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DUALITY OF SCHRAMM-LOEWNER EVOLUTIONS

BY JULIEN DUBÉDAT

ABSTRACT. – In this note, we prove a version of the conjectured duality for Schramm-Loewner Evolutions, by establishing exact identities in distribution between some boundary arcs of chordal SLE_κ, $\kappa > 4$, and appropriate versions of $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$.

RÉSUMÉ. – On démontre dans cette note une version de la dualité conjecturée pour les évolutions de Schramm-Loewner, en établissant des identités en distribution exactes entre certains arcs de SLE_κ chordal, $\kappa > 4$, et des versions appropriées de $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$.

1. Introduction

Schramm-Loewner Evolutions (or SLE), introduced by Schramm in 1999, are probability distributions, parameterized by $\kappa > 0$, on non-self traversing curves (the trace) connecting two boundary points in a planar, simply connected domain. They are characterized by a conformal invariance condition and a domain Markov property. See [7, 12, 16] for general SLE background.

The geometric properties of the trace vary with the parameter κ. In particular, when $\kappa \leq 4$, the trace is a.s. a simple curve; this is no longer the case if $\kappa > 4$ ([12]). The trace stopped at some finite time is then distinct from its boundary. The duality conjecture for SLE, roughly stated, is that a boundary arc of SLE_κ is locally absolutely continuous w.r.t. to (some version) of $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$. This was suggested by Duplantier. In the case $(\kappa, \hat\kappa) = (8, 2)$, this follows from the exact combinatorial relation between Loop-Erased Random Walks and Uniform Spanning Trees and the identification of their scaling limits in terms of SLE ([8]). In the case $(\kappa, \hat\kappa) = (6, 8/3)$, it follows from the locality/restriction framework ([6]). An approach based on a relation with the free field has been proposed by Sheffield. A precise duality conjecture is stated in [1] and elaborated on in [3]; we prove slightly different versions here.

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In [3], it is shown that duality shares common features with reversibility and the question of defining multiple SLE strands in a common domain. This *local commutation* property states that two SLE strands can be grown in a domain to a positive size, in a way that does not depend on the order in which the SLE's are growing. Such systems of commuting SLE's are classified in [3]; in particular, two versions of SLE_κ, $\mathrm{SLE}_{\hat\kappa}$ can commute only if $\hat\kappa \in \{\kappa, 16/\kappa\}$.

While it is rather easy to check directly the local commutation identities implied by reversibility and some duality conjectures, the crucial difficulty consists in working backward and proving reversibility or duality from these local identities. One may think of this as a "local to global" problem.

Decisive progress was achieved by Zhan in [18], where he proves reversibility of chordal SLE_κ, $\kappa \leq 4$, i.e. that the range of the trace of an SLE_κ in D going from x to y has the same distribution as the range of the trace of SLE going from y to x in D. This was previously known for $\kappa \in \{2, 8/3, 4, 6, 8\}$. The argument involves a sequence of couplings of an $\mathrm{SLE}_\kappa(D, x, y)$ with an $\mathrm{SLE}_\kappa(D, y, x)$, such that each coupling in the sequence is absolutely continuous w.r.t. the trivial (independent) coupling, and the limiting coupling is exact (the ranges of the two traces are identical). The fact that similar techniques may be used to prove duality is also mentioned in [18]. The present article stems in part from an effort to clarify and extend this "local to global" argument.

After the present work appeared as a preprint, the manuscript [17] was brought to our attention. There, as here, ideas and techniques from [3, 18] are combined to obtain a certain number of duality identities, with some overlap with those stated here (in Proposition 10). Subsequently, a different construction of duality identities was given in [4], via the free field; this allows to establish "strong duality" identities, in which the conditional law of an SLE given a boundary arc is also specified. Such identities were first conjectured in [1].

Let $\gamma, \hat\gamma$ be traces of two SLE's satisfying the local commutation condition. Then, for U, V disjoint open subsets of the domain, one has a coupling of $(\gamma, \hat\gamma)$ which is "correct" on the time set $\{(s,t) : s \leq \tau, t \leq \hat\tau\}$, where τ, $\hat\tau$ are stopping times for the two SLE's, such that $\gamma^\tau \subset U$, $\hat\gamma^{\hat\tau} \subset V$. We construct a coupling of $(\gamma, \hat\gamma)$, which is "correct" on the time set $\{(s,t) : \gamma_{[0,s]} \cap \hat\gamma_{[0,t]} = \varnothing\}$. See Theorem 6 for a precise statement.

The duality identities follow from applying Theorem 6 to appropriate pairs of commuting SLE's, together with some *a priori* geometric information on the traces. Plainly, many identities may be generated in this fashion.

The identities considered here involve variants of SLE_κ: the $\mathrm{SLE}_\kappa(\underline\rho)$ processes ($\underline\rho = \rho_1, \dots, \rho_n$). They satisfy a domain Markov property when keeping track of n marked points $z_1, \dots, z_n$ (in addition of the origin and the target of chordal SLE). The influence of z_i on the SLE trace is quantified by the real parameter ρ_i; this influence is attractive for $\rho_i < 0$ and repulsive for $\rho_i > 0$.

Let us consider a chordal SLE in the upper half-plane $\mathbb{H}$, going from 0 to infinity. In the phase $4 < \kappa < 8$, a boundary point, say 1, is "swallowed", *i.e.* gets disconnected from infinity by the trace at a random time τ_1 when the trace hits some point in $(1, \infty)$. The boundary arc straddling 1 is the boundary arc seen by 1 at time τ_1^-.

THEOREM 1. – *Consider a chordal* SLE_κ *in* $(\mathbb{H}, 0, \infty)$*,* $4 < \kappa < 8$*; let* D *be the leftmost visited point on* $(1, \infty)$*. Conditionally on* D*, the boundary arc straddling 1 is distributed as an* $\mathrm{SLE}_{\hat\kappa}(-\frac{\hat\kappa}{2}, \hat\kappa - 4, \hat\kappa - 2)$ *in* $(\mathbb{H}, D, \infty, 0, 1, D^+)$*, stopped when it hits* $(0, 1)$*.*

In the phase $\kappa \geq 8$, a.s. every point in $\overline{\mathbb{H}}$ is visited by the trace. We isolate a boundary arc in a different way. Let G be the leftmost point on $(-\infty, 0)$ visited by the trace before τ_1. We consider the boundary of K_{τ_G}, the hull of the SLE stopped when it first visits G; this boundary is an arc between G and a point in $(0, 1)$.

THEOREM 2. – *Consider a chordal* SLE_κ *in* $(\mathbb{H}, 0, \infty)$*,* $\kappa \geq 8$*. Let* G *be the leftmost visited point on* $(-\infty, 0)$ *before* τ_1*. Conditionally on* G*, the boundary of* K_{τ_G} *is distributed as an* $\mathrm{SLE}_{\hat\kappa}(\frac{\hat\kappa}{2}, \frac{\hat\kappa}{2} - 2, -\frac{\hat\kappa}{2}, \hat\kappa - 4)$ *in* $(\mathbb{H}, G, \infty, G^-, G^+, 0, 1)$*, stopped when it hits* $(0, 1)$*.*

The distributions of D and G are well known and easy to derive.

The article is organized as follows. Section 2 recalls some absolute continuity properties of chordal SLE. Local commutation is discussed in Section 3. Maximal couplings of commuting SLE's are constructed in Section 4. Geometric consequences (in particular duality) are drawn in Section 5. Some technical lemmas are postponed to Section 6.

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2. Absolute continuity for chordal SLE

In this section we consider some absolute continuity properties of chordal SLE, mostly based on [6]. Chordal SLE will also serve as a reference measure for variants we will study later; some familiarity with chordal SLE is assumed (see, e.g., [7, 12, 16]).

We adopt the following notation: $c = (D, x, y)$ is a configuration where D is a simply connected domain and x, y are distinct boundary points. Unless there is an ambiguity, the configuration is simply denoted by D. The chordal SLE_κ measure on $c = (D, x, y)$ is denoted by μ_c (κ is fixed). It is seen as a measure on Loewner chains up to increasing time change; or as a configuration-valued continuous process (up to time change); or as a measure on non self-traversing paths ([12]). This path (the SLE "trace") is denoted by γ, while the hull it generates is denoted by K ($D \setminus K_t$ is the connected component of $D \setminus \gamma_{[0,t]}$ having y on its boundary). Let U be a subdomain of D, agreeing with D in a neighborhood of x, and not containing y on its boundary. Then μ_c^U denotes the measure on paths induced by chordal SLE starting from x and stopped on exiting U; this happens at a random time τ, at which the hull is K_τ, the tip of the trace is γ_τ, and the configuration c_τ is $(D_\tau = D \setminus K_\tau, \gamma_\tau, y)$. More generally, for τ a stopping time, γ^τ denotes the trace stopped at τ (*i.e.* the process up to time τ), μ_c^τ the measure induced by stopping at τ. We will use γ to denote both the trace as a process and as a subset of $\overline{D}$ (the range of the process).

For some computations, it is convenient to fix a particular time parameterization, typically half-plane capacity of the hull (mapping conformally the domain to the upper half-plane). Otherwise we will reason up to bicontinuous (progressive, increasing) change of time. The class of stopping times is invariant under such time reparameterizations.

Later on, we will use tightness conditions, so we shall review some technical points now. Let (D, x, y) be a configuration, K a hull such that $(D \setminus K, x', y)$ is a configuration for some $x' \in \partial K$. By the Riemann mapping theorem, there is a conformal equivalence $\phi_K : D \setminus K \to D$; one can specify it uniquely by requiring its 2-jet at y to be trivial ($\phi_K(y) = y$, $\phi_K'(y) = 1$, $\phi_K''(y) = 0$ if ϕ_K extends smoothly at y; this condition is coordinate independent, so one can first "straighten" the boundary at y). One defines a topology on hulls as follows: (K_n) converges to K if $\phi_{K_n}^{-1}$ converges to ϕ_K^{-1} uniformly on compact sets of D. This is a version of Carathéodory convergence. A topology on chains $(K_t)_{t\geq 0}$ is given by the condition: $(K_t^n)_t$ converges to $(K_t)_t$ if $(t, w) \mapsto \phi_{K_t^n}^{-1}(w)$ converges uniformly on compact sets of $[0, T] \times D$. Then the Loewner equation maps continuously $C(\mathbb{R}^+, \mathbb{R})$ (with the usual topology of uniform convergence on compact sets) to the space of chains endowed with this topology. Thus the induced measure on chains is a Radon measure. From [12], we know that the chain is a.s. generated by a continuous non self-traversing path γ. For clarity, we will think of SLE as a measure on such paths, with the topology on chains described above.

To express densities, we need to define some conformal invariants. Let (D, x, y) be a configuration, z_x, z_y analytic local coordinates at the boundary (z_x mapping a neighborhood of x in D to the neighborhood of 0 in the upper semidisk). The Poisson excursion kernel is defined as

$$H_D(x, y) = \lim_{X\to x, Y\to y} \frac{G_D(X, Y)}{\Im(z_x(X))\Im(z_y(Y))}$$

where G_D is the Green function in D (with Dirichlet boundary conditions); this depends on the choice of z_x (or z_y) as a 1-form. (If z_x' is another local coordinate at x, dz_x'/dz_x is positive). If D and D' agree in a neighborhood of x, we choose the same local coordinate z_x, so that $H_{D'}(x, y')/H_D(x, y)$ does not depend on a choice of local coordinate at x. Similarly for $i, j = 1, 2$, consider configurations (D_{ij}, x_i, y_j) such that D_{ij} agrees with $D_{i,3-j}$ in a neighborhood of x_i and with $D_{3-i,j}$ in a neighborhood of y_j. Then the ratio:

$$\frac{H_{D_{11}}(x_1, y_1)H_{D_{22}}(x_2, y_2)}{H_{D_{12}}(x_1, y_2)H_{D_{21}}(x_2, y_1)}$$

is defined independently of any (coherent) choice of local coordinates at x_i, y_j. To simplify the notation, if $c = (D, x, y)$ is a configuration, we set $H(c) = H_D(x, y)$.

There is a σ-finite measure μ^{loop} on unrooted loops in $\mathbb{C}$, the Brownian loop measure ([6, 9]). As in [5], let us denote

$$m(D; K, K') = \mu^{\text{loop}}\{\delta : \delta \subset D, \delta \cap K \neq \varnothing, \delta \cap K' \neq \varnothing\}.$$

In accordance with [6], set $\alpha = \alpha_\kappa = \frac{6-\kappa}{2\kappa}$, $\lambda = \lambda_\kappa = \frac{(6-\kappa)(8-3\kappa)}{2\kappa}$.

PROPOSITION 3. – *Assume that $c = (D, x, y)$ and $c' = (D', x, y')$ are configurations agreeing in a neighborhood U of x such that $\overline{U}$ is compact and at positive distance to the* *symmetric difference $D\Delta D'$, and $\partial U\cap\partial D=\partial U\cap\partial D'$ is a connected arc containing x at positive distance of y,y'. Then μ_c^U and $\mu_{c'}^U$ are mutually absolutely continuous, with density*

$$\frac{d\mu^U_{c'}}{d\mu^U_c}(\gamma)=\left(\frac{H(c'_\tau)H(c)}{H(c_\tau)H(c')}\right)^\alpha\exp(-\lambda m(D;K_\tau,D\setminus D')+\lambda m(D';K_\tau,D'\setminus D))$$

uniformly bounded above and below.

Proof. – The bound on densities can be derived as in Lemma 14 (or actually seen as a particular case of Lemma 14, when D and D' can be obtained by removing two different hulls from a common larger domain D''). We will reduce the statement to two known cases.

