficients.

We use the notation $\mathcal{X}^+ := \mathcal{X}(\mathbb{R}_{>0})$. Recall that for a given seed $\mathbf{i}$, there are canonical coordinates $\{X_i\}$ and $\{A_i\}$ on the $\mathcal{X}$ and $\mathcal{A}$ spaces. By the very definition, their restrictions to $\mathcal{X}^+$ and $\mathcal{A}^+$ are positive, so we have the corresponding logarithmic coordinates $x_i := \log X_i$ and $a_i := \log A_i$. The coordinates on the tropicalizations $\mathcal{X}(\mathbb{A}^t)$ and $\mathcal{A}(\mathbb{A}^t)$ are also denoted by $x_i$ and $a_i$.

A convex function on a lattice $L$ is a function $F(l)$ such that $F(l_1 + l_2) \leq F(l_1) + F(l_2)$. Let $\mathcal{X}$ be a positive space. A convex function on $\mathcal{X}(\mathbb{A}^t)$ or $\mathcal{X}(\mathbb{Q})$ is a function which is convex in each of the coordinate systems from the defining atlas of $\mathcal{X}$.

## 4.1. Canonical maps in the classical setting

We consider a partial order on monomials $X^a = \prod X_i^{a_i}$ such that $X^a \geq X^b$ if $a_i \geq b_i$ for all $i \in I$. We say that $X^a$ is the highest term of a Laurent polynomial $F$ if $X^a$ is bigger (i.e. $\geq$) than any other monomial in $F$. Recall the map $p : \mathcal{A} \to \mathcal{X}$.

CONJECTURE 4.1. – *There exist $\Gamma$-equivariant isomorphisms of sets*

$$\mathcal{A}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{X}^\vee), \qquad \mathcal{X}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{A}^\vee). \tag{62}$$

*These isomorphisms give rise to $\Gamma$-equivariant isomorphisms*

$$\mathbb{I}_\mathcal{A} : \mathbb{Z}_+\{\mathcal{A}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}_+(\mathcal{X}^\vee), \qquad \mathbb{I}_\mathcal{X} : \mathbb{Z}_+\{\mathcal{X}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}_+(\mathcal{A}^\vee). \tag{63}$$

*These maps have the following properties:*

1. *Let $(a_1, \ldots, a_n)$ be the coordinates of $l \in \mathcal{A}(\mathbb{Z}^t)$. Then the highest term of $\mathbb{I}_\mathcal{A}(l)$ is $\prod_i X_i^{a_i}$.*
2. *If in a certain cluster coordinate system the coordinates $(x_1, \ldots, x_n)$ of an element $l \in \mathcal{X}(\mathbb{Z}^t)$ are non negative numbers, then*

$$\mathbb{I}_\mathcal{X}(l) = \prod_{i \in I} A_i^{x_i}.$$

3. *Let $l \in \mathcal{A}(\mathbb{Z}^t)$ and $m$ be a point of $\mathcal{A}^\vee$. Then*

$$\mathbb{I}_\mathcal{A}(l)(p(m)) = \mathbb{I}_\mathcal{X}(p(l))(m),$$

*where $\mathbb{I}_\mathcal{A}(l)(p(m))$ means the value of the function $\mathbb{I}_\mathcal{A}(l)$ on $p(m)$*

4. *Extending the coefficients from $\mathbb{Z}_+$ to $\mathbb{Z}$, we arrive at isomorphisms*

$$(64) \qquad \mathbb{I}_{\mathcal{A}} : \mathbb{Z}\{\mathcal{A}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}(\mathcal{X}^\vee), \qquad \mathbb{I}_{\mathcal{X}} : \mathbb{Z}\{\mathcal{X}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}(\mathcal{A}^\vee),$$

The isomorphisms (62) imply that one should have

$$(65) \qquad \mathbb{I}_*(l_1)\mathbb{I}_*(l_2) = \sum_l c_*(l_1, l_2; l)\mathbb{I}_*(l),$$

where the coefficients $c_*(l_1, l_2; l)$ are positive integers and the sum is finite. Here $*$ stands for either $\mathcal{A}$ or $\mathcal{X}$. It follows from part 1 of Conjecture 4.1 and (65) that $c_*(l_1, l_2; l_1 + l_2) = 1$.

In addition, the canonical maps are expected to satisfy the following additional properties.

i) *Convexity conjecture.* The structural constants $c_*(l_1, l_2; l)$ can be viewed as maps

$$c_{\mathcal{A}} : \mathcal{A}(\mathbb{Z}^t)^3 \longrightarrow \mathbb{Z}, \quad c_{\mathcal{X}} : \mathcal{X}(\mathbb{Z}^t)^3 \longrightarrow \mathbb{Z}.$$

CONJECTURE 4.2. – *In any cluster coordinate system, the supports of the functions $c_{\mathcal{A}}$ and $c_{\mathcal{X}}$ are convex polytopes.*

ii) *Frobenius Conjecture.* We conjecture that in every cluster coordinate system $\{X_i\}$, for every prime $p$ one should have the congruence

$$(66) \qquad \mathbb{I}_{\mathcal{A}}(p \cdot l)(X_i) = \mathbb{I}_{\mathcal{A}}(l)(X_i^p) \quad \text{modulo } p.$$

EXAMPLE. – Take an element $\delta_i^{\mathbf{i}} \in \mathcal{X}(\mathbb{Z}^t)$ whose coordinates $(x_1, \dots, x_n)$ in the coordinate system related to a seed $\mathbf{i}$ are $x_j = 0$ for $j \neq i$, $x_i = 1$. Setting $\mathbb{I}_{\mathcal{X}}(\delta_i^{\mathbf{i}}) := A_i^{\mathbf{i}}$ we get a universally Laurent polynomial by the Laurent phenomenon theorem of [13]. Its positivity was conjectured in *loc. cit.* Therefore the cluster algebra sits inside of the algebra of universally Laurent polynomials $\mathbb{L}(\mathcal{A})$. However the latter can be strictly bigger than the former.

The set of points of a positive space $\mathcal{X}$ with values in a semifield is determined by a single positive coordinate system on $\mathcal{X}$. Contrary to this, the semiring of universally positive Laurent polynomials depends on the whole collection of positive coordinate systems on $\mathcal{X}$. So the source of the canonical map is determined by a single coordinate system of a positive atlas on the source space, while its image depends on the choice of a positive atlas on the target space. This shows, for instance, that the set of positive coordinate systems on the target space cannot be "too big" or "too small".

CONJECTURE 4.3. – *There exist canonical $\Gamma$-equivariant pairings between the real tropical and real positive spaces*

$$(67) \qquad \mathrm{I}_{\mathcal{A}} : \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R}, \quad \mathrm{I}_{\mathcal{X}} : \mathcal{X}(\mathbb{R}^t) \times \mathcal{A}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R},$$

*as well as a canonical $\Gamma$-equivariant intersection pairing between the real tropical spaces*

$$(68) \qquad \mathcal{I} : \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}^\vee(\mathbb{R}^t) \to \mathbb{R}.$$

*We expect the following properties of these pairings:*

1. *All the pairings are convex functions in each of the two variables, and in each of the cluster coordinate systems.*
2. *All the pairings are homogeneous with respect to the tropical variable(s): for any $\alpha > 0$ one has $\mathrm{I}_*(\alpha l, m) = \alpha \mathrm{I}_*(l, m)$, and the same holds for $\mathcal{I}$.*

3. *Let $l \in \mathcal{A}(\mathbb{R}^t)$ and $m$ be a point of $\mathcal{A}^\vee(\mathbb{R})$. Then*

$$\mathrm{I}_\mathcal{A}(l, p(m)) = \mathrm{I}_\mathcal{X}(p(l), m).$$

*Similarly, $\mathcal{I}(l, p(m)) = \mathcal{I}(p(l), m)$.*

4. *If the coordinates $(x_1, \ldots, x_n)$ of a point $l \in \mathcal{X}(\mathbb{R}^t)$ are positive numbers, and a point $m \in \mathcal{A}^+$ has the logarithmic coordinates $(a_1, \ldots, a_n)$, then*

$$\mathrm{I}_\mathcal{X}(l, m) = \sum_{i \in I} x_i a_i,$$

*and the same holds for $\mathcal{I}$.*

5. *Let $m$ be a point of either $\mathcal{X}^\vee(\mathbb{R}_{>0})$ or $\mathcal{A}^\vee(\mathbb{R}_{>0})$ with logarithmic coordinates $u_1, \ldots, u_n$. Let $C \in \mathbb{R}$. Denote by $C \cdot m$ the point with coordinates $Cu_1, \ldots, Cu_n$. Let $m_L$ be the point of either $\mathcal{A}(\mathbb{R}^t)$ or $\mathcal{X}(\mathbb{R}^t)$ with the coordinates $u_1, \ldots, u_n$. Then, for both $* = \mathcal{A}$ and $* = \mathcal{X}$,*

$$\tag{69} \lim_{C \to \infty} \mathrm{I}_*(l, C \cdot m)/C = \mathcal{I}(l, m_L).$$

6. *The intersection pairing (68) restricts to a pairing*

$$\tag{70} \mathcal{I} : \mathcal{A}(\mathbb{Q}^t) \times \mathcal{X}^\vee(\mathbb{Q}^t) \to \mathbb{Q}.$$

REMARK. – The convexity property implies that the intersection pairing is continuous. So an intersection pairing (70) satisfying the convexity property 1 determines the pairing (68). Similarly convex pairings

$$\tag{71} \mathrm{I}_\mathcal{A} : \mathcal{A}(\mathbb{Q}^t) \times \mathcal{X}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R}, \quad \mathrm{I}_\mathcal{X} : \mathcal{X}(\mathbb{Q}^t) \times \mathcal{A}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R}$$

can be uniquely extended to continuous pairings (67).

