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REAL SINGULAR DEL PEZZO SURFACES AND 3-FOLDS FIBRED BY RATIONAL CURVES, II

BY Fabrizio CATANESE AND Frédéric MANGOLTE

ABSTRACT. – Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves such that $W(\mathbb{R})$ is orientable. J. Kollár proved that a connected component N of $W(\mathbb{R})$ is essentially either Seifert fibred or a connected sum of lens spaces.

Answering three questions of Kollár, we give sharp estimates on the number and the multiplicities of the Seifert fibres (resp. the number and the torsions of the lens spaces) when X is a geometrically rational surface.

When N is Seifert fibred over a base orbifold F, our result generalizes Comessatti's theorem on smooth real rational surfaces: F cannot be simultaneously orientable and of hyperbolic type. We show as a surprise that, unlike in Comessatti's theorem, there are examples where F is non orientable, of hyperbolic type, and X is minimal.

RÉSUMÉ. – Soit $W \to X$ une variété projective réelle non singulière munie d'une fibration en courbes rationnelles et telle que $W(\mathbb{R})$ soit orientable. J. Kollár a montré qu'une composante connexe N de $W(\mathbb{R})$ est essentiellement une variété de Seifert ou une somme connexe d'espaces lenticulaires.

Répondant à trois questions de Kollár, nous donnons une estimation optimale du nombre et des multiplicités des fibres de Seifert (resp. du nombre et des torsions des espaces lenticulaires) lorsque X est une surface géométriquement rationnelle.

Lorsque N admet une fibration de Seifert au-dessus d'un orbifold F, nos résultats généralisent le théorème de Comessatti sur les surfaces rationnelles réelles lisses : F ne peut pas être à la fois orientable et de type hyperbolique. Nous montrons, ce qui est une surprise, qu'à la différence du théorème de Comessatti, il existe des exemples où F est non orientable, de type hyperbolique, et X est minimale.

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Introduction

Given a smooth real projective variety W of dimension n, we consider the topology of a connected component N of the set $W(\mathbb{R})$ of its real points.

John Nash proved in [12] that any compact connected differentiable manifold N is obtained in this way, and went over to ask whether the same would hold if one assumes W to be geometrically rational.

However, when W is a surface of negative Kodaira dimension, one is able, after the work of Comessatti [2] for geometrically rational surfaces, to deduce drastical restrictions for the topology of N. Namely, if N is orientable, then it is diffeomorphic to a sphere or to a torus: in other words, N cannot be simultaneously oriented and of hyperbolic type. In this note, we make a step towards a complete classification of the topological types for N when W is a rationally connected 3-fold fibred by rational curves (this is one of the higher dimensional analogues of Comessatti's theorem).

This study was initiated by János Kollár, in the third paper [8] of a ground-breaking series of articles applying the minimal model program to the study of the topology of real algebraic 3-folds.

Kollár's philosophy is that a very important condition in order to obtain restrictions upon the topological type of $W(\mathbb{R})$ is that W has terminal singularities and K_W is Cartier along $W(\mathbb{R})$.

Kollár proved in particular that if W is a smooth 3-fold fibred by rational curves (in particular, W has negative Kodaira dimension) and such that $W(\mathbb{R})$ is orientable, then a connected component N of $W(\mathbb{R})$ is essentially a Seifert fibred 3-manifold or the connected sum of a finite number of lens spaces. Note that in [5, 6] it was shown that conversely all the above manifolds N do occur for some smooth 3-fold W fibred by rational curves.

When W belongs to the subclass of rationally connected 3-folds fibred by rational curves, Kollár proved some additional restrictions upon N and made three further conjectures. In our first note [1] we proved two of the optimal estimates that Kollár conjectured to hold. In the present note we prove the third estimate, which is the most important one since it allows us to conclude in particular that, if N is a Seifert fibred 3-manifold, then the base orbifold cannot be simultaneously oriented and of hyperbolic type.

Let us now introduce our results in more detail.

Let N be an oriented three dimensional compact connected topological manifold without boundary. Take a decomposition $N = N' \#^a \mathbb{P}^3(\mathbb{R}) \#^b (S^1 \times S^2)$ with $a + b$ maximal and observe that this decomposition is unique by a theorem of Milnor [10].

We shall focus our attention on the case where N' is Seifert fibred or a connected sum of lens spaces. We consider the integers $k := k(N)$ and $n_l := n_l(N)$, $l = 1 \dots k$ defined as follows:

1. if $g \colon N' \to F$ is a Seifert fibration, k denotes the number of multiple fibres of g and $n_1 \leq n_2 \leq \cdots \leq n_k$ denote the respective multiplicities;
2. if N' is a connected sum of lens spaces, k denotes the number of lens spaces and $n_1 \leq n_2 \leq \cdots \leq n_k$, $n_l \geq 3$, $\forall l$, the orders of the respective fundamental groups (thus we have a decomposition $N' = \#_{l=1}^k (L(n_l, q_l))$ for some $1 < q_l < n_l$ relatively prime to n_l).

Observe that when N' is a connected sum of lens spaces, the number k and the numbers n_l, $l = 1, \dots k$, are well defined (again by Milnor's theorem). In the case of a Seifert fibred manifold N', these integers may a priori depend upon the choice of a Seifert fibration.

Three results of our two notes are summarized by the following.

THEOREM 0.1. – *Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational*[(1)] *surface X. Suppose that $W(\mathbb{R})$ is orientable. Then, for each connected component $N \subset W(\mathbb{R})$, $k(N) \leq 4$ and $\sum_l (1 - \frac{1}{n_l(N)}) \leq 2$. Furthermore, if N' is Seifert fibred over $S^1 \times S^1$, then $k(N) = 0$.*

This theorem answers, as we already said, some questions posed by Kollár, see [8, Remark 1.2 (1,2,3)]. In the first note, we proved the estimate $k(N) \leq 4$ and we showed that $k(N) = 0$ if N' is Seifert fibred over the torus. Thus Theorem 0.1 follows from [1, Corollary 0.2, and Theorem 0.3] and from Theorem 0.2 of the present paper using results of [8] as in [1]. The present note is mainly devoted to the proof of the inequality $\sum_l (1 - \frac{1}{n_l(N)}) \leq 2$, see Lemma 6.1.

The proof of this inequality goes as follows: let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Using the same arguments as in [1, Sec. 3], we reduce the proof of the estimate for the integers $n_l(N)$ to an inequality depending on the indices of certain singular points of a real component M of the topological normalization of $X(\mathbb{R})$ (see Definition 1.1). In this process, the number $k(N)$ can be made to correspond to the number of real singular points on M which are of type A_μ^+, and globally separating when μ is odd; each number $n_l(N) - 1$ corresponds to the index μ_l of the singularity $A_{\mu_l}^+$ of M. The main part of the paper is devoted to the proof of the following.

THEOREM 0.2. – *Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of the topological normalization $\overline{X(\mathbb{R})}$ contains at most 4 singular points x_l of type $A_{\mu_l}^+$ which are globally separating for μ_l odd. Furthermore, their indices satisfy*

$$\sum \left(1 - \frac{1}{\mu_l + 1}\right) \leq 2 \, .$$

Let us now give an interpretation of the above results in terms of geometric topology (see e.g. [13] for the basic definitions and classical results). Suppose that N' admits a Seifert fibration with base orbifold F. From our main Theorem 0.1 we infer that, if the underlying manifold $|F|$ is orientable, then the Euler characteristic of the compact 2-dimensional orbifold F is nonnegative (see Proposition 7.1). Thus, by the uniformization theorem for compact 2-dimensional orbifolds, F admits a spherical structure or an euclidean structure.

In general, a 3-manifold N does not possess a geometric structure, but, if it does, then the geometry involved is unique. Moreover, it turns out that every Seifert fibred manifold admits a geometric structure. The geometry of N is modeled on one of the six following models (see [13] for a detailed description of each geometry):

$$S^3, S^2 \times \mathbb{R}, E^3, \mathrm{Nil}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}_2 \, \mathbb{R}},$$

[(1)] By [4] these assumptions are equivalent to: W rationally connected and fibred by rational curves.

where E^3 is the 3-dimensional euclidean space and $\mathbb{H}^2$ is the hyperbolic plane. The six above geometries are called the Seifert geometries. The appropriate geometry for a Seifert fibration is determined by the Euler characteristic of the base orbifold and by the Euler number of the Seifert bundle [13, Table 4.1].

Let W be a real projective 3-fold fibred by rational curves and such that $W(\mathbb{R})$ is orientable, let $N \subset W(\mathbb{R})$ be a connected component and let N' be the manifold defined as above. Suppose moreover that N' possesses a geometric structure. By Theorem [8, Th. 1.1], the geometry of N' is one of the six Seifert geometries. Conversely, by [6], any orientable three dimensional manifold endowed with any Seifert geometry is diffeomorphic to a real component of a real projective 3-fold fibred by rational curves. But, when W is rationally connected, the following corollary of our main theorem gives further restrictions.

Corollary 0.3. – *Let W be a real smooth projective rationally connected 3-fold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable and let N be a connected component of $W(\mathbb{R})$. Then neither N nor N' can be endowed with a $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$ structure or with a $\mathbb{H}^2 \times \mathbb{R}$ structure whose base orbifold F is orientable.*

Observe moreover that in [8] all compact 3-manifolds with S^3 or E^3 geometry, and some manifolds with Nil geometry, are realized as a real component of a real smooth projective rationally connected 3-fold fibred by rational curves.

There remains of course the question about what happens when N is Seifert fibred over a non-orientable orbifold F: is the orbifold still not of hyperbolic type? In the last section we show that the answer to this question is negative. We produce indeed an example of a smooth 3-fold W, fibred by rational curves over a Du Val Del Pezzo surface X, where $W(\mathbb{R})$ is orientable, and contains a connected component which is Seifert fibred over a non-orientable base orbifold of hyperbolic type.

The striking fact is here that X is a real minimal surface: this contrasts with Comessatti's theorem: since indeed a real minimal nonsingular geometrically rational surface cannot have a component which is of hyperbolic type.

