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Michael HITRIK & Johannes SJÖSTRAND

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RATIONAL INVARIANT TORI, PHASE SPACE TUNNELING, AND SPECTRA FOR NON-SELFADJOINT OPERATORS IN DIMENSION 2*

BY MICHAEL HITRIK AND JOHANNES SJÖSTRAND

ABSTRACT. – We study spectral asymptotics and resolvent bounds for non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2, assuming that the classical flow of the unperturbed part is completely integrable. Spectral contributions coming from rational invariant Lagrangian tori are analyzed. Estimating the tunnel effect between strongly irrational (Diophantine) and rational tori, we obtain an accurate description of the spectrum in a suitable complex window, provided that the strength of the non-selfadjoint perturbation $\gg h$ (or sometimes $\gg h^2$) is not too large.

RÉSUMÉ. – Nous étudions des asymptotiques spectrales et des estimations de la résolvante des perturbations non-autoadjointes d'opérateurs h-pseudodifférentiels autoadjoints en dimension 2, en supposant que le flot classique de la partie non-perturbée soit complètement intégrable. Les contributions spectrales parvenant des tores invariants lagrangiens rationnels sont analysées. En estimant l'effet tunnel entre des tores diophantiens et rationnels, nous obtenons une description précise du spectre dans une région convenable du plan complexe spectral, sous l'hypothèse que la force de la perturbation non-autoadjointe $\gg h$ (ou parfois $\gg h^2$) ne soit pas trop grande.

1. Introduction

In [24], A. Melin and the second author observed that for large and stable classes of non-selfadjoint analytic (pseudo)differential operators in two dimensions, the individual eigenvalues can be determined up to arbitrarily high powers of the semiclassical parameter by a complex Bohr-Sommerfeld quantization condition. This is quite analogous to known results in dimension one in the selfadjoint case [8], [3], [7], and remarkable in the sense that corresponding results for selfadjoint operators in higher dimensions are known only in very special situations. Applications to resonances were also given in [24].

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A natural continuation of [24] was to study non-selfadjoint perturbations of selfadjoint operators in the semiclassical limit, of the form

$$P_\varepsilon(x, hD_x) = P(x, hD_x) + i\varepsilon Q(x, hD_x),\ 0 < h \ll 1, \tag{1.1}$$

with principal symbol (classical Hamiltonian)

$$p_\varepsilon(x, \xi) = p(x, \xi) + i\varepsilon q(x, \xi), \tag{1.2}$$

either on $\mathbb{R}^2$ or on a compact analytic manifold of dimension 2. Here P is selfadjoint so p is real, and we may assume to fix the ideas that q is real. Both p and q are assumed to be analytic, at least in the cases when $\varepsilon \gg h$.

In [14]–[16] we studied the case when the classical flow of p is periodic and showed that the spectrum has a lattice structure as in [24], with an eigenvalue separation of the order of h in the real ("horizontal") direction and of the order of εh in the imaginary ("vertical") direction. (In [16] we got richer phenomena near branching point levels.) The methods in [14]–[16] were based on a reduction to a one-dimensional operator. Again it should be noticed that the results obtained are more precise than what is currently known in the case of selfadjoint perturbations. Applications to resonances and the damped wave equation were given. See also [11].

As in classical works of A. Weinstein [35] and Y. Colin de Verdière [34], the trajectory averages of q play an important role in the precise formulation of the results. Under much more general assumptions they allow to estimate the width of the spectrum in the imaginary directions (see also [19], [29]). It should also be recalled that the real parts of the eigenvalues distribute according to the same Weyl law as for the unperturbed operator P (see Markus and Matsaev [22], [21]).

The next step was taken by the authors together with S. Vũ Ngọc in [17], where we studied the case when p is classically completely integrable or close to being completely integrable. In the integrable case, the energy surface $p = E_0$ foliates into invariant Lagrangian tori and possibly some more complicated sets. The classical flow on each invariant torus has a rotation number which "most of the time" is Diophantine (i.e. poorly approximated by rational numbers). On such a torus Λ (or more generally on an irrational one), the time averages

$$\langle q\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp tH_p dt$$

of q along the classical trajectories of p all converge to the space average $\langle q\rangle(\Lambda)$ of q over Λ, when $T \to \infty$. When Λ is a torus with a rational rotation number, or a more general "singular" invariant set in the foliation of the energy surface $p^{-1}(E_0)$, then we need to consider the whole interval $Q_\infty(\Lambda)$ of limits of flow averages as above, and in the rational torus case we have $\langle q\rangle(\Lambda) \in Q_\infty(\Lambda)$.

In the completely integrable case, the main result of [17] says very roughly that if $F_0 \in \mathbb{R}$ is a value such that $F_0 = \langle q\rangle(\Lambda_j)$ for finitely many Diophantine tori $\Lambda_1, ..., \Lambda_{N_0}$ in $p^{-1}(E_0)$, and F_0 does not belong to $Q_\infty(\Lambda)$ for any other invariant set Λ in the energy surface, then the spectrum can be completely determined in a rectangle $[E_0 - h^\delta/C, E_0 + h^\delta/C] + i\varepsilon[F_0 - h^\delta/C, F_0 + h^\delta/C]$ modulo $\mathcal{O}(h^\infty)$, where δ is a positive exponent that can be chosen arbitrarily small, and ε may vary in any interval of the form $h^K < \varepsilon \ll 1$. Again the eigenvalues

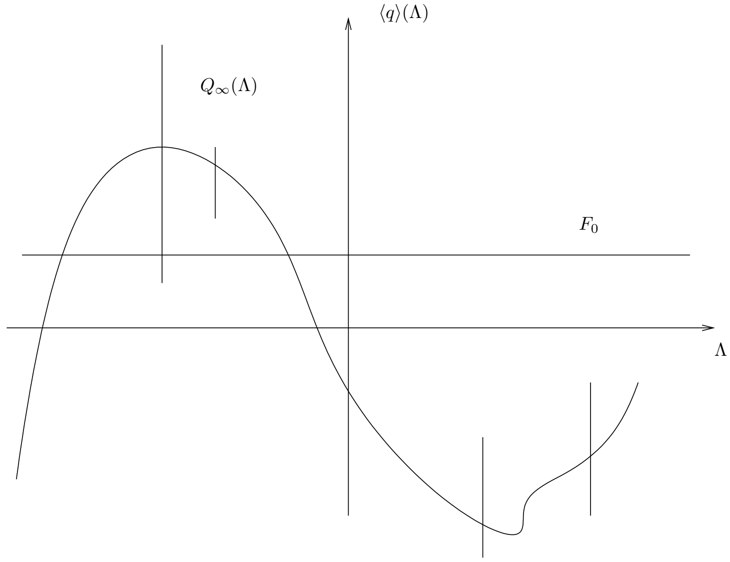

FIGURE 1. The figure represents the graph of the function $\Lambda \mapsto \langle q\rangle(\Lambda)$ as Λ varies over the set of flow invariant Lagrangian tori in the energy surface $p^{-1}(E_0)$. The vertical segments in the figure correspond to the intervals $Q_\infty(\Lambda)$ of limits of flow averages $\langle q\rangle_T$, as $T \to \infty$, when Λ is a rational invariant torus.

form a (superposition of finitely many) distorted lattice(s), with horizontal spacing $\sim h$ and vertical spacing $\sim \varepsilon h$. The proofs were based on the use of suitable exponentially weighted spaces and Birkhoff normal forms near the Diophantine tori.

Notice that the intervals $Q_\infty(\Lambda)$ shrink very fast if Λ are rational tori converging to a Diophantine torus Λ_0. For that reason, there may be plenty of levels F_0 satisfying the assumptions of the result above, and we may cover a substantial fraction of the energy band $[E_0 - 1/C, E_0 + 1/C] + i\varepsilon[\liminf\langle q\rangle_T, \limsup\langle q\rangle_T]$ with such rectangles.

Nevertheless, the intervals $Q_\infty(\Lambda)$ for the rational tori form sets of positive measure of forbidden values for the main result of [17]. In the present paper we shall study what happens when F_0 belongs to finitely many such intervals. Our first attempt was to use secular perturbation theory to analyze the individual eigenvalues produced by rational tori. However this leads to possibly quite serious pseudospectral phenomena for certain one-dimensional operators, and it is doubtful whether such a program can succeed completely. Instead we estimated the number of eigenvalues that can be created by such tori and showed that it is much smaller than the number of eigenvalues created by the Diophantine ones.

Very roughly, the main result of the present paper is as follows: assume that F_0 is equal to $\langle q\rangle(\Lambda_{j,d})$ for finitely many Diophantine tori $\Lambda_{j,d}$ (as in the main result of [17]), and that $F_0 \in Q(\Lambda_{k,r}) \setminus \langle q\rangle(\Lambda_{k,r})$ for finitely many rational tori $\Lambda_{k,r}$. We further assume that F_0 belongs to no other set $Q_\infty(\Lambda)$, for Λ in the foliation of $p^{-1}(E_0)$, and we restrict ε to the interval

$$h \ll \varepsilon \le h^{\frac{2}{3}+\delta}, \tag{1.3}$$

where $\delta > 0$ is any fixed parameter. Then the spectrum of P_ε in the rectangle

$$R(\varepsilon) = [E_0 - \varepsilon/C, E_0 + \varepsilon/C] + i\varepsilon[F_0 - \varepsilon^\delta, F_0 + \varepsilon^\delta] \tag{1.4}$$

is of the form $E_d \cup E_r$, where

- E_d is the union of finitely many distorted lattices as in the main result of [17] (see above)
- E_r is a set of cardinality $\mathcal{O}(\varepsilon^{3/2}/h^2)$.

Here we notice that E_d is of cardinality $\sim \varepsilon^{1+\delta}/h^2$, so choosing δ small enough we see that most eigenvalues in the rectangle $R(\varepsilon)$ belong to E_d and can be asymptotically determined.

Using secular theory arguments ("partial Birkhoff normal forms") we simplify the operator near each rational torus and conclude roughly that the eigenvalues in $R(\varepsilon)$ produced near the rational tori must come from a set in phase space of volume $\mathcal{O}(\varepsilon^{3/2})$. In the absence of Diophantine tori, this leads to the bound $\mathcal{O}(\varepsilon^{3/2}h^{-2})$ on the total number of eigenvalues in $R(\varepsilon)$. When Diophantine tori are present this has to be combined with the analysis of [17], via an auxiliary so called Grushin problem. Near the Diophantine tori we have a nice control on the solution operator, while near the rational tori, we only have the bound $\mathcal{O}(\exp{(C\varepsilon^{3/2}/h^2)})$. Luckily, by means of phase space exponential weights we are able to estimate the tunnel effect between the tori by $\mathcal{O}(\exp{(-1/(Ch))})$, and thanks to the condition (1.3) the Grushin problem can be solved globally, leading to the result above.

In a parallel work [13], the first author and San Vũ Ngọc are currently investigating the case of larger real perturbations. Here the strategy is quite different and uses KAM theory to show that the rational tori split into Diophantine ones under the effect of the perturbation.

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2. Statement of the main results

2.1. General assumptions

We shall start by describing the general assumptions on our operators, which will be the same as in [17], as well as in the earlier papers mentioned above. Let M denote either the space $\mathbb{R}^2$ or a real analytic compact manifold of dimension 2. We shall let $\widetilde{M}$ stand for a complexification of M, so that $\widetilde{M} = \mathbb{C}^2$ in the Euclidean case, and in the compact case, we let $\widetilde{M}$ be a Grauert tube of M — see [6] for the definition and further references.

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \quad 0 < h \leq 1, \tag{2.1}$$

be the h–Weyl quantization on $\mathbb{R}^2$ of a symbol $P(x,\xi,\varepsilon;h)$ (i.e. the Weyl quantization of $P(x,h\xi,\varepsilon;h)$), depending smoothly on $\varepsilon \in \text{neigh}(0,\mathbb{R})$ and taking values in the space of holomorphic functions of (x,ξ) in a tubular neighborhood of $\mathbb{R}^4$ in $\mathbb{C}^4$, with

$$|P(x,\xi,\varepsilon;h)| \leq \mathcal{O}(1)m(\mathrm{Re}\,(x,\xi)), \tag{2.2}$$

there. Here $m \geq 1$ is an order function on $\mathbb{R}^4$, in the sense that

$$m(X) \leq C_0\langle X-Y\rangle^{N_0}m(Y), \quad X,Y\in\mathbb{R}^4, \tag{2.3}$$

for some $C_0, N_0 > 0$. We shall assume, as we may, that m belongs to its own symbol class, so that $m \in C^\infty(\mathbb{R}^4)$ and $\partial^\alpha m = \mathcal{O}_\alpha(m)$ for each $\alpha \in \mathbb{N}^4$. Then for $h > 0$ small enough and when equipped with the domain $H(m) := (m^w(x,hD))^{-1}\left(L^2(\mathbb{R}^2)\right)$, P_ε becomes a closed densely defined operator on $L^2(\mathbb{R}^2)$.

Assume furthermore that

$$P(x,\xi,\varepsilon;h) \sim \sum_{j=0}^{\infty} h^j p_{j,\varepsilon}(x,\xi) \tag{2.4}$$

in the space of holomorphic functions satisfying (2.2) in a fixed tubular neighborhood of $\mathbb{R}^4$. We assume that $p_{0,\varepsilon}$ is elliptic near infinity,

$$|p_{0,\varepsilon}(x,\xi)| \geq \frac{1}{C}m(\mathrm{Re}\,(x,\xi)), \quad |(x,\xi)| \geq C, \tag{2.5}$$

for some $C > 0$.

When M is a compact manifold, for simplicity we shall take P_ε to be a differential operator on M, such that for every choice of local coordinates, centered at some point of M, it takes the form

$$P_\varepsilon = \sum_{|\alpha|\leq m} a_{\alpha,\varepsilon}(x;h)(hD_x)^\alpha, \tag{2.6}$$

where $a_\alpha(x;h)$ is a smooth function of $\varepsilon \in \text{neigh}(0,\mathbb{R})$ with values in the space of bounded holomorphic functions in a complex neighborhood of $x=0$. We further assume that

$$a_{\alpha,\varepsilon}(x;h) \sim \sum_{j=0}^{\infty} a_{\alpha,\varepsilon,j}(x)h^j, \quad h\to 0, \tag{2.7}$$

in the space of such functions. The semiclassical principal symbol $p_{0,\varepsilon}$, defined on T^*M, takes the form

$$p_{0,\varepsilon}(x,\xi) = \sum a_{\alpha,\varepsilon,0}(x)\xi^\alpha, \tag{2.8}$$

if (x,ξ) are canonical coordinates on T^*M. We make the ellipticity assumption,

$$|p_{0,\varepsilon}(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^m, \quad (x,\xi)\in T^*M, \quad |\xi|\geq C, \tag{2.9}$$

for some large $C > 0$. Here we assume that M has been equipped with some real analytic Riemannian metric so that $|\xi|$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ are well-defined.

Sometimes, we write p_ε for $p_{0,\varepsilon}$ and simply p for $p_{0,0}$. We make the assumption that

$$P_{\varepsilon=0} \quad \text{is formally selfadjoint.}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M,\mu(dx))$ where $\mu(dx)$ is the Riemannian volume element.

The assumptions above imply that the spectrum of P_ε in a fixed neighborhood of $0 \in \mathbb{C}$ is discrete, when $0 < h \leq h_0$, $0 \leq \varepsilon \leq \varepsilon_0$, with $h_0 > 0$, $\varepsilon_0 > 0$ sufficiently small. Moreover, if $z \in \mathrm{neigh}(0, \mathbb{C})$ is an eigenvalue of P_ε then $\mathrm{Im}\, z = \mathcal{O}(\varepsilon)$.

We furthermore assume that the real energy surface $p^{-1}(0) \cap T^*M$ is connected and that

$$dp \neq 0 \quad \text{along} \quad p^{-1}(0) \cap T^*M.$$

In what follows we shall write

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2), \tag{2.10}$$

in a neighborhood of $p^{-1}(0) \cap T^*M$, and for simplicity we shall assume throughout this paper that q is real valued on the real domain. (In the general case, we should simply replace q below by $\mathrm{Re}\, q$.)

Let $H_p = p'_\xi \cdot \partial_x - p'_x \cdot \partial_\xi$ be the Hamilton field of p. In [17], it was assumed that the energy surface $p^{-1}(0) \cap T^*M$ contains finitely many H_p–invariant analytic Lagrangian tori satisfying a Diophantine condition. Let us recall that according to a classical theorem of Kolmogorov [1], the existence of such tori is assured when p is a small perturbation of a completely integrable symbol satisfying suitable non-degeneracy assumptions. Since our primary purpose here is to examine the rôle of the rational tori, which are in general destroyed by perturbing a completely integrable system, throughout this paper we shall work under the assumption that the H_p–flow itself is completely integrable. We proceed therefore to discuss the precise assumptions on the geometry of the energy surface $p^{-1}(0) \cap T^*M$ in this case.

2.2. Assumptions related to the complete integrability

As in [17], let us assume that there exists an analytic real valued function f on T^*M such that $H_p f = 0$, with the differentials df and dp being linearly independent almost everywhere. For each $E \in \mathrm{neigh}(0, \mathbb{R})$, the level sets $\Lambda_{a,E} = f^{-1}(a) \cap p^{-1}(E) \cap T^*M$ are invariant under the H_p–flow and form a singular foliation of the 3-dimensional hypersurface $p^{-1}(E) \cap T^*M$. At each regular point, the leaves of this foliation are 2-dimensional Lagrangian submanifolds, and each regular leaf is a finite union of tori. In what follows we shall use the word "leaf" and notation Λ for a connected component of some $\Lambda_{a,E}$. Let J be the set of all leaves in $p^{-1}(0) \cap T^*M$. Then we have a disjoint union decomposition

$$p^{-1}(0) \cap T^*M = \bigcup_{\Lambda \in J} \Lambda, \tag{2.11}$$

where Λ are compact connected H_p–invariant sets. The set J has a natural structure of a graph whose edges correspond to families of regular leaves and the set S of vertices is composed of singular leaves. The union of edges $J \backslash S$ possesses a natural real analytic structure and the corresponding tori depend analytically on $\Lambda \in J \backslash S$ with respect to that structure. See section 7 for an explicit description of the Lagrangian foliation in the case when M is an analytic surface of revolution in $\mathbb{R}^3$.

As in [17], we shall require J to be a finite connected graph. We identify each edge of J analytically with a real bounded interval and this determines a distance on J in the natural way. Assume the continuity property

(2.12) For every $\Lambda_0 \in J$ and every $\varepsilon > 0$, $\exists\, \delta > 0$, such that

$$\text{if } \Lambda \in J \text{ and } \operatorname{dist}_J(\Lambda, \Lambda_0) < \delta, \text{ then } \Lambda \subset \{\rho \in p^{-1}(0);\ \operatorname{dist}(\rho, \Lambda_0) < \varepsilon\}.$$

These assumptions are satisfied, for instance, when f is a Morse-Bott function restricted to $p^{-1}(0) \cap T^*M$, as in this case the structure of the singular leaves is known [25].

Each torus $\Lambda \in J\backslash S$ carries real analytic coordinates x_1, x_2 identifying Λ with $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, so that along Λ, we have

$$H_p = a_1\partial_{x_1} + a_2\partial_{x_2}, \tag{2.13}$$

where $a_1, a_2 \in \mathbb{R}$. The rotation number is defined as the ratio

$$\omega(\Lambda) = [a_1 : a_2] \in \mathbb{RP}^1,$$

and it depends analytically on $\Lambda \in J\backslash S$. We assume that

$$\omega(\Lambda) \text{ is not identically constant on any open edge.}$$

Recall that the leading perturbation q has been introduced in (2.10). For each torus $\Lambda \in J\backslash S$, we define the torus average $\langle q\rangle(\Lambda)$ obtained by integrating $q|_\Lambda$ with respect to the natural smooth measure on Λ, and assume that the analytic function $J\backslash S \ni \Lambda \mapsto \langle q\rangle(\Lambda)$ is not identically constant on any open edge.

We introduce

$$\langle q\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp(tH_p)\, dt, \quad T > 0, \tag{2.14}$$

and consider the compact intervals $Q_\infty(\Lambda) \subset \mathbb{R}$, $\Lambda \in J$, defined as in [17],

$$Q_\infty(\Lambda) = \left[\lim_{T\to\infty} \inf_{\Lambda} \langle q\rangle_T, \lim_{T\to\infty} \sup_{\Lambda} \langle q\rangle_T\right]. \tag{2.15}$$

Notice that when $\Lambda \in J\backslash S$ and $\omega(\Lambda) \notin \mathbb{Q}$ then $Q_\infty(\Lambda) = \{\langle q\rangle(\Lambda)\}$. In the rational case, we write $\omega(\Lambda) = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are relatively prime, and where we may assume that $m = \mathcal{O}(n)$. When $k(\omega(\Lambda)) := |m| + |n|$ is the height of $\omega(\Lambda)$, we recall from Proposition 7.1 in [17] that

$$Q_\infty(\Lambda) \subset \langle q\rangle(\Lambda) + \mathcal{O}\left(\frac{1}{k(\omega(\Lambda))^\infty}\right)[-1, 1]. \tag{2.16}$$

Remark. As $J\backslash S \ni \Lambda \to \Lambda_0 \in S$, the set of all accumulation points of $\langle q\rangle(\Lambda)$ is contained in the interval $Q_\infty(\Lambda_0)$. Indeed, when $\Lambda \in J\backslash S$ and $T > 0$, there exists $\rho = \rho_{T,\Lambda} \in \Lambda$ such that $\langle q\rangle(\Lambda) = \langle q\rangle_T(\rho)$. Therefore, each accumulation point of $\langle q\rangle(\Lambda)$ as $\Lambda \to \Lambda_0 \in S$, belongs to $[\inf_{\Lambda_0} \langle q\rangle_T, \sup_{\Lambda_0} \langle q\rangle_T]$. The conclusion follows if we let $T \to \infty$.

Let $\Lambda_0 \in J\backslash S$ be a rational invariant Lagrangian torus, so that as above, $\omega_0 := \omega(\Lambda_0) = \frac{m}{n} \in \mathbb{Q}$, $m = \mathcal{O}(n)$. For future reference, we shall finish this subsection by considering the behavior of the interval $Q_\infty(\Lambda)$ when $\Lambda \neq \Lambda_0$ is a rational torus in a neighborhood of Λ_0.

Writing $\omega(\Lambda) = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are relatively prime, $p = \mathcal{O}(q)$, we get, using that $\omega(\Lambda) \neq \omega_0$,

$$|\omega(\Lambda) - \omega_0| \geq \frac{1}{nq} \geq \frac{1}{nk(\omega(\Lambda))}, \tag{2.17}$$

and therefore, in view of (2.16),

$$Q_\infty(\Lambda) \subset \langle q\rangle(\Lambda) + \mathcal{O}(\mathrm{dist}(\omega(\Lambda), \omega_0)^\infty)[-1, 1]. \tag{2.18}$$

This estimate is uniform in ω_0 provided that we have a uniform upper bound on the height of the rotation number $\omega_0 \in \mathbb{Q}$.

2.3. Statement of the main results

From Theorem 7.6 in [17] we recall that

$$\frac{1}{\varepsilon}\mathrm{Im}\ (\mathrm{Spec}(P_\varepsilon) \cap \{z; |\mathrm{Re}\, z| \leq \delta\}) \subset \left[\inf \bigcup_{\Lambda\in J} Q_\infty(\Lambda) - o(1), \sup \bigcup_{\Lambda\in J} Q_\infty(\Lambda) + o(1)\right], \tag{2.19}$$

as ε, h, $\delta \to 0$. Let us also recall from [17] that a torus $\Lambda \in J\backslash S$ is said to be Diophantine if representing $H_p|_\Lambda = a_1\partial_{x_1} + a_2\partial_{x_2}$, as in (2.13), we have

$$|a \cdot k| \geq \frac{1}{C_0\,|k|^{N_0}}, \quad 0 \neq k \in \mathbb{Z}^2, \tag{2.20}$$

for some fixed C_0, $N_0 > 0$.

Let $F_0 \in \cup_{\Lambda\in J} Q_\infty(\Lambda)$ be such that there exist finitely many Lagrangian tori

$$\Lambda_{1,d}, \dots, \Lambda_{L,d} \in J\backslash S \tag{2.21}$$

that are uniformly Diophantine as in (2.20), and such that

$$\langle q\rangle(\Lambda_{j,d}) = F_0 \quad \text{for } 1 \leq j \leq L, \tag{2.22}$$

with

$$d_{\Lambda=\Lambda_{j,d}}\langle q\rangle(\Lambda) \neq 0, \quad 1 \leq j \leq L. \tag{2.23}$$

Moreover, assume also that there exist tori $\Lambda_{1,r}, \dots, \Lambda_{L',r} \in J\backslash S$ with $\omega(\Lambda_{j,r}) \in \mathbb{Q}$, $1 \leq j \leq L'$, and such that the isoenergetic condition

$$d_{\Lambda=\Lambda_{j,r}}\omega(\Lambda) \neq 0 \tag{2.24}$$

is satisfied for each j, $1 \leq j \leq L'$. Assume next that the length $|Q_\infty(\Lambda_{j,r})|$ of each interval $Q_\infty(\Lambda_{j,r})$ satisfies

$$|Q_\infty(\Lambda_{j,r})| > 0, \ \ j = 1, \dots, L', \tag{2.25}$$

and that

$$F_0 \in Q_\infty(\Lambda_{j,r}), \quad 1 \leq j \leq L'. \tag{2.26}$$

We shall assume that

$$|\langle q\rangle(\Lambda_{j,r}) - F_0| \geq \frac{1}{\mathcal{O}(1)}, \quad 1 \leq j \leq L'. \tag{2.27}$$

Let us finally make the following global assumption:

$$F_0 \notin \bigcup_{\Lambda \in J \setminus \{\Lambda_{1,d},\dots,\Lambda_{L,d},\Lambda_{1,r},\dots,\Lambda_{L',r}\}} Q_\infty(\Lambda). \tag{2.28}$$

Here we notice that the earlier assumptions imply that $F_0 \notin Q_\infty(\Lambda)$ for $\Lambda_{j,d} \neq \Lambda \in \mathrm{neigh}(\Lambda_{j,d}, J)$, $1 \le j \le L$, and $\Lambda_{j,r} \neq \Lambda \in \mathrm{neigh}(\Lambda_{j,r}, J)$, $1 \le j \le L'$.

THEOREM 2.1. – *Let $F_0 \in \cup_{\Lambda\in J} Q_\infty(\Lambda)$ be such that the assumptions (2.22), (2.23), (2.24), (2.25), (2.26), (2.27), and (2.28) are satisfied. For $1 \le j \le L$, we fix a basis for the first homology group of each Diophantine torus $\Lambda_{j,d}$ given by the cycles $\alpha_{k,j} \subset \Lambda_{j,d}$, $k = 1, 2$, and let $S_j \in \mathbb{R}^2$ be the actions and $k_j \in \mathbb{Z}^2$ be the Maslov indices of $\alpha_{k,j}$. Let*

$$\kappa_j : \mathrm{neigh}(\Lambda_{j,d}, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2) \tag{2.29}$$

be a canonical transformation given by the action-angle variables near $\Lambda_{j,d}$, $1 \le j \le L$, and such that $\kappa_j(\alpha_{k,j}) = \{x \in \mathbb{T}^2; x_{3-k} = 0\}$, $k = 1, 2$. Let $\delta > 0$ be small and assume that

$$h \ll \varepsilon \le h^{\frac{2}{3}+\delta}.$$

Let $C > 0$ be sufficiently large. Then there exists a bijection b between the spectrum of P_ε in the rectangle

$$R(F_0, C, \varepsilon, \delta) := \left[-\frac{\varepsilon}{C}, \frac{\varepsilon}{C}\right] + i\varepsilon\left[F_0 - \frac{\varepsilon^\delta}{C}, F_0 + \frac{\varepsilon^\delta}{C}\right] \tag{2.30}$$

and the union of two sets of points, E_d and E_r, such that $b(\mu) - \mu = \mathcal{O}(h^{N_0})$. Here N_0 is fixed but can be taken arbitrarily large. The elements of the set E_d, $z(j,k)$, are described by Bohr-Sommerfeld type conditions,

$$z(j,k) = P_j^{(\infty)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \varepsilon; h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbb{Z}^2, \quad 1 \le j \le L, \tag{2.31}$$

with precisely one element for each $k \in \mathbb{Z}^2$ such that the corresponding $z(j,k)$ belongs to the rectangle (2.30). Here $P_j^{(\infty)}(\xi, \varepsilon; h)$ is smooth in $\xi \in \mathrm{neigh}(0, \mathbb{R}^2)$ and $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$, real-valued for $\varepsilon = 0$. We have

$$P_j^{(\infty)}(\xi, \varepsilon; h) \sim \sum_{\ell=0}^{\infty} h^\ell p_{j,\ell}^{(\infty)}(\xi, \varepsilon), \quad 1 \le j \le L, \tag{2.32}$$

and

$$p_{j,0}^{(\infty)}(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2). \tag{2.33}$$

Here p and q have been expressed in terms of the action-angle variables near $\Lambda_{j,d}$ given by κ_j in (2.29), and $\langle q\rangle$ is the torus average of q in these coordinates. The cardinality of the set E_r is

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right). \tag{2.34}$$

Remark. It follows from Theorem 2.1 that the total number of elements of the set E_d is $\sim \varepsilon^{1+\delta}/h^2$. Therefore, from (2.34) we see that for $0 < \delta < 1/2$, the contribution E_r to the spectrum of P_ε in $R(F_0, C, \varepsilon, \delta)$, coming from the rational region, is much weaker than that of the Diophantine tori $\Lambda_{j,d}$, $1 \le j \le L$. As will be seen in the proof, for this result the assumption (2.27) is important.