### 4.2. Conjecture 4.1 essentially implies Conjecture 4.3

Conjecture 4.1 is an algebraic cousin of Conjecture 4.3. Roughly speaking, it is obtained by replacing in the pairings (67) the tropical semifield $\mathbb{R}^t$ by its integral version $\mathbb{Z}^t$, and the real manifolds $\mathcal{X}^\vee(\mathbb{R}_{>0})$ and $\mathcal{A}^\vee(\mathbb{R}_{>0})$ by the corresponding cluster varieties $\mathcal{X}^\vee$ and $\mathcal{A}^\vee$. It is handy to think about these algebraic pairings as of maps from the sets of integral tropical points to positive regular functions on the corresponding scheme. Observe that the positive structure of the corresponding schemes has been used to define the positive regular functions on them.

One can interpret the map $\mathbb{I}_*$ as a pairing $\mathbb{I}_*(*, *)$ between laminations and points of the corresponding space: $\mathbb{I}_*(l, z) := \mathbb{I}_*(l)(z)$. We are going to show how the canonical pairings $\mathcal{I}(*, *)$ and $\mathrm{I}_*(*, *)$ emerge from the one $\mathbb{I}_*(*, *)$ in the tropical and scaling limits.

*The tropical limit and the intersection pairing $\mathcal{I}(*, *)$.* Let $F(X_i)$ be a positive integral Laurent polynomial. Let $F^t(x_i)$, where $x_i$ belong to a semifield, be the corresponding tropical polynomial.

EXAMPLE. – If $F(X_1, X_2) = X_1^2 + 3X_1 X_2$, then $F^t(x_1, x_2) = \max\{2x_1, x_1 + x_2\}$.
Observe that one has

$$\tag{72} \lim_{C \to \infty} \frac{\log(e^{Cx_1} + \cdots + e^{Cx_n})}{C} = \max\{x_1, \ldots, x_n\}.$$

Therefore the Laurent polynomial $F$ and its tropicalization $F^t$ are related as follows:

$$(73) \qquad \lim_{C \to \infty} \frac{\log F(e^{Cx_1}, \ldots, e^{Cx_n})}{C} = F^t(x_1, \ldots, x_n), \quad x_i \in \mathbb{R}.$$

DEFINITION 4.4. – *Let us assume that we have the canonical maps $\mathbb{I}_*$ from Conjecture 4.1. Then, for an integral tropical point $l$ of the $*$-space, where $*$ stands for either $\mathcal{A}$ or $\mathcal{X}$, the function $\mathcal{I}_*(l, \bullet)$ is the tropicalization of the positive integral Laurent polynomial $\mathbb{I}_*(l)$:*

$$(74) \qquad \mathcal{I}_*(l, \bullet) := \mathbb{I}_*^t(l)(\bullet).$$

Therefore (73) implies that

$$\mathcal{I}_*(l, m) = \lim_{C \to \infty} \frac{\log \mathbb{I}_*(l)(e^{C \cdot m})}{C}.$$

*The scaling limit.* Observe that the restriction of $\mathbb{I}_\mathcal{A}(l)$ to $\mathcal{X}^\vee(\mathbb{R}_{>0})$, as well as $\mathbb{I}_\mathcal{X}(l)$ to $\mathcal{A}^\vee(\mathbb{R}_{>0})$, are positive valued functions, so the logarithm $\log \mathbb{I}_*(l)$ makes sense. Recall that the group $\mathbb{Q}_+^*$ acts by automorphisms of the semiring $\mathbb{Q}^t$, and hence acts on the set of $\mathbb{Q}^t$-points of a positive space $\mathcal{X}$. Namely, if $l \in \mathcal{X}(\mathbb{Q}^t)$ and $C \in \mathbb{Q}_+^*$, we denote by $C \cdot l$ the element obtained by multiplying all coordinates of $l$ by $C$, in any of the coordinate systems. Let us define pairings

$$(75) \qquad \mathrm{I}_\mathcal{A} : \mathcal{A}(\mathbb{Q}^t) \times \mathcal{X}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R}, \quad \mathrm{I}_\mathcal{X} : \mathcal{X}(\mathbb{Q}^t) \times \mathcal{A}^\vee(\mathbb{R}_{>0}) \longrightarrow \mathbb{R}$$

by taking the scaling limit, where, as usual, $*$ stands for either $\mathcal{X}$ or $\mathcal{A}$.

$$(76) \qquad \mathrm{I}_*(l, u) := \lim_{C \to \infty} \frac{\log \mathbb{I}_*(C \cdot l)(u)}{C}.$$

CONJECTURE 4.5. – *The scaling limit (76) exists for both $* = \mathcal{A}$ and $* = \mathcal{X}$.*

THEOREM 4.6. – *Let us assume Conjectures 4.1 and 4.5. Then the pairings $\mathrm{I}_*(l, u)$ and $\mathcal{I}(l, m)$ satisfy all conditions of Conjecture 4.3.*

*Proof.* – Definition 4.4 and (76) provide the pairings for integral tropical laminations $l$. First of all we have to extend them to rational and real tropical points.

LEMMA 4.7. – *Under the assumptions of Theorem 4.6, $\mathrm{I}_*(l, u)$ enjoys the following properties:*

i) *It is homogeneous in $l$.*

ii) *It is convex in the $l$-variable, and it is convex in the $u$-variable.*

*Proof.* – The property i) is clear. To check ii) for the Teichmüller, $u$-variable, observe that

$$\mathbb{I}_*(l, e^x) \cdot \mathbb{I}_*(l, e^y) \geq \mathbb{I}_*(l, e^{x+y}).$$

Indeed, $(\sum a_I e^{x_I})(\sum a_I e^{y_I}) \geq (\sum a_I e^{x_I + y_I})$ since $a_I$ are positive integers. So taking the logarithm we get a convex function, and a limit of convex functions is convex. Similarly the property ii) for the tropical, $l$-variable, follows from $\mathbb{I}_*(l_1)\mathbb{I}_*(l_2) \geq \mathbb{I}_*(l_1 + l_2)$, which is an immediate corollary of $c_*(l_1, l_2; l_1 + l_2) = 1$ and the property (65) in Conjecture 4.1. The lemma is proved. $\square$

Using homogeneity in $l$ we extend the pairings to rational tropical points $l$. Then convexity of the pairings (75) implies that they are continuous in the natural topology on the set of rational tropical points, and thus can be uniquely extended to real pairings (67). So we get the pairings $\mathrm{I}_*$ and $\mathcal{I}$ defined for any real tropical point $l$ of the corresponding space, which satisfy Properties 1 and 2. Property 6 is evidently valid. Properties 3 and 4 follow immediately from Properties 3 and 2 in Conjecture 4.1. Finally, Property 5 follows from the very definition:

$$\lim_{C_1, C_2 \to \infty} \frac{\log \mathbb{I}_*(C_1 \cdot l)(C_2 \cdot u)}{C_1 C_2} = \lim_{C_1 \to \infty} \frac{i_*(C_1 \cdot l)(u_L)}{C_1} \overset{1.}{=} \mathcal{I}_*(l, u_L).$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

### 4.3. Quantum canonical map

We define a *universally positive Laurent polynomial* related to the quantum space $\mathcal{X}_q$ as an element of the (non commutative) fraction field $\mathbb{T}_{\mathbf{i}}^q$ which for any seed $\mathbf{i}'$ is a Laurent polynomial, with positive integral coefficients, in $q$ and the corresponding quantum $\mathcal{X}$-coordinates $X_i$. We denote by $\mathbb{L}_+(\mathcal{X}_q)$ the *semiring of universally positive Laurent polynomials* on the quantum space $\mathcal{X}_q$. Dropping the condition of positivity of the coefficients we get the *ring of universally Laurent polynomials* $\mathbb{L}(\mathcal{X}_q)$ on the quantum space $\mathcal{X}_q$.

Let us make a few remarks preceding the conjecture.

1. Given an order on the set $I$, an element $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ provides a monomial

$$(77) \qquad\qquad\qquad\qquad X_a := q^{-\sum_{i<j} \widehat{\varepsilon}_{ij} a_i a_j} \prod_i X_i^{a_i}.$$

It is $*$-invariant and does not depend on the choice of ordering of $I$ used to define it (Section 3.1).

2. For any positive space $\mathcal{X}$ and a positive integer $N$ there is a subset $\mathcal{X}(N\mathbb{Z}^t) \subset \mathcal{X}(\mathbb{Z}^t)$. It consists of the points whose coordinates in one, and hence in any positive coordinate system for $\mathcal{X}$ are integers divisible by $N$. The canonical map of positive spaces $p : \mathcal{A} \to \mathcal{X}$ induces the map

$$(78) \qquad\qquad\qquad\qquad p : \mathcal{A}(\mathbb{Z}^t) \to \mathcal{X}(\mathbb{Z}^t).$$

Let us take the inverse image of the subset $\mathcal{X}(N\mathbb{Z}^t)$ under this map, and consider the positive abelian semigroup it generates. We denote it by $Z_{\mathcal{A}}(N)$:

$$Z_{\mathcal{A}}(N) := \mathbb{Z}_+ \left\{ p^{-1}\left( \mathcal{X}(N\mathbb{Z}^t) \right) \right\}.$$

Observe that $\alpha \in \mathrm{Ker}_L[*, *]$ provides an element $l(\alpha) \in \mathcal{A}(\mathbb{Z}^t)$ which lies in $Z_{\mathcal{A}}(N)$ for all $N$.