1. Assume that $D=D'$. Then the statement follows from Lemma 3.2 in [3]; see also [14]. More precisely, consider the following situation: $\mathbb{H}$ is the upper half-plane, three boundary points x,y,y' are marked; K is a hull around x, x' its tip, ϕ a conformal equivalence $\mathbb{H}\setminus K\to\mathbb{H}$. Then $H_{\mathbb{H}}(x,y)=(x-y)^{-2}$, $H_{\mathbb{H}}(x,y')=(x-y')^{-2}$, computing in the natural local coordinate. It follows that $H_{\mathbb{H}\setminus K}(x',y)=\frac{\phi'(y)}{(\phi(y)-\phi(x'))^2}$, $H_{\mathbb{H}\setminus K}(x',y')=\frac{\phi'(y')}{(\phi(y')-\phi(x'))^2}$, for an appropriate (common) local coordinate at x'. Then the ratio

$$\frac{H_{\mathbb{H}\setminus K}(x',y')H_{\mathbb{H}}(x,y)}{H_{\mathbb{H}\setminus K}(x',y)H_{\mathbb{H}}(x,y')}=\phi'(y')\left(\frac{y'-x'}{\phi(y')-\phi(x')}\right)^2\phi'(y)^{-1}\left(\frac{y'-x'}{\phi(y')-\phi(x')}\right)^{-2}$$

is independent of (coherent) choices. One concludes by identifying the density of an $\mathrm{SLE}_\kappa(\mathbb{H},x,y)$ and an $\mathrm{SLE}_\kappa(\mathbb{H},x,y')$ with respect to the common reference measure $\mathrm{SLE}_\kappa(\mathbb{H},x,\infty)$.

2. Assume that $D'\subset D$, D' and D agree in a neighborhood of $y=y'$. Then the statement is essentially a consequence of Proposition 5.3 in [6] and results on loop measures in [9]. More precisely, it is shown in [6] that

$$M_t=\left(\frac{H(c'_t)H(c)}{H(c_t)H(c')}\right)^\alpha\exp(-\lambda m(D;K_t,D\setminus D'))$$

is a (positive) local martingale under μ_c. If we stop the chain at τ (first exit of U), then the stopped process M^τ is a bounded martingale. The Girsanov transform of μ_c^τ by M^τ is then $\mu^\tau_{c'}$. See, e.g., Proposition 3.1 in [5] for a similar interpretation of the results in [6].

3. The general case reduces to 1,2 as follows. Let y'' be a point on the connected boundary arc of x in $\partial D\cap\partial D'$, which is not on $\overline{\partial U}$; and V the connected component of $D\cap D'$ having x on its boundary. Then apply 1 to go from (D,x,y) to (D,x,y''); then 2 to go from (D,x,y'') to (V,x,y''); then 2 to go from (V,x,y'') to (D',x,y''); then 1 to go from (D',x,y'') to (D',x,y'). Cancellations occur due to the “inclusion exclusion” form of the ratios $(H(c'_\tau)H(c)/H(c_\tau)H(c'))$ and the restriction property of the loop measure ([9]).

Explicitly, assume that the statement holds for two pairs of configurations (c_1,c_2) and (c_2,c_3), $c_i=(D_i,x,y_i)$, all three configurations agreeing in a neighborhood U of x. We are thus assuming:

$$\frac{d\mu^U_{c_{i+1}}}{d\mu^U_{c_i}}(\gamma)=\left(\frac{H(c^\tau_{i+1})H(c_i)}{H(c^\tau_i)H(c_{i+1})}\right)^\alpha\exp(-\lambda m(D_i;K_\tau,D_i\setminus D_{i+1})+\lambda m(D_{i+1};K_\tau,D_{i+1}\setminus D_i))$$

for $i=1,2$. Since trivially

$$\frac{d\mu^U_{c_3}}{d\mu^U_{c_1}}(\gamma)=\frac{d\mu^U_{c_3}}{d\mu^U_{c_2}}(\gamma)\cdot\frac{d\mu^U_{c_2}}{d\mu^U_{c_1}}(\gamma)$$

we need only show that

$$\left(\frac{H(c_3^\tau)H(c_1)}{H(c_1^\tau)H(c_3)}\right)=\left(\frac{H(c_2^\tau)H(c_1)}{H(c_1^\tau)H(c_2)}\right)\cdot\left(\frac{H(c_3^\tau)H(c_2)}{H(c_2^\tau)H(c_3)}\right)$$

$$\begin{aligned}m(D_1;K_\tau,D_1\setminus D_3)-m(D_3;K_\tau,D_3\setminus D_1))&=m(D_1;K_\tau,D_1\setminus D_2)-m(D_2;K_\tau,D_2\setminus D_1))\\&\quad+m(D_2;K_\tau,D_2\setminus D_3)-m(D_3;K_\tau,D_3\setminus D_2)).\end{aligned}$$

The first equation is clear, as one can evaluate these ratios with a common choice of local coordinate for each of the marked points $x,x'=\gamma_\tau,y_i$. For the second relation, consider the set of loops that intersect K_τ and exactly one or two of the three sets D_1^c,D_2^c,D_3^c. Then the measure of this set of loops under $\mu^{\rm loop}$ can be expressed as

$$m(D_1;K_\tau,D_1\setminus D_3)+m(D_2;K_\tau,D_2\setminus D_1)+m(D_3;K_\tau,D_3\setminus D_2)$$

so that this quantity does not change under a permutation of the indices 1, 2, 3, which yields the second relation. □

3. Local commutation

3.1. Reversibility

Following the discussion in Section 2.1 of [3], we phrase and then check a necessary condition for reversibility.

Consider a configuration $c=c_{0,0}=(D,x,y)$, γ an SLE from x to y and $\hat\gamma$ an SLE from y to x. Denote $c_{s,t}=(D\backslash(K_s\cup\hat K_t),\gamma_s,\hat\gamma_t)$. Let $U,\hat U$ be disjoint neighborhoods of x,y respectively; $\tau,\hat\tau$ denote first exits of $U,\hat U$ by $\gamma,\hat\gamma$ respectively. Assume that $\gamma,\hat\gamma$ can be coupled so that one is the reversal of the other. Then an application of the Markov property for $\hat\gamma$ shows that the distribution of γ^τ conditional on $\hat\gamma^{\hat\tau}$ is (stopped) SLE in $c_{0,\hat\tau}=(D\setminus\hat K_{\hat\tau},x,\hat\gamma_{\hat\tau})$. Indeed, the distribution of $\hat\gamma$ after $\hat\tau$ conditional on $\hat\gamma^{\hat\tau}$ is that of chordal SLE in $c_{0,\hat\tau}$ (from $\hat\gamma_{\hat\tau}$ to x). Applying the (assumed) reversal coupling in $c_{0,\hat\tau}$, one obtains the conditional distribution of γ^τ given $\hat\gamma^{\hat\tau}$. Symmetrically, the conditional distribution of $\hat\gamma^{\hat\tau}$ given γ^τ is SLE in $c_{\tau,0}=(D\setminus K_\tau,\gamma_\tau,y)$. (The identities in distribution considered here are up to time reparameterization).

By integration, this gives the identity of measures:

$$\int\hat f(\hat\gamma^{\hat\tau})\Big(\int f(\gamma^\tau)d\mu^U_{c_{0,\hat\tau}}(\gamma^\tau)\Big)d\hat\mu_c^{\hat U}(\hat\gamma^{\hat\tau})=\int f(\gamma^\tau)\Big(\int\hat f(\hat\gamma^{\hat\tau})d\hat\mu^{\hat U}_{c_{\tau,0}}(\hat\gamma^{\hat\tau})\Big)d\mu_c^U(\gamma^\tau)\tag{3.1}$$

for arbitrary positive Borel functions $f,\hat f$. This is the *local commutation* condition studied in [3]. Disintegrating and inserting densities (that exist from absolute continuity properties) yields the condition:

$$\left(\frac{d\mu^U_{c_{0,\hat\tau}}}{d\mu^U_c}\right)(\gamma^\tau)=\left(\frac{d\hat\mu^{\hat U}_{c_{\tau,0}}}{d\hat\mu^{\hat U}_c}\right)(\hat\gamma^{\hat\tau})\tag{3.2}$$

almost everywhere in $\gamma^\tau,\hat\gamma^{\hat\tau}$. This is an identity between two (continuous) functions of the paths $\gamma,\hat\gamma$. From the above results on absolute continuity of SLE (Proposition 3), we see that both sides are indeed equal, and their common value is the explicit quantity:

$$\ell_D(\gamma^\tau,\hat\gamma^{\hat\tau})=\left(\frac{H(c_{\tau,\hat\tau})H(c)}{H(c_{\tau,0})H(c_{0,\hat\tau})}\right)^\alpha\exp(-\lambda m(D;K_\tau,\hat K_{\hat\tau})) \tag{3.3}$$

which is manifestly symmetric in $\gamma^\tau,\hat\gamma^{\hat\tau}$. (We use ℓ for *likelihood ratio*, somewhat abusively.)

Using the expression of ℓ as Radon-Nikodym derivatives of probability measures, we see that:

$$\begin{aligned}\int \ell_D(\gamma^\tau,\hat\gamma^{\hat\tau})d\hat\mu_c^{\hat U}(\hat\gamma^{\hat\tau})&=\int d\hat\mu_{c_{\tau,0}}(\hat\gamma^{\hat\tau})=1\quad\forall\gamma^\tau\\ \int \ell_D(\gamma^\tau,\hat\gamma^{\hat\tau})d\mu_c^{U}(\gamma^{\tau})&=\int d\mu_{c_{0,\hat\tau}}(\gamma^{\tau})=1\quad\forall\hat\gamma^{\hat\tau}\end{aligned} \tag{3.4}$$

This also applies to any pair of stopping times $\sigma,\hat\sigma$ dominated by $\tau,\hat\tau$ (*i.e.* σ is a stopping time for γ such that $\sigma\le\tau$ a.s.).

Such local commutation identities (without relying on explicit densities) are proved in greater generality in [3] under an infinitesimal commutation condition, which is easily checked in the present case. In the next subsection, we include a simple construction for the cases needed in this article.

Let us also point out a (deterministic) tower property, dictated by compatibility with the SLE Markov property. Let (D,x,y) be a domain, (K_s) a Loewner chain growing at x (with trace γ_s), and $(\hat K_t)$ a Loewner chain growing at x (with trace $\hat\gamma_t$). Let $c_{s,t}=(D\setminus(K_s\cup\hat K_t),\gamma_s,\hat\gamma_t)$. Let us denote, for $0\le s_1\le s_2$, $0\le t_1\le t_2$:

$$\ell_{s_1,t_1}^{s_2,t_2}=\ell_{c_{s_1,t_1}}(\gamma^{s_2},\hat\gamma^{t_2}).$$

Then, for $0\le s_1\le s_2\le s_3$, $0\le t_1\le t_2\le t_3$:

$$\ell_{s_1,t_1}^{s_2,t_2}\ell_{s_2,t_1}^{s_3,t_2}=\ell_{s_1,t_1}^{s_3,t_2},\qquad \ell_{s_1,t_1}^{s_2,t_2}\ell_{s_1,t_2}^{s_2,t_3}=\ell_{s_1,t_1}^{s_2,t_3}. \tag{3.5}$$

For fixed γ, the first relation has to hold a.e. in γ to ensure compatibility of (3.4) with the Markov property of γ; the second relation corresponds to the Markov property of $\hat\gamma$. Alternatively, this can be checked directly from the explicit expression (3.3), by telescopic cancellations and the restriction property of the loop measure $\mu^{\rm loop}$ ([9]).

3.2. The general case

We now move to the general case (in simply connected domains) of local commutation, following Theorem 7.1 of [3], which we rephrase in the present context. For now we think of local commutation as a condition on pairs of (collections of) measures on paths. In Section 4, we will use this condition to construct pairings, *i.e.* (collections of) measures on pairs of paths.

A configuration consists of a simply connected domain D with marked points: $c=(D,z_0,z_1,\dots,z_n,z_{n+1})$; the marked points are distinct and in some prescribed order on the boundary. The question is to classify pairs of SLE measures that satisfy local commutation (3.1), (3.2). The first measure is on paths growing at z_0 (hulls (K_s), trace γ); the second measure is on paths growing at z_{n+1} (hulls $(\hat K_t)$, trace $\hat\gamma$). We assume that, at least up to a positive stopping time, the first (resp. second) SLE is absolutely continuous w.r.t. the reference measure $\mathrm{SLE}_\kappa(D,z_0,z)$ (resp. $\mathrm{SLE}_{\hat\kappa}(D,z_{n+1},z)$); here z is another marked boundary point, used solely for normalization, and chains are considered up to time change.

We also require conformal invariance and the Markov property (for configurations with $n+2$ marked points) for each SLE. In the upper half-plane model, the driving process of each SLE differs from a Brownian motion by a drift term which is a function of the position of the marked points; we assume that this dependence is smooth.

As before, we denote $c_{s,t} = (D_{s,t} = D \setminus (K_s \cup \hat{K}_t), \gamma_s, z_1, \dots, z_n, \hat{\gamma}_t)$ when this is still a configuration (*i.e.* before swallowing of any marked point). Let σ be a stopping time for the first SLE such that K_σ is in a fixed compact set at positive distance of a boundary arc containing $z_1, \dots, z_{n+1}, z$. Consider the measures induced on $(K_u)_{u \leq s \wedge \sigma}$ by a) the SLE under consideration and b) the reference $\mathrm{SLE}_\kappa(D, z_0, z)$. The Radon-Nikodym derivative, denoted by M_s, is a measurable function of $(K_u)_{u \leq s \wedge \sigma}$. It is also a martingale under the reference measure. We define similarly $(\hat{M}_t)$, the density of the second SLE w.r.t. its reference measure $\mathrm{SLE}_{\hat{\kappa}}(D, z_{n+1}, z)$.