Remark. Assume that the subprincipal symbol of $P_{\varepsilon=0}$ in (2.1) and (2.6) vanishes. Then it follows from the discussion in the body of the paper that Theorem 2.1 is valid in the larger range

$$h^2 \ll \varepsilon = \mathcal{O}(h^{\frac{2}{3}+\delta}), \quad 0<\delta \ll 1. \tag{2.35}$$

Remark. In section 7, following [17], we shall introduce the notion of uniformly good values $F_0 \in \mathbb{R}$, for which the conclusion of Theorem 2.1 will be valid uniformly, so that in particular, the (explicit or implicit) constants in Theorem 2.1 are independent of the choice of a uniformly good value.

Theorem 2.1 can be viewed as a partial generalization of one of the main results of [17], where energy levels corresponding only to Diophantine tori have been considered. In that paper, instead of the upper bound $\varepsilon \leq h^{2/3+\delta}$, it was merely required that $\varepsilon = \mathcal{O}(h^{\delta})$ for some small fixed $\delta > 0$. (Also the lower bounds there were considerably weaker than $h \ll \varepsilon$.) As will be seen in the proof, here the strengthened upper bound on ε is required in order to compensate for the exponential growth of the resolvent of P_ε in the rational region, when considering the tunnel effect between the Diophantine and the rational tori — see also the discussion in the next section.

In the case when there are no Diophantine tori corresponding to the energy level $(0, \varepsilon F_0) \in \mathbb{C}$, the result of Theorem 2.1 can be improved in two ways: we can put $\delta = 0$ in (2.30), and also, the upper bound $\varepsilon \leq h^{2/3+\delta}$ can be replaced by $\varepsilon = \mathcal{O}(h^{\widetilde{\delta}})$, $\widetilde{\delta} > 0$.

THEOREM 2.2. – *Let us keep all the assumptions of Theorem 2.1, and assume that $L = 0$ in (2.21). Assume furthermore that $\varepsilon = \mathcal{O}(h^{\widetilde{\delta}})$, $\widetilde{\delta} > 0$, satisfies $\varepsilon \gg h$. There exists a constant $C > 0$ such that the number of eigenvalues of P_ε in the rectangle*

$$|\mathrm{Re}\, z| < \frac{\varepsilon}{C}, \quad \left| \frac{\mathrm{Im}\, z}{\varepsilon} - F_0 \right| < \frac{1}{C} \tag{2.36}$$

does not exceed

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right). \tag{2.37}$$

Remark. As will be explained in the beginning of section 4, the isoenergetic assumption (2.24) implies that associated with each rational torus $\Lambda_{j,r}$, $1 \leq j \leq L'$, there is an analytic family of rational Lagrangian tori $\Lambda_{E,j,r} \subset p^{-1}(E) \cap T^*M$ for $E \in \mathrm{neigh}(0, \mathbb{R})$, depending analytically on E, and with $\Lambda_{E=0,j,r} = \Lambda_{j,r}$, $1 \leq j \leq L'$. Theorem 2.2 can therefore be interpreted as saying that only an $\varepsilon^{1/2}$-neighborhood of the set

$$\bigcup_{j=1}^{L'} \bigcup_{E=\mathcal{O}(\varepsilon)} \Lambda_{E,j,r} \tag{2.38}$$

contributes to the spectrum in the region (2.36).

For notational simplicity only, when proving Theorem 2.1 and Theorem 2.2, we shall assume that $L = 2$, $L' = 1$, and that $\Lambda_{1,d}$, $\Lambda_{2,d}$, and $\Lambda_{1,r}$ all belong to the same open edge of J, so that, when identifying the edge with a real bounded interval, we have

$$\Lambda_{1,d} < \Lambda_{1,r} < \Lambda_{2,d}. \tag{2.39}$$

See also Figure 1.

The structure of the paper is as follows. In section 3 we present a general outline of the proof of Theorem 2.1. Section 4 is devoted to a formal microlocal Birkhoff normal form construction for P_ε near $\Lambda_{1,r}$, and in section 5 the formal argument of the previous section is justified by constructing a microlocal Hilbert space in a full neighborhood of $\Lambda_{1,r}$, realizing the normal form reduction there. In the beginning of section 6 we construct the global Hilbert space where we study our operator P_ε, and introduce two reference operators, associated with the Diophantine and the rational regions, respectively. Section 6 is concluded by constructing the resolvent for P_ε globally, and we obtain Theorem 2.1 by comparing the spectral projections of P_ε and of the reference operators. In section 7, we apply Theorem 2.1 to a small complex perturbation of the semiclassical Laplacian on a convex analytic surface of revolution, and give a partial generalization of the corresponding discussion in [17]. The appendix contains a proof of a simple trace class estimate for the Toeplitz operator with a compactly supported smooth symbol, acting on a weighted L^2–space of holomorphic functions. This estimate, which seems to be of an independent interest, is used in section 6 in the main text.

3. Outline of the proof

The purpose of this section is to provide a broad outline of the proof of Theorem 2.1. Compared with the previous work [17], addressing only the case of Diophantine tori, here the essential new difficulties will be concerned with the analysis in the rational region. We shall begin by presenting an outline of the argument in this case.

Working microlocally in the rational region and introducing action-angle variables in a neighborhood of $\Lambda_{1,r} \simeq \mathbb{T}^2$, we are led to consider an operator, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2_{x,\xi}$, with the leading symbol given by

$$p_\varepsilon(x,\xi) = p(\xi) + i\varepsilon q(x,\xi) + \mathcal{O}(\varepsilon^2), \quad p(\xi) = \omega\cdot\xi + \mathcal{O}(\xi^2). \tag{3.1}$$

Here $\omega = (k,l) \in \mathbb{Z}^2$ and to fix the ideas, let us restrict the attention to the model case where $\omega = (0,1)$ and the $\mathcal{O}(\xi^2)$–term in (3.1) reduces to ξ_1^2 — this choice of the nonlinearity in p is in agreement with the isoenergetic condition (2.24). Following the general ideas of a Birkhoff normal form construction, we would like to eliminate, as much as possible, the x-dependence in the symbol in (3.1). Performing first successive averaging procedures along the closed orbits of the H_p–flow comprising the rational tori $\Lambda_{E,1,r}$, $E \in \mathrm{neigh}(0,\mathbb{R})$, we achieve that the leading symbol in (3.1) becomes

$$\widetilde{p}_\varepsilon(x,\xi) = \xi_2 + \xi_1^2 + \mathcal{O}(\varepsilon) + \mathcal{O}((\varepsilon,\xi_1)^\infty), \tag{3.2}$$

where the $\mathcal{O}(\varepsilon)$–term is independent of x_2. In the terminology of classical mechanics, this initial reduction is based on a secular perturbation theory — see [20]. Carrying out the reduction on the operator level, we obtain an operator of the form $\widetilde{P}_\varepsilon = \widetilde{P}_\varepsilon(x_1, hD_{x_1}, hD_{x_2}; h)$, which may also be viewed as a family of one-dimensional non-selfadjoint operators acting in x_1, with a leading symbol of the form

$$hk + \xi_1^2 + \mathcal{O}(\varepsilon), \quad k \in \mathbb{Z}.$$

For this family, we cannot exclude the occurrence of a pseudospectral phenomenon [2], leading to the exponential growth of the resolvent norms in the spectral regions of interest. This makes it difficult to exploit the secular perturbation theory and to simplify the operator further.

Nevertheless, in section 4 we show that working in a region where

$$|\xi_1| \gg \varepsilon^{1/2}, \tag{3.3}$$

and so away from an $\varepsilon^{1/2}$–neighborhood of the set

$$\bigcup_{|E|<\delta_0} \Lambda_{E,1,r}, \quad 0 < \delta \ll 1, \tag{3.4}$$

the x_1-dependence in the symbol (3.2) can be eliminated completely, and in particular, here the leading perturbation q in (3.1) becomes replaced by its torus average. When approaching the region where $\xi_1 = \mathcal{O}(\varepsilon^{1/2})$, the normal form construction breaks down and no additional simplification of the operator P_ε is obtained.

To implement the complete reduction in the region (3.3) requires an introduction of a microlocal Hilbert space of functions in a sufficiently small but fixed neighborhood of $\Lambda_{1,r}$. Because of the degeneration of the normal form construction very close to the rational torus, when defining the Hilbert space in a full neighborhood of $\Lambda_{1,r}$, it becomes convenient and indeed, natural, to perform a second microlocalization — in this case, it amounts to considering our operators in $\tilde{h} = h/\sqrt{\varepsilon}$–quantization with respect to the x_1–variable and performing an $\tilde{h}$–Bargmann transformation in x_1. In section 5 we show that, on the transform side, the microlocal Hilbert space in question becomes a well-defined weighted space of holomorphic functions in a region $(\mathrm{Re}\, x_1, \mathrm{Re}\, x_2) \in \mathbb{T}^2$, $|\mathrm{Im}\, x_1| \ll \frac{1}{\sqrt{\varepsilon}}$, $|\mathrm{Im}\, x_2| \ll 1$, with the corresponding strictly plurisubharmonic weight being uniformly well behaved and close to the standard quadratic one — see Proposition 5.1 for the precise statement and also, the discussion in subsection 5.4.

The idea now is to use the assumption (2.27) to show that $(1/\varepsilon)\,(P_\varepsilon - z)$ becomes elliptic (viewed as an $\tilde{h}$-pseudodifferential operator) near $\Lambda_{1,r}$, when away from an $\mathcal{O}(\varepsilon^{1/2})$–neighborhood of the set in (2.38), while the invertibility away from the tori $\Lambda_{1,r} \cup \Lambda_{1,d} \cup \Lambda_{2,d}$ should follow from (2.28). Here the spectral parameter z varies in the domain (2.36).

To handle the remaining phase space region near $\Lambda_{1,r}$, in section 6 we construct a trace class perturbation K, whose trace class norm does not exceed

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right), \tag{3.5}$$

such that if $P_d := P_\varepsilon + i\varepsilon K$ then $P_d - z$ becomes invertible, when away from the Diophantine quasi-eigenvalues $z(j,k)$ in (2.31). Moreover, we obtain a sufficiently good control on the norm of the inverse of P_d, when the latter is considered in a global Hilbert space, obtained by gluing together the microlocal Hilbert space near $\Lambda_{1,r}$ and the space away from $\Lambda_{1,r}$, defined using the Diophantine analysis of [17]. The trace class perturbation K is constructed as a Toeplitz operator on the FBI–Bargmann transform side and when deriving the trace class norm bound (3.5), we use a general estimate of Proposition A.1 in the appendix.

It will be fruitful to think of the Diophantine and the rational tori in question as of microlocal wells to which the main difficulties of our problem are localized. From this point of view we may think of the operator P_d as a reference operator associated to the Diophantine region. Proceeding in the spirit of tunneling problems, in section 6 we next define and study a reference operator associated to the rational region, P_r, obtained by modifying P_ε away from the rational region and such that $P_r - z$ is invertible outside of a small neighborhood of $\Lambda_{1,r}$. Because of the pseudospectral difficulties in the normal form construction for P_ε in an $\mathcal{O}(\varepsilon^{1/2})$–neighborhood of $\Lambda_{1,r}$, when estimating the resolvent of P_r, we are only able to show that it enjoys an exponential upper bound, with the exponent there being given, roughly speaking, by the phase space volume of the region near the rational torus, not covered by the normal form, multiplied by h^{-2}, or, equivalently, by the trace class norm of the perturbation K in (3.5).

Using the operators P_d and P_r, together with an additional reference operator corresponding to the elliptic region, we next construct and study an approximate, and then exact, resolvent of P_ε. To obtain the main result of Theorem 2.1 we would like to compare the spectral projections of P_ε with those of the reference operators. Due to the exponential growth of the resolvent of P_ε near $\Lambda_{1,r}$, at this point it becomes very important to estimate the tunnel effect between the Diophantine and the rational tori and to show that it is small enough to overrule the pseudospectral growth of the resolvent in the rational region. This tunneling analysis is carried out at the end of section 6 and it involves an additional modification of phase space exponential weights near the invariant tori. Imposing the upper bound $\varepsilon = \mathcal{O}\left(h^{2/3+\delta}\right)$, $0 < \delta \ll 1$, assures that our perturbative argument goes through, and we can conclude the proof by comparing the spectral projections, as indicated above.

Remark. The idea of using auxiliary trace class perturbations to create a gap in the spectrum of a non-selfadjoint operator has a long tradition in abstract non-selfadjoint spectral theory and seems to go back to the work of Markus and Matsaev [22], see also [21]. It has been used by the second author in the theory of resonances [28], [30], and when studying spectral asymptotics for damped wave equations on compact domains [29] (see also [12]). In the present paper, in the absence of the Diophantine tori, once the trace class perturbation K, alluded to above, has been constructed, we can conclude the proof of Theorem 2.2, in section 6, by relying upon some standard Fredholm determinant estimates [5].

4. The normal form construction near $\Lambda_{1,r}$

For simplicity, we shall concentrate throughout the following discussion on the case when $M = \mathbb{R}^2$, the compact real analytic case being analogous — see also the appendix in [14] for the basic facts about FBI transforms on manifolds. We shall keep all the assumptions made in the introduction, and consider an operator P_ε in (2.1) with a principal symbol

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2), \tag{4.1}$$

in a neighborhood of $p^{-1}(0) \cap \mathbb{R}^4$. In order to simplify the presentation, we shall furthermore assume that the order function m introduced in (2.2) belongs to $L^\infty(\mathbb{R}^4)$. It will be clear that the analysis below extends to the case of a general order function $m \geq 1$. From the introduction, let us also recall the simplifying assumption that $L' = 1$ so that $\Lambda_{1,r}$ is the only rational torus corresponding to the level $(0, \varepsilon F_0)$.

In this section, we shall work microlocally near $\Lambda_{1,r} \subset p^{-1}(0) \cap \mathbb{R}^4$. Let

$$\kappa_0 : \mathrm{neigh}(\Lambda_{1,r}, \mathbb{R}^4) \to \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2), \tag{4.2}$$

be a real and analytic canonical transformation, given by the action-angle variables, and such that $\kappa_0(\Lambda_{1,r})$ is the zero section in $T^*\mathbb{T}^2$. Then $p \circ \kappa_0^{-1}$ is a function of ξ only, and to simplify the notation we shall write $p \circ \kappa_0^{-1} = p(\xi)$. We have $p(0) = 0$ and without loss of generality we may assume that

$$\partial_{\xi_1} p(0) = 0, \quad \partial_{\xi_2} p(0) > 0. \tag{4.3}$$

The isoenergetic assumption (2.24) takes the following form,

$$\partial_{\xi_1}^2 p(0) \neq 0. \tag{4.4}$$

In order to fix the ideas, we assume that $\partial_{\xi_1}^2 p(0) > 0$.

By the implicit function theorem, the equation $\partial_{\xi_1} p(\xi) = 0$ has a unique analytic local solution $\xi_1 = f(\xi_2)$ with $f(0) = 0$. The function $\xi_2 \mapsto p(f(\xi_2), \xi_2)$ has a positive derivative near 0, and therefore the equation $p(f(\xi_2), \xi_2) = E$ has a unique solution $\xi_2(E)$ close to 0 for $E \in \mathrm{neigh}(0, \mathbb{R})$. We obtain a family of rational Lagrangian tori $\Lambda_{E,1,r} \subset p^{-1}(E)$, defined by

$$\xi_2 = \xi_2(E), \quad \xi_1 = f(\xi_2(E)). \tag{4.5}$$

By construction, $\partial_{\xi_1} p = 0$ on $\Lambda_{E,1,r}$, and hence,

$$\partial_{\xi_1} p(\xi_1, \xi_2) = \mathcal{O}(\xi_1 - f(\xi_2)), \partial_{\xi_2} p(\xi_1, \xi_2) = \partial_{\xi_2} p(f(\xi_2), \xi_2) + \mathcal{O}((\xi_1 - f(\xi_2))). \tag{4.6}$$

Implementing κ_0 in (4.2) by means of a microlocally unitary Fourier integral operator with a real phase as in Theorem 2.4 in [14], and conjugating P_ε by this operator, we obtain a new h-pseudodifferential operator, still denoted by P_ε, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$. The full symbol of P_ε is holomorphic in a fixed complex neighborhood of $\xi = 0$, and the leading symbol is given by

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + \mathcal{O}(\varepsilon^2), \tag{4.7}$$

with

$$p(\xi_1, \xi_2) = p(f(\xi_2), \xi_2) + g(\xi_1, \xi_2)(\xi_1 - f(\xi_2))^2. \tag{4.8}$$

Here $g(0) > 0$, since we have assumed that $\partial_{\xi_1}^2 p(0) > 0$, and the function q in (4.7) is real on the real domain. On the operator level, P_ε acts in the space of microlocally defined Floquet periodic functions on $\mathbb{T}^2$, $L^2_\theta(\mathbb{T}^2) \subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$, elements u of which satisfy

$$u(x - \nu) = e^{i\theta\cdot\nu} u(x), \quad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}, \quad \nu \in 2\pi\mathbb{Z}^2. \tag{4.9}$$

Here $S = (S_1, S_2)$ is given by the classical actions,

$$S_j = \int_{\alpha_j} \eta\, dy, \quad j = 1, 2,$$

with α_j forming a system of fundamental cycles in $\Lambda_{1,r}$, such that

$$\kappa_0(\alpha_j) = \beta_j, \quad j = 1, 2, \quad \beta_j = \{x \in \mathbb{T}^2; x_{3-j} = 0\}.$$

The tuple $k_0 = (k_0(\alpha_1), k_0(\alpha_2)) \in \mathbb{Z}^2$ stands for the Maslov indices of the cycles α_j, $j = 1, 2$.

As a first step in the normal form construction for P_ε, we shall apply the secular perturbation theory to the principal symbol p_ε in (4.7) — see also [20].

Let

$$\langle q\rangle_2(x_1,\xi) = \frac{1}{2\pi}\int_0^{2\pi} q(x,\xi)\,dx_2 \tag{4.10}$$

denote the average of q with respect to x_2. Using the assumption (4.3) and proceeding as in section 4 of [14] (see also section 2 of [17]), it is straightforward to construct, by successive averagings in x_2, a symbol $G_1(x,\xi) = G_1^{(N)}(x,\xi)$, analytic in (x,ξ), such that

$$H_pG_1 = q - \widetilde{\langle q\rangle_2}(x_1,\xi) + \mathcal{O}((\xi_1 - f(\xi_2))^N), \tag{4.11}$$

where $\widetilde{\langle q\rangle_2}(x_1,\xi) = \langle q\rangle_2(x_1,\xi) + \mathcal{O}(\xi_1 - f(\xi_2))$ is independent of x_2. Here $N \in \mathbb{N}$ can be taken arbitrarily large but fixed. We get from (4.11), by a Taylor expansion,

$$\begin{aligned} p_\varepsilon\left(\exp\left(i\varepsilon H_{G_1}\right)(x,\xi)\right) &= p(\xi) + i\varepsilon\widetilde{\langle q\rangle_2}(x_1,\xi) + \mathcal{O}\left(\varepsilon^2 + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N\right)\\ &= p(\xi) + i\varepsilon\widetilde{\langle q\rangle_2}(x_1,\xi) + i\varepsilon^2\widetilde{q} + \mathcal{O}\left(\varepsilon^3 + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N\right), \end{aligned}$$

where $\widetilde{q} = \widetilde{q}(x,\xi)$. We next construct G_2, analytic in (x,ξ) and such that

$$H_pG_2 = \widetilde{q} - \widetilde{\langle \widetilde{q}\rangle_2}(x_1,\xi) + \mathcal{O}((\xi_1 - f(\xi_2))^N).$$

Then

$$\begin{aligned} &p_\varepsilon\left(\exp\left(i\varepsilon H_{G_1}\right)\left(\exp\left(i\varepsilon^2 H_{G_2}\right)(x,\xi)\right)\right)\\ &\qquad\qquad = p(\xi) + i\varepsilon\widetilde{\langle q\rangle_2} + i\varepsilon^2\widetilde{\langle \widetilde{q}\rangle_2} + \mathcal{O}\left(\varepsilon^3 + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N\right). \end{aligned}$$

It is clear that this procedure can be iterated, and after N steps, we define

$$\kappa_\varepsilon := \exp\left(i\varepsilon H_{G_1}\right) \circ \exp\left(i\varepsilon^2 H_{G_2}\right) \circ \cdots \circ \exp\left(i\varepsilon^N H_{G_N}\right). \tag{4.12}$$

It follows that

$$\begin{aligned} p_\varepsilon\left(\kappa_\varepsilon(x,\xi)\right) &= p(\xi) + i\varepsilon\widetilde{\langle q\rangle_2}(x_1,\xi) + \varepsilon^2 r_\varepsilon(x_1,\xi)\\ &+ \mathcal{O}\left(\varepsilon^{N+1} + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N\right) = p'_\varepsilon(x_1,\xi) + \mathcal{O}\left(\varepsilon^{N+1} + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N\right). \end{aligned} \tag{4.13}$$

Here the last equality defines $p'_\varepsilon(x_1,\xi)$.

Using the same averaging procedure as above also on the level of lower order symbols, as in section 4 of [14] and section 3 of [17], we conclude that there exists an analytic elliptic Fourier integral operator $F = F_\varepsilon^{(N)}$ in the complex domain, quantizing the holomorphic canonical transformation κ_ε in (4.12), such that

$$F^{-1}P_\varepsilon F = P'_\varepsilon(x_1, hD_x; h) + R_\varepsilon(x, hD_x; h). \tag{4.14}$$

Here the full symbol of P'_ε is independent of x_2 and

$$R_\varepsilon(x,\xi;h) = \mathcal{O}\left(\varepsilon^{N+1} + \varepsilon\left(\xi_1 - f(\xi_2)\right)^N + h^{N+1}\right). \tag{4.15}$$

The leading symbol of $P'_\varepsilon(x_1, hD_x; h)$ is $p'_\varepsilon(x_1,\xi)$ in (4.13). As in section 6 of [14] and section 2 of [24], the operator F is defined by working on the FBI–Bargmann transform side.

When discussing further reductions of P'_ε, it is natural to exploit the fact that this operator is independent of x_2, and hence, at least formally, by taking a Fourier series expansion in x_2, we can reduce the study of P'_ε to the study of a family of one-dimensional operators $P'_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$, with

$$\xi_2 = h\left(k - \frac{k_0(\alpha_2)}{4}\right) - \frac{S_2}{2\pi} \in \operatorname{neigh}(0, \mathbb{R}), \quad k \in \mathbb{Z}. \tag{4.16}$$

The family $P'_\varepsilon(x_1, hD_{x_1}, \xi_2, h)$ acts on the microlocal space of Floquet periodic functions $L^2_{\theta_1}(\mathbb{T}^1)$, $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\theta_1 = S_1/2\pi h + k_0(\alpha_1)/4$, defined similarly to (4.9). We would like to eliminate the x_1–dependence in the symbol of $P'_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$ by means of an additional conjugation by an elliptic Fourier integral operator. Using (4.8) and (4.13) we get

$$\begin{aligned} P'_\varepsilon(x_1, hD_{x_1}, \xi_2; h) = p(f(\xi_2), \xi_2) + g(hD_{x_1}, \xi_2)\,(hD_{x_1} - f(\xi_2))^2 \\ + i\varepsilon\widetilde{\langle q\rangle}_2(x_1, hD_{x_1}, \xi_2) + \varepsilon^2 r_\varepsilon(x_1, hD_{x_1}, \xi_2) + \mathcal{O}(h) + \mathcal{O}(h^2). \end{aligned} \tag{4.17}$$

Let us recall that $g(0) > 0$, and the $\mathcal{O}(h)$–contribution in (4.17) is the subprincipal term in the full symbol of P'_ε. After a conjugation by $\exp\left(\frac{i}{h} f(\xi_2) x_1\right)$, modifying the Floquet condition on $\mathbb{T}^1$, we get

$$\begin{aligned} & e^{-\frac{i}{h} f(\xi_2) x_1} P'_\varepsilon\,(x_1, hD_{x_1}, \xi_2; h)\, e^{\frac{i}{h} f(\xi_2) x_1} \\ &= p(f(\xi_2), \xi_2) + g\,(f(\xi_2) + hD_{x_1}, \xi_2)\,(hD_{x_1})^2 \\ &\quad + \left(i\varepsilon\widetilde{\langle q\rangle}_2 + \varepsilon^2 r_\varepsilon + \mathcal{O}(h) + \mathcal{O}(h^2)\right)(x_1, f(\xi_2) + hD_{x_1}, \xi_2). \end{aligned} \tag{4.18}$$

In section 4 of [14], it is explained how to eliminate the x_1–dependence in (4.18) by means of a Fourier integral operator conjugation. Here we shall follow the procedure there after a suitable change of Planck's constant. Let us work microlocally in a region

$$|\xi_1| \sim \mu, \quad (\varepsilon + h)^{1/2} \ll \mu \ll 1. \tag{4.19}$$

We write

$$hD_{x_1} = \mu\tilde{h}D_{x_1}, \quad \tilde{h} = \frac{h}{\mu} \ll 1.$$

If ξ_1, $\widetilde{\xi_1}$ denote the cotangent variables corresponding to hD_{x_1} and $\tilde{h}D_{x_1}$, respectively, we have

$$\xi_1 = \mu\widetilde{\xi_1}.$$

Then (4.18) gives

$$\begin{aligned} & \mu^{-2} e^{-\frac{i}{h} f(\xi_2) x_1} P'_\varepsilon(x_1, hD_{x_1}, \xi_2; h) e^{\frac{i}{h} f(\xi_2) x_1} \\ &= \frac{1}{\mu^2} p(f(\xi_2), \xi_2) + g(f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2)\left(\tilde{h}D_{x_1}\right)^2 \\ &\quad + \left(\frac{\varepsilon}{\mu^2} i\widetilde{\langle q\rangle}_2 + \frac{\mathcal{O}(h)}{\mu^2} + \frac{\varepsilon^2}{\mu^2} r_\varepsilon\right)(x_1, f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2) \\ &\quad + \mathcal{O}(\tilde{h}^2)\left(x_1, f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2\right), \end{aligned} \tag{4.20}$$

which can be viewed as an $\tilde{h}$–pseudodifferential operator. The symbol associated to the second term in the right hand side of (4.20) is then

$$g(f(\xi_2) + \mu\widetilde{\xi_1}, \xi_2)\widetilde{\xi_1}^2, \tag{4.21}$$

and it follows from (4.19) that we work in a region where

$$\left|\widetilde{\xi_1}\right| \sim 1. \tag{4.22}$$

Notice that in this region, the $\widetilde{\xi}_1$–gradient of (4.21) is of the order of magnitude 1.

We set next

$$r_0\left(x_1, \widetilde{\xi_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = g(f(\xi_2) + \mu\widetilde{\xi_1}, \xi_2)\widetilde{\xi_1}^2 + \mathcal{O}\left(\frac{\varepsilon + h}{\mu^2}\right), \tag{4.23}$$

where the $\mathcal{O}\left(\frac{\varepsilon+h}{\mu^2}\right)$–term stands for the third term in the right hand side of (4.20). Following the argument of section 4 of [14], we shall now recall how the x_1–dependence in r_0 can be eliminated by means of a suitable canonical transformation.

We look for $\varphi_0 = \varphi_0\left(x_1, \widetilde{\xi_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right)$, such that

$$r_0\left(x_1, \widetilde{\xi_1} + \partial_{x_1}\varphi_0, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = \left\langle r_0\left(\cdot, \widetilde{\xi_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right)\right\rangle_1. \tag{4.24}$$

Here, for a smooth function $f(x, \xi)$ defined near $\xi = 0$ in $T^*\mathbb{T}^2$, the expression $\langle f\rangle_1$ stands for the average with respect to x_1,

$$\langle f\rangle_1(x_2, \xi) = \frac{1}{2\pi}\int_0^{2\pi} f(x, \xi)\, dx_1.$$

By the implicit function theorem, (4.24) has an analytic solution with $\partial_{x_1}\varphi_0$ single-valued and $\mathcal{O}((\varepsilon + h)/\mu^2)$. Taking a Taylor expansion of (4.24) and using (4.23), we get

$$\left(\partial_{\widetilde{\xi_1}} r_0\right)\left(x_1, \widetilde{\xi_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right)\partial_{x_1}\varphi_0 + (r_0 - \langle r_0\rangle_1) = \mathcal{O}\left(\left(\frac{\varepsilon + h}{\mu^2}\right)^2\right),$$

and using also that the $\widetilde{\xi}_1$–gradient of (4.21) is ~ 1, we conclude that

$$\varphi_0 = \varphi_\mu + x_1\widetilde{\zeta_1},$$

where

$$\widetilde{\zeta_1} = \widetilde{\zeta_1}\left(\widetilde{\xi_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = \mathcal{O}\left(\left(\frac{\varepsilon + h}{\mu^2}\right)^2\right),$$

and $\varphi_\mu = \mathcal{O}((\varepsilon + h)/\mu^2)$ is periodic in x_1. We set $\widetilde{\eta_1} = \widetilde{\xi_1} + \widetilde{\zeta_1}$, and view φ_μ as a function of $\widetilde{\eta_1}$ rather than $\widetilde{\xi_1}$.