3. Recall that for a split torus $H$ the set $H(\mathbb{Z}^t)$ is the group of cocharacters $X_*(H)$ of $H$. In particular it is an abelian group. The canonical isomorphism

$$X_*(H_{\mathcal{A}}) = X^*(H_{\mathcal{X}^\vee})$$

allows us to consider an element $\alpha \in H_{\mathcal{A}}(\mathbb{Z}^t)$ as a character $\chi_\alpha$ of the torus $H_{\mathcal{X}^\vee}$. The inverse image of this character under the projection $\theta_q^\vee$ delivers a central element

$$(79) \qquad\qquad\qquad\qquad \widetilde{\chi}_\alpha^q := (\theta_q^\vee)^* \chi_\alpha \in \mathrm{Center}\left( \mathbb{L}_+(\mathcal{X}_q^\vee) \right).$$

4. The torus $H_{\mathcal{A}}$ acts on the $\mathcal{A}$-space. So the abelian group $H_{\mathcal{A}}(\mathbb{Z}^t)$ acts on the set $\mathcal{A}(\mathbb{Z}^t)$. We denote this action by $*$. In coordinates it looks as follows. Choose a seed $\mathbf{i}$. Then an element $\alpha \in H_{\mathcal{A}}(\mathbb{Z}^t)$ is given by $\{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_i \in \mathbb{Z}$ and $\sum_j \varepsilon_{ij}\alpha_j = 0$ for all $i \in I$. The element $\{\alpha_1, \ldots, \alpha_n\} \in H_{\mathcal{A}}(\mathbb{Z}^t)$ acts on $\{\beta_1, \ldots, \beta_n\} \in \mathcal{A}(\mathbb{Z}^t)$ by

$$\{\alpha_1, \ldots, \alpha_n\} * \{\beta_1, \ldots, \beta_n\} = \{\alpha_1 + \beta_1, \ldots, \alpha_1 + \beta_n\}.$$

5. Let $p$ be a prime, $q$ a primitive $p$-th root of unity. Then $\mathbb{Z}[q]$ is the cyclotomic ring. Let $(1-q)$ be the ideal generated $1-q$. It is the kernel of the surjective map

$$\pi_p : \mathbb{Z}[q] \longrightarrow \mathbb{Z}/p\mathbb{Z}, \quad q \longmapsto 1.$$

For a $\mathbb{Z}[q]$-module $M$ denote by $\pi_p(M)$ the reduction of $M$ modulo the ideal $(1-q)$. So $\pi_p(M)$ is an $F_p$-vector space.

The following conjecture is the main conjecture in our paper.

CONJECTURE 4.8. – *There exists a quantum canonical map, that is, a $\Gamma$-equivariant isomorphism*

$$(80) \qquad \widehat{\mathbb{I}}^q : \mathbb{Z}_+\{\mathcal{A}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}_+(\mathcal{X}_q^\vee)$$

*satisfying the following properties:*

1. $\widehat{\mathbb{I}}^1(l) = \mathbb{I}_{\mathcal{A}}(l)$.
2. *Let $a = (a_1, \ldots, a_n)$ be the coordinates of $l \in \mathcal{A}(\mathbb{Z}^t)$ in a cluster coordinate system. Then the highest term of $\widehat{\mathbb{I}}^q(l)$ is $X_a = q^{-\sum_{i<j} \widehat{\varepsilon}^{ij} a_i a_j} \prod_i X_i^{a_i}$.*
3. *Self-duality: $*\widehat{\mathbb{I}}^q(l) = \widehat{\mathbb{I}}^q(l)$.*
4. $\widehat{\mathbb{I}}^q(l_1)\widehat{\mathbb{I}}^q(l_2) = \sum_l c^q(l_1, l_2; l)\widehat{\mathbb{I}}^q(l)$, *where the $c^q(l_1, l_2; l)$ are Laurent polynomials of $q$ with positive integral coefficients. Moreover $c^q(l_1, l_2; l_1 + l_2) = 1$.*
5. *The center at roots of unity: Let $N$ be a positive integer and $q$ a primitive $N$-th root of unity. Then the map $\widehat{\mathbb{I}}^q$ induces an isomorphism*

$$(81) \qquad \widehat{\mathbb{I}}^q\left(Z_{\mathcal{A}}(N)\right) \xrightarrow{\sim} \mathrm{Center}\left(\mathbb{L}_+(\mathcal{X}_q^\vee)\right).$$

6. *The quantum Frobenius on $\mathbb{L}_+(\mathcal{X})$: Let $N \in \mathbb{Z}_{>0}$ and $q$ be a primitive $N$-th root of unity. Then the quantum Frobenius map $\mathbb{F}_N^*$ (see Theorem 3.11) is related to the map $\widehat{\mathbb{I}}^q$ as follows: $\mathbb{F}_N^*\mathbb{I}_{\mathcal{A}}(l) = \widehat{\mathbb{I}}^q(N \cdot l)$, i.e. in every cluster coordinate system $\{X_i\}$ one has*

$$(82) \qquad \mathbb{F}_N^*\mathbb{I}_{\mathcal{A}}(l)(X_i) := \mathbb{I}_{\mathcal{A}}(l)(X_i^N) = \widehat{\mathbb{I}}^q(N \cdot l)(X_i).$$

7. *Let $p$ be a prime, $q$ a primitive $p$-th root of unity. Then there is a canonical isomorphism*

$$(83) \qquad \pi_p\left(\mathbb{L}_+(\mathcal{X}_q)\right) = \mathbb{L}_+(\mathcal{X}) \otimes \mathbb{Z}/p\mathbb{Z}.$$

8. *The map $\widehat{\mathbb{I}}^q$ transforms the action of an element $\alpha \in H_{\mathcal{A}}(\mathbb{Z}^t)$ on $\mathcal{A}(\mathbb{Z}^t)$ to the multiplication by the corresponding central element $\widetilde{\chi}_\alpha^q$, given by (79):*

$$(84) \qquad \widehat{\mathbb{I}}^q(\{\alpha * \beta\}) = \widetilde{\chi}_\alpha^q \cdot \widehat{\mathbb{I}}^q(\{\beta\}).$$

9. *Restricting the scalars from $\mathbb{Z}_+$ to $\mathbb{Z}$ we get an isomorphism $\widehat{\mathbb{I}}^q : \mathbb{Z}\{\mathcal{A}(\mathbb{Z}^t)\} \xrightarrow{\sim} \mathbb{L}(\mathcal{X}_q^\vee)$.*

REMARKS. – 1. The map $\mathbb{F}_N^*$ was defined only for those $N$ which satisfy the condition formulated in Theorem 3.11. Formula (82) suggests a definition of the quantum Frobenius map $\mathbb{F}_N^*$ on the algebra $\mathbb{L}(\mathcal{X})$ for all $N$. Indeed, the first equality in (82) serves as a definition of $\mathbb{F}_N^*$ in a given cluster coordinate system $\{X_i\}$, and the second would imply that it is independent of the choice of the coordinate system.

2. The isomorphism (83) plus (82) obviously imply Frobenius Conjecture (66).

3. The property (84) in the coordinate form looks as follows: assuming $\sum_j \varepsilon_{ij}\alpha_j = 0$,

$$\widehat{\mathbb{I}}^q(\{\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n\}) = q^{-\sum_{i<j} \widehat{\varepsilon_{ij}}\alpha_i\alpha_j} \prod_i X_i^{\alpha_i} \cdot \widehat{\mathbb{I}}^q(\{\beta_1, \ldots, \beta_n\}).$$

Property (84) cannot even be stated if we skip the Langlands dual (defined at the end of Section 1.2) in (80), replacing $\mathcal{X}_q^\vee$ by $\mathcal{X}_q$. Indeed, there is no canonical isomorphism between $X_*(H_\mathcal{A})$ and $X^*(H_\mathcal{X})$.

## 4.4. An example: the map $\mathbb{I}_\mathcal{A}$ for the cluster ensemble of type $A_2$

Let $\varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the cluster modular group is $\mathbb{Z}/5\mathbb{Z}$. Its generator acts on the $\mathcal{X}$-space by

$$(X, Y) \mapsto (Y(1 + X^{-1})^{-1}, X^{-1}),$$

and on the tropical $\mathcal{A}$-space and by $(a, b) \mapsto (b, \max(b, 0) - a)$. The canonical map $\mathbb{I}_\mathcal{A}$ is given by:

$$\mathbb{I}_\mathcal{A}(a, b) = \begin{cases} X^a Y^b & \text{for } a \leq 0 \text{ and } b \geq 0 \\ \left(\frac{1+X}{XY}\right)^{-b} X^a & \text{for } a \leq 0 \text{ and } b \leq 0 \\ \left(\frac{1+X+XY}{Y}\right)^a \left(\frac{1+X}{XY}\right)^{-b} & \text{for } a \geq 0 \text{ and } b \leq 0 \\ ((1+Y)X)^b \left(\frac{1+X+XY}{Y}\right)^{a-b} & \text{for } a \geq b \geq 0 \\ Y^{b-a}((1+Y)X)^a & \text{for } b \geq a \geq 0. \end{cases}$$

Or equivalently

$$\mathbb{I}_\mathcal{A}(a, b) = \begin{cases} X^a Y^b & \text{for } a \leq 0 \text{ and } b \geq 0 \\ X^a Y^b (1 + X^{-1})^{-b} & \text{for } a \leq 0 \text{ and } b \leq 0 \\ X^a Y^b (1 + X^{-1})^{-b}(1 + Y^{-1} + X^{-1}Y^{-1})^a & \text{for } a \geq 0 \text{ and } b \leq 0 \\ X^a Y^b (1 + Y^{-1})^b (1 + Y^{-1} + X^{-1}Y^{-1})^{a-b} & \text{for } a \geq b \geq 0 \\ X^a Y^b (1 + Y^{-1})^a & \text{for } b \geq a \geq 0. \end{cases}$$

One can easily verify, that the formulae agree on the overlapping domains of values of $a$ and $b$. For a more elaborate and geometric discussion of this example, see [20].

REMARK. – By freezing some of the variables of a cluster ensemble, we prohibit mutations in the direction of these variables, and therefore change the spaces of universally positive Laurent polynomials for the corresponding $\mathcal{A}$ and $\mathcal{X}$ spaces. However we do not change the sets of the points of these spaces with values in any semifield. So by freezing some of the variables we do not change the source of a canonical map, we do change its target.