Then:

THEOREM 4 ([3]). – *Local commutation* (3.1)*,* (3.2) *is satisfied iff the following properties hold:*

1. $\hat{\kappa} \in \{\kappa, 16/\kappa\}$ *, and*
2. *There exist a conformally invariant function ψ on the configuration space and exponents ν_{ij} satisfying $\sum_{j=1}^{n+1} \nu_{0,j} = \alpha_\kappa$, $\sum_{i=0}^{n} \nu_{i,n+1} = \alpha_{\hat{\kappa}}$ such that if we define*

$$Z(c) = \psi(c) \prod_{0 \leq i < j \leq n+1} H_D(z_i, z_j)^{\nu_{ij}}$$

then, up to multiplicative constants and before some positive stopping times, the densities M_s, $\hat{M}_t$ can be expressed as:

$$M_s = H_{c_{s,0}}(\gamma_s, z)^{-\alpha_\kappa} Z(c_{s,0})$$
$$\hat{M}_t = H_{c_{0,t}}(\hat{\gamma}_t, z)^{-\alpha_{\hat{\kappa}}} Z(c_{0,t}).$$

In the theorem, notice that $M_s = H_{c_{s,0}}(\gamma_s, z)^{-\alpha_\kappa} Z(c_{s,0})$ is defined via a choice of local coordinates at $z, z_1, \dots, z_{n+1}$, but not at γ_t, where the evolution occurs (the choice of local coordinates is arbitrary but fixed under evolution, and contributes a multiplicative constant to the martingales).

Proof. – The delicate part is necessity, for which we refer to [3]. However, in this article we shall need only sufficiency, which is rather straightforward. A proof is included for the sake of self-containedness.

Assume 1,2. Let μ_c be the measure on the first SLE in the configuration c, and $\bar{\mu}_c$ the associated reference measure, viz. chordal $\mathrm{SLE}_\kappa(D, z_0, z)$. Consider $\tau, \hat{\tau}$ stopping times such that γ^τ (resp. $\hat{\gamma}^{\hat{\tau}}$) are in compact neighborhoods of z_0, z_{n+1} at positive distance of each other and all other marked points. We can assume that the two SLE's are defined at least up to $\tau, \hat{\tau}$.

We compute the density:

$$\begin{aligned}
\left(\frac{d\mu_{c_{0,\hat\tau}}}{d\mu_c}\right)(\gamma^\tau) &= \left(\frac{d\mu_{c_{0,\hat\tau}}}{d\bar\mu_{c_{0,\hat\tau}}}\right)\cdot\left(\frac{d\bar\mu_{c_{0,\hat\tau}}}{d\bar\mu_c}\right)\cdot\left(\frac{d\bar\mu_c}{d\mu_c}\right)(\gamma^\tau)\\
&= \left(\frac{H_{c_{\tau,\hat\tau}}(\gamma_\tau,z)^{-\alpha_\kappa}Z(c_{\tau,\hat\tau})}{H_{c_{0,\hat\tau}}(z_0,z)^{-\alpha_\kappa}Z(c_{0,\hat\tau})}\right)\\
&\quad\cdot\left(\left(\frac{H_{c_{\tau,\hat\tau}}(\gamma_\tau,z)H_{c_{0,0}}(z_0,z)}{H_{c_{\tau,0}}(\gamma_\tau,z)H_{c_{0,\hat\tau}}(z_0,z)}\right)^{\alpha_\kappa}\exp(-\lambda_\kappa m(D;K_\tau,\hat K_{\hat\tau}))\right)\\
&\quad\cdot\left(\frac{H_{c_{0,0}}(z_0,z)^{-\alpha_\kappa}Z(c_{0,0})}{H_{c_{\tau,0}}(\gamma_\tau,z)^{-\alpha_\kappa}Z(c_{\tau,0})}\right)\\
&= \left(\frac{Z(c_{\tau,\hat\tau})Z(c_{0,0})}{Z(c_{\tau,0})Z(c_{0,\hat\tau})}\right)\exp(-\lambda_\kappa m(D;K_\tau,\hat K_{\hat\tau})).
\end{aligned}$$

The outer densities are obtained by applying 2 in $c_{0,\hat\tau}$ and in $c_{0,0}$, while the inner density follows from Proposition 3. Notice that when $\hat\kappa\in\{\kappa,16/\kappa\}$, $\lambda_\kappa=\lambda_{\hat\kappa}$. The final expression is plainly symmetric, which yields (3.2):

$$\left(\frac{d\mu^\tau_{c_{0,\hat\tau}}}{d\mu^\tau_c}\right)(\gamma^\tau) = \left(\frac{d\hat\mu^{\hat\tau}_{c_{\tau,0}}}{d\hat\mu^{\hat\tau}_c}\right)(\hat\gamma^{\hat\tau}) = \left(\frac{Z(c_{\tau,\hat\tau})Z(c_{0,0})}{Z(c_{\tau,0})Z(c_{0,\hat\tau})}\right)\exp(-\lambda_\kappa m(D;K_\tau,\hat K_{\hat\tau})) \stackrel{\text{def}}{=} \ell_c(\gamma^\tau,\hat\gamma^{\hat\tau}),$$

and by integration (3.1). □

A good example of the situation is the following: $\hat\kappa=\kappa=6$, with four marked points (z_0,z_1,z_2,z_3), Z the probability that there is a percolation crossing from (z_0z_3) to (z_1z_2).

The definition of Z involves 1-jets of local coordinates at marked points; one could imagine more complicate dependences, say on k-jets at marked points; this is ruled out by the theorem. We refer to such functions, defined on the space of simply connected domains with marked boundary points and 1-jets of local coordinates at the marked points, as *partition functions*. (For other purposes, it is convenient to allow a dependence on the metric, see [4].)

The theorem gives a recipe to construct pairs of SLE satisfying local commutation: we simply have to find Z such that the associated processes $M,\hat M$ are (positive, local) martingales under the respective reference measures. Taking the Girsanov transform of the first reference measure by M, we obtain an SLE distribution that satisfies the Markov property w.r.t. configurations with $n+2$ marked points (up to a stopping time); one proceeds similarly for the second measure. This yields two collections of measures (indexed by configurations) satisfying the commutation identity (3.2).

An easy way to generate systems of commuting SLE's is to look for partition functions in the simple form $Z(c)=\prod_{i<j}H_D(z_i,z_j)^{\nu_{ij}}$. (This situation is studied in [3], Section 3.2.) To get a system of commuting SLE's, we simply need to find a choice of exponents ν_{ij} for which the processes $(M_s),(\hat M_t)$ defined from Z are (local) martingales under the respective reference measures, and thus can be used as densities to define a probability measure on (stopped) chains. Then the martingale transform of the first measure is an $\text{SLE}_\kappa(\underline\rho,\rho)$ in $(\mathbb{H},z_0,\dots,z_{n+1})$ with $\rho_i=-2\kappa\nu_{0,i}$, $i=1\dots n$, and $\rho=\rho_{n+1}=-2\kappa\nu_{0,n+1}$, as is easily seen from Girsanov's theorem (see Lemma 12). Similarly, the other SLE is an $\text{SLE}_{\hat\kappa}(\hat\rho,\underline{\hat\rho})$ with $\hat\rho=\hat\rho_0=-2\hat\kappa\nu_{0,n+1}$, and $\hat\rho_i=-2\hat\kappa\nu_{i,n+1}$. The following systems are found to solve the local commutation condition:

1. $\kappa = \hat\kappa$; $\rho = \hat\rho = \kappa - 6$; $\rho_i = \hat\rho_i = 0$ (two chordal SLE's aiming at each other, the reversibility setup)
2. $\kappa = \hat\kappa$; $\rho = \hat\rho = 2$; $\rho_i = \hat\rho_i$, $\rho + \sum_i \rho_i = \kappa - 6$ ($n-1$ arbitrary parameters)
3. $\kappa\hat\kappa = 16$, $\rho = -\kappa/2$, $\hat\rho = -\frac{\hat\kappa}{2}$, $\hat\rho_i = -(\hat\kappa/4)\rho_i = -(4/\kappa)\rho_i$, $\rho + \sum_i \rho_i = \kappa - 6$ ($n-1$ arbitrary parameters).

In these cases, we have the following expressions for the common partition function:

$$Z(c) = H(z_0, z_{n+1})^{\frac{6-\kappa}{2\kappa}}$$
$$Z(c) = H(z_0, z_{n+1})^{-\frac{1}{\kappa}} \prod_i H(z_0, z_i)^{-\frac{\rho_i}{2\kappa}} H(z_{n+1}, z_i)^{-\frac{\rho_i}{2\kappa}} \prod_{i<j} H(z_i, z_j)^{-\frac{\rho_i\rho_j}{4\kappa}}$$
$$Z(c) = H(z_0, z_{n+1})^{-\frac{1}{4}} \prod_i H(z_0, z_i)^{-\frac{\rho_i}{2\kappa}} H(z_{n+1}, z_i)^{\frac{\rho_i}{8}} \prod_{i<j} H(z_i, z_j)^{-\frac{\rho_i\rho_j}{4\kappa}}.$$

In the third case ($\kappa\hat\kappa = 16$), notice that $-\frac{\rho_i}{2\kappa} = \frac{\hat\rho_i}{8}$, $-\frac{\rho_i\rho_j}{4\kappa} = -\frac{\hat\rho_i\hat\rho_j}{4\hat\kappa}$.

Another explicit situation is when four points are marked, so that there is a single cross-ratio. Consider a configuration $c = (D, x, y, z_1, z_2)$, the marked points in some prescribed order. Let ν be a parameter and β a solution of the quadratic equation $\frac{\kappa}{2}\beta(\beta-1) + 2\beta = 2\nu$. Define a partition function

$$Z(c) = H_D(x, y)^{\frac{6-\kappa}{2\kappa}} H_D(z_1, z_2)^{\nu}\psi(u)$$

where u is the cross-ratio $u = \frac{(z_1-x)(z_2-y)}{(y-x)(z_2-z_1)}$ (in the upper half-plane) and

$$\psi(u) = (u(1-u))^{\beta}{}_2F_1\left(2\beta, 2\beta + \frac{8}{\kappa} - 1; 2\beta + \frac{4}{\kappa}; u\right)$$

where ${}_2F_1$ designates a solution of the hypergeometric equation with parameters $2\beta, \dots$ (this equation is invariant under $u \leftrightarrow 1-u$). If the solution is chosen so that it is positive on the configuration space, then a computation shows that Z satisfies the condition of the theorem and drives two locally commuting SLE's starting from x, y. This covers for instance the following situations: a chordal SLE_κ from x to y conditioned not to intersect the interval $[z_1, z_2]$, $4 < \kappa < 8$; a chordal $\mathrm{SLE}_{8/3}$ from x to y conditioned not to intersect a restriction measure (with exponent ν) from z_1 to z_2; and the marginal of a system of two SLE strands $x \leftrightarrow y$, $z_1 \leftrightarrow z_2$ ([2], Section 4.1; this corresponds to $\nu = \alpha_\kappa$).

The situation in other (non simply connected) topologies is quite involved, though an important part of the analysis carries through (see [3]). For n SLE's in a simply connected domain, the local commutation relation for the system of n SLE's reduces to $\frac{n(n-1)}{2}$ pairwise commutation conditions.

4. Coupling

Let $c = (D, z_0, z_1, \dots, z_n, z_{n+1})$ be a configuration, where D is a simply connected, bounded domain with $n+2$ distinct marked points on the boundary in some prescribed order. We consider a system of two SLE's satisfying local commutation, one originating at z_0, the other at z_{n+1}. These two SLE's have the SLE Markov property for domains with $n+2$ marked points. The first one is absolutely continuous (up to a disconnection event) w.r.t. $\mathrm{SLE}_\kappa(D,z_0,z_n)$; the Loewner chain is (K_s), the trace γ, the measure μ_c. The second one is absolutely continuous (up to a disconnection event) w.r.t. $\mathrm{SLE}_{\hat\kappa}(D,z_{n+1},z_n)$; the Loewner chain is $(\hat K_t)$, the trace $\hat\gamma$, the measure $\hat\mu_c$. We also denote $c_{s,t}=(D\setminus(K_s\cup\hat K_t),\gamma_s,z_1,\dots,z_n,\hat\gamma_t)$.

From Theorem 4, we know that $\hat\kappa\in\{\kappa,16/\kappa\}$ and there is a positive conformally invariant function ψ on configurations, weights ν_{ij}, such that for $\tau,\hat\tau$ a pair of stopping times (before disconnection events)

$$\begin{aligned}\ell_c(\gamma^\tau,\hat\gamma^{\hat\tau})&=\left(\frac{Z(c_{\tau,\hat\tau})Z(c_{0,0})}{Z(c_{\tau,0})Z(c_{0,\hat\tau})}\right)\exp(-\lambda m(D;K_\tau,\hat K_{\hat\tau}))\\&=\left(\frac{d\mu^\tau_{c_{0,\hat\tau}}}{d\mu^\tau_c}\right)(\gamma^\tau)=\left(\frac{d\hat\mu^{\hat\tau}_{c_{\tau,0}}}{d\hat\mu^{\hat\tau}_c}\right)(\hat\gamma^{\hat\tau})\end{aligned}\tag{4.6}$$

where $Z(c)=\psi(c)\prod_{i<j}H_D(z_i,z_j)^{\nu_{ij}}$.

Let $\hat\gamma^{\hat\tau}$ be a fixed stopped path. Let τ be a stopping time for γ such that a.s. γ^τ is at distance at least $\eta>0$ from $(\hat\gamma^{\hat\tau}\cup\partial)$, where ∂ is a connected boundary arc of D containing marked points other than z_0. Then we deduce immediately from (4.6) that:

$$\int\ell_c(\gamma^\tau,\hat\gamma^{\hat\tau})d\mu^\tau_c(\gamma^\tau)=\int d\mu^\tau_{c_{0,\hat\tau}}(\gamma^\tau)=1\quad\forall\hat\gamma^{\hat\tau}.\tag{4.7}$$

The symmetric statement holds for the same reason.