Summarizing the discussion above, we see that there exists a holomorphic phase function

$$\varphi_\mu(x_1, \widetilde{\eta_1}) = \varphi_\mu\left(x_1, \widetilde{\eta_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = \mathcal{O}\left(\frac{\varepsilon + h}{\mu^2}\right) \tag{4.25}$$

defined in a fixed complex neighborhood of $x_1 \in \mathbb{T}^1$, $|\widetilde{\eta_1}| \sim 1$, such that if

$$\psi(x_1, \widetilde{\eta_1}) = x_1\widetilde{\eta_1} + \varphi_\mu(x_1, \widetilde{\eta_1}),$$

then the canonical transformation

$$\kappa_{\mu,\varepsilon,h} : (y_1, \widetilde{\eta_1}) = (\psi'_{\widetilde{\eta_1}}, \widetilde{\eta_1}) \mapsto (x_1, \psi'_{x_1}) = (x_1, \widetilde{\xi_1}) \tag{4.26}$$

is $\mathcal{O}\left(\frac{\varepsilon+h}{\mu^2}\right)$–close to the identity, and

$$\left(g(f(\xi_2) + \mu\widetilde{\xi_1}, \xi_2)\widetilde{\xi_1}^2 + \left(\frac{\varepsilon}{\mu^2} i\widetilde{\langle q\rangle_2} + \frac{\mathcal{O}(h)}{\mu^2} + \frac{\varepsilon^2}{\mu^2} r_\varepsilon\right)(x_1, f(\xi_2) + \mu\widetilde{\xi_1}, \xi_2)\right) \circ \kappa_{\mu,\varepsilon,h}$$

is independent of y_1 and is equal to

$$g(f(\xi_2)+\mu\widetilde{\eta_1})\widetilde{\eta_1}^2 + \left(\frac{\varepsilon}{\mu^2} i\langle\widetilde{\langle q\rangle_2}\rangle_1 + \frac{\mathcal{O}(h)}{\mu^2} + \frac{\varepsilon^2}{\mu^2}\langle r_\varepsilon\rangle_1\right)(f(\xi_2)+\mu\widetilde{\eta_1},\xi_2) + \mathcal{O}\left(\left(\frac{\varepsilon+h}{\mu^2}\right)^2\right). \tag{4.27}$$

In what follows, we shall fix the choice of φ_μ by requiring that $(\varphi_\mu)_{x_1=0}=0$.

Associated to $\kappa_{\mu,\varepsilon,h}$, we can construct an elliptic $\tilde{h}$–Fourier integral operator of the form

$$Gu(x_1) = \frac{1}{2\pi\tilde{h}} \iint e^{\frac{i}{\tilde{h}}(\varphi_\mu(x_1,\widetilde{\eta_1},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\eta_2)+(x_1-y_1)\widetilde{\eta_1})} a(x_1,\widetilde{\eta_1},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\eta_2;\tilde{h})u(y_1)\,dy_1 d\widetilde{\eta_1}, \tag{4.28}$$

such that the full symbol of the $\tilde{h}$-pseudodifferential operator

$$G^{-1}\mu^{-2}e^{-\frac{i}{h}f(\xi_2)x_1}P'_\varepsilon(x_1,\mu\tilde{h}D_{x_1},\xi_2;h)e^{\frac{i}{h}f(\xi_2)x_1}G \tag{4.29}$$

is independent of x_1 (and of x_2), with the principal symbol given by (4.27). For the amplitude in (4.28), we shall require that $(a)_{x_1=0}=1$.

Remark. Working microlocally in a region

$$|\xi_1|\sim\mu,$$

where $\mu\ll 1$ is such that

$$\frac{(\varepsilon+h)^{1/2}}{\mu} \le h^{\delta_1}, \quad \delta_1>0, \tag{4.30}$$

and following some further arguments of section 4 of [14], we see that the canonical transformation $\kappa_{\mu,\varepsilon,h}$ and the $\tilde{h}$–Fourier integral operator G in (4.28) can be constructed by a formal Taylor series in the asymptotically small parameter $(\varepsilon+h)/\mu^2=\mathcal{O}(h^{2\delta_1})$.

Remark. Assume that the subprincipal symbol of $P_{\varepsilon=0}$ in (2.1) vanishes. Then it follows from some arguments in sections 2 and 4 in [14] that the x_1–dependence in $P'_\varepsilon(x_1,hD_{x_1},\xi_2;h)$ in (4.17) can be eliminated microlocally in a region $|\xi_1|\sim\mu$, where

$$(\varepsilon+h^2)^{1/2}\ll\mu\ll 1.$$

By rescaling, we can express G in (4.28) as an h–Fourier integral operator. Indeed, using that $\frac{d\widetilde{\eta_1}}{\tilde{h}}=\frac{d\eta_1}{h}$, we get

$$Gu(x_1) = \frac{1}{2\pi h}\iint e^{\frac{i}{h}\left(\mu\varphi_\mu(x_1,\frac{\eta_1}{\mu},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\xi_2)+(x_1-y_1)\eta_1\right)} \times a(x_1,\frac{\eta_1}{\mu},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\xi_2;\frac{h}{\mu})u(y_1)\,dy_1\,d\eta_1. \tag{4.31}$$

Moreover, the introduction of the small parameter μ in (4.19) was artificial, and therefore we can carry out the constructions in such a way that the phase function $\mu\varphi_\mu(x_1,\frac{\eta_1}{\mu},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\xi_2)$ and the amplitude $a(x_1,\frac{\eta_1}{\mu},\frac{\varepsilon}{\mu^2},\frac{h}{\mu^2},\xi_2;\frac{h}{\mu})$ in (4.31) are independent of μ. We write then

$$Gu(x_1) = \frac{1}{2\pi h}\iint e^{\frac{i}{h}(\varphi_{\text{new}}(x_1,\eta_1,\varepsilon,h,\xi_2)+(x_1-y_1)\eta_1)} \times a_{\text{new}}(x_1,\eta_1,\varepsilon,h,\xi_2;h)u(y_1)\,dy_1\,d\eta_1, \tag{4.32}$$

with $\varphi_{\rm new}, a_{\rm new}$ defined for $\varepsilon + h \ll |\eta_1|^2 \ll 1$ and satisfying

$$\varphi_{\rm new} = \mathcal{O}\left(\frac{\varepsilon+h}{|\eta_1|}\right), \tag{4.33}$$

and

$$a_{\rm new} \sim \sum_{j=0}^{\infty} a_{{\rm new},j} h^j, \quad a_{{\rm new},j} = \mathcal{O}(|\eta_1|^{-2j}). \tag{4.34}$$

Here $a_{{\rm new},j}$ do not depend on h. Since we work in the complex domain, we can estimate the derivatives of $\varphi_{\rm new}$ and $a_{{\rm new},j}$ using the Cauchy inequalities. In particular, when $(\alpha_1, \beta_1) \in \mathbb{N}^2$, we get using (4.33),

$$\partial_{x_1}^{\alpha_1}\partial_{\eta_1}^{\beta_1}\varphi_{\rm new} = \mathcal{O}_{\alpha_1\beta_1}\left(\frac{(\varepsilon+h)}{|\eta_1|^{1+|\beta_1|}}\right). \tag{4.35}$$

Since, as we have just observed,

$$\mu\varphi_\mu\left(x_1, \frac{\eta_1}{\mu}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = \varphi_{\rm new}\left(x_1, \eta_1, \varepsilon, h, \xi_2\right),$$

where $\varphi_{\rm new}$ satisfies (4.33), it follows that the phase φ_μ in (4.25) extends to a region $1 \ll |\widetilde{\eta_1}| \ll \frac{1}{\mu}$ and satisfies there

$$\varphi_\mu\left(x_1, \widetilde{\eta_1}, \frac{\varepsilon}{\mu^2}, \frac{h}{\mu^2}, \xi_2\right) = \mathcal{O}\left(\frac{(\varepsilon+h)}{\mu^2\,|\widetilde{\eta_1}|}\right).$$

Similarly, the normal form (4.29) corresponds, after a multiplication by μ^2, to an operator which is independent of μ,

$$\begin{aligned} P''_\varepsilon(hD_{x_1}, \xi_2; h) &= G^{-1}e^{-\frac{i}{h}f(\xi_2)x_1}P'_\varepsilon(x_1, hD_{x_1}, \xi_2; h)e^{\frac{i}{h}f(\xi_2)x_1}G \\ &= p(f(\xi_2), \xi_2) + g(f(\xi_2) + hD_{x_1})(hD_{x_1})^2 \\ &\quad + \left(i\varepsilon\langle\widetilde{\langle q\rangle_2}\rangle_1 + \mathcal{O}(h) + \varepsilon^2\langle r_\varepsilon\rangle_1\right)(f(\xi_2) + hD_{x_1}, \xi_2) \\ &\quad + \mathrm{Op}_h\left(\mathcal{O}\left(\frac{(\varepsilon+h)^2}{\xi_1^2}\right)\right) + R(hD_{x_1}, \xi_2, \varepsilon; h), \end{aligned} \tag{4.36}$$

where

$$R \sim \sum_{j=2}^{\infty} h^j R_j(\xi), \quad R_j(\xi) = \mathcal{O}\left(\frac{1}{|\xi_1|^{2j-2}}\right).$$

For future reference we remark that we can also view the operator G in (4.28) as acting on (Floquet periodic) functions on $\mathbb{T}^2$. If we maintain the scaling, we get

$$Gu(x) = \frac{1}{(2\pi\tilde{h})(2\pi h)}\int e^{\frac{i}{\tilde{h}}(\varphi_\mu(x,\widetilde{\eta_1}) + (x_1-y_1)\widetilde{\eta_1}) + \frac{i}{h}(x_2-y_2)\eta_2} a(x_1, \widetilde{\eta_1}, \eta_2; \tilde{h}, h)u(y)\, dy d\widetilde{\eta_1}\, d\eta_2, \tag{4.37}$$

where η_2 is the same variable as ξ_2. Without the scaling, we have a similar formula by adding a y_2, η_2–integration to (4.31) (after replacing ξ_2 there by η_2), and adding a phase factor $e^{\frac{i}{h}(x_2-y_2)\eta_2}$.

Naturally, the argument so far is formal, with the various normal forms computed by formal stationary phase expansions. Also, let us recall that the phase $\varphi_{\rm new}$ in (4.32) is defined only for $(\varepsilon+h)^{1/2} \ll |\eta_1| \ll 1$.

We summarize the discussion in this section in the following proposition.

PROPOSITION 4.1. – *Let P_ε be an h-pseudodifferential operator defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, and assume that the principal symbol of P_ε,*

$$p_\varepsilon(x,\xi) = p(\xi) + i\varepsilon q(x,\xi) + \mathcal{O}(\varepsilon^2),$$

is such that $p(\xi)$ satisfies (4.3), (4.4). Then we write

$$p(\xi_1,\xi_2) = p(f(\xi_2),\xi_2) + g(\xi_1,\xi_2)(\xi_1 - f(\xi_2))^2, \quad f(0) = 0, \tag{4.38}$$

where $g(0,0) > 0$. For each $N \in \mathbb{N}$ there exists an elliptic Fourier integral operator in the complex domain $F = F_\varepsilon^{(N)}$ such that the symbol of $M^{-1}F^{-1}P_\varepsilon FM$ is of the form

$$P'_\varepsilon(x_1,\xi_1 + f(\xi_2),\xi_2;h) + \mathcal{O}\left(\varepsilon^{N+1} + \varepsilon\xi_1^N + h^{N+1}\right). \tag{4.39}$$

Here M is the operator of multiplication by $e^{\frac{i}{h}f(\xi_2)x_1}$, and $P'_\varepsilon(x_1,hD_{x_1},hD_{x_2};h)$ is defined in (4.17).

Furthermore, let $(\varepsilon+h)^{1/2} \ll \mu \ll 1$, and let us view $\mu^{-2}P'_\varepsilon(x_1,hD_{x_1} + f(\xi_2),\xi_2;h)$ as an $\tilde{h}$–pseudodifferential operator in x_1, with $\tilde{h} = h/\mu$. There exists an elliptic $\tilde{h}$–Fourier integral operator G in x_1, defined in (4.28), microlocally in $\left|\widetilde{\xi}\right| \sim 1$, such that the full symbol of $G^{-1}\mu^{-2}P'_\varepsilon(x_1,hD_{x_1} + f(\xi_2),\xi_2;h)G$ is independent of x_1. The operator G quantizes a holomorphic canonical transformation whose generating function is of the form $x_1\widetilde{\eta_1} + \varphi_\mu(x_1,\widetilde{\eta_1})$, where φ_μ is defined in $1 \ll |\widetilde{\eta_1}| \ll \frac{1}{\mu}$ and satisfies there

$$\varphi_\mu(x_1,\widetilde{\eta_1}) = \mathcal{O}\left(\frac{\varepsilon+h}{\mu^2\,|\widetilde{\eta_1}|}\right). \tag{4.40}$$

In this region we have, when $(\alpha_1,\beta_1) \in \mathbb{N}^2$,

$$\partial_{x_1}^{\alpha_1}\partial_{\widetilde{\eta_1}}^{\beta_1}\varphi_\mu = \mathcal{O}_{\alpha_1\beta_1}\left(\frac{\varepsilon+h}{\mu^2\,|\widetilde{\eta_1}|^{1+|\beta_1|}}\right). \tag{4.41}$$

5. Microlocal Hilbert spaces near the rational torus

Let P_ε be as in section 2. In section 4, we have constructed a microlocal normal form for P_ε near the rational Lagrangian torus $\Lambda_{1,r} \subset p^{-1}(0) \cap \mathbb{R}^4$, but away from an $\mathcal{O}((\varepsilon+h)^{1/2})$–neighborhood of this set — see (4.36). The purpose of this section is to follow up the preceding formal constructions with suitable function spaces and to construct a microlocal Hilbert space in a sufficiently small but fixed neighborhood of $\Lambda_{1,r}$, implementing the reduction scheme of Proposition 4.1.

5.1. Microlocal Hilbert spaces outside of a tiny neighborhood of $\Lambda_{1,r}$

Let us consider an operator P_ε, microlocally defined near $\xi = 0$ in $T^*\mathbb{T}^2$, with the leading symbol given by (4.7), (4.8). We shall work as much as possible with functions on $\mathbb{T}^2$, and with corresponding Fourier integral operators operating in 2 variables. Adopting this point of view, we see that the multiplication by $e^{\frac{i}{h}f(\xi_2)x_1}$, introduced in (4.18), can be viewed as the semiclassical Fourier integral operator

$$
\begin{aligned}
(5.1) \qquad Mu(x) &= \frac{1}{2\pi h}\iint e^{\frac{i}{h}(f(\eta_2)x_1+(x_2-y_2)\eta_2)}u(x_1,y_2)\,dy_2\,d\eta_2 \\
&= \frac{1}{(2\pi h)^2}\iint e^{\frac{i}{h}(f(\eta_2)x_1+(x-y)\cdot\eta)}u(y)\,dy\,d\eta,
\end{aligned}
$$

associated to the canonical transformation

$$
(5.2) \qquad \kappa_M : (x_1, x_2 + f'(\eta_2)x_1; \eta_1, \eta_2) \mapsto (x_1, x_2; \eta_1 + f(\eta_2), \eta_2).
$$

Let us recall now the operators F and G, introduced in (4.14) and (4.37), respectively. In the previous section we have obtained that formally,

$$
(5.3) \qquad G^{-1}M^{-1}F^{-1}P_\varepsilon FMG = P''_\varepsilon(hD_x,h) + (MG)^{-1}R_\varepsilon MG,
$$

with P''_ε and R_ε given in (4.36) and (4.15), respectively. The fact that the phase φ_μ in (4.37) (see also Proposition 4.1) is only defined for $1 \ll |\widetilde{\eta_1}| \ll \frac{1}{\mu}$, $(\varepsilon+h)^{1/2} \ll \mu \ll 1$, is a difficulty that we shall address later in this section. Ignoring that problem for a moment and still arguing formally, we would like to consider P''_ε acting on the space $L^2_\theta(\mathbb{T}^2)$, microlocally defined near the zero section, but away from the exceptional region

$$
|\xi_1| = \mathcal{O}((\varepsilon+h)^{1/2}).
$$

Consequently, the natural formal Hilbert space for considering P_ε should be given by $FMG(L^2_\theta(\mathbb{T}^2))$. When realizing the latter, it is going to be convenient to work on the FBI transform side.

We shall work with the standard FBI–Bargmann transform,

$$
(5.4) \qquad Tu(x) = T_{h,h}u(x) = Ch^{-3/2}\int e^{-\frac{1}{2h}(x-y)^2}u(y)\,dy, \quad C>0,
$$

acting on $L^2_\theta(\mathbb{T}^2)$, and mapping this space to a weighted space of Floquet periodic holomorphic functions on $\mathbb{C}^2$. Associated to T, there is a canonical transformation

$$
(5.5) \qquad \kappa_{T_{h,h}} = \kappa_T : (y,\eta) \mapsto (x,\xi) = (y - i\eta, \eta),
$$

mapping the real phase space $T^*\mathbb{T}^2$ to the IR-manifold

$$
(5.6) \qquad \Lambda_{\Phi_0} : \ \xi = \frac{2}{i}\frac{\partial\Phi_0}{\partial x} = -\mathrm{Im}\,x,\ \ \Phi_0(x) = \frac{1}{2}(\mathrm{Im}\,x)^2.
$$

Let us also recall that the transformation

$$
(5.7) \qquad T : L^2(\mathbb{T}^2) \to H_{\Phi_0}(\mathbb{C}^2/2\pi\mathbb{Z}^2)
$$

is unitary, for a suitable choice of $C>0$ in (5.4), and it has been verified in section 3 of [24] that it remains unitary when acting on the Floquet space $L^2_\theta(\mathbb{T}^2)$. Here and in what follows, when $\Omega \subset \mathbb{C}^2/2\pi\mathbb{Z}^2 = \mathbb{T}^2 + i\mathbb{R}^2$ is open and Φ is a suitable strictly plurisubharmonic weight, close to Φ_0 in (5.6), we shall let $H_\Phi(\Omega)$ stand for the closed subspace of $L^2(\Omega; e^{-\frac{2\Phi}{h}}L(dx))$, consisting of functions that are holomorphic in Ω — see also the appendix.

Neglecting the Floquet conditions for the time being, we should have,

$$
(5.8) \qquad TFMG(L^2(\mathbb{T}^2)) = H_\Phi,
$$

where the weight Φ is such that

$$\Lambda_\Phi := \left\{\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}\right)\right\} = \kappa_T \circ \kappa_\varepsilon \circ \kappa_M \circ \kappa_{\mu,\varepsilon,h}(T^*\mathbb{T}^2). \tag{5.9}$$

Here κ_ε and $\kappa_{\mu,\varepsilon,h}$ are the canonical transformations corresponding to F and G, and introduced in (4.12) and (4.26), respectively. The weight Φ in (5.8) should be a small perturbation of Φ_0 since κ_ε, $\kappa_{\mu,\varepsilon,h}$ are small perturbations of the identity, and κ_M in (5.2) is a real canonical transformation.

We shall assume from now on that

$$\varepsilon \gg h, \tag{5.10}$$

and abusing the previous notation slightly, we shall take

$$\mu = \sqrt{\varepsilon}. \tag{5.11}$$

Because of the blow-up of the normal form construction in the region where $\eta_1 = \mathcal{O}(\varepsilon^{1/2})$ (see (4.33)), when realizing the formal space in (5.8), we shall have to make some modifications. First, the operator G should be written as in (4.37) with

$$\tilde{h} = \frac{h}{\sqrt{\varepsilon}} \ll 1,$$

and correspondingly, in order to define a microlocal space corresponding to the formal space $G(L^2(\mathbb{T}^2))$, we shall consider the mixed transform

$$T_{\tilde{h},h}u(x) = C\tilde{h}^{-3/4}h^{-3/4} \iint e^{-\frac{1}{2\tilde{h}}(x_1-y_1)^2 - \frac{1}{2h}(x_2-y_2)^2} u(y_1,y_2)\, dy_1\, dy_2. \tag{5.12}$$

Here $C > 0$ is the same constant as in (5.4). For future reference, we notice that when viewed as an h–Fourier integral operator, the transform $T_{\tilde{h},h}$ is associated with the canonical transformation

$$\kappa_{T_{\tilde{h},h}}(y_1,\eta_1;y_2,\eta_2) = \left(y_1 - i\frac{\eta_1}{\sqrt{\varepsilon}}, \eta_1; y_2 - i\eta_2, \eta_2\right). \tag{5.13}$$

Here we have written $(y_1,\eta_1;y_2,\eta_2)$ rather than (y,η).

We shall show that $T_{\tilde{h},h}G(L^2(\mathbb{T}^2))$ becomes a well-defined exponentially weighted space of holomorphic functions $u(x_1,x_2)$ in a region $1 \ll |\mathrm{Im}\, x_1| \ll \frac{1}{\mu}$, $|\mathrm{Im}\, x_2| \ll 1$. Once this has been done and the basic properties of the weight have been investigated, we shall extend the definition of the weight to the entire domain $|\mathrm{Im}\, x_1| \ll \frac{1}{\mu}$, $|\mathrm{Im}\, x_2| \ll 1$ — this will then lead to a definition of a microlocal Hilbert space corresponding to a formal space $G(L^2(\mathbb{T}^2))$, in a full neighborhood of the rational torus, and we shall be able to proceed as indicated above.

Let us compute $T_{\tilde{h},h}Gu$, when $u \in L^2$. In doing so, it will be convenient to do the computation first in the x_1-variable alone, and as in (4.28), we introduce, with $\mu = \sqrt{\varepsilon}$,

$$Gu(x_1,y_2,\eta_2) = \frac{1}{2\pi\tilde{h}} \iint e^{\frac{i}{\tilde{h}}\left(\varphi_\mu\left(x_1,\widetilde{\eta_1},\frac{h}{\varepsilon},\eta_2\right)+(x_1-y_1)\widetilde{\eta_1}\right)} a(x_1,\widetilde{\eta_1},\frac{h}{\varepsilon},\eta_2;\tilde{h})u(y_1,y_2)\, dy_1\, d\widetilde{\eta_1}. \tag{5.14}$$

Composing this expression with the one-variable transform $T_{\tilde{h}}$, we get

$$(5.15)\qquad T_{\tilde{h}}Gu(z_1, y_2, \eta_2)$$
$$= \frac{C\tilde{h}^{-\frac{3}{4}}}{2\pi\tilde{h}} \iiint e^{\frac{i}{\tilde{h}}\left(\frac{i}{2}(z_1-x_1)^2+\varphi_\mu(x_1,\widetilde{\eta_1},\frac{h}{\varepsilon},\eta_2)+(x_1-y_1)\widetilde{\eta_1}\right)} a(x_1, \widetilde{\eta_1}, \frac{h}{\varepsilon}, \eta_2; \tilde{h}) u(y_1, y_2)\, dy_1 d\widetilde{\eta_1} dx_1$$
$$= C_1 \tilde{h}^{-\frac{3}{4}} \int e^{\frac{i}{\tilde{h}}\left(\frac{i}{2}(z_1-y_1)^2+\widetilde{\varphi}_\mu(z_1,y_1,\frac{h}{\varepsilon},\eta_2)\right)} b(z_1, y_1, \frac{h}{\varepsilon}, \eta_2; \tilde{h}) u(y_1, y_2)\, dy_1, \quad C_1 > 0.$$

Here the last expression follows from the stationary phase method in the variables $x_1, \widetilde{\eta_1}$ [26], whereby we notice that the critical point of the phase in (5.15), $(x_1^c, \widetilde{\eta_1}^c)$, satisfies

$$x_1^c = y_1 + \mathcal{O}\left(\frac{1}{|z_1 - y_1|^2}\right), \quad \widetilde{\eta_1}^c = i(z_1 - y_1) + \mathcal{O}\left(\frac{1}{|z_1 - y_1|}\right).$$

It follows that the phase $\widetilde{\varphi}_\mu$ in (5.15) is a well-defined holomorphic function of z_1 in a region $1 \ll |z_1 - y_1| \ll 1/\varepsilon^{1/2}$, $|\mathrm{Im}\, z_1| \gg |\mathrm{Re}\, z_1 - y_1|$, and enjoys the same estimates as φ_μ in (4.40), (4.41),

$$(5.16)\qquad \widetilde{\varphi}_\mu = \mathcal{O}\left(\frac{1}{|\mathrm{Im}\, z_1|}\right), \quad \partial_{y_1}^l \partial_{z_1}^m \widetilde{\varphi}_\mu = \mathcal{O}_{lm}\left(\frac{1}{|\mathrm{Im}\, z_1|^{1+m}}\right).$$

Here, as before, the estimates on the derivatives of $\widetilde{\varphi}_\mu$ follow from the Cauchy inequalities.

It follows from (5.15) that

$$(5.17)\qquad T_{\tilde{h}}Gu(z_1, y_2, \eta_2) \in H_{\Phi_1(\cdot,\eta_2),\tilde{h}},$$

in the region $1 \ll |\mathrm{Im}\, z_1| \ll \frac{1}{\varepsilon^{1/2}}$, where

$$(5.18)\qquad \Phi_1(z_1, \eta_2) = \sup_{y_1\in\mathbb{R}} \left(-\frac{1}{2}\mathrm{Re}\,(z_1 - y_1)^2 - \mathrm{Im}\, \widetilde{\varphi}_\mu(z_1, y_1, \frac{h}{\varepsilon}, \eta_2)\right)$$
$$= \frac{1}{2}(\mathrm{Im}\, z_1)^2 + \Phi_2(z_1, \eta_2),$$

and

$$(5.19)\qquad \Phi_2(z_1, \eta_2) = \mathcal{O}\left(\frac{1}{|\mathrm{Im}\, z_1|}\right).$$

The critical point y_1^c corresponding to the supremum in (5.18) satisfies

$$(5.20)\qquad y_1^c = \mathrm{Re}\, z_1 + \mathcal{O}\left(\frac{1}{|\mathrm{Im}\, z_1|}\right).$$

Using (5.16) together with a scaling argument very similar to the one described in detail in the proof of Proposition 5.3 below, we see that the $\mathcal{O}$–term in (5.20) satisfies

$$\partial_{\mathrm{Re} z_1}^k \partial_{\mathrm{Im} z_1}^l \mathcal{O}\left(\frac{1}{|\mathrm{Im}\, z_1|}\right) = \mathcal{O}\left(\frac{1}{|\mathrm{Im}\, z_1|^{1+l}}\right).$$

It follows that for $\Phi_2(z_1, \eta_2)$ in (5.18) we have

$$\partial_{\mathrm{Re} z_1}^k \partial_{\mathrm{Im} z_1}^l \Phi_2(z_1, \eta_2) = \mathcal{O}_{kl}\left(\frac{1}{|\mathrm{Im}\, z_1|^{1+l}}\right).$$

If we now let G denote the full 2-variable operator in (4.37), we get from (5.15),

(5.21) $$\begin{aligned} &T_{\tilde{h},h}Gu \\ &= C_2\tilde{h}^{-\frac{3}{4}}h^{-\frac{3}{4}-1}\iiiint e^{\frac{i}{\tilde{h}}\left(\frac{i}{2}(z_1-y_1)^2+\widetilde{\varphi}_\mu(z_1,y_1,\frac{h}{\varepsilon},\eta_2)\right)+\frac{i}{h}\left(\frac{i}{2}(z_2-x_2)^2+(x_2-y_2)\eta_2\right)} \\ &\quad\times b(z_1,y_1,\frac{h}{\varepsilon},\eta_2;\tilde{h})u(y_1,y_2)\,dy_1\,dy_2\,d\eta_2\,dx_2 \\ &= C_3\tilde{h}^{-\frac{3}{4}}h^{-\frac{3}{4}}\iint e^{\frac{i}{h}\left(\frac{i}{2}\sqrt{\varepsilon}(z_1-y_1)^2+\frac{i}{2}(z_2-y_2)^2+\sqrt{\varepsilon}\widehat{\varphi}_\mu(z_1,y_1,z_2-y_2,\frac{h}{\varepsilon})\right)} \\ &\quad\times c(z_1,y_1,z_2-y_2;\tilde{h},h)u(y_1,y_2)\,dy_1\,dy_2, \end{aligned}$$

where the last equality follows from stationary phase in x_2, η_2, and $\widehat{\varphi}_\mu$ satisfies the same estimates as $\widetilde{\varphi}_\mu$,

(5.22) $$\partial_{z_2}^k\partial_{z_1}^l\widehat{\varphi}_\mu = \mathcal{O}_{kl}\left(\frac{1}{|\mathrm{Im}\,z_1|^{1+l}}\right).$$

It follows that

(5.23) $$T_{\tilde{h},h}Gu \in H_{\Phi_3,h},\quad \Phi_3(z_1,\mathrm{Im}\,z_2) = \frac{\sqrt{\varepsilon}}{2}(\mathrm{Im}\,z_1)^2+\frac{1}{2}(\mathrm{Im}\,z_2)^2+\Phi_4(z_1,\mathrm{Im}\,z_2),$$

and

(5.24) $$\partial^k_{\mathrm{Re}z_1,\mathrm{Im}z_2}\partial^l_{\mathrm{Im}z_1}\Phi_4(z_1,\mathrm{Im}\,z_2) = \mathcal{O}_{kl}\left(\frac{\sqrt{\varepsilon}}{|\mathrm{Im}\,z_1|^{1+l}}\right).$$

Here Φ_3, Φ_4 are independent of $\mathrm{Re}\,z_2$.

The discussion above is summarized in the following, somewhat informal, proposition.

