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LOCAL GRADIENT ESTIMATES OF p-HARMONIC FUNCTIONS, $1/H$-FLOW, AND AN ENTROPY FORMULA

BY BRETT KOTSCHWAR AND LEI NI

ABSTRACT. – In the first part of this paper, we prove local interior and boundary gradient estimates for p-harmonic functions on general Riemannian manifolds. With these estimates, following the strategy in recent work of R. Moser, we prove an existence theorem for weak solutions to the level set formulation of the $1/H$ (inverse mean curvature) flow for hypersurfaces in ambient manifolds satisfying a sharp volume growth assumption. In the second part of this paper, we consider two parabolic analogues of the p-harmonic equation and prove sharp Li-Yau type gradient estimates for positive solutions to these equations on manifolds of nonnegative Ricci curvature. For one of these equations, we also prove an entropy monotonicity formula generalizing an earlier such formula of the second author for the linear heat equation. As an application of this formula, we show that a complete Riemannian manifold with nonnegative Ricci curvature and sharp L^p-logarithmic Sobolev inequality must be isometric to Euclidean space.

RÉSUMÉ. – Dans la première partie de cet article, nous établissons des estimées locales de gradient pour les fonctions p-harmoniques à l'intérieur et au bord, sur les variétés riemanniennes générales. Grâce à ces estimations et suivant une idée récente de R. Moser, nous obtenons un théorème d'existence de solutions faibles au sens de la formulation d'ensemble de niveau pour le flot $1/H$ (inverse de la courbure moyenne) des hypersurfaces dans les variétés ambiantes ayant la propriété de la croissance optimale du volume. Dans la deuxième partie, nous considérons deux types d'équations paraboliques pour les fonctions p-harmoniques et nous établissons une estimation optimale du type de Li-Yau pour les solutions positives pour ces équations sur les variétés à courbure de Ricci non-négative. Nous montrons aussi une formule de monotonie des entropies associées à ces équations. Cette formule généralise un résultat antérieur du deuxième auteur pour l'équation de la chaleur linéaire. Comme application, nous montrons que toute variété riemannienne complète à courbure de Ricci positive ou nulle et admettant une inégalité logarithmique L^p optimale est isométrique à l'espace euclidien.

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1. Introduction

Recently, in [26], an interesting connection between the p-harmonic functions and the $1/H$ hypersurface flow (also called the inverse mean curvature flow) was established. Let v be a positive p-harmonic function, i.e., a function satisfying

$$\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0, \tag{1.1}$$

and let $u \doteqdot -(p-1)\log v$. It is easy to see that u satisfies

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{p}. \tag{1.2}$$

The objective in [26] is to obtain a weak solution to the $1/H$ flow in the level-set formulation of [21] on a co-compact subdomain Ω of $\mathbb{R}^n$. Where it is sufficiently regular, such a solution u will satisfy

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|, \tag{1.3}$$

that is, (1.2) with $p = 1$. Moser's strategy in [26] is to produce weak solutions to (1.3) as limits of solutions $u^{(p)}$ to (1.2) as $p \searrow 1$. He obtains the $u^{(p)}$ via the above correspondence by using explicit barriers to first solve (1.1) with appropriate Dirichlet boundary conditions.

The key ingredient in the proof of the convergence and, consequently, to the success of this strategy, is a uniform gradient estimate on the $u = u^{(p)}$. The estimate is aided by the following observation. If

$$f \doteqdot |\nabla u|^2 \doteqdot (p-1)^2\frac{|\nabla v|^2}{v^2},$$

then, expressed in terms of f, (1.2) has the equivalent form

$$f^{p/2-1}\Delta u + \left(\frac{p}{2}-1\right) f^{p/2-2}\langle \nabla f, \nabla u\rangle = f^{p/2}. \tag{1.4}$$

Using this equation, the gradient estimate in [26] follows from a boundary estimate, which in turn is derived from a C^0 estimate by way of explicit barriers, certain integral estimates on $|\nabla v|$, and a Harnack inequality for p-harmonic functions on Euclidean spaces. The construction of the barriers in particular relies on the underlying Euclidean structure of the manifold. Some precedence for both the regularization procedure and the reduction of the convergence argument to a uniform gradient estimate can be found, for example, in [14], in another context.

In the first part of this paper, we derive interior and boundary gradient estimates on a general Riemannian manifold (M, g) via the gradient estimate technique of [25, 5] and use it to establish an existence result for the $1/H$-flow on a class of complete Riemannian manifolds. A new feature of our derivation of the local estimate is a nonlinear Bochner type formula relating the nonlinear operator with its linearization.

We first start with an interior/local estimate for positive p-harmonic functions, which is of independent interest.

THEOREM 1.1. – *Assume that v is a positive p-harmonic function on the ball $B(x_0,R)$, and that on the ball $B(x_0,R)$ the sectional curvature of (M,g), $K_M \geq -K^2$. Then for any $\epsilon > 0$,*

$$\sup_{B(x_0,\frac{R}{2})} |\nabla u|^2 \leq \frac{20(n-1)}{R^2(1-\epsilon)}\left(c_{p,n} + \frac{(n-1)b_{p,n}^2}{8\epsilon}\right) + C(n,K,p,R,\epsilon) \tag{1.5}$$

where

$$\begin{aligned}
b_{p,n} &= \left(\frac{2(p-1)}{n-1}-2\right),\\
c_{p,n} &= \left(\frac{2(p/2-1)^2}{n-1}+\frac{(p-2)p}{2}\right)_- + \left(\left(\frac{p}{2}+1\right)\max\{p-1,1\}\right),\\
C(n,K,p,R,\epsilon) &= \frac{(n-1)^2}{1-\epsilon}K^2+\frac{40(n+p-2)(n-1)}{1-\epsilon}\frac{1+KR}{R^2}\\
&\quad +\frac{20\max\{p-1,1\}(n-1)}{(1-\epsilon)R^2}.
\end{aligned}$$

Note that $b_{p,n}$, $c_{p,n}$ and $C(n,K,p,R,\epsilon)$ all stay finite as $p\to 1$. Hence Theorem 1.1 effectively gives an estimate for the gradient of the solution to $1/H$-flow. Also, if v is defined globally, by taking $R\to\infty$, then $\epsilon\to 0$, Theorem 1.1 implies that for any positive p-harmonic function v, $u=-(p-1)\log v$ satisfies

$$|\nabla u|^2 \le (n-1)^2K^2. \tag{1.6}$$

The constant $(n-1)^2$ is sharp in light of the results of [24, 30] for the case $p=2$. From the proof it is evident that when $p=2$, one can relax the assumption $K_M\ge -K^2$ to $\mathrm{Ric}\ge -(n-1)K^2$. A direct consequence of this is that if (M,g) is a complete manifold with nonnegative sectional curvature, then any positive p-harmonic function must be a constant. In fact, using the gradient technique of this paper, one can prove that any positive p-harmonic function on a manifold with nonnegative Ricci curvature and a sectional curvature lower bound must be a constant. Indeed, this result can be obtained by only assuming that the manifold has the so-called volume doubling property and satisfies a Poincaré type inequality (see, for example, [20] as well as Section 4 for more details). Hence it holds in particular on any Riemannian manifold with nonnegative Ricci curvature. On the other hand, it is not clear whether one can obtain an estimate on $|\nabla u|^2$ such as (1.6) under the weaker assumption of a Ricci curvature lower bound.

With the help of the interior gradient estimate, by constructing suitable (local) barrier functions we can establish the following boundary estimate:

For every $\epsilon>0$, there exists $p_0=p(\epsilon)>1$ such that if u satisfies equation (1.2) on Ω for some $1<p\le p_0$, then we have the estimate

$$|\nabla u|\le H_+ + \epsilon \tag{1.7}$$

where H denotes the mean curvature of $\partial\Omega$ and $H_+(x)=\max\{H(x),0\}$.

A similar boundary estimate was first proved in [21] for solutions to a different equation approximating (1.3). Our method is a modification of theirs.

With the help of the interior and boundary estimates above, and following the general scheme of [26], one can prove the existence of a *proper solution* (please see Section 4 for the definition) for a class of Riemannian manifolds which includes the asymptotically Euclidean manifolds considered in [21]. The following is a special case implied by our general existence theorem.

THEOREM 1.2. – *Let M be a complete Riemannian manifold such that its sectional curvature $K_M(x) \geq -k(r(x))$ for some nonincreasing function $k(t)$ with $\int_0^\infty tk(t)\,dt < \infty$. Let Ω be an end of M. Assume that for some $p_0 > 1$,*

$$\int_1^\infty \left(\frac{t}{V(\Omega\cap B(o,t))}\right)^{1/(p_0-1)} dt < \infty$$

and

$$\lim_{r\to\infty}\sup_{2r\leq t<\infty}\frac{t}{V(\Omega\cap B(o,t))} = 0.$$

Then (1.3) has a proper weak solution u with $\lim_{x\to\infty}|\nabla u|(x) = 0$.

The volume growth conditions in the theorem are optimal for the existence of *proper* solutions. This is shown in Section 4.

After the uniform gradient estimate, the key of the proof of the existence theorem is to construct certain bounded p-harmonic functions and obtain effective C^0-estimates of such p-harmonic functions at infinity which hold up as $p \to 1$. The estimate is somewhat delicate as we need to ensure that the limit as $p \to 1$ is a nonconstant function which tends to $+\infty$ at the spatial infinity. Here we rely crucially on an early work of Holopainen [20]. One may expect more general existence results, for example, for manifolds with a Laplacian whose L^2-spectrum has a positive lower bound [24]. However, it seems that perhaps more refined estimates may be needed.

In the second part of the paper, beginning in Section 5, we consider some nonlinear parabolic equations motivated by Theorem 1.1. First, we prove sharp gradient estimates of Li-Yau type for two nonlinear parabolic equations associated with (1.1). Both estimates proved are sharp in the case that M has nonnegative Ricci curvature and provide nonlinear generalizations of Li-Yau's estimate for the heat equation. One of these estimates, in the case $M = \mathbb{R}^n$, was obtained earlier in [11, 12] (see also [29]) in the study of the regularity of nonnegative weak solutions. We refer the readers to Section 5 and 6 for the more detailed discussions on these results.

A little surprisingly, we also obtain the following entropy formula for a class of nonlinear parabolic equations, generalizing the earlier formula for the linear heat equation in [27].

THEOREM 1.3. – *Let (M, g) be a complete Riemannian manifold. For any $p > 1$, let v be a positive solution to the equation*

$$\frac{\partial v^{p-1}}{\partial t} = (p-1)^{p-1}\operatorname{div}(|\nabla v|^{p-2}\nabla v) \tag{1.8}$$

satisfying $\int v^{p-1}d\mu = 1$. Then

$$\frac{d}{dt}\mathcal{W}_p(v,t) = -tp\int_M\left(\left|f^{p/2-1}\nabla_i\nabla_j u - \frac{1}{tp}a_{ij}\right|_A^2 + f^{p-2}R_{ij}u_iu_j\right)v^{p-1}\,d\mu$$

with $u = -(p-1)\log v$, $f = |\nabla u|^2$, $a_{ij} = g_{ij} - \frac{p-2}{p-1}\frac{v_iv_j}{|\nabla v|^2}$, $|T|_A^2 = A^{ik}A^{jl}T_{ij}T_{kl}$ for any 2-tensor T where (A^{ij}) is the inverse of (a_{ij}). The entropy

$$\mathcal{W}_p(v,t) = \int_M (t|\nabla\varphi|^p + \varphi - n)v^{p-1}\,d\mu$$

is defined with $v^{p-1} = \frac{1}{\pi^{n/2}(p^{*p-1}p)^{\frac{n}{p}}} \frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)} \frac{e^{-\varphi}}{t^{\frac{n}{p}}}$, *where* $p^* = \frac{p}{p-1}$, *and is assumed to be finite.*

The nonlinear heat equation (1.8) has been the object of some previous study. See, for example, [29] for a nice survey on the subject. When $p = 1$, the above theorem limits to the following form.

THEOREM 1.4. – *Let* (M, g) *be a complete Riemannian manifold. Let* u *be a solution to*

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{-1}\nabla u) + |\nabla u| = 0 \tag{1.9}$$

satisfying that $\int_M e^{-u}\, d\mu = 1$. *Then*

$$\frac{d}{dt}\mathcal{W}_1(u,t) = -t\int_M \left(\left| f^{-1/2}\nabla_\alpha\nabla_\beta u - \frac{1}{t}g_{\alpha\beta}\right|^2 + \frac{1}{t^2} + f^{-1}R_{ij}u_iu_j \right) e^{-u}\, d\mu$$

with $f = |\nabla u|^2$, $2 \le \alpha, \beta \le n$ *are with respect to the orthonormal frame* $e_1 = \frac{\nabla u}{|\nabla u|}, e_2, \cdots, e_n$. *The entropy*

$$\mathcal{W}_1(u,t) = \int_M (t|\nabla u| + u - n\log t - \log \omega_n - n)\, e^{-u}\, d\mu,$$

where ω_n *is the volume of unit ball in* $\mathbb{R}^n$, *is assumed to be finite.*

Note that (1.9) is the parabolic equation associated to (1.2), the level-set formulation of the $1/H$-flow equation; it has been studied in [19]. In view of the importance of the entropy formula of Perelman [28], we expect that the above result will also play a role in understanding the analytical properties of this nonlinear parabolic equation. In fact, as a simple consequence of the entropy formula, one can conclude that on manifolds with nonnegative Ricci curvature, *any positive ancient solution to* (1.8) *must be constant*. As it is in the case $p = 2$, the entropy $\mathcal{W}(\varphi, t)$ is closely related to the optimal L^p logarithmic Sobolev inequality (cf. [16, 8]) of Euclidean space. When $M = \mathbb{R}^n$, the optimal L^p logarithmic Sobolev inequality implies that $\mathcal{W}(\varphi, t) \ge 0$ for any v^{p-1} with $\int v^{p-1}\, d\mu = 1$. As another application, we prove the following statement:

Let (M, g) *be a complete Riemannian manifold with nonnegative Ricci curvature. Assume that the* L^p*-logarithmic Sobolev inequality* (6.22) *holds with the sharp constant on* M *for some* $p > 1$. *Then* M *is isometric to* $\mathbb{R}^n$.

This generalizes the $p = 2$ case which was originally proved in [1] (see also [27] for a different proof via the entropy formula for the linear heat equation).

The entropy formula coupled with Ricci flow is of special interest. This will be the subject of a forthcoming paper.

Finally we study the localized version of the sharp gradient estimates of Li-Yau type mentioned previously for manifolds with lower bound on the sectional curvature. This is carried out in the last section.

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2. The proof of Theorem 1.1

Let u, f be as above. Assume that $f > 0$ over some region of M. As in [26], define

$$\mathcal{L}(\psi) \doteqdot \operatorname{div}\left(f^{p/2-1}A(\nabla\psi)\right) - pf^{p/2-1}\langle\nabla u, \nabla\psi\rangle.$$

Here

$$A = \operatorname{id} + (p-2)\frac{\nabla u \otimes \nabla u}{f}$$

which can be checked easily to be nonnegative definite in general and positive definite for $p > 1$. Note that the operator $\mathcal{L}$ is the linearized operator of the nonlinear equation (1.2).