Theorem 0.4. – *There exists a minimal real Du Val Del Pezzo surface X of degree 1 having exactly two singular points, of type A_2^+, and such that the real part $X(\mathbb{R})$ has a connected component containing the two singular points and which is homeomorphic to a real projective plane.*

Let W' be the projectivized tangent bundle of X: then W' has terminal singularities, $W'(\mathbb{R})$ is contained in the smooth locus of W', in particular if W is obtained resolving the singular points of W', then $W(\mathbb{R}) = W'(\mathbb{R})$.

Moreover $W(\mathbb{R})$ is orientable and contains a connected component N which is Seifert fibred over a non orientable orbifold of hyperbolic type (the real projective plane with two points of multiplicity 3).

Briefly, now, the contents of the paper.

Sections 1 and 2 are devoted to the reduction of the proof of the main theorem to the assertion of non existence of seven configurations of singular points on a real component of a Du Val Del Pezzo surface of degree 1.

Two main methods used here are borrowed from [1]: namely, the generalization of Brusotti's theorem to the effect that one can independently take any smoothing of the singularities of a Du Val Del Pezzo surface, and also the use of the plane model where the family of hyperplane sections of the quadric cone Q is represented by the family of parabolae in the plane with a fixed asymptotic direction. These methods combine with a delicate argument, suggested by E. Brugalle, excluding the possibility of an intersection of Q with a cubic surface yielding an irreducible curve B with four real cusps (see 2.1).

Section 3 introduces the main tools used in the proof (the topological classification of real smooth Del Pezzo surfaces of degree 1, and the choice of the appropriate partial smoothings), and ends with the exclusion of two configurations via complicated although elementary topological considerations.

Section 4 uses a classification of critical points for the projection of B and a precise table for the local contributions to the multiplicity of the discriminant and for the local contribution to the Euler number in order to exclude two more cases.

Section 5 proves Theorem 0.2 by excluding the three remaining cases by combining all the previous tools with an ad hoc analysis and with two new tools, namely: the use of the Comessatti characteristic, relating the total Betti number of the real part with the one of the complex part, and the calculation of the contributions of the singularities to the Picard and to the various Euler numbers.

Finally, Section 6 is devoted to the proof of Lemma 6.1 and in Section 7, after showing that the base orbifold cannot be oriented and hyperbolic, we exhibit the example of a projectivized tangent bundle over a Du Val Del Pezzo surface for which a component N is Seifert fibred with base orbifold of hyperbolic type.

In the course of this complicated construction we give a quite general method to construct Seifert fibrations as projectivized tangent bundles of surfaces with singularities of type A_n.

We want to thank E. Brugalle for pointing out the statement of Lemma 2.1 and suggesting the main idea of the proof , and Ingrid Bauer for helping us to understand the configuration of lines on Del Pezzo surfaces of degree 1.

1. Singular geometrically rational surfaces

Using the results and notation of [1, Section 1], we reduce the proof of Theorem 0.2 to the proof of a statement about singular Del Pezzo surfaces of degree 1 with small Picard number ρ.

Recall that a surface singularity which is a rational double point is also called a Du Val singularity and that a projective surface X is called a *Du Val* surface if X has only Du Val singularities. A surface singularity is *of type* A_μ^+ if it is real analytically equivalent to $x^2 + y^2 - z^{\mu+1} = 0,\ \mu \geq 1$; and *of type* A_μ^- if it is equivalent to $x^2 - y^2 - z^{\mu+1} = 0,\ \mu \geq 1$. The type A_1^+ is real analytically isomorphic to A_1^-; otherwise, singularities with different names are not isomorphic.

We recall some definitions due to Kollár (see [1, Section 1]).

DEFINITION 1.1. – *Let V be a simplicial complex with only a finite number of points $x \in V$ where V is not a manifold. Define the* topological normalization

$$\overline{n}\colon \overline{V} \to V$$

as the unique proper continuous map such that $\overline{n}$ is a homeomorphism over the set of points where V is a manifold and $\overline{n}^{-1}(x)$ is in one-to-one correspondence with the connected components of a good punctured neighborhood of x in V otherwise.

Observe that if V is pure of dimension 2, then $\overline{V}$ is a topological manifold (since each point of $\overline{V}$ has a neighbourhood which is a cone over S^1).

DEFINITION 1.2. – *Let X be a real Du Val surface, and let $x \in X(\mathbb{R})$ be a singular point of type $A_\mu^\pm$ with μ odd. The topological normalization $\overline{X(\mathbb{R})}$ has two connected components locally near x. We will say that x is* globally separating *if these two local components lie on different connected components of $\overline{X(\mathbb{R})}$ and* globally nonseparating *otherwise. Let*

$$\mathcal{P}_X := \operatorname{Sing} X \setminus \{x \text{ of type } A_\mu^-,\ \mu \text{ even}\} \setminus \{x \text{ of type } A_\mu^-,\ \mu \text{ odd and } x \text{ is globally nonseparating}\}\ .$$

Let X be a real Du Val surface, let $\overline{n}\colon \overline{X(\mathbb{R})} \to X(\mathbb{R})$ be the topological normalization, and let $M_1, M_2, \ldots, M_r$ be the connected components of $\overline{X(\mathbb{R})}$. By [8, Cor. 9.7], the unordered sequence of numbers

$$m_i := \#(\overline{n}^{-1}(\mathcal{P}_X) \cap M_i),\ i = 1, 2, \ldots, r$$

is an invariant for extremal birational contractions of Du Val surfaces.

We will now reduce the proof of Theorem 0.2 to the proof of the following.

THEOREM 1.3. – *Let X be a real Du Val Del Pezzo surface of degree 1 with $\rho(X) \leq 2$. Then $m_i \leq 4$, $i = 1, 2, \ldots, r$, and moreover for any $M := M_i$ such that $\overline{n}(M)$ contains $A^+_{\mu_1} + A^+_{\mu_2} + \cdots + A^+_{\mu_{m_i}}$ where $A^+_{\mu_l}$ is globally separating for μ_l odd, we have:*

$$\sum_{l=1}^{m_i} \left(1 - \frac{1}{\mu_l + 1}\right) \leq 2\ .$$

Up to Section 6, the sequel of this paper is devoted to the proof of Theorem 1.3.

2. Reducing to seven configurations

Numerically, the following configurations of A_μ^+ singularities are the only ones allowed by the inequality

$$\sum_{l=1}^{m_i} \left(1 - \frac{1}{\mu_l + 1}\right) \leq 2\ . \tag{1}$$

- $m_i = 4$ and the configuration is $4A_1^+$,
- $m_i = 3$ and the configuration is
    - $2A_1^+ + A_\mu^+$, any μ, or
    - $A_1^+ + A_2^+ + A_\mu^+$, $\mu \leq 5$, or
    - $A_1^+ + 2A_3^+$,

  - $3A_2^+$,
- $m_i = 2$.

Recall that a Du Val Del Pezzo surface X is by definition a Du Val surface (i.e., a surface with only rational double points as singularities) whose anticanonical divisor is ample, see [1, Section 2]. The anticanonical model of a Del Pezzo surface X of degree 1 is a ramified double covering $q\colon X \to Q$ of a quadric cone $Q \subset \mathbb{P}^3$ whose branch locus is the union of the vertex of the cone with a curve B not passing through the vertex and which is the complete intersection of the cone with a cubic surface.

Let X be a real Du Val Del Pezzo surface of degree 1 and let X' be the singular surface obtained from X by blowing up the pull-back by q of the vertex of the cone (which is a smooth point of X). The surface X' is a ramified double covering of the Hirzebruch surface $\mathbb{F}_2$ whose branch curve is the union of the unique section of negative selfintersection, the section at infinity Σ_∞, and the trisection B of the ruling $p\colon \mathbb{F}_2 \to \mathbb{P}^1$, which is disjoint from Σ_∞. The composition $X' \to \mathbb{F}_2 \to \mathbb{P}^1$ is a real elliptic fibration.

The different cases that we shall now consider are distinguished by the number of irreducible components of the trisection B. Notice that if all the singular points are of type A_1, the conclusion of Theorem 1.3 follows from [1, Proposition 2.1].

2.1. Three components

If B has strictly more than 4 real singular points, all the possible cases are enumerated in [1, Section 2], and an inspection of [Ibid., Figures 1, 2, 3] shows that for any connected component of the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$, the configuration is $4A_1$ or $A_3 + 2A_1$. Thus the inequality (1) holds except possibly in the situation where two irreducible components of B are tangent to the third one. It turns out that there is only one normal form for this situation, see Figure 1. Indeed, the affine part of B is a union of three parabolae and without loss of generality, these three parabolae are given by $y = 0$, $y = x^2$ and $y = \alpha(x-a)^2$, $\alpha,\ a \in \mathbb{R}$ [Ibid.]. We have $a \neq 0$, else B has a triple point with an infinitely near triple point, contradicting the fact that X has only Du Val singularities. Furthermore, in order to get at least three real intersection points, α has to be positive. Up to reflection $x \leftrightarrow -x$, this leads to one possibility.

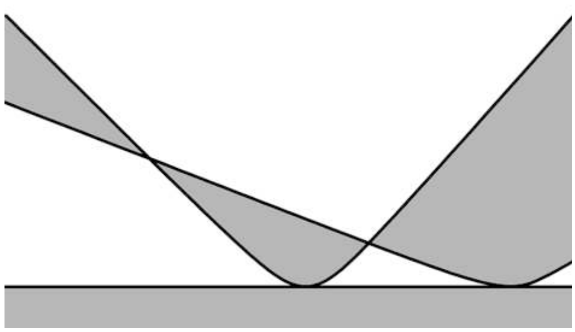


FIGURE 1. Three parabolae with two tacnodes.

Recalling that in this figure two components are connected at infinity if their boundaries have two unbounded arcs belonging to the same pair of parabolae, we see that none of the connected components of $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ contains more than 3 singular points and at most two of them are tacnodes. Thus (1) holds also in this case.

2.2. Two components

Then $B = L \cup C$ where C is a bisection of the ruling p and L is a section. The bisection C has arithmetic genus one, hence it has at most one double point A_1 or A_2 and at most 4 intersection points with the section L.