## 5. Canonical pairings in the finite type case

### 5.1. The canonical pairing between the tropical spaces

Given a seed $\mathbf{i}$, let us consider the following function [8] $P_{\mathbf{i}}$ on $\mathcal{A}(\mathbb{R}^t) \times \mathcal{X}(\mathbb{R}^t)$:

$$(85) \qquad\qquad P_{\mathbf{i}} := \sum_{i \in I} d_i a_i x_i.$$

For a finite type cluster ensemble $(\mathcal{A}, \mathcal{X})$ we define a version $\mathcal{I}'$ of the canonical pairing between the tropical spaces by maximizing this function over the (finite) set of all seeds:

$$(86) \qquad \mathcal{I}' : \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}(\mathbb{R}^t) \longrightarrow \mathbb{R}, \qquad \mathcal{I}'(a,x) := \max_{\{\text{seeds } \mathbf{i}\}} \sum_i d_i a_i x_i.$$

In the duality conjectures, however, we are looking for a canonical pairing involving the tropical points of the Langlands dual cluster $\mathcal{X}$-variety $\mathcal{X}^\vee$:

$$(87) \qquad\qquad \mathcal{I} : \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}^\vee(\mathbb{R}^t) \longrightarrow \mathbb{R}.$$

This apparent contradiction is resolved by the following lemma.

LEMMA 5.1. – *Let $x = \{x_i\} \in \mathcal{X}(\mathbb{Z}^t)$. Then the coordinates $\{d_i x_i\}$ behave under the mutations in the tropical space $\mathcal{X}(\mathbb{Z}^t)$ just like the coordinates of a point $\delta(x)$ of the Langlands dual tropical space $\mathcal{X}^\vee(\mathbb{Z}^t)$. So there is a canonical $\Gamma$-equivariant inclusion*

$$\delta : \mathcal{X}(\mathbb{Z}^t) \hookrightarrow \mathcal{X}^\vee(\mathbb{Z}^t),$$

*given in any cluster coordinate system by $\{x_i\} \longmapsto \{d_i x_i\}$. The map $\delta$ is an isomorphism for rational or real tropical points.*

*Proof.* – The tropical mutation formulae are

$$x_i^\sharp = x_i - [\varepsilon_{ik}]_+ \max(0, x_k), \qquad x_i' = \begin{cases} -x_k & \text{if } i = k, \\ x_i + [\varepsilon_{ik}]_+ x_k & \text{if } i \neq k. \end{cases}$$

So we get the claim by changing $x_j \longmapsto d_j x_j$ and using positivity of $d_k$ as well as the formula (19). $\qquad\square$

Lemma 5.1 tells that the pairing $\mathcal{I}'$ determines the restriction of the canonical pairing (87) to the subset $\delta(\mathcal{X}(\mathbb{Z}^t))$ of points of $\mathcal{X}^\vee(\mathbb{Z}^t)$ with the coordinates $\{d_i x_i\}$. Let us check now that the canonical pairing $\mathcal{I}(a, \delta(x))$ defined this way satisfies the crucial part 4) of Conjecture 4.3.

The latter tells that if $y \in \mathcal{X}^\vee(\mathbb{Z}^t)$ has non-negative coordinates $\{y_i\}$ in a cluster coordinate system, then one should have $\mathcal{I}(a, y) = \sum_i a_i y_i$. Theorem 5.2 guarantees that this is the case for the points $y$ of $\delta(\mathcal{X}(\mathbb{Z}^t))$. Indeed, if $y = \delta(x)$ then $y_i = d_i x_i$, so $\mathcal{I}(a, y) = \sum_i d_i a_i x_i$.

THEOREM 5.2. – *Suppose that $(\mathcal{A}, \mathcal{X})$ is a finite type cluster ensemble. Then, for any $(a, x) \in \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}(\mathbb{R}^t)$, there exists a seed $\mathbf{i}$ at which the maximum in (85) is attained, such that all coordinates $x_i$ of $x$ at $\mathbf{i}$ are non-negative.*

---

[8] We will see below that it is the tropicalization of the more fundamental element (90).

*Proof.* – Let us investigate how the function $P_{\mathbf{i}}$ behaves under mutations. Let $\{a_i\}$ be the coordinates of $a \in \mathcal{A}(\mathbb{R}^t)$. Set

$$(88) \qquad \varphi_k(a) := (p^*x_k)(a) = \sum_j \varepsilon_{kj} a_j.$$

PROPOSITION 5.3. – *Given a mutation $\mu_k : \mathbf{i} \to \mathbf{i}'$, we have*

$$(P_{\mathbf{i}'} - P_{\mathbf{i}})(a, x) = \begin{cases} d_k |x_k| \varphi_k(a) & \text{if } x_k \varphi_k(a) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

*This implies the following:*

a) *If $x_k = 0$, the mutation at the vertex $k$ does not change the function $P_{\mathbf{i}}$.*

b) *If $x_k > 0$, the mutation at the vertex $k$ strictly decreases the function $P_{\mathbf{i}}$ if $\varphi_k(a) < 0$, and does not change it otherwise.*

c) *If $x_k < 0$, the mutation at the vertex $k$ strictly increases the function $P_{\mathbf{i}}$ if $\varphi_k(a) > 0$, and does not change it otherwise.*

We show in Section 6.1 that this is an easy consequence of Basic Lemma 6.1.

LEMMA 5.4. – *Let $\mathcal{X}$ be an arbitrary cluster $\mathcal{X}$-variety. Let $\mathbf{i_0} \to \mathbf{i_1} \to \cdots \to \mathbf{i_n} \to \mathbf{i_0}$ be a sequence of mutations. Denote by $k_s$ the direction of the mutation $\mathbf{i_s} \to \mathbf{i_{s+1}}$.*

*Let us assume that there exists a tropical point $x \in \mathcal{X}(\mathbb{R}^t)$ such that for every $s = 0, \ldots, n$ the $x_{k_s}$-coordinate of the point $x$ for the seed $\mathbf{i_s}$ is negative. Then the sequence is trivial, i.e. $n = 0$.*

*Proof.* – Take a point $a \in \mathcal{A}(\mathbb{R}^t)$ with $\varphi_{k_0}(a) > 0$. Then, by Proposition 5.3, after the cluster transformation $\mathbf{i_0} \to \mathbf{i_1} \to \cdots \to \mathbf{i_n} \to \mathbf{i_0}$ the sum $\sum d_i a_i x_i$ will strictly increase. This contradiction proves the lemma. □

Take a point $(a, x) \in \mathcal{A}(\mathbb{R}^t) \times \mathcal{X}(\mathbb{R}^t)$. Take a seed $\mathbf{i}$ which realizes the maximum of sum (85) evaluated at this point. If all coordinates $x_i$ of $x$ in this seed are non-negative, we are done. If not, there exists a vertex $k$ such that $x_k < 0$. Let us perform a mutation at $k$. If the new coordinate system is non-negative for $x$, we are done. If not, we perform a mutation at a vertex $k'$ such that $x_{k'} < 0$, and so on. Since $\mathcal{X}$ is of finite type, after a finite number of mutations we get to a certain coordinate system for the second time. This contradicts Lemma 5.4, and thus proves the theorem. □

COROLLARY 5.5. – *Let $\mathcal{X}$ be a finite type cluster $\mathcal{X}$-variety. Then for every $x \in \mathcal{X}(\mathbb{R}^t)$ there exists a cluster coordinate system such that the coordinates $x_i$ of the point $x$ are non-negative.*

5.1.1. *The positive part of a tropical space*. – Given a positive space $\mathcal{X}$, the set of the points of the tropical space $\mathcal{X}(\mathbb{R}^t)$ which have non-negative coordinates in a given coordinate system is a convex cone. We call it the *positive cone* assigned to the coordinate system. The union of the positive cones forms the *positive part* $\mathcal{X}(\mathbb{R}^t)_+$ of $\mathcal{X}(\mathbb{R}^t)$.

DEFINITION 5.6. – *A positive space $\mathcal{X}$ is of definite (semi-definite, indefinite) type if the subset $\mathcal{X}(\mathbb{R}^t)_+$ is equal to (respectively dense in, not dense in) $\mathcal{X}(\mathbb{R}^t)$.*

Corollary 5.5 tells that a finite type cluster $\mathcal{X}$-variety is of definite type.

CONJECTURE 5.7. – *A cluster $\mathcal{X}$-variety is of finite type if and only if it is of definite type.*

EXAMPLES. – 1. Let $S$ be a surface with holes and marked points on the boundary. Denote by $h$ the number of holes without marked points on its boundary. There is an action of the group $(\mathbb{Z}/2\mathbb{Z})^h$ by birational automorphisms of the moduli space $\mathcal{X}_{PGL_2,S}$, see [6]. It acts by cluster transformations if and only if $h > 1$. Therefore, by Theorem 12.2 in *loc. cit.*, the cluster atlas on $\mathcal{X}_{PGL_2,S}$ is of semi-definite type if and only if $h > 1$.

However even if $h = 1$, the group $\mathbb{Z}/2\mathbb{Z}$ acts by positive transformations which leave invariant the semiring $\mathbb{L}_+(\mathcal{X}_{PGL_2,S})$ of positive regular functions. Therefore it is natural to extend the cluster atlas on $\mathcal{X}_{PGL_2,S}$ by adding the images of the cluster coordinate systems by the action of the group $\mathbb{Z}/2\mathbb{Z}$. We call the obtained positive atlas the *extended cluster atlas*. The moduli space $\mathcal{X}_{PGL_2,S}$ is of semi-definite type for this atlas.

2. The canonical pairing $\mathcal{I}$ for the cluster ensemble $(\mathcal{A}_{SL_2,S}, \mathcal{X}_{PGL_2,S})$ coincides with the intersection pairing between the $\mathcal{A}$- and $\mathcal{X}$-laminations defined in Section 12 of *loc. cit.*.