The first is a computational lemma.

Lemma 2.1. – *We have*

$$\mathcal{L}(f) = 2f^{p/2-1}\left(u_{ij}^2 + R_{ij}u_iu_j\right) + \left(\frac{p}{2} - 1\right)|\nabla f|^2 f^{p/2-2}. \tag{2.1}$$

Here u_{ij} is the Hessian of u. R_{ij} is the Ricci curvature of M.

Proof. – Direct calculation shows that

$$\begin{aligned}
\mathcal{L}(f) \;=\; & \left(\frac{p}{2} - 1\right) f^{p/2-2}|\nabla f|^2 + f^{p/2-1}\Delta f + (p-2)\Delta u\langle\nabla u, \nabla f\rangle f^{p/2-2} \\
& +(p-2)\left(\frac{p}{2} - 1\right) f^{p/2-3}\langle\nabla u, \nabla f\rangle^2 \\
& +(p-2)\left(u_{ij}f_iu_jf^{p/2-2} + f_{ij}u_iu_jf^{p/2-2} - \langle\nabla u, \nabla f\rangle^2 f^{p/2-3}\right) \\
& -pf^{p/2-1}\langle\nabla u, \nabla f\rangle.
\end{aligned}$$

Using

$$\Delta f = 2u_{ij}^2 + 2\langle\nabla\Delta u, \nabla u\rangle + 2R_{ij}u_iu_j$$

and combining terms we have that

$$\begin{aligned}
\mathcal{L}(f) \;=\; & f^{p/2-1}\left(2u_{ij}^2 + 2\langle\nabla\Delta u, \nabla u\rangle + 2R_{ij}u_iu_j\right) + \left(\frac{p}{2} - 1\right) f^{p/2-2}|\nabla f|^2 \\
& +(p-2)\Delta u\langle\nabla u, \nabla f\rangle f^{p/2-2} + (p-2)\left(\frac{p}{2} - 2\right) f^{p/2-3}\langle\nabla u, \nabla f\rangle^2 \\
& +(p-2)\left(u_{ij}f_iu_jf^{p/2-2} + f_{ij}u_iu_jf^{p/2-2}\right) - pf^{p/2-1}\langle\nabla u, \nabla f\rangle.
\end{aligned}$$

Taking the gradient of both sides of (1.4) and computing its product with ∇u, we have that

$$\begin{aligned}
\frac{p}{2}f^{p/2-1}\langle\nabla f, \nabla u\rangle = & \left(\frac{p}{2} - 1\right) f^{p/2-2}\Delta u\langle\nabla f, \nabla u\rangle + f^{p/2-1}\langle\nabla\Delta u, \nabla u\rangle \\
& + \left(\frac{p}{2} - 1\right)\left(\frac{p}{2} - 2\right) f^{p/2-3}\langle\nabla f, \nabla u\rangle^2 \\
& + \left(\frac{p}{2} - 1\right) f^{p/2-2}\left(f_{ij}u_iu_j + u_{ij}f_iu_j\right)
\end{aligned}$$

Combining the above two, we arrive at (2.1). □

Now let $\eta(x) = \theta\left(\frac{r(x)}{R}\right)$, where $\theta(t)$ is a cut-off function such that $\theta(t) \equiv 1$ for $0 \le t \le \frac{1}{2}$ and $\theta(t) \equiv 0$ for $t \ge 1$. Furthermore, take the derivatives of θ to satisfy $\frac{(\theta')^2}{\theta} \le 40$ and $\theta'' \ge -40\theta \ge -40$. Here $r(x)$ denotes the distance from some fixed x_0. Let $Q = \eta f$, which vanishes outside $B(x_0, R)$. At the maximum point of Q, it is easy to see that

$$\nabla Q = (\nabla\eta)f + (\nabla f)\eta = 0 \tag{2.2}$$

and

$$0 \geq \mathcal{L}(Q).$$

On the other hand, at the maximum point,

$$\begin{aligned}\mathcal{L}(Q) &= \eta \operatorname{div}\left(f^{p/2-1} A\nabla f\right) - \eta p f^{p/2-1}\langle \nabla u, \nabla f\rangle \\ &\quad + f^{p/2-1}\langle A(\nabla f), \nabla\eta\rangle + \operatorname{div}(f^{p/2}A\nabla\eta) - pf^{p/2}\langle\nabla u, \nabla\eta\rangle \\ &= \eta\mathcal{L}(f) - \left(\frac{p}{2}+1\right) f^{p/2}\frac{\langle A(\nabla\eta), \nabla\eta\rangle}{\eta} + f^{p/2}\operatorname{div}(A\nabla\eta) - pf^{p/2}\langle\nabla u, \nabla\eta\rangle.\end{aligned}$$

If $1 < p \leq 2$,

$$\frac{\langle A(\nabla\eta), \nabla\eta\rangle}{\eta} \leq \frac{|\nabla\eta|^2}{\eta}.$$

For $p \geq 2$ case we have that

$$\frac{\langle A(\nabla\eta), \nabla\eta\rangle}{\eta} \leq (p-1)\frac{|\nabla\eta|^2}{\eta}.$$

The next lemma estimates $\operatorname{div}(A\nabla\eta)$.

LEMMA 2.2. – *Assume that, on $B(x_0, R)$, the sectional curvature of (M, g) satisfies $K_M \geq -K^2$. Then, at the maximum point of Q,*

$$\begin{aligned}\operatorname{div}(A\nabla\eta) &\geq -80(n+p-2)\frac{1+KR}{R^2} - \frac{40\max\{p-1,1\}}{R^2} + (p-2)\langle\nabla u, \nabla\eta\rangle \\ &\quad + (p-2)\frac{p}{2}\frac{\langle\nabla u, \nabla\eta\rangle^2}{\eta f} - \left(\frac{p}{2}-1\right)\frac{|\nabla\eta|^2}{\eta}.\end{aligned} \tag{2.3}$$

Proof. – Direct computation yields

$$\begin{aligned}\operatorname{div}(A\nabla\eta) &= \Delta\eta + (p-2)\frac{\eta_{ij}u_iu_j}{f} + (p-2)\Delta u\frac{\langle\nabla u, \nabla\eta\rangle}{f} \\ &\quad -(p-2)\frac{\langle\nabla u, \nabla f\rangle\langle\nabla u, \nabla\eta\rangle}{f^2} + (p-2)\frac{u_{ij}u_i\eta_j}{f}.\end{aligned}$$

Now using (2.2), (1.4) and that $f_j = 2u_{ij}u_i$, we can eliminate Δu and ∇f to arrive at

$$\begin{aligned}\operatorname{div}(A\nabla\eta) &= \Delta\eta + (p-2)\frac{\eta_{ij}u_iu_j}{f} + (p-2)\langle\nabla u, \nabla\eta\rangle \\ &\quad + (p-2)\frac{p}{2}\frac{\langle\nabla u, \nabla\eta\rangle^2}{f\eta} - \left(\frac{p}{2}-1\right)\frac{|\nabla\eta|^2}{\eta}.\end{aligned}$$

We only need to estimate the first two terms, for which we compute

$$\eta_{ij} = \theta'\frac{r_{ij}}{R} + \theta''\frac{r_ir_j}{R^2}.$$

Using the Hessian comparison theorem [17], which states that $r_{ij} \leq \frac{1+Kr}{r}g_{ij}$, and the Laplacian comparison theorem, we have that

$$A_{ij}r_{ij} \leq (n+p-2)\left(\frac{1+Kr}{r}\right).$$

Noting that $\theta' = 0$ if $r \leq \frac{R}{2}$, we have that

$$\begin{aligned}\Delta\eta + (p-2)\frac{\eta_{ij}u_i u_j}{f} &= A_{ij}\eta_{ij} \\ &\geq -80(n+p-2)\frac{1+KR}{R^2} - 40\max\{p-1,1\}\frac{1}{R^2}.\end{aligned}$$

Taken together, these estimates prove the lemma. □

REMARK 2.3. – The above lemma is the only place we need to assume that the sectional curvature of M is bounded from below by $-K^2$. We expect that by some judicious choice of cut-off function, one may be able to prove Theorem 1.1 only assuming that the Ricci curvature is bounded from below.

To prove the theorem, we first estimate $\mathcal{L}(f)$ from below. We only need to consider the region where $f > 0$ for our purpose of estimating f from above. Choose a local orthonormal frame $\{e_i\}$ near any such given point so that at the given point $\nabla u = |\nabla u| e_1$. Then $f_1 = 2u_{j1}u_j = 2u_{11}u_1$ and for $j \geq 2$, $f_j = 2u_{j1}u_1$, which implies that

$$2u_{k1} = \frac{f_k}{f^{1/2}}. \tag{2.4}$$

Now (1.4) becomes

$$\sum_{j\geq 2} u_{jj} = f - \left(\frac{p}{2}-1\right)\frac{f_1 u_1}{f} - u_{11}.$$

Hence

$$\begin{aligned}\sum_{i,j=1}^{n} u_{ij}^2 &\geq u_{11}^2 + 2\sum_{j\geq 2} u_{j1}^2 + \sum_{j\geq 2} u_{jj}^2 \geq u_{11}^2 + 2\sum_{j\geq 2} u_{j1}^2 + \frac{1}{n-1}\left(\sum_{j\geq 2} u_{jj}\right)^2 \\ &= \frac{n}{n-1}u_{11}^2 + 2\sum_{j\geq 2} u_{j1}^2 + \frac{1}{n-1}f^2 + \frac{1}{n-1}\left(\frac{p}{2}-1\right)^2\frac{(f_1u_1)^2}{f^2} \\ &\quad - \frac{2}{n-1}\left(\frac{p}{2}-1\right)f_1u_1 - \frac{2}{n-1}fu_{11} + \frac{2}{n-1}\left(\frac{p}{2}-1\right)\frac{f_1u_1u_{11}}{f}.\end{aligned}$$

Using (2.4), we can replace all the second derivatives of u and arrive at

$$\begin{aligned}\sum_{i,j=1}^{n} u_{ij}^2 &\geq \frac{1}{n-1}f^2 + \frac{1}{n-1}\left(\frac{n}{4}+\frac{p}{2}-1\right)\frac{f_1^2}{f} + \frac{1}{2}\sum_{j\geq 2}\frac{f_j^2}{f} \\ &\quad + \frac{1}{n-1}\left(\frac{p}{2}-1\right)^2\frac{\langle\nabla f,\nabla u\rangle^2}{f^2} - \frac{p-1}{n-1}\langle\nabla f,\nabla u\rangle \\ &\geq \frac{1}{n-1}f^2 + a_{n,p}\frac{|\nabla f|^2}{f} + \frac{1}{n-1}\left(\frac{p}{2}-1\right)^2\frac{\langle\nabla f,\nabla u\rangle^2}{f^2} - \frac{p-1}{n-1}\langle\nabla f,\nabla u\rangle \qquad (2.5)\end{aligned}$$

where

$$a_{n,p} \doteqdot \min\{\frac{1}{n-1}\left(\frac{n}{4}+\frac{p}{2}-1\right), \frac{1}{2}\} \geq 0.$$

Hence by (2.1), (2.2) we have that

$$f^{p/2-1}\mathcal{L}(f) \geq \frac{2}{n-1}f^p + 2a_{n,p}f^{p-3}|\nabla f|^2 + \frac{2(p/2-1)^2}{n-1}f^{p-2}\frac{\langle\nabla\eta,\nabla u\rangle^2}{\eta^2}$$
$$+\frac{2(p-1)}{n-1}f^{p-1}\frac{\langle\nabla\eta,\nabla u\rangle}{\eta} - 2(n-1)K^2f^{p-1} + \left(\frac{p}{2}-1\right)f^{p-1}\frac{|\nabla\eta|^2}{\eta^2}. \tag{2.6}$$

Now combining the previous estimates, we have that

$$0 \geq f^{p/2-1}\eta^{p-1}\mathcal{L}(Q) \geq \frac{2}{n-1}Q^p + Q^{p-2}\left(\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2}\right)\langle\nabla u,\nabla\eta\rangle^2$$
$$+\left(\frac{2(p-1)}{n-1}-2\right)Q^{p-1}\langle\nabla u,\nabla\eta\rangle - \left(\left(\frac{p}{2}+1\right)\max\{p-1,1\}\right)Q^{p-1}\frac{|\nabla\eta|^2}{\eta}$$
$$-\left(2(n-1)K^2 + 20(n+p-2)\frac{1+KR}{R^2} + \frac{10\max\{p-1,1\}}{R^2}\right)Q^{p-1}.$$

Since

$$Q^{p-2}\left(\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2}\right)\langle\nabla u,\nabla\eta\rangle^2 \geq -\left(\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2}\right)_- Q^{p-1}\frac{|\nabla\eta|^2}{\eta}$$

and

$$\left(\frac{2(p-1)}{n-1}-2\right)Q^{p-1}\langle\nabla u,\nabla\eta\rangle \geq -\frac{2\epsilon}{n-1}Q^p - \frac{b_{p,n}^2(n-1)}{8\epsilon}Q^{p-1}\frac{|\nabla\eta|^2}{\eta}$$

with $b_{p,n} = \left(\frac{2(p-1)}{n-1}-2\right)$, we have that

$$0 \geq f^{p/2-1}\eta^{p-1}\mathcal{L}(Q) \geq \frac{2(1-\epsilon)}{n-1}Q^p - \frac{40}{R^2}Q^{p-1}\left(c_{p,n} + \frac{b_{p,n}^2(n-1)}{8\epsilon}\right)$$
$$-\left(2(n-1)K^2 + 80(n+p-2)\frac{1+KR}{R^2} + \frac{40\max\{p-1,1\}}{R^2}\right)Q^{p-1}$$

where

$$c_{p,n} = \left(\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2}\right)_- + \left(\left(\frac{p}{2}+1\right)\max\{p-1,1\}\right).$$

Here we have used $\frac{|\nabla\eta|^2}{\eta} \leq \frac{40}{R^2}$. Theorem 1.1 then follows easily from the above inequality.

3. Boundary estimate

Let $\Omega \subset M$ be an open subset such that Ω^c is compact and $\partial\Omega$ is C^∞. Again, for $p > 1$, we consider the p-harmonic equation:

$$\widehat{\Lambda}_p(v) := \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0 \tag{3.1}$$

Any positive solution v of (3.1) gives rise to a solution u of

$$\Lambda_p(u) := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - |\nabla u|^p = 0 \tag{3.2}$$

via the relationship $u = (1-p)\log v$.

Our primary objective in this section is to prove the following boundary estimate, which corresponds to a similar result in [21].

Proposition 3.1. – *For every $\epsilon > 0$, there exists $p_0 = p(\epsilon) > 1$ such that if u satisfies Equation* (3.2) *on Ω for some $1 < p \le p_0$, then we have the estimate*

$$|\nabla u| \le H_+ + \epsilon \tag{3.3}$$

where H denotes the mean curvature of $\partial\Omega$ and $H_+(x) = \max\{H(x), 0\}$.

We begin by recording the result of a basic computation.