Proposition 5.1. – *Let us assume that $\varepsilon \gg h$ and set $\tilde{h} = h/\sqrt{\varepsilon}$. Via the $(\tilde{h},h)$–Bargmann transform $T_{\tilde{h},h}$ defined in (5.12), the formal space $G(L^2(\mathbb{T}^2))$ corresponds to the weighted space of holomorphic functions $H_{\Phi_3,h}$ in the region $1 \ll |\mathrm{Im}\,z_1| \ll \frac{1}{\sqrt{\varepsilon}}$, $|\mathrm{Im}\,z_2| \ll 1$. The weight $\Phi_3 = \Phi_3(z_1,\mathrm{Im}\,z_2)$ is such that*

(5.25) $$\Phi_3(z_1,\mathrm{Im}\,z_2) = \frac{\sqrt{\varepsilon}}{2}(\mathrm{Im}\,z_1)^2+\frac{1}{2}(\mathrm{Im}\,z_2)^2+\Phi_4(z_1,\mathrm{Im}\,z_2),$$

where the perturbation Φ_4 satisfies

(5.26) $$\partial^k_{\mathrm{Re}z_1,\mathrm{Im}z_2}\partial^l_{\mathrm{Im}z_1}\Phi_4(z_1,\mathrm{Im}\,z_2) = \mathcal{O}_{kl}\left(\frac{\sqrt{\varepsilon}}{|\mathrm{Im}\,z_1|^{1+l}}\right).$$

The corresponding statement also holds when considering the formal space $G(L^2_\theta(\mathbb{T}^2))$ of Floquet periodic functions.

Remark. Let us remark that the cutoff and remainder errors not written out explicitly in the stationary phase expansions above are all of the size $\mathcal{O}(1)\exp(-1/C\tilde{h}) = \mathcal{O}(1)\exp(-\frac{\sqrt{\varepsilon}}{Ch})$ [26], while the deviation of the weight, due to Φ_4, corresponds to an exponential factor

$$\exp\left(\mathcal{O}(1)\frac{\sqrt{\varepsilon}}{h\,|\mathrm{Im} z_1|}\right) \ll \exp\left(\frac{\sqrt{\varepsilon}}{Ch}\right),$$

since we work in a region where $|\mathrm{Im}\,z_1| \gg 1$.

Remark. Constructing and working with the $\tilde{h}$–Fourier integral operator G in the domain where

$$\frac{\sqrt{\varepsilon}}{h^{\delta_1}} \leq |\xi_1| \ll 1,$$

for some $\delta_1 > 0$ small (see also (4.30)), we find that the formal space $G(L^2(\mathbb{T}^2))$ corresponds, via the $(\tilde{h}, h)$–Bargmann transform, to the space $H_{\Phi_3,h}$, as in Proposition 5.1, now viewed in the region $h^{-\delta_1} \leq |\mathrm{Im}\, z_1| \ll \frac{1}{\sqrt{\varepsilon}}$, $|\mathrm{Im}\, z_2| \ll 1$.

5.2. Fourier series expansions in H_Φ–spaces

The purpose of this subsection is to obtain a relation between the 1-variable weight $\Phi_1(z_1, \eta_2)$ and the 2-variable weight $\Phi_3(z_1, \mathrm{Im}\, z_2)$, introduced in (5.18) and (5.23), respectively. The starting point for us will be the following remark concerning Fourier series on the FBI transform side. Let us rewrite (5.14) with slightly different notation, now for a function $u \in L^2$ of one variable only:

$$(5.27)\quad G_{\eta_2} u(x_1) = \frac{1}{2\pi\tilde{h}} \iint e^{\frac{i}{\tilde{h}}\left(\varphi_\mu(x_1, \widetilde{\eta_1}, \frac{h}{\varepsilon}, \eta_2) + (x_1 - y_1)\widetilde{\eta_1}\right)} a(x_1, \widetilde{\eta_1}, \frac{h}{\varepsilon}, \eta_2; \tilde{h}) u(y_1)\, dy_1\, d\widetilde{\eta_1}.$$

If $u = u(y_1, y_2) \in L^2(\mathbb{T}^2)$ depends on 2 variables and we introduce the Fourier series expansion in y_2,

$$(5.28)\qquad u(y_1, y_2) = \sum_{k\in\mathbb{Z}} e^{\frac{i}{h} y_2 kh} \widehat{u}(y_1, kh),$$

then

$$Gu(x_1, x_2) = \sum_{k\in\mathbb{Z}} e^{\frac{i}{h} x_2 kh} \left(G_{kh} \widehat{u}(\cdot, kh)\right)(x_1),$$

and therefore, applying $T_{\tilde{h},h}$ of (5.12), we get

$$(5.29)\qquad T_{\tilde{h},h} Gu(z_1, z_2) = \sum_{k\in\mathbb{Z}} T_h^{(2)}(e^{\frac{i}{h} x_2 kh})(z_2) T_{\tilde{h}}^{(1)} G_{kh} \widehat{u}(\cdot, kh)(z_1).$$

Here the superscripts (1), (2) in (5.29) indicate the variable in which the corresponding operators are applied. A straightforward computation shows that

$$(5.30)\qquad T_h^{(2)}\left(e^{\frac{i}{h}(\cdot)\xi_2}\right)(z_2) = C h^{-\frac{1}{4}} e^{-\frac{\xi_2^2}{2h}} e^{\frac{i}{h} z_2 \xi_2} =: e_{\xi_2}(z_2), \quad C > 0,$$

and clearly, as can also be verified directly, this function is normalized in the space $H_{\Phi_0}(\mathbb{C}/2\pi\mathbb{Z})$, $\Phi_0(z_2) = \frac{1}{2}(\mathrm{Im}\, z_2)^2$. The functions e_{kh}, $k \in \mathbb{Z}$, form an orthonormal basis in this space, and hence a general element of $H_{\Phi_0}(\mathbb{C}/2\pi\mathbb{Z})$ has an expansion

$$(5.31)\qquad v = \sum_{k\in\mathbb{Z}} \widetilde{v}_k e_{kh}, \quad \widetilde{v}_k = (v|e_{kh})_{H_{\Phi_0}}.$$

We shall now pause to review Fourier series expansions in $H_\Phi(\mathbb{C}/2\pi\mathbb{Z})$, where $\Phi = \Phi(\mathrm{Im}\, z)$ is a general smooth weight such that $t \mapsto \Phi(t)$ is strictly convex:

$$(5.32)\qquad v(z) = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{\frac{i}{h} zkh}.$$

Here the scalar product

$$\int_{\mathbb{C}/2\pi\mathbb{Z}} e^{\frac{i}{h} zkh} \overline{e^{\frac{i}{h} z\ell h}} e^{-\frac{2\Phi(\mathrm{Im}\, z)}{h}} L(dz), \quad k, l \in \mathbb{Z}$$

vanishes for $\ell \neq k$ and for $k = \ell$ it is equal to

$$2\pi \int e^{-\frac{2}{h}(\Phi(\mathrm{Im} z)+kh\mathrm{Im} z)}\, d\mathrm{Im}\, z, \tag{5.33}$$

which can be evaluated by the method of stationary phase. The critical point $t = \mathrm{Im}\, z$ in (5.33) is given by $\Phi'(t) + kh = 0$ and $\widetilde{\Phi}(kh) := \inf(kht + \Phi(t)) = -\sup((-kh)t - \Phi(t)) = -\mathcal{L}\Phi(-kh)$, where $\mathcal{L}$ is the Legendre transformation. Notice also that the critical point t can be characterized by

$$\frac{2}{i}\frac{\partial \Phi}{\partial z}(x+it) = kh, \quad x \in \mathbb{R},$$

when identifying $\Phi(z) = \Phi(\mathrm{Im}\, z)$. Thus, by stationary phase (the Laplace method), we get

$$\| e^{\frac{i}{h}(\cdot)kh} \|^2_{H_\Phi} = h^{1/2} a_\Phi(kh;h) e^{\frac{2}{h}\mathcal{L}\Phi(-kh)},$$

where $a_\Phi(t;h) \sim a_0(t) + ha_1(t) + \ldots$ is a positive elliptic symbol.

For the Fourier series expansion (5.32) we therefore have the Parseval relation,

$$\| v \|^2_{H_\Phi} = \sum_{k\in\mathbb{Z}} h^{\frac{1}{2}} a_\Phi(kh;h) e^{\frac{2}{h}\mathcal{L}\Phi(-kh)} \left|\widehat{v}_k\right|^2, \tag{5.34}$$

telling us to which weighted l^2-space the Fourier coefficients $\widehat{v}_k$ belong.

Applying (5.34) to (5.29), (5.30) viewed as a Fourier series in z_2 with z_1 as a parameter, we get

$$\begin{aligned} &\| T_{\tilde{h},h} G u(z_1,\cdot) \|^2_{H_{\Phi_3(z_1,\cdot),h}} \\ &\quad = |C|^2 \sum_{k\in\mathbb{Z}} a_{\Phi_3(z_1,\cdot)}(kh;h) e^{\frac{2}{h}\mathcal{L}\Phi_3(z_1,-kh)-\frac{(kh)^2}{h}} \left| T^{(1)}_{\tilde{h}} G_{kh} \widehat{u}(\cdot,kh)(z_1)\right|^2. \end{aligned} \tag{5.35}$$

Now recall that the weights Φ_3 and Φ_1 have been chosen so that

$$\begin{aligned} \| T_{\tilde{h},h} G u \|^2_{H_{\Phi_3,h}} \sim \| u \|^2_{L^2(\mathbb{T}^2)} &= \sum_{k\in\mathbb{Z}} \| \widehat{u}_k \|^2_{L^2(\mathbb{T}^1)} \\ &= \sum_{k\in\mathbb{Z}} \| T_{\tilde{h}} G_{kh} \widehat{u}_k \|^2_{H_{\Phi_1(\cdot,kh),\tilde{h}}}, \end{aligned} \tag{5.36}$$

where as in (5.28), $u(y_1,y_2) = \sum_{k\in\mathbb{Z}} e^{iky_2}\widehat{u}_k(y_1)$, and thus we want the last member of (5.36) to coincide with that of (5.35) after an integration with respect to z_1. This means that

$$\frac{2}{h}(\mathcal{L}\Phi_3)(z_1,-kh) - \frac{(kh)^2}{h} = -\frac{2\sqrt{\varepsilon}}{h}\Phi_1(z_1,kh),$$

so that

$$(\mathcal{L}\Phi_3)(z_1,-\eta_2) = \frac{1}{2}\eta_2^2 - \sqrt{\varepsilon}\Phi_1(z_1,\eta_2). \tag{5.37}$$

When verifying (5.37), we recall from (5.18) that

$$\Phi_1(z_1,\eta_2) = \sup_{y_1\in\mathbb{R}} \left(-\frac{1}{2}\mathrm{Re}\,(z_1-y_1)^2 - \mathrm{Im}\,\widetilde{\varphi}_\mu(z_1,y_1,\frac{h}{\varepsilon},\eta_2)\right). \tag{5.38}$$

We need a similar formula for Φ_3. To that end, let us notice that in (5.21) we can insert an intermediate step, where we only integrate with respect to x_2, and exploiting that

$$\int e^{\frac{i}{h}(x_2-y_2)\eta_2 - \frac{1}{2h}(x_2-z_2)^2}\, dx_2 = \sqrt{2\pi h} e^{\frac{i}{h}(z_2-y_2)\eta_2 - \frac{1}{2h}\eta_2^2},$$

we get

(5.39) $$T_{\tilde{h},h}Gu(z) = C_3\tilde{h}^{-\frac{3}{4}}h^{-\frac{3}{4}-\frac{1}{2}}\iiint e^{-\frac{1}{2\tilde{h}}(z_1-y_1)^2+\frac{i}{\tilde{h}}\widetilde{\varphi}_\mu(z_1,y_1,\frac{h}{\varepsilon},\eta_2)+\frac{i}{h}(z_2-y_2)\eta_2-\frac{1}{2h}\eta_2^2} \times b(z_1,y_1,\frac{h}{\varepsilon},\eta_2;\tilde{h})u(y_1,y_2)\,dy_1\,dy_2\,d\eta_2.$$

The formula for Φ_3 becomes

(5.40) $$\Phi_3(z_1,\mathrm{Im}\,z_2) = \sup_{y_1\in\mathbb{R}}\sup_{y_2\in\mathbb{R}} \mathrm{vc}_{\eta_2} - \frac{\sqrt{\varepsilon}}{2}\mathrm{Re}\,(z_1-y_1)^2 - \sqrt{\varepsilon}\mathrm{Im}\,\widetilde{\varphi}_\mu - \mathrm{Im}\,(z_2-y_2)\eta_2 - \frac{1}{2}\mathrm{Re}\,\eta_2^2.$$

For y_1 fixed, the $\sup_{y_2\in\mathbb{R}}\mathrm{vc}_{\eta_2}$ corresponds to taking the critical value with respect to y_2, η_2 and the criticality with respect to y_2 requires η_2 to be real, making the right hand side independent of y_2. Thus "vc_{η_2}" in (5.40) can be replaced by "$\sup_{\eta_2\in\mathbb{R}}$" and we get

(5.41) $$\begin{aligned}\Phi_3(z_1,\mathrm{Im}\,z_2) &= \sup_{y_1\in\mathbb{R}}\sup_{\eta_2\in\mathbb{R}} -\frac{\sqrt{\varepsilon}}{2}\mathrm{Re}\,(z_1-y_1)^2 - \sqrt{\varepsilon}\mathrm{Im}\,\widetilde{\varphi}_\mu(z_1,y_1,\frac{h}{\varepsilon},\eta_2) - \frac{1}{2}\eta_2^2 - \eta_2\mathrm{Im}\,z_2\\ &= \sup_{\eta_2\in\mathbb{R}} \sqrt{\varepsilon}\Phi_1(z_1,\eta_2) - \frac{1}{2}\eta_2^2 - \eta_2\mathrm{Im}\,z_2\\ &= \mathcal{L}_{\eta_2\to\mathrm{Im}z_2}\left(\frac{1}{2}\eta_2^2 - \sqrt{\varepsilon}\Phi_1(z_1,\eta_2)\right)(-\mathrm{Im}\,z_2).\end{aligned}$$

With $f(\eta_2) = \frac{1}{2}\eta_2^2 - \sqrt{\varepsilon}\Phi_1(z_1,\eta_2)$, $Ju(t) = u(-t)$, and z_1 treated as a parameter, (5.37) reads

$$J\mathcal{L}\Phi_3 = f,$$

while (5.41) tells us that $J\mathcal{L}f = \Phi_3$. Since $J^2 = \mathcal{L}^2 = 1$ and $J\mathcal{L} = \mathcal{L}J$, we then see that (5.37) follows from (5.41).

Proposition 5.2. – *Let the strictly plurisubharmonic weight functions $\Phi_1(z_1,\eta_2)$ and $\Phi_3(z_1,\mathrm{Im}\,z_2)$ be defined in (5.18) and (5.40), respectively. Then we have the relation*

(5.42) $$(\mathcal{L}_{\mathrm{Im}z_2\to\eta_2}\Phi_3)(z_1,-\eta_2) = \frac{1}{2}\eta_2^2 - \sqrt{\varepsilon}\Phi_1(z_1,\eta_2).$$

Here $\mathcal{L}f(\xi) = \sup_x(x\xi - f(x))$ is the Legendre transform of a strictly convex smooth function $f:\mathbb{R}\to\mathbb{R}$.

It follows that if we have an expansion of $u\in H_{\Phi_3,h}$,

(5.43) $$u(z_1,z_2) = \sum_{k\in\mathbb{Z}}\widetilde{u}_k(z_1)e_{kh}(z_2),\quad e_{kh}(z_2) = Ch^{-\frac{1}{4}}e^{-\frac{(kh)^2}{2h}}e^{\frac{i}{h}z_2kh},\quad C>0,$$

then

(5.44) $$||\,u\,||^2_{H_{\Phi_3,h}} \sim \sum_{k\in\mathbb{Z}}||\,\widetilde{u}_k\,||^2_{H_{\Phi_1(\cdot,kh),\tilde{h}}}.$$

5.3. Comparison with the ordinary transform away from $\Lambda_{1,r}$

This subsection is a preparation for defining the global Hilbert space by gluing together the local constructions near $\Lambda_{1,r}$ to the weighted spaces that we used in [17]. This discussion will be continued in section 6.

In subsection 5.1 we have analyzed the space $T_{\tilde{h},h}G(L^2(\mathbb{T}^2))$, $\tilde{h} = \frac{h}{\sqrt{\varepsilon}}$, and identified it with a weighted space $H_{\Phi_3,h}$ of holomorphic functions defined in a region where $1 \ll |\mathrm{Im}\, z_1| \ll \frac{1}{\sqrt{\varepsilon}}$, $|\mathrm{Im}\, z_2| \ll 1$. We shall now see that restricting the attention to a region where $|\mathrm{Im}\, z_1| \gg \varepsilon^{-1/6}$, we can identify this space with a weighted space of holomorphic functions on the $T_{h,h}$–transform side. Specifically, when studying $T_{h,h}G(L^2(\mathbb{T}^2))$ as a weighted space, we shall show that the region $|\mathrm{Im}\, z_1| \gg \varepsilon^{-\frac{1}{6}}$ on the $T_{\tilde{h},h}$–side corresponds to a region $|\mathrm{Im}\, z_1| \gg \varepsilon^{\frac{1}{3}}$ on the $T_{h,h}$–side.

All the work will concern the variables of index 1, and therefore we shall restrict the attention to the one-dimensional situation for a while and consider (as appears also in the discussion of the second microlocalization in Chapter 16 of [26]),

$$T_h T_{\tilde{h}}^{-1} u(x) = C h^{-\frac{3}{4}} \tilde{h}^{-\frac{1}{4}} \iint e^{-\frac{1}{2h}(x-t)^2 + \frac{1}{2\tilde{h}}(t-y)^2} u(y)\, dy\, dt.$$

Eliminating the t–integration by exact stationary phase, we get

$$T_h T_{\tilde{h}}^{-1} u(x) = C(1+\mathcal{O}(\sqrt{\varepsilon})) h^{-\frac{1}{4}} \tilde{h}^{-\frac{1}{4}} \int e^{\frac{1}{2h}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(x-y)^2} u(y)\, dy. \tag{5.45}$$

Let us consider first the operator $T_h T_{\tilde{h}}^{-1}$ as a map from $H_{\Phi_0,\tilde{h}}$ to $H_{\Phi_0,h}$ with $\Phi_0(x) = \frac{1}{2}(\mathrm{Im}\, x)^2$. Considering the reduced kernel of (5.45), we then want to look at the real part of the phase,

(5.46)

$$\begin{aligned} &-\frac{1}{2}(\mathrm{Im}\, x)^2 + \mathrm{Re}\, \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(x-y)^2 + \frac{\sqrt{\varepsilon}}{2}(\mathrm{Im}\, y)^2 \\ &= \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(\mathrm{Re}\, x - \mathrm{Re}\, y)^2 - \frac{1}{2}\left((\mathrm{Im}\, x)^2 + \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(\mathrm{Im}\, x - \mathrm{Im}\, y)^2 - \sqrt{\varepsilon}(\mathrm{Im}\, y)^2\right) \\ &= \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\left(\mathrm{Re}\, x - \mathrm{Re}\, y\right)^2 - \frac{1}{2(1-\sqrt{\varepsilon})}\left(\sqrt{\varepsilon}\mathrm{Im}\, y - \mathrm{Im}\, x\right)^2. \end{aligned}$$

This means that we can choose the integration contour $\mathrm{Re}\, y = \mathrm{Re}\, x$ in (5.45), and using Schur's lemma we see that

$$\begin{aligned} \|\, T_h T_{\tilde{h}}^{-1}\, \|_{H_{\Phi_0,\tilde{h}} \to H_{\Phi_0,h}} &= \mathcal{O}(1) h^{-\frac{1}{4}} \tilde{h}^{-\frac{1}{4}} \left(\int e^{-\frac{1}{2h(1-\sqrt{\varepsilon})}(\sqrt{\varepsilon}\mathrm{Im} y - \mathrm{Im} x)^2}\, d\mathrm{Im}\, y\right)^{1/2} \\ &\quad \times \left(\int e^{-\frac{1}{2h(1-\sqrt{\varepsilon})}(\sqrt{\varepsilon}\mathrm{Im} y - \mathrm{Im} x)^2}\, d\mathrm{Im}\, x\right)^{1/2} \\ &= \mathcal{O}(1) h^{-\frac{1}{4}} \tilde{h}^{-\frac{1}{4}} \left(\frac{h}{\varepsilon}\right)^{\frac{1}{4}} h^{\frac{1}{4}} = \mathcal{O}(1)\varepsilon^{\frac{1}{8}-\frac{1}{4}} = \mathcal{O}(1)\varepsilon^{-\frac{1}{8}}. \end{aligned}$$

Here $H_{\Phi_0,\tilde{h}} = H_{\sqrt{\varepsilon}\Phi_0,h}$. We notice that the factor $\varepsilon^{-1/8}$ here represents a loss, since we know that $T_h T_{\tilde{h}}^{-1} : H_{\Phi_0,\tilde{h}} \to H_{\Phi_0,h}$ is unitary, modulo exponentially small errors. The loss is due to the fact that here we are using contour integrals as a preparation for the next case when the weights are no longer the standard quadratic ones.

Let us pass from $\sqrt{\varepsilon}\Phi_0(y) = \frac{\sqrt{\varepsilon}}{2}(\operatorname{Im} y)^2$ to

$$\Phi_3(y) = \frac{\sqrt{\varepsilon}}{2}(\operatorname{Im} y)^2 + \Phi_4(y) \tag{5.47}$$

in (5.23), (5.24), with

$$\partial^k_{\operatorname{Re} y}\partial^l_{\operatorname{Im} y}\Phi_4(y) = \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} y|^{1+l}}\right), \quad 1 \ll |\operatorname{Im} y| \ll \frac{1}{\sqrt{\varepsilon}}. \tag{5.48}$$

(Here we continue to neglect the dependence on the variable of index 2.) When defining $T_h T_{\tilde{h}}^{-1}$ on $H_{\Phi_3,h}$ we need to choose the integration contour in (5.45) passing through the critical point of

$$\begin{aligned} y \mapsto \operatorname{Re} \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(x-y)^2 + \Phi_3(y) \\ = \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(\operatorname{Re} x - \operatorname{Re} y)^2 - \frac{\varepsilon}{2(1-\sqrt{\varepsilon})}\left(\operatorname{Im} y - \frac{\operatorname{Im} x}{\sqrt{\varepsilon}}\right)^2 + \frac{1}{2}(\operatorname{Im} x)^2 + \Phi_4(y). \end{aligned} \tag{5.49}$$

We shall now discuss the estimates on the critical point $y(x)$ in (5.49). Using (5.48), we see first that the criticality with respect to $\operatorname{Im} y$ means that

$$\operatorname{Im} y(x) + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}(\operatorname{Im} y(x))^2}\right) = \frac{\operatorname{Im} x}{\sqrt{\varepsilon}}. \tag{5.50}$$

Working in a region where

$$\frac{1}{\sqrt{\varepsilon}\,|\operatorname{Im} y|^2} \ll |\operatorname{Im} y| \quad \text{so that} \quad |\operatorname{Im} y| \gg \varepsilon^{-\frac{1}{6}}, \tag{5.51}$$

we then see that

$$\operatorname{Im} y(x) = \frac{\operatorname{Im} x}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|^2}\right). \tag{5.52}$$

Considering the $\operatorname{Re} y$-gradient of the phase in (5.49), we get

$$\operatorname{Re} y(x) = \operatorname{Re} x + \mathcal{O}\left(\frac{1}{|\operatorname{Im} y(x)|}\right), \tag{5.53}$$

and in view of (5.52),

$$\operatorname{Re} y(x) = \operatorname{Re} x + \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|}\right). \tag{5.54}$$

PROPOSITION 5.3. – *The critical point $y(x)$ in (5.49) satisfies*

$$\operatorname{Re} y = \operatorname{Re} x + \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|}\right), \quad \operatorname{Im} y = \frac{\operatorname{Im} x}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|^2}\right),$$

where the remainders enjoy the following symbolic estimates: for each $k, l \in \mathbb{N}$, we have

$$\partial^k_{\operatorname{Re} x}\partial^l_{\operatorname{Im} x}\mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|}\right) = \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|^{1+l}}\right), \quad \partial^k_{\operatorname{Re} x}\partial^l_{\operatorname{Im} x}\mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|^2}\right) = \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{|\operatorname{Im} x|^{2+l}}\right). \tag{5.55}$$

Proof. – The proof is a rescaling argument. With $\mathrm{Re}\, x = t$, $\frac{\mathrm{Im} x}{\sqrt{\varepsilon}} = s$, $|s| \gg \varepsilon^{-1/6}$, the equations (5.50) and (5.53) become, if we write $\mathrm{Re}\, y = u$, $\mathrm{Im}\, y = v$,

$$\begin{cases} u = t + f(u,v) \\ v = s + g(u,v), \end{cases} \tag{5.56}$$

with

$$\partial_u^k \partial_v^l f(u,v) = \mathcal{O}\left(\frac{1}{|v|^{1+l}}\right), \quad \partial_u^k \partial_v^l g(u,v) = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|v|^{2+l}}\right).$$

Assume that $s \simeq s_0$, $|s_0| \gg \varepsilon^{-1/6}$, and write $\tilde{v} = \frac{v}{s_0}$. Then (5.56) gives

$$\begin{cases} u = t + f(u, s_0\tilde{v}) \\ \tilde{v} = \tilde{s} + \frac{1}{s_0} g(u, s_0\tilde{v}). \end{cases} \tag{5.57}$$

Here we have written $\tilde{s} = s/s_0$. Now

$$\partial_u^k \partial_{\tilde{v}}^l f(u, s_0\tilde{v}) = \mathcal{O}\left(\frac{1}{|s_0|}\right) \ll 1,$$

$$\partial_u^k \partial_{\tilde{v}}^l \frac{1}{s_0} g(u, s_0\tilde{v}) = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|s_0|^3}\right) \ll 1,$$

and we conclude that $u = u(t,\tilde{s})$ and $\tilde{v} = \tilde{v}(t,\tilde{s})$ with $\partial_t^k \partial_{\tilde{s}}^l u = \mathcal{O}(1)$, $\partial_t^k \partial_{\tilde{s}}^l \tilde{v} = \mathcal{O}(1)$. Reinjecting this information into (5.57), we get

$$u = t + a(t,\tilde{s}), \quad \tilde{v} = \tilde{s} + \tilde{b}(t,\tilde{s}),$$

with

$$\partial_t^k \partial_{\tilde{s}}^l a = \mathcal{O}\left(\frac{1}{|s_0|}\right), \quad \partial_t^k \partial_{\tilde{s}}^l \tilde{b} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|s_0|^3}\right).$$

Using that $\partial_{\tilde{s}} = s_0 \partial_s$, we get

$$u = t + \mathcal{O}\left(\frac{1}{|s|}\right), \quad \partial_t^k \partial_s^l \mathcal{O}\left(\frac{1}{|s|}\right) = \mathcal{O}\left(\frac{1}{|s|^{1+l}}\right),$$

and

$$v = s + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|s|^2}\right), \quad \partial_t^k \partial_s^l \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|s|^2}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\,|s|^{2+l}}\right).$$

The symbolic estimates (5.55) follow and this completes the proof. □

Choosing the integration contour $\{y; \mathrm{Re}\, y = \mathrm{Re}\, y(x)\}$ in (5.45) passing through the critical point $y(x)$, and noticing that the y-Hessian of the phase occurring in (5.49) along the contour is negative definite, we obtain that $T_h T_{\tilde{h}}^{-1}$ becomes a well-defined operator of norm $\mathcal{O}(\varepsilon^{-\frac{1}{8}})$ from $H_{\Phi_3,h}$ to $H_{\Phi_5,h}$, where $\Phi_5(x)$ is given by

$$\mathrm{vc}_y \left(\frac{1}{2}(\mathrm{Im}\, x)^2 + \frac{1}{2}\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}(\mathrm{Re}\, x - \mathrm{Re}\, y)^2 - \frac{\varepsilon}{2(1-\sqrt{\varepsilon})}\left(\mathrm{Im}\, y - \frac{\mathrm{Im}\, x}{\sqrt{\varepsilon}}\right)^2 + \Phi_4(y)\right). \tag{5.58}$$

Using the estimates (5.52) and (5.53), we see that

$$\Phi_5(x) = \frac{1}{2}(\mathrm{Im}\, x)^2 + \mathcal{O}\left(\frac{\varepsilon}{|\mathrm{Im}\, x|} + \frac{\varepsilon^{3/2}}{|\mathrm{Im}\, x|^2} + \frac{\varepsilon^2}{|\mathrm{Im}\, x|^4}\right). \tag{5.59}$$

In view of (5.51), the strictly subharmonic function $\Phi_5(x)$ is naturally defined in a region $\varepsilon^{1/3} \ll |\mathrm{Im}\, x| \ll 1$, and therefore we get

$$\Phi_5(x) = \frac{1}{2}(\mathrm{Im}\, x)^2 + \mathcal{O}\left(\frac{\varepsilon}{|\mathrm{Im}\, x|}\right). \tag{5.60}$$

Estimating the derivatives of Φ_5 using Proposition 5.3 and adding the dependence on the variable x_2, we get the following result.