In Section 5.2 we show that for a finite type cluster $\mathcal{X}$-variety the positive cones give a finite decomposition of the space $\mathcal{X}(\mathbb{R}^t)$. It is dual to the generalized associahedron. Combining this with the Laurent Phenomenon Theorem [13] we construct the canonical map $\mathbb{I}_{\mathcal{X}}$ in the finite type case. Its tropicalization provides the canonical pairing (86).

## 5.2. The canonical map $\mathbb{I}_{\mathcal{X}}$

Any subset of vertices of a seed $\mathbf{i}$ provides us with a *subseed* $\mathbf{j} \subset \mathbf{i}$. Mutating the seed $\mathbf{i}$ at a vertex of $\mathbf{j}$ we get a new seed with a subseed canonically identified with $\mathbf{j}$. So we can mutate at its vertices, and so on. The collection of seeds obtained this way is called *the set of seeds $\mathbf{j}$-equivalent to a seed $\mathbf{i}$*. We use the notation $\mathbf{i_1} \sim_{\mathbf{j}} \mathbf{i_2}$ for $\mathbf{j}$-equivalent seeds.

Let $l \in \mathcal{X}(\mathbb{Z}^t)$. Let $\mathbf{i}$ be a non-negative seed for $l$, i.e. the $x$-coordinates of $l$ in this seed are non-negative. Let $\mathbf{j}(l)$ be the subseed of $\mathbf{i}$ determined by the zero $x$-coordinates of $l$, i.e. by the set of coordinates $x_i$ such that $x_i(l) = 0$. We call it the *zero subseed for $l$*.

THEOREM 5.8. – *Let $\mathcal{X}$ be a finite type cluster $\mathcal{X}$-variety, and $l \in \mathcal{X}(\mathbb{Z}^t)$. Let $\mathbf{i}$ be a non-negative seed for $l$, and $\mathbf{j}(l) \subset \mathbf{i}$ the zero subseed for $l$. Let $\mathbf{i}'$ be another non-negative seed for $l$. Then the seeds $\mathbf{i}$ and $\mathbf{i}'$ are $\mathbf{j}(l)$-equivalent.*

*Proof.* – Choose a cluster transformation **c** connecting the seeds **i** and **i′**. Among the sequences of mutations realizing **c**, choose the subset of sequences maximizing the minimal value of $P_{\mathbf{s}}(l)$, and in this subset choose a sequence where this minimum is attained a minimal number of times. Due to the finite type assumption, the number of possible values is finite, so such a subset exists. Our goal is to show that for any such a sequence the $x$-coordinates of all mutating vertices are zero. Let $\mathbf{i} = \mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_g = \mathbf{i}'$ be our sequence of seeds

Consider the first seed $\mathbf{s}_r$ where the minimum of $P_{\mathbf{s}_r}(l)$ is attained for the first time. Suppose that we come to this seed by a mutation $\mu_j$ at the vertex $j$ and leave it by a mutaion $\mu_i$ at the vertex $i$. Recall that if the pair $(i, j)$ generates the standard $(h+2)$-gon, i.e. (16) holds, then the cluster transformation $\mu_j\mu_i$ equals the one given by a sequence $\mu_i\mu_j \ldots$ of length $h$, where $h + 2$ is the period of the sequence $\mu_j\mu_i \ldots$, times $\sigma_{ij}$ in the $A_2$ case. We shall show that if we replace the subsequence of mutations $\mu_i\mu_j$ in our cluster transformation by the sequence $\mu_j\mu_i \ldots$ of length $h$, then in the new sequence either the minimum of $P_{\mathbf{s}_i}(l)$ will be bigger or it will be attained a smaller number of times, contradicting our assumption.

To show this it is sufficient to prove the following

LEMMA 5.9. – *For the exchange functions $\varepsilon_{ij}$ corresponding to Dynkin diagrams $A_1 \times A_1, A_2, B_2, G_2$, the values $P_{\mathbf{s}}(l), P_{\mu_j\mathbf{s}}(l), P_{\mu_i\mu_j\mathbf{s}}(l), \ldots$ change growth to decay only once per period.*

To prove this we will show that in this sequence the $x$-coordinates at the mutated vertices change their signs only once per period, and then use Proposition 5.3.

The claim we have to prove is an immediate corollary of the following observation. Recall the mutations $\mu$ (types $A_2$ and $A_1 \times A_1$) or $\mu_{\pm}$ (types $B_2$ and $G_2$). Consider the action of the sequence of cluster transformations $\mu, \mu^2, \mu^3, \ldots$ or, respectively, $\mu_{\pm}, \mu_{\mp}\mu_{\pm}, \mu_{\pm}\mu_{\mp}\mu_{\pm}, \ldots$ on the tropical plane $(x_1, x_2)$. The sequence of $x_1$-coordinates of the points on the orbit of a point is the set we were looking for. Now Lemmas 2.17, 2.19 and 2.20 imply that the $x_1$-coordinate changes the sign just once per period. The theorem is proved. $\qquad\square$

By the Laurent Phenomenon Theorem [13], $\mathbb{I}_{\mathcal{X}}(l)$ is a universally Laurent polynomial.

CONJECTURE 5.10. – *Theorem 5.8 is valid for an arbitrary cluster $\mathcal{X}$-variety.*

Theorems 5.2 and 5.8 tell that the space $\mathcal{X}(\mathbb{R}^t)$ has a canonical decomposition into cones. The cones are parametrised by the cluster $\mathcal{X}$-coordinate systems. Namely, such a cone is given by the set of all points with non-negative coordinates in a given cluster coordinate system.

PROPOSITION 5.11. – *The decomposition of $\mathcal{X}(\mathbb{R}^t)$ is dual to the generalized associahedron.*

*Proof.* – Follows immediately from the combinatorial description of the $(n - k)$-dimensional faces of the cones implied by Theorem 5.8: the cones are parametrised by the **j**-equivalence classes of seeds, where $|\mathbf{j}| = k$. Furthermore, the subcones of a given cone are parametrised by the subseeds **j′** squeezed between **j** and **i**. $\qquad\square$

CONJECTURE 5.12. – *In any cluster coordinate system the decomposition of $\mathcal{X}(\mathbb{R}^t)$ is a decomposition into convex cones.*

5.2.1. *Construction of the canonical map $\mathbb{I}_{\mathcal{X}}$ for finite type cluster ensembles.* – It is provided by Theorems 5.2 and 5.8 as follows. Given $l \in \mathcal{X}(\mathbb{Z}^t)$, Theorem 5.2 tells that there exists a seed $\mathbf{i}$ such that all coordinates $(x_1, \ldots, x_n)$ of $l$ for this seed are non-negative. Set

$$(89) \qquad \mathbb{I}_{\mathcal{X}}(l) := A_1^{x_1} \cdots A_n^{x_n}$$

where $(A_1, \ldots, A_n)$ are the $\mathcal{A}$-coordinates for the Langlands dual seed $\mathbf{i}^\vee$. Such a seed $\mathbf{i}$ may not be unique. However Theorem 5.8 guaranties that expression (89) does not depend on the choice of $\mathbf{i}$. Indeed, it tells that any other seed $\mathbf{i}'$ is $\mathbf{j}(l)$-equivalent to $\mathbf{i}$, and hence the expression (89) for $\mathbf{i}'$ is the same as for $\mathbf{i}$. So $\mathbb{I}_{\mathcal{X}}(l)$ is well defined.

## 6. Motivic avatar of the form $\Omega$ on the $\mathcal{A}$-space and the dilogarithm

Given a seed $\mathbf{i}$, we introduce a Casimir element $\mathbf{P_i}$. It does depend on the choice of a seed $\mathbf{i}$. Basic Lemma 6.1 provides a transformation formula for the element $\mathbf{P_i}$. Its applications include a proof of Proposition 5.3 as well as the key properties of the motivic dilogarithm class introduced in Section 6.4.

### 6.1. The basic lemma

Given a seed $\mathbf{i}$, the seed cluster tori $\mathcal{A}_\mathbf{i}$ and $\mathcal{X}_\mathbf{i}$ are dual to each other. Recall the cluster $\mathcal{A}$- and $\mathcal{X}$-coordinates $\{A_i\}$ and $\{X_i\}$ related to the seed $\mathbf{i}$. The set of characters $\{X_i^{d_i}\}$ of the torus $\mathcal{X}_\mathbf{i}$ is dual to the basis of characters $\{A_i\}$ of the torus $\mathcal{A}_\mathbf{i}$, see (4). So there is a natural Casimir element

$$(90) \qquad \mathbf{P_i} := \sum_{i \in I} d_i \cdot A_i \otimes X_i \in \mathbb{Q}(\mathcal{A})^* \otimes \mathbb{Q}(\mathcal{X})^*.$$

Here $\mathbb{Q}(\mathcal{A})^*$ is the multiplicative group of the field of rational functions on $\mathcal{A}$, similarly $\mathbb{Q}(\mathcal{X})^*$. We denote by $d \cdot *$ the multiplication of an element $*$ of the tensor product by an integer $d$.

