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TAMELY RAMIFIED SUPERCUSPIDAL REPRESENTATIONS

BY LAWRENCE MORRIS (*)

ABSTRACT. – Let **G** be a connected reductive group defined over a complete local non archimedean field F, and let G denote its F-valued points. Let $(\pi, \mathcal{V})$ be an irreducible admissible representation of G, and let (σ, W) be a representation of a parahoric subgroup P which is trivial on the prounipotent radical U of P. We say that $(\pi, \mathcal{V})$ contains (σ, W) if, when restricted to P, the σ-isotypic part $\mathcal{V}_\sigma$ is nontrivial. Assume that (σ, W) is irreducible cuspidal on the finite group of Lie type P/U, and that $\mathcal{V}_\sigma \neq 0$.

We show that $(\pi, \mathcal{V})$ is supercuspidal if and only if P is maximal; in this case π is compactly induced from the normaliser of P. We then classify supercuspidal representations containing unipotent cuspidal representations, provided **G** is an inner form of a split adjoint group, following a conjecture of G. Lusztig.

Introduction

Let G be the group of rational points of a reductive group defined over a local non archimedean field F. In [M] we described the structure of the intertwining algebra $\mathcal{H}(\sigma) = \mathrm{End}_G(c - \mathrm{Ind}_P^G(\sigma))$ when σ was an irreducible cuspidal representation of the Levi component of a parabolic subgroup P; it is closely related to a standard affine Iwahori-Hecke algebra.

Let P^+ denote the normaliser in G of P, and suppose that P is maximal in the sense of Bruhat-Tits. In section 1 of this paper we show that if ρ is an irreducible smooth (hence finite dimensional) representation of P^+ which contains σ on restriction to P then $c - \mathrm{Ind}_{P^+}^G(\rho)$ is an irreducible supercuspidal representation of G. Previously, results of this type were known only for (hyper)special parahoric groups. Such groups are easy to treat because of the associated Cartan decomposition; in the general case one is obliged to use the affine BN-pair structure and the associated Bruhat decomposition for P^+. For this we rely on the results of [M] Section 3. In Section 2 we show conversely that any irreducible supercuspidal representation of G which contains σ, must be of this form. All of these results generalise and amplify those of [M1].

Now suppose that P is not maximal. In Section 3 we show that any irreducible smooth representation of G containing σ can *never* be supercuspidal. This generalises a result proved for $\mathbb{GL}_n$ in [K1], and the underlying principle is similar: the structure of $\mathcal{H}(\sigma)$

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implies the existence of many invertible operators, which in turn implies the existence of matrix coefficients with non compact support. This result was believed to be true for groups other than $\mathbb{GL}_n$ for some time; another proof of it is given in [M2] using Jacquet functors.

In Section 4 we examine the special case when σ is unipotent cuspidal and G is split simple adjoint. In this situation one can describe the algebra $\mathcal{H}(\sigma)$ in more detail: it contains a (generally) large affine Iwahori-Hecke algebra which can be explicitly described, and a (small, but typically non trivial) group algebra arising from diagram automorphisms.

We then proceed in Section 5 to consider more particularly the case where G is simple split adjoint, σ is unipotent cuspidal and P is maximal. The preceding results guarantee that any irreducible admissible representation of G which contains σ must be supercuspidal, and induced from P^{+}; according to Langlands' philosophy as refined by Lusztig, there should be a bijection between such representations and a certain subset of those triples (s, N, ρ) where s is a semisimple isolated element in the dual group ${}^{L}G$, $N \in \mathrm{Lie}({}^{L}G)$ such that $\mathrm{Ad}(s) N = q N$ and ρ is an irreducible representation of the group

$$(Z_{{}^{L}G}(s, N)/Z^{0}_{{}^{L}G}(s, N)\, Z({}^{L}G)).$$

In fact the relevant triples correspond to irreducible admissible cuspidal complexes [L3] on ${}^{L}G$ which have trivial central character. (Such complexes were originally introduced and studied in [L3] to account for the missing component representations in the Springer correspondence.) These have been classified by Lusztig, and we help ourselves liberally to his results to produce a bijection. We emphasise that this is all we do; we hope that the bijection we produce is natural, in some yet to be determined sense. As one might expect, our result is obtained via a case by case analysis. There is some overlap between our investigations and some recent work of M. Reeder [R]. For example, in the case of G_2 he computes the corresponding L-packets and shows that the formal degrees in each packet are integer multiples of a unique generic representation; he shows that the multiple is always the degree of the corresponding ρ. These L-packets always contain both non supercuspidal square integrable representations, and supercuspidal representations.

In Section 6 we pursue this further, by sketching how the analogue of Section 5 works for inner forms: for each non split inner form of a split adjoint group G we produce a bijection similar to those in Section 5 between irreducible *unramified* (=level zero, containing a unipotent cuspidal representation) supercuspidal representations of the inner form and certain admissible homomorphisms of the Weil-Deligne group.

The results in Section 5 and 6 support some recent conjectures of D. Vogan [V] which refine Langlands' philosophy. In this particular case they were directly motivated by a lecture of Lusztig, given at the institute for Advanced Study in November 1988; see also [L0] and [L2]. It is also worth noting that the bijections in Sections 5 and 6 can be interpreted as bijections between orbits of certain isomorphic finite groups, which arise in more refined versions of Langlands' philosophy. This will appear elsewhere.