Lemma 3.2. – *Suppose $\phi : (0,\infty) \to \mathbb{R}$ is a smooth function and $r(x) = d(x, x_0)$ for some fixed $x_0 \in M$. Fix $p \ge 1$. If $\Sigma = \mathrm{cut}(x_0)$, then for $x \in M \setminus \Sigma$, we have the following formula for $w(x) = \phi(r(x))$*

$$\begin{aligned}\nabla_i\nabla_j w &= \phi'\nabla_i\nabla_j r + \phi''\nabla_i r\nabla_j r\\ \widehat{\Lambda}_p(w) &= |\nabla w|^{p-2}\left(\Delta w + (p-2)\frac{(\nabla\nabla w)(\nabla w, \nabla w)}{|\nabla w|^2}\right)\\ &= |\phi'|^{p-2}\left(\phi'\Delta r + (p-1)\phi''\right) \end{aligned}\tag{3.4}$$

Now we prove a preliminary estimate that will assist us in the proof of Proposition 3.1.

Proposition 3.3. – *Suppose u is a solution to* (3.2) *on Ω for some $1 < p < n$ with $u = 0$ on $\partial\Omega$. Define*

$$\beta(x) := \sup\{\, r > 0 \mid \exists y \in \Omega^c \text{ with } B(y,r) \subset \Omega^c \text{ and } x \in \partial B(y,r)\,\},$$

$$\iota = \min_{x\in\Omega^c} \mathrm{inj}(x)$$

and put

$$R = \min\left\{\inf_{x\in\partial\Omega}\beta(x), \frac{\iota}{3}\right\}.$$

Choose $K \ge 0$ so that $\mathrm{Rc} \ge -(n-1)Kg$ *on*

$$S_{2R} := \{\, x \in M \mid d(x, \Omega^c) \le 2R\,\}$$

Then

$$|\nabla u| \le \frac{n-1}{R}\left(1 + 2\sqrt{K}R\right)\left(1 - 2^{\frac{n-p}{1-p}}\right)^{-1} \tag{3.5}$$

on $\partial\Omega$.

We should remark that in the case that M has nonnegative Ricci curvature, the above estimate can be sharpened to

$$|\nabla u| \le \frac{n-1}{R},$$

which generalizes the one proved in [26].

Proof. – Consider the function $\sigma : (0,\infty) \to \mathbb{R}$ defined by

$$\sigma(r) := \left(\frac{Re^{-\sqrt{K}(r-R)}}{r}\right)^{\frac{n-1}{p-1}}.$$

If we put

$$\phi(r) := 1 - \alpha\int_R^r \sigma(\rho)\,d\rho$$

for $\alpha > 0$ to be determined later, we have $\phi(R) = 1$ and

$$\frac{\phi''}{\phi'} = -\left(\frac{n-1}{p-1}\right)\left(\sqrt{K} + \frac{1}{r}\right). \tag{3.6}$$

Now let u be a solution of (3.2) for some $1 < p < n$. Fix $x_0 \in \partial\Omega$, and choose $y_0 \in \Omega^c$ such that $B(y_0, R) \subset \Omega^c$ and $x_0 \in \partial B(y_0, R)$. Setting $r(x) = d(y_0, x)$ and $\widetilde{w}(x) = \phi(r(x))$, we have, from (3.4) and (3.6)

$$\begin{aligned}\widehat{\Lambda}_p(\widetilde{w}) &= |\phi'|^{p-2}\,\phi'\left(\Delta r + (p-1)\frac{\phi''}{\phi'}\right)\\ &= |\phi'|^{p-2}\,\phi'\left(\Delta r - (n-1)\left(\sqrt{K} + \frac{1}{r}\right)\right)\end{aligned} \tag{3.7}$$

away from the cut locus of y_0. Since $\phi' < 0$, applying the Laplacian comparison theorem to (3.7) implies that $\widehat{\Lambda}_p(\widetilde{w}) \geq 0$, i.e., that $\widetilde{w}$ is a subsolution to (3.1) on $B(y_0, 2R) \setminus \{y_0\}$ (recall that $3R \leq \mathrm{inj}(y_0)$). Now choose α to be

$$\alpha := \left(\int_R^{2R} \sigma(\rho)\, d\rho\right)^{-1}. \tag{3.8}$$

Then we have $\widetilde{w}(x) = \phi\,(2R) \equiv 0$ on $\partial B(y_0, 2R)$ and $\widetilde{w}(x) \leq 1$ on $\partial\Omega$. Extend $\widetilde{w}$ to a subsolution on all of $M \setminus \{y_0\}$ by

$$w(x) := \begin{cases}\widetilde{w}(x) & \text{if } \ x \in B(y_0, 2R) \setminus \{y_0\}\\ 0 & \text{if } \ x \in B(y_0, 2R)^c.\end{cases}$$

Since the function $v = \exp(u/(1-p))$ satisfies $v \equiv 1$ on $\partial\Omega$ and $v > 0$ in Ω, it follows from the comparison principle that $w \leq v$ on Ω. In particular, we have the inequalities $0 < w \leq v$ on $\overline{B(y_0, (3/2)R)} \cap \Omega$, Thus if we define z by $z(x) := (1-p)\log w(x)$, z is a nonnegative supersolution to (3.2) on the annular region

$$A := A_{R, \frac{3R}{2}} := \overline{B\left(y_0, 3R/2\right)} \setminus B\left(y_0, R\right),$$

satisfying $z(x) > 0$. If we extend u to be identically 0 in Ω^c, we have $z \geq u$ on ∂A, and hence in all of A by the comparison principle. In particular, at x_0, we have, for all smooth paths $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x_0$ and $\gamma'(0) = -\nu(x_0)$ (where ν is the unit normal to $\partial\Omega$ at x_0 pointing into Ω^c),

$$\frac{u(\gamma(h)) - u(x_0)}{h} = \frac{u(\gamma(h))}{h} \leq \frac{z(\gamma(h))}{h} = \frac{z(\gamma(h)) - z(\gamma(x_0))}{h}. \tag{3.9}$$

Hence

$$|\nabla u|(x_0) \leq |\nabla z|(x_0).$$

It remains, then, just to estimate the gradient of z at x_0.

Recall from the definition of z that

$$|\nabla z|(x_0) = (p-1)\frac{|\phi'(R)|}{\phi(R)} = (p-1)\alpha. \tag{3.10}$$

Integrating by parts, we see

$$\int_R^{2R} \sigma(\rho)\, d\rho = \left(\frac{p-1}{n-1}\right)(R - 2R\sigma(2R)) - \sqrt{K}\frac{n-1}{n-p}\int_R^{2R} \rho\sigma(\rho)\, d\rho.$$

So

$$\left(1+2\left(\frac{n-1}{n-p}\right)\sqrt{K}R\right)\int_R^{2R}\sigma(\rho)d\rho \geq \int_R^{2R}\left(1+\left(\frac{n-1}{n-p}\right)\sqrt{K}\rho\right)\sigma(\rho)d\rho$$
$$= R\left(\frac{p-1}{n-p}\right)(1-2\sigma(2R)).$$

Since

$$2\sigma(2R) \leq 2^{\frac{n-p}{1-p}},$$

we have

$$\alpha = \left(\int_R^{2R}\sigma(\rho)\,d\rho\right)^{-1} \leq \frac{n-p}{(p-1)R}\left(1+2\frac{n-1}{n-p}\sqrt{K}R\right)\left(1-2^{\frac{n-p}{1-p}}\right)^{-1},$$

and the desired inequality (3.5) follows from (3.10). □

Now we turn to the proof of Proposition 3.1. We use the device in Lemma 3.4 of [21] to obtain a supersolution for the solution u, using the rudimentary control near the boundary we have already obtained in the previous proposition in concert with the interior gradient estimates of Theorem 1.1 to ensure that the growth of our supersolution is sufficiently rapid.

Proof of Proposition 3.1. – Let H denote the mean curvature of $\partial\Omega$ and set $H_+ = \max\{H, 0\}$. Choose a smooth function $\widetilde{w}$ with $\widetilde{w} \equiv 0$ on $\partial\Omega$ such that

$$H_+ < \frac{\partial\widetilde{w}}{\partial\nu} \leq H_+ + \epsilon \tag{3.11}$$

on $\partial\Omega$. Then $|\nabla\widetilde{w}| > 0$ and

$$\Lambda_1(\widetilde{w}) = \frac{\Delta\widetilde{w}}{|\nabla\widetilde{w}|} - \frac{(\nabla\nabla\widetilde{w})(\nabla\widetilde{w},\nabla\widetilde{w})}{|\nabla\widetilde{w}|^3} - |\nabla\widetilde{w}| = H - \frac{\partial\widetilde{w}}{\partial\nu} < 0$$

on $\partial\Omega$; indeed, for suitably small $\delta > 0$, both inequalities hold strictly on U_δ, which we define to be the components of $\{0 \leq \widetilde{w} < \delta\}$ containing $\partial\Omega$. Now introduce

$$w := \frac{\widetilde{w}}{1-\widetilde{w}/\delta}.$$

Then also $w \equiv 0$ on $\partial\Omega$ and $w(x) \to \infty$ as x approaches $\partial U_\delta \setminus \partial\Omega$. Moreover, a computation shows

$$\Lambda_1(w) = \Lambda_1(\widetilde{w}) + \left(1 - \frac{1}{(1-\widetilde{w}/\delta)^2}\right)|\nabla\widetilde{w}|,$$

so $\Lambda_1(w) < 0$ on U_δ as well. By Proposition 3.3, and the interior estimates in Theorem 1.1, for any precompact open set V containing Ω^c, there exists a constant $C = C(K_M, V)$ depending on the geometry of $\partial\Omega$ and the lower bound of the sectional curvature in V (but independent of p!) such that any u satisfying (3.2) with $u \equiv 1$ on $\partial\Omega$ satisfies $u(x) \leq C$ on V. In particular, if we choose such a $C = C(\delta)$ for U_δ, we see that $w \geq u$ for any such u on ∂U_δ. More precisely we can define $\widetilde{U}_C$ to be the components of $\{0 \leq w \leq C\}$ in U_δ. Clearly $w \geq u$ on $\partial\widetilde{U}_{C+1}$.

But we may also compute

$$\Lambda_p(w) = |\nabla w|^{p-1}\left(\Lambda_1(w) + (p-1)\frac{(\nabla\nabla w)(\nabla w,\nabla w)}{|\nabla w|^3}\right),$$

which shows that choosing p_0 sufficiently close to 1, depending on $C(\delta)$ (and implicitly on our original ϵ), as well as bounds on our (fixed) barrier w, we have that $\Lambda_p(w) < 0$ for all $1 < p \le p_0$ on $\widetilde{U}_{C+1}$. Therefore, it follows from the comparison principle for Equation (3.2) that $u \le w$ on $\widetilde{\widetilde{U}}_{C+1}$. Hence, recalling (3.11), we have

$$\frac{\partial u}{\partial \nu} \le \frac{\partial w}{\partial \nu} = \frac{\partial \widetilde{w}}{\partial \nu} \le H_+ + \epsilon$$

on $\partial\Omega$ as claimed. □

4. $1/H$-flow

The $1/H$ (or *inverse mean curvature*) flow is a parabolic evolution equation for hypersurfaces. Given an initial embedding $X_0 : N^{n-1} \to M^n$, the flow may be defined parametrically by $N_t = X_t(N)$ where

$$\frac{\partial X_t}{\partial t} = \frac{1}{H}\nu.$$

Here ν denotes the outward normal. Alternatively, the flow may be defined in terms of a level set formulation, in which case N_t is given by $\partial\{u < t\}$ for a u satisfying (1.3). A theory of weak solutions to (1.3) was established in [21], based on a variational principle involving the functional

$$J_u(w\,;K) \doteqdot \int_K (|\nabla w| + w|\nabla u|)\, d\mu$$

for any precompact subset $K \subset \Omega$; the reader is encouraged to consult this foundational paper of Huisken and Ilmanen for further details and the motivation behind this theory. In this section, our interest is in the general problem of the existence of weak solutions to (1.3), particularly in the case in which N_0 is the compact boundary of an end Ω of M.

Definition 4.1. – A function $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$ is called a weak solution of (1.3) if for every precompact set $K \subset \Omega$ and every $w \in C^{0,1}_{\mathrm{loc}}(\Omega)$ with $w = u$ in $\Omega \setminus K$, the inequality

$$J_u(u\,;K) \le J_u(w\,;K) \tag{4.1}$$

holds. A weak solution is called proper if $\lim_{x\to\infty} u(x) = +\infty$.

In [26], the functional

$$J^p_u(w\,;K) \doteqdot \int_K \left(\frac{1}{p}|\nabla w|^p + w|\nabla u|^p\right) d\mu$$

was introduced for every precompact set $K \subset \Omega$. It was shown that if v is p-harmonic and $u = -(p-1)\log v$, then

$$J^p_u(u\,;K) \le J^p_u(w\,;K) \tag{4.2}$$

for every $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying $w = u$ in $\Omega \setminus K$. By a compactness-type argument which originated in [21], it was also shown (p. 82 of [26]) that once one has a uniform estimate on $|\nabla u|$ that is independent of p as $p \to 1$, one can obtain a weak solution to (1.3) from the limit of u as $p \to 1$. To ensure that the solution one obtains is proper, one needs further estimates. In the case of [26], these are furnished by comparison with explicit solutions to (1.2) on $\mathbb{R}^n$. In our case, we derive suitable estimates from results of [20] (see also [23]).

First we need to recall the notions of p-Green's functions and p-nonparabolicity. Let $G(x, z)$ be the Green's function centered at $z \in M$. One can construct G via a compact exhaustion as in [23] for the case $p = 2$; this was done in [20]. To ensure that the Green's function so obtained is positive, one must impose conditions on the end Ω. For the statement of these conditions, we will need to recall the notion of *homogeneous* ends from [20], and the associated *volume doubling* (VD) condition and *weak Neumann Poincaré* (WNP) inequality.

Let $o \in M$ be a fixed point. A manifold M (or an end Ω of M) is said to have property (VD) if there exists $C_1 > 0$ such that for any $x \in M$ (or $\in \Omega$)

$$V(x, 2r) \leq C_1 V(x, r).$$

Here $V(x, r)$ denotes the volume of the ball $B(x, r)$. It is said to have property (WNP) if there exists $C_2(p) > 0$ such that

$$\frac{1}{V(x,r)} \int_{B(x,r)} |u - \bar{u}| \leq C_2 r \left(\frac{1}{V(x,r)} \int_{B(x,2r)} |\nabla u|^p \right)^{1/p}$$

where $\bar{u} = \frac{1}{V(x,r)} \int_{B(x,r)} u$. For any subset Ω we define $\Omega(t) \doteqdot \Omega \setminus B(o, t)$.

Definition 4.2. – Assume that (VD) and (WNP) hold on Ω. The end Ω is called p-homogeneous if there exists C_3 such that

$$V(\Omega \cap B(o,t)) \leq C_3 V(x, \frac{t}{8}) \tag{4.3}$$

for every $x \in \partial\Omega(t)$. Ω is called uniformly homogeneous if $C_2(p) \leq C_2'$ for some C_2' independent of p as $p \to 1$.

As in [23], we denote the above volume comparison condition (4.3) by (VC).

Definition 4.3. – An end Ω is called p-nonparabolic, for some $p > 1$, if

$$\int_1^\infty \left(\frac{t}{V(\Omega \cap B(o,t))} \right)^{1/(p-1)} dt < \infty. \tag{4.4}$$

Tracing the proof of Proposition 5.7 in [20] we have the following estimate on p-Green's function. Note that the uniformly homogeneous condition implies a uniform Harnack constant for positive p-harmonic functions in the Harnack inequality (2.23) of [20].