If C is non singular, we have $4A_1$ or $2A_1 + A_3$ or only two singular points. In each case we get an allowed configuration.

Assume C is singular: if B has 5 singular points, we are done since either all singular points are of type A_1, see [1, Figures 4, 5, 6], or we are in the situation depicted in [1, Figure 7] and then the A_2^+ is on a component with only two other singularities, of type A_1. If B has 4 singular points, the possibilities are $A_1 + A_3 + 2A_1$, or $A_2 + A_3 + 2A_1$. If B has 3 singular points, the possibilities are $A_1 + 2A_3$, or $A_2 + 2A_3$, or $A_2 + A_1 + A_5$.

Thus if B has two irreducible components, we get the conclusion of Theorem 1.3 unless the configuration of singular points is $A_3 + 3A_1$, or $A_3 + A_2 + 2A_1$, or $2A_3 + A_2$.

2.3. One component

If the trisection is irreducible, then it has at most 4 singular points, since $B(\mathbb{C})$ has genus 4.

LEMMA 2.1. – *The real curve B cannot have 4 real cusps.*

Proof. – Suppose that B is irreducible with 4 real cusps. Choose three of them. Let L' be a section of the ruling p corresponding to a plane section of Q passing through these three points; for an appropriate choice of the plane model of Q (see [1], beginning of Section 2) we may assume L' to be the horizontal x-axis $y = 0$ in the plane

Since the intersection number $L' \cdot B = 6$, we get that L' intersects B exactly at the three chosen cusps, and transversally. This means that, w.l.o.g., B lies in the upper halfplane: in fact, since B is rational and irreducible, then its real part $B(\mathbb{R})$ is homeomorphic to S^1, in particular it is connected.

Observe moreover that none of the cusps is tangent to a fibre, since each cusp gives a contribution at least 3 to the local multiplicity of the discriminant of B, and this contribution becomes 4 if the cusp is tangent to the fibre: and the order of the discriminant is 12.

In fact, we get more from this calculation: the projection p has no further critical points on B.

It follows that the projective line with coordinate x is divided into 4 open intervals, such that the cardinality of the fibre of $p\colon B(\mathbb{R}) \to \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ varies alternatingly from 3 to 1.

On the intervals where we have 3 counterimages, it makes sense to talk about first, second and third branch (ordered according increasing value of the y-coordinate), on each interval it makes sense to talk about the highest and the lowest branch.

Whenever one moves on $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ and goes across a cusp lying on the x-axis, the highest branch continues to be the highest branch.

Since three of the cusps lie on the x-axis, we may assume that the fourth cusp is located at $x = \infty$, and the three cusps with $y = 0$ occur for $x = A, B, C$ where $A < B < C$. Then the highest branch over the interval $(-\infty, A)$ remains the highest branch on the whole real line by virtue of the previous remark. By compactness of $B(\mathbb{R})$ we get a connected component of $B(\mathbb{R})$ mapping to $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ homeomorphically, contradicting our previous assertion about the cardinalities of the fibres. □

Thus, if B is irreducible, we observe that B has arithmetic genus 4, and nonnegative geometric genus: hence the 'number of double points' δ is at most 4. But each point of type A_n contributes exactly $[\frac{n+1}{2}]$ double points. Therefore an elementary calculation shows that we get the conclusion of Theorem 1.3 unless the configuration of singular points is one of the following : $A_4 + 2A_2$, $A_3 + 2A_2$, or $3A_2 + A_1$, or $2A_2 + 2A_1$, or $A_2 + 3A_1$.

We are going now to exclude the first case by an argument similar to the one of Lemma 2.1, even if it could also be treated by the same methods used in Section 4.

Lemma 2.2. – *The real curve B cannot have 2 real A_2 singularities and an A_4 singularity.*

Proof. – We already know that B is irreducible and we argue as in Lemma 2.1, assuming that the three singular points lie on the horizontal x-axis $\{y = 0\} := L'$ in the plane and that, since $B(\mathbb{R})$ is homeomorphic to S^1, B and L' intersect exactly at the three chosen points, and transversally, hence $B(\mathbb{R})$ lies in the upper halfplane.

If none of the cusps is tangent to a fibre, since each cusp A_{2n} gives a contribution $2n + 1$ to the local multiplicity of the discriminant of B, and the order of the discriminant is 12, there is exactly another critical point for the restriction of the projection p to B, and the same argument as in Lemma 2.1 provides the same contradiction.

There remains the case where exactly one cusp is vertical, and there are no further critical points.

It follows that the projective line with coordinate x is divided into 3 open intervals, and the cardinality of the fibre of $p\colon B(\mathbb{R}) \to \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ must be equal to 1 on the two intervals neighbouring the vertical cusp. At the two other cusps the highest branch remains the highest, and we get the usual contradiction (since over the third interval we have three branches). □

In any case, regardless of the difference between A^+_μ and A^-_μ, we have reduced the problem to the exclusion of 7 configurations. For any of these configurations, we can suppose that all singular points are of type A^+_μ with $A^+_{\mu_l}$ globally separating for μ_l is odd. Indeed, if one of the points is not of this type, the sum $\sum(1 - \frac{1}{\mu_l+1})$ restricted to the remaining points if less than or equal to 2.

Summarizing, we get seven remaining configurations to be excluded:

1. $2A^+_3 + A^+_2$ (Section 5)
2. $A^+_3 + 2A^+_2$ (Section 4)
3. $A^+_3 + A^+_2 + 2A^+_1$ (Section 3)
4. $A^+_3 + 3A^+_1$ (Section 3)
5. $3A^+_2 + A^+_1$ (Section 4)
6. $2A^+_2 + 2A^+_1$ (Section 5)
7. $A^+_2 + 3A^+_1$ (Section 5)

3. Smoothings of Du Val Del Pezzo surfaces

We recall that our problem consists in giving an estimate concerning the configurations of certain singular points lying on a component of the topological normalization of a real Du Val Del Pezzo surface X. For this purpose, we want to understand as much as possible the topology of $X(\mathbb{R})$, and we do this by taking a global smoothing of X, and then using the known topological classification of smooth real Del Pezzo surfaces of degree 1.

The best strategy is to choose a global smoothing realizing certain local smoothings of the singularities chosen a priori. That this can be done for all choices of the local smoothings holds true by a generalization of the theorem of Brusotti which was proven in our preceding paper.

THEOREM 3.1 ([1, Th. 4.3]). – *Let X be a Du Val Del Pezzo surface. One can obtain, by a global small deformation of X, all the possible local smoothings of the singular points of X.*

PROPOSITION 3.2 (Global). – *Let X be a real smooth Del Pezzo surfaces of degree 1: then the real part $X(\mathbb{R})$ is diffeomorphic to one of the surfaces in the following list:*

- $\mathbb{P}^2(\mathbb{R}) \sqcup p\mathcal{S},\ p = 1, \dots, 4$;
- $\mathbb{P}^2(\mathbb{R}) \sqcup \mathcal{K}$;
- $\#^3\mathbb{P}^2(\mathbb{R}) \sqcup \mathcal{S}$;
- $\#^{2p+1}\mathbb{P}^2(\mathbb{R}),\ p = 0, \dots, 4$.

Here $\#^l\mathbb{P}^2(\mathbb{R})$ denotes the connected sum of l copies of the real projective plane, $\mathcal{K} = \mathbb{P}^2(\mathbb{R})\#\mathbb{P}^2(\mathbb{R})$ denotes the Klein bottle and $p\mathcal{S}$ denotes the disjoint union of p copies of the 2-sphere.

Proof. – It is the well-known classification of real smooth Del Pezzo surfaces, see e.g. [3]. $\square$

LEMMA 3.3 (Local). – *Consider a real singular point of a surface X of type A_μ^+, of local equation $z^2 = f(x, y)$ where f vanishes at the origin. Then for each case $\mu \in \{1, 2, 3\}$, there exist local smoothings X_ε with equation $z^2 = f_\varepsilon(x, y)$, such that $X_\varepsilon(\mathbb{R})$ is represented by one of the Figures 2, 3, or 4.*

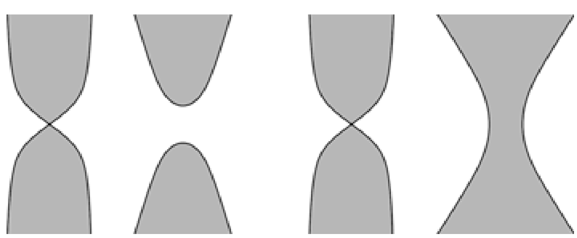


FIGURE 2. The *cut* and the *cylinder* smoothings of the node A_1^+.

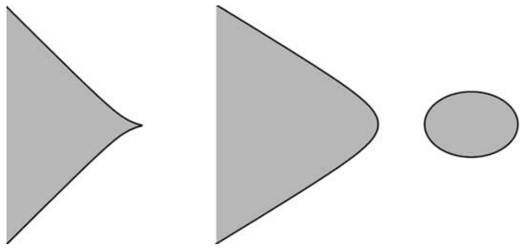


FIGURE 3. The +*sphere* smoothing of the cusp A_2^+.

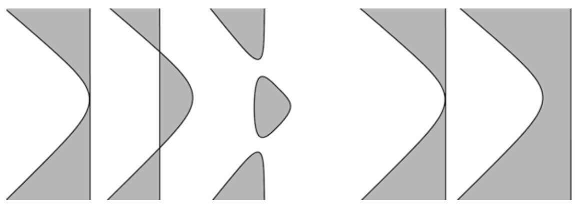


FIGURE 4. The *cut+sphere* and the *cylinder* smoothings of the tacnode A_3^+.

3.1. Topology of connected components

Let X be a real Du Val Del Pezzo surface of degree 1. Recall that X' denotes the singular elliptic surface obtained from X by blowing up a smooth point. We denote by $\overline{n}\colon \overline{X'(\mathbb{R})} \to X'(\mathbb{R})$ the topological normalization of the real part and we assume that there is a connected component M_0 of $\overline{X'(\mathbb{R})}$ whose image by $\overline{n}$ contains at least three singular points of X'. Furthermore, we assume that the singular points are only of type A_μ^+, with A_μ^+ globally separating for μ odd.