PROPOSITION 5.4. – *Let us consider the weight Φ_3, defined in (5.40) and satisfying (5.47), (5.48). Working in a region $\varepsilon^{-1/6} \ll |\mathrm{Im}\, y_1| \ll \varepsilon^{-1/2}$, $|\mathrm{Im}\, y_2| \ll 1$, we have*

$$T_{h,h} T^{-1}_{\tilde{h},h} = \mathcal{O}(1)\varepsilon^{-\frac{1}{8}} : H_{\Phi_3,h} \to H_{\Phi_5,h}, \tag{5.61}$$

where the strictly plurisubharmonic function Φ_5 is given in (5.58) and is defined for $\varepsilon^{1/3} \ll |\mathrm{Im}\, x_1| \ll 1$, $|\mathrm{Im}\, x_2| \ll 1$. We have

$$\Phi_5(x) = \frac{1}{2}(\mathrm{Im}\, x)^2 + \Phi_6(x_1, \mathrm{Im}\, x_2), \tag{5.62}$$

where

$$\partial^k_{\mathrm{Re} x_1, \mathrm{Im} x_2} \partial^l_{\mathrm{Im} x_1} \Phi_6(x_1, \mathrm{Im}\, x_2) = \mathcal{O}_{kl}\left(\frac{\varepsilon}{|\mathrm{Im}\, x_1|^{l+1}}\right) \tag{5.63}$$

Remark. Let us notice that we would also obtain the weighted space $H_{\Phi_5,h}$ more directly by studying $T_h G u$ in the x_1-variable, with G given in (4.32) and $u \in L^2$. Notice also that the canonical transformation associated to $T_{h,h} T^{-1}_{\tilde{h},h}$ in (5.61) is $\kappa_{T_{h,h}} \circ \kappa^{-1}_{T_{\tilde{h},h}}$, and since in view of (5.13),

$$\kappa^{-1}_{T_{\tilde{h},h}}(x_1, \xi_1; x_2, \xi_2) = \left(x_1 + i\frac{\xi_1}{\sqrt{\varepsilon}}, \xi_1; x_2 + i\xi_2, \xi_2\right),$$

we get, using (5.5),

$$\kappa_{T_{h,h}} \circ \kappa^{-1}_{T_{\tilde{h},h}}(x_1, \xi_1; x_2, \xi_2) = \left(x_1 + i\xi_1 \frac{1-\sqrt{\varepsilon}}{\sqrt{\varepsilon}}, \xi_1; x_2, \xi_2\right). \tag{5.64}$$

The weights Φ_3 in (5.47) and Φ_5 in (5.58) are related through the formula

$$\kappa_{T_{h,h}} \circ \kappa^{-1}_{T_{\tilde{h},h}}(\Lambda_{\Phi_3}) = \Lambda_{\Phi_5}. \tag{5.65}$$

Here the manifolds Λ_{Φ_3} and Λ_{Φ_5} are defined as in (5.9).

5.4. Microlocal Hilbert space in a full neighborhood of $\Lambda_{1,r}$

Let us return to the situation discussed in section 4, and recall from (4.14) that the action of P_ε on $F(L^2_\theta(\mathbb{T}^2))$ is, microlocally near $\xi = 0$, equivalent to the action of

$$F^{-1} P_\varepsilon F = P'_\varepsilon(x_1, hD_x; h) + R_\varepsilon(x, hD_x; h)$$

on $L^2_\theta(\mathbb{T}^2)$. Recall also from (5.3) that

$$G^{-1} M^{-1} P'_\varepsilon M G = P''_\varepsilon(hD_x; h),$$

the operator $P''_\varepsilon(hD_x; h)$ being defined in (4.36). With $\tilde{h} = \frac{h}{\sqrt{\varepsilon}}$, in Proposition 5.1 we have identified $T_{\tilde{h},h} G(L^2(\mathbb{T}^2))$ with a space of holomorphic functions $H_{\Phi_3,h}$ in the region $1 \ll |\mathrm{Im}\, z_1| \ll \frac{1}{\sqrt{\varepsilon}}$, $|\mathrm{Im}\, z_2| \ll 1$, with Φ_3 as in (5.25), (5.26). Now let us extend $\Phi_3(\cdot, \mathrm{Im}\, z_2)$ from the region $R < |\mathrm{Im}\, z_1| \ll \frac{1}{\sqrt{\varepsilon}}$ for $R \gg 1$, to the entire domain $|\mathrm{Im}\, z_1| \ll \frac{1}{\sqrt{\varepsilon}}$ so that we still have as in Proposition 5.1,

$$\Phi_3(z_1, \mathrm{Im}\, z_2) = \frac{\sqrt{\varepsilon}}{2}(\mathrm{Im}\, z_1)^2 + \frac{1}{2}(\mathrm{Im}\, z_2)^2 + \Phi_4(z_1, \mathrm{Im}\, z_2)$$

with

$$\partial^k_{\mathrm{Re} z_1, \mathrm{Im} z_2}\partial^l_{\mathrm{Im} z_1}\Phi_4(z_1, \mathrm{Im}\, z_2) = \mathcal{O}_{kl}\left(\frac{\sqrt{\varepsilon}}{(R + |\mathrm{Im}\, z_1|)^{1+l}}\right), \quad R \gg 1, \tag{5.66}$$

and with Φ_3, Φ_4 still independent of $\mathrm{Re}\, z_2$. It follows that when

$$(x_1, \widetilde{\xi_1}, x_2, \xi_2) \in \Lambda_{\Phi_3} = \left\{\left(x, \frac{2}{i}\frac{\partial \Phi_3}{\partial x}\right)\right\},$$

then $\mathrm{Im}\, \xi_2 = -\frac{\partial \Phi_3}{\partial \mathrm{Re}\, x_2} = 0$.

Recall that in subsection 5.1 we have defined a microlocal Hilbert space in a fixed neighborhood of

$$\bigcup_{|E|<\delta_0} \Lambda_{E,1,r} \quad 0 < \delta_0 \ll 1,$$

but away from a $\sqrt{\varepsilon}$–neighborhood of that set as $FMG(L^2_\theta(\mathbb{T}^2))$. Here the tori $\Lambda_{E,1,r}$ have been introduced in (4.5). Having extended Φ_3, we now fill the gap by replacing $G(L^2_\theta(\mathbb{T}^2))$ by $T^{-1}_{\tilde{h},h}H_{\Phi_3,h}$, and introduce a microlocal Hilbert space defined in a full neighborhood of $\Lambda_{1,r}$ and given by

$$FMT^{-1}_{\tilde{h},h}H_{\Phi_3,h}. \tag{5.67}$$

Here it will be understood that the elements of $H_{\Phi_3,h}$ are Floquet periodic as in (4.9). In what follows, in order to simplify the presentation, we shall neglect the Floquet conditions and work under the assumption that the elements of the weighted space $H_{\Phi_3,h}$ are $2\pi\mathbb{Z}^2$–periodic functions. It will be clear that the discussion below will extend to the Floquet periodic case. Also, in (5.67) we are identifying a neighborhood of $\Lambda_{1,r}$ with a neighborhood of the zero section in $T^*\mathbb{T}^2$ by means of the canonical transformation κ_0 in (4.2).

Microlocally near $\Lambda_{1,r}$, the action of P_ε on the space (5.67) can be identified with that of

$$T_{\tilde{h},h}M^{-1}P'_\varepsilon MT^{-1}_{\tilde{h},h} + T_{\tilde{h},h}M^{-1}R_\varepsilon MT^{-1}_{\tilde{h},h} =: \widetilde{P}_\varepsilon + \widetilde{R}_\varepsilon \tag{5.68}$$

on $H_{\Phi_3,h}$, in view of (4.14). The operator $\frac{1}{\varepsilon}M^{-1}P'_\varepsilon M$ is given by (4.20) with ξ_2 replaced by hD_{x_2}. The operator $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon$ therefore becomes, with $\mu = \sqrt{\varepsilon}$,

$$\begin{aligned}\frac{1}{\varepsilon}\widetilde{P}_\varepsilon = & \frac{1}{\varepsilon}p(f(\xi_2), \xi_2) + g(f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2)(\tilde{h}D_{x_1})^2 \\ & + \left(i\widetilde{\langle q\rangle_2} + \frac{\mathcal{O}(h)}{\varepsilon} + \varepsilon r\right)(x_1 + i\tilde{h}D_{x_1}, f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2) \\ & + \mathcal{O}(\tilde{h}^2)(x_1 + i\tilde{h}D_{x_1}, f(\xi_2) + \mu\tilde{h}D_{x_1}, \xi_2),\end{aligned} \tag{5.69}$$

where we replace ξ_2 by hD_{x_2}, since the $\tilde{h}$–Fourier integral operator $T^{(1)}_{\tilde{h}}$ is a convolution operator with the associated canonical transformation $(y_1, \eta_1) \mapsto (y_1 - i\eta_1, \eta_1)$, and similarly for $T^{(2)}_h$. From (4.15) we also find that

$$\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \widetilde{\xi_1}, x_2, \xi_2; h) = \mathcal{O}\left(\varepsilon^N + h^N + \varepsilon^{\frac{N}{2}}\widetilde{\xi_1}^N\right). \tag{5.70}$$

To study the operator in (5.68), we take a Fourier series expansion in x_2 of a general element $u \in H_{\Phi_3,h}$,

$$u(x_1,x_2) = \sum_{k\in\mathbb{Z}} \widetilde{u}_k(x_1) e_{kh}(x_2). \tag{5.71}$$

Here the functions $e_{kh}(x_2)$ have been introduced in (5.43). From Proposition 5.2 we recall that

$$\|\, u \,\|^2_{H_{\Phi_3,h}} \sim \sum_{k\in\mathbb{Z}} \|\, \widetilde{u}_k \,\|^2_{H_{\Phi_1(\cdot,kh),\tilde{h}}},$$

and correspondingly,

$$\frac{1}{\varepsilon}\widetilde{P}_\varepsilon u = \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon}\widetilde{P}_\varepsilon\left(x_1, \tilde{h}D_{x_1}, kh; h\right) \widetilde{u}_k(x_1) e_{kh}(x_2). \tag{5.72}$$

Therefore we have to study

$$\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, kh; h) : H_{\Phi_1(\cdot,kh),\tilde{h}} \to H_{\Phi_1(\cdot,kh),\tilde{h}}. \tag{5.73}$$

Here we recall from (5.18) and (5.19) that

$$\Phi_1(x_1,\eta_2) = \frac{1}{2}(\operatorname{Im} x_1)^2 + \Phi_2(x_1,\eta_2), \quad \Phi_2(x_1,\eta_2) = \mathcal{O}\left(\frac{1}{|\operatorname{Im} x_1|}\right) \tag{5.74}$$

is defined in the region $R < |\operatorname{Im} x_1| \ll \frac{1}{\sqrt{\varepsilon}}$, and when extending the definition to the domain $|\operatorname{Im} x_1| \leq R$, we use Proposition 5.2. Using also (5.66), we see that the representation (5.74) holds in the entire region $|\operatorname{Im} x_1| \ll \frac{1}{\sqrt{\varepsilon}}$, with

$$\partial^k_{\operatorname{Re} x_1,\eta_2} \partial^l_{\operatorname{Im} x_1} \Phi_2(x_1,\eta_2) = \mathcal{O}\left(\frac{1}{(R + |\operatorname{Im} x_1|)^{l+1}}\right), \quad R \gg 1. \tag{5.75}$$

In the region where $|\operatorname{Im} x_1| \gg 1$, we have, from Proposition 4.1,

$$\widetilde{P}_\varepsilon = T_{\tilde{h},h} G P''_\varepsilon G^{-1} T^{-1}_{\tilde{h},h},$$

and correspondingly for 1-variable pseudodifferential operators:

$$\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, kh; h) = \frac{1}{\varepsilon} T_{\tilde{h}} G_{kh} P''_\varepsilon(\tilde{h}D_{x_1}, kh; h) G^{-1}_{kh} T^{-1}_{\tilde{h}}.$$

Here G_{kh} is defined in (5.27) and $P''_\varepsilon(\tilde{h}D_{x_1}, kh; h)$ is given in (4.36):

$$\begin{aligned} \frac{1}{\varepsilon} P''_\varepsilon(\tilde{h}D_{x_1}, \xi_2; h) = {} & \frac{1}{\varepsilon} p(f(\xi_2), \xi_2) + g(f(\xi_2) + \sqrt{\varepsilon}\tilde{h}D_{x_1})(\tilde{h}D_{x_1})^2 \\ & + \left(i\widetilde{\langle\langle q\rangle_2\rangle}_1 + \mathcal{O}\left(\frac{h}{\varepsilon}\right) + \varepsilon\langle r_\varepsilon\rangle_1 \right) (f(\xi_2) + \sqrt{\varepsilon}\tilde{h}D_{x_1}, \xi_2) \\ & + \operatorname{Op}_{\tilde{h}}\left(\mathcal{O}\left(\frac{1}{\widetilde{\xi}_1^{\,2}}\right)\right) + \widetilde{R}(\tilde{h}D_{x_1}, \xi_2, \varepsilon; h), \end{aligned} \tag{5.76}$$

where

$$\widetilde{R} \sim \sum_{j=2}^{\infty} \tilde{h}^j \varepsilon^{-j/2} \widetilde{R}_j, \quad \widetilde{R}_j = \mathcal{O}\left(\frac{1}{\left|\widetilde{\xi}_1\right|^{2j-2}}\right).$$

An application of Egorov's theorem then shows that in the region where $|\mathrm{Im}\, x_1| \gg 1$, the symbol of $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, kh; h)$ restricted to

$$\Lambda_{\Phi_1(\cdot,kh)} = \left\{\left(x_1, \frac{\partial \Phi_1(x_1,kh)}{\partial x_1}\right)\right\},$$

can be identified with the symbol of (5.76) restricted to $T^*\mathbb{T}^2$, modulo an error $\mathcal{O}(\tilde{h})$. Let us notice also that if $(x_1, \widetilde{\xi_1}) \in \Lambda_{\Phi_1(\cdot,kh)}$ then from (5.74), (5.75),

$$\mathrm{Re}\,\widetilde{\xi_1} = -\frac{\partial \Phi_1}{\partial \mathrm{Im} x_1}(x_1, kh) = -\mathrm{Im}\, x_1 + \mathcal{O}\left(\frac{1}{(R+|\mathrm{Im} x_1|)^2}\right), \tag{5.77}$$

and

$$\mathrm{Im}\,\widetilde{\xi_1} = -\frac{\partial \Phi_1}{\partial \mathrm{Re} x_1}(x_1, kh) = \mathcal{O}\left(\frac{1}{R+|\mathrm{Im}\, x_1|}\right), \quad R \gg 1, \tag{5.78}$$

so that the imaginary part of the term $g(f(\xi_2)+\sqrt{\varepsilon}\widetilde{\xi_1}, \xi_2)\widetilde{\xi_1}^2$, occurring in the symbol in (5.69), restricted to $\Lambda_{\Phi_1(\cdot,\eta_2=kh)}$, is small, when $|\mathrm{Im}\, x_1| = \mathcal{O}(1)$.

We shall finish this section by discussing the action of the remainder in (5.68), $\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, x_2, hD_{x_2}; h)$, on $H_{\Phi_3,h}$. In doing so, we shall work, as we may, with the classical rather than the Weyl quantization. We shall study the scalar product $(\frac{1}{\varepsilon}\widetilde{R}_\varepsilon u_k|u_\ell)_{H_{\Phi_3,h}}$, where

$$u_k(x_1,x_2) = \widetilde{u}_k(x_1)e_{kh}(x_2), \quad u_\ell(x_1,x_2) = \widetilde{u}_\ell(x_1)e_{\ell h}(x_2), \quad k,\ell \in \mathbb{Z},\ k \neq \ell.$$

Here we have, in view of (4.16),

$$|kh| \leq \frac{1}{\widetilde{C}}, \quad |\ell h| \leq \frac{1}{\widetilde{C}}, \tag{5.79}$$

for some $\widetilde{C} \gg 1$. Let us consider first

$$\frac{1}{2\pi}\int_{\mathbb{C}/2\pi\mathbb{Z}} \frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, x_2, kh; h)\widetilde{u}_k(x_1)e^{\frac{i}{h}(kh)x_2}e^{-\frac{i}{h}(\ell h)\overline{x_2}}e^{-\frac{2\Phi_3(x_1,\mathrm{Im} x_2)}{h}}\, L(dx_2), \tag{5.80}$$

which is equal to

$$\int \widehat{\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, \cdot + i\mathrm{Im} x_2, kh; h)\widetilde{u}_k(x_1)}(\ell-k)e^{-\frac{2}{h}(\Phi_3(x_1,\mathrm{Im} x_2)+\frac{(k+\ell)}{2}h\mathrm{Im} x_2)}\, d\mathrm{Im} x_2, \tag{5.81}$$

where $\widehat{\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, \cdot + i\mathrm{Im}\, x_2, kh; h)\widetilde{u}_k(x_1)}(\ell-k)$ is the Fourier coefficient of

$$\mathbb{R}/2\pi\mathbb{Z} \ni \mathrm{Re}\, x_2 \mapsto \frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, \mathrm{Re}\, x_2 + i\mathrm{Im}\, x_2, kh; h)\widetilde{u}_k(x_1)$$

at the point $\ell - k$, and is therefore equal to the Fourier coefficient of

$$\mathbb{R}/2\pi\mathbb{Z} \ni \mathrm{Re}\, x_2 \mapsto \frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, \mathrm{Re}\, x_2, kh; h)\widetilde{u}_k(x_1) \tag{5.82}$$

at the same point times $e^{(k-\ell)\mathrm{Im} x_2}$. It follows that (5.80) is equal to

$$\left(\widehat{\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1, \tilde{h}D_{x_1}, \cdot, kh; h)\widetilde{u}_k(x_1)}\right)(\ell-k)\int e^{-\frac{2}{h}(\Phi_3(x_1,\mathrm{Im} x_2)+\ell h\mathrm{Im} x_2)}\, d\mathrm{Im} x_2, \tag{5.83}$$

and evaluating the integral in (5.83) by the method of stationary phase, as in subsection 5.2, we get

$$\left(\widehat{\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1,\tilde{h}D_{x_1},\cdot,kh;h)\widetilde{u}_k(x_1)}\right)(\ell-k)h^{\frac{1}{2}}a_{\Phi_3(x_1,\cdot)}(\ell h;h)e^{\frac{2}{h}\mathcal{L}\Phi_3(x_1,-\ell h)}. \tag{5.84}$$

Here the amplitude $a_{\Phi_3(x_1,\cdot)}$ is as in (5.35).

When estimating the first factor in (5.84), we recall that as in [26], modulo an error that is $\mathcal{O}(e^{-1/C\tilde{h}})$, $C>0$, we may write

$$\begin{aligned}\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1,\tilde{h}D_{x_1},x_2,kh;h)\widetilde{u}_k(x_1)\\ =\frac{1}{2\pi\tilde{h}}\iint e^{\frac{i}{\tilde{h}}(x_1-y_1)\widetilde{\xi}_1}\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1,\widetilde{\xi}_1,x_2,kh;h)\chi(x_1-y_1)\widetilde{u}_k(y_1)\,dy_1\,d\widetilde{\xi}_1,\end{aligned} \tag{5.85}$$

where χ is a suitable cutoff in a neighborhood of 0, and in (5.85) we choose a good contour adapted to the weight $\Phi_1(\cdot,\ell h)$ and given by

$$\widetilde{\xi}_1=\frac{2}{i}\frac{\partial\Phi_1(x_1,\ell h)}{\partial x_1}+iC\overline{(x_1-y_1)},\quad C\gg 1.$$

It follows, using also (5.70) and (5.74) that the absolute value of the kernel of

$$e^{-\Phi_1(\cdot,\ell h)/\tilde{h}}\frac{1}{\varepsilon}\widetilde{R}_\varepsilon e^{\Phi_1(\cdot,\ell h)/\tilde{h}}$$

does not exceed

$$\frac{\mathcal{O}(1)}{\tilde{h}}e^{-|x_1-y_1|^2/\tilde{h}}\left(\varepsilon^N+h^N+\varepsilon^{\frac{N}{2}}\left|\operatorname{Im}x_1\right|^N\right),$$

and since $\Phi_1(x_1,\eta_2)$ is defined for $|\operatorname{Im}x_1|\le\frac{1}{R\sqrt{\varepsilon}}$, $R\gg 1$, it follows that the $H_{\Phi_1(\cdot,\ell h),\tilde{h}}$–norm of (5.85) does not exceed, uniformly in x_2, $|\operatorname{Im}x_2|\ll 1$,

$$\mathcal{O}\left(\varepsilon^N+h^N+\frac{1}{R^N}\right)||\,\widetilde{u}_k\,||_{H_{\Phi_1(\cdot,\ell h),\tilde{h}}}.$$

Shifting also the contour of integration in x_2, we conclude that the $H_{\Phi_1(\cdot,\ell h),\tilde{h}}$–norm of

$$x_1\mapsto\left(\widehat{\frac{1}{\varepsilon}\widetilde{R}_\varepsilon(x_1,\tilde{h}D_{x_1},\cdot,kh;h)\widetilde{u}_k(x_1)}\right)(\ell-k) \tag{5.86}$$

can be estimated by

$$\mathcal{O}\left(\varepsilon^N+h^N+\frac{1}{R^N}\right)e^{-|k-\ell|/\mathcal{O}(1)}||\,\widetilde{u}_k\,||_{H_{\Phi_1(\cdot,\ell h),\tilde{h}}}. \tag{5.87}$$

Combining (5.84), (5.87), and Proposition 5.2 we see that the scalar product

$$(\frac{1}{\varepsilon}\widetilde{R}_\varepsilon u_k|u_\ell)_{H_{\Phi_3,h}} \tag{5.88}$$

can be estimated by

$$\mathcal{O}\left(\varepsilon^N+h^N+\frac{1}{R^N}\right)e^{-|k-\ell|/\mathcal{O}(1)}e^{\frac{h}{2}(\ell^2-k^2)}||\,\widetilde{u}_k\,||_{H_{\Phi_1,(\cdot,\ell h),\tilde{h}}}||\,\widetilde{u}_l\,||_{H_{\Phi_1,(\cdot,\ell h),\tilde{h}}}. \tag{5.89}$$

Here when considering $\widetilde{u}_k$, we want to replace $\Phi_1(\cdot,\ell h)$ by $\Phi_1(\cdot,kh)$, and according to (5.75), we can do it at the expense of the exponential factor $\exp\left(\mathcal{O}(1)\frac{\sqrt{\varepsilon}|k-\ell|}{R+|\operatorname{Im}x_1|}\right)$, which is permissible due to the presence of the factor $\exp\left(-|k-\ell|/\mathcal{O}(1)\right)$ in (5.89). Taking into account also (5.79) and (5.44), we may summarize this discussion in the following result.

PROPOSITION 5.5. – *Assume that $k,\ell \in \mathbb{Z}$ are such that (5.79) holds, and make the assumption (5.70). Then the scalar product*

$$\left(\frac{1}{\varepsilon}\widetilde{R}_\varepsilon u_k | u_\ell\right)_{H_{\Phi_3,h}},$$

where

$$u_k(x_1,x_2) = Ch^{-\frac{1}{4}} e^{-\frac{(kh)^2}{2h}} e^{\frac{i}{h}(kh)x_2}\widetilde{u}_k(x_1), \quad \widetilde{u}_k(x_1) \in H_{\Phi_1(\cdot,kh),\tilde{h}}, \ C>0,$$

and u_ℓ is defined similarly, can be estimated by

$$\mathcal{O}\left(\varepsilon^N + h^N + \frac{1}{R^N}\right) e^{-|k-\ell|/\mathcal{O}(1)} || u_k ||_{H_{\Phi_3,h}} || u_\ell ||_{H_{\Phi_3,h}}, \quad R \gg 1. \tag{5.90}$$

6. Global Hilbert space and spectral asymptotics for P_ε

6.1. Behavior of the Diophantine weight near $\Lambda_{1,r}$

Let us recall from section 2 that our spectral parameter z varies in a rectangle of the form

$$|\mathrm{Re}\, z| < \frac{\varepsilon}{\mathcal{O}(1)}, \quad \left|\frac{\mathrm{Im}\, z}{\varepsilon} - F_0\right| < \frac{1}{\mathcal{O}(1)},$$

where $F_0 \in Q_\infty(\Lambda_{1,r})$ satisfies (2.22), (2.26), (2.27) and (2.28). Recall also that we assume for simplicity that $L=2$ in (2.21) and $L'=1$ in (2.25).

In the absence of rational tori corresponding to the energy level $(0,\varepsilon F_0)$, the global weight that we used in [17] when away from a small but fixed neighborhood of $\cup_{j=1}^2 \Lambda_{j,d}$, was coming from an averaging procedure along the H_p–flow, and it is the weight that we should use in the present case, also when away from a neighborhood of $\Lambda_{1,r}$. Following [17], we shall now recall the definition of the weight in question.

Let $0 \le K \in C_0^\infty(\mathbb{R})$ be even and such that $\int K(t)\,dt = 1$. When $T>0$, we introduce the smoothed out flow average of q,

$$\langle q\rangle_{T,K} = \int K_T(t) q\circ \exp(tH_p)\,dt, \quad K_T(t) = \frac{1}{T}K\left(\frac{t}{T}\right), \tag{6.1}$$

the standard flow average in (2.14) corresponding to taking $K = 1_{[-1/2,1/2]}$. Let G_T be an analytic function defined near $p^{-1}(0)\cap\mathbb{R}^4$, such that

$$H_p G_T = q - \langle q\rangle_{T,K}. \tag{6.2}$$

As in [17], we solve (6.2) by setting

$$G_T = \int T J_T(-t) q\circ \exp(tH_p)\,dt, \quad J_T(t) = \frac{1}{T}J\left(\frac{t}{T}\right), \tag{6.3}$$

where the function J is compactly supported, smooth away from 0, and with

$$J'(t) = \delta(t) - K(t). \tag{6.4}$$

The behavior of G_T near the Diophantine tori $\Lambda_{j,d}$, $j=1,2$, as $T\to\infty$, has been analyzed in [17]. We shall now consider the behavior of G_T near $\Lambda_{1,r}$. Passing to the torus side by means of the canonical transformation in (4.2) and composing $p=p(\xi)$ in (4.8) with κ_M in (5.2), we may reduce ourselves to the case when

$$p(\xi_1,\xi_2) = p(f(\xi_2),\xi_2) + g(\xi_1 + f(\xi_2),\xi_2)\xi_1^2, \quad f(0)=0, \tag{6.5}$$

where $g(0,0)>0$. The expression (6.3) gives

$$G_T(x,\xi)=\int J\left(-\frac{t}{T}\right)q(x+tp'(\xi),\xi)\,dt,$$

and expanding $q(\cdot,\xi)$ in a Fourier series, we get

$$G_T(x,\xi)=\sum_{k=(k_1,k_2)\neq 0,k\in\mathbb{Z}^2}T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}, \tag{6.6}$$

since it follows from (6.4) and the fact that K is even that $\widehat{J}(0)=0$. Here $\widehat{q}(k,\xi)$ are the Fourier coefficients of $q(x,\xi)$ and $\widehat{J}(\tau)=\int e^{-it\tau}J(t)\,dt$ is the Fourier transform of J.

We write

$$G_T(x,\xi)=\sum_{k_2\neq 0}T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}+\sum_{k_2=0}T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}=\mathrm{I}+\mathrm{II}, \tag{6.7}$$

with the natural definitions of I and II. When estimating I, we notice that when $k_2\neq 0$, $|p'(\xi)\cdot k|\geq\left|p'_{\xi_2}k_2\right|-C\,|\xi_1|\,|k_1|\geq 1/2$, $C>0$, provided that $2C\,|\xi_1|\,|k_1|\leq 1$. (Here for notational simplicity we assume that $\left|p'_{\xi_2}\right|\geq 1$.) Let now $0\leq\chi\in C_0^\infty((-1,1))$ be such that $\chi=1$ on $[-1/2,1/2]$ and write, using also (6.4),

$$\begin{aligned}\mathrm{I}=&\sum_{k_2\neq 0}\chi(2C\,|\xi_1|\,|k_1|)T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}\\&+\sum_{k_2\neq 0}(1-\chi(2C\,|\xi_1|\,|k_1|))T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}\\=&\sum_{k_2\neq 0}\chi(2C\,|\xi_1|\,|k_1|)\frac{1-\widehat{K}(Tp'(\xi)\cdot k)}{ip'(\xi)\cdot k}\widehat{q}(k,\xi)e^{ix\cdot k}\\&+\sum_{k_2\neq 0}(1-\chi(2C\,|\xi_1|\,|k_1|))T\widehat{J}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}.\end{aligned} \tag{6.8}$$

It is easy to see that

$$\mathrm{I}=\mathcal{O}\left(1+T\,|\xi_1|^\infty\right),\quad T\geq 1. \tag{6.9}$$

When considering the contribution coming from II, we notice that

$$\begin{aligned}\mathrm{II}&=\sum_{k_2=0,k_1\neq 0}T\widehat{J}(Tp'_{\xi_1}k_1)e^{ix_1k_1}\widehat{q}(k,\xi)\\&=\sum_{k_2=0,k_1\neq 0}\frac{1-\widehat{K}(Tp'_{\xi_1}k_1)}{ip'_{\xi_1}k_1}e^{ix_1k_1}\widehat{q}(k,\xi),\end{aligned} \tag{6.10}$$

and therefore, since $\left|p'_{\xi_1}\right|\sim|\xi_1|$, in view of (6.5), we get uniformly in $T\geq 1$,

$$\mathrm{II}=\mathcal{O}(1)\frac{1}{|\xi_1|}. \tag{6.11}$$

Combining (6.11) with the bound $\mathrm{II}=\mathcal{O}(T)$, we get

$$\mathrm{II}=\mathcal{O}(1)\frac{T}{T\,|\xi_1|+1}. \tag{6.12}$$

PROPOSITION 6.1. – *Let G_T be defined as in (6.3), (6.4), so that it satisfies* (6.2)*. Assume that near $\xi = 0$ we have*

$$p(\xi_1,\xi_2) = p(f(\xi_2),\xi_2) + g(\xi_1 + f(\xi_2),\xi_2)\xi_1^2, \quad f(0) = 0, \quad g(0,0) > 0.$$

Then

$$G_T(x,\xi) = \mathcal{O}\left(1 + T\,|\xi_1|^\infty + \frac{T}{T\,|\xi_1| + 1}\right), \quad T \geq 1. \tag{6.13}$$

6.2. Global Hilbert space and the reference operators

In the first part of this subsection, we shall construct a global h-dependent Hilbert space where we shall study resolvent bounds for P_ε. The Hilbert space will be associated to a globally defined IR-manifold $\Lambda_\varepsilon \subset \mathbb{C}^4$, which in a complex neighborhood of $p^{-1}(0) \cap \mathbb{R}^4$, away from a sufficiently small but fixed neighborhood of

$$\bigcup_{|E|<\delta_0} \Lambda_{E,1,r} \quad 0 < \delta_0 \ll 1, \tag{6.14}$$

and away from a small neighborhood of $\bigcup_{j=1}^2 \Lambda_{j,d}$, will be given by

$$\Lambda_\varepsilon = \Lambda_{\varepsilon G_T} := \{\exp(i\varepsilon H_{G_T})(\rho);\ \rho \in \mathbb{R}^4\} \subset \mathbb{C}^4. \tag{6.15}$$

Here the function G_T has been defined in (6.3). In view of the assumption (2.28) and Lemma 2.4 of [17], the imaginary part of p_ε in (2.10) along Λ_ε in this region avoids the value εF_0, provided that T is taken sufficiently large but fixed.