Define:

$$\ell^{s_2,t_2}_{s_1,t_1}=\ell_{c_{s_1,t_1}}(\gamma^{s_2},\hat\gamma^{t_2})=\left(\frac{Z(c_{s_2,t_2})Z(c_{s_1,t_1})}{Z(c_{s_2,t_1})Z(c_{s_1,t_2})}\right)\exp(-\lambda m(D\setminus(K_{s_1}\cup\hat K_{t_1});K_{s_2},\hat K_{t_2})).$$

We recall the (deterministic) tower property: for $0\le s_1\le s_2\le s_3$, $0\le t_1\le t_2\le t_3$ such that c_{s_3,t_3} is a configuration

$$\ell^{s_2,t_2}_{s_1,t_1}\ell^{s_3,t_2}_{s_2,t_1}=\ell^{s_3,t_2}_{s_1,t_1},\qquad\ell^{s_2,t_2}_{s_1,t_1}\ell^{s_2,t_3}_{s_1,t_2}=\ell^{s_2,t_3}_{s_1,t_1}.\tag{4.8}$$

The goal of this section is to use these properties to extend a local coupling of $\mu_c,\hat\mu_c$, that exists from local commutation, to a maximal coupling “up to disconnection”. The idea, introduced in [18] in the case of reversibility, is to consider a sequence of couplings absolutely continuous w.r.t. the independent coupling $\mu_c\otimes\hat\mu_c$ that converges to a maximal coupling. The construction of the coupling presented here somewhat differs from that in [18]. The existence of such a maximal coupling relies solely on (4.7), (4.8), and the SLE Markov property.

4.1. Local coupling

We briefly discuss here the interpretation of local commutation (4.6) in terms of couplings.

A coupling of $\mu_c,\hat\mu_c$ is a measure on pairs of paths $(\gamma,\hat\gamma)$ (or chains $(K,\hat K)$) such that the first marginal is μ_c, the second marginal is $\hat\mu_c$. The trivial (independent) coupling is $\mu_c\otimes\hat\mu_c$.

For simplicity, consider $U,\hat U$ neighborhoods of x,y respectively in D with disjoint closures. Assume that U is at positive distance of a boundary arc of D containing marked points other than x, and the symmetric condition holds for $\hat U$. Let $\tau,\hat\tau$ be the first exit of $U,\hat U$ by $\gamma,\hat\gamma$. Then we can consider the measure on pairs of stopped paths $(\gamma^\tau,\hat\gamma^{\hat\tau})$ given by:

$$\ell_c(\gamma^\tau,\hat\gamma^{\hat\tau})d\mu^\tau_c(\gamma^\tau)d\hat\mu^{\hat\tau}_c(\hat\gamma^{\hat\tau})$$

From (4.7), we know that this is a coupling of the stopped measures $\mu_c^\tau, \hat{\mu}_c^{\hat{\tau}}$. One can extend it to a coupling of μ_c, $\hat{\mu}_c$ as follows: after τ, γ is continued as an SLE in $c_{\tau,0}$ (that is, following $\mu_{c_{\tau,0}}$) independent of the rest conditionally on $c_{\tau,0}$; $\hat{\gamma}$ is continued in a similar way. This describes a coupling of μ_c, $\hat{\mu}_c$, using the (strong) Markov property. It is clear that this procedure describes the measure:

$$\ell_c(\gamma^\tau, \hat{\gamma}^{\hat{\tau}}) d\mu_c(\gamma) d\hat{\mu}_c(\hat{\gamma})$$

on pairs of paths $(\gamma, \hat{\gamma})$. This coupling is local in the sense that the interaction is restricted to the time set $[0,\tau] \times [0,\hat{\tau}]$. In the next subsection, we will extend the interaction to the (random) time set $\{(s,t) : K_s \cap \hat{K}_t = \varnothing\}$, to obtain "maximal couplings".

4.2. Maximal coupling

The goal here is to construct explicit couplings of $\mu_c, \hat{\mu}_c$ parameterized by a small parameter $\eta > 0$ such that any subsequential limit as $\eta \searrow 0$ is a maximal coupling. As in the local case, we start from the independent coupling $\mu_c \otimes \hat{\mu}_c$ and introduce a density that preserves the marginal distributions.

We assume $\eta \ll \mathrm{dist}(z_0, z_{n+1}) \leq \mathrm{diam}(D) < \infty$. (The domain D is embedded in the plane and distances are measured in the ambient plane.)

Define by induction the stopping times: $\tau_0 = 0$;

$$\tau_{i+1} = \inf\{t \geq 0 : \gamma_t \notin (K_{\tau_i})^\eta\}$$

where $A^\eta = \{x \in D : \mathrm{dist}(x, A) < \eta\}$. Then τ_i is a function of the path γ (taking a modification where traces exist and are continuous). The sequence τ_i is strictly increasing until it reaches ∞.

If $\tau_n < \infty$, then for all $i < j \leq n$, $\mathrm{dist}(\gamma_{\tau_i}, \gamma_{\tau_j}) \geq \eta$, since γ_{τ_j} is outside of $(K_{\tau_{j-1}})^\eta$. Thus $n\pi(\frac{\eta}{2})^2 \leq area(D + B(0,\eta))$. This gives a fixed $N = \lfloor 4area(D + B(0,\eta))/\pi \rfloor$ such that $\tau_N = \infty$.

Similarly, define $\hat{\tau} = 0$, and by induction:

$$\hat{\tau}_{j+1} = \inf\{t \geq 0 : \hat{\gamma}_t \notin (\hat{K}_{\hat{\tau}_j})^\eta\}.$$

We now introduce a dependence on the other path. Let ∂ be the smallest connected boundary arc of D containing all marked points except z_0; symmetrically, let $\hat{\partial}$ be the smallest connected boundary arc of D containing all marked points except z_{n+1}. These two arcs overlap in general.

We define a (random) set $G \subset \mathbb{N}^2$ of good pairs of indices as follows:

$$\{(i,j) \in G\} = \{\mathrm{dist}(K_{\tau_i}, \hat{K}_{\hat{\tau}_j}) > 3\eta\} \cap \{\mathrm{dist}(K_{\tau_i}, \partial) > 2\eta\} \cap \{\mathrm{dist}(\hat{K}_{\hat{\tau}_j}, \hat{\partial}) > 2\eta\}.$$

(Here, $\mathrm{dist}(K, \hat{K}) = \inf_{x \in K, \hat{x} \in \hat{K}}(\mathrm{dist}(x, \hat{x}))$.) This is a separation condition. If $i' \leq i$, $j' \leq j$, $(i,j) \in G$, then a fortiori $(i',j') \in G$. We take η small enough so that $(0,0) \in G$.

For conciseness, set

$$\ell_{ij}^{i'j'} = \ell_{\tau_i, \hat{\tau}_j}^{\tau_{i+1}, \hat{\tau}_{j+1}} = \ell_{c_{\tau_i, \hat{\tau}_j}}(\gamma^{\tau_{i'}}, \hat{\gamma}^{\hat{\tau}_{j'}})$$

for $i \leq i'$, $j \leq j'$. If $\tau_i = \tau_{i'}$, set $\ell_{ij}^{i'j'} = 1$ (this is the case if $i = i'$ or $\tau_i = \tau_{i'} = \infty$).

Let $(i,j) \in G$. Then K_{τ_i} is at distance at least 3η from $\hat{K}_{\hat{\tau}_j}$ and at least 2η from ∂. Thus $K_{\tau_{i+1}} \subset \overline{(K_{\tau_i})^\eta}$ is at distance at least 2η from $\hat{K}_{\hat{\tau}_j}$, at least η from $\hat{K}_{\hat{\tau}_{j+1}}$, and at least η

from ∂. Symmetric statements hold for $\hat{K}_{\hat{\tau}_j}$. Thus $\ell_{i,j}^{i+1,j+1}$ is well defined and by Lemma 14 uniformly bounded by some $C = C(D, \eta)$.

Define

$$I(j) = \inf\{i \in \mathbb{N} : (i,j) \notin G\}$$
$$J(i) = \inf\{j \in \mathbb{N} : (i,j) \notin G\}.$$

Plainly, I and J are non-increasing. If $\mathcal{F}$ (resp. $\hat{\mathcal{F}}$) is the filtration generated by γ (resp. $\hat{\gamma}$), τ_i is an $\mathcal{F}$-stopping time. Also, the event $\{(i,j) \in G\}$ is $(\mathcal{F}_{\tau_i} \vee \hat{\mathcal{F}}_{\hat{\tau}_j})$-measurable. It follows that $\tau_{I(j)}$ is a stopping time in the enlarged filtration $(\mathcal{F}_t \vee \hat{\mathcal{F}}_{\hat{\tau}_j})_t$. Symmetric statements hold for $\hat{\tau}_j$, $\hat{\tau}_{J(i)}$.

Consider the measure Θ_c^η on pairs $(\gamma, \hat{\gamma})$:

$$d\Theta_c^\eta(\gamma, \hat{\gamma}) = \left[\prod_{(i,j)\in G} \ell_{i,j}^{i+1,j+1}\right] d\mu_c(\gamma) d\hat{\mu}_{\hat{c}}(\hat{\gamma}).$$

The density is a function on pairs of paths $(\gamma, \hat{\gamma})$. From the tower property (3.5), one can rewrite this density as:

$$L = \prod_{(i,j)\in G} \ell_{i,j}^{i+1,j+1} = \prod_{0\leq i\leq N} \ell_{i,0}^{i+1,J(i)} = \prod_{0\leq j\leq N} \ell_{0,j}^{I(j),j+1}$$

showing in particular that L is bounded by C^N. The last two expressions of L will behave well in combination with the Markov property of γ, $\hat{\gamma}$ respectively.

Lemma 5. – *The measure $\Theta_c^\eta = L(\mu_c \otimes \hat{\mu}_c)$ is a coupling of $\mu_c, \hat{\mu}_c$.*

Proof. – The statement can be rephrased as

$$\forall \hat{\gamma}, \mathbb{E}(L) = 1 \qquad\qquad \forall \gamma, \hat{\mathbb{E}}(L) = 1$$

where $\mathbb{E}$, $\hat{\mathbb{E}}$ refer to integration w.r.t. $d\mu_c(\gamma)$, $d\hat{\mu}_{\hat{c}}(\hat{\gamma})$ respectively (in other terms $\mathbb{E}(.) = \mathbb{E} \otimes \hat{\mathbb{E}}(.|\hat{\mathcal{F}}_\infty)$). The situation is completely symmetric, so we shall consider only the distribution of the second marginal; that is, we have to check that given any $\hat{\gamma}$, $\mathbb{E}(L) = 1$.

For this, the relevant expression of the density is: $L = \prod_{0\leq i\leq N} \ell_{i,0}^{i+1,J(i)}$. Notice that $\hat{\gamma}^{\hat{\tau}_{J(i)}}$ is $(\mathcal{F}_{\tau_i} \vee \hat{\mathcal{F}}_\infty)$-measurable. This implies that a term $\ell_{i,0}^{i+1,J(i)}$ is $(\mathcal{F}_{\tau_n} \vee \hat{\mathcal{F}}_\infty)$-measurable for any $i < n$. Thus for fixed n:

$$\mathbb{E}\left(\prod_{i=0}^{n} \ell_{i,0}^{i+1,J(i)} | \mathcal{F}_{\tau_n}\right) = \left[\prod_{i=0}^{n-1} \ell_{i,0}^{i+1,J(i)}\right] \mathbb{E}(\ell_{n,0}^{n+1,J(n)} | \mathcal{F}_{\tau_n})$$

and $\mathbb{E}(\ell_{n,0}^{n+1,J(n)} | \mathcal{F}_{\tau_n}) = 1$ by (4.7); the point is that given $(\mathcal{F}_{\tau_n} \vee \hat{\mathcal{F}}_\infty)$, γ^{τ_n} and $\hat{\gamma}^{\hat{\tau}_{J(n)}}$ are fixed. This is saying that

$$M_n = \prod_{i=0}^{n-1} \ell_{i,0}^{i+1,J(i)}$$

is a (discrete time, bounded) martingale in the filtration $(\mathcal{F}_{\tau_n} \vee \hat{\mathcal{F}}_\infty)_{n\geq 0}$. In particular, $\mathbb{E}(L) = \mathbb{E}(M_N) = M_0 = 1$. □

We can actually get a more precise statement. Let τ be an arbitrary $\mathcal{F}$-stopping time; let $n=\inf\{i\in\mathbb{N}:\tau_i\geq\tau\}$, a random integer. We assume that $\mathrm{dist}(\gamma^\tau,\partial)\geq 3\eta$ a.s., so that $(n,0)\in G$. Then τ_n is a stopping time approximating τ (as $\eta\searrow 0$). Consider the joint distribution of $(\gamma^{\tau_n},\hat\gamma^{\hat\tau_{J(n)}})$ under Θ^η. Then γ^{τ_n} has distribution $\mu_c^{\tau_n}$. Moreover:

$$\begin{aligned}
\mathbb{E}\otimes\hat{\mathbb{E}}(L|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}) &= \Bigg[\prod_{\substack{i<n\\ j<J(n)}}\ell_{i,j}^{i+1,j+1}\Bigg]\\
&\quad\cdot\ \mathbb{E}\otimes\hat{\mathbb{E}}\Bigg(\prod_{\substack{i\geq n\\ j<J(i)}}\ell_{i,j}^{i+1,j+1}\prod_{\substack{j\geq J(n)\\ i<I(j)}}\ell_{i,j}^{i+1,j+1}|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\Bigg)\\
&=\left[\ell_{0,0}^{n,J(n)}\right]\mathbb{E}\otimes\hat{\mathbb{E}}\left(\prod_{i\geq n}\ell_{i,0}^{i+1,J(i)}\prod_{j\geq J(n)}\ell_{0,j}^{I(j),j+1}|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right).
\end{aligned}$$

The term $\prod_{i\geq n}\ell_{i,0}^{i+1,J(i)}$ involves γ after τ_n and $\hat\gamma$ before $\hat\tau_{J(n)}$ (if $i\geq n$, $J(i)\leq J(n)$); the other term, $\prod_{j\geq J(n)}\ell_{0,j}^{I(j),j+1}$, involves $\hat\gamma$ after $\hat\tau_{J(n)}$ and γ before τ_n. Hence these two terms are independent conditionally on $(\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}})$ (under the independent measure $\mu_c\otimes\hat\mu_c$). Moreover

$$\mathbb{E}(\prod_{i\geq n}\ell_{i,0}^{i+1,J(i)}|\mathcal{F}_{\tau_n})=\mathbb{E}(\frac{M_N}{M_n}|\mathcal{F}_{\tau_n})=1$$

where $(M_.)$ is the discrete-time, bounded martingale considered in the previous lemma, since n is a stopping time for its discrete filtration. Similarly, $\hat{\mathbb{E}}(\prod_{j\geq J(n)}\ell_{0,j}^{I(j),j+1}|\hat{\mathcal{F}}_{\hat\tau_{J(n)}})=1$, and consequently:

$$\begin{aligned}
\mathbb{E}\otimes\hat{\mathbb{E}}\left(L|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right) &= (\ell_{0,0}^{n,J(n)})\mathbb{E}\otimes\hat{\mathbb{E}}\left(\prod_{i\geq n}\ell_{i,0}^{i+1,J(i)}|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right)\mathbb{E}\\
&\quad\otimes\hat{\mathbb{E}}\left(\prod_{j\geq J(n)}\ell_{0,j}^{I(j),j+1}|\mathcal{F}_{\tau_n}\vee\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right)\\
&=(\ell_{0,0}^{n,J(n)})\hat{\mathbb{E}}\left[\mathbb{E}\left[\prod_{i\geq n}\ell_{i,0}^{i+1,J(i)}|\mathcal{F}_{\tau_n}\right]|\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right]\\
&\quad\cdot\mathbb{E}\left[\hat{\mathbb{E}}\left[\prod_{j\geq J(n)}\ell_{0,j}^{I(j),j+1}|\hat{\mathcal{F}}_{\hat\tau_{J(n)}}\right]|\mathcal{F}_{\tau_n}\right]\\
&=\ell_{0,0}^{n,J(n)}=\frac{d\hat\mu_{c_{\tau_n,0}}^{\hat\tau_{J(n)}}}{d\hat\mu_c^{\hat\tau_{J(n)}}}(\hat\gamma^{\hat\tau_{J(n)}}).
\end{aligned}$$

This proves that under Θ^η, the conditional distribution of $\hat\gamma^{\hat\tau_{J(n)}}$ given γ^{τ_n} is $\hat\mu_{c_{\tau_n,0}}^{\hat\tau_{J(n)}}$, where $\hat\tau_{J(n)}$ is a stopping time conditionally on γ^{τ_n}.