Let M_j, $j = 1, \dots c$ be the other components of $\overline{X'(\mathbb{R})}$ such that $\overline{n}(M_j)$ and $\overline{n}(M_0)$ intersect (in some singular point of $X'(\mathbb{R})$). Any singular point A_μ^+ of M_0 with μ odd is globally separating, while the ones with μ even are not, thus in particular the number c satisfies $1 \leq c \leq \#\{P \in \overline{n}(M_0) | P \text{ of type } A_\mu^+,\ \mu \text{ odd}\}$.

Let us denote by M_∞ the connected component of $\overline{X'(\mathbb{R})}$ which meets the section at infinity, i.e., $\overline{n}(M_\infty) \bigcap \Sigma_\infty \neq \varnothing$. In the proof of the main theorem we will often use the distinction between the cases $M_\infty = M_0$ and $M_\infty = M_j$ for some $j \neq 0$.

LEMMA 3.4. – *The component $M_\infty \subset \overline{X'(\mathbb{R})}$ of the topological normalization is a Klein bottle unless the elliptic fibration has two white returns (see Table 1). In the latter situation, $\overline{X'(\mathbb{R})}$ contains at most another component which is then a sphere.*

Proof. – If the fibre of the double covering $q'\colon X' \to \mathbb{F}_2$ over a real point P contains a real point, we shall say that P belongs to the region of positivity, which we denote by $\mathbb{F}_{2+}$.

The section Σ_∞ is part of the branch locus and is bilateral in $\mathbb{F}_2$.

Consider $U := \mathbb{F}_2 \setminus \Sigma_\infty$ which is an oriented $\mathbb{A}^1$-bundle over $\mathbb{P}^1$, and indeed homeomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{R}$. Hence we take corresponding coordinates $(x, y) \in \mathbb{P}^1_{\mathbb{R}} \times \mathbb{R}$ for the points of U.

We may assume without loss of generality that $(x, y) \in \mathbb{F}_{2+}$ for $y \gg 0$.

Consider now the function $\eta\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}, \eta(x) := \inf\{y \mid \{x\}\times[y, \infty[\subset \mathbb{F}_{2+}\}$. Therefore, if η is a continuous function on $\mathbb{P}^1(\mathbb{R})$, then we have $M_\infty = \mathcal{K}$.

For further use, we notice that:

LEMMA 3.5. – *Let $M \subset \overline{X'(\mathbb{R})}$ be a connected component of the topological normalization of $X'(\mathbb{R})$, and consider x as a function on the boundary of M: then the number of changes of monotonicity of x is even.*

Let $\Delta(x)$ be the discriminant of the elliptic fibre over x (i.e., the discriminant of the degree three polynomial in y whose zero set is the trisection). In view of Table 1 (p. 544), the only root of $\Delta(x)$ which can break the continuity of η corresponds to a white return $(A_0, e = -1)$.

If the lower part of the white return branch continues and meets as first critical point of $p\colon B(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ a point which contributes one change of monotonicity of x (that is, a black return, or a black node, or a tangent node, or a transversal cusp or tacnode), then we can topologically deform to the case where η is continuous. The flex and the tangent cusp are clearly irrelevant and if the first met critical point is a white node, we can perform a cut smoothing and pass to the next critical point. The only obstacle is then the case when we meet another white return singularity on the branch curve. In this case, one sees easily that there is another component D of $B(\mathbb{R})$ disjoint from the component D' containing the white return branches, hence $b_1(M_\infty) \geq 4$ (take the 4 cycles respective inverse images of the section at infinity Σ_∞, of D, and of two segments, one joining D' with Σ_∞, the other joining D' with D). Recall that the topological normalization $\overline{X(\mathbb{R})}$ of the real Del Pezzo surface X can be realised by a global smoothing of X, see [1, Lemma 4.4 and Theorem 4.3]. Thus the component of $\overline{X(\mathbb{R})}$ corresponding to M_∞ has $b_1 \geq 3$ and, by 3.2, either we have $\overline{X(\mathbb{R})} = \#^3\mathbb{P}^2(\mathbb{R}) \sqcup \mathcal{S}$ or $\overline{X(\mathbb{R})} = \#^{2p+1}\mathbb{P}^2(\mathbb{R})$ for some $p = 1, \ldots, 4$. □

REMARK 3.6. – *More generally, by 3.2, the real part of any global smoothing X'_ε of X', including the case when $X'_\varepsilon(\mathbb{R}) = \overline{X'(\mathbb{R})}$, is diffeomorphic to one of the surfaces in the following list,*

- $\mathcal{K} \sqcup p\mathcal{S},\ p = 1, \ldots, 4$;
- $\mathcal{K} \sqcup \mathcal{K}$;
- $\mathcal{K}\#\mathcal{K} \sqcup \mathcal{S}$;
- $\#^q\mathcal{K},\ q = 1, \ldots, 5$.

LEMMA 3.7. – *Let X be a real Du Val Del Pezzo surface of degree one, and let X' be the corresponding rational elliptic surface. Let X'_ε be a global smoothing of X'. Then we have the following estimates for the Betti numbers*

$$b_i(X'_\varepsilon(\mathbb{R})) := \operatorname{rank} H_i(X'_\varepsilon(\mathbb{R}), \mathbb{Z}/2)$$

- $b_0(X'_\varepsilon(\mathbb{R})) \geq 3 \Rightarrow b_1(X'_\varepsilon(\mathbb{R})) = 2$.
- $b_0(X'_\varepsilon(\mathbb{R})) \geq 2 \Rightarrow b_1(X'_\varepsilon(\mathbb{R})) \leq 4$.
- *In any case,* $b_0(X'_\varepsilon(\mathbb{R})) \leq 5$.

3.2. Exclusion of $A_3^+ + A_2^+ + 2A_1^+$ and $A_3^+ + 3A_1^+$

For each node of $\overline{n}(M_0)$ connecting $\overline{n}(M_0)$ with some $\overline{n}(M_j)$, we choose the cut smoothing if this point is the only singular point on $\overline{n}(M_j)$. Otherwise, we choose the cylinder smoothing. We do the +sphere smoothing for the cusp. For the tacnode, connecting $\overline{n}(M_0)$ with some $\overline{n}(M_j)$, we choose the cut+sphere smoothing if this point is the only singular point on $\overline{n}(M_j)$ or if we are in the last two cases in the next list. Otherwise, we choose the cylinder smoothing. Recalling that $b_1\left(\overline{X'(\mathbb{R})}\right) \geq 2$, we obtain the following inequalities for the Betti numbers of $X'_\varepsilon(\mathbb{R})$. The different cases are distinguished by the number c defined above.

- (3) $A_3^+ + A_2^+ + 2A_1^+$
  $c = 3$: $b_0 \geq 6$;
  $c = 2$: $b_0 \geq 3$, and $b_1 \geq 4$;
  $c = 1$: $b_0 \geq 2$, and $b_1 \geq 6$.
- (4) $A_3^+ + 3A_1^+$
  $c = 4$: $b_0 \geq 6$;
  $c = 3$: $b_0 \geq 3$, and $b_1 \geq 4$;
  $c = 2$: (cut+sphere smoothing for the tacnode) $b_0 \geq 3$, and $b_1 \geq 4$.
  $c = 1$: (cut+sphere smoothing for the tacnode) $b_0 \geq 2$, and $b_1 \geq 6$.

In each case, these inequalities contradict Lemma 3.7. Thus cases (3) and (4) are excluded.

4. The Euler number of an elliptic fibration

Recall that X' is a singular surface obtained from the singular degree 1 Del Pezzo surface X by blowing up a smooth point. It is a ramified double covering of the Hirzebruch surface $\mathbb{F}_2$ whose branch locus is the union $\Sigma_\infty \cup B$ where B is a trisection of the ruling $p\colon \mathbb{F}_2 \to \mathbb{P}^1$. The composition $X' \to \mathbb{F}_2 \to \mathbb{P}^1$ is a real elliptic fibration and $\Delta(x)$ denotes the discriminant of the elliptic fibre over x (i.e., the discriminant of the degree three polynomial in y whose zero set is the trisection). Table 1 gives a local topological description of the fibration over a neighbourhood of a real zero of Δ, in terms of two basic numerical invariants, namely the multiplicity of the zero of Δ, and the Euler number of the real part of the singular fibre of the elliptic surface. The table considers only the singular points that we have to deal with, and introduces a name for each case, which will be used in the course of the forthcoming proofs. Observe finally that, in drawing as black the region of positivity, we have used the convention introduced in Lemma 3.4. Finally, a point of type A_0 is here a smooth point of B which is a critical point for the restriction of p to B.

We use now Table 1 in order to proceed with our case by case exclusion.

4.1. Exclusion of $A_3^+ + 2A_2^+$ and $3A_2^+ + A_1^+$

- (2) $A_3^+ + 2A_2^+$,
  Here $c = 1$, and $\overline{n}(M_1), \overline{n}(M_0)$ meet in the tacnode A_3^+. By doing the +sphere smoothing for each cusp and the cut+sphere smoothing for the tacnode, we get

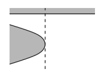
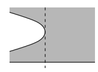
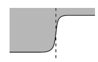
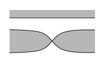
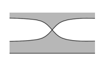
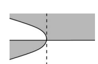
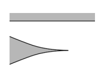
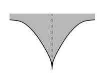
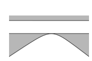
Type	Fibre type	Picture	Multiplicity of Δ	Euler number e
A_0	black return		1	1
A_0	white return		1	-1
A_0	flex		2	0
A_1^+	black node		2	1
A_1^+	white node		2	-1
A_1^+	tangent node		3	0
A_2^+	transversal cusp		3	1
A_2^+	tangent cusp		4	0
A_3^+	tacnode		4	1

TABLE 1. Singular points of the elliptic fibration and contributions to the Euler number.

at least 5 connected components, hence a Klein bottle and 4 spheres by 3.7, and $\overline{X'(\mathbb{R})} = \mathcal{K} \sqcup \mathcal{S}$. We conclude that $e(X'(\mathbb{R})) = 1$.