When defining the global IR-manifold Λ_ε near the union of the Diophantine tori $\Lambda_{j,d}$, $j = 1, 2$, we follow the procedure of [17], implementing a Birkhoff normal form construction there. Therefore, it only remains to discuss the definition of Λ_ε in a full neighborhood of $\Lambda_{1,r}$, and how to extend it further to $\Lambda_{\varepsilon G_T}$ in (6.15).

From the discussion in section 5, we know that near (6.14), on the torus side, $H(\Lambda_\varepsilon)$ should agree with the microlocal Hilbert space

$$FMT_{\tilde{h},h}^{-1} H_{\Phi_3,h}, \tag{6.16}$$

introduced in (5.67). Now let us recall from Proposition 5.4 that in the region where

$$\varepsilon^{-1/6} \ll |\operatorname{Im} x_1| \ll \varepsilon^{-1/2}, \quad |\operatorname{Im} x_2| \ll 1, \tag{6.17}$$

on the $T_{\tilde{h},h}$–transform side, we have an identification $T_{\tilde{h},h}^{-1} H_{\Phi_3,h} \simeq T_{h,h}^{-1} H_{\Phi_5,h}$, with the weight Φ_5 having the properties described in (5.62), (5.63). Moreover, on the $T_{h,h}$–transform side, the region in (6.17) corresponds to a region where $\varepsilon^{1/3} \ll |\operatorname{Im} x_1| \ll 1$, $|\operatorname{Im} x_2| \ll 1$. In this region we may therefore identify the microlocal Hilbert space in (6.16) with

$$FMT_{h,h}^{-1} H_{\Phi_5,h} = MT_{h,h}^{-1} H_{\Phi_7,h}, \tag{6.18}$$

where the smooth strictly plurisubharmonic function $\Phi_7(x)$ is such that

$$\kappa_{T_{h,h}} \circ \kappa_M^{-1} \circ \kappa_\varepsilon \circ \kappa_M \circ \kappa_{T_{h,h}}^{-1} (\Lambda_{\Phi_5}) = \Lambda_{\Phi_7}.$$

Here $\kappa_{T_{h,h}} : (y,\eta) \mapsto (y - i\eta, \eta)$ is the canonical transformation associated to the Bargmann transform $T_{h,h}$ on $\mathbb{T}^2$, given in (5.4). The transform κ_ε corresponding to the operator F has been introduced in (4.12).

The transformation $\kappa_M^{-1} \circ \kappa_\varepsilon \circ \kappa_M$ is $\mathcal{O}(\varepsilon)$–close to the identity in the C^∞–sense and hence it follows from Proposition 5.4 that

$$\Phi_7(x) = \Phi_0(x) + \Phi_8(x), \quad \Phi_0(x) = \frac{1}{2}(\operatorname{Im} x)^2, \tag{6.19}$$

where the perturbation $\Phi_8(x)$ satisfies

$$\partial^k_{\mathrm{Re}x_1,x_2} \partial^l_{\mathrm{Im}x_1} \Phi_8(x) = \mathcal{O}_{kl}\left(\frac{\varepsilon}{|\mathrm{Im}x_1|^{1+l}}\right). \tag{6.20}$$

In particular, the Hessian of Φ_7 is uniformly bounded in a region where $\varepsilon^{1/3} \ll |\operatorname{Im} x_1| \ll 1$, $|\operatorname{Im} x_2| \ll 1$.

We conclude that near (6.14) but away from an $\mathcal{O}(\varepsilon^{1/3})$-neighborhood of that set, we should choose

$$\Lambda_\varepsilon = \kappa_0^{-1} \circ \kappa_\varepsilon \circ \kappa_M \circ \kappa^{-1}_{T_{h,h}}(\Lambda_{\Phi_5}) = \kappa_0^{-1} \circ \kappa_M \circ \kappa^{-1}_{T_{h,h}}(\Lambda_{\Phi_7}) \tag{6.21}$$

where κ_0 is the action-angle transform defined in (4.2).

We shall now glue the manifolds $\Lambda_{\varepsilon G_T}$ in (6.15) and Λ_ε in (6.21). To that end, from subsection 6.1 we recall that we have simplified the symbol p in (4.8) by composing it with the transformation κ_M in (5.2). Hence

$$\Lambda_{\varepsilon G_T} = \kappa_0^{-1} \circ \kappa_M \left(\Lambda_{\varepsilon G_T \circ \kappa_0^{-1} \circ \kappa_M}\right),$$

where $G_T := G_T \circ \kappa_0^{-1} \circ \kappa_M$ is given in Proposition 6.1. Recall next for example from [4] that if Φ_d is such that $\kappa_{T_{h,h}}(\Lambda_{\varepsilon G_T}) = \Lambda_{\Phi_d}$, then

$$\Phi_d(x) = \Phi_0(x) + \varepsilon G_T(\operatorname{Re} x, -\operatorname{Im} x) + \mathcal{O}(\varepsilon^2 |\nabla G_T|^2). \tag{6.22}$$

Let $\chi = \chi(\operatorname{Im} x_1) \in C_0^\infty$, $0 \leq \chi \leq 1$, be a standard cut-off function in a sufficiently small but fixed neighborhood of 0, and consider

$$\widetilde{\Phi}(x) = \chi(\operatorname{Im} x_1)\Phi_7(x) + (1 - \chi(\operatorname{Im} x_1))\Phi_d(x). \tag{6.23}$$

The function $\widetilde{\Phi}$ is strictly plurisubharmonic in a region $\varepsilon^{1/3} \ll |\operatorname{Im} x_1| \leq \frac{1}{\mathcal{O}(1)}$, $|\operatorname{Im} x_2| \leq \frac{1}{\mathcal{O}(1)}$. Moreover, it follows from (6.19), (6.20), (6.22), and Proposition 6.1 that

$$\widetilde{\Phi}(x) = \Phi_0(x) + \Phi_9(x), \tag{6.24}$$

where Φ_9 and its derivatives satisfy the same estimates as $\Phi_8(x)$ in (6.20). It follows that in a fixed neighborhood of the set in (6.14) but away from its $\varepsilon^{1/3}$–neighborhood, the IR-manifold Λ_ε is defined as

$$\Lambda_\varepsilon = \kappa_0^{-1} \circ \kappa_M \circ \kappa^{-1}_{T_{h,h}}\left(\Lambda_{\widetilde{\Phi}}\right), \tag{6.25}$$

and we need to fill the remaining gap. To that end, it will be convenient to go back to (6.16) and to work on the $T_{\tilde{h},h}$-transform side. Let us recall the relation (5.65) between the weights Φ_3 and Φ_5,

$$\kappa_{T_{h,h}} \circ \kappa^{-1}_{T_{\tilde{h},h}}(\Lambda_{\Phi_3}) = \Lambda_{\Phi_5},$$

with the transform $\kappa_{T_{h,h}} \circ \kappa_{T_{\tilde{h},h}}^{-1}$ defined in (5.64). Corresponding to the weight $\widetilde{\Phi}$ in (6.23), on the $T_{h,h}$–transform side, we introduce a weight $\widehat{\Phi}(x)$ on the $T_{\tilde{h},h}$-transform side given by the analogous relation

$$\kappa_{T_{h,h}} \circ \kappa_{T_{\tilde{h},h}}^{-1}\left(\Lambda_{\widehat{\Phi}}\right) = \Lambda_{\widetilde{\Phi}}. \tag{6.26}$$

We have

$$\widehat{\Phi}(x) = \frac{\sqrt{\varepsilon}}{2}(\operatorname{Im} x_1)^2 + \frac{1}{2}(\operatorname{Im} x_2)^2 + \Phi_{10}(x), \tag{6.27}$$

where Φ_{10} and its derivatives satisfy the same estimates as Φ_4 in Proposition 5.1. Moreover, in a region where $\varepsilon^{-1/6} \ll |\operatorname{Im} x_1| \ll \varepsilon^{-1/2}$, $|\operatorname{Im} x_2| \ll 1$, the weight $\widehat{\Phi}$ is an $\mathcal{O}(\sqrt{\varepsilon})$-perturbation of Φ_3, and as such it extends to the entire region $|\operatorname{Im} x_1| \ll \varepsilon^{-1/2}$, $|\operatorname{Im} x_2| \ll 1$, in the same way as in subsection 5.4.

The definition of $\Lambda_\varepsilon \subset \mathbb{C}^4$ in a full neighborhood of $\Lambda_{1,r}$, including the gluing region, is then as follows,

$$\Lambda_\varepsilon = \kappa_0^{-1} \circ \kappa_M \circ \kappa_{T_{\tilde{h},h}}^{-1}\left(\Lambda_{\widehat{\Phi}}\right). \tag{6.28}$$

where the transform $\kappa_{T_{\tilde{h},h}}$ has been defined in (5.13). Further away from $\Lambda_{1,r}$, we have $\Lambda_\varepsilon = \Lambda_{\varepsilon G_T}$ in (6.15), and when approaching the Diophantine region $\Lambda_{1,d} \cup \Lambda_{2,d}$, we define Λ_ε as in [17]. This gives a global definition of the IR–manifold $\Lambda_\varepsilon \subset \mathbb{C}^4$, which agrees with $\mathbb{R}^4$ outside a bounded set.

Let T be the standard FBI–Bargmann transform, defined as in (5.4), acting on $L^2(\mathbb{R}^2)$, and with the associated canonical transformation $\kappa_T : T^*\mathbb{C}^2 \to T^*\mathbb{C}^2$, defined as in (5.5). From [17] we know that away from a neighborhood of the rational region, we have

$$\kappa_T(\Lambda_\varepsilon) = \Lambda_{\Phi_\varepsilon} := \left\{(x,\xi) \in \mathbb{C}^2 \times \mathbb{C}^2;\, \xi = \frac{2}{i}\frac{\partial \Phi_\varepsilon}{\partial x}\right\}, \tag{6.29}$$

where Φ_ε is strictly plurisubharmonic with $\Phi_\varepsilon - \Phi_0 = \mathcal{O}(\varepsilon)$, $\nabla(\Phi_\varepsilon - \Phi_0) = \mathcal{O}(\varepsilon)$, $\Phi_0(x) = \frac{1}{2}(\operatorname{Im} x)^2$. Associated to Λ_ε, we then introduce a global h–dependent Hilbert space $H(\Lambda_\varepsilon)$, which agrees with $L^2(\mathbb{R}^2)$ as a set, and which is equipped with the norm

$$||\, u\,|| := ||\, T\,(1-\chi)\,u\,||_{H_{\Phi_\varepsilon}} + ||\, T_{\tilde{h},h} M^{-1} F^{-1} U^{-1} \chi u\,||_{H_{\Phi_3,h}}. \tag{6.30}$$

Here $\chi \in C_0^\infty(\Lambda_\varepsilon)$ is a cut-off to a small neighborhood of the rational region, which we quantize as a Toeplitz operator on the FBI–Bargmann transform side — see also the following discussion in this section. The elliptic Fourier integral operator U quantizes the action-angle symplectomorphism κ_0^{-1} in (4.2).

We shall now introduce a more precise description of the spectral window to which the spectral parameter z is confined. In doing so, let us recall the assumption (2.27), and assume, in order to fix the ideas, that $F_0 < \langle q\rangle(\Lambda_{1,r})$. Introduce a rectangle

$$R_\ell = \left[-\frac{\varepsilon}{C_0}, \frac{\varepsilon}{C_0}\right] + i\varepsilon\left[F_0 - \frac{1}{C_1}, F_0 + \frac{1}{C_2}\right], \tag{6.31}$$

where $C_0 > 0$ is large enough. Moreover, we shall take $C_2 > 1$ so large that

$$\frac{\operatorname{Im} z}{\varepsilon} < \langle q\rangle(\Lambda_{1,r}), \quad z \in R_\ell. \tag{6.32}$$

We further take $C_1 > 0$ so that

$$F_0 - \frac{1}{C_1} < \inf Q_\infty(\Lambda_{1,r}). \tag{6.33}$$

Our goal now is to construct a trace class Toeplitz operator $K : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon)$ such that the operator

$$\frac{1}{\varepsilon}\left(P_\varepsilon + i\varepsilon K - z\right)$$

becomes elliptic, in the $\tilde{h}$–pseudodifferential operator sense, in a full neighborhood of $\Lambda_{1,r}$, for z varying in (6.31). To this end, we shall restrict the attention to the rational region.

When constructing the operator K, we recall that microlocally near $\Lambda_{1,r}$, the action of P_ε on $H(\Lambda_\varepsilon)$ can be identified with the action of the operator in (5.68) on the weighted space $H_{\Phi_3,h}$. In what follows, as in (5.72), (5.73), we shall consider the one-parameter family of operators $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, \xi_2; h)$ acting on $H_{\Phi_1(\cdot,\xi_2),\tilde{h}}$, where ξ_2 is given in (4.16) and

$$|\xi_2| = \left| h\left(k - \frac{k_0(\alpha_2)}{4}\right) - \frac{S_2}{2\pi}\right| \ll 1.$$

We now claim that for $z \in \mathbb{C}$ in the domain (6.31) and in the region where $|\xi_2| \gg \varepsilon$, the elliptic bound

$$\left|\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \widetilde{\xi}_1, \xi_2; h) - \frac{z}{\varepsilon}\right| \geq \frac{1}{\mathcal{O}(1)} \tag{6.34}$$

holds true. Here $\widetilde{\xi}_1 = \frac{2}{i}\frac{\partial \Phi_1}{\partial x_1}(x_1, \xi_2)$, so that $(x_1, \widetilde{\xi}_1) \in \Lambda_{\Phi_1(\cdot,\xi_2)}$. When verifying (6.34), we recall from subsection 4.4 that in the region where $|\mathrm{Im}\, x_1| \gg 1$, the symbol of $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, \xi_2; h)$, restricted to $\Lambda_{\Phi_1(\cdot,\xi_2)}$, is identified with the symbol of (5.76) restricted to $T^*\mathbb{T}^2$, modulo $\mathcal{O}(\tilde{h})$, and (6.34) follows by considering the imaginary part of $\frac{1}{\varepsilon}\left(\widetilde{P}_\varepsilon(x_1, \widetilde{\xi}_1, \xi_2; h) - z\right)$, and using (6.32).

It remains therefore to check (6.34) in the region where $|\mathrm{Im}\, x_1| = \mathcal{O}(1)$. Here it follows by considering the real part of $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \widetilde{\xi}_1, \xi_2; h) - \frac{z}{\varepsilon}$ in (5.69) and using that $p(f(\xi_2), \xi_2) = a(\xi_2)\xi_2$, $a(\xi_2) > 0$, and that $g(0,0) > 0$, together with (5.77), (5.78).

In what follows, when considering the one-parameter family $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon\left(x_1, \tilde{h}D_{x_1}, \xi_2; h\right)$, we shall therefore restrict the attention to the quantum numbers $k \in \mathbb{Z}$ given by the condition

$$\xi_2 = h\left(k - \frac{k_0(\alpha_2)}{4}\right) - \frac{S_2}{2\pi} = \mathcal{O}(\varepsilon). \tag{6.35}$$

When $\widetilde{\xi}_1 = \frac{2}{i}\frac{\partial \Phi_1}{\partial x_1}(x_1, \xi_2)$, using (5.69) together with (5.77), (5.78), we obtain that for $|\mathrm{Im}\, x_1| = \mathcal{O}(1)$,

$$\mathrm{Im}\, \frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \widetilde{\xi}_1, \xi_2; h) = \widetilde{\langle q\rangle}_2(\mathrm{Re}\, x_1, -\mu \mathrm{Im} x_1 + f(\xi_2), \xi_2) + \mathcal{O}\left(\frac{h}{\varepsilon} + \varepsilon + \frac{1}{R + |\mathrm{Im}\, x_1|}\right). \tag{6.36}$$

Here we recall that $\mu = \sqrt{\varepsilon}$ and $R \gg 1$. Furthermore, as already exploited above, in the region where $|\mathrm{Im}\, x_1| \gg 1$, the closure of the range of the imaginary part of the symbol of $\frac{1}{\varepsilon}\widetilde{P}_\varepsilon(x_1, \tilde{h}D_{x_1}, \xi_2; h)$, restricted to $\Lambda_{\Phi_1(\cdot,\xi_2)}$, avoids the value $F_0 \in Q_\infty(\Lambda_{1,r})$.

For each $k \in \mathbb{Z}$ satisfying (6.35), let $0 \leq r_k = r_k(\operatorname{Im} x_1) \in C_0^\infty(\mathbb{R})$ be such that r_k vanishes for $|\operatorname{Im} x_1| \gg 1$ and such that the value F_0 is away from the closure of the range of

$$\operatorname{Im} \frac{1}{\varepsilon} \widetilde{P}_\varepsilon \left(x_1, \frac{2}{i} \frac{\partial \Phi_1}{\partial x_1}(x_1, \xi_2), \xi_2; h \right) + r_k(\operatorname{Im} x_1), \tag{6.37}$$

when $|\operatorname{Im} x_1| \leq R_1$, R_1 large enough. We notice that we can take r_k to be a suitably large multiple of some standard cutoff function. Associated with r_k we then have a Toeplitz operator

$$\operatorname{Top}(r_k) : H_{\Phi_1(\cdot,\xi_2),\tilde{h}} \to H_{\Phi_1(\cdot,\xi_2),\tilde{h}}, \tag{6.38}$$

defined as in the appendix. Using the one-dimensional operators $\operatorname{Top}(r_k)$, we introduce an operator $\mathcal{F}_{x_2}^{-1} \operatorname{Top}(r_k) \mathcal{F}_{x_2} : H_{\Phi_3,h} \to H_{\Phi_3,h}$ given by

$$\mathcal{F}_{x_2}^{-1} \operatorname{Top}(r_k) \mathcal{F}_{x_2} u(x_1, x_2) = \sum_{\xi_2 = \mathcal{O}(\varepsilon)} \left(\operatorname{Top}(r_k) \widetilde{u}_k \right)(x_1) e_{\xi_2}(x_2), \quad u \in H_{\Phi_3,h}, \tag{6.39}$$

with ξ_2 as in (6.35). Here, as in (5.71), we have written

$$u(x_1, x_2) = \sum_{k \in \mathbb{Z}} \widetilde{u}_k(x_1) e_{\xi_2}(x_2).$$

Combining (6.34) together with Proposition 5.5, and the construction of $\operatorname{Top}(r_k)$, for $k \in \mathbb{Z}$ satisfying (6.35), we conclude that for z in the domain (6.31), we have an elliptic estimate

$$\left|\left| \left(\frac{1}{\varepsilon} \widetilde{P}_\varepsilon + \frac{1}{\varepsilon} \widetilde{R}_\varepsilon + i \mathcal{F}_{x_2}^{-1} \operatorname{Top}(r_k) \mathcal{F}_{x_2} - \frac{z}{\varepsilon} \right) u \right|\right|_{H_{\Phi_3,h}} \geq \frac{1}{\mathcal{O}(1)} || u ||_{H_{\Phi_3,h}}. \tag{6.40}$$

Here we are also using the basic formula relating quantization and symbol multiplication on the FBI–Bargmann transform side, established in Theorem 1.3 in [27] (see also section 3 of [10]).

Back on the globally defined manifold Λ_ε, we let now $0 \leq \chi_0 \in C_0^\infty(\Lambda_\varepsilon)$ be such that $\chi_0 = 1$ near the rational torus and with supp χ_0 contained in a small neighborhood of the torus. We then take $0 \leq \chi_1 \in C_0^\infty(\Lambda_\varepsilon)$ supported near $\Lambda_{1,r}$, such that $\chi_1 = 1$ in a neighborhood of supp χ_0, and consider

(6.41)
$$K := \chi_1 U F M T_{\tilde{h},h}^{-1} \mathcal{F}_{x_2}^{-1} \operatorname{Top}(r_k) \mathcal{F}_{x_2} T_{\tilde{h},h} M^{-1} F^{-1} U^{-1} \chi_0 = \mathcal{O}(1) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon).$$

Here, as in (6.30), U is a unitary Fourier integral operator quantizing the action-angle transformation κ_0^{-1} in (4.2). When defining the operators corresponding to the functions χ_0 and χ_1 in (6.41), we identify $H(\Lambda_\varepsilon)$ with $F M T_{\tilde{h},h}^{-1} H_{\Phi_3,h}$ and use the Toeplitz quantization on the FBI–Bargmann transform side.

Now it is clear that the operator in (6.39) is of trace class on $H_{\Phi_3,h}$, with its trace class norm not exceeding

$$\mathcal{O}\left(\frac{\varepsilon}{h} \right) \sup_k || \operatorname{Top}(r_k) ||_{\mathrm{tr}} \leq \mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2} \right), \tag{6.42}$$

since an application of Proposition A.1 shows that the trace class norm of the Toeplitz operator (6.38) is

$$\mathcal{O}\left(\frac{1}{\tilde{h}}\right) = \mathcal{O}\left(\frac{\sqrt{\varepsilon}}{h}\right). \tag{6.43}$$

It follows that K in (6.41) is of trace class on $H(\Lambda_\varepsilon)$, its trace class norm not exceeding

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right).$$

PROPOSITION 6.2. – *Let us keep all the general assumptions from the introduction, and assume that* $F_0 \in \cup_{\Lambda \in J} Q_\infty(\Lambda)$ *satisfies the assumption (2.22)–(2.28). Assume also that* $h \ll \varepsilon = \mathcal{O}(h^\delta)$, *for some* $\delta > 0$. *Then there exist a globally defined IR-manifold* $\Lambda_\varepsilon \subset \mathbb{C}^4$ *and smooth Lagrangian tori* $\widehat{\Lambda}_{1,d}$, $\widehat{\Lambda}_{2,d}$, $\widehat{\Lambda}_{1,r} \subset \Lambda_\varepsilon$ *such that when* $\rho \in \Lambda_\varepsilon$ *is away from a small neighborhood of* $\widehat{\Lambda}_{1,d} \cup \widehat{\Lambda}_{2,d} \cup \widehat{\Lambda}_{1,r}$ *we have*

$$|\mathrm{Re}\, P_\varepsilon(\rho)| \geq \frac{1}{\mathcal{O}(1)} \quad or \quad |\mathrm{Im}\, P_\varepsilon(\rho) - \varepsilon F_0| \geq \frac{\varepsilon}{\mathcal{O}(1)}. \tag{6.44}$$

The estimates (6.44) remain valid for $\rho \in \Lambda_\varepsilon$ *near* $\widehat{\Lambda}_{1,r}$ *when away from an* $\mathcal{O}(\varepsilon^{1/2})$*–neighborhood of this set. The manifold* Λ_ε *is close to* $\mathbb{R}^4$ *and agrees with it outside a bounded set. We have*

$$P_\varepsilon = \mathcal{O}(1) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon).$$

For $j = 1, 2$ *there exists an elliptic Fourier integral operator*

$$U_j = \mathcal{O}(1) : H(\Lambda_\varepsilon) \to L^2_\theta(\mathbb{T}^2)$$

such that microlocally near $\widehat{\Lambda}_{j,d}$, $j = 1, 2$, *we have*

$$U_j P_\varepsilon = \left(P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h)\right) U_j.$$

Here $P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h)$ *is defined microlocally near* $\xi = 0$ *in* $T^*\mathbb{T}^2$, *the full symbol of* $P_j^{(N)}(hD_x, \varepsilon; h)$ *is independent of* x, *and*

$$R_{N+1,j}(x, \xi, \varepsilon; h) = \mathcal{O}\left((\xi, \varepsilon, h)^{N+1}\right).$$

Here N *is arbitrarily large but fixed. The leading symbol of* $P_j^{(N)}(hD_x, \varepsilon; h)$ *is of the form*

$$p_j(\xi) + i\varepsilon\langle q_j\rangle(\xi) + \mathcal{O}(\varepsilon^2),$$

with the differentials of p_j *and* $\langle q_j\rangle$ *being linearly independent when* $\xi = 0$, $j = 1, 2$.

Furthermore, there exists a trace class Toeplitz operator

$$K = \mathcal{O}(1) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon),$$

which has the following properties:

- K *is concentrated to the torus* $\widehat{\Lambda}_{1,r}$ *in the sense that if* $\psi \in C_0^\infty(\Lambda_\varepsilon)$ *is supported away from* $\widehat{\Lambda}_{1,r}$ *then*

$$\psi K = K\psi = \mathcal{O}(h^\infty) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.45}$$

- *The trace class norm of K satisfies*

$$\| K \|_{\mathrm{tr}} = \mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right).$$

- *For $\rho \in \Lambda_\varepsilon$ near $\widehat{\Lambda}_{1,r}$, we have*

$$|P_\varepsilon(\rho) + i\varepsilon K(\rho) - z| \geq \frac{\varepsilon}{\mathcal{O}(1)},$$

provided that the spectral parameter $z \in \mathbb{C}$ belongs to the domain (6.31), assuming (6.32), (6.33).

Remark. It follows from the discussion preceding Proposition 6.2 that the operator K enjoys better localization properties than (6.45), and is in fact concentrated to an $\mathcal{O}(\varepsilon^{1/2})$–neighborhood of $\widehat{\Lambda}_{1,r} \subset \Lambda_\varepsilon$.

We shall now derive resolvent bounds for the perturbed operator $P_\varepsilon + i\varepsilon K$ in the space $H(\Lambda_\varepsilon)$. To this end, let us recall the set E_d, defined in Theorem 2.1, which consists of the quasi-eigenvalues $z(j,k)$, $1 \leq j \leq 2$, $k \in \mathbb{Z}^2$, introduced in (2.31). We introduce an additional small parameter $0 < \widetilde{\varepsilon} = \mathcal{O}(h^\delta)$ such that $\widetilde{\varepsilon} \gg \varepsilon^{1/2}$, $\widetilde{\varepsilon} > h^{1/2-\delta}$. Then it follows from Proposition 6.2 (see also Proposition 5.1 in [17]) that when $\rho \in \Lambda_\varepsilon$ is away from an $\widetilde{\varepsilon}$–neighborhood of $\widehat{\Lambda}_{1,d} \cup \widehat{\Lambda}_{2,d} \cup \widehat{\Lambda}_{1,r}$, we have

$$|\mathrm{Re}\, P_\varepsilon(\rho; h)| \geq \frac{\widetilde{\varepsilon}}{\mathcal{O}(1)} \quad \text{or} \quad |\mathrm{Im}\, P_\varepsilon - \varepsilon F_0| \geq \frac{\varepsilon\widetilde{\varepsilon}}{\mathcal{O}(1)}. \tag{6.46}$$

In what follows, we shall let $z \in \mathbb{C}$ vary in the rectangle

$$\left[-\frac{\varepsilon}{C}, \frac{\varepsilon}{C}\right] + i\varepsilon\left[F_0 - \frac{\widetilde{\varepsilon}}{C}, F_0 + \frac{\widetilde{\varepsilon}}{C}\right], \tag{6.47}$$

for some $C > 0$ sufficiently large but fixed. Let $N_0 \geq 1$ be arbitrarily large but fixed. When z in the rectangle (6.47) avoids the union of $\varepsilon h^{N_0}/\mathcal{O}(1)$-neighborhoods of the $z(j,k)$'s, we would like to show that $P_\varepsilon + i\varepsilon K - z$ is invertible and to estimate the inverse in $H(\Lambda_\varepsilon)$. When doing so, to be able to exploit the Birkhoff normal form in the Diophantine region, as in [17], we shall use a partition of unity involving cutoff functions to small h–dependent neighborhoods of the Lagrangian tori.