Proposition 4.4. – *Assume that Ω is uniformly homogeneous and p-nonparabolic. Then there exists a positive Green's function $G(x, z)$ on M. Furthermore, there exists a positive constant C_4 independent of p (as $p \to 1$) such that*

$$G(x,z) \leq \frac{C_4}{(p-1)^2} \int_{2r}^\infty \left(\frac{t}{V(\Omega \cap B(o,t))} \right)^{1/(p-1)} dt \tag{4.5}$$

for every $x \in \Omega(r)$.

The factor $\frac{1}{(p-1)^2}$ comes from applying Young's inequality in Lemma 5.6 of [20]. In [23] and [20], it can be seen that, by solving the Dirichlet problem on a compact exhaustion, one can obtain a positive p-harmonic function v with $v = 1$ on $\partial\Omega$ and $\lim_{x\to\infty} v(x) = 0$, provided the end is p-nonparabolic. It is easy to see from the proof of [20] that the above estimate (4.5) also holds for such v. This is what is needed for our purposes. Alternatively, one can obtain such an estimate from (4.5), using the Green's function as an upper barrier and applying the comparison principle.

Combining the gradient estimates of the previous sections and the above result we have the following existence theorem.

THEOREM 4.5. – *Let M be a complete Riemannian manifold. Assume that Ω is a uniformly homogeneous end and is p_0-nonparabolic for some $p_0 > 1$. Additionally, assume that* $\lim_{r\to\infty} \mathcal{V}(r) = 0$ *where*

$$\mathcal{V}(r) = \sup_{2r\le t<\infty} \frac{t}{V(\Omega\cap B(o,t))}.$$

Then there exists a proper solution u to (1.2) *on Ω with $u = 0$ on $\partial\Omega$.*

Before turning to the proof of the theorem, let us first comment on its assumptions. The uniformly homogeneous assumption holds in particular for manifolds of so-called asymptotically nonnegative sectional curvature. These are manifolds M for which there exists a continuous non-increasing function $k(t) : [0,\infty) \to [0,\infty)$ such that $K_M \ge -k(t)$ and $\int_0^\infty tk(t)\,dt < \infty$. The reader is referred to [23] and [20] for the verification of the fact that on such manifolds the first two assumptions of the theorem are satisfied. Notice that in this case Theorem 1.1 implies that $|\nabla u|(x) \to 0$ as $x \to \infty$. Also notice that if

$$V(\Omega\cap B(o,t)) \ge \delta t^{1+\epsilon}$$

for some $\delta > 0$ and $\epsilon > 0$, then Ω is p-nonparabolic for sufficient small p, and satisfies $\lim_{r\to\infty} \mathcal{V}(r) = 0$. In particular the above theorem can be applied to the asymptotically *locally* Euclidean (ALE) manifolds, which include the cases considered in [21]. We refer the reader to [23] and [20] for other *partially homogenous* examples, including the manifolds with finite first Betti number and nonnegative Ricci curvature outside a compact set.

In a sense, the extra volume growth condition ($\lim_{r\to\infty} \mathcal{V}(r) = 0$) imposed in the theorem is optimal. Let (M,g) be Hamilton's 2-dimensional 'cigar' [18], i.e, $M = \mathbb{R}^2$ and $g = ds^2 + \tanh^2 s d\theta^2$. Then M has linear volume growth and nonnegative Ricci curvature, and is homogeneous, however

$$u \doteqdot \ln\left(\frac{\tanh s}{\tanh 1}\right)$$

is a solution to (1.2) on $\{u \ge 0\} \subset M$ but is not proper. Moreover, we can show that in fact M admits no proper solution. Indeed suppose that u is such a solution. Let φ be a cut-off function. Integration by parts yields that

$$\int_M \varphi|\nabla u|\,d\mu \le \int_M |\nabla\varphi|\,d\mu.$$

Using the fact that (M,g) is of linear volume growth, by choosing suitable φ we can conclude that

$$\int_M |\nabla u|\,d\mu \le C$$

for some C depending on the geometry of M. On the other hand,

$$\int_0^{2\pi} u(R,\theta)d\theta \to \infty$$

as $R \to \infty$, since u is proper. In particular, we have that

$$\int_1^R \int_0^{2\pi} u_s(s,\theta)\, d\theta, ds \to \infty.$$

But, if we denote the point where $s = 0$ by o, it is easy to see that there exists C_1 such that

$$\int_1^R \int_0^{2\pi} u_s(s,\theta)\, d\theta \le C_1 \int_{B(o,R)\setminus B(o,1)} |\nabla u|\, d\mu \le C_1 C,$$

which is a contradiction. This shows that the volume condition $\lim_{r\to\infty} \mathcal{V}(r) \to 0$ is necessary for the existence of a proper solution. Note that the example can be adapted to high dimensions.

Proof of Theorem 4.5. – We first construct p-harmonic functions $v^{(p)}$ for $p \le p_0$ with $v^{(p)} = 1$ on $\partial\Omega$ and $v^{(p)}(x) \to 0$ as $x \to \infty$. This can be done by solving the Dirichlet problem on a compact exhaustion and then taking a limit. In view of the regularity result of [22], the p-harmonic function $v^{(p)}$ will be $C^{1,\alpha}$. On the set where $|\nabla u| \ne 0$, elliptic regularity theory implies that both Theorem 1.1 and the boundary estimate of Section 3 are valid and can be applied. The existence of a non-trivial limit is then ensured by the estimate below. For $p \le p_0$ it is easy to see that

$$\left(\int_{2r}^{\infty} \left(\frac{t}{V(\Omega\cap B(o,t))}\right)^{1/(p-1)} dt\right)^{p-1} \le \mathcal{V}(r)^{\frac{p_0-p}{p_0-1}} \left(\mathcal{A}_{p_0}(r)\right)^{p-1}$$

where

$$\mathcal{A}_{p_0}(r) = \int_{2r}^{\infty} \left(\frac{t}{V(\Omega\cap B(o,t))}\right)^{1/(p_0-1)} dt.$$

Hence Ω is also p-nonparabolic for $p \le p_0$. This ensures that there exist $v^{(p)}$ satisfying

$$v^{(p)}(x) \le \frac{C_4}{(p-1)^2} \int_{2r(x)}^{\infty} \left(\frac{t}{V(\Omega\cap B(o,t))}\right)^{1/(p-1)} dt.$$

Combining the previous inequalities, we have that for $r(x) \gg 1$

$$\begin{aligned} u^{(p)}(x) \ \ge\ & 2(p-1)\log(p-1) - (p-1)\log C_4 - (p-1)\log(\mathcal{A}_{p_0}(r(x))) \\ & -\frac{p_0-p}{p_0-1}\log(\mathcal{V}(r(x))). \end{aligned}$$

By the gradient estimates in the previous sections we can conclude that $u^{(p)}$ converges locally uniformly to a limit function $u(x) \in C^{0,1}_{\mathrm{loc}}(\Omega)$. Moreover, $u(x)$ satisfies that

$$u(x) \ge -\log(\mathcal{V}(r(x)))$$

for $r(x) \gg 1$. By the compactness argument from [26], Theorem 1.1, and Proposition 3.1, we can conclude that u is a weak proper solution to (1.2). □

5. The p-Laplacian heat equation

Motivated by Theorem 1.1, we consider smooth solutions to the parabolic analog of (1.1), namely

$$\frac{\partial v}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \tag{5.1}$$

for $p > 1$. This nonlinear evolution equation is the gradient flow for the p-energy functional

$$E_p(v) = \int_M |\nabla v|^p d\mu$$

and has been studied rather extensively – see, for example, [2], [29], and the references therein. For a given smooth solution v of (5.1), it will be useful for us to consider the linearization of the operator $\widehat{\Lambda}_p$ at v, given by

$$\widehat{\mathcal{L}}(\psi) = \operatorname{div}(h^{p/2-1}A(\nabla\psi)).$$

Here $h = |\nabla v|^2$ and A is the tensor introduced in Section 2, namely

$$A_{ij} = \delta_{ij} + (p-2)\frac{v_i v_j}{h}.$$

The main result in this section is essentially a consequence of the following calculation.

Lemma 5.1. – *Suppose $v : M \times [0,T) \to \mathbb{R}$ is a smooth, positive solution to (5.1) with $p > 1$. For any $\alpha > 0$, define*

$$F_\alpha := \frac{|\nabla v|^p}{v^2} - \alpha\frac{v_t}{v}.$$

Then, on the region $|\nabla v| > 0$, the following estimate holds

$$\begin{aligned}\left(\widehat{\mathcal{L}} - \frac{\partial}{\partial t}\right) F_\alpha = ph^{p-2}\left(\left|\frac{v_{ij}}{v} - \frac{1}{p-1}\frac{v_i v_j}{v^2}\right|_A^2 + \frac{R_{ij}v_i v_j}{v^2}\right) + (\alpha-1)(p-2)\frac{v_t^2}{v^2}\\ + (p-2)F_1^2 - 2(p-1)\frac{h^{p/2-1}}{v}\langle\nabla F_\alpha, \nabla v\rangle.\end{aligned} \tag{5.2}$$

In particular, if $\mathrm{Rc} \geq -Kg$ *for some $K > 0$, we have*

$$\begin{aligned}\left(\widehat{\mathcal{L}} - \frac{\partial}{\partial t}\right) F_\alpha \geq \left(\frac{p+n(p-2)}{n}\right) F_1^2 + (\alpha-1)(p-2)\frac{v_t^2}{v^2}\\ - pK\frac{h^{p-1}}{v^2} - 2(p-1)\frac{h^{p/2-1}}{v}\langle\nabla F_\alpha, \nabla v\rangle.\end{aligned} \tag{5.3}$$

Here, for a two-tensor T, we write $|T|_A^2 := A^{ik}A^{jl}T_{ij}T_{kl}$.

Proof. – First we have the following formula resembling Lemma 2.1.

Lemma 5.2. – *Let $h = |\nabla v|^2$. Then*

$$\left(\frac{\partial}{\partial t} - \widehat{\mathcal{L}}\right) h = -2h^{p/2-1}(|\nabla\nabla v|^2 + R_{ij}v_i v_j) - \left(\frac{p}{2} - 1\right) h^{p/2-2}|\nabla h|^2. \tag{5.4}$$

The proof of the above identity is a straightforward calculation and very similar to that of Lemma 2.1. Hence we leave the details to interested readers.

By the definition of $\widehat{\mathcal{L}}$, we have that

$$\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right) v_t=0, \tag{5.5}$$

and with the help of Lemma 5.2, we find that

$$\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right) h^{p/2}=ph^{p-2}\left[|\nabla\nabla v|_A^2+R_{ij}v_iv_j\right]. \tag{5.6}$$

Using the general formula

$$\begin{aligned}\widehat{\mathcal{L}}\left(\frac{Q}{v^\gamma}\right)&=\frac{1}{v^\gamma}\widehat{\mathcal{L}}(Q)-\gamma\frac{Q}{v^{\gamma+1}}\widehat{\mathcal{L}}(v)\\&\quad-\frac{2\gamma(p-1)}{v^{\gamma+1}}h^{p/2-1}\langle\nabla Q,\nabla v\rangle+\gamma(\gamma+1)(p-1)h^{p/2}\frac{Q}{v^{\gamma+2}},\end{aligned}$$

and the identity

$$\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right) v=(p-2)v_t$$

together with (5.6) and (5.5), we then obtain

$$\begin{aligned}\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right)\frac{h^{p/2}}{v^2}&=\frac{p}{v^2}h^{p-2}\left[|\nabla\nabla v|_A^2+R_{ij}v_iv_j)\right]-2(p-2)h^{p/2}\frac{v_t}{v^3}\\&\quad-4p(p-1)\frac{h^{p-2}}{v^3}v_{ij}v_iv_j+6(p-1)\frac{h^p}{v^4}\end{aligned} \tag{5.7}$$

and

$$\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right)\frac{v_t}{v}=-(p-2)\frac{v_t^2}{v^2}-2(p-1)\frac{h^{p/2-1}}{v^2}\langle\nabla v_t,\nabla v\rangle+2(p-1)\frac{h^{p/2}}{v^3}v_t. \tag{5.8}$$

Now,

$$\nabla_kF_\alpha=ph^{p/2-1}\frac{v_{ik}v_i}{v^2}-2\frac{v_k}{v^3}-\alpha\frac{\nabla_kv_t}{v}+\alpha\frac{v_t}{v^2}v_k$$

so, after multiplying both sides by α, (5.8) becomes

$$\begin{aligned}\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right)\left(\alpha\frac{v_t}{v}\right)&=-\alpha(p-2)\left(\frac{v_t}{v}\right)^2-2p(p-1)\frac{v_{ij}v_iv_j}{v^3}\\&\quad+4(p-1)\frac{h^p}{v^4}+2(p-1)\frac{h^{p/2-1}}{v}\langle\nabla F_\alpha,\nabla v\rangle.\end{aligned}$$

Combining this with (5.7), we have

$$\begin{aligned}\left(\widehat{\mathcal{L}}-\frac{\partial}{\partial t}\right)F_\alpha&=p\frac{h^{p-2}}{v^2}\left[|\nabla\nabla v|_A^2+R_{ij}v_iv_j\right]-2(p-2)\frac{h^{p/2}}{v^3}v_t\\&\quad-2p(p-1)\frac{h^{p-2}}{v^3}v_{ij}v_iv_j+2(p-1)\frac{h^p}{v^4}\\&\quad+\alpha(p-2)\frac{v_t^2}{v^2}-2(p-1)\frac{h^{p/2-1}}{v}\langle\nabla F_\alpha,\nabla v\rangle.\end{aligned} \tag{5.9}$$

Equation (5.2) then follows from (5.9) and the identity

$$ph^{p-2}\left|\frac{v_{ij}}{v}-\frac{1}{p-1}\frac{v_iv_j}{v^2}\right|_A^2=p\frac{h^{p-2}}{v^2}|\nabla\nabla v|_A^2-2p(p-1)\frac{h^{p-2}}{v^3}v_{ij}v_iv_j+p\frac{h^p}{v^4}.$$

For (5.3), we observe that

$$F_1 = h^{p/2-1}\operatorname{tr}_A\left(\frac{1}{(p-1)}\frac{v_iv_j}{v^2} - \frac{v_{ij}}{v}\right)$$

where

$$\operatorname{tr}_A\left(\frac{1}{(p-1)}\frac{v_iv_j}{v^2} - \frac{v_{ij}}{v}\right) = A_{ij}\left(\frac{1}{(p-1)}\frac{v_iv_j}{v^2} - \frac{v_{ij}}{v}\right),$$

so that

$$ph^{p-2}\left|\frac{1}{(p-1)}\frac{v_iv_j}{v^2} - \frac{v_{ij}}{v}\right|_A^2 \geq \frac{p}{n}F_1^2$$

by the standard inequality $n|T|_A^2 \geq (\operatorname{tr}_A T)^2$ for a two-tensor T. □

In the case $\alpha = 1$, it is convenient to consider the expression of the above equations in terms of the "pressure" quantity