If $M_0 \neq M_\infty$ (thus $M_0 = \mathcal{S}$ and $M_1 = \mathcal{K}$), the cusps are transversal. Since the total multiplicity of Δ is 12, the fibration has at most two real A_0 singular fibers. But any white return stays on $\overline{n}(M_\infty)$. Then on the boundary of $\overline{n}(M_0)$ the monotonicity of the function x has three changes, a contradiction.

If $M_0 = M_\infty$, then the component $M_1 = \mathcal{S}$ must contain a black return (recall that the two cusps belong to $\overline{n}(M_0)$). The existence of the tacnode on $\overline{n}(M_\infty)$ forces a white return. The contributions to the multiplicity of Δ are then $4+3+3+1+1 = 12$ which implies that the two cusps are transversal. The contributions to the Euler characteristic of $X'(\mathbb{R})$ are then $1+1+1+1-1 = 3$, a contradiction.

Thus case (2) is excluded.

- (5) $3A_2^+ + A_1^+$,

  Here also $c = 1$ and the same argument as above shows that $\overline{X'(\mathbb{R})} = \mathcal{K} \sqcup \mathcal{S}$ and $e(X'(\mathbb{R})) = 1$.

  If $M_0 \neq M_\infty$, the three cusps are transversal. By Lemma 3.5, if we had a white node, we would have a black return on $\overline{n}(M_0)$, thus $e(X'(\mathbb{R})) = 3-1+1 = 3$, a contradiction. If we did not have a white node, then $e(X'(\mathbb{R})) \geq 3+0-1 = 2$, a contradiction again.

  If $M_0 = M_\infty$, then the node on $\overline{n}(M_1)$ forces a black return. Since M_1 is a sphere, the contributions to the multiplicity of Δ impose that the fibration has at most one real A_0 singular fiber and then that all singular points are of transversal type. Whence $e(X'(\mathbb{R})) \geq 3-1+1 = 3$, a contradiction.

  Thus case (5) is excluded.

5. The Euler number of a real singular Del Pezzo surface

Recall that X' is a singular rational elliptic surface obtained from the Del Pezzo surface X by blowing up a smooth point; and that a singular point A_μ^+ is real analytically equivalent to $x^2 + y^2 - z^{\mu+1} = 0,\ \mu \geq 1$.

Let us denote by $S' \to X'$ the minimal resolution of singularities.

Definition 5.1. – *In this paper, we define, for a real variety X, $\rho(X)$ to be the Picard number of the surface X* over $\mathbb{R}$. *It must not be confused with the Picard number of the complexification $X_\mathbb{C}$ of X. We have always $\rho(X) \leq \rho(X_\mathbb{C})$ but, generally, $\rho(X) < \rho(X_\mathbb{C})$.*

Lemma 5.2. – *Suppose that the singularities of X (and then of X') are only of type A_μ^+. Then, denoting by $\#A_\mu^+$ the number of singular points which are of type A_μ^+, we have:*

$$2\rho(S') + e\,(S'(\mathbb{R})) - 2\rho(X') - e\,(X'(\mathbb{R})) = \sum_\mu \mu \cdot (\#A_\mu^+)\ .$$

Proof. – A local computation shows that

$$\rho(S') - \rho(X') = \sum_{\mu \text{ odd}} \left(1 + \frac{\mu-1}{2}\right) + \sum_{\mu \text{ even}} \frac{\mu}{2}$$

and $e\,(S'(\mathbb{R})) - e\,(X'(\mathbb{R})) = -\#\{A_\mu,\ \mu \text{ odd}\}$. □

Lemma 5.3. – *Let X be a real Du Val Del Pezzo surface of degree 1. Suppose that $\rho(X) \leq 2$. Suppose moreover that the singularities are only of type A_μ^+, $\mu \in \{1,2,3\}$. Then we get for the rational elliptic surface X':*

$$e(X'(\mathbb{R})) = (8 \quad or \quad 6) - \sum_{\mu=1}^{3} \mu(\#A_\mu^+)\ .$$

Proof. – Denote by λ the Comessatti characteristic of S' given by $2\lambda = b_*(S'(\mathbb{C})) - b_*(S'(\mathbb{R}))$ (see [14, Chap. I], and recall that our Betti numbers are taken with coefficients $\mathbb{Z}/2$).

The nonsingular rational elliptic surface S' has total Betti number $b_*(S'(\mathbb{C})) = e(S'(\mathbb{C})) = 12$.

Moreover, for a nonsingular surface S' with $p_g(S') = 0$ and with $S'(\mathbb{C})$ simply connected, we have $b_1(S'(\mathbb{R})) = \rho(S') - \lambda$, see [9] or [14]. Since however $2b_1(S'(\mathbb{R})) = b_*(S'(\mathbb{R})) - e(S'(\mathbb{R}))$ we get

$$e(S'(\mathbb{R})) + 2\rho(S') = b_*(S'(\mathbb{R})) - 2b_1(S'(\mathbb{R})) + 2\rho(S') = b_*(S'(\mathbb{R})) + 2\lambda = b_*(S'(\mathbb{C})) = 12.$$

By our hypothesis on the Picard number of the singular Del Pezzo surface X, we have $2 \leq \rho(X') \leq 3$, thus the formula follows from Lemma 5.2. □

5.1. Exclusion of $2A_3^+ + A_2^+$, $2A_2^+ + 2A_1^+$ and $A_2^+ + 3A_1^+$

In the first case, the branch curve has 2 irreducible components which are rational. Indeed one of them is smooth rational and the other has genus 1 and one singular point. Furthermore the two irreducible components intersect in a real point, thus the real part of the branch curve is connected. In the last two cases, the branch curve B is irreducible, has genus 4, and has 4 singular points. Thus the curve is rational and its real part $B(\mathbb{R})$ is connected. It follows that every connected component M of the topological normalization has the property that

$$\overline{n}(M) \cap \overline{n}(M_0) \neq \varnothing.$$

Hence $(c+1)$ is the number of connected components of the normalization.

- (1) $2A_3^+ + A_2^+$, $\sum \mu(\#A_\mu^+) = 8$ and $e(X'(\mathbb{R})) = 0$ or -2 by Lemma 5.3.

  The total multiplicity of Δ is 12, thus there is at most one fibre A_0 and the contributions to the Euler characteristic are $1+1+1+1$, or $1+1+1-1$, or $1+1+0$ when the cusp is tangent to a fibre. Thus $e(X'(\mathbb{R}))$ would be greater than or equal to 2, a contradiction. Thus case (1) is excluded.

- (6) $2A_2^+ + 2A_1^+$, $\sum \mu(\#A_\mu^+) = 6$ and $e(X'(\mathbb{R})) = 2$ or 0.

  Here, the number of components of $\overline{X'(\mathbb{R})}$ such that $\overline{n}(M_j)$ and $\overline{n}(M_0)$ belong to the same connected component of $X'(\mathbb{R})$ satisfies $1 \leq c \leq 2$.

  Assume $c = 1$, and do the cylinder smoothing for the nodes, and the +sphere smoothing for the cusps. We obtain $b_0 = 3$, and $b_1 \geq 4$, a contradiction.

  Assume $c = 2$, then there are two cases: $M_0 = M_\infty$ or $M_0 \neq M_\infty$. The topological normalization has $3 = c + 1$ components, hence $\overline{X'(\mathbb{R})} = \mathcal{K} \sqcup 2\mathcal{S}$.

  Assume $M_0 \neq M_\infty$, then any cusp is transversal and yields a $(+1)$ contribution to the Euler number. For the component M_1, which is distinct from M_0 and from M_∞, we must have a black node, therefore on it there is also a black return. In order to get $e(X'(\mathbb{R})) \leq 2$, there must be a white return, and then we should have a white node to make $e(X'(\mathbb{R})) \leq 2$. But a white return is necessarily on $\overline{n}(M_\infty)$, and its existence implies the existence of other critical points at ∞, a contradiction.

  Assume $M_0 = M_\infty$, consider the two components not at ∞, M_1 and M_2. On them, a white node implies at least two black returns, while a black or tangent node implies at least one black return. Since Δ has degree 12, there are exactly two black returns (on each respective M_j) and two black nodes (on each respective M_j). At ∞, there are as critical points only the singular points, and these are transversal, whence we get 2 transversal cusps, thus $e(X'(\mathbb{R})) = 6$, a contradiction.

Thus the configuration $2A_2^+ + 2A_1^+$ does not exist.

- (7) $A_2^+ + 3A_1^+$, $\sum \mu(\#A_\mu^+) = 5$ and $e(X'(\mathbb{R})) = 3$ or 1.

  In this case, we have $1 \leq c \leq 3$.

  If $c = 1$ we have two components M_0, M_1 and each node on $\overline{n}(M_0)$ connects with $\overline{n}(M_1)$. We do 3 cylinder smoothings and the +sphere smoothing for the cusp. We obtain $b_0 = 2$, and $b_1 \geq 6$, a contradiction.

  If $c = 2$ we have three components M_0, M_1, M_2 and we let M_1 be the component such that there are two nodes on $\overline{n}(M_0)$ connecting with $\overline{n}(M_1)$. We perform 2 cylinder smoothings at these nodes. For the remaining node connecting $\overline{n}(M_0)$ with $\overline{n}(M_2)$, we choose the cut smoothing. We do the +sphere smoothing for the cusp. This gives $b_0 = 3$, and $b_1 \geq 4$, a contradiction.

  Assume $c = 3$, and take the normalization. There are 4 components, whence they are $\overline{X'(\mathbb{R})} = \mathcal{K} \sqcup 3\mathcal{S}$ with $\mathcal{K} = M_\infty$. Notice in particular that $e(X'(\mathbb{R})) = 3$.

  If $M_0 = M_\infty$, assume that we have a black node. For the corresponding $\mathcal{S}$ component this requires a black return. For a white node, we need at least two black returns. For a tangent node, one needs one black return. However, the number of A_0 fibres is ≤ 3, thus there is no white node, and we have exactly 3 black returns. The contribution to the Euler number is then $e(X'(\mathbb{R})) \geq 4$. This is a contradiction which excludes the case $M_0 = M_\infty$.