In what follows we shall write that a function $a = a(\rho; h) \in C^\infty(\Lambda_\varepsilon)$ is in the symbol class $S_{\widetilde{\varepsilon}}(1)$ if uniformly on Λ_ε, we have

$$\nabla^m a = \mathcal{O}_m(\widetilde{\varepsilon}^{-m}), \quad m \geq 0.$$

We take a smooth partition of unity on the manifold Λ_ε,

$$1 = \sum_{j=1}^{2} \chi_j + \psi_{1,+} + \psi_{1,-} + \psi_{2,+} + \psi_{2,-} + \psi_3. \tag{6.48}$$

Here $0 \leq \chi_j \in C_0^\infty(\Lambda_\varepsilon) \cap S_{\widetilde{\varepsilon}}(1)$ is a cut-off function to an $\widetilde{\varepsilon}$–neighborhood of $\widehat{\Lambda}_{j,d}$, $j = 1,2$, and as in [17] we arrange so that

$$[P_\varepsilon, \chi_j] = \mathcal{O}(h^{(N+1)\delta}) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.49}$$

The functions $0 \le \psi_{1,\pm} \in S_{\widetilde{\varepsilon}}(1)$ are such that $\pm \mathrm{Re}\, P_\varepsilon \ge \widetilde{\varepsilon}/\mathcal{O}(1)$ in the support of $\psi_{1,\pm}$, respectively. Next, the functions $0 \le \psi_{2,\pm} \in C_0^\infty(\Lambda_\varepsilon) \cap S_{\widetilde{\varepsilon}}(1)$ are supported in regions invariant under the H_p–flow, where $\pm\,(\mathrm{Im}\, P_\varepsilon - \varepsilon F_0) \ge \varepsilon\widetilde{\varepsilon}/\mathcal{O}(1)$, respectively. We also arrange so that $\psi_{2,\pm}$ Poisson commute with p on Λ_ε. Here we have written p to denote the leading symbol of $P_{\varepsilon=0}$ acting on $H(\Lambda_\varepsilon)$. Finally, the function $0 \le \psi_3 \in C_0^\infty(\Lambda_\varepsilon) \cap S_{\widetilde{\varepsilon}}(1)$ is a cut-off to an $\widetilde{\varepsilon}$–neighborhood of $\widehat{\Lambda}_{1,r}$ such that $H_p \psi_3 = 0$. Moreover, we can arrange that

$$\psi_3 K = K + \mathcal{O}(h^\infty) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon).$$

At this point, we may follow the arguments of section 5 of [17] (see also [14]) to prove, using (6.46) together with the sharp Gårding inequality, that when

$$(P_\varepsilon + i\varepsilon K - z)\, u = v, \quad u \in H(\Lambda_\varepsilon), \tag{6.50}$$

with $z \in \mathbb{C}$ varying in (6.47), we have

$$\|\, \left(1 - \sum_{j=1}^{2} \chi_j - \psi_3\right) u \,\| \le \frac{\mathcal{O}(1)}{\varepsilon\widetilde{\varepsilon}} \|\, v \,\| + \mathcal{O}(h^\infty) \|\, u \,\|, \tag{6.51}$$

provided that

$$\frac{h}{\widetilde{\varepsilon}^5} \le h^\delta. \tag{6.52}$$

Here $\|\cdot\|$ is the norm in $H(\Lambda_\varepsilon)$. Let us also remark that when establishing (6.51), following [14], we use, in particular, that, on the operator level,

$$[P_\varepsilon, \psi_{2,\pm}] = [P_{\varepsilon=0}, \psi_{2,\pm}] + \mathcal{O}\left(\frac{\varepsilon h}{\widetilde{\varepsilon}^2}\right) = \mathcal{O}\left(\frac{h^2}{\widetilde{\varepsilon}^4}\right) + \mathcal{O}\left(\frac{\varepsilon h}{\widetilde{\varepsilon}^2}\right) = \mathcal{O}\left(\frac{\varepsilon h}{\widetilde{\varepsilon}^4}\right),$$

since $h \le \varepsilon$. Furthermore, since z belonging to (6.47) is such that $\mathrm{dist}(z, E_d) \ge \varepsilon h^{N_0}/\mathcal{O}(1)$, directly from section 5 in [17] we see, using also (6.49), that for $j = 1, 2$,

$$\|\, \chi_j u \,\| \le \frac{\mathcal{O}(1)}{\varepsilon h^{N_0}} \|\, v \,\| + \mathcal{O}(h^{(N+1)\delta - N_0 - 1}) \|\, u \,\|, \quad (N+1)\delta - N_0 - 1 \gg 1. \tag{6.53}$$

Combining (6.51) and (6.53), we get

$$\|\, (1 - \psi_3) u \,\| \le \frac{\mathcal{O}(1)}{\varepsilon h^{N_0}} \|\, v \,\| + \mathcal{O}(h^{(N+1)\delta - N_0 - 1}) \|\, u \,\|. \tag{6.54}$$

It remains to derive an estimate for $\psi_3 u$. When doing so, we write

$$(P_\varepsilon + i\varepsilon K - z)\, \psi_3 u = \psi_3 v + [P_\varepsilon + i\varepsilon K, \psi_3] u. \tag{6.55}$$

Here

$$[P_\varepsilon + i\varepsilon K, \psi_3] = \mathcal{O}\left(\frac{\varepsilon h}{\widetilde{\varepsilon}^4}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon),$$

and using (6.54) with a cut-off closer to $\widehat{\Lambda}_{1,r}$ we see that the $H(\Lambda_\varepsilon)$–norm of the commutator term in the right hand side of (6.55) is controlled by

$$\mathcal{O}\left(\frac{\varepsilon h}{\widetilde{\varepsilon}^4}\right) \frac{1}{\varepsilon h^{N_0}} \|\, v \,\| + \mathcal{O}(h^{(N+1)\delta - N_0 - 1}) \|\, u \,\| = \frac{\mathcal{O}(1)}{\widetilde{\varepsilon}^4 h^{N_0 - 1}} \|\, v \,\| + \mathcal{O}(h^{(N+1)\delta - N_0 - 1}) \|\, u \,\|. \tag{6.56}$$

Using (6.40) together with (6.55) and (6.56), we get

$$\|\, \psi_3 u \,\| \le \frac{\mathcal{O}(1)}{\varepsilon \widetilde{\varepsilon}^4 h^{N_0 - 1}} \|\, v \,\| + \mathcal{O}(h^{(N+1)\delta - N_0 - 2}) \|\, u \,\|. \tag{6.57}$$

Combining (6.54) and (6.57), and using also (6.52), we obtain the resolvent bounds, summarized in the following proposition.

PROPOSITION 6.3. – *Assume that $\widetilde{\varepsilon} = \mathcal{O}(h^{\delta})$, $\delta > 0$, is such that $\widetilde{\varepsilon} \gg \varepsilon^{1/2}$ and that (6.52) holds. Let*

$$z \in \left[-\frac{\varepsilon}{C}, \frac{\varepsilon}{C}\right] + i\varepsilon \left[F_0 - \frac{\widetilde{\varepsilon}}{C}, F_0 + \frac{\widetilde{\varepsilon}}{C}\right], \quad C \gg 1, \tag{6.58}$$

be such that $\mathrm{dist}(z, E_d) \geq \varepsilon h^{N_0}/\mathcal{O}(1)$, *for some* $N_0 \geq 1$. *Then, with the norm being the operator norm on* $H(\Lambda_\varepsilon)$, *we have*

$$|| \, (P_\varepsilon + i\varepsilon K - z)^{-1} \, || \leq \frac{\mathcal{O}(1)}{\varepsilon h^{N_0}}. \tag{6.59}$$

Remark. Continuing to argue as in [17] and solving a suitable Grushin problem as in that paper, we see that the eigenvalues of $P_\varepsilon + i\varepsilon K$ in the domain (6.58) are given by the elements of the set E_d in (2.31), modulo $\mathcal{O}(h^\infty)$, their total number being $\sim \varepsilon\widetilde{\varepsilon}/h^2$. We may think therefore of $P_\varepsilon + i\varepsilon K$ as a reference operator associated to the Diophantine region, and hereafter we shall often write

$$P_d = P_\varepsilon + i\varepsilon K. \tag{6.60}$$

Remark. Applying a simplified version of the argument above, we see that in the absence of Diophantine tori corresponding to the level $(0, \varepsilon F_0)$, the reference operator

$$P_\varepsilon + i\varepsilon K - z : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon)$$

is globally invertible, with

$$(P_\varepsilon + i\varepsilon K - z)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.61}$$

for z belonging to the rectangle (2.36). Indeed, when checking the injectivity, and hence the invertibility, of $P_\varepsilon + i\varepsilon K - z$, together with (6.60), we may use a partition of unity of the form (6.48), without the χ_j's, with all the terms there being symbols of class $S_{\widetilde{\varepsilon}=1}(1)$. The bound (6.61) is relevant for the proof of Theorem 2.2.

In the following discussion, we shall let $z \in \mathbb{C}$ vary in the rectangle (6.31), so that in particular (6.32) and (6.33) hold.

We shall now introduce a reference operator associated to the rational region. In doing so, we let $0 \leq \widetilde{\chi}_d \in C_0^\infty(\Lambda_\varepsilon)$ be such that $\widetilde{\chi}_d = 0$ in a small but fixed neighborhood of $\widehat{\Lambda}_{1,r}$ while $\widetilde{\chi}_d = 1$ away from a slightly larger neighborhood of this set, when restricting the attention to the region where $|\mathrm{Re}\, P_\varepsilon| \leq 1/\mathcal{O}(1)$. Also, $\widetilde{\chi}_d$ vanishes outside of a slightly larger set of the form $|\mathrm{Re}\, P_\varepsilon| \leq 1/\mathcal{O}(1)$. When $C > 1$ is large enough, let us consider the operator

$$P_r = P_\varepsilon + i\varepsilon C \widetilde{\chi}_d. \tag{6.62}$$

We may view P_r as a reference operator associated to the rational region. Notice that the trace class norm of the perturbation $P_r - P_\varepsilon$ on $H(\Lambda_\varepsilon)$ is $\mathcal{O}(\varepsilon h^{-2})$.

Our purpose is to study the spectrum of P_ε in the domain (6.47) in terms of the spectral information about the reference operators P_d and P_r in this region. In particular, Proposition 6.3 gives a polynomial in $1/h$ control on the resolvent of P_d, and we also know that the eigenvalues of P_d in (6.47) are given, modulo $\mathcal{O}(h^\infty)$, by the elements of the set E_d. While the spectral information available for the rational reference operator P_r is not going to be as precise, as a next step in our analysis, we shall derive resolvent bounds on P_r in $H(\Lambda_\varepsilon)$, when z in (6.31) is not too close to the spectrum of this operator.

Using the same arguments as earlier and choosing $C > 0$ sufficiently large, it is easily seen that the operator $P_r + i\varepsilon K - z = P_\varepsilon + i\varepsilon C\widetilde{\chi}_d + i\varepsilon K - z$ is globally invertible on $H(\Lambda_\varepsilon)$, with

$$(P_r + i\varepsilon K - z)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.63}$$

Here z varies in the rectangle (6.31). Write

$$P_r - z = (P_r + i\varepsilon K - z)\left(1 - i\varepsilon\,(P_r + i\varepsilon K - z)^{-1}\,K\right). \tag{6.64}$$

Proposition 6.2 together with (6.63) implies that $i\varepsilon\,(P_r + i\varepsilon K - z)^{-1}\,K$ is of trace class on $H(\Lambda_\varepsilon)$, and the corresponding trace class norm satisfies

$$||\, i\varepsilon\,(P_r + i\varepsilon K - z)^{-1}\,K\,||_{\rm tr} = \mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right). \tag{6.65}$$

It follows from (6.65) together with a basic estimate of [5] that the holomorphic function

$$D(z) = \det\left(I - i\varepsilon(P_r + i\varepsilon K - z)^{-1}K\right), \tag{6.66}$$

defined for z in the rectangle (6.31), satisfies

$$|D(z)| \leq \exp\left(\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right)\right). \tag{6.67}$$

The zeros of the perturbation determinant $D(z)$ in the domain (6.31) are precisely the eigenvalues of P_r in this region. To estimate the number of the zeros in such a domain, with slightly increased values of C_0, C_1, and C_2 in (6.31), it suffices, in view of Jensen's formula (see for example [21]) to establish a lower bound on $D(z)$ at a single point $z = z_0$ in (6.31). To this end we notice that the condition (6.33) allows us to find z_0 in the domain (6.31) such that

$$\frac{\operatorname{Im} z_0}{\varepsilon} < \inf Q_\infty(\Lambda_{1,r}). \tag{6.68}$$

As before, it follows that $P_r - z_0$ is invertible with

$$(P_r - z_0)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.69}$$

We get, using (6.64),

$$\begin{aligned}(I - i\varepsilon\,(P_r + i\varepsilon K - z_0)\,K)^{-1} &= (P_r - z_0)^{-1}\,(P_r + i\varepsilon K - z_0)\\ &= I + i\varepsilon\,(P_r - z_0)^{-1}\,K,\end{aligned} \tag{6.70}$$

and it follows, using (6.69), that the absolute value of the determinant of the right hand side of (6.70) is $\mathcal{O}(\varepsilon^{3/2}/h^2)$. Therefore,

$$|D(z_0)| \geq \exp\left(-\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right)\right), \tag{6.71}$$

and combining this bound together with (6.67) and Jensen's formula, we conclude that the number of eigenvalues of P_r in the rectangle (6.31), after an arbitrarily small decrease of the constants C_0, C_1, and C_2, is

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right).$$

The proof of Theorem 2.2 is now complete, in view of the second remark following Proposition 6.3.

Continuing with the proof of Theorem 2.1, we now come to derive resolvent estimates for the reference operator P_r. Let z in the rectangle (6.31) be such that

$$\operatorname{dist}(z, \operatorname{Spec}(P_r)) \geq g(h) > 0, \quad g(h) \ll \varepsilon. \tag{6.72}$$

An application of Theorem 5.1 from chapter 5 in [5] together with (6.65) shows that

$$|| \left(1 - i\varepsilon \left(P_r + i\varepsilon K - z\right)^{-1} K\right)^{-1} || \leq \frac{1}{|D(z)|} \exp\left(\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right)\right), \tag{6.73}$$

and in view of (6.63) and (6.64), it suffices to estimate $|D(z)|$ from below, away from its zeros. At this point, rather than recalling the details of the now well established argument for that, based on Cartan's lemma (or, alternatively, on Lemma 4.3 in [30]) and the Harnack inequality together with the maximum principle, we shall merely refer to [21] and [28], [30]. We obtain that if z in the domain (6.31), with increased values of the constants there, satisfies (6.72), then

$$|D(z)| \geq \exp\left(-\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2} \log \frac{1}{g(h)}\right)\right). \tag{6.74}$$

Combining (6.63), (6.64), (6.73), and (6.74), we get the following result.

PROPOSITION 6.4. – *Assume that $z \in \mathbb{C}$ is such that*

$$|\operatorname{Re} z| < \frac{\varepsilon}{C}, \quad |\operatorname{Im} z - \varepsilon F_0| < \frac{\varepsilon}{C}, \quad C \gg 1, \tag{6.75}$$

with $\operatorname{dist}(z, \operatorname{Spec}(P_r)) \geq g(h)$, $0 < g(h) \ll \varepsilon$. *Then*

$$|| \left(P_r - z\right)^{-1} || \leq \frac{\mathcal{O}(1)}{\varepsilon} \exp\left(\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right) \log \frac{1}{g(h)}\right). \tag{6.76}$$

Here $P_r = P_\varepsilon + i\varepsilon C \widetilde{\chi}_d$, where $\widetilde{\chi}_d \in C_0^\infty(\Lambda_\varepsilon; [0,1])$ is such that $\widetilde{\chi}_d = 0$ near the rational torus $\widehat{\Lambda}_{1,r}$ and $\widetilde{\chi}_d = 1$ further away from this set, in the region where $|\operatorname{Re} P_\varepsilon| \leq 1/\mathcal{O}(1)$.

Relying upon the resolvent estimates for the reference operators P_d and P_r, given in Propositions 6.3 and 6.4, we shall address the invertibility properties of $P_\varepsilon - z$. This is the subject of the next subsection.

6.3. Exponentially weighted estimates and bounds on spectral projections

Let us recall the reference operators

$$P_d = P_\varepsilon + i\varepsilon K \quad \text{and} \quad P_r = P_\varepsilon + i\varepsilon C\widetilde{\chi}_d, \quad C \gg 1, \tag{6.77}$$

introduced in (6.60) and (6.62). In the present subsection, as in Proposition 6.3, we shall let $z \in \mathbb{C}$ vary in the domain

$$R_s := \left[-\frac{\varepsilon}{C}, \frac{\varepsilon}{C}\right] + i\varepsilon\left[F_0 - \frac{\widetilde{\varepsilon}}{C}, F_0 + \frac{\widetilde{\varepsilon}}{C}\right], \quad C \gg 1, \tag{6.78}$$

where we recall that $\widetilde{\varepsilon}$ here can be chosen so that $\widetilde{\varepsilon} \sim \varepsilon^\delta$ for any small $\delta > 0$. Let us assume that

$$\operatorname{dist}(z, \operatorname{Spec}(P_d) \cup \operatorname{Spec}(P_r)) \geq g(h), \tag{6.79}$$

where, as in Proposition 6.3, we take

$$g(h) = \varepsilon h^{N_0},$$

for some arbitrarily large but fixed $N_0 \geq 2$. We know from (6.59) that $(P_d - z)^{-1}$ enjoys polynomial upper bounds as a bounded operator on $H(\Lambda_\varepsilon)$, while Proposition 6.4 provides an exponential estimate for the resolvent of P_r.

Next we shall introduce a reference operator associated with the elliptic region, where $|\mathrm{Re}\, P_\varepsilon|$ is bounded away from zero. To this end, let $\psi \in C_0^\infty(\Lambda_\varepsilon; [0,1])$ be such that $\psi = 1$ in a region where $|\mathrm{Re}\, P_\varepsilon| \leq 1/\mathcal{O}(1)$, and assume that ψ vanishes outside of a slightly larger region of the same form. When $C \gg 1$ and z varies in the domain (6.78), we see that the operator

$$P_\varepsilon + i\varepsilon C\psi - z : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon) \tag{6.80}$$

is invertible, with

$$(P_\varepsilon + i\varepsilon C\psi - z)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.81}$$

Let us consider a smooth partition of unity on the manifold Λ_ε,

$$1 = \chi_r + \chi_d + \chi_0. \tag{6.82}$$

Here $0 \leq \chi_r \in C_0^\infty(\Lambda_\varepsilon; [0,1])$ is $= 1$ near $\widehat{\Lambda}_{1,r}$ and $\operatorname{supp} \chi_r$ is contained in a small but fixed neighborhood of this set. The function $\chi_d \in C_0^\infty(\Lambda_\varepsilon; [0,1])$ is $= 1$ near $\operatorname{supp} \widetilde{\chi}_d$, while $\chi_0 \in C_b^\infty(\Lambda_\varepsilon; [0,1])$ is such that $\operatorname{supp} \chi_0$ is contained in a region where $|\mathrm{Re}\, P_\varepsilon| \geq 1/\mathcal{O}(1)$ and $\chi_0 = 1$ further away from the region where $|\mathrm{Re}\, P_\varepsilon|$ is small. We furthermore arrange so that the functions ψ and χ_0 have disjoint supports.

Recall that $z \in R_s$ in (6.78) satisfies (6.79). As an approximation to the inverse of $P_\varepsilon - z$, we consider

$$R_0(z) = (P_r - z)^{-1} \chi_r + (P_d - z)^{-1} \chi_d + (P_\varepsilon + i\varepsilon C\psi - z)^{-1} \chi_0. \tag{6.83}$$

Using the definitions (6.60) and (6.62) of the operators P_d and P_r, we see that

$$(P_\varepsilon - z)\, R_0(z) = 1 + L, \tag{6.84}$$

where

$$L = -i\varepsilon C\widetilde{\chi}_d\, (P_r - z)^{-1} \chi_r - i\varepsilon K\, (P_d - z)^{-1} \chi_d - i\varepsilon C\psi\, (P_\varepsilon + i\varepsilon C\psi - z)^{-1} \chi_0. \tag{6.85}$$

The key step will consist of establishing the following result.

PROPOSITION 6.5. –

$$L = \mathcal{O}(e^{-\frac{1}{\widetilde{C}h}}) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.86}$$

for some $\widetilde{C} > 0$.

When proving Proposition 6.5, we shall introduce additional modifications of the exponential weight corresponding to the IR-manifold Λ_ε. The various modifications of the weight will take place only in regions away from a small neighborhood of the rational torus $\Lambda_{1,r}$.

We start by considering the term

$$L_1 = -i\varepsilon C\widetilde{\chi}_d\,(P_r - z)^{-1}\,\chi_r : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.87}$$

occurring in (6.85), and notice that the compact sets supp $\widetilde{\chi}_d$ and supp χ_r are disjoint. From (6.29) let us recall that away from $\widehat{\Lambda}_{1,r} \subset \Lambda_\varepsilon$, we have

$$\kappa_T\,(\Lambda_\varepsilon) = \left\{(x,\xi) \in T^*\mathbb{C}^2; \xi = \frac{2}{i}\frac{\partial \Phi_\varepsilon}{\partial x}\right\}, \tag{6.88}$$

with $\Phi_\varepsilon - \Phi_0 = \mathcal{O}(\varepsilon)$ and $\nabla\,(\Phi_\varepsilon - \Phi_0) = \mathcal{O}(\varepsilon)$, $\Phi_0(x) = (1/2)(\operatorname{Im} x)^2$. Here, as usual,

$$\kappa_T(y,\eta) = (y - i\eta, \eta) = (x,\xi). \tag{6.89}$$

In the following discussion, we shall often identify an open set $\Omega \subset \Lambda_\varepsilon$ whose closure is away from $\widehat{\Lambda}_{1,r}$, with $\pi_x\,(\kappa_T(\Omega)) \subset \mathbb{C}^2$. Here $\pi_x : T^*\mathbb{C}^2 \to \mathbb{C}^2$ is the natural projection given by $\pi_x(x,\xi) = x$. Correspondingly, a function $F : \Omega \to \mathbb{C}$ may be identified with $F \circ (\pi_x \circ \kappa_T)^{-1} : \mathbb{C}^2 \to \mathbb{C}$.

Let us recall from section 2 that we assume, for simplicity of the exposition only, that the tori $\Lambda_{j,d}$, $j = 1,2$ and $\Lambda_{1,r}$ belong to the same open edge of J in (2.11) so that (2.39) holds. Let $\widetilde{\Lambda}_{j,d} \subset \Lambda_\varepsilon$, $j = 1, 2$, be "intermediate" Diophantine tori belonging to the same open edge of J as $\Lambda_{j,d}$ and $\Lambda_{1,r}$, away from supp $\widetilde{\chi}_d$, with

$$\Lambda_{1,d} < \widetilde{\Lambda}_{1,d} < \Lambda_{1,r}, \quad \Lambda_{1,r} < \widetilde{\Lambda}_{2,d} < \Lambda_{2,d}.$$

Here, in order to simplify the notation, we are identifying the real tori $\widetilde{\Lambda}_{j,d} \subset p^{-1}(0) \cap \mathbb{R}^4$ with their images in Λ_ε, by means of the canonical transformation $\exp\,(i\varepsilon H_{G_T}) : \mathbb{R}^4 \to \Lambda_\varepsilon$ — see also (6.15). We shall introduce a new weight $G \in C_0^\infty(\mathbb{C}^2)$ supported in a region where $|\operatorname{Re} P_\varepsilon| \leq 1/\mathcal{O}(1)$, such that $G = 0$ in a fixed neighborhood of supp χ_r, while $G = -\eta < 0$ in a fixed neighborhood of supp $\widetilde{\chi}_d$. Here $\eta > 0$ is very small but fixed and we shall have $|\nabla G| \ll 1$, $\left|\nabla^2 G\right| \ll 1$ everywhere. Moreover, G will be chosen so that, when restricting the attention to the region where $|\operatorname{Re} P_\varepsilon| \leq 1/\mathcal{O}(1)$, the support of ∇G is contained in a sufficiently small but fixed neighborhood of $\widetilde{\Lambda}_{1,d} \cup \widetilde{\Lambda}_{2,d}$.

We shall now define G near $\widetilde{\Lambda}_{j,d}$, say, when $j = 1$. When doing so, take a smooth canonical diffeomorphism

$$\widetilde{\kappa} : \operatorname{neigh}(\widetilde{\Lambda}_{1,d}, \Lambda_\varepsilon) \to \operatorname{neigh}(\xi = 0, T^*\mathbb{T}^2), \tag{6.90}$$

mapping $\widetilde{\Lambda}_{1,d}$ to the zero section in $T^*\mathbb{T}^2$ and obtained by composing the action-angle canonical transformation near the real torus with the holomorphic transformation

$\exp\left(-i\varepsilon H_{G_T}\right)$. Composing p_ε in (4.1) with $\widetilde{\kappa}^{-1}$, we obtain a new symbol, still denoted by p_ε, defined near the zero section $\xi = 0$ in $T^*\mathbb{T}^2$, which is of the form

$$p_\varepsilon(x,\xi) = p(\xi) + i\varepsilon\langle q\rangle_T(x,\xi) + \mathcal{O}_T(\varepsilon^2). \tag{6.91}$$

Here, as already observed in the beginning of subsection 6.2, we take $T > 0$ sufficiently large but fixed, so that $\langle q\rangle_T(x,\xi)$ avoids the value F_0 in this region. In view of the implicit function theorem, we may assume that the energy surface $p^{-1}(0)$ is given by an equation

$$\xi_2 = f(\xi_1), \quad |\xi_1| \leq a, \quad 0 < a \ll 1, \tag{6.92}$$

where the analytic function f satisfies $f(0) = 0$, $f'(0) \neq 0$.

When defining the weight G near $\xi = 0$ we shall require that it should be constant on each invariant torus $\xi = \mathrm{Const}$. In doing so, we shall first define G on $p^{-1}(0)$, and to that end we introduce the tori $\Lambda_\mu \subset p^{-1}(0)$, $|\mu| \leq a$, given by $\xi_1 = \mu$, $\xi_2 = f(\mu)$. In order to fix the ideas, let us assume that when $\mu < 0$, then the tori $\widetilde{\kappa}^{-1}\left(\Lambda_\mu\right)$ satisfy

$$\widetilde{\Lambda}_{1,d} < \widetilde{\kappa}^{-1}\left(\Lambda_\mu\right) < \widehat{\Lambda}_{1,r},$$

and for $\mu > 0$, we have

$$\widehat{\Lambda}_{1,d} < \widetilde{\kappa}^{-1}\left(\Lambda_\mu\right) < \widetilde{\Lambda}_{1,d}.$$

When $\delta > 0$ is very small but fixed, we then let $G_0 = G_0(\xi_1) \in C^\infty([-a,a];[0,\delta])$ be increasing and such that $G_0 = 0$ near $-a$, $G_0 = \delta$ near a, and with G_0' having a compact support in a small neighborhood of $\xi_1 = 0$. Taking $\delta > 0$ small enough, we achieve that $|G_0'| \ll 1$ and $|G_0''| \ll 1$. Setting $G(\xi_1, f(\xi_1)) = -G_0(\xi_1)$, we see that we have defined G on $p^{-1}(0)$. We then extend G suitably to a full neighborhood of $\xi = 0$ in $\mathbb{R}^2$ so that it still depends on ξ only and $|\nabla G| \ll 1$ is different from zero only in a small neighborhood of $\xi = 0$.

Introduce next the IR-manifold

$$\left(T^*\mathbb{T}^2\right)_G = \left\{(x + iG_\xi'(\xi), \xi);\ (x,\xi) \in T^*\mathbb{T}^2\right\}, \tag{6.93}$$

defined in a complex neighborhood of the zero section $\xi = 0$. Then the imaginary part of the symbol of p_ε in (6.91), along $\left(T^*\mathbb{T}^2\right)_G$, still avoids the value εF_0.

Similarly, working in the action-angle variables, we define $G = G(\xi)$ in a neighborhood of the Diophantine torus $\widetilde{\Lambda}_{2,d}$. It is then clear that we can define the new global IR–manifold $\widetilde{\Lambda}_\varepsilon \subset \mathbb{C}^4$ so that near $\widetilde{\Lambda}_{1,d}$, it is given by $\widetilde{\kappa}^{-1}\left(\left(T^*\mathbb{T}^2\right)_G\right)$, and away from $\widetilde{\Lambda}_{1,d} \cup \widetilde{\Lambda}_{2,d} \cup \widehat{\Lambda}_{1,r}$, we define $\widetilde{\Lambda}_\varepsilon$ so that the representation

$$\kappa_T\left(\widetilde{\Lambda}_\varepsilon\right) = \Lambda_{\widetilde{\Phi}_\varepsilon}$$

holds true. Here $\widetilde{\Phi}_\varepsilon - \Phi_\varepsilon \in C_0^\infty(\mathbb{C}^2)$ and its gradient is supported in a small neighborhood of $\widetilde{\Lambda}_{1,d} \cup \widetilde{\Lambda}_{2,d}$, when restricting the attention to the region where $|\mathrm{Re}\, P_\varepsilon| \leq 1/\mathcal{O}(1)$. We have $\widetilde{\Phi}_\varepsilon = \Phi_\varepsilon - \eta$ in a fixed neighborhood of $\mathrm{supp}\, \widetilde{\chi}_d$, while $\widetilde{\Lambda}_\varepsilon = \Lambda_\varepsilon$ near $\widehat{\Lambda}_{1,r}$.

The discussion above is summarized in the following proposition.