Let us investigate how the Casimir element $\mathbf{P_i}$ changes under a mutation $\mu_k : \mathbf{i} \longrightarrow \mathbf{i}'$. Denote by $\{A_i'\}$ and $\{X_i'\}$ the cluster coordinates related to the seed $\mathbf{i}'$. Recall the notation

$$p^* X_k = \prod_{j \in I} A_j^{\varepsilon_{kj}} = \frac{\mathbb{A}_k^+}{\mathbb{A}_k^-} \in \mathbb{Q}(\mathcal{A})^*.$$

LEMMA 6.1. – *Given a mutation $\mu_k : \mathbf{i} \longrightarrow \mathbf{i}'$ at the vertex $k$, one has*

$$\mathbf{P_{i'}} - \mathbf{P_i} = d_k \Big( p^* X_k \otimes (1 + X_k) - (1 + p^* X_k) \otimes X_k \Big).$$

*Proof.* – We decompose the mutation $\mu_k$ into the composition $\mu_k = \mu'_k \circ \mu_k^\sharp$ (Section 2.1). The map $\mu'_k$ clearly preserves the element $\mathbf{P}_i$. Let us calculate the effect of the automorphism $\mu_k^\sharp$. We have

$$\sum_i d_i \cdot A'_i \otimes X'_i - \sum_i d_i \cdot A_i \otimes X_i$$

$$\overset{(26)-(27)}{=} \sum_i d_i \cdot A_i \otimes X_i(1+X_k)^{-\varepsilon_{ik}} + (1+p^*X_k)^{-1} \otimes X_k - \sum_i d_i \cdot A_i \otimes X_i$$

$$= \sum_i d_i \cdot A_i \otimes (1+X_k)^{-\varepsilon_{ik}} - d_k \cdot (1+p^*X_k) \otimes X_k$$

$$= \sum_i -d_i\varepsilon_{ik} \cdot A_i \otimes (1+X_k) - d_k \cdot (1+p^*X_k) \otimes X_k.$$

Notice that $-d_i\varepsilon_{ik} = -\varepsilon_{ik} = \varepsilon_{ki} = d_k\varepsilon_{ki}$. So the first term is equal to

$$\sum_i d_k\varepsilon_{ki} \cdot A_i \otimes (1+X_k) = d_k \cdot p^*X_k \otimes (1+X_k).$$

The basic lemma is proved. □

*Proof of Proposition 5.3.* – Notice that $P_i$ is nothing else but the tropicalization of the element $\mathbf{P}_i$. Furthermore, $\max(0,x) = [x]_+$. So the basic lemma implies

$$(P_{i'} - P_i)(a,x) = d_k\Big(\varphi_k^*(a)\,[x_k]_+ - [\varphi_k^*(a)]_+\,x_k\Big).$$

If $x_k > 0$, we get

$$(P_{i'} - P_i)(a,x) = d_k x_k\Big(\varphi_k^*(a) - [\varphi_k^*(a)]_+\Big) = \begin{cases} 0 & \text{if } \varphi_k^*(a) \geq 0 \\ d_k x_k \varphi_k^*(a) < 0 & \text{if } \varphi_k^*(a) < 0. \end{cases}$$

If $x_k < 0$, we get

$$(P_{i'} - P_i)(a,x) = -d_k x_k[\varphi_k^*(a)]_+ = \begin{cases} -d_k x_k \varphi_k^*(a) > 0 & \text{if } \varphi_k^*(a) > 0 \\ 0 & \text{if } \varphi_k^*(a) \leq 0. \end{cases}$$

The proposition is proved. □

## 6.2. The group $K_2$, the Bloch complex and the dilogarithm

6.2.1. *The Milnor group $K_2$ of a field.* – Let $A$ be an abelian group. Recall that $\Lambda^2 A$ is the quotient of $\otimes^2 A$ modulo the subgroup generated by the elements $a \otimes b + b \otimes a$. We denote by $a \wedge b$ the projection of $a \otimes b$ to the quotient.

Let $F$ be an arbitrary field. The Milnor group $K_2(F)$ is an abelian group given as the quotient of the group $F^* \otimes F^*$ by the subgroup generated by the *Steinberg relations* $(1-x) \otimes x$ where $x \in F^* - \{1\}$. The image of the generator $x \otimes y$ in $F^* \otimes F^*$ is denoted by $\{x,y\}$. It is well known [23] that the Steinberg relations imply that $\{x,y\} = -\{y,x\}$. So one has

$$(91) \qquad\qquad K_2(F) = \frac{\Lambda^2 F^*}{\text{Steinberg relations}}.$$

6.2.2. *A regulator map on $K_2$.* – Let $X$ be a smooth algebraic variety. Denote by $\Omega^2_{\log}(X)$ the space of 2-forms with logarithmic singularities on $X$. Denote by $\mathbb{Q}(X)$ the field of rational functions on $X$. One has $d\log \wedge d\log((1-f) \wedge f) = 0$. So there is a group homomorphism

$$d\log \wedge d\log : K_2(\mathbb{Q}(X)) \longrightarrow \Omega^2_{\log}(X), \qquad f \wedge g \longmapsto d\log(f) \wedge d\log(g).$$

6.2.3. *The Bloch complex.* – It is clear from (91) that the Milnor group $K_2(F)$ is the cokernel of the map

$$\delta : \mathbb{Z}[F^* - \{1\}] \longrightarrow \bigwedge\nolimits^2 F^*, \qquad \{x\} \longmapsto (1-x) \wedge x,$$

where $\{x\}$ is the generator of $\mathbb{Z}[F^* - \{1\}]$ corresponding to $x \in F^* - \{1\}$.

It turns out that one can describe nicely some elements in the kernel of this map parametrised by connected varieties of dimension bigger than zero. Namely, let $r(*, *, *, *)$ be the cross-ratio of four points on $\mathbb{P}^1$, normalized by $r(\infty, 0, 1, x) = x$. Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^* - \{1\}]$ generated by the following elements (the "five term relations"):

$$\sum_{i=1}^{5} (-1)^i \{r(x_1, \ldots, \widehat{x}_i, \ldots, x_5)\}, \qquad x_i \in P^1(F), \quad x_i \neq x_j.$$

Then one can check that $\delta(R_2(F)) = 0$ (see [19] for a conceptual proof). Let us set

$$B_2(F) := \frac{\mathbb{Z}[F^* - \{1\}]}{R_2(F)}.$$

Then the map $\delta$ gives rise to a homomorphism of $F$:

$$\delta : B_2(F) \longrightarrow \bigwedge\nolimits^2 F^*.$$

We view it as a complex, called the Bloch complex. Let $\{x\}_2$ be the projection of $\{x\}$ to $B_2(F)$. It is handy to add the elements $\{0\}_2, \{1\}_2, \{\infty\}_2$ and put $\delta\{0\}_2 = \delta\{1\}_2 = \delta\{\infty\}_2 = 0$.

Recall that $K_1(F) = F^*$. The product in Quillen's K-theory provides a map

$$K_1(F) \otimes K_1(F) \otimes K_1(F) \longrightarrow K_3(F).$$

The cokernel of this map is the *indecomposable part* $K_3^{\mathrm{ind}}(F)$ of $K_3(F)$. By Suslin's theorem [26] there is an isomorphism

$$(92) \qquad \mathrm{Ker}\Big(B_2(F) \xrightarrow{\delta} \bigwedge\nolimits^2 F^*\Big) \otimes \mathbb{Q} \xrightarrow{\cong} K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q}.$$

6.2.4. *The dilogarithm.* – Recall the classical dilogarithm function

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-t) d\log t.$$

The single-valued cousin of the dilogarithm, the Bloch-Wigner function

$$\mathcal{L}_2(z) := \mathrm{Im}\Big(\mathrm{Li}_2(z) + \arg(1-z)\log|z|\Big)$$

satisfies the Abel five term functional equation

$$\sum_{i=0}^{4} (-1)^i \mathcal{L}_2(r(x_0, \ldots, \widehat{x}_i, \ldots, x_4)) = 0.$$

Equivalently, it provides a group homomorphism

$$\mathcal{L}_2 : B_2(\mathbb{C}) \to \mathbb{R}, \qquad \sum n_i \{z\}_2 \longmapsto \sum n_i \mathcal{L}_2(z_i).$$

### 6.3. The $W$–class in $K_2$

Recall that a seed $\mathbf{i}$ provides cluster coordinates $\{A_i\}$ on $\mathcal{A}$. Set

$$(93) \qquad W_{\mathbf{i}} := \sum_{i,j \in I} \widetilde{\varepsilon}_{ij} \cdot A_i \wedge A_j \in \bigwedge^2 \mathbb{Q}(\mathcal{A})^*, \quad \widetilde{\varepsilon}_{ij} := d_i \varepsilon_{ij}.$$

Let us consider a map

$$\pi : \mathcal{A} \longrightarrow \mathcal{X} \times \mathcal{A}$$

given as a composition

$$\mathcal{A} \hookrightarrow \mathcal{A} \times \mathcal{A} \xrightarrow{p \otimes \mathrm{Id}} \mathcal{X} \times \mathcal{A}.$$

Here the first map is the diagonal map, and the second is the map $p$ on the first factor.