(5.10) $$\phi \doteqdot \frac{p-1}{p-2}v^{\frac{p-2}{p-1}}.$$

Then

$$\nabla\nabla\phi = v^{\frac{p-2}{p-1}}\left(\frac{\nabla\nabla v}{v} - \frac{1}{p-1}\frac{\nabla v\otimes\nabla v}{v^2}\right)$$

and

$$\widehat{\Lambda}_p(\phi) = |\nabla\phi|^{p-2}\operatorname{tr}_A(\nabla\nabla\phi) = -F_1$$

so that in the case $\alpha = 1$, the identity (5.2) has the equivalent form

(5.11) $$\begin{aligned}\left(\frac{\partial}{\partial t} - \widehat{\mathcal{L}}\right)\widehat{\Lambda}_p(\phi) \geq p|\nabla\phi|^{2(p-2)}\left(|\nabla\nabla\phi|_A^2 + R_{ij}\phi_i\phi_j\right) + (p-2)\widehat{\Lambda}_p(\phi)^2\\ - 2(p-1)\left\langle\nabla\widehat{\Lambda}_p(\phi),\nabla\phi\right\rangle.\end{aligned}$$

When $\mathrm{Rc} \geq 0$, (5.3) suggests, by way of the maximum principle, the global estimate

(5.12) $$\widehat{\Lambda}_p(\phi) = \frac{v_t}{v} - \frac{|\nabla v|^p}{v^2} \geq -\frac{n\beta}{t}$$

for $\beta = 1/(p+n(p-2))$ and $p > 2n/(n+1)$. Such an estimate, analogous to the differential Harnack estimate in [25] for the heat equation, would indeed be sharp in view of the explicit source-type solutions (see, e.g., [2])

$$H_p(x,t) = \frac{1}{t^{n\beta}}\left(1 + \frac{\beta^{\frac{1}{p-1}}(2-p)}{p}\left(\frac{|x|}{t^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$$

to Equation (5.1) on $\mathbb{R}^n$ for which (5.12) is an equality. However, (5.1) is degenerate where $|\nabla v| = 0$, and our calculations, carried out in the region $|\nabla v| > 0$, are thus as yet insufficient to draw such a conclusion. Following [12], we therefore consider a family of strictly parabolic equations which approximate (5.1) and by proving analogous estimates for the corresponding quantities of the solutions to the approximate equations, we may obtain an estimate of the above form via a limiting procedure. The precise statement of the result, which was obtained in [12] in the case $M = \mathbb{R}^n$ (albeit for a broader class of solutions), is the following. We restrict our attention to smooth solutions.

THEOREM 5.3. – *Suppose (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature, and suppose v is a smooth, nonnegative solution to (5.1) with $p > 2n/(n+1)$ for $t \in [0, \Omega)$. Then, for all $t \in (0, \Omega)$, one has*

$$\frac{|\nabla v|^p}{v^2} - \frac{1}{v}\frac{\partial v}{\partial t} \leq \frac{n\bar{\beta}}{t}. \tag{5.13}$$

where

$$\bar{\beta} = \begin{cases} \beta & \text{if} \quad \frac{2n}{n+1} < p < 2 \\ (p-1)\beta & \text{if} \quad p > 2. \end{cases}$$

REMARK 5.4. – In the case $M = \mathbb{R}$, the above result – with the sharp constant $\beta = 1/2(p-1)$ – was obtained in [11] for all $p > 1$.

Proof. – The argument follows almost exactly as in that of Section 2 of [12], where the equation satisfied by the pressure $\phi = (p-1)/(p-2)v^{(p-2)/(p-1)}$, namely,

$$\frac{\partial \phi}{\partial t} = \frac{p-2}{p-1}\phi \operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi|^p,$$

is approximated by the equation

$$\frac{\partial \phi_\epsilon}{\partial t} = \frac{p-2}{p-1}\phi_\epsilon \operatorname{div}(\varphi_\epsilon(|\nabla\phi_\epsilon|)\nabla\phi_\epsilon) + \psi_\epsilon(|\nabla\phi_\epsilon|) \tag{5.14}$$

for judicious choices of φ_ϵ and ψ_ϵ depending on $\epsilon > 0$. For completeness, we outline the argument below.

First, take $\zeta_\epsilon \in C^\infty([0,\infty))$ to satisfy

$$\zeta_\epsilon(r) = \begin{cases} p-1 & \text{if} \quad r \geq \epsilon \\ 1 & \text{if} \quad r \in [0, a_\epsilon] \end{cases} \qquad \text{where} \qquad a_\epsilon = \begin{cases} \epsilon(p-1)^{1/\epsilon}/2 & \text{if} \quad p < 2 \\ \epsilon(p-1)^{-1/\epsilon}/2 & \text{if} \quad p > 2. \end{cases}$$

On the interval (a_ϵ, r), choose ζ_ϵ to be non-decreasing if $p < 2$, non-increasing if $p > 2$, and to satisfy $|r\zeta_\epsilon'(r)| \leq \epsilon\zeta_\epsilon(r)$. Then let φ_ϵ satisfy

$$r\frac{\varphi_\epsilon'(r)}{\varphi_\epsilon(r)} = \zeta_\epsilon(r) - 1,$$

so that $\varphi_\epsilon(r)$ is constant for small r and equal to r^{p-2} for $r \geq \epsilon$. Finally, define

$$\psi_\epsilon(r) = \frac{p}{p-1}\left(r^2\varphi_\epsilon(r) - \int_0^r s\varphi_\epsilon(s)\,ds\right).$$

With these choices, (5.14) is strictly parabolic and we obtain a smooth solution ϕ_ϵ for each $\epsilon > 0$.

Next, we derive an equation analogous to (5.11) for $F_{1,\epsilon} := -\operatorname{div}(\varphi_\epsilon(|\nabla\phi_\epsilon|)\nabla\phi_\epsilon)$. Define

$$B_\epsilon^{ij} = g^{ij} + \frac{\varphi_\epsilon'}{\rho\varphi_\epsilon}\phi_\epsilon^i\phi_\epsilon^j \qquad \text{and} \qquad \tilde{B}_\epsilon^{ij} = g^{ij} - \frac{\phi_\epsilon^i\phi_\epsilon^j}{\rho^2},$$

where $\rho = |\nabla\phi_\epsilon|$. Introducing the operator

$$\widehat{\mathcal{L}}_\epsilon(\Phi) \doteqdot \operatorname{div}\left(\frac{p-2}{p-1}\phi_\epsilon\varphi_\epsilon B_\epsilon\nabla\Phi\right),$$

we compute

$$
\begin{aligned}
\left(\widehat{\mathcal{L}}_\epsilon - \frac{\partial}{\partial t}\right) F_{1,\epsilon} &= 2\zeta_\epsilon\varphi_\epsilon \left\langle \nabla F_{1,\epsilon}, \nabla\phi_\epsilon\right\rangle + \frac{p-2}{p-1}\zeta_\epsilon F_{1,\epsilon}^2 \\
&\quad + \frac{p}{p-1}\zeta_\epsilon\varphi_\epsilon^2 \left[|\nabla\nabla\phi_\epsilon|^2_{\tilde{B}_\epsilon} + \mathrm{Rc}(\nabla\phi_\epsilon, \nabla\phi_\epsilon)\right] \\
&\quad + \frac{p}{p-1}\zeta_\epsilon\varphi_\epsilon^2 \left(\frac{\rho\zeta_\epsilon'}{\zeta_\epsilon} + 2\zeta_\epsilon\right)(S - H^2) \\
&\quad + \zeta_\epsilon^2\varphi_\epsilon^2 \left(\frac{p}{p-1}\zeta_\epsilon + 2\frac{\rho\zeta_\epsilon'}{\zeta_\epsilon}\right) H^2 + \frac{p-2}{p-1}\zeta_\epsilon'\varphi_\epsilon^2 H(\Delta\phi_\epsilon - H),
\end{aligned}
\tag{5.15}
$$

where

$$
S \doteqdot \frac{1}{\rho^2} g^{ik}\nabla_i\nabla_j\phi_\epsilon\nabla_k\nabla_l\phi_\epsilon\nabla^j\phi_\epsilon\nabla^l\phi_\epsilon \qquad \text{and} \qquad H \doteqdot \frac{1}{\rho^4}\left[\nabla\nabla\phi_\epsilon(\nabla\phi_\epsilon, \nabla\phi_\epsilon)\right]^2.
$$

Now, $F_{1,\epsilon} = -\varphi_\epsilon B_\epsilon^{ij}\nabla_i\nabla_j\phi_\epsilon$, and when $\rho \in [0, a_\epsilon] \cup [\epsilon, \infty)$, the final five terms on the left side of (5.15) reduce to

$$
\begin{aligned}
\frac{p-2}{p-1}\zeta_\epsilon F_{1,\epsilon}^2 + \frac{p}{p-1}\zeta_\epsilon\varphi_\epsilon^2 &\left[|\nabla\nabla\phi_\epsilon|^2_{B_\epsilon} + \mathrm{Rc}(\nabla\phi_\epsilon, \nabla\phi_\epsilon)\right] \\
&\geq \frac{\zeta_\epsilon}{p-1}\left(p - 2 + \frac{p}{n}\right) F_{1,\epsilon}^2 \\
&\geq \frac{1}{\max\{1, p-1\} n\beta} F_{1,\epsilon}^2.
\end{aligned}
\tag{5.16}
$$

When $\rho \in (a_\epsilon, \epsilon)$, we need to do a little extra estimation. First, note that $S \geq H^2$ (so that the fourth term in (5.15) is nonnegative for sufficiently small ϵ by the assumptions on ζ_ϵ'). Also, one has

$$
\begin{aligned}
F_{1,\epsilon}^2 &\leq \frac{n}{n-1}\varphi_\epsilon^2 (\Delta\phi_\epsilon - H)^2 + n\zeta_\epsilon^2\varphi_\epsilon^2 H^2 \\
&\leq n\varphi_\epsilon^2 |\nabla\nabla\phi_\epsilon|^2_{\tilde{B}_\epsilon} + n\zeta_\epsilon^2\varphi_\epsilon^2 H^2,
\end{aligned}
\tag{5.17}
$$

and similarly for any $\mu > 0$, that

$$
H(\Delta\phi_\epsilon - H) \geq -\frac{\mu}{2}H^2 - \frac{n-1}{2\mu}|\nabla\nabla\phi_\epsilon|^2_{\tilde{B}_\epsilon}. \tag{5.18}
$$

With these inequalities, equation (5.15), and the assumptions on ζ_ϵ', we can find a constant C, independent of ϵ, such that

$$
\begin{aligned}
&\left(\widehat{\mathcal{L}}_\epsilon - 2\zeta_\epsilon\varphi_\epsilon\nabla\phi_\epsilon\cdot\nabla - \frac{\partial}{\partial t}\right) F_{1,\epsilon} \\
&\qquad \geq \frac{\zeta_\epsilon}{p-1}\left[(p-2)F_{1,\epsilon}^2 + (p - C\epsilon)\varphi_\epsilon^2\left(|\nabla\nabla\phi_\epsilon|^2_{\tilde{B}_\epsilon} + \zeta_\epsilon^2 H^2\right)\right] \\
&\qquad \geq \frac{1}{\max\{1, p-1\}}\left(p - 2 + \frac{p - C\epsilon}{n}\right) F_{1,\epsilon}^2.
\end{aligned}
\tag{5.19}
$$

Since we have already noted that the stronger inequality (5.16) holds when $\rho \leq a_\epsilon$ or $\rho \geq \epsilon$, one may appeal to the maximum principle to deduce in general that

$$
F_{1,\epsilon} \leq \frac{n\beta}{(1 - C\epsilon)t}.
$$

Letting $\epsilon \to 0$, and making use of appropriate energy estimates, as in [12], we can obtain a limiting ϕ, and consequently a solution v to (5.1), for which the inequality (5.13) holds. □

6. Another nonlinear parabolic equation

Note that with the above parabolic estimate, one can not recover the elliptic result proved in Section 2, thus in this section, we consider instead the parabolic equation associated to the operator Λ_p. Let u be a solution to the equation

$$\frac{\partial u}{\partial t} - \Lambda_p(u) = \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^p = 0. \tag{6.1}$$

This nonlinear parabolic equation has also been studied in the literature. See, for example, [29] and the references therein.

The corresponding equation for $v = \exp(-\frac{u}{p-1})$ is

$$\frac{\partial v^{p-1}}{\partial t} = (p-1)^{p-1} \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right). \tag{6.2}$$

Recall from Section 2 the operator $\mathcal{L}$ defined as

$$\mathcal{L}(\psi) = \operatorname{div}(f^{p/2-1}A(\nabla\psi)) - pf^{p/2-1}\langle\nabla u, \nabla\psi\rangle,$$

which is the linearized operator of Λ_p. Note that

$$A^{ij} = g^{ij} + (p-2)\frac{v^i v^j}{|\nabla v|^2} = g^{ij} + (p-2)\frac{u^i u^j}{|\nabla u|^2}.$$

For any u, letting $f = |\nabla u|^2$, the proof of Lemma 2.1 yields the nonlinear Bochner formula

$$\mathcal{L}(f) = 2f^{p/2-1}(u_{ij}^2 + R_{ij}u_iu_j) + 2\langle\nabla u, \nabla(\Lambda_p(u))\rangle + \left(\frac{p}{2}-1\right)|\nabla f|^2 f^{p/2-2}. \tag{6.3}$$

A bit of computation together with this formula yields the following lemma.

Lemma 6.1. – *Let $f = |\nabla u|^2$. Then*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) u_t = 0 \tag{6.4}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) f = -2f^{p/2-1}(u_{ij}^2 + R_{ij}u_iu_j) - \left(\frac{p}{2}-1\right) f^{p/2-2}|\nabla f|^2. \tag{6.5}$$

An immediate consequence is the following result.

Corollary 6.2. – *For any $\alpha > 0$, let $F_\alpha = |\nabla u|^p + \alpha u_t = f^{p/2} + \alpha u_t$. Then*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) F_\alpha = -pf^{p-2}\left(|\nabla\nabla u|_A^2 + R_{ij}u_iu_j\right). \tag{6.6}$$

Notice that (6.1) implies that

$$F_1 = f^{p/2-1}\,\mathrm{tr}_A(\nabla\nabla u).$$

Hence

$$f^{p-2}|\nabla\nabla u|_A^2 \geq \frac{1}{n}F_1^2.$$

In the case that M is compact with nonnegative Ricci curvature, we obtain at once the following global estimate.

THEOREM 6.3. – *Let M be a compact manifold with nonnegative Ricci curvature. Let v be a positive solution to* (6.2). *Then for any $p > 1$,*

$$(p-1)^p\frac{|\nabla v|^p}{v^p} - (p-1)\frac{v_t}{v} \leq \frac{n}{pt}. \tag{6.7}$$

The result also holds on noncompact manifolds assuming that the left hand side is bounded. Note that when $p = 2$, the above reduces to Li-Yau's estimate. Hence the corollary provides another nonlinear analogue of Li-Yau's estimate for the heat equation.