To phrase the following theorem, it is convenient to introduce:

$$\sigma = \sup\{t \geq 0 : K_t \cap \partial = \varnothing\}$$
$$\hat{\sigma} = \sup\{t \geq 0 : \hat{K}_t \cap \hat{\partial} = \varnothing\}.$$

The measures μ_c, $\hat{\mu}_c$ are defined on paths $(\gamma, \hat{\gamma})$ up to $\sigma, \hat{\sigma}$. Let Θ be a coupling of μ_c, $\hat{\mu}_c$. If τ is an $\mathcal{F}$-stopping time, $\tau \leq \sigma$, define the $(\mathcal{F}_\tau \vee \hat{\mathcal{F}}_t)_t$-stopping time $\hat{\tau}$:

$$\hat{\tau} = \sup\{t \geq 0 : \hat{K}_t \cap (K_\tau \cup \hat{\partial}) = \varnothing\}.$$

We say that the coupling Θ satisfies the property $P(\tau)$ if the conditional distribution of $\hat{\gamma}^{\hat{\tau}}$ given γ^τ (under Θ) is $\hat{\mu}^{\hat{\tau}}_{c_{\tau,0}}$. The symmetric property (starting from an $\hat{\mathcal{F}}$-stopping time $\hat{\tau}$) is denoted $\hat{P}(\hat{\tau})$. A coupling Θ is said to be *maximal* if $P(\tau)$ (resp. $\hat{P}(\hat{\tau})$) is satisfied for any $\mathcal{F}$- stopping time $\tau \leq \sigma$ (resp. for any $\hat{\mathcal{F}}$- stopping time $\hat{\tau} \leq \hat{\sigma}$).

THEOREM 6. – *Let $\mu_c, \hat{\mu}_c$ be* SLE *measures in a configuration $(D, z_0, \ldots, z_{n+1})$ satisfying local commutation. Then there exists a maximal coupling Θ of μ_c, $\hat{\mu}_c$.*

Proof. – The two marginal distributions of Θ^η are the fixed (*i.e.* independent of $\eta > 0$) Radon measures μ_c^σ and $\mu_c^{\hat{\sigma}}$ (in the topology of Carathéodory convergence of Loewner chains). Thus the family $(\Theta^\eta)_{\eta>0}$ is tight, and Prokhorov's theorem ensures existence of subsequential limits. Let $(\eta_k)_k$ be a sequence $\eta_k \searrow 0$ along which Θ^{η_k} has a weak limit Θ. Then Θ is a coupling of μ_c and $\hat{\mu}_c$. We can consider a probability space with sample $((\gamma^1, \hat{\gamma}^1), \ldots, (\gamma^k, \hat{\gamma}^k), \ldots)$ such that the distribution of $(\gamma^k, \hat{\gamma}^k)$ is Θ^{η_k} and $(\gamma^k, \hat{\gamma}^k) \to (\gamma, \hat{\gamma})$ a.s., where the distribution of $(\gamma, \hat{\gamma})$ is Θ.

Let τ be an $\mathcal{F}$-stopping time; we approximate τ in a convenient way. Firstly, τ can be approximated by τ' taking values in some discrete countable sequence $(t_i)_{i\geq 0}$ (e.g., dyadic times). Hence there are Borel sets B_i such that $\mathbf{1}_{B_i}$ is a Borel function of K^{t_i} and $\tau' = \inf\{t_i : K_{t_i} \in B_i\}$. Replace the Borel set B_i by a larger open set U_i such that the measure of $U_i \setminus B_i$ is very small. Then $\tau'' = \inf\{t_i : K_{t_i} \in U_i\}$ is a stopping time equal to τ' with probability arbitrarily close to 1. Finally, let $\tau''' = \tau'' \wedge \sup\{t : \mathrm{dist}(\gamma_t, \partial) \geq \varepsilon'\}$ for some fixed $\varepsilon' > 0$. Let us assume for now that τ is of type τ'''. This gives a common stopping rule for all the chains $K^k_.$: stop the first time that $K^k_{t_i}$ is in U_i or at distance ε of ∂. We denote τ^k this stopping time for the chain $K^k_.$. In particular, $\tau^k \to \tau$ a.s. (using that the U_i's are open).

For $\eta > 0$, τ^k_n is an approximation of τ^k as above: $\tau^k_n = \inf\{\tau_i : \tau_i \geq \tau^k\}$; then

$$K^k_{\tau^k_{n-1}} \subset K^k_{\tau^k} \subset K^k_{\tau^k_n} \subset (K^k_{\tau^k_{n-1}})^{\eta_k}.$$

It is easy to see that $\tau^k_n \to \tau$, $\tau^k_{n-1} \to \tau$, $K^k_{\tau^k_n} \to K_\tau$ and $\gamma^k_{\tau^k_n} \to \gamma_\tau$ (since $\gamma_\tau = \cap_{s>0}\overline{K_{\tau+s} \setminus K_\tau}$) as $k \to \infty$. We have seen that the conditional distribution of $\hat{\gamma}_k^{\hat{\tau}_{J(n)}}$ is $\hat{\mu}^{\hat{\tau}_{J(n)}}_{c_{\tau_n,0}}$. Notice that $\hat{\tau}_{J(n)}$ occurs after first entrance in $(K^k_{\tau^k_n})^{3\eta_k}$ and before entrance in $(K^k_{\tau^k_n})^{\eta_k}$.

For fixed $\varepsilon > 0$, $K^k_{\tau^k_n} \subset (K_\tau)^\varepsilon$ for k large enough. The configuration $\hat{c}^k_{\tau^k_n,0}$ converges in the Carathéodory topology to $c_{\tau,0}$ (with also convergence of $\gamma^k_{\tau^k_n}$ to γ_τ); this implies weak convergence of the conditional distribution of $\hat{\gamma}_k$ stopped when entering $(K_\tau)^\varepsilon$ to the corresponding stopped SLE in $c_{\tau,0}$. This gives the correct conditional distribution of $\hat{\gamma}$ stopped when entering $(K_\tau)^\varepsilon$, conditional on γ^τ. One concludes by taking $\varepsilon \searrow 0$.

This proves the result for a dense set of stopping times of type τ''' as above (this will be enough to draw geometric consequences). A general stopping time $\tau \le \sigma$ is the limit of a sequence of stopping times τ'''_m; for each m, the conditional distribution of $\hat\gamma$ stopped upon entering $(K_{\tau'''_m})^\varepsilon$ is correct. One concludes by taking $m \to \infty$ and then $\varepsilon \searrow 0$. □

One may want to consider couplings that extend after collisions of the two SLE's; this requires additional arguments. Some examples are treated in [4].

There are some obvious extensions of this result. One involves radial SLE's (not necessarily aiming at the same bulk point). Another involves systems of n (pairwise) commuting SLE's. Let us discuss this case briefly.

Consider a configuration $c = (D, z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m})$, with n SLE's starting at $z_1, \dots, z_n$, driven by the same partition function Z. One can reason as above (sampling the SLE's at discrete times $\tau^1_{i_1}, \dots, \tau^n_{i_n}$). In a maximal coupling, one can stop the first SLE at a stopping time τ^1, the second at τ^2 (first time it ceases to be defined or meets $K^1_{\tau^1}$), …, the n-th at τ^n (first time it ceases to be defined or meets $\cup_{i=1}^{n-1} K^i_{\tau^i}$) and get the appropriate joint distribution. This works for any permutation of indices.

Let us describe the local coupling in this case: let $U_1, \dots, U_n$ be disjoint neighborhoods of $z_1, \dots, z_n$, $\mu^1_\cdot, \dots, \mu^n_\cdot$ the commuting SLE measures, Z their common partition function. Let K_i be the hull of the stopped i-th SLE; $c_{\varepsilon_1 \dots \varepsilon_n}$, $\varepsilon_i \in \{0,1\}$, is the configuration where the i-th SLE has grown (until stopped) if $\varepsilon_i = 1$. Consider the density

$$L = \frac{d\mu^n_{c_{1\dots 10}}}{d\mu^n_{c_{0\dots 0}}} \cdot \frac{d\mu^{n-1}_{c_{1\dots 100}}}{d\mu^{n-1}_{c_{0\dots 0}}} \cdots \frac{d\mu^2_{c_{10\dots 0}}}{d\mu^2_{c_{0\dots 0}}} = \frac{Z(c_{1\dots 1}) Z(c_{0\dots 0})^{n-1}}{Z(c_{10\dots 0}) \dots Z(c_{0\dots 01})} \exp\left(-\lambda \sum_{j=2}^n m(D, \cup_{i=1}^{j-1} K_i, K_j)\right).$$

Then it is clear from the first expression that the first marginal of $L(\mu^1_c \otimes \cdots \otimes \mu^n_c)$ is μ^1_c (integrating out K_n, then K_{n-1}, …); the second expression shows that the construction is symmetric (for a discussion of the loop measure contribution, see Section 3.4 of [2]).

5. Geometric consequences

We have proved (Theorem 6) existence of maximal couplings under a local commutation assumption. On the other hand, the systems of SLE's satisfying this assumption are classified; we also checked directly the condition in the few cases we will need (combining Theorem 4 and Lemma 12). So we can now apply the existence of maximal couplings to appropriate systems of commuting SLE's to extract information on the geometry of SLE curves.

5.1. Reversibility

Reversibility for $\kappa \in (0, 4]$ is proved in [18]. We review the result for the reader's convenience.

THEOREM 7. – *If* $\kappa \le 4$, SLE *is reversible; any maximal coupling* Θ *of chordal* SLE_κ *in* (D, x, y) *with* SLE_κ *in* (D, y, x) *is the coupling of* SLE *with its reversed trace.*

Proof. – Let (D, x, y) be a configuration, μ_c the chordal SLE measure from x to y, $\hat{\mu}_c$ the chordal SLE measure from y to x. They satisfy local commutation, hence there exists a maximal coupling Θ.

Take a countable dense sequence of $\mathcal{F}$-stopping times (τ^m) (e.g., capacity of the hull reaches a rational number); denote simply by τ an element in this sequence. Then in the maximal coupling Θ, the conditional distribution of $\hat{\gamma}^{\hat{\tau}}$ is SLE in $(D \setminus K_\tau, y, \gamma_\tau)$ stopped upon hitting K_τ. For $\kappa \leq 4$, the SLE trace intersects the boundary only at its endpoints. Hence $\hat{\gamma}_{\hat{\tau}} = \gamma_\tau$. This proves that under Θ, the intersection $\gamma \cap \hat{\gamma}$ is a.s. dense in γ. Since both $\gamma, \hat{\gamma}$ are a.s. closed, $\gamma \subset \hat{\gamma}$; symmetrically, $\hat{\gamma} \subset \gamma$. Hence the occupied sets of $\gamma, \hat{\gamma}$ are equal. Again, as the paths are simple, the occupied set determines the parameterized trace. Hence in any maximal coupling Θ, $\hat{\gamma} = \gamma^r$ (the reverse trace) a.s.; this determines the coupling uniquely. □

Besides local commutation, the argument uses only qualitative properties of the paths. So we can phrase at no additional cost:

Corollary 8. – *Let $\kappa \leq 4$, μ_c, $\hat{\mu}_c$ a system of commuting SLE in the configuration $c = (D, z_0, z_1, \ldots, z_n, z_{n+1})$. Assume that μ_c, $\hat{\mu}_c$ are supported on simple paths that meet the boundary of D only at z_0, z_{n+1}. Then γ^r and $\hat{\gamma}$ are identical in distribution.*

A simple example of the situation is as follows: let $(z_1, \ldots, z_4)$ be four marked points on the arc $(z_0 z_5)$. Then we can consider chordal SLE from z_1 to z_5 weighted by any, say, bounded above and below function of the cross-ratio of $(z_1, \ldots, z_4)$ in $D \setminus \gamma$. This plainly preserves both local commutation and reversibility.