LEMMA 6.2. – *The element $W_{\mathbf{i}}$ is the projection of $\pi^* \mathbf{P}_{\mathbf{i}}$ to $\bigwedge^2 \mathbb{Q}(\mathcal{A})^*$.*

*Proof.* – One can write the element $W_{\mathbf{i}}$ as follows:

$$(94) \qquad W_{\mathbf{i}} = \sum_{i \in I} d_i \cdot A_i \wedge p^* X_i.$$

Indeed,

$$\sum_{i,j \in I} \widetilde{\varepsilon}_{ij} \cdot A_i \wedge A_j = \sum_{i,j \in I} d_i \cdot A_i \wedge A_j^{\varepsilon_{ij}} = \sum_{i \in I} d_i \cdot A_i \wedge p^* X_i.$$

The lemma follows immediately from this. $\qquad \square$

PROPOSITION 6.3. – *Let $\mu_k : \mathbf{i} \to \mathbf{i}'$ be a mutation. Then one has*

$$\mu_k^* W_{\mathbf{i}'} - W_{\mathbf{i}} = -2d_k \cdot p^* \Big( (1 + X_k) \wedge X_k \Big) = -2d_k \cdot \delta\{-p^* X_k\}_2.$$

*Proof.* – Follows from Basic Lemma 6.1 and Lemma 6.2. $\qquad \square$

COROLLARY 6.4. – *The element*

$$W = \sum_{i,j \in I} \widetilde{\varepsilon}_{ij} \cdot \{A_i, A_j\} \in K_2(\mathbb{Q}(\mathcal{A}))$$

*does not depend on the choice of the cluster coordinate system $\{A_i\}$.*

*Proof.* – Observe that $2 \cdot (1+x) \wedge x = 2 \cdot (1-(-x)) \wedge (-x)$ is twice a Steinberg relation. $\quad \square$

COROLLARY 6.5. – *The 2–form*

$$\Omega = \sum_{i,j \in I} \widetilde{\varepsilon}_{ij} \cdot d\log A_i \wedge d\log A_j$$

*does not depend on the choice of cluster coordinates $\{A_i\}$.*

LEMMA 6.6. – *The element $W$ is lifted from $\mathcal{U}$, i.e. one has $W \in p^* \bigwedge^2 \mathbb{Q}(\mathcal{U})^*$.*

*Proof.* – This is equivalent to the following. Let $\{b_j\}$ be a set of integers such that $\sum_j \varepsilon_{ij} b_j = 0$. Then replacing $A_i$ by $\lambda^{b_i} A_i$ we do not change the class $W_{\mathbf{i}}$. Let us check this claim. We have, using the skew symmetry of $\varepsilon_{ij}$:

$$\sum_{i,j\in I} \widetilde{\varepsilon}_{ij} \cdot (\lambda^{b_i} A_i) \wedge (\lambda^{b_j} A_j) - \sum_{i,j\in I} \widetilde{\varepsilon}_{ij} \cdot A_i \wedge A_j = 2 \sum_{i\in I} \sum_{j\in I} \widetilde{\varepsilon}_{ij} b_j \cdot A_i \wedge \lambda = 0.$$

The last equality follows from the condition $\sum_j \widetilde{\varepsilon}_{ij} b_j = 0$, which is equivalent to $\sum_j \varepsilon_{ij} b_j = 0$.

The lemma is proved.                                                                                            □

A cluster coordinate $A_k$ provides a valuation $v_{A_k}$ of the field $\mathbb{Q}(\mathcal{A})$ given by $v_{A_k}(A_j) = \delta_{jk}$.

LEMMA 6.7. – *The element $W$ has zero tame symbol with respect to the valuation $v_{A_k}$.*

*Proof.* – The tame symbol for the valuation $v_{A_k}$ is equal to

$$\prod_{j\in I} A_j^{\widetilde{\varepsilon}_{kj}} = \left( \prod_{j\in I} A_j^{\varepsilon_{kj}} \right)^{d_k} = p^* X_k^{d_k}.$$

One has $1 + X_k = (\mathbb{A}_k^+ + \mathbb{A}_k^-)/\mathbb{A}_k^- = A_k A_k'/\mathbb{A}_k^-$. Thus $1 + X_k = 0$ if $A_k = 0$. The lemma is proved.                                                                                            □

REMARK. – *Over local fields.* The $K_2$-class $W$ has the following amusing application. Let $F$ be a local field. Let $\mu_F$ be the group of roots of unity contained in $F$. Let $\alpha : K_2(F) \longrightarrow \mu_F$ be the norm residue symbol ([Mi], ch. 15). It is a (weakly) continuous function on $F \times F$ ([Mi], ch. 16). Clearly there is a restriction $i_x^* W \in K_2(F)$ of $W$ to any $F$-point $x$ of the union of the cluster tori. So we get a continuous function

$$h_F : \mathcal{A}(F) \longrightarrow \mu_F; \quad x \in \mathcal{A}(F) \longmapsto \alpha(i_x^* W) \in \mu_F.$$

If $F = \mathbb{R}$, we get a continuous function $h_{\mathbb{R}} : \mathcal{A}(\mathbb{R}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$. Its value on $\mathcal{A}(\mathbb{R}_{>0})$ is $+1$.

LEMMA 6.8. – *The action of the torus $H_{\mathcal{A}}$ on the space $\mathcal{A}$ preserves the class $W$ in $K_2$. The orbits of this action coincide with the null-foliation of the 2-form $\Omega$.*

*Proof.* – Follows from the very definitions and Lemma 1.5.                                                      □

COROLLARY 6.9. – *The symplectic structure on the positive space $\mathcal{U}$ provided by the form $\Omega$ on the space $\mathcal{A}$ coincides with the symplectic structure induced by the Poisson structure on the space $\mathcal{X}$.*

*Proof.* – This is an immediate consequence of Lemma 1.5, Proposition 2.9, and Lemma 6.8.                                                                                            □

*A degenerate symplectic structure on the space of real tropical $\mathcal{A}$-space.* Let $\{a_i\}$ be the coordinates corresponding to a seed $\mathbf{i}$. Consider the 2-form $\omega := \sum_{i,j\in I} \widetilde{\varepsilon}_{ij} da_i \wedge da_j$ on the space $\mathcal{A}(\mathbb{R}^t)$. Since mutations are given by piece-wise linear transformations, it makes sense to ask whether this form is invariant under mutations. The following easy lemma follows from Proposition 6.3.

LEMMA 6.10. – *The form $\omega$ does not depend on the choice of a cluster coordinate system.*

So the 2-form $\omega$ provides the real tropical space $\mathcal{A}(\mathbb{R}^t)$ with a degenerate symplectic structure invariant under the modular group.

## 6.4. The motivic dilogarithm class

6.4.1. *The weight two motivic complexes.* – Let $X^{(k)}$ be the set of all codimension $k$ irreducible subvarieties of $X$. Recall the tame symbol map

$$(95) \qquad \text{Res} : \bigwedge^2 \mathbb{Q}(X)^* \longrightarrow \prod_{Y \in X^{(1)}} \mathbb{Q}(Y)^*; \quad f \wedge g \longmapsto \text{Rest}_Y \left( f^{v_Y(g)}/g^{v_Y(f)} \right)$$

where $v_Y(f)$ is the order of zero of a rational function $f$ at the generic point of an irreducible divisor $Y$, and $\text{Rest}_Y$ denotes restriction to $Y$. The *weight two motivic complex* $\Gamma(X; 2)$ of a regular irreducible variety $X$ with the field of functions $\mathbb{Q}(X)$ is the following complex of abelian groups:

$$(96) \qquad \Gamma(X; 2) := \qquad B_2(\mathbb{Q}(X)) \xrightarrow{\delta} \bigwedge^2 \mathbb{Q}(X)^* \xrightarrow{\text{Res}} \prod_{Y \in X^{(1)}} \mathbb{Q}(Y)^* \xrightarrow{\text{div}} \prod_{Y \in X^{(2)}} \mathbb{Z}$$

where the first group is in degree 1 and Res is the tame symbol map (95). If $Y \in X^{(1)}$ is normal, the last map is given by the divisor $\text{div}(f)$ of $f$. If $Y$ is not normal, we take its normalization $\widetilde{Y}$, compute $\text{div}(f)$ on $\widetilde{Y}$, and then push it down to $Y$. The rational cohomology of this complex of groups is the weight two motivic cohomology of the scheme $X$:

$$H^i(X, \mathbb{Q}_{\mathcal{M}}(2)) := H^i \Gamma(X; 2) \otimes \mathbb{Q}.$$

The complex (96) is a complex of global sections of a complex of acyclic sheaves in the Zariski topology. We denote this complex of sheaves by $\underline{\Gamma}(2)$.

6.4.2. *Background on equivariant cohomology.* – Let $P$ be an oriented polyhedron, possibly infinite. We denote by $V(P)$ the set of its vertices. Suppose that we have a covering $\mathcal{U} = \{U_v\}$ of a scheme $X$ by Zariski open subsets, parametrized by the set $V(P)$ of vertices $v$ of $P$. Then for every face $F$ of $P$ there is a Zariski open subset $U_F := \cap_{v \in V(F)} U_v$. For every inclusion $F_1 \hookrightarrow F_2$ of faces of $P$ there is an embedding $j_{F_1,F_2} : U_{F_1} \hookrightarrow U_{F_2}$. In particular there is an embedding $j_F : U_F \hookrightarrow X$. So we get a diagram of schemes $\{\mathcal{U}_F\}$, whose objects are parametrized by the faces of $P$, and arrows correspond to codimension one inclusions of the faces.

Given a complex of sheaves $\mathcal{F}^\bullet$ on $X$, the diagram $\{\mathcal{U}_F\}$ provides a bicomplex $\mathcal{F}_{\mathcal{U},P}^{\bullet,\bullet}$ defined as follows. For any integer $k \geq 0$, let $\mathcal{F}_{\mathcal{U},P}^{\bullet,k}$ be the following direct sum of the complexes of sheaves:

$$\mathcal{F}_{\mathcal{U},P}^{\bullet,k} := \oplus_{F:\text{codim}F=k} \, j_{F*} j_F^* \mathcal{F}^\bullet.$$

The second differential $d_2 : \mathcal{F}_{\mathcal{U},P}^{\bullet,k} \longrightarrow \mathcal{F}_{\mathcal{U},P}^{\bullet,k+1}$ is a sum of restriction morphisms $j_{F_1,F_2}^*$ where codim $F_1 = k$ and codim$F_2 = k+1$ with the signs reflecting the orientations of the faces of $F$.