Lemma 6.6. – *There exists an IR-manifold $\widetilde{\Lambda}_\varepsilon \subset T^*\mathbb{C}^2$ which coincides with Λ_ε near $\widehat{\Lambda}_{1,r}$, such that away from $\widehat{\Lambda}_{1,r}$, after applying the canonical transformation κ_T, defined in (6.89), so that Λ_ε becomes $\Lambda_{\Phi_\varepsilon}$, with*

$$\Phi_\varepsilon = \Phi_0 + \mathcal{O}(\varepsilon), \quad \Phi_0(x) = \frac{(\mathrm{Im}\, x)^2}{2},$$

$\widetilde{\Lambda}_\varepsilon$ becomes $\Lambda_{\widetilde{\Phi}_\varepsilon}$, where $\widetilde{\Phi}_\varepsilon - \Phi_\varepsilon$ is compactly supported and for some $\eta > 0$ small enough but fixed we have $\widetilde{\Phi}_\varepsilon = \Phi_\varepsilon - \eta$ near $\pi_x(\kappa_T(\operatorname{supp}\widetilde{\chi}_d))$. Furthermore,

$$P_\varepsilon = \mathcal{O}(1) : H(\widetilde{\Lambda}_\varepsilon) \to H(\widetilde{\Lambda}_\varepsilon),$$

and the resolvent bounds (6.59) and (6.76) hold true in the sense of bounded linear operators on $H(\widetilde{\Lambda}_\varepsilon)$.

It is now easy to estimate the norm of the term (6.87) as a bounded operator on $H(\Lambda_\varepsilon)$. First notice that

$$\chi_r = \mathcal{O}(1) : H(\Lambda_\varepsilon) \to H(\widetilde{\Lambda}_\varepsilon),$$

where we use, as before, the Toeplitz quantization of χ_r on the FBI–Bargmann side. Combining this with Proposition 6.4 and Lemma 6.6, we get

$$(P_r - z)^{-1}\chi_r = \frac{\mathcal{O}(1)}{\varepsilon}\exp\left(\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right)\log\frac{1}{h}\right) : H(\Lambda_\varepsilon) \to H(\widetilde{\Lambda}_\varepsilon). \tag{6.94}$$

Now $\operatorname{supp}\widetilde{\chi}_d$ is contained in a region where $\widetilde{\Phi}_\varepsilon - \Phi_\varepsilon = -\eta < 0$ and hence,

$$\widetilde{\chi}_d = \mathcal{O}\left(e^{-\frac{\eta}{h}}\right) : H(\widetilde{\Lambda}_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.95}$$

Here $\widetilde{\chi}_d$ is quantized as a Toeplitz operator in the weighted space H_{Φ_ε}, by working on the transform side. Using (6.94) and (6.95) together with the upper bound $\varepsilon = \mathcal{O}(h^{2/3+\delta})$, $\delta > 0$, we conclude that

$$L_1 = -i\varepsilon C\widetilde{\chi}_d\,(P_r - z)^{-1}\chi_r = \mathcal{O}\left(e^{-1/\widetilde{C}h}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \quad \widetilde{C} > 0. \tag{6.96}$$

When estimating the operator norm of the expression

$$L_2 = -i\varepsilon K\,(P_d - z)^{-1}\chi_d : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.97}$$

we argue similarly and introduce a weak but h-independent weight, supported in a region where $|\operatorname{Re} P_\varepsilon| \le 1/\mathcal{O}(1)$, which is equal to a very small strictly positive constant $\eta > 0$ in a fixed neighborhood of $\operatorname{supp}\chi_d$. We then obtain a new microlocally weighted space $H(\widehat{\Lambda}_\varepsilon)$ associated to an IR–manifold $\widehat{\Lambda}_\varepsilon$ defined similarly to $\widetilde{\Lambda}_\varepsilon$, such that if $\kappa_T(\widehat{\Lambda}_\varepsilon) = \Lambda_{\widehat{\Phi}_\varepsilon}$, then $\widehat{\Phi}_\varepsilon = \Phi_\varepsilon + \eta$, $0 < \eta \ll 1$, in a fixed neighborhood of $\operatorname{supp}\chi_d$. Then

$$\chi_d = \mathcal{O}\left(e^{-\frac{\eta}{h}}\right) : H(\Lambda_\varepsilon) \to H(\widehat{\Lambda}_\varepsilon), \tag{6.98}$$

and combining this estimate together with the fact that $K = \mathcal{O}(1) : H(\widehat{\Lambda}_\varepsilon) \to H(\Lambda_\varepsilon)$ and with Lemma 6.6, we infer that

$$L_2 = -i\varepsilon K\,(P_d - z)^{-1}\chi_d = \mathcal{O}\left(e^{-\frac{1}{Ch}}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \quad C > 0. \tag{6.99}$$

To finish the proof of Proposition 6.5, we only need to estimate the norm of the operator

$$L_3 = i\varepsilon C\psi\,(P_\varepsilon + i\varepsilon C\psi - z)^{-1}\chi_0 : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.100}$$

and this requires an introduction of a new weight on the FBI–Bargmann transform side, that we shall still denote by G. We take $G \in C_b^\infty(\mathbb{C}^2)$ with $|\nabla G| \ll 1$, $\left|\nabla^2 G\right| \ll 1$, such that $G = 0$ in a fixed neighborhood of $\operatorname{supp}\psi$. We shall furthermore choose G so that it is equal to a very small but strictly positive constant in a fixed neighborhood of $\operatorname{supp}\chi_0$, and hence in a neighborhood of infinity. Here we may recall that the support of χ_0 does not intersect the compact set supp ψ. We also choose G so that the support of ∇G is contained in a thin domain included in a region where $|\mathrm{Re}\, P_\varepsilon| \geq 1/\mathcal{O}(1)$. It is then easy to see that

$$L_3 = \mathcal{O}\left(e^{-1/Ch}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \quad C > 0, \tag{6.101}$$

and combining this estimate together with (6.96), (6.99), and (6.85), we complete the proof of Proposition 6.5.

Combining Proposition 6.5 with (6.84) we see that for z satisfying (6.79), the operator $P_\varepsilon - z : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon)$ is invertible, with

$$(P_\varepsilon - z)^{-1} = R_0(z)\,(1 + L)^{-1}. \tag{6.102}$$

Writing $(1 + L)^{-1} = 1 - (1 + L)^{-1} L$ we get

$$(P_\varepsilon - z)^{-1} = R_0(z) - R_0(z)\,(1 + L)^{-1}\, L. \tag{6.103}$$

Let now γ be a simple positively oriented closed C^1–contour contained in the domain (6.78), of length $\mathcal{O}(\varepsilon)$, such that (6.79) holds for each z along γ. Let

$$\Pi = -\frac{1}{2\pi i} \int_\gamma (P_\varepsilon - z)^{-1}\, dz \tag{6.104}$$

be the spectral projection of P_ε associated to the spectrum of P_ε inside γ. The finite-dimensional space $\Pi(H(\Lambda_\varepsilon))$ is spanned by the generalized eigenfunctions of P_ε corresponding to the eigenvalues of P_ε in the interior of γ. Define also

$$\Pi_0 = -\frac{1}{2\pi i} \int_\gamma R_0(z)\, dz, \tag{6.105}$$

and notice that the last term in the right hand side of (6.83) does not contribute to the integral in (6.105), since $(P_\varepsilon + i\varepsilon C\psi - z)^{-1}$ is holomorphic in $z \in R_s$. Let us also introduce the finite-dimensional space $E \subset H(\Lambda_\varepsilon)$ spanned by the generalized eigenfunctions of the operators P_d and P_r, corresponding to their spectra inside γ. Notice that the range of Π_0 in (6.105) is contained in E.

Now (6.103) gives that

$$\Pi = \Pi_0 + \frac{1}{2\pi i} \int_\gamma R_0(z)\,(1 + L)^{-1}\, L\, dz, \tag{6.106}$$

and combining Proposition 6.3, Proposition 6.4 and Proposition 6.5 together with the fact that $\varepsilon = \mathcal{O}\left(h^{2/3+\delta}\right)$, $\delta > 0$, we see that the operator norm of the contour integral in the right hand side of (6.106), is $\mathcal{O}(\exp(-1/\widehat{C}h))$, for some $\widehat{C} > 0$. In particular, if $u \in H(\Lambda_\varepsilon)$, $||\, u\, || = 1$, belongs to the range of Π, then

$$\Pi_0 u = u + \mathcal{O}(e^{-1/\widehat{C}h}).$$

Using the basic properties of the non–symmetric distance between two closed subspaces of a Hilbert space, introduced and studied in [9] (see also [3]), we conclude that

$$\dim \Pi(H(\Lambda_\varepsilon)) \leq \dim E. \tag{6.107}$$

When proving the opposite inequality, we write, using (6.83),

$$\Pi_0 = \Pi_r \chi_r + \Pi_d \chi_d,$$

where

$$\Pi_d = -\frac{1}{2\pi i}\int_\gamma (P_d - z)^{-1}\, dz = \mathcal{O}\left(\frac{1}{h^{N_0}}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon), \tag{6.108}$$

and

$$\Pi_r = -\frac{1}{2\pi i}\int_\gamma (P_r - z)^{-1}\, dz \tag{6.109}$$

satisfies

$$\Pi_r = \exp\left(\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right)\log\frac{1}{h}\right) : H(\Lambda_\varepsilon) \to H(\Lambda_\varepsilon). \tag{6.110}$$

Here we have also used Propositions 6.3 and 6.4.

Let $u \in E$ be a normalized generalized eigenfunction of, say, P_d, corresponding to an eigenvalue of this operator inside γ. Then using exponentially weighted estimates, in the same way as in the proof of Proposition 6.5, together with (6.110) and the upper bound $\varepsilon = \mathcal{O}(h^{2/3+\delta})$, we see that

$$\Pi_r \chi_r u = \mathcal{O}(e^{-1/Ch}), \quad C > 0.$$

Similarly, we find that $\Pi_d \chi_d u = u + \mathcal{O}(e^{-1/Ch})$, and therefore,

$$\Pi_0 u = u + \mathcal{O}(e^{-1/Ch}). \tag{6.111}$$

We get the same conclusion also when $u \in E$ is a normalized generalized eigenfunction of P_r.

Let now $u \in E$ be such that $\|u\| = 1$. Using (6.106) and (6.111), we infer that

$$\Pi u = u + \mathcal{O}(e^{-1/Ch}),$$

and it follows that the dimension of E does not exceed that of $\Pi(H(\Lambda_\varepsilon))$. This together with (6.107) implies that the spaces $\Pi(H(\Lambda_\varepsilon))$ and E have the same dimension, and from here it is easy to see how to get the full statement of Theorem 2.1.

7. An application to surfaces of revolution

The purpose of this section is to illustrate how Theorem 2.1 applies to the case when M is an analytic surface of revolution in $\mathbb{R}^3$, and

$$P_\varepsilon = -h^2\Delta + i\varepsilon q, \tag{7.1}$$

where Δ is the Laplace-Beltrami operator and q is an analytic function on M. We shall consider the same class of surfaces of revolution as in [17], and begin by recalling the assumptions made on M in that paper.

Let us normalize M so that the x_3-axis is its axis of revolution, and parametrize it by the cylinder $[0, L] \times S^1$, $L > 0$,

$$[0, L] \times S^1 \ni (s, \theta) \mapsto (u(s)\cos\theta, u(s)\sin\theta, v(s)), \tag{7.2}$$

assuming, as we may, that the parameter s is the arclength along the meridians, so that $(u'(s))^2 + (v'(s))^2 = 1$. In the coordinates (s, θ), the Euclidean metric on M takes the form

$$g = ds^2 + u^2(s)d\theta^2. \tag{7.3}$$

The functions u and v are assumed to be real analytic on $[0, L]$, and we shall assume that for each $k \in \mathbb{N}$,

$$u^{(2k)}(0) = u^{(2k)}(L) = 0,$$

and that $u'(0) = 1$, $u'(L) = -1$. As we recalled in [17], these assumptions guarantee the regularity of M at the poles.

Assume furthermore that M is a simple surface of revolution, in the sense that $0 \leq u(s)$ has precisely one critical point $s_0 \in (0, L)$, and that this critical point is a non-degenerate maximum, $u''(s_0) < 0$. To fix the ideas, we shall assume that $u(s_0) = 1$. Notice that s_0 corresponds to the equatorial geodesic $\gamma_E \subset M$ given by $s = s_0$, $\theta \in S^1$. This is an elliptic orbit.

Writing

$$T^*\left(M \backslash \{(0,0,v(0)), (0,0,v(L))\}\right) \simeq T^*\left((0, L) \times S^1\right),$$

and using (7.3) we see that the leading symbol of $P_0 = -h^2\Delta$ on M is given by

$$p(s, \theta, \sigma, \theta^*) = \sigma^2 + \frac{(\theta^*)^2}{u^2(s)}. \tag{7.4}$$

Here σ and θ^* are the dual variables to s and θ, respectively. Since the function p in (7.4) does not depend on θ, it follows that $\{p, \theta^*\} = 0$, and we recover the well-known fact that the geodesic flow on M is completely integrable.

Let $E > 0$ and $|F| < E^{1/2}$, $F \neq 0$. Then the set

$$\Lambda_{E,F} : p = E, \;\; \theta^* = F,$$

is an analytic Lagrangian torus contained inside the energy surface $p^{-1}(E)$. Geometrically, the torus $\Lambda_{E,F}$ consists of geodesics contained between and intersecting tangentially the parallels $s_\pm(E, F)$ on M defined by the equation

$$u(s_\pm(E, F)) = \frac{|F|}{E^{1/2}}.$$

For $F = 0$, the parallels reduce to the two poles and we obtain a torus consisting of a family of meridians. The case $|F| = E^{1/2}$ is degenerate and corresponds to the equator $s = s_0$, traversed with the two different orientations. Writing $\Lambda_a := \Lambda_{1,a}$, we get a decomposition as in (2.11),

$$p^{-1}(1) = \bigcup_{a \in J} \Lambda_a,$$

with $J = [-1, 1]$, $S = \{\pm 1\}$.

In [17], we have derived an explicit expression for the rotation number $\omega(\Lambda_a)$ of the torus Λ_a, $0 \neq a \in (-1, 1)$,

$$\omega(\Lambda_a) = \frac{a}{\pi} \int_{s_-(a)}^{s_+(a)} \frac{1}{u^2(s)} \left(1 - \frac{a^2}{u^2(s)}\right)^{-1/2} ds, \quad u(s_\pm(a)) = |a|. \tag{7.5}$$

We are going to assume that the analytic function $(-1, 1) \ni a \mapsto \omega(\Lambda_a)$ is not identically constant.

Let $\alpha > 0$, $d > 0$. In what follows we shall say that a torus $\Lambda_a \subset p^{-1}(1)$, $a \in (-1,1)$, is (α, d)–Diophantine if the rotation number $\omega(\Lambda_a)$ satisfies

$$\left|\omega(\Lambda_a) - \frac{p}{q}\right| \geq \frac{\alpha}{q^{2+d}}, \quad p \in \mathbb{Z},\ q \in \mathbb{N}. \tag{7.6}$$

From the introduction let us also recall that if a torus $\Lambda_a \subset p^{-1}(1)$ is rational, so that $\omega(\Lambda_a) = \frac{m}{n}$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ relatively prime and $m = \mathcal{O}(n)$, then we define the height of $\omega(\Lambda_a)$ as $k(\omega(\Lambda_a)) = |m| + |n|$.

Let $q = q(s, \theta)$ be a real-valued analytic function on M which we shall view as a function on T^*M. Associated to each $a \in J$, we introduce the compact interval $Q_\infty(\Lambda_a) \subset \mathbb{R}$ defined as in (2.15). We also define an analytic function

$$(-1,1) \ni a \mapsto \langle q \rangle(\Lambda_a),$$

obtained by averaging q over the invariant tori Λ_a. Assume that $a \mapsto \langle q \rangle(\Lambda_a)$ is not identically constant. From the introduction, let us recall that as $a \to a_0 \in S$, the set of the accumulation points of $\langle q \rangle(\Lambda_a)$ is contained in $Q_\infty(\Lambda_{a_0})$.

Following [17], we now come to introduce uniformly good values in $\mathbb{R}$, for which the conclusion of Theorem 2.1 will be valid uniformly. In doing so, let us notice that the following discussion is not restricted to the case of surfaces of revolution.

Let $d > 0$ be fixed. Given $\alpha, \beta, \gamma > 0$ we say that $F_0 \in \mathbb{R}$ is (α, β, γ)–good if the following conditions hold:

- F_0 is not in the union of all $Q_\infty(\Lambda_a)$ with $\mathrm{dist}(\Lambda_a, S) \leq \alpha$.
- If $F_0 \in Q_\infty(\Lambda_a)$ and $\omega(\Lambda_a) \notin \mathbb{Q}$ then Λ_a is (α, d)–Diophantine and $|d_a \langle q \rangle(\Lambda_a)| \geq \alpha$.
- If $F_0 \in Q_\infty(\Lambda_a)$ and $\omega(\Lambda_a) \in \mathbb{Q}$ then $k(\omega(\Lambda_a)) = \mathcal{O}(\frac{1}{\alpha})$, $|d_a \omega(\Lambda_a)| \geq \alpha$, and $|F_0 - \langle q \rangle(\Lambda_a)| \geq \alpha$.
- Let $\langle q \rangle^{-1}(F_0) = \{\Lambda_{a_1,d}, \ldots \Lambda_{a_L,d}\}$, $\omega(\Lambda_{a_j,d}) \notin \mathbb{Q}$, $1 \leq j \leq L$, and $F_0 \in Q_\infty(\Lambda_{a_j,r})$, $\omega(\Lambda_{a_j,r}) \in \mathbb{Q}$, $j = 1, \ldots L'$. Then the distance in $\mathbb{R}$ from F_0 to the union

  $$\bigcup_{\Lambda_a \in J;\, \mathrm{dist}_J\left(\Lambda_a, \left(\cup_{j=1}^{L} \Lambda_{a_j,d}\right) \cup \left(\cup_{k=1}^{L'} \Lambda_{a_k,r}\right)\right) > \beta} Q_\infty(\Lambda_a)$$

  is $> \gamma$.

Remark. This definition of an (α, β, γ)–good value is less restrictive than in our previous work [17], since we now allow such a value F_0 to belong to an interval $Q_\infty(\Lambda_a)$ corresponding to a rational torus satisfying the isoenergetic condition, provided that F_0 is not too close to the torus average $\langle q \rangle(\Lambda_a)$.

In the following proposition we shall make use of the fact, observed in the introduction, that in the case when the subprincipal symbol of $P_{\varepsilon=0}$ in (2.6) vanishes, the validity of Theorem 2.1 extends to the range $h^2 \ll \varepsilon = \mathcal{O}(h^{2/3+\delta})$.

PROPOSITION 7.1. – *Assume that M is a simple analytic surface of revolution with a parametrization (7.2), for which the rotation number $\omega(\Lambda_a)$ defined in (7.5) is not identically constant. Consider an operator of the form $P_\varepsilon = -h^2\Delta + i\varepsilon q$, where q is a real valued analytic function on M, such that the torus averages function $a \mapsto \langle q\rangle(\Lambda_a)$ is not identically constant. Let α, β, $\gamma > 0$, and fix $0 < \delta \ll 1$. There exists $C > 0$ such that if F_0 is (α,β,γ)–good, $0 < h \leq \frac{1}{C}$, and $h^2/C \leq \varepsilon \leq h^{2/3+\delta}$, then Theorem 2.1 applies uniformly to describe the spectrum of P_ε in the rectangle*

$$\left[-\frac{\varepsilon}{C}, \frac{\varepsilon}{C}\right] + i\varepsilon\left[F_0 - \frac{\varepsilon^\delta}{C}, F_0 + \frac{\varepsilon^\delta}{C}\right].$$

Remark. If $\varepsilon = h$, then the operator P_ε in Proposition 7.1 is a semiclassical version of the stationary damped wave operator [19], [29], [11].

Remark. In the corresponding discussion in subsection 7.2 of [17], it has been assumed that the complex perturbation q in Proposition 7.1 is close to a rotationally symmetric one. This additional assumption has now been removed, thanks to Theorem 2.1, at the expense of weakening the final result and restricting the bounds on the strength ε of the non-selfadjoint perturbation.

Appendix

Trace class estimates for Toeplitz operators

The purpose of this appendix is to derive a simple estimate on the trace class norm of a Toeplitz operator with a compactly supported smooth symbol acting in a weighted L^2–space of holomorphic functions on $\mathbb{C}^n$. Indeed, the result will be seen to be a straightforward consequence of the analysis of [23].

Let $\Phi_0(x)$ be a real quadratic form on $\mathbb{C}^n$ and assume that Φ_0 is strictly plurisubharmonic. (In what follows we may think of the special case when $\Phi_0(x) = \frac{1}{2}(\operatorname{Im} x)^2$.) Let

$$H_{\Phi_0} := \operatorname{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n; e^{-\frac{2\Phi_0}{h}} L(dx)), \tag{A.1}$$

where $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $\operatorname{Hol}(\mathbb{C}^n)$ is the space of entire holomorphic functions on $\mathbb{C}^n$. Then H_{Φ_0} is a closed subspace of the space $L^2_{\Phi_0} := L^2(\mathbb{C}^n; e^{-\frac{2\Phi_0}{h}} L(dx))$, and from [23] we recall the following expression for the orthogonal projection $\Pi_{\Phi_0} : L^2_{\Phi_0} \to H_{\Phi_0}$,

$$\Pi_{\Phi_0} u(x) = \frac{C}{h^n} \int e^{\frac{2}{h}\psi_0(x,y)} u(y) e^{-\frac{2}{h}\Phi_0(y)}\, L(dy), \tag{A.2}$$

where the constant C is real and $\psi_0(x,y)$ is the unique quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ which is holomorphic in x, anti-holomorphic in y, and satisfies

$$\psi_0(x,x) = \Phi_0(x). \tag{A.3}$$

In the case when $\Phi_0(x) = \frac{1}{2}(\operatorname{Im} x)^2$, we have $\psi_0(x,y) = -\frac{1}{8}(x-\overline{y})^2$.

Now let $\Phi \in C^\infty(\mathbb{C}^n;\mathbb{R})$ be such that $\Phi - \Phi_0$ is bounded and $\sup\left|\frac{\partial\Phi}{\partial x} - \frac{\partial\Phi_0}{\partial x}\right|$ small enough. Assume also that $\nabla^k\Phi$ is bounded for each $k \geq 2$ and that Φ is uniformly strictly plurisubharmonic, so that the set

$$\Lambda_\Phi = \left\{\left(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}(x)\right); x \in \mathbb{C}^n\right\} \tag{A.4}$$

is an IR-manifold. Associated with the weight Φ we have the orthogonal projection

$$\Pi_\Phi : L^2_\Phi \to H_\Phi, \tag{A.5}$$

where $L^2_\Phi = L^2(\mathbb{C}^n; e^{-\frac{2\Phi}{h}}L(dx))$ and $H_\Phi = \mathrm{Hol}(\mathbb{C}^n) \cap L^2_\Phi$. If now $p \in C_0^\infty(\mathbb{C}^n)$, we introduce the corresponding Toeplitz operator

$$\mathrm{Top}(p) = \Pi_\Phi p \Pi_\Phi = \mathcal{O}(1) : H_\Phi \to H_\Phi. \tag{A.6}$$

Our goal is to show that $\mathrm{Top}(p)$ is of trace class as an operator on H_Φ and to estimate its trace class norm. In doing so, it is convenient to recall from [23] the asymptotic description of the Bergman projection Π_Φ, as $h \to 0$.

Let $\psi(x,y) \in C^\infty(\mathbb{C}^n_x \times \mathbb{C}^n_y)$ be almost holomorphic in x and almost anti-holomorphic in y at the diagonal $\mathrm{diag}(\mathbb{C}^n_x \times \mathbb{C}^n_y)$, such that $\nabla^k\psi$ is bounded on $\mathbb{C}^{2n}$ for each $k \geq 2$ and with

$$\psi(x,x) = \Phi(x). \tag{A.7}$$

Then we know that

$$\Phi(x) + \Phi(y) - 2\mathrm{Re}\,\psi(x,y) \sim |x-y|^2, \tag{A.8}$$

uniformly for $|x-y| \leq 1/C$, for $C > 0$ large enough.

It follows from [23] that there exists $f(x,y;h) \sim \sum_{j=0}^\infty f_j(x,y)h^j$ in $C_b^\infty(\mathbb{C}^{2n})$, with $\mathrm{supp} f \subset \{(x,y); |x-y| \leq 1/C\}$, $C \gg 1$, with $f(x,x;h)$ real, $1/C \leq f_0(x,x) \leq C$, and with

$$\partial_{\overline{x},y} f = \mathcal{O}\left(|x-y|^\infty + h^\infty\right), \tag{A.9}$$

such that if

$$\widetilde{\Pi}_\Phi u(x) = \frac{1}{(\pi h)^n}\int e^{\frac{2}{h}(\psi(x,y)-\Phi(y))} f(x,y;h) u(y)\, L(dy), \tag{A.10}$$

then

$$\Pi_\Phi = \widetilde{\Pi}_\Phi + R. \tag{A.11}$$

Here

$$R = e^{\frac{\Phi}{h}}\widetilde{R}e^{-\frac{\Phi}{h}}, \tag{A.12}$$

where $\widetilde{R}$ is a negligible integral operator in the sense of section 3 of [23]. In particular, it follows from [23] that $R = \mathcal{O}(h^\infty) : L^2_\Phi \to L^2_\Phi$. It follows furthermore from the results of Section 3 of [23] that the operators Rp and pR are of trace class as operators $H_\Phi \to L^2_\Phi$, with the trace class norm $\mathcal{O}(h^\infty)$.

When estimating the trace class norm of $\mathrm{Top}(p)$ on H_Φ, we may therefore replace Π_Φ by $\widetilde{\Pi}_\Phi$, and consider the corresponding operator $\widetilde{\mathrm{Top}}(p) = \widetilde{\Pi}_\Phi p \widetilde{\Pi}_\Phi$. The factorization

$$\widetilde{\mathrm{Top}}(p) = \left(\widetilde{\Pi}_\Phi p_1\right)\left(p_2\widetilde{\Pi}_\Phi\right), \tag{A.13}$$

where $p = p_1 p_2$ and $p_2 = |p|^{1/2}$, shows that it suffices to prove that the operators $\widetilde{\Pi}_\Phi p_1 : L^2_\Phi \to H_\Phi$ and $p_2 \Pi_\Phi : H_\Phi \to L^2_\Phi$ are of Hilbert-Schmidt class. Now the reduced kernel of $\widetilde{\Pi}_\Phi p_1$, in view of (A.10), is equal to

$$\frac{1}{(\pi h)^n} f(x,y;h) p_1(y) e^{-\frac{\Phi(x)}{h}} e^{\frac{2}{h}(\psi(x,y)-\Phi(y))} e^{\frac{\Phi(y)}{h}}, \tag{A.14}$$

and using also (A.8) we immediately see that the square of its L^2–norm over $\mathbb{C}^{2n}$ is bounded by

$$\frac{\mathcal{O}(1)}{h^{2n}} \iint |p(y)|\, e^{-c|x-y|^2/h}\, L(dy)\, L(dx) = \frac{\mathcal{O}(1)}{h^n} ||\, p \,||_{L^1}, \quad c > 0. \tag{A.15}$$

It follows that $\widetilde{\Pi}_\Phi p_1 : L^2_\Phi \to H_\Phi$ is of Hilbert-Schmidt class with

$$||\, \widetilde{\Pi}_\Phi p_1 \,||_{\mathrm{HS}} = \frac{\mathcal{O}(1)}{h^{n/2}} ||\, p \,||^{1/2}_{L^1}. \tag{A.16}$$

Since a similar argument applies to $p_2 \widetilde{\Pi}_\Phi$, we get the following result.

PROPOSITION A.1. – *When $\Phi \in C^\infty(\mathbb{C}^n;\mathbb{R})$ is a strictly plurisubharmonic function satisfying the general assumptions of the beginning of this section, let $\Pi_\Phi : L^2_\Phi \to H_\Phi$ be the orthogonal projection. If $p \in C_0^\infty(\mathbb{C}^n)$, then the Toeplitz operator* $\mathrm{Top}(p) = \Pi_\Phi p \Pi_\Phi : H_\Phi \to H_\Phi$ *is of trace class and we have*

$$||\, \mathrm{Top}(p) \,||_{\mathrm{tr}} \le \frac{\mathcal{O}(1)}{h^n} ||\, p \,||_{L^1} + \mathcal{O}(h^\infty), \tag{A.17}$$

where the implicit constant in $\mathcal{O}(h^\infty)$ is a continuous seminorm of p on the Schwartz space $\mathcal{S}(\mathbb{C}^n)$.

Remark. Rather than working on all of $\mathbb{C}^n$, we could also consider an open domain $\Omega \subset \mathbb{C}^n$, with $p \in C_0^\infty(\Omega)$. Then Proposition A.1 remains valid if we replace $\mathbb{C}^n$ by Ω in (A.1), with Π_Φ still being the orthogonal projection on all of $\mathbb{C}^n$. In the main text, we work on $H_\Phi(\Omega)$, where $\Omega \subset \mathbb{C}^n/2\pi\mathbb{Z}^n$ is open, and Proposition A.1 then still holds.

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Michael Hitrik
Department of Mathematics
University of California
Los Angeles CA 90095-1555, U.S.A.
E-mail: hitrik@math.ucla.edu

Johannes Sjöstrand
CMLS
École polytechnique
91128 Palaiseau, France
E-mail: johannes@math.polytechnique.fr