Another setup where the corollary applies is the following: $c = (D, z_0, z_1, \ldots, z_{n+1})$ with points in counterclockwise order. Let $\rho_1, \ldots, \rho_n$ be such that $\rho_1 + \cdots + \rho_i \geq 0$ for $1 \leq i < n$ and $\rho_1 + \cdots + \rho_n = 0$. Then the traces of $\mathrm{SLE}_4(\underline{\rho}, -2)$ starting from z_0 and $\mathrm{SLE}_4(-2, -\underline{\rho})$ starting from z_{n+1} are the reverse of each other in distribution. This describes the scaling limit of the zero level line of a discrete free field ([13]) with piecewise constant boundary conditions (with jump at z_i proportional to ρ_i). A version with marked points on both sides of z_0 also holds.

One also obtains reversibility identities for the pairs of commuting SLE's (aiming at each other) with four marked points described at the end of Section 3.2. By degenerating two points into one, this describes the reversal of $\mathrm{SLE}_\kappa(\rho)$, $\kappa \leq 4$, $\rho \geq \frac{\kappa}{2} - 2$. For instance, if $\kappa = 8/3$, one can represent an $\mathrm{SLE}_{8/3}(\rho)$ in $(\mathbb{H}, 0, 1, \infty)$ as the limit of a chordal $\mathrm{SLE}_{8/3}$ in $(\mathbb{H}, 0, \infty)$ conditioned not to intersect a restriction measure with exponent $\nu = \nu(\rho)$ from 1 to $z \gg 1$ ([15]; reversibility in this case follows from [6]). For general κ, it is unclear whether there is a simple probabilistic interpretation, but one still gets an exact (if unwieldy in general) description of the reversal.

Corollary 9. – *Let $\kappa \leq 4$, $\rho \geq \frac{\kappa}{2} - 2$, (D, x, y) a configuration. Then $\mathrm{SLE}_\kappa(\rho)$ in (D, x, y, x^+) and in (D, y, x, y^-) have the same occupied set in distribution, where x, x^+, y^-, y are in this order on the boundary.*

Proof. – We sketch the argument. The result follows from reversibility in the regular situation with four marked points described at the end of Section 3.2. Indeed, if x, z_1, z_2, y are in this order on the boundary, one has a pair of commuting SLE's starting at x, y with common partition function: $Z(c) = H_D(x,y)^{\frac{6-\kappa}{2\kappa}} H_D(z_1,z_2)^\nu \psi(u)$ where u is the cross-ratio $u = \frac{(z_1-x)(z_2-y)}{(y-x)(z_2-z_1)}$ (in the upper half-plane), $\frac{\kappa}{2}\beta(\beta-1)+2\beta = 2\nu$ and

$$\psi(u) = (u(1-u))^\beta {}_2F_1(2\beta, 2\beta + \frac{8}{\kappa} - 1; 2\beta + \frac{4}{\kappa}; u).$$

When $\kappa \le 4$, $\rho = \kappa\beta \ge \frac{\kappa}{2} - 2$, the processes do not hit $[z_1, z_2]$, by comparison arguments. Thus the local commutation extends to a maximal coupling, and in this coupling the occupied sets coincide.

The first SLE is the martingale transform of chordal SLE_κ in (D, x, y) by the martingale (in upper half-plane coordinates):

$$t \longmapsto \left(\frac{g_t'(z_1)g_t'(z_2)}{(g_t(z_1)-g_t(z_2))^2}\right)^\nu \psi(u_t)$$

where u_t is the cross-ratio at time t. Take $z_2 = y - \varepsilon$. Then the leading term of the expansion of the martingale as $\varepsilon \searrow 0$ is:

$$t \longmapsto \left(\frac{g_t'(z_1)g_t'(y)}{(g_t(z_1)-g_t(y))^2}\right)^\nu \left(\frac{(g_t(z_1)-X_t)g_t'(y)}{(g_t(y)-X_t)(g_t(y)-g_t(z_1))}\right)^\beta$$

so that this limiting process is identified from Lemma 12 as $\mathrm{SLE}_\kappa(\rho)$ in (D, x, y, z_1), $\rho = \kappa\beta$. A symmetric result holds for the other SLE. □

The same arguments can be used to establish reversibility of systems of multiple SLE's considered in [2] (this also follows from the symmetry of the density of the system w.r.t. independent chordal SLE's when the pairing of endpoints is fixed).

5.2. Duality

The question of SLE duality is to describe boundaries of SLE_κ, $\kappa > 4$, in terms of $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$.

There are various parametric situations we can consider. Let us start with the simplest setting: a configuration $c = (D, x, z_1, y, z_2)$ has four marked points x, y, z_1, z_2 on the boundary. We consider two SLE's (inducing the measures μ_c, $\hat\mu_c$, with traces γ, $\hat\gamma$), see Table 1 ($[\kappa]$ represents an SLE_κ "seed", the other entries are the ρ parameters).

TABLE 1.

x	z_1	y	z_2
$[\kappa]$	ρ_1	$-\frac{\kappa}{2}$	ρ_2
$-\frac{\hat\kappa}{2}$	$\hat\rho_1$	$[\hat\kappa]$	$\hat\rho_2$

The additional conditions for local commutation are $\rho_1 + \rho_2 = \frac{3}{2}(\kappa - 4)$, $\hat\rho_i = -\frac{4}{\kappa}\rho_i$, consequently $\hat\rho_1 + \hat\rho_2 = \frac{3}{2}(\hat\kappa - 4)$. This leaves one free parameter, say $\rho_1 = \rho$. We need to put conditions on ρ so that paths have a correct geometry. Take $\rho \in [\frac{\kappa-4}{2}, \kappa - 4]$, a nonempty interval when $\kappa \ge 4$. Consequently, $\rho_2 \in [\frac{\kappa-4}{2}, \kappa - 4]$, $\hat\rho_1, \hat\rho_2 \in [\frac{\hat\kappa-4}{2}, \hat\kappa - 4]$. Then the first

SLE will first intersect $[z_1,z_2]$ at y (see Lemma 15). To restrict even more the situation, take $\hat\rho_1 = \hat\kappa - 4$ (or symmetrically $\hat\rho_2 = \hat\kappa - 4$). Then the second SLE cannot hit (z_2,x), nor (z_1,z_2) except at y; and it hits (x,z_1) since $\hat\rho_1 = \hat\kappa - 4 < \frac{\hat\kappa}{2} - 2$.

PROPOSITION 10. – *In a maximal coupling Θ of μ_c, $\hat\mu_c$, the range of $\hat\gamma$ is contained in that of γ. If $\rho_1 = \kappa - 4$, $\hat\gamma$ is the right boundary of K; if $\rho_2 = \kappa - 4$, $\hat\gamma$ is the left boundary of K.*

Proof. – As before, take $\hat\tau$ a stopping time for the second SLE. The first SLE in $c_{0,\hat\tau}$ is defined until it exits at $\hat\gamma_{\hat\tau}$ at time τ. More precisely, it is defined up to a time where it accumulates at $\hat\gamma_{\hat\tau}$ and at no other point of the boundary arc $[z_1,z_2]$ of $c_{0,\hat\tau}$; this follows from maximality of the coupling and Lemma 15. But γ is continuous away from $[z_1,z_2]$ in c_0; so if $\hat\tau$ is positive, γ stopped when exiting $c_{0,\hat\tau}$ has a limit, which is $\hat\gamma_{\hat\tau}$. Hence $\hat\gamma_{\hat\tau}$ is on γ. Taking countably many stopping times, this shows that $\hat\gamma$ is included in (the range of) γ. Moreover, if $\sigma(t) = \inf\{u \geq 0 : \gamma_u \in \hat\gamma_{[0,t]}\}$, then σ is a.s. strictly decreasing (if $\hat\tau$ stopping time and $t < \hat\tau$, then $\sigma(t) > \sigma(\hat\tau)$, and stopping times are dense), and a.s. $\gamma_{\sigma(t)} = \hat\gamma_t$ for all t (again by density of stopping times).

Set $\rho_1 = \kappa - 4$. Then the range of γ is partitioned into points on the range of $\hat\gamma$, points to the left of $\hat\gamma$, and points to the right of $\hat\gamma$; here left and right are from the perspective of a particle moving along γ, starting from x. We want to prove that actually no point in the range of γ is (strictly) to the right of $\hat\gamma$. Take a stopping time τ. Then $\hat\gamma$ first hits $K_\tau \cup [x,z_1]$ on the arc $[\gamma_\tau, z_1]$ at time $\hat\tau$, again by maximality of the coupling and exit properties of the second SLE. If $\hat\gamma_t$ is in K_τ for all $t > \hat\tau$, then γ_τ is on $\hat\gamma$ or to its left. If there exists $t > \hat\tau$ such that $\hat\gamma_t \notin K_\tau$, then $\sigma(\hat\tau) \leq \tau < \sigma(t)$, which contradicts the monotonicity of σ.

Hence a generic point γ_τ lies on $\hat\gamma$ or to its left. This implies that $\hat\gamma$ is contained in the right boundary of the range of γ. Indeed, a point z on γ is on the right boundary of γ if there is a continuous curve δ with endpoints z_1, z that lies in $D \setminus \gamma$ except at its endpoints. For any t, since $\hat\gamma$ is simple, one can find such a crosscut δ from z_1 to $\hat\gamma_t$ in $D \setminus \hat\gamma$. This crosscut lies to the right of $\hat\gamma$ (except at its endpoint), where there is no other point of γ; it is thus also a crosscut in $D \setminus \gamma$. Hence the right boundary of γ contains $\hat\gamma$.

Since $\hat\gamma$ starts at y (where γ ends) and ends on (x,z_1), this shows that $\hat\gamma$ is the full right boundary of γ. Indeed, $\hat\gamma$ disconnects z_1 from points (strictly) to the left of $\hat\gamma$. As no point on γ lies to the right of $\hat\gamma$, the endpoint of $\hat\gamma$ on (x,z_1) is also the rightmost point on (x,z_1) visited by γ. □

REMARK 11. – *The situation where ρ varies in $[\frac{\kappa-4}{2}, \kappa-4]$ is of some independent interest and seems related to pivotal points questions.*

We consider now versions where the non simple SLE is actually chordal SLE$_\kappa$, at the expense of some complication for the dual simple SLE$_{\hat\kappa}$.

Proof of Theorem 1. – Assume that $\kappa \in (4,8)$. Consider chordal SLE$_\kappa$, say in $(\mathbb{H}, 0, \infty)$. The point 1 is swallowed at time τ_1; $D = \gamma_{\tau_1}$ is on $(1,\infty)$ with distribution given by:

$$\mathbb{P}(D \in (1,z)) = F(z) = c\int_1^z u^{-\frac{4}{\kappa}}(u-1)^{\frac{8}{\kappa}-2}du$$

where $c = B(1-4/\kappa, 8/\kappa - 1)^{-1}$. In other words, D^{-1} has a Beta$(1-4/\kappa, 8/\kappa-1)$ distribution. The function F is such that $t \mapsto F((g_t(z)-W_t)/(g_t(1)-W_t))$ is a martingale. Let us disintegrate the SLE measure w.r.t. D (see [1] for related questions). It is easy to see that up to τ_1, the SLE conditional on $D \in dz$ is the Girsanov transform of chordal SLE by:

$$t \mapsto \partial_z F\left(\frac{g_t(z)-W_t}{g_t(1)-W_t}\right) = c\frac{g_t'(z)}{g_t(1)-W_t}(g_t(z)-W_t)^{-\frac{4}{\kappa}}(g_t(z)-g_t(1))^{\frac{8}{\kappa}-2}(g_t(1)-W_t)^{2-\frac{4}{\kappa}}$$

and this is readily identified with $\mathrm{SLE}_\kappa(\kappa-4,-4)$ in $(\mathbb{H},0,\infty,1,z)$ (Lemma 12). To get a regular situation, we split the point z into two points y and z_2, while setting $x=0$, $z_1=1$, $z_3=\infty$. Consider the system of commuting SLE's given by Table 2.

TABLE 2. $4<\kappa<8$

x	z_1	y	z_2	z_3
$[\kappa]$	$\kappa-4$	$-\frac{\kappa}{2}$	$\frac{\kappa}{2}-4$	2
$-\frac{\hat\kappa}{2}$	$\hat\kappa-4$	$[\hat\kappa]$	$\hat\kappa-2$	$-\frac{\hat\kappa}{2}$

The first SLE hits $[z_1,z_3]$ at y, while the second SLE will not hit $[z_2,z_3]$ and exits $[x,z_1]$ somewhere in (x,z_1) (Lemma 15). Arguing as in Proposition 10, this shows that $\hat\gamma$ is the right boundary of K. Finally, one takes $z_2 \searrow y$, so that the first SLE_κ becomes chordal SLE_κ conditional on $D=y$. This yields Theorem 1. □

When $\kappa \geq 8$, the trace is a.s. space filling, and we have to proceed differently to isolate a boundary arc.

Proof of Theorem 2. – Consider now the case of a chordal SLE_κ in $(\mathbb{H},0,\infty)$, $\kappa\geq 8$ (thus $\hat\kappa\leq 2$). Then $\gamma_{\tau_1}=1$ a.s. There is a leftmost point G on $(\infty,0)$ visited by the trace before τ_1. We are interested in the boundary of K_{τ_G}, a simple curve from G to some point in $(0,1)$. Then the distribution of G is given by:

$$\mathbb{P}(G\in(z,0)) = c\int_z^0 (-u)^{-4/\kappa}(1-u)^{\frac{8}{\kappa}-2}du$$

where $c=B(1-4/\kappa,1-4/\kappa)$. In other words, G is such that $G/(G-1)$ has a $\mathrm{Beta}(1-4/\kappa,1-4/\kappa)$ distribution (generalized arcsine distribution). The disintegrated SLE measure w.r.t. G is again $\mathrm{SLE}_\kappa(-4,\kappa-4)$ in $(\mathbb{H},0,\infty,G,1)$, up to hitting G. In order to get a regular situation, we need to split the point G into three points z_1,y,z_2; we also set $x=0$, $z_3=1$, $z_4=\infty$. Consider the system of two commuting SLE's in $(\mathbb{H},y,z_1,x,z_2,z_3)$ given by Table 3.