  If $M_0 \neq M_\infty$, the cusp is transversal hence it contributes 1 to the Euler number. Consider the two nodes not involving the component M_∞. Necessarily they are black nodes since the other two types of nodes involve the component M_∞. These singularities each involve a black return as before. We get a contribution 5 to $e(X'(\mathbb{R})) = 3$. Hence the remaining node and return must contribute twice a (-1). A (-1) contribution is white and involves the component M_∞. But because the white return gives for the boundary of $\mathbb{F}_{2+} \cap M_\infty$ some x for which the degree is 3, and others for which it is 1, there must be another critical point on $\mathbb{F}_{2+} \cap M_\infty$, a contradiction.

  Thus case (7) is excluded. This concludes the proof of Theorem 1.3. $\square$

We end this section with the

Proof of Theorem 0.2. – First of all, the reduction from the case of a geometrically rational Du Val surface to the case of a Du Val Del Pezzo surface of degree 1 is done precisely as in [1], Proposition 2.4. and the subsequent proof of Theorem 0.1. The same argument given in the proof of Cor. 0.2 ([1], end of the third section) shows that it suffices to consider the singular points of type A_μ^+ which are globally separating when locally separating. Finally, by Lemma 1.8 of [1], it remains only to check the case where X is a real Du Val Del Pezzo surface of degree 1 with $\rho(X) \leq 2$.

Then our assertion is exactly reduced to the main assertion of Theorem 1.3. $\square$

6. Real rationally connected threefolds

This short section explains how Theorem 0.2 implies the following.

LEMMA 6.1. – *Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Suppose that $W(\mathbb{R})$ is orientable. Then for each connected component $N \subset W(\mathbb{R})$, we have*

$$\sum_l \left(1 - \frac{1}{n_l(N)}\right) \leq 2 \, .$$

Proof. – Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Suppose that $W(\mathbb{R})$ is orientable. Let $N \subset W(\mathbb{R})$ be a connected component. Kollár proved (see also [1, 3.3, 3.4, and proof of Cor. 0.2]), that there is

1) a pair of birational contractions $c : W \to W'$, $r : X \to X'$, where

2) W' is a real projective 3-fold W' with terminal singularities such that $K_{W'}$ is Cartier along $W'(\mathbb{R})$,

3) X' is a Du Val surface

4) a rational curve fibration $f' \colon W' \to X'$ such that $-K_{W'}$ is f'-ample and with

5) $f' \circ c = r \circ f$.

Let N'' be the connected component of the topological normalization $\overline{W'(\mathbb{R})}$ such that N'' maps onto $c(\overline{n}(N))$.

The main property of this construction is that

6) $N'' = N' \#^{a'} \mathbb{P}^3(\mathbb{R})$.

Thanks to [8, Theorem 8.1], and [1, Proof of Cor. 0.2, end of Section 3], there is a small perturbation $g\colon N'' \to F$ of $f'|_{\overline{n}(N'')}$ such that $g|_{g^{-1}(F \setminus \partial F)}$ is a Seifert fibration, and an injection from the set of multiple fibres of $g|_{g^{-1}(F \setminus \partial F)}$ to the set of singular points of X' contained in $f'(\overline{n}(N''))$ which are of type A^+ and globally separating when locally separating. Under this injection, the multiplicity of the Seifert fibre equals $\mu + 1$ if the singular point is of type A^+_μ. Hence, the desired inequality follows from Theorem 0.2. □

7. Two-dimensional orbifolds

In this section we derive first some consequences from our main result on the components of the topological normalization of a geometrically rational Du Val surface. Then we construct a real smooth algebraic 3-fold whose real part contains a connected component which is Seifert fibred over the real projective plane, with two multiple fibres of multiplicity 3.

The first consequence is the following corollary, already mentioned in the introduction.

COROLLARY 7.1 (0.3). – Let W be a real smooth projective rationally connected 3-fold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable and let N be a connected component of $W(\mathbb{R})$. Then neither N nor N' can be endowed with a $\widetilde{\mathrm{SL}_2 \mathbb{R}}$ structure or with a $\mathbb{H}^2 \times \mathbb{R}$ structure whose base orbifold F is orientable.

Proof. – As we already mentioned,

1) if a 3-manifold possesses a geometric structure, then the corresponding geometry is unique,

2) every Seifert fibred manifold admits a geometric structure.

Moreover,

3) if N or N' can be endowed with a $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$ structure or with a $\mathbb{H}^2 \times \mathbb{R}$ structure, then N' is Seifert fibred and, by the cited theorem of Milnor, we have that N' is Seifert fibred by the given rational curve fibration.

Now, the six geometries for Seifert fibrations are distinguished by negativity, nullity or positivity of the Euler characteristic $\chi_{top}(F)$ of the base orbifold and by the vanishing or non vanishing of the Euler number of the Seifert bundle [13, Table 4.1]. In particular the $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$ and the $\mathbb{H}^2 \times \mathbb{R}$ geometry correspond exactly to the 'hyperbolic' case, where $\chi_{top}(F)$ is negative.

We conclude then by virtue of Theorem 0.1. □

PROPOSITION 7.1. – *Let N be as in Corollary 0.3. Suppose moreover that N admits a Seifert fibration with base orbifold F such that $|F|$ is orientable. Then the Euler characteristic $\chi_{top}(F)$ of the compact 2-dimensional orbifold F is nonnegative.*

Proof. – By [1, Theorem 4.3 and Lemma 4.4], the topological normalization $\overline{X(\mathbb{R})}$ can be realized as the real part of a real perturbation X_ε of X. Thanks to Comessatti's Theorem, an orientable connected component of $X_\varepsilon(\mathbb{R})$ is a sphere or a torus. In the last case, the Seifert fibration $N \to F$ has no singular fibre and F is a manifold, hence the Euler characteristic of F is zero. In the latter case, the Euler characteristic of F is positive. □

7.1. A Seifert fibration with base orbifold of hyperbolic type.

As announced in the introduction, we are going to construct a real smooth 3-fold W, fibred by rational curves over a Du Val Del Pezzo surface X , with the property that $W(\mathbb{R})$ is connected and enjoys the following properties:

i) $W(\mathbb{R})$ is orientable,

ii) $W(\mathbb{R})$ has a connected component which is Seifert fibred over a base orbifold F,

iii) F is non orientable and of hyperbolic type

iv) the Du Val Del Pezzo surface X is minimal over $\mathbb{R}$.

Our method of construction is based on a rather general procedure which produces Seifert fibrations as projectivized tangent bundles of Du Val surfaces, so we start with some easy lemmas, the first one being well known.

LEMMA 7.2. – *Let M be a real differentiable manifold. Then the tangent TM is always orientable, while $\mathbb{P}(TM)$ is orientable if $n := dim_{\mathbb{R}}(M)$ is even.*

Proof. – Let $p : TM \to M$ be the natural projection.

By the exact sequence $0 \to p^*(TM) \to T(TM) \to p^*(TM) \to 0$ we get that $\bigwedge^{2n}(T(TM)) \cong \bigwedge^n(p^*(TM))^{\otimes 2}$ is trivial.

Let $\pi : \mathbb{P}(TM) \to M$ be the natural projection.

Then by the exact sequences

$0 \to VT(\mathbb{P}(TM)) \to T(\mathbb{P}(TM)) \to \pi^*(TM) \to 0$,

(here VT denotes the subbundle of vertical vectors) and

$0 \to (\mathbb{R} \times \mathbb{P}(TM)) \to \pi^*(TM) \otimes U^{-1} \to VT(\mathbb{P}(TM)) \to 0$,

where U is the tautological line subbundle, we get

$\bigwedge^{2n-1} T(\mathbb{P}(TM)) \cong \bigwedge^n(\pi^*(TM))^{\otimes 2} \otimes U^{\otimes n}$, thus we have a trivial line bundle if n is even. □

Next, we consider the projectivized tangent bundle of Du Val surfaces with A_n singularities.

This 3-fold is simply obtained by glueing together the projectivized tangent bundle of the smooth part with the μ_{n+1} quotient

$$Y_n := (\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})/\mu_{n+1}$$

of the projectivized tangent bundle of the affine plane via the action of the $(n+1)$-th roots of unity induced by the action on $\mathbb{A}^2_{\mathbb{C}}$ yielding the quotient $A_n := \mathbb{A}^2_{\mathbb{C}}/\mu_{n+1}$.

LEMMA 7.3. – *Y_n has isolated singularities if and only if n is even. If n is even, these singularities are terminal quotient singularities $Z_n := \frac{1}{n+1}(1,-1,2)$ where the canonical divisor is not Cartier.*

Proof. – $\mu_{n+1} := \{\zeta | \zeta^{n+1} = 1\}$ acts on the affine plane $\mathbb{A}^2_{\mathbb{C}}$ by $(x,y) \mapsto (\zeta x, \zeta^{-1} y)$, whence its action on $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$,

$$(x,y)(\xi : \eta) \mapsto (\zeta x, \zeta^{-1} y)(\zeta \xi : \zeta^{-1} \eta).$$

If n is odd, $n+1 = 2k$ and ζ^k acts trivially on $\mathbb{P}^1_{\mathbb{C}}$; we see that we get a corresponding 1-dimensional singular locus, analytically isomorphic to $A_1 \times \mathbb{A}^1_{\mathbb{C}}$.

Assume now that n is even, so that each nontrivial group element has only two fixed points, namely, for $x = y = \xi = 0$, respectively for $x = y = \eta = 0$. At each point, passing to local coordinates, we see that we have a singularity of type Z_n, the quotient $Z_n := \mathbb{A}^3_{\mathbb{C}}/\mu_{n+1}$ by the action where $(x,y,z) \mapsto (\zeta x, \zeta^{-1} y, \zeta^2 z)$. This singularity is well known to be terminal (see [11]), and the Zariski canonical divisor K_Z there is not Cartier because the differential form $dx \wedge dy \wedge dz$ is not invariant, being multiplied by ζ^2 (only $(n+1)K_Z$ is Cartier). □

Over the real numbers, however, we have different forms of the A_n singularities, as we mentioned in the beginning.

The following two lemmas are essentially known by [8] (the case A_n^+ is contained in Theorem 8.1) but we state and prove them explicitly for the reader's benefit.