REMARK 6.4. – It is easy to check that for $p > 1$, the function

$$v_0 = \left(\pi^{-n/2}(p^{*p-1}pt)^{-n/p}\frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)}\exp\left(-\frac{|x-x_0|^{p^*}}{(tp^{*p-1}p)^{\frac{1}{p-1}}}\right)\right)^{\frac{1}{p-1}} \tag{6.8}$$

is a fundamental solution of (6.2) on $\mathbb{R}^n$, and achieves equality in the estimate (6.7). This demonstrates the sharpness of the estimate. Here $p^* = \frac{p}{p-1}$.

REMARK 6.5. – It seems that an estimate in the above sharp form is not known except in dimension one [11]. It should be useful in the study of the regularity of weak solutions of (6.2).

As in [25], integration on minimizing path gives the following Harnack inequality for the positive solutions.

COROLLARY 6.6. – *Let v be a positive solution to* (6.2). *For any (x_2, t_2) and (x_1, t_1) with $t_2 > t_1$, then*

$$v(x_1,t_1)t_1^{\frac{n}{(p-1)p}} \leq v(x_2,t_2)t_2^{\frac{n}{(p-1)p}}\exp\left(\frac{d(x_1,x_2)^{p^*}}{p^{p^*}(t_2-t_1)^{1/(p-1)}}\right).$$

When $p = 1$ we have the following result.

THEOREM 6.7. – *Let M be a compact manifold with nonnegative Ricci curvature. Let u be a smooth solution to* (6.1). *Then*

$$|\nabla u| + u_t \leq \frac{n-1}{t}. \tag{6.9}$$

Since the equation becomes degenerate when $\nabla u = 0$, the above computation needs extra justification. First we introduce the notion of a weak solution to (6.1) as in [19]. For any compact subset K, we define the functional

$$J_u^p(w\,;K) \doteqdot \int_{K\times\{t\}} \frac{1}{p}|\nabla w|^p + w(|\nabla u|^p + u_t)\,d\mu$$

for $w \in \mathcal{C} \doteqdot \{\varphi \in C^0(M\times[0,\infty))\,|\,\varphi(\cdot,t)\in C^{0,1}(M)$, and for every $x\in M, \varphi(x,\cdot)\in C^{0,1}_{loc}(0,\infty)\}$ such that $\{w\neq u\}\subset\subset M$. A function u is called a *continuous weak solution* to (6.1) if

$$J_u^p(u\,;K) \le J_u^p(w\,;K)$$

for any $w\in\mathcal{C}$. Note that when u is smooth (locally), satisfying (6.1) with $|\nabla u|\neq 0$, we have that

$$\begin{aligned} J_u^p(u\,;K) - J_u^p(w\,;K) &= \int_K \frac{1}{p}(|\nabla u|^p - |\nabla w|^p) + (u-w)\operatorname{div}(|\nabla u|^{p-2}\nabla u)\,d\mu \\ &\le \int_K \frac{1}{p}(|\nabla u|^p - |\nabla w|^p) - |\nabla u|^p + |\nabla u|^{p-1}|\nabla w|\,d\mu \le 0 \end{aligned}$$

by Young's inequality. To obtain weak solutions more generally, we use an ϵ-regularization process, replacing the equation (6.1) by the approximate version

$$\frac{\partial u_\epsilon}{\partial t} = \Lambda_{p,\epsilon}(u_\epsilon) \doteqdot \operatorname{div}(f_\epsilon^{p/2-1}\nabla u_\epsilon) - f_\epsilon^{p/2} \tag{6.10}$$

where $f_\epsilon \doteqdot |\nabla u_\epsilon|^2 + \epsilon$. Since (6.10) is strictly parabolic, one can apply the established theory of parabolic equations to obtain a solution u_ϵ and take the limit $\epsilon\to 0$ to obtain a *continuous weak solution* to (6.1) as in [19]. We next show that an estimate analogous to (in fact, somewhat stronger than) (6.9) can be obtained for solutions u_ϵ to (6.10), and with this estimate recover (6.9) in the limit as $u_\epsilon \to u$.

It will be convenient to introduce the notation

$$\mathcal{L}_\epsilon(\psi) \doteqdot D\Lambda_{p,\epsilon}[u_\epsilon](\psi) = \operatorname{div}(f_\epsilon^{p/2-1}A_\epsilon\nabla\psi) - pf_\epsilon^{p/2-1}\langle\nabla u_\epsilon,\nabla\psi\rangle$$

for the linearization of the operator $\Lambda_{p,\epsilon}$, where

$$A_\epsilon = \operatorname{id} + (p-2)\frac{\nabla u_\epsilon\otimes\nabla u_\epsilon}{f_\epsilon}.$$

It is easy to check that

$$\left(\mathcal{L}_\epsilon - \frac{\partial}{\partial t}\right)\frac{\partial u_\epsilon}{\partial t} = 0,$$

and a computation as before establishes the following Bochner-type formula.

LEMMA 6.8. – *We have*

$$\begin{aligned} \mathcal{L}_\epsilon(f_\epsilon) &= 2f_\epsilon^{p/2-1}\left(|\nabla\nabla u_\epsilon|^2 + \operatorname{Ric}(\nabla u_\epsilon,\nabla u_\epsilon)\right) + 2\langle\nabla u_\epsilon, \nabla(\Lambda_{p,\epsilon}(u_\epsilon))\rangle \\ &\quad + \left(\frac{p}{2}-1\right)|\nabla f_\epsilon|^2 f_\epsilon^{p/2-2}. \end{aligned} \tag{6.11}$$

In particular,

$$\left(\mathcal{L}_\epsilon - \frac{\partial}{\partial t}\right)f_\epsilon^{p/2} = pf_\epsilon^{p-2}\left(\operatorname{Ric}(\nabla u_\epsilon,\nabla u_\epsilon) + |\nabla\nabla u_\epsilon|^2_{A_\epsilon}\right).$$

Thus we obtain the analog of Corollary 6.2 for solutions to our approximate equation (6.10).

LEMMA 6.9. – *For all* $\alpha > 0$,

$$F_{\alpha,\epsilon} \doteqdot f_\epsilon^{p/2} + \alpha \frac{\partial u_\epsilon}{\partial t}$$

satisfies

$$\left(\mathcal{L}_\epsilon - \frac{\partial}{\partial t}\right) F_{\alpha,\epsilon} = p f_\epsilon^{p-2}\left(\mathrm{Ric}(\nabla u_\epsilon, \nabla u_\epsilon) + |\nabla\nabla u_\epsilon|^2_{A_\epsilon}\right). \tag{6.12}$$

In the same way as before, we note

$$F_{1,\epsilon} = \mathrm{div}(f_\epsilon^{p/2-1}\nabla u_\epsilon) = f_\epsilon^{p/2-1}\,\mathrm{tr}_{A_\epsilon}(\nabla\nabla u_\epsilon)$$

so

$$f_\epsilon^{p-2}|\nabla\nabla u|^2_{A_\epsilon} \geq \frac{1}{n} F^2_{1,\epsilon},$$

which, with the above lemma, implies the following global estimate.

THEOREM 6.10. – *Suppose* u_ϵ *is a positive solution to* (6.10) *on the compact manifold* M *of nonnegative Ricci curvature. Then, for all* $p > 1$ *and all* $t > 0$,

$$(|\nabla u_\epsilon|^2 + \epsilon)^{p/2} + \frac{\partial u_\epsilon}{\partial t} \leq \frac{n}{pt}. \tag{6.13}$$

Hence Theorem 6.3 and Theorem 6.7 hold for the weak solutions obtained via the ϵ-regularization process.

The Bochner-type formula (6.6) is effective enough to give a nonlinear entropy formula (see [27] for the entropy formula for the heat equation). For this purpose, we first observe that $\int_M v^{p-1}\, d\mu$ is preserved under the equation (6.2). A little less obvious, perhaps, is the following conservation law.

PROPOSITION 6.11. – *For any smooth* ψ,

$$\frac{d}{dt}\int_M \psi v^{p-1}\, d\mu = \int_M \left(\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\psi\right) v^{p-1}\, d\mu. \tag{6.14}$$

In particular, if ψ *satisfies*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\psi = 0 \tag{6.15}$$

then $\int_M \psi v^{p-1}\, d\mu$ *is a constant along the equation* (6.2) *and* (6.15).

Proof. – Notice that $\nabla v = -\frac{v}{p-1}\nabla u$, hence $(p-1)^{p-1}|\nabla v|^{p-2}\nabla v = -v^{p-1}|\nabla u|^{p-2}\nabla u$. Direct computation shows that

$$\begin{aligned}\frac{d}{dt}\int_M \psi v^{p-1}\, d\mu &= \int_M \psi_t v^{p-1} + \psi (v^{p-1})_t \\ &= \int_M \left(\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\psi\right) v^{p-1} - \int_M |\nabla u|^{p-2}\langle A(\nabla\psi), \nabla v^{p-1}\rangle \\ &\quad -(p-1)\int_M |\nabla u|^{p-2}\langle \nabla u, \nabla\psi\rangle v^{p-1}.\end{aligned}$$

The result follows from the observations that $\nabla v^{p-1} = -v^{p-1}\nabla u$ and

$$
\begin{aligned}
-\int_M |\nabla u|^{p-2}\langle A(\nabla\psi), \nabla v^{p-1}\rangle = -&\int_M |\nabla u|^{p-2}\langle \nabla\psi, \nabla v^{p-1}\rangle \\
&- (p-2)\int_M |\nabla u|^{p-4}\langle \nabla\psi, \nabla u\rangle\langle \nabla u, \nabla v^{p-1}\rangle \\
&= (p-1)\int_M |\nabla u|^{p-2} v^{p-1}\langle \nabla\psi, \nabla u\rangle. \qquad \square
\end{aligned}
$$

REMARK 6.12. – In the above, we do not really need that ψ is smooth. The argument carries over assuming only the summability of the integrands.

Let (a_{ij}) be the inverse of (A^{ij}). Explicitly, $a_{ij} = g_{ij} - \frac{p-2}{p-1}\frac{u_i u_j}{f}$, and can be viewed as a metric tensor. Let v be a positive solution to (6.2) satisfying $\int_M v^{p-1}\, d\mu = 1$. Define

$$
\mathcal{N}_p(v,t) = \int_M v^{p-1} u\, d\mu - \frac{n}{p}\log t
$$

and denote the first term on the right hand side by $N(v,t)$. Note that $u = -\log(v^{p-1})$. Following [27] (see also [15]), define $\mathcal{F}_p(v,t) = \frac{d}{dt}\mathcal{N}_p(v,t)$ and $\mathcal{W}_p(v,t) = \frac{d}{dt}(t\mathcal{N}_p(v,t))$. More explicitly, motivated by (6.8), if we write

$$
v^{p-1} = \frac{1}{\pi^{n/2}(p^{*p-1}p)^{\frac{n}{p}}}\frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)}\frac{e^{-\varphi}}{t^{\frac{n}{p}}},
$$

then define

$$
\mathcal{W}_p(\varphi,t) = \int_M (t|\nabla\varphi|^p + \varphi - n)v^{p-1}\, d\mu.
$$

For the case $p=1$, it is helpful to write $\mathcal{W}$ in terms of u,

$$
\mathcal{W}_p(u,t) = \int_M \left(t|\nabla u|^p + u - \frac{n}{p}\log t + \log\left(\frac{1}{\pi^{n/2}(p^{*p-1}p)^{\frac{n}{p}}}\frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)}\right) - n\right)e^{-u}\, d\mu,
$$

which becomes

$$
\mathcal{W}_1(u,t) = \int_M \left(t|\nabla u| + u - n\log t + \log\frac{\Gamma(n/2+1)}{\pi^{n/2}} - n\right)e^{-u}\, d\mu
$$

when $p=1$. Note that when $p=2$, this is precisely the entropy defined in [27].

For the entropy quantity $\mathcal{W}_p(v,t)$ we have the following nonlinear entropy formula.

THEOREM 6.13. – *Let v be a positive solution to* (6.2) *satisfying $\int_M v^{p-1}\, d\mu = 1$. Then*

$$
\frac{d}{dt}\mathcal{W}_p(v,t) = -tp\int_M \left(\left|f^{p/2-1}\nabla_i\nabla_j u - \frac{1}{tp}a_{ij}\right|_A^2 + f^{p-2}R_{ij}u_iu_j\right)v^{p-1}\, d\mu. \tag{6.16}
$$

When $p=2$, this recovers the entropy formula of [27].For the proof we need the following result.

PROPOSITION 6.14. –

$$
\frac{d}{dt}N(v,t) = \int_M F_1 v^{p-1}\, d\mu = \int_M f^{p/2-1}\operatorname{tr}_A(\nabla\nabla u)v^{p-1}\, d\mu, \tag{6.17}
$$

$$
\frac{d}{dt}\int_M F_1 v^{p-1}\, d\mu = -p\int_M f^{p-2}\left(|\nabla\nabla u|_A^2 + R_{ij}u_iu_j\right)v^{p-1}\, d\mu. \tag{6.18}
$$

Proof of Proposition 6.14. – Direct calculation shows that

$$\left(\frac{\partial}{\partial t}-\mathcal{L}\right)u=(p-1)|\nabla u|^p-(p-2)\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Hence (6.17) follows from (6.14) together with the observation

$$\int_M F_1 v^{p-1}=\int_M \operatorname{div}(|\nabla u|^{p-2}\nabla u)v^{p-1}=\int_M |\nabla u|^p v^{p-1}.$$

The second identity of Proposition 6.14 follows from (6.6) and (6.14). □

Note that in the case that the Ricci curvature of M is nonnegative, (6.18) yields the monotonicity of 'energy'

$$\overline{\mathcal{F}}(v,t)\doteqdot\int_M |\nabla u|^p v^{p-1}.$$

Now Theorem 6.13 follows from the above proposition similarly as in [27] (also [15]), by observing that $\frac{d\mathcal{W}_p}{dt}=t\frac{d\mathcal{F}_p}{dt}+2\mathcal{F}_p$ and completing the square. This point of view is taken from physics [28, 13]. The case $p=1$ (Theorem 1.4) can be shown similarly.

By ODE considerations and the Cauchy-Schwarz inequality, we also have the following result.

Corollary 6.15. – *Assume that M has nonnegative Ricci curvature. Then*

$$\frac{d\mathcal{N}_p}{dt}=\mathcal{F}_p=\overline{\mathcal{F}}-\frac{n}{pt}\leq 0. \tag{6.19}$$

In particular, any positive ancient solution to (6.2) must be a constant.

Summarizing we have that *if M has nonnegative Ricci curvature then $\mathcal{N}_p(v,t)$ is a monotone non-decreasing concave function in $\frac{1}{t}$.*

To address the potential vanishing of ∇u, we appeal to the approximation scheme as before. Note that (6.17) can be justified in view of Remark 6.12, but (6.18) requires additional argument. Let u_ϵ be a solution to (6.10) and define φ_ϵ similarly as above. We have the following pointwise computations.