Assume that a group $\Gamma$ acts on $X$, and it acts freely on a contractible polyhedron $P$, preserving its polyhedral structure. Moreover, we assume that for any $\gamma \in \Gamma$ and any face $F$ of $P$ one has $\gamma(\mathcal{U}_F) = \mathcal{U}_{\gamma(F)}$. Finally, let us assume that $\mathcal{F}^\bullet$ is a $\Gamma$-equivariant complex of sheaves on $X$. Then the group $\Gamma$ acts freely on the bicomplex $\mathcal{F}_{\mathcal{U},P}^{\bullet,\bullet}$. If the sheaves $j_{U_F}^* \mathcal{F}^i$

are acyclic – this will be the case below – then the $\Gamma$-equivariant hypercohomology of $X$ with coefficients in $\mathcal{F}^\bullet$ are computed as follows:

$$(97) \qquad H_\Gamma^*(X, \mathcal{F}^\bullet) = H^* \left( \mathrm{Tot}(\mathcal{F}_{\mathcal{U},P}^{\bullet,\bullet})^\Gamma \right).$$

Here the right hand side has the following meaning: we take the total complex of the bicomplex $\mathcal{F}_{\mathcal{U},P}^{\bullet,\bullet}$, take its subcomplex of $\Gamma$-invariants and compute its cohomology. Formula (97) holds modulo $N$-torsion if the group $\Gamma$ acts on $P$ with finite stabilisers, whose orders divide $N$.

6.4.3. *The motivic dilogarithm class.* – The special cluster modular group $\widehat{\Gamma}$ acts on $\mathcal{U}$. Let us apply the above construction in the case when $X = \mathcal{U}$ and the cover $\mathcal{U}$ is given by (the images of) the cluster tori:

DEFINITION 6.11. – *The second integral weight two $\widehat{\Gamma}$-equivariant motivic cohomology group of $\mathcal{U}$, denoted $H_{\widehat{\Gamma}}^2(\mathcal{U}, \mathbb{Z}_\mathcal{M}(2))$, is obtained by the construction of Section 5.4 in the following situation:*

*1. The scheme $X$ is $\mathcal{U}$.*

*2. The group $\Gamma$ is the special cluster modular group $\widehat{\Gamma}$.*

*3. The polyhedral complex $P$ is the modular complex $\widehat{M}$ to which we glue cells of dimension $\geq 3$ to make it contractible.*

*4. The complex of sheaves $\mathcal{F}^\bullet$ is the complex of sheaves $\underline{\Gamma}(2)$ on $\mathcal{U}$.*

REMARKS. – i) One easily sees from the construction that the group $H_{\widehat{\Gamma}}^2(\mathcal{U}, \mathbb{Z}_\mathcal{M}(2))$ does not depend on the choice of the cells of dimension $\geq 3$ glued to the universal cover on $M$.

ii) One cannot define the $H_{\widehat{\Gamma}}^i(\mathcal{U}, \mathbb{Z}_\mathcal{M}(2))$ for $i < 2$ in a similar way using the complex (96), since it computes only the rational, but not the integral motivic cohomology in the degree 1.

THEOREM 6.12. – *There is a class $\mathbb{W} \in H_{\widehat{\Gamma}}^2(\mathcal{U}, \mathbb{Z}_\mathcal{M}(2))$.*

*Proof.* – A construction of a cocycle representing a cohomology class on the right boils down to the following procedure.

i) For each seed $\mathbf{i}$ exhibit a class $W \in \bigwedge^2 \mathbb{Q}(\mathcal{U})^*$ such that for every irreducible divisor $D$ in $\mathcal{U}$ the tame symbol of $W_\mathbf{i}$ at $D$ vanishes.

ii) To any mutation $\mathbf{i} \to \mathbf{i}'$ in $\widehat{M}$ find an element

$$(98) \qquad \beta_{\mathbf{i} \to \mathbf{i}'} \in B_2(\mathbb{Q}(\mathcal{U})) \quad \text{such that} \quad \delta\left(\beta_{\mathbf{i} \to \mathbf{i}'}\right) = W_\mathbf{i} - W_{\mathbf{i}'}.$$

iii) Prove that for any 2-dimensional cell of $\widehat{M}$ the sum of the elements (98) assigned to the oriented edges of the boundary of this cell is zero.

Recall the element $W_\mathbf{i} \in \bigwedge^2 \mathbb{Q}(\mathcal{U})^*$ assigned to a seed $\mathbf{i}$, see (93). For a mutation $\mu_k : \mathbf{i} \to \mathbf{i}'$ set

$$\beta_{\mathbf{i} \to \mathbf{i}'} := -2d_k \cdot \{-X_k\}_2 \in B_2(\mathbb{Q}(\mathcal{U})).$$

Then Proposition 6.3 is equivalent to (98). The 2-cells in $\widehat{M}$ are the standard $(h+2)$-gons. Therefore we have to check the statement only for the cluster transformations (16), which are given by an $(h+2)$-fold composition of the recursion (59). Therefore iii) reduces to the following

LEMMA 6.13. – *Let us define elements $x_i$ of a field $F$ by recursion* (59). *Then if the sequence* $\{x_i\}$ *is periodic with the period $h + 2$, one has*

$$(99) \qquad \sum_{i=1}^{h+2} d_i\{-x_i\}_2 = 0 \quad in \quad B_2(F) \otimes \mathbb{Q}, \qquad d_i = \begin{cases} b & \text{for } i \text{ even,} \\ c & \text{for } i \text{ odd.} \end{cases}$$

*Proof.* – Suppose that we have rational functions $f_i$ such that $\sum(1 - f_i) \wedge f_i = 0$. Then (see [19]) if there is a point $a \in F$ such that $\sum\{f_i(a)\}_2 = 0$, then $\sum\{f_i(x)\}_2 = 0$ for any $x \in F$.

Formula (99) in the case $A_2$ is equivalent to the famous five term relation for the dilogarithm. In the case $A_1 \times A_1$ it is the well known inversion relation $2(\{x\}_2 + \{x^{-1}\}_2) = 0$ in the group $B_2(F)$. It is easy to check using the recursion (59) that $\delta \sum_{i=1}^{h+2} d_i\{-x_i\}_2 = 0$ in $\bigwedge^2 F^*$. (Modulo 2-torsion this follows from Proposition 6.3.) Specializing $x_1 = -1$ in the $G_2$ case we get

$$3\{1\}_2 + \{-x_2\}_2 + 3\{1 + x_2\}_2 + \{x_2^2\}_2 + 3\{1 - x_2\}_2 + \{x_2\}_2 + 3\{1\}_2 + \{0\}_2.$$

Specializing further $x_2 = 0$ we get $12\{1\}_2 + 4\{0\}_2 = 0$. The $B_2$ case is similar. The lemma and hence the theorem are proved. □

EXERCISE. – The lemma asserts that the element on the left of (99) can be presented as linear combinations of the five term relations. Write them down in the $B_2$ and $G_2$ cases.

COROLLARY 6.14. – *Let us define elements $x_i \in \mathbb{C}$ by recursion* (59). *Then one has*

$$(100) \qquad \sum_{i=1}^{h+2} d_i \mathcal{L}_2(-x_i) = 0.$$

*Proof.* – Follows from Lemma 6.13 and and the five-term functional equation for the dilogarithm. □

COROLLARY 6.15. – *A path $\alpha : \mathbf{i} \to \mathbf{j}$ in $\widehat{\mathcal{G}}$ provides an element $\beta_\alpha \in B_2(\mathbb{C})$ such that* $\delta(\beta_\alpha) = W_{\mathbf{i}} - W_{\mathbf{j}}$.

*Proof.* – Decomposing the path $\alpha$ as a composition of mutations $\mathbf{i} = \mathbf{i}_1 \to \mathbf{i}_2 \to \cdots \to \mathbf{i}_n = \mathbf{j}$, set

$$\beta_\alpha := \sum_{i=1}^{n-1} \beta_{\mathbf{i}_i \to \mathbf{i}_{i+1}} \in B_2(\mathbb{C}).$$

The relations in the groupoid $\widehat{\mathcal{G}}$ are generated by the ones corresponding to the standard $(h + 2)$-gons. Thus Lemma 6.13 implies that this element does not depend on the choice of a decomposition. It evidently satisfies formula (98). The corollary is proved. □

## 6.5. Invariant points of the modular group and $K_3^{\mathrm{ind}}(\overline{\mathbb{Q}})$

Let $g$ be an element of the group $\widehat{\Gamma}$. It acts by an automorphism of the scheme $\mathcal{U}$. It follows that any stable point of $g$ is defined over $\overline{\mathbb{Q}}$: it is determined by a set of equations with rational coefficients. Let $p \in \mathcal{U}(\overline{\mathbb{Q}})$ be a stable point of $g$. The element $g$ can be presented by a loop $\alpha(g)$ based at $v$.

PROPOSITION 6.16. – *Let $p \in \mathcal{U}(\overline{\mathbb{Q}})$ be a stable point of an element $g \in \widehat{\Gamma}$. Then there is an invariant*

$$\beta_{g,p} := \beta_{\alpha(g)} \in K_3^{\mathrm{ind}}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}.$$

*Proof.* – Since $p \in \mathcal{U}$, the element $W_{\mathbf{i}}$ can be evaluated at $p$. Corollary 6.15, applied to the loop $\alpha(g)$ based at $v$, implies $\delta\beta_{g,p} = W_{\mathbf{i}}(p) - W_{\mathbf{i}}(p) = 0$. So $\beta_{\alpha(g)} \in H^1(B(\overline{\mathbb{Q}}; 2))$. Using (92) we get an element in $K_3^{\mathrm{ind}}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$. $\square$

Here is how we compute the invariant $\beta_{g,p}$. Let us present $g$ as a composition of mutations: $g = \gamma_1 \cdots \gamma_n$. Each mutation $\gamma$ determines the corresponding rational function $X_\gamma$ on $\mathcal{U}$. Then

$$\beta_{g,p} = \sum_{i=1}^{n} 2d_\gamma \{X_{\gamma_i}(p)\}_2.$$

Here $d_\gamma$ is the multiplier assigned to the cluster coordinate $X_\gamma$. A similar procedure can be applied to any stable point $p \in \mathcal{U}$ of $g$, assuming that the functions $X_{\gamma_i}$ can be evaluated at $p$.

REMARK. – According to Borel's theorem, the rank of $K_3^{\mathrm{ind}}(F)$ for a number field $F$ equals to the number $r_2(F)$ of embeddings $F \subset \mathbb{C}$ up to complex conjugation.

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