LEMMA 7.4. – *Let n be an even number and define Y_n^- to be the projectivized tangent bundle of a singularity of type A_n^-, and define analogously Y_n^+ for a singularity of type A_n^+. $Y_n^{\pm}$ has terminal isolated singularities and the real part $Y_n^-(\mathbb{R})$ is a PL-manifold of real dimension 3, while the real part $Y_n^+(\mathbb{R})$ is contained in the smooth locus of Y_n^+.*

The natural projection $Y_n^+(\mathbb{R}) \to A_n^+(\mathbb{R})$ is a Seifert fibration with a multiple fibre of multiplicity $(n+1)$ over the origin, while $Y_n^-(\mathbb{R}) \to A_n^-(\mathbb{R})$ is a topologically trivial S^1-bundle.

Proof. – We treat first the A_n^--case. We consider the real group scheme $\mu_{n+1}^- := \{\zeta | \zeta^{n+1} = 1\}$ which acts on the affine plane $\mathbb{A}^2_{\mathbb{R}}$ by $(x,y) \mapsto (\zeta x, \zeta^{-1} y)$, whence its action on $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$,

$$(x,y)(\xi : \eta) \mapsto (\zeta x, \zeta^{-1} y)(\zeta \xi : \zeta^{-1} \eta)$$

is such that each nontrivial group element has only two fixed points, namely, the point where $x = y = \xi = 0$, respectively the one where $x = y = \eta = 0$. At each point, passing to local coordinates, we see that we have a singularity of type Z_n^-, the quotient $Z_n^- := \mathbb{A}^3_{\mathbb{R}}/\mu_{n+1}^-$ by the action where $(x,y,z) \mapsto (\zeta x, \zeta^{-1} y, \zeta^2 z)$.

Let us now observe that Z_n^- sits inside a Galois sandwich

$$\mathbb{A}^3_{\mathbb{R}} \xrightarrow{\psi_2} Z_n^- \xrightarrow{\psi_1} \mathbb{A}^3_{\mathbb{R}}$$

where ψ_2 is the quotient morphism and the composition $\Phi := \psi_1 \circ \psi_2$ is given by

$$\Phi(x,y,z) := (x^{n+1}, y^{n+1}, z^{n+1})$$

(the coordinates of ψ_2 are just a set of invariant monomials including $x^{n+1}, y^{n+1}, z^{n+1}, xy, y^2 z$). Since Φ induces a homeomorphism $\Phi(\mathbb{R})\colon \mathbb{A}^3_{\mathbb{R}} \to \mathbb{A}^3_{\mathbb{R}}$, our claim is established if we show that in the real part of the sandwich

$$\mathbb{R}^3 \xrightarrow{\psi_2(\mathbb{R})} Z_n^-(\mathbb{R}) \xrightarrow{\psi_1(\mathbb{R})} \mathbb{R}^3$$

the polynomial map $\psi_2(\mathbb{R})$ is surjective.

Take a point $P \in Z_n^-(\mathbb{R})$: since it maps under $\psi_1(\mathbb{R})$ to $\mathbb{R}^3$, there exist a real point $(x,y,z) \in \mathbb{R}^3$ and elements $\zeta_i \in \mu_{n+1}$, for $i = 1,2,3$, such that $P = \psi_2(\zeta_1 x, \zeta_2 y, \zeta_3 z)$. Since however $\zeta_1 x \zeta_2 y \in \mathbb{R}$ and $(\zeta_2 y)^2 \zeta_3 z \in \mathbb{R}$, we get: $\zeta_1 \zeta_2 \in \mathbb{R}$, $(\zeta_2)^2 \zeta_3 \in \mathbb{R}$. Since $n+1$ is odd, then $\zeta_2 = \zeta_1^{-1}$ and $\zeta_3 = \zeta_2^{-2} = \zeta_1^2$: we have thus proven that $P = \psi_2(x,y,z)$.

Similarly, we see that the quotient morphism $\mathbb{R}^2 \to A_n^-(\mathbb{R})$ is a homeomorphism. Hence, the product fibration $\mathbb{R}^2 \times \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ descends to a topologically trivial S^1-bundle over $A_n^-(\mathbb{R})$.

The case of the A_n^+-case is simpler but more interesting. The action of $\mu_{n+1}(\mathbb{C}) := \{\zeta \in \mathbb{C} | \zeta^{n+1} = 1\}$ on the affine plane $\mathbb{A}^2_{\mathbb{C}}$ is given by

$$(x+iy, x-iy) \mapsto (\zeta(x+iy), \zeta^{-1}(x-iy)).$$

The action is defined over $\mathbb{R}$ since

$$(x,y) \mapsto (Re(\zeta)x - Im(\zeta)y, Im(\zeta)x + Re(\zeta)y),$$

and it defines an action of the real group scheme $\mu_{n+1}^+ := \{(a,b) | (a+ib)^{n+1} = 1\}$ on $\mathbb{A}^2_{\mathbb{R}}$ given by

$$(x,y) \mapsto (ax - by, bx + ay).$$

The ring of real invariant polynomials is generated, if we set $P := (x+iy)^{n+1}$, by $u := (P + \bar{P}), v := \frac{1}{i}(P - \bar{P}), w := (x^2 + y^2)$, which satisfy the equation of A_n^+, $u^2 + v^2 = w^{n+1}$.

The cyclic group stabilizes $\mathbb{R}^2$, and the origin is the only fixed point, while the action on $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ has no real fixed points, hence $Y_n^+(\mathbb{R}) \to A_n^+(\mathbb{R})$ is a Seifert bundle and the multiplicity over the origin is $n+1$. □

REMARK 7.5. – *As a consequence of the previous lemma, given any real Du Val surface X with only A_n^+ singularities with n even, the projectivized tangent bundle of X, $W' := \mathbb{P}(TX)$ is a 3-fold with terminal singularities, such that*

i) *the real part $W'(\mathbb{R})$ is contained in the smooth locus of W',*

ii) *$W'(\mathbb{R})$ is orientable.*

The previous remark allows us to construct the desired real 3-fold, thus proving Theorem 0.4.

Observe that in the course of the proof we construct some interesting examples of real Del Pezzo surfaces of degree 1 such that the corresponding Bertini involution on the plane (the involution associated to the bianticanonical map) is conjugate to a linear transformation.

Proof of Theorem 0.4. – We begin by constructing a family of weak Del Pezzo surfaces as the blow-up of the real projective plane in 8 real points. We obtain then a family of Del Pezzo surfaces Y, having two real A_2^- singularities, and two real and non isolated A_1 singularities. For certain values of the parameters, once we represent the Del Pezzo surface Y as the double cover of the quadric cone Q branched on the vertex of the cone and on a real branch curve B, then the two A_1 points give rise to two isolated real points of the real part $B(\mathbb{R})$ of the branch curve.

Using then the generalization of Brusotti's theorem given in Theorem 4.3 of [1], we can take a small deformation which leaves unchanged the two real A_2^- singularities, but deforms the two A_1 points replacing the two isolated points of $B(\mathbb{R})$ by two small ovals.

We obtain then a real Del Pezzo surface Z of degree 1 with exactly two real A_2^- singularities, and our desired real Del Pezzo surface X is finally constructed as the same complex surface Z, but with a new real structure $\sigma' := \sigma \circ i$, where σ is the real involution of Z, and $i : Z \to Z$ is the *Bertini involution*, the covering involution of the bianticanonical morphism, yielding Z as a double cover of the quadric cone Q.

In terms of this last representation, this simply amounts to exchanging the region of positivity with the region of negativity. For this reason, X has now two A_2^+ singularities, and the inside of the two ovals are now regions of positivity; we conclude that the connected components of $X(\mathbb{R})$ consist of two spheres S^2, and of a component homeomorphic to the real projective plane, and containing the two A_2^+ singularities.

In order to provide the defining equations for these Del Pezzo surfaces, consider now the plane $\mathbb{R}^2$ with coordinates (u,v) and in it the two pairs of lines

$$L_+ \cup L_- := \{u^2 - v^2 = 0\}$$
$$L_1 \cup L_{-1} := \{u^2 - 1 = 0\}.$$

Each pair of lines shall yield a respective A_2 configuration on our Del Pezzo surfaces of degree 1. We choose in fact eight points in the plane, such that each of the four lines contains four of them, namely the set

$$\{(1,1),(1,-1),(-1,1),(-1,-1),(1+\delta,1+\delta),(-1-\delta,1+\delta),(1,1+\epsilon),(-1,1+\epsilon)\}.$$

We see easily that the configuration is symmetric with respect to the real involution i such that $i(u,v) = (-u,v)$, and this symmetry is responsible of the fact that there is a conic D containing the following symmetrical set of six points: $\{(1,1),(-1,1),(1+\delta,1+\delta),(-1-\delta,1+\delta),(1,1+\epsilon),(-1,1+\epsilon)\}$.

Similarly, there is a conic D' containing the following symmetrical set of six points: $\{(1,-1),(-1,-1),(1+\delta,1+\delta),(-1-\delta,1+\delta),(1,1+\epsilon),(-1,1+\epsilon)\}$.

We let $\tilde{Y}$ be the blow-up of the real projective plane in the above eight points. We claim that $\tilde{Y}$ is a weak Del Pezzo surface of degree 1, i.e., that its anticanonical divisor is nef (it is big since it clearly satisfies $K_{\tilde{Y}}^2 = 1$).

This claim follows right away from the fact that, if we take homogeneous coordinates (u,v,t) on $\mathbb{P}^2$, the system of cubics through the eight points is the pencil spanned by the two cubics

$$x_0 := (u^2-v^2)(v-(1+\epsilon)t),\ x_1 := (u^2-t^2)(v-(1+\delta)t),$$

whose proper transforms meet only (transversally) in the point 'at infinity' $v = t = 0$.

The bianticanonical morphism ϕ of $\tilde{Y}$ is the double covering of the quadric cone $Q = \mathbb{P}(1,1,2)$ given by (x_0, x_1, y_2), where

$$y_2 := (u^2-v^2)(u^2-t^2)((\delta v - \epsilon(1+\delta)t)^2 - (\delta-\epsilon)^2 u^2)$$

has as set of zeros the union of six lines and passes doubly through the 8 points, but does not vanish on the base point of the anticanonical pencil (hence, y_2 is not a linear combination of $x_0^2, x_0 x_1, x_1^2$).