TABLE 3. $\kappa\geq 8$

z_1	y	z_2	x	z_3	z_4
-2	$-\frac{\kappa}{2}$	$\frac{\kappa}{2}-2$	$[\kappa]$	$\kappa-4$	2
$\frac{\hat\kappa}{2}$	$[\hat\kappa]$	$\frac{\hat\kappa}{2}-2$	$-\frac{\hat\kappa}{2}$	$\hat\kappa-4$	$-\frac{\hat\kappa}{2}$

The first SLE exits at y, the second one exits in (x, z_3) (Lemma 15). As in Proposition 10, this shows that one can couple the two SLE's in such a way that the second one is the boundary arc of the first one between y and a point of (x, z_3). Taking $z_1 \nearrow y$, $z_2 \searrow y$ gives Theorem 2. □

At the expense of some complications, one can consider more symmetric situations. Let (D, x, y, z, z', y', x') be a configuration (points are in that order). There is a system of four commuting SLE's attached to this configuration (where $a + b = 2$), see Table 4.

TABLE 4.

x	y	z	z'	y'	x'
$[\kappa]$	$a(\kappa-4)$	$-\frac{\kappa}{2}$	$-\frac{\kappa}{2}$	$b(\kappa-4)$	2
2	$a(\kappa-4)$	$-\frac{\kappa}{2}$	$-\frac{\kappa}{2}$	$b(\kappa-4)$	$[\kappa]$
$-\frac{\hat\kappa}{2}$	$a(\hat\kappa-4)$	$[\hat\kappa]$	2	$b(\hat\kappa-4)$	$-\frac{\hat\kappa}{2}$
$-\frac{\hat\kappa}{2}$	$a(\hat\kappa-4)$	2	$[\hat\kappa]$	$b(\hat\kappa-4)$	$-\frac{\hat\kappa}{2}$

6. Some technical results

6.1. Absolute continuity for variants of SLE

In this subsection, we phrase similar absolute continuity results for different versions of SLE. In the context of duality, it is useful to consider SLE-type measures in a parametric family $\mathrm{SLE}_\kappa(\underline{\rho})$ ([1, 6]), as acknowledged in [1].

An $\mathrm{SLE}_\kappa(\underline{\rho})$, $\underline{\rho} = \rho_1, \dots, \rho_n$, in the configuration $(\mathbb{H}, x, \infty, z_1, \dots, z_n)$ is an SLE the driving process of which satisfies the SDE:

$$dW_t = \sqrt{\kappa} dB_t + \sum_{i=1}^{n} \frac{\rho_i}{W_t - g_t(z_i)} dt$$

and $W_0 = x$, up to swallowing of a z_i; as usual (g_t) denotes the solution to the Loewner equation: $dg_t(z) = 2dt/(g_t(z) - W_t)$. See Lemma 3.2 of [3] for homographic change of coordinates. In particular, if $\sum_i \rho_i = \kappa - 6$, the point at infinity is used for normalization only.

The following lemma is a change of measure result (see also [15]).

LEMMA 12. – *Consider an* SLE_κ *starting from* x *in* $\mathbb{H}$, $\underline{\rho} = \rho_1, \dots, \rho_n$; *let* $Z_t^i = g_t(z_i) - W_t$. *Then:*

$$M_t = \prod_i g_t'(z_i)^{\alpha_i} |Z_t^i|^{\beta_i} \prod_{i<j} |Z_t^j - Z_t^i|^{\eta_{ij}}$$

is a local martingale if $2\alpha_i = \frac{\kappa}{2}\beta_i(\beta_i - 1) + 2\beta_i$, $2\eta_{ij} = \kappa\beta_i\beta_j$. *Before the swallowing of any marked point,* M_t/M_0 *is the density of an* $\mathrm{SLE}_\kappa(\underline{\rho})$ *starting from* $(x, z_1, \dots z_n)$ *w.r.t.* SLE_κ, *where* $\underline{\rho} = \kappa\beta_1, \dots, \kappa\beta_n$.

Proof. – This is a standard computation relying on:

$$dZ_t^i = \frac{2}{Z_t^i}dt - \sqrt{\kappa}dB_t \qquad \frac{dg_t'(z_i)}{g_t'(z_i)} = -\frac{2}{(Z_t^i)^2}dt \qquad \frac{d(Z_t^j - Z_t^i)}{(Z_t^j - Z_t^i)} = -\frac{2}{Z_t^i Z_t^j}dt$$

so that:

$$\frac{dM_t}{M_t} = \sum_i \frac{\beta_i}{Z_t^i}\left(\frac{2}{Z_t^i}dt - \sqrt{\kappa}dB_t\right) + \frac{\kappa}{2}\frac{\beta_i(\beta_i-1)}{(Z_t^i)^2}dt - \frac{2\alpha_i}{(Z_t^i)^2}dt + \sum_{i<j}(\kappa\beta_i\beta_j - 2\eta_{ij})\frac{dt}{Z_t^i Z_t^j}.$$

The statement on the density follows from the Girsanov theorem (e.g., [11]), observing that:

$$\frac{d\langle M_t, W_t\rangle}{M_t} = -\sum_i \frac{\kappa\beta_i}{Z_t^i}dt$$

that is, the drift term of an $\mathrm{SLE}_\kappa(\underline{\rho})$ with $\rho_i = \kappa\beta_i$. More precisely, under the original measure, $W = \sqrt{\kappa}B$, B a standard Brownian motion. Under the transformed measure (via the local martingale M_t stopped away from swallowing a z_i), $\hat{W} = W - \langle W, M\rangle/M$ is a (local) martingale with the same quadratic variation as W; *i.e.*, from Lévy's theorem, a Brownian motion $\sqrt{\kappa}\hat{B}$. Hence $W = \sqrt{\kappa}\hat{B} + \langle W, M\rangle/M$. □

Let $c = (D, z_0, \dots, z_n)$ be a configuration. As in Theorem 4, we consider a variant of SLE of the following type: let ψ be a positive, continuous, conformally invariant function on the configuration space and exponents ν_{ij} such that if

$$Z(c) = \psi(c)\prod_{0\leq i<j\leq n+1} H_D(z_i, z_j)^{\nu_{ij}}$$

then $\sum_{j=1}^{n+1}\nu_{0,j} = \alpha_\kappa$ and

$$M_s = H_D(z_0, z)^{-\alpha_\kappa} Z(c_{s,0})$$

is a local martingale under the reference measure $\mathrm{SLE}_\kappa(D, z_0, z)$, z an auxiliary marked point on the boundary. For short, let us denote by $\mathrm{SLE}_\kappa(Z)$ the measure obtained by Girsanov transform of the reference measure $\mathrm{SLE}_\kappa(D, z_0, z)$ by M (up to a disconnection event).

For example, from Lemma 12, it is easy to see that $\mathrm{SLE}_\kappa(\underline{\rho})$ in $c = (D, z_0, z_1, \dots, z_n)$, $\rho_1 + \cdots + \rho_n = \kappa - 6$, is $\mathrm{SLE}_\kappa(Z)$ with:

$$Z(c) = \prod_{i=1}^n H_D(z_0, z_i)^{-\frac{\rho_i}{2\kappa}} \prod_{1\leq i<j\leq n} H_D(z_i, z_j)^{-\frac{\rho_i\rho_j}{4\kappa}}.$$

The following is an extension of Proposition 3.

LEMMA 13. – *Let* $c = (D, z_0, z_1, \dots, z_n)$ *be a configuration consisting of a simply connected domain* D *with* $n+1$ *marked points on the boundary;* $c' = (D', z_0, z_1', \dots, z_n')$ *is another configuration agreeing with* D *in a neighborhood* U *of* z_0*;* U *is at positive distance of a connected boundary arc containing marked points other than* z_0*. Let* $\overline{\mu}_c^U$ *denote the distribution of an* $\mathrm{SLE}_\kappa(Z)$ *in c, stopped upon exiting* U*. Then:*

$$\frac{d\overline{\mu}_{c'}^U}{d\overline{\mu}_c^U}(\gamma) = \left(\frac{Z(c_\tau')Z(c)}{Z(c_\tau)Z(c')}\right)\exp(-\lambda m(D; K_\tau, D\setminus D') + \lambda m(D'; K_\tau, D'\setminus D))$$

where $c_\tau = (D\setminus K_\tau, \gamma_\tau, z_1, \dots, z_n)$*, similarly for* c_τ'.

Proof. – One can proceed as follows: let μ_c^U denote chordal SLE_κ in the configuration c (aiming at an auxiliary point z), stopped upon exiting U. Then trivially:

$$\frac{d\overline{\mu}_{c'}^U}{d\overline{\mu}_c^U} = \frac{d\overline{\mu}_{c'}^U}{d\mu_{c'}^U} \cdot \frac{d\mu_{c'}^U}{d\mu_c^U} \cdot \frac{d\mu_c^U}{d\overline{\mu}_c^U}.$$

The middle term is studied in Proposition 3, while the outer terms are, from the definition of $\mathrm{SLE}_\kappa(Z)$:

$$\frac{d\mu_c^U}{d\overline{\mu}_c^U} = \frac{M_\tau}{M_0} = \frac{Z(c_\tau)}{Z(c)} \cdot \frac{H_D(z_0,z)^\alpha}{H_{D_\tau}(\gamma_\tau,z)^\alpha}$$

where τ is the first exit of U, and similarly for the other term. Under the assumptions above, M^τ is uniformly bounded (see Lemma 14). ☐

Recall that $Z(c)$ depends on a choice of local coordinates at the marked points as a power of a 1-form; but the ratio $\frac{Z(c'_\tau)Z(c)}{Z(c_\tau)Z(c')}$ does not depend on the choices.

6.2. A bound on densities

We give an upper bound on Radon-Nikodym derivatives that appear in the coupling argument. This is a crude estimate that is sufficient for our purposes.

A configuration $c = (D, x, y, z_1, \dots, z_n)$ consists in a bounded simply connected Jordan domain D, with distinct marked points on its boundary; ∂ (resp. $\hat\partial$) is the smallest connected boundary arc containing all marked points except x (resp. y); K (resp. $\hat K$) is a chain growing at x (resp. y) generated by the continuous trace γ (resp. $\hat\gamma$). We denote $c_{s,t} = (D \setminus (K_s \cup \hat K_t), \gamma_s, z_1, \dots z_n, \hat\gamma_t)$; also $Z(c) = \psi(c)\prod_{i<j} H_D(z_i,z_j)^{\nu_{ij}}$, ψ a positive, continuous, conformally invariant function. For $0 \le s' \le s, 0 \le t' \le t$, define:

$$\ell_{s',t'}^{s,t} = \left(\frac{Z(c_{s,t})Z(c_{s',t'})}{Z(c_{s',t})Z(c_{s,t'})}\right)\exp(-\lambda m(D\setminus(K_{s'}\cup \hat K_{t'}); K_s, \hat K_t)).$$

LEMMA 14. – *For any $\eta > 0$ small enough, there exists $C = C(D,\eta) > 0$ such that for all chains $K, \hat K$, $0 \le s' \le s, 0 \le t' \le t$ with* $\mathrm{dist}(K_s, \hat K_t) \ge \eta$, $\mathrm{dist}(K_s, \partial) \ge \eta$, $\mathrm{dist}(\hat K_t, \hat\partial) \ge \eta$,

$$C^{-1} < \ell_{s',t'}^{s,t} < C.$$

Proof. – From the identity: $\ell_{s',t'}^{s,t} = \ell_{0,0}^{s',t'}\ell_{0,0}^{s,t}(\ell_{0,0}^{s',t}\ell_{0,0}^{s,t'})^{-1}$, it is enough to prove the bound for $s' = t' = 0$. From, e.g., Corollary 2.8 in [10], it is enough to prove it in any reference Jordan domain, say the upper semidisk, with all marked points on the segment $(-1,1)$. Also without loss of generality, one can assume there is at least one marked point z_1.

In the bounded domain D, the total mass of loops of diameter at least η in the loop measure μ^{loop} is finite; this gives uniform bounds above and below for the factor $\exp(-\lambda m(\dots))$.

Consider the set S of quadruplets $(K, x', \hat K, y')$ where $K, \hat K$ are compact subsets of $\overline{D}$, $K, \hat K$ connected, with x', y' on their respective boundaries, $\mathrm{dist}(K, K') \ge \eta$, $\mathrm{dist}(K, \partial) \ge \eta$, $\mathrm{dist}(\hat K, \hat\partial) \ge \eta$, and x' (resp. y') corresponds to a single prime end on $D\setminus(K\cup\hat K)$. (This last condition is always satisfied "at the tip"). The set S is compact (for Hausdorff convergence of compact subsets of $\overline{D}$). To such a quadruplet are associated four configurations: $c_{0,0} = c$, $c_{1,0} = (D\setminus K, x', y, z_1, \dots, z_n)$, $c_{0,1} = (D\setminus \hat K, x, y', \dots)$, $c_{1,1} = (D\setminus(K\cup\hat K), x', y', \dots)$. Then the ratio $\frac{Z(c_{11})Z(c_{00})}{Z(c_{10})Z(c_{01})}$ defines a positive function on the compact set S. It is enough to prove that this function is continuous.

Let $(K_n, x'_n, \hat{K}_n, y'_n)_n$ converge to $(K, x', \hat{K}, y')$. By Schwarz reflection across $[-1,1]$ and the Carathéodory convergence theorem ([10], Theorem 1.8), the conformal equivalence ϕ^n_{11} between $D^n_{11} = D\backslash(K_n \cup \hat{K}_n)$ and $D_{11} = D\backslash(K \cup \hat{K})$, extended by reflection and normalized by $\phi^n_{11}(z_1) = z_1$, $\phi^n_{11}(z_1) > 0$, converges locally uniformly away from the unit circle and $(K \cup \overline{K} \cup \hat{K} \cup \overline{\hat{K}})$ (here $\overline{K}$ is the conjugate of K). The same holds for $D^n_{10} = D \setminus K_n$ and $D^n_{01} = D \setminus \hat{K}_n$.