Proposition 6.16. – *Let u_ϵ, φ_ϵ and f_ϵ be as before. Let*

$$V_\epsilon=2\operatorname{div}(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon)-|\nabla u_\epsilon|^p,\quad W_\epsilon=t(2\operatorname{div}(|\nabla\varphi_\epsilon|^{p-2}\nabla\varphi_\epsilon)-|\nabla\varphi_\epsilon|^p)+\varphi_\epsilon-n.$$

Then

$$\left(\frac{\partial}{\partial t}-\mathcal{L}_\epsilon\right)V_\epsilon=-pf_\epsilon^{p-2}\left(|\nabla\nabla u_\epsilon|_A^2+\operatorname{Ric}(\nabla u_\epsilon,\nabla u_\epsilon)\right) \tag{6.20}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\mathcal{L}_\epsilon\right)W_\epsilon=-tp&\left(\left|f_\epsilon^{p/2-1}\nabla_i\nabla_j u_\epsilon-\frac{1}{tp}(a_\epsilon)_{ij}\right|^2_{A_\epsilon}+\operatorname{Ric}(\nabla u_\epsilon,\nabla u_\epsilon)\right)\\&+(p-2)\left(|\nabla u_\epsilon|^p-\operatorname{div}(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon)\right).\end{aligned} \tag{6.21}$$

Notice that

$$\int_M V_\epsilon v^{p-1} = 2\int_M \langle \nabla u_\epsilon, \nabla u\rangle |\nabla u_\epsilon|^{p-1} v^{p-1} - |\nabla u_\epsilon|^p v^{p-1} \to \int_M |\nabla u|^p v^{p-1}$$
$$= \overline{\mathcal{F}}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_M V_\epsilon v^{p-1} &= -p\int_M f_\epsilon^{p-2}\left(|\nabla\nabla u_\epsilon|^2_{A_\epsilon} + \mathrm{Ric}(\nabla u_\epsilon, \nabla u_\epsilon)\right) v^{p-1}\\
&\quad - \int_M f_\epsilon^{p/2-1}\left(\langle \nabla V_\epsilon, \nabla v^{p-1}\rangle + (p-2)\frac{\langle \nabla u_\epsilon, \nabla V_\epsilon\rangle}{f_\epsilon}\langle \nabla u_\epsilon, \nabla v^{p-1}\rangle\right)\\
&\quad -p\int_M f_\epsilon^{p/2-1}\langle \nabla u_\epsilon, \nabla V_\epsilon\rangle v^{p-1} + \int_M f^{p/2-1} v^{p-1}\langle \nabla u, \nabla V_\epsilon\rangle\\
&\to -p\int_M f^{p-2}\left(|\nabla\nabla u|^2_A + \mathrm{Ric}(\nabla u, \nabla u)\right) v^{p-1}
\end{aligned}$$

as $\epsilon \to 0$. Hence (6.18) can be established for weak solutions which can be properly approximated by the regularization process.

Strictly speaking, the above proof can be applied only for the case when M is compact. When M is non-compact, further work is needed to justify the integrations by parts involved. On complete noncompact manifolds with nonnegative Ricci curvature (or for which the Ricci curvature is bounded from below) the entropy formula has been established rigorously in the book [7] for the case $p = 2$, with full justification for such manipulations. The argument there can be adapted to the $p > 1$ case, and we refer the readers to [7] for the technical details.

In [27], the entropy formula for the heat equation was used to show that a manifold with nonnegative Ricci curvature and sharp logarithmic Sobolev inequality must be isometric to $\mathbb{R}^n$. It is interesting to study the relation between the L^p-logarithmic Sobolev inequality and the entropy formula (6.16). First recall the sharp L^p-logarithmic Sobolev inequality (see for example [9]).

Theorem 6.17. – *Let M be Euclidean space $\mathbb{R}^n$. Then for any $p > 1$, and any w with $\int |w|^p dx = 1$,*

$$\int |w|^p \log |w|^p \, dx \le \frac{n}{p}\log\left(\mathcal{C}_{p,n}\int |\nabla w|^p \, dx\right) \tag{6.22}$$

where

$$\mathcal{C}_{p,n} = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-p/2}\left(\frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)}\right)^{p/n}.$$

The above inequality with sharp constant was established after the work of [16, 8] (see also [10]), and has the following form connected with the entropy quantity. (For the L^1-version, see [4] as well as [3].)

Proposition 6.18. – *If on (M, g), (6.22) holds then*

$$\mathcal{W}_p(\varphi, t) \ge 0$$

for

$$v^{p-1} = \frac{1}{\pi^{n/2}(p^{*p-1}p)^{\frac{n}{p}}}\frac{\Gamma(n/2+1)}{\Gamma(n/p^*+1)}\frac{e^{-\varphi}}{t^{\frac{n}{p}}},$$

with $\int_M v^{p-1} = 1$, *where* t *is just a scaling factor.*

Proof. – The proof is a technically more complicated version of the $p = 2$ case shown in [6], pages 247–249. We leave the detailed checking to the interested reader. □

We shall next prove the following characterization of $\mathbb{R}^n$ among manifolds with non-negative Ricci curvature by the sharp L^p-logarithmic Sobolev inequality.

THEOREM 6.19. – *Let* (M, g) *be a complete, connected Riemannian manifold with non-negative Ricci curvature. Assume that* (6.22) *holds for some* $p > 1$ *with the sharp constant* $\mathcal{C}_{p,n}$ *on* M. *Then* M *is isometric to* $\mathbb{R}^n$.

Proof. – By an argument along the lines of [28], Section 3, one can show that for the fundamental solution H to (6.2), $\lim_{t\to 0} \mathcal{W}_p(H, t) = 0$. Then, the entropy formula ensures that $\mathcal{W}_p(H, t) \leq 0$. However, in the presence of a sharp L^p-logarithmic Sobolev inequality, one may prove, as in Corollary 6.18, that $\mathcal{W}_p(H, t) \geq 0$ for $t > 0$. Hence we can conclude that $\frac{d}{dt}\mathcal{W}_p(H, t) \equiv 0$. Writing $u = -(p-1)\log H$, and $f = |\nabla u|^2$ we have that

$$u_{ij} = \frac{1}{tpf^{p/2-1}}\left(g_{ij} - \frac{p-2}{p-1}\frac{u_i u_j}{|\nabla u|^2}\right) \tag{6.23}$$

off of the set of critical points of u, which we shall denote by Σ.

Equation (6.23) implies in particular that $\nabla\nabla u > 0$ on $M \setminus \Sigma$, so $\nabla\nabla u \geq 0$ on all of M. It follows, then, that $p \in \Sigma$ if and only if $u(p) = \inf_{q\in M} u(q) \doteqdot m$. We shall prove later that Σ consists of precisely one point; for the present we note that these facts imply at least that Σ is a connected (in fact, convex) subset of M. Indeed, if $p_1, p_2 \in \Sigma$, and $\gamma : [0, l] \to M^n$ is a unit-speed geodesic with $\gamma(0) = p_1$ and $\gamma(l) = p_2$, then we have

$$\frac{d^2}{ds^2}u(\gamma(s)) = (\nabla_{\dot\gamma}\nabla_{\dot\gamma}u)(\gamma(s)).$$

Since $u(\gamma(s)) \geq u(\gamma(0)) = u(\gamma(l)) = m$, the weak convexity of u implies $u(\gamma(s)) \equiv m$ and so $\gamma([0, l]) \subset \Sigma$.

Let us now consider the structure of an arbitrary connected component M_1 of $M \setminus \Sigma$. We may write $M_1 \approx I \times N$ for an interval I and a hypersurface N defined by

$$N = M_1 \cap \{u = u(p_0)\}$$

for some arbitrary $p_0 \in M_1$. Now we choose a orthonormal frame such that $e_1 = \frac{\nabla u}{|\nabla u|}, e_2, \ldots, e_n$. Then the equality (6.23) implies that

$$\begin{aligned} u_{11} &= \frac{1}{tp(p-1)u_1^{p-2}}, \\ u_{1\beta} &= 0, \\ u_{\alpha\alpha} &= \frac{1}{tpu_1^{p-2}} \end{aligned}$$

for $2 \leq \alpha, \beta \leq n$. This shows that $|\nabla u|$ is constant along the level set of u. We shall use the above information to find the explicit form of the metric. Recall that $u_{\alpha\beta} = h_{\alpha\beta}u_1$, where

$h_{\alpha\beta}$ is the second fundamental form of the level set hypersurface of u. Let $U = p|\nabla u|^{p-1}$. It is easy to see that the mean curvature of this hypersurface is $H = \frac{n-1}{U}$. Hence

$$U_1 = 1, \quad \Delta U = U_{11} + HU_1 = \frac{n-1}{U}.$$

If we fix a level set and parametrize the other ones using s, the oriented distance from this fixed one, we have that

$$u_{ss} = \frac{1}{p(p-1)u_s^{p-2}}$$

which implies that $(u_s^{p-1})_s = \frac{1}{p}$. Thus

$$u_1 = \left(\frac{s}{p} + B\right)^{\frac{1}{p-1}}, \tag{6.24}$$

where B is the value of $|\nabla u|^{p-1}$ on the originally fixed hypersurface. This implies that the level hypersurfaces at distance s have the second fundamental form

$$h_{\alpha\beta} = \frac{1}{s+pA} g_{\alpha\beta}.$$

This further implies that the metric on $N \times I$ is of the form

$$g_{ij} = ds^2 + (s+pB)^2 g_N \tag{6.25}$$

where g_N is the metric of the hypersurface N. Since M is complete, and M_1 a connected component of $M \setminus \Sigma$, we see that $I = (-pB, \infty)$.

Now, from Equation (6.25), one may compute that $K_M(X, \nabla u, X, \nabla u) = 0$ along N for any X tangential to N and hence (either again from (6.25) or from the Gauss equations) that

$$\mathrm{Rc}_N(X, X) = \mathrm{Rc}_M(X, X) + (n-1)g_N(X, X). \tag{6.26}$$

Since the Ricci curvature of M is assumed to be nonnegative, it follows immediately from Myer's theorem that N is compact and, moreover, that the diameter of the hypersurface N_s of signed distance $s \in I$ from N satisfies the bound

$$\mathrm{diam}(N_s) \leq \pi(s+pB). \tag{6.27}$$

Thus $\lim_{s\to -pB} \mathrm{diam}(N_s) = 0$ and ∂M_1 must consist of a single point $\mathcal{O} \in \Sigma$. In fact, we must have $M = M_1 \cup \{\mathcal{O}\}$ as otherwise the removal of $\mathcal{O}$ would disconnect M – an impossibility if $\dim(M) \geq 2$. Thus $M = M_1 \cup \{\mathcal{O}\}$ in this case. (Note that since the foregoing considerations imply that M must be non-compact, the theorem has already been established in the case $\dim(M) = 1$.) So N (and hence all N_s) are geodesic spheres about the point $\mathcal{O}$, and the representation (6.25) on $M \setminus \{\mathcal{O}\}$ implies that M is isometric to $\mathbb{R}^n$. □

REMARK 6.20. – Here we assumed the existence and regularity of the fundamental solution, together with the finiteness of all the integrals involved in the entropy. These assumptions require justification, although they are similar to the heat equation case treated in [7].

7. Localization

In this section, carrying on in the notation of the last, we localize the estimate (6.7) of Theorem 6.3. Here, the nonlinearity/degeneracy of the parabolic version of the equation (6.1) introduces some technical complications that prove to be somewhat thornier than those encountered in Section 2 in the derivation of the local estimate for the elliptic version.

Since the case $p > 2$ is very similar, we shall focus on the case $1 < p < 2$. As in Section 2, assume that u is a solution on $B(x_0, R)$ and let $\theta(s) : [0, \infty) \to [0, 1]$ be a cut-off function satisfying $\frac{(\theta')^2}{\theta} \leq 40$ and $\theta'' \geq -40\theta \geq -40$. Now let $\eta(x) = \theta^p\left(\frac{r(x)}{R}\right)$. It is easy to see that

$$\frac{|\nabla \eta|}{\eta^{1-\frac{1}{p}}} \leq \frac{80}{R}. \tag{7.1}$$

The proof of Lemma 2.2 implies the following estimate.

Lemma 7.1. – *Assume that the sectional curvature is bounded from below on $B(x_0, R)$ by $-K^2$. Then*

$$\Delta \eta + (p-2)\frac{\eta_{ij}u_i u_j}{f} \geq -\frac{C_1(n, KR)}{R^2} \tag{7.2}$$

where $C_1(n, KR) = 160(n + p - 2)(1 + KR) - 80$.

Theorem 7.2. – *Assume that the sectional curvature of M on $B(x_0, R) \times [0, T]$ satisfies $K_M \geq -K^2$ and u is a solution of (6.1). Assume further that $1 < p < 2$ and $u_t \leq 0$ on $B(x_0, R) \times [0, T]$. Then for any $\alpha > 1$, there exist $C_2' = C_2(n, p, KR)$ and $C_3 = C_3(n, p)$ such that on $B(x_0, \frac{R}{2}) \times [0, T]$,*

$$|\nabla u|^p + \alpha u_t \leq \max\left\{\left(8n\alpha^2\left(\frac{C_2'}{R^2} + \frac{C_3}{t^{\frac{2}{p}}}\right)\right)^{\frac{p}{2}}, \quad (2pn(n-1)K^2)^{p/2}\left(\frac{\alpha}{\alpha-1}\right)^p\right\}. \tag{7.3}$$

Here $C_2'(n, p, \alpha, KP) = C_1(n, KR) + 6400 + 6400\left(2 + (2-p)\frac{\alpha-1}{\alpha}\right)^2 \frac{n\alpha^2}{4(\alpha-1)} + 6400\frac{2-p}{p-1} + 6400n(2-p)^2$, $C_3 = \frac{p}{2}\left(2pn\alpha^2(2-p)\right)^{\frac{2-p}{2}}$.

Remark 7.3. – The above result is a parabolic generalization of Theorem 1.1. Indeed, if u is a solution to the elliptic problem, we have $u_t = 0$, hence one can apply the theorem and recover the elliptic estimate (albeit with a worse constant).

The assumption $u_t \leq 0$ is preserved under the flow due to (6.4), provided with some growth conditions. Hence it is true if for $t = 0$ the condition holds. From the proof, it can be seen that if the assumption $u_t \leq 0$ is replaced by that $u_t \leq A|\nabla u|^p$ for some $A > 0$, on $B(x_0, R) \times [0, T]$, the result remains true (with modified constants).

Proof. – Clearly, $tF_\alpha\eta$ achieves a maximum over $B(x_0, R) \times [0, T]$ somewhere. For the purpose of bounding F_α from above, we may assume that $\eta F_\alpha > 0$ at this maximum point (y_0, t_0); this implies in particular that $t_0 > 0$. Applying the maximum principle at (y_0, t_0), we have that $\left(\frac{\partial}{\partial t} - \mathcal{L}\right)(tF_\alpha\eta) \geq 0$.