The morphism ϕ clearly factors through the quotient of $\tilde{Y}$ by the involution i ($\phi(u,z,t) = \psi(u^2,v,t)$) and we have then a factorization $\phi = \psi \circ \pi$, where $\psi : \tilde{Y}/i \to Q$, and π is the quotient projection. Hence it follows that ψ contracts the images under π of the (-2)-curves on $\tilde{Y}$, and that the branch curve $B \subset Q$ is the image under ϕ of the projective line $u = 0$.

The branch curve B is irreducible of arithmetic genus 4, and it has 4 singular points, corresponding to the blow-down of the curves $D, D', L_+ \cup L_-, L_1 \cup L_{-1}$: hence we conclude that the only (-2)-curves on $\tilde{Y}$ lie on the corresponding fibres of the anticanonical pencil, and a direct inspection shows that there are no other (-2)-curves on $\tilde{Y}$.

We want now to find a choice of the parameters such that the curves D, D' are real and do not intersect the line $u = 0$, thereby yielding two real isolated double points of the branch curve $B(\mathbb{R})$.

These conditions lead to some inequalities holding among δ, ϵ, but a simple solution is obtained choosing

$$\delta = 1, \epsilon = -1.$$

For this choice we get

$$x_0 = (u^2-v^2)v,\ x_1 = (u^2-t^2)(v-2t), y_2 = (u^2-v^2)(u^2-t^2)((2t+v)^2-4u^2)$$

$$D = \{-2(u^2-t^2)+3v(v-t)=0\}, D' = \{-2(u^2-t^2)+v(v+t)=0.$$

Whence, the line $u = 0$ meets D and D' in two pairs of complex conjugate points.

Using now the plane representation of the quadric cone (a main tool in [1]) given by the rational map (x_0^2, x_0x_1, y_2), we see that the line $u = 0$ maps to $x := x_1/x_0, y :=$

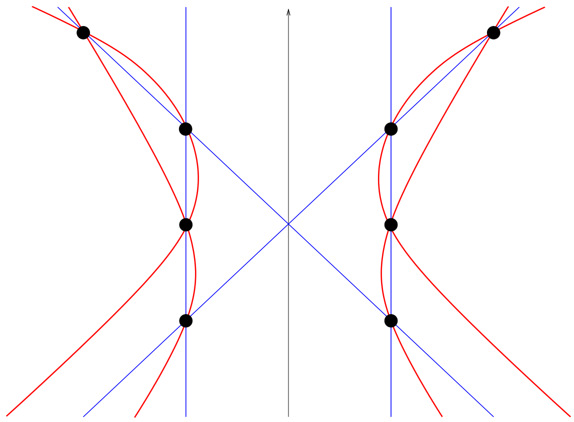


FIGURE 5.

y_2/x_0^2, and using the inhomogeneous coordinate v on the line $u = 0$, we obtain $x = \frac{v-2}{v^3}$, $y = \frac{(v+2)^2}{v^4} \geq 0$.

We get $y \geq 0$, $y = 0$ for $v = \infty, v = -2$, and the corresponding points of B are: $(0,0)(1/2,0)$.

Choosing t as local coordinate at $v = \infty$, we get $x = t^2(1-2t), y = t^2(2t+1)^2$, whence at $(0,0)$ B has an ordinary cusp with non vertical tangent.

The rational function x only ramifies for $v = 0, 3, \infty$, and for $v = 3$ we get the point $x = \frac{1}{27}, y = \frac{25}{81}$. For $v = 0$ we must go to the other chart of our Segre-Hirzebruch surface $\mathbb{F}_2$, and setting $x' = x_0/x_1, y' = y_2/x_1^2$, we get the parametrization $x' = \frac{v^3}{v-2}, y' = \frac{v^2(v+2)^2}{(v-2)^2}$, showing that the point $x' = y' = 0$ of B is an ordinary cusp with vertical tangent.

Observe however that, since on our surface $\tilde{Y}$

$$y = v^{-2}(u^2 - v^2)^{-1}(u^2 - 1)((2+v)^2 - 4u^2),$$

and as $u \to \infty$ $y \to -\infty$, the image of $\tilde{Y}(\mathbb{R})$ (region of positivity for the double cover) is the one beneath the curve $B(\mathbb{R})$. Moreover, the isolated real double points of the branch curve $B(\mathbb{R})$ are not isolated points of the real Del Pezzo surface $\tilde{Y}(\mathbb{R})$, whence they lie in the region of positivity. Since we blew up 8 real points in the plane, we have for the Euler characteristic (which we calculate through the $\mathbb{Z}/2$ Betti numbers): $e(\tilde{Y}(\mathbb{R})) = 1 - 8 = -7$.

After contracting the six (-2)-curves to the two A_2^- singularities and the two A_1-singularities, we obtain a Du Val Del Pezzo surface Y with $e(Y(\mathbb{R})) = -1$.

As already announced, we obtain a Du Val Del Pezzo surface Z with two A_2^- singularities using a small deformation on Y which realizes the following local deformations: no local deformation at the A_2^- singularities, and, at the A_1 singularities, which are of the form $z^2 = a^2 + b^2$, for suitable local coordinates (a, b) on the quadric cone Q, a small deformation of type $z^2 = a^2 + b^2 - r^2$, where r is a small real number. The existence of this global deformation is guaranteed by [1, Theorem 4.3]; in practice, some simple calculations show that this deformation is obtained by simply moving the 8 points on the four lines L_+, L_-, L_1, L_{-1} but breaking the symmetry $u \mapsto -u$.

This deformation produces on the one side a real Du Val Del Pezzo Z with $e(Z(\mathbb{R})) = -3$, on the other side it produces two real ovals in the branch curve $B(\mathbb{R})$, whose respective interiors are now in the region of negativity.

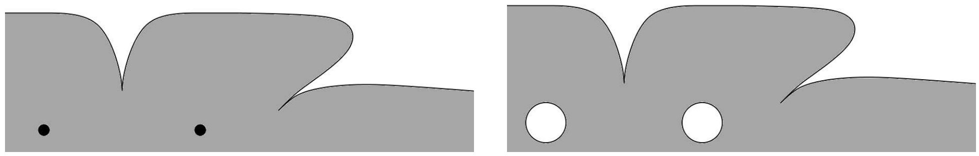


FIGURE 6.

We define now X as the same complex surface as Z, but with real structure $\sigma \circ i$, i being the Bertini involution. Now the region of positivity and negativity are exchanged, since the old real function z is replaced by $iz := w$; hence, instead of having $z^2 = f(x, y)$, we obtain $w^2 = -f(x, y)$.

Clearly $X(\mathbb{R})$ consists of a real projective plane with two A_2^+ singularities, together with two spheres. The ($\mathbb{R}$-) minimality of X (see e.g. [7]) is a consequence of the following

LEMMA 7.6. – *X has real Picard number $\rho(X) = 1$.*

Proof. – Let S be the minimal resolution of singularities of X, and observe that $\rho(S) = \rho(X) + 2$, since the blow-up of a real singular point of type A_2^+ yields a pair of complex conjugate (-2)-curves. Arguing as in Lemma 5.3 we calculate

$$\rho(S) = b_1(S(\mathbb{R})) + \lambda(S) = 1 + \lambda(S).$$

Here, by the definition of λ,

$$2\lambda(S) = b_*(S(\mathbb{C})) - b_*(S(\mathbb{R})) = 11 - (2 + 2 + 3) = 4.$$

Whence $\lambda(S) = 2$, $\rho(S) = 3$, $\rho(X) = 1$. This concludes the proof of the lemma. □

The remaining assertions follow from Lemma 7.4, and most of them were already mentioned in the previous remark: observe finally that a real projective plane with two points of multiplicity 3 is an orbifold of hyperbolic type, since $1 - \frac{2}{3} - \frac{2}{3} = -\frac{1}{3} < 0$. □

REMARK 7.7. – *János Kollár (e-mail communication) suggested an alternative way to construct such Del Pezzo surfaces.*

He constructs degree 4 curves $C \subset \mathbb{P}^2$, having 3 components C_1, C_2, C_3 such that C_1 has a cusp and moreover a hyperflex tangent at $p \in C_1$.

Assume without loss of generality that the cusp is at $(0, 1, 0)$ with tangent $z = 0$ and the hyperflex tangent line at $p = (0, 0, 1)$ is the x-axis. Thus p is the origin for affine coordinates (x, y). Then C is defined by an equation $y^2 + a(x)y + cx^4 = 0$ where $a(x)$ is a cubic polynomial. This gives 3 components iff the discriminant $a(x)^2 - 4cx^4$ has 6 real roots. This holds if $a(x)$ has 3 positive roots and c is small enough (depending on $a(x)$). Let us choose now $c < 0$. Then the cusp and the point p are on the same component since the discriminant $a(x)^2 - 4cx^4 > 0$ for x real and $x < 0$.

Consider now the degree 2 Del Pezzo surface Y which is the double cover of the plane branched on C, and blow up the point p' lying over p, obtaining a weak singular Del Pezzo surface S of degree 1.

The inverse image of the tangent line in Y splits into two complex conjugate components, which are tangent in p' ($w^2 - cx^4 = 0$ in local real holomorphic coordinates).

After the blow-up, we obtain two complex conjugate -2-curves, and contracting them we obtain finally a real Del Pezzo surface X of degree 1 with two A_2^+ singularities.

The real part $Y(\mathbb{R})$ has three connected components, all homeomorphic to a sphere. After the blow-up of p the connected component of $S(\mathbb{R})$ containing p' is homeomorphic to the real projective plane. Thus $X(\mathbb{R})$, which is homeomorphic to $S(\mathbb{R})$, has two smooth components homeomorphic to spheres, and a third component homeomorphic to the real projective plane, and containing the two A_2^+ singularities; this component yields an orbifold of hyperbolic type, and X has real Picard number 1 by Lemma 7.6.

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