Fix small semidisks $D(z_i, \eta/2)$ around the marked points z_i's; the choice of D as the upper semidisk gives a choice of local coordinates at the z_i's. Then the $H_{D^n_{..}}(z_i, z_j)$ are numbers; they can be decomposed into: excursion harmonic measure in the semidisks $D(z_i, \eta/2)$, $D(z_j, \eta/2)$, and the Green function at points on $C(z_i, \eta/2)$, $C(z_j, \eta/2)$. The excursion harmonic measures are fixed and the Green function converges due to the conformal invariance of the Green function and uniform convergence of the $\phi^n_{..}$ near the z_i's. This proves continuity of the $H_{D^n_{..}}(z_i, z_j)$.

The treatment of ratios of type $H_{D^n_{1,1}}(x'_n, z_i)/H_{D^n_{1,1}}(x'_n, z_j)$ is similar: take a crosscut δ at positive distance of K, separating it from $\hat{K}$ and the marked points. Then the Poisson excursion kernel can be decomposed w.r.t. the first crossing (by a Brownian motion starting near x'_n) of δ and the last crossing of $C(z_i, \eta/2)$. It is easy to see that the excursion harmonic measure on δ converges. This gives continuity of ratios of type $H_{D^n_{1,1}}(x'_n, z_i)/H_{D^n_{1,1}}(x'_n, z_j)$. Assuming without loss of generality that there are at least two marked points z_1, z_2, the term $H(x, y)^{\cdot}$ can be eliminated from the partition function.

The only remaining thing to check is the convergence of the cross ratios. This is immediate for those not involving x, y, from the convergence of the $\phi^n_{..}$ as above. This can be done also for those involving x, y; though for our purposes it is enough to prove that all cross-ratios (between marked points) are uniformly bounded. By comparison arguments, it is enough to prove it for cross-ratios involving x^-, x^+, y^-, y^+ (instead of x, y, x', y') , where these new points are on $[-1,1]$ and are such that the interval (x^-, x^+) (resp. (y^-, y^+)) contains $K_n \cap [-1,1]$ (resp. $\hat{K}_n \cap [-1,1]$) for n large enough, and no other marked point. This then reduces to the previous situation. □

6.3. First exit of $\mathrm{SLE}_\kappa(\underline{\rho})$

We need to establish some simple qualitative properties of $\mathrm{SLE}_\kappa(\underline{\rho})$ in, say, a reference configuration $c = (\mathbb{H}, 0, \infty, z_1, \dots, z_n)$. In particular, we are interested in the position of the trace the first time a marked point is swallowed.

Assume that $n = 2$, $0 < z_1 < z_2 < \infty$. Then the SLE is well defined up to swallowing of z_1 at time $\tau_1 = \tau_{z_1}$. There are several possibilities: $\tau_1 = \infty$; $\gamma_{\tau_1} = z_1$; $\gamma_{\tau_1} \in (z_1, z_2)$; $\gamma_{\tau_1} = z_2$; $\gamma_{\tau_1} \in (z_2, \infty)$; or γ_{τ_1} does not exist. (This last case is unlikely to ever happen, though delicate to rule out in general).

More precisely, let $Y_t = \frac{g_t(z_1)-W_t}{g_t(z_2)-W_t}$ and $ds = \frac{dt}{(g_t(z_2)-W_t)^2} = -\frac{1}{2} d\log g'_t(z_2)$. Then:

$$dY_s = (1 - Y_s)\left[\sqrt{\kappa} dB_s + (\frac{\rho_1 + 2}{Y_s} + \rho_2 + 2 - \kappa)dt\right]$$

where B is a standard Brownian motion. This is a diffusion on $[0,1]$. Notice that $g_t'(z_2)$ is positive before τ_2, and goes to zero at $t \nearrow \tau_2$. A scale function of this diffusion is F:

$$F(y) = \int_{1/2}^{y} u^{-\frac{2}{\kappa}(2+\rho_1)}(1-u)^{\frac{2}{\kappa}(4+\rho_1+\rho_2-\kappa)}du.$$

It blows up at 0 if $\rho_1 \geq \frac{\kappa}{2} - 2$. This means that Y does not reach 0 in finite time, so that $\tau_1 = \tau_2$ a.s. (possibly infinite). If $\rho_1 < \frac{\kappa}{2} - 2$, $\rho_1 + \rho_2 \leq \frac{\kappa}{2} - 4$, the scale function blows up at 1, not 0, meaning that $\tau_1 < \tau_2$ a.s.

Assume that the trace has an accumulation point in $[z_1, z_2)$ as $t \nearrow \tau_2$. Then (Y_t) accumulates at 0 as $t \nearrow \tau_2$. This can be seen by interpreting $(g_t(z) - W_t)$ as the limit of the probability divided by y that a Brownian motion started at iy, $y \gg 1$, exits $\mathbb{H} \setminus K_t$ on the boundary arc $[\gamma_t, z]$. Consider a time where the trace is near an accumulation point in $[z_1, z_2)$ In order to exit on $[\gamma_t, z_1]$, the Brownian motion has to get near z_2, and then move through a strait where the trace accumulates; this conditional probability is controlled by Beurling's estimate.

The following lemma gives conditions under which the exit point of an $\mathrm{SLE}_\kappa(\underline{\rho})$ process (first disconnection of a marked point) can be located (at least on a segment). The statement is in terms of accumulation points, which will be enough for our purposes. It is likely that a stronger statement (in terms of limits) holds.

LEMMA 15. – *Consider an* $\mathrm{SLE}_\kappa(\underline{\rho})$ *in* $(\mathbb{H}, 0, \infty, z_1, \dots, z_n)$, $0 < z_1 < \cdots < z_n$. *Let* $\overline{\rho}_k = \rho_1 + \cdots + \rho_k$, $\overline{\rho}_n = \kappa - 6$.

1. *Assume that for some* k, $\overline{\rho}_i \geq \frac{\kappa}{2} - 2$ *for* $i < k$ *and* $\overline{\rho}_i \leq \frac{\kappa}{2} - 4$ *for* $k \leq i < n$. *Then a.s. as* $t \nearrow \tau_1$, γ_t *accumulates at* z_k *and at no other point in* $[z_1, z_n]$.
2. *Assume that for some* k, $\overline{\rho}_i \geq \frac{\kappa}{2} - 2$ *for* $i < k$; $\overline{\rho}_k \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$; *and* $\overline{\rho}_i \leq \frac{\kappa}{2} - 4$ *for* $k < i < n$. *Then a.s. as* $t \nearrow \tau_1$, γ_t *accumulates at a point in* $[z_k, z_{k+1}]$ *and at no point in* $[z_1, z_n] \setminus [z_k, z_{k+1}]$.

Proof. – While the results are fairly intuitive from a simple Bessel dimension count, complete arguments are a bit involved.

1. a) Case $k = n = 2$. By a change of coordinates, one can send z_2 to ∞. Then the SLE is defined for all times, $\tau_1 = \tau_2 = \infty$. This implies that the trace is unbounded (accumulates at $z_2 = \infty$). Moreover, for any $z \in (z_1, z_2)$, the (time changed) diffusion $Y_t = (g_t(z) - W_t)/(g_t(z_1) - W_t)$ goes to 1 as $t \nearrow \tau_1 = \infty$, by a study of its scale function. In particular, it does not accumulate at 0; hence the trace does not accumulate in $[z_1, z_3)$. So the only point of accumulation of the trace in $[z_1, z_2]$ is z_2.

b) Case $k = n \geq 2$. Again, we change coordinates so that $z_n = \infty$. Let $\rho_1 = \frac{\kappa}{2} - 2 + \rho_1'$, $\rho_i = \rho_i' - \rho_{i-1}'$. By assumption, $\rho_i' \geq 0$, $i < n$. Consider the SDE (notations as in Lemma 12):

$$dZ_t^1 = \frac{2}{Z_t^1}dt - \sqrt{\kappa}dB_t + \sum_{i=1}^{n-1} \frac{\rho_i}{Z_t^i}dt = -\sqrt{\kappa}dB_t + \frac{\kappa}{2}.\frac{dt}{Z_t^1} + \sum_{i=1}^{n-1} \rho_i' \frac{Z_t^{i+1} - Z_t^i}{Z_t^i Z_t^{i+1}}dt$$

the last sum being nonnegative. By a stochastic domination argument (comparison with a Bessel process, $\delta = 2$), this shows that $\tau_1 = \infty$. Hence the process is defined for all times and the trace is unbounded.

Next we prove that there is no point of accumulation of the trace in $[z_1, z_{n-1}]$. Take a small neighbourhood U of $[z_1, z_{n-1}]$. Let σ_n be the first time the trace goes at distance n (an a.s. finite stopping time). Let U_n be the connected component of 1 in $(g_{\sigma_n}(U)-W_t)/(g_{\sigma_n}(z_{n-1})-W_t)$. By harmonic measure estimates, it is easy to see that U_n is contained in an arbitrarily small neighborhood of 1 as $n \to \infty$, say $D(1, \varepsilon_n)$. By a), with probability $1-o(1)$, an $\mathrm{SLE}_\kappa(\overline{\rho}_{n-1})$ in $(\mathbb{H}, 0, \infty, 1)$ does not intersect $D(1, \sqrt{\varepsilon_n})$. On the other hand, the density of the $\mathrm{SLE}_\kappa(\underline{\rho})$ starting with all marked points in $D(1, \varepsilon_n)$ w.r.t. to $\mathrm{SLE}_\kappa(\overline{\rho}_{n-1})$ in $(\mathbb{H}, 0, \infty, 1)$ is $1+o(1)$ on the event that the trace does not intersect $D(1, \sqrt{\varepsilon_n})$. This follows from an inspection of the densities (Lemma 12) and the fact that on the event $\{\gamma \cap D(0, \sqrt{\varepsilon_n}) = \varnothing\}$, $(g_t'(z)/g_t'(1) - 1)$ is small uniformly in t and $z \in [1-\varepsilon_n, 1+\varepsilon_n]$. Indeed, Brownian excursions starting from 1 and z couple with high probability before exiting $D(1, \sqrt{\varepsilon_n})$.

This proves that with probability $1-o(1)$, the original SLE does not return to U after σ_n. Notice that one can insert a point z_n' between z_{n-1} and z_n with $\rho_n = 0$ and the result still applies. This shows that there is no point of accumulation in $[z_1, z_n)$.

c) Case $n = 3$, $k = 2$. We prove that the trace does not accumulate at z_3 (similarly, at z_1). By sending z_3 at infinity, it is easy to see that the half-plane capacity of the hull at τ seen from z_1 is finite a.s. (one can even compute its Laplace transform). We have to rule out that the hull is unbounded while having finite half-plane capacity. It is enough to prove that the driving process stopped at τ stays bounded. Since $Z_t^1 = g_t(z_1) - W_t$ goes to zero as $t \nearrow \tau$, it is enough to prove that $\int^\tau \frac{ds}{Z_s^1}$ is finite. Consider the SDE for Z_t^1:

$$dZ_t^1 = -\sqrt{\kappa} dB_t + \frac{\rho_1 + \rho_2 + 2 + \varepsilon_t}{Z_t^1}$$

where $\varepsilon_t = \rho_2(1 - Z_t^1/Z_t^2)$; ε_t goes to zero as $t \nearrow \tau$ (by studying the time changed diffusion (Z_t^1/Z_t^2)). One can proceed with a comparison with Bessel processes. On the event $\{\varepsilon_t \in [0, \varepsilon], t \geq t_0\}$, for $t \geq t_0$, Z_t^1 is between a $\mathrm{Bessel}(\delta - \varepsilon)$ and a $\mathrm{Bessel}(\delta)$, $\delta = 1 + 2\frac{\rho_1+\rho_2+2}{\kappa} \leq 2 - \frac{8}{\kappa}$ (both hit zero in finite time). Let t_1 be the first time the ratio of the two bounding Bessel processes X^- and X^+ is 2; restart them at t_1 from the same position $Z_{t_1}^1$, and define inductively t_i, $i > 1$. One can think of restarting the majorizing Bessel process at a lower level at t_1 as waiting for the Bessel to reach level $Z_{t_1}^1$. This proves that $\int_{t_0}^\tau \frac{ds}{Z_s^1} \leq 2 \int_{t_0}^{\tau_0} \frac{ds}{X_s^+}$, which is finite.

d) General case. Send z_k to infinity by a change of coordinate. The conditions on the ρ_i's are rephrased as:

$$\rho_1, \rho_1 + \rho_2, \ldots, \rho_1 + \cdots + \rho_{k-1} \geq \frac{\kappa}{2} - 2$$
$$\rho_n, \rho_n + \rho_{n-1}, \ldots, \rho_n + \cdots + \rho_{k+1} \geq \frac{\kappa}{2} - 2.$$

Hence the situation to the left and to the right of 0 are identical. It is easy to see from b) that the trace is defined for all times and is unbounded. Let σ_n be the time of first exit of $D(0, n)$ by γ. Rescale the process so that $g_t(z_1)$ (resp. $g_t(z_n)$, W_t) is sent to -1 (resp. 1, w_t) at $t = \sigma_n$. If w_t is away from ± 1, one can reason as in b) from the result of c). If w is close to 1, say, one can rescale by sending w to 0 (and 1) is fixed. The resulting process has density very close to 1 with a process of type b) as long as w_t stays close to 1. When w_t separates from 1, one can apply c) with a density argument.

2. a) Case $n = 2, k = 1$. It is easily seen by sending z_2 (or z_1) to infinity that the trace is defined for a finite time. Reasoning as in 1c) shows that the trace is bounded. Hence it accumulates somewhere in (z_1, z_2), but not at z_2 (and by symmetry z_1).

b) General case. Send z_k to infinity (so that $z_{k+1} < 0$). A stochastic domination argument as in 1b) shows that the driving process is dominated by the one corresponding to $z_2, \dots, z_{k-1}$ being sent to infinity while $z_{k+2}, \dots, z_n$ are sent to z_{k+1}. It is easily seen that for that process, $\tau_{k+1} < \tau_1$. Consequently, this is also the case for the original process, viz. the trace accumulates on $[z_k, z_n]$ without accumulating on $[z_1, z_k)$. As in 1d), the situation is symmetric, so there is also no accumulation on $(z_{k+1}, z_n]$. □

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