Direct computation shows that

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)(tF_\alpha\eta) = t\eta\left(\frac{\partial}{\partial t} - \mathcal{L}\right)F_\alpha + tF_\alpha\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\eta - 2tf^{p/2-1}\langle\nabla\eta, \nabla F_\alpha\rangle_A + \eta F_\alpha.$$

Here $\langle \cdot, \cdot \rangle_A$ denotes the product with respect to A^{ij}. Using that $\nabla(\eta F_\alpha) = 0$ at the maximum point, with formula (6.6) and the simple estimate on $\frac{|\nabla \eta|_A^2}{\eta}$ from Section 2 we find that

$$
\begin{aligned}
0 &\le \left(\frac{\partial}{\partial t} - \mathcal{L}\right)(tF_\alpha \eta) \\
&\le -pt\eta f^{p-2}\left(|\nabla\nabla u|_A^2 - (n-1)K^2 f\right) + tF_\alpha \left(\frac{\partial}{\partial t} - \mathcal{L}\right)\eta + 2tf^{p/2-1}F_\alpha \frac{|\nabla \eta|^2}{\eta} + \eta F_\alpha.
\end{aligned}
$$

We first derive an effective estimate on $F_\alpha\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\eta$. Direct calculation using (6.1) yields

$$
\begin{aligned}
F_\alpha\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\eta = {} & (2-p)F_\alpha f^{p/2-2}\left(\langle \nabla \eta, \nabla f\rangle - \frac{\langle \nabla u, \nabla \eta\rangle \langle \nabla u, \nabla f\rangle}{f}\right) \\
& - \left(\Delta \eta + (p-2)\frac{\eta_{ij}u_i u_j}{f}\right) f^{p/2-1}F_\alpha + pf^{p/2-1}F_\alpha \langle \nabla u, \nabla \eta\rangle \\
& + (2-p)F_1 F_\alpha \frac{\langle \nabla u, \nabla \eta\rangle}{f}.
\end{aligned}
$$

We shall estimate each term in turn. Define $\tilde{A}^{ij} = g^{ij} - \frac{u^i u^j}{f}$. It is easy to see that

$$
\begin{aligned}
\mathrm{I} &= (2-p)F_\alpha f^{p/2-2}\left(\langle \nabla \eta, \nabla f\rangle - \frac{\langle \nabla u, \nabla \eta\rangle \langle \nabla u, \nabla f\rangle}{f}\right) \\
&\le (2-p)F_\alpha f^{p/2-2}\langle \nabla \eta, \nabla f\rangle_{\tilde{A}} \le (2-p)F_\alpha f^{p/2-2}|\nabla \eta||\nabla f|_{\tilde{A}}.
\end{aligned}
$$

Choose a normal frame so that $e_1 = \frac{u_1}{|\nabla u|}$. Then $u_j = 0$ for all $j \ge 2$, and $f_k = 2u_{1k}u_1$ for $k \ge 1$. Hence

$$
F_\alpha f^{p/2-2}|\nabla f|_{\tilde{A}} = 2F_\alpha f^{p/2-2}\left(\sum_{j\ge 2} u_{j1}^2 u_1^2\right)^{1/2} = 2F_\alpha f^{p/2-3/2}\left(\sum_{j\ge 2} u_{j1}^2\right)^{1/2}.
$$

On the other hand,

$$
\begin{aligned}
|\nabla\nabla u|_A^2 &= u_{ik}^2 + 2(p-2)u_{k1}^2 + (p-2)^2 u_{11}^2 \\
&= (p-1)^2 u_{11}^2 + 2(p-1)\sum_{j\ge 2} u_{j1}^2 + \sum_{j,k\ge 2} u_{jk}^2 \\
&\ge 2(p-1)\sum_{j\ge 2} u_{j1}^2 + \frac{1}{n}\left(\mathrm{tr}_A(\nabla\nabla u)\right)^2
\end{aligned}
$$

since $\mathrm{tr}_A \nabla\nabla u = (p-1)u_{11} + \sum_{k\ge 2} u_{kk}$. Hence by Cauchy-Schwarz we have that

$$
\mathrm{I} \le \frac{2-p}{2}\eta f^{p-2}|\nabla\nabla u|_A^2 + \frac{2-p}{p-1}\frac{|\nabla \eta|^2}{\eta}\frac{F_\alpha^2}{f} - \frac{2-p}{2n}F_1^2 \eta. \tag{7.4}
$$

Lemma 7.1 implies that

$$
\mathrm{II} = -\left(\Delta \eta + (p-2)\frac{\eta_{ij}u_i u_j}{f}\right) f^{p/2-1}F_\alpha \le \frac{C_1(n, KR)}{R^2} f^{p/2-1}F_\alpha. \tag{7.5}
$$

Furthermore, it is easy to see that

$$
\mathrm{III} = pf^{p/2-1}F_\alpha \langle \nabla u, \nabla \eta\rangle \le pf^{\frac{p-1}{2}}F_\alpha |\nabla \eta| \tag{7.6}
$$

and

$$\mathrm{IV} = (2-p)F_1 F_\alpha \frac{\langle \nabla u, \nabla \eta\rangle}{f} \le (2-p)F_1 F_\alpha |\nabla \eta| f^{-1/2}. \tag{7.7}$$

Since $f^{p-2}|\nabla\nabla u|_A^2 \ge \frac{1}{n}F_1^2$ we split $\eta F_\alpha = y - \alpha z$ with $y = \eta|\nabla u|^p$ and $z = -\eta u_t$. Also we can write $\eta^2 F_1^2$ as

$$(y-z)^2 = \frac{1}{\alpha^2}(y-\alpha z)^2 + 2\frac{\alpha-1}{\alpha^2}(y-\alpha z)y + \left(\frac{\alpha-1}{\alpha}\right)^2 y^2. \tag{7.8}$$

Note that $y - z = \frac{1}{\alpha}(y-\alpha z) + (1-\frac{1}{\alpha})y > 0$ since $F_\alpha > 0$ (assumed for the purpose of the upper estimate). Combining (7.4)–(7.7), we arrive at

$$\begin{aligned}
0 \le & -\frac{3p-2}{2}\eta^2 f^{p-2}|\nabla\nabla u|_A^2 + (n-1)p\eta^2 f^{p-1}K^2 + \frac{2-p}{p-1}|\nabla\eta|^2 F_\alpha^2 f^{-1} \\
& + \frac{C_1(n,KR)}{R^2}\eta f^{p/2-1}F_\alpha + pf^{\frac{p-1}{2}}F_\alpha|\nabla\eta|\eta + (2-p)F_1F_\alpha|\nabla\eta|\eta f^{-1/2} \\
& + f^{p/2-1}F_\alpha|\nabla\eta|^2 + \frac{\eta^2 F_\alpha}{t} \\
\le & -\frac{3p-2}{2n}(y-z)^2 + p(n-1)y^{2-\frac{2}{p}}K^2 + \frac{2-p}{p-1}(y-\alpha z)^2 y^{-\frac{2}{p}}\frac{|\nabla\eta|^2}{\eta^{2-\frac{2}{p}}} \\
& + \frac{C_1(n,KR)}{R^2}(y-\alpha z)y^{1-\frac{p}{2}} + p(y-\alpha z)y^{1-\frac{1}{p}}\frac{|\nabla\eta|}{\eta^{1-\frac{1}{p}}} \\
& + (2-p)(y-z)(y-\alpha z)y^{-\frac{1}{p}}\frac{|\nabla\eta|}{\eta^{1-\frac{1}{p}}} + (y-\alpha z)y^{1-\frac{2}{p}}\frac{|\nabla\eta|^2}{\eta^{2-\frac{2}{p}}} + \frac{y-\alpha z}{t} \\
\le & -\frac{1}{2n}\left(\frac{1}{\alpha^2}(y-\alpha z)^2 + 2\frac{\alpha-1}{\alpha^2}(y-\alpha z)y + \left(\frac{\alpha-1}{\alpha}\right)^2 y^2\right) \\
& + p(n-1)y^{2-\frac{2}{p}}K^2 + \frac{6400}{R^2}\frac{2-p}{p-1}(y-\alpha z)^2 y^{-\frac{2}{p}} + \frac{C_1(n,KR)}{R^2}(y-\alpha z)y^{1-\frac{2}{p}} \\
& + \frac{160}{R}(y-\alpha z)y^{1-\frac{1}{p}} + \frac{80(2-p)}{\alpha R}(y-\alpha z)^2 y^{-\frac{1}{p}} + \frac{80(2-p)(\alpha-1)}{R\alpha}(y-\alpha z)y^{1-\frac{1}{p}} \\
& + \frac{6400}{R^2}(y-\alpha z)y^{1-\frac{2}{p}} + \frac{y-\alpha z}{t}.
\end{aligned}$$

The above shall be enough to give a local estimate if we assume $u_t \le 0$ (namely $z \ge 0$). (Assuming that $u_t \le 0$ initially, this can be established by the maximum principle, in view of (6.4).) In this case, we have that $y - \alpha z \le y$.

We make use of the above computation as follows. If we do not have

$$y^{2/p} \le 2pn(n-1)K^2\left(\frac{\alpha}{\alpha-1}\right)^2, \tag{7.9}$$

which implies that $y - \alpha z$ is bounded by the same bound, we can drop the terms only involving the powers of y to still have the last estimate on the functions of $y - \alpha z$.

By the elementary inequality $2ab \leq a^2 + b^2$, we have that

$$(y-\alpha z)\left(-\frac{\alpha-1}{n\alpha^2}y + \frac{80(2+(2-p)\frac{\alpha-1}{\alpha})}{R}y^{1-\frac{1}{p}} + \frac{C_1(n,KR)+6400}{R^2}y^{1-\frac{2}{p}}\right)$$
$$\leq (y-\alpha z)\left(C_1 + 6400 + 6400\left(2+(2-p)\frac{\alpha-1}{\alpha}\right)^2\frac{n\alpha^2}{4(\alpha-1)}\right)\frac{1}{R^2}y^{1-\frac{2}{p}}$$
$$\leq \frac{1}{R^2}\left(C_1 + 6400 + 6400\left(2+(2-p)\frac{\alpha-1}{\alpha}\right)^2\frac{n\alpha^2}{4(\alpha-1)}\right)(y-\alpha z)^{2-\frac{2}{p}}.$$

Here we have used $y-\alpha z \leq y$. This takes care of all but the last of the terms in the above inequality in which $(y-\alpha z)$ appears linearly.

Making use of $y - \alpha z \leq y$ again, we have that

$$0 \leq -\frac{1}{2n\alpha^2}(y-\alpha z)^2 + \frac{C_2}{R^2}(y-\alpha z)^{2-\frac{2}{p}} + \frac{80(2-p)}{\alpha R}(y-\alpha z)^{2-\frac{1}{p}} + \frac{y-\alpha z}{t}$$

with

$$C_2 = C_1(n,KR) + 6400 + 6400\left(2+(2-p)\frac{\alpha-1}{\alpha}\right)^2\frac{n\alpha^2}{4(\alpha-1)} + 6400\frac{2-p}{p-1}.$$

Another use of the above elementary inequality reduces it to

$$0 \leq -\frac{1}{4n\alpha^2}(y-\alpha z)^2 + \frac{C_2'}{R^2}(y-\alpha z)^{2-\frac{2}{p}} + \frac{y-\alpha z}{t} \tag{7.10}$$

with $C_2' = C_2 + 6400n(2-p)^2$. Using Young's inequality $ab \leq \frac{a^{p'}}{p'} + \frac{b^{q'}}{q'}$ with $\frac{1}{p'} + \frac{1}{q'} = 1$, we finally have that

$$0 \leq -\frac{1}{8n\alpha^2}(y-\alpha z)^2 + \left(\frac{C_2'}{R^2} + \frac{C_3}{t^{\frac{2}{p}}}\right)(y-\alpha z)^{2-\frac{2}{p}} \tag{7.11}$$

where

$$C_3 = \frac{p}{2}\left(2pn\alpha^2(2-p)\right)^{\frac{2-p}{2}}.$$

From (7.11) it is then easy to derive the theorem. □

REFERENCES

[1] D. Bakry, D. Concordet, M. Ledoux, Optimal heat kernel bounds under logarithmic Sobolev inequalities, *ESAIM Probab. Statist.* **1** (1995/97), 391–407.

[2] G. I. Barenblatt, On self-similar motions of a compressible fluid in a porous medium, *Akad. Nauk SSSR. Prikl. Mat. Meh.* **16** (1952), 679–698.

[3] W. Beckner, Geometric asymptotics and the logarithmic Sobolev inequality, *Forum Math.* **11** (1999), 105–137.

[4] S. Bobkov, A functional form of the isoperimetric inequality for the Gaussian measure, *J. Funct. Anal.* **135** (1996), 39–49.

[5] S. Y. Cheng, S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975), 333–354.

[6] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *The Ricci flow: techniques and applications. Part I: Geometric aspects*, Mathematical Surveys and Monographs **135**, Amer. Math. Soc., 2007.

[7] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *The Ricci flow: techniques and applications. Part II: Analytic aspects*, Mathematical Surveys and Monographs **144**, Amer. Math. Soc., 2008.

[8] M. Del Pino, J. Dolbeault, The optimal Euclidean L^p-Sobolev logarithmic inequality, *J. Funct. Anal.* **197** (2003), 151–161.

[9] M. Del Pino, J. Dolbeault, I. Gentil, Nonlinear diffusions, hypercontractivity and the optimal L^p-Euclidean logarithmic Sobolev inequality, *J. Math. Anal. Appl.* **293** (2004), 375–388.

[10] K. Ecker, Logarithmic Sobolev inequalities on submanifolds of Euclidean space, *J. reine angew. Math.* **522** (2000), 105–118.

[11] J. R. Esteban, J. L. Vázquez, Homogeneous diffusion in **R** with power-like nonlinear diffusivity, *Arch. Rational Mech. Anal.* **103** (1988), 39–80.

[12] J. R. Esteban, J. L. Vázquez, Régularité des solutions positives de l'équation parabolique p-laplacienne, *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), 105–110.

[13] L. C. Evans, Entropy and partial differential equations, lecture notes at UC Berkeley.

[14] L. C. Evans, H. Ishii, A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2** (1985), 1–20.

[15] M. Feldman, T. Ilmanen, L. Ni, Entropy and reduced distance for Ricci expanders, *J. Geom. Anal.* **15** (2005), 49–62.

[16] I. Gentil, The general optimal L^p-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations, *J. Funct. Anal.* **202** (2003), 591–599.

[17] R. E. Greene, H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Math. **699**, Springer, 1979.

[18] R. S. Hamilton, The formation of singularities in the Ricci flow, in *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, Int. Press, Cambridge, MA, 1995, 7–136.

[19] B. Hein, A homotopy approach to solving the inverse mean curvature flow, *Calc. Var. Partial Differential Equations* **28** (2007), 249–273.

[20] I. Holopainen, Volume growth, Green's functions, and parabolicity of ends, *Duke Math. J.* **97** (1999), 319–346.

[21] G. Huisken, T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, *J. Differential Geom.* **59** (2001), 353–437.

[22] J. L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.* **32** (1983), 849–858.

[23] P. Li, L.-F. Tam, Green's function, harmonic functions, and volume comparison, *J. Differential Geom.* **41** (1995), 277–318.

[24] P. Li, J. Wang, Complete manifolds with positive spectrum. II, *J. Differential Geom.* **62** (2002), 143–162.

[25] P. Li, S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 153–201.

[26] R. Moser, The inverse mean curvature flow and p-harmonic functions, *J. Eur. Math. Soc.* **9** (2007), 77–83.

[27] L. Ni, The entropy formula for linear heat equation, *J. Geom. Anal.* **14** (2004), 87–100; addenda *J. Geom. Anal.* **14** (2004), 369–374.

[28] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, preprint arXiv:math.DG/0211159.

[29] J. L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type*, Oxford Lecture Series in Mathematics and its Applications **33**, Oxford University Press, 2006.

[30] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228.

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