Added in proof: A version of this paper has been available since August 1992. In the meantime the preprint "Classification of unipotent representations of simple p-adic groups" (1995) by G. Lusztig has appeared in which the author proves his conjecture completely. Finally, it is a pleasure to thank the referee of this journal for a careful reading of the original manuscript.

Notation and Convention

In general the notation and conventions in this paper continue that of [M]; we have also attempted to keep the reference listing compatible.

In particular, F will always denote a non archimedean local field with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p}$; we denote its residue field by $\mathbb{F}_q$. We write $\overline{F}$ for a fixed algebraic closure of F and $\Gamma = \mathrm{Gal}(\overline{F}/F)$ for the Galois group.

If $\mathbf{V}$ is an algebraic variety defined over F, we write $V = \mathbf{V}(F)$, in particular if $\mathbf{G}$ is a connected reductive F-group, G is naturally endowed with the structure of a second countable totally disconnected locally compact Hausdorff group.

►Let $\mathbf{G}$ be a connected reductive F-group. From Section 4 on we shall need the identity component of the Langlands dual group of $\mathbf{G}$; we shall denote it by ${}^{L}G$. This notation is not conventional.◄

In Sections 6 and 7 we shall employ Galois cohomology sets/groups; we denote them by $H^i(F, -) = H^i(\Gamma, -)$.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have their customary meanings.

1. Some supercuspidal representations

1.1. We begin with a lemma which first appears in [Cy] and which by now is well known. To state it we retake the notation and framework of Section 4.1 of [M]; thus the group G is locally compact, totally disconnected and unimodular. We denote by Z the centre of G. Let P be an open subgroup of G which contains Z, and which is compact mod Z; let σ be an irreducible admissible representation of P. Just as in [M] 4.3 we can define the representation $c\text{-}\mathrm{Ind}_P^G(\sigma)$, except that the functions in question must now have compact support mod P. As in [M] 4.2 we choose a base of neighbourhoods P_i $(i \in \mathbb{N})$ of the identity, such that each P_i is normal in P, and compact open in G. The analogue of [M] 4.2 in this situation then asserts that $\sigma|Z$ is a quasicharacter, and then that σ is finite dimensional; this follows from a well known version ([Ca] 1.4 (c)) of Schur's lemma. The analogues of [M] 4.3 and [M] 4.4 then follow; in particular $c\text{-}\mathrm{Ind}_P^G(\sigma)$ is smooth.

Let H be a closed subgroup of G and let ρ be a smooth representation of H; we then have the following Mackey decomposition formula (cf. [K])

$$\mathrm{Hom}_G(c - \mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_H^G(\rho)) \simeq \bigoplus_x \mathrm{Hom}_{P \cap^x H}(x\,\sigma, \rho) \tag{1.1.1}$$

where x runs through a set of double coset representatives for $P\backslash G/H$. In particular take $G = H$ in the above; we obtain Frobenius reciprocity for compact induction:

$$\mathrm{Hom}_G(c - \mathrm{Ind}_P^G(\sigma), \rho) \simeq \mathrm{Hom}_P(\sigma, \rho|P). \tag{1.1.2}$$

In addition there is also Frobenius reciprocity for ordinary smooth induction

(1.1.3) $$\mathrm{Hom}_G\,(\rho,\, \mathrm{Ind}_P^G\,(\sigma) \simeq \mathrm{Hom}_P\,(\rho|P,\, \sigma)$$

In (1.1.2) take $\rho = c - \mathrm{Ind}_P^G\,(\sigma)$; this is a smooth representation, and we then have the following results.

LEMMA (cf. [Cy], 1.5). – *If* $\dim\,(\mathrm{Hom}_G\,(c - \mathrm{Ind}_P^G\,(\sigma),\, c - \mathrm{Ind}_P^G\,(\sigma))) = 1$, *then* $c - \mathrm{Ind}_P^G\,(\sigma)$ *is irreducible.*

Proof. – Suppose that $\rho = c - \mathrm{Ind}_P^G\,(\sigma)$ is not irreducible. Let $\mathcal{V}$ be the space of ρ; we can then find an invariant subspace $\mathcal{U}$ fitting into a short exact sequence

$$0 \to \mathcal{U} \to \mathcal{V} \to \mathcal{W} \to 0$$

of smooth G-spaces. There is then a non zero map of G-modules

$$\mathcal{U} \to \mathcal{V} \to \mathcal{V}'$$

where $\mathcal{V}$ denotes the space of $\mathrm{Ind}_P^G\,(\sigma)$, hence (1.1.3) implies a non zero map

$$\mathcal{U}|P \to V$$

where V denotes the space of σ. By semisimplicity of smooth representations on P, this implies that V occurs as a direct summand in $\mathcal{U}|P$.

There is also a non trivial projection map $\mathcal{V} \to \mathcal{W} \to 0$, so by (1.1.2) we obtain a non zero map $V \to \mathcal{W}|P$, and by semisimplicity again this implies that V occurs as a direct summand of $\mathcal{W}|P$.

Again by semisimplicity we have $\mathcal{V}|P = \mathcal{U}|P \oplus \mathcal{W}|P$. It follows from this and the above that σ occurs in $\mathcal{V}|P$ with multiplicity at least two. By (1.1.2), this contradicts the hypothesis of the lemma.

1.2. For the result below recall that if $(\pi,\, \mathcal{V})$ is an irreducible smooth representation of G, we say that it is *supercuspidal* if its matrix coefficients have compact support mod Z.

COROLLARY. – *With the assumptions of the lemma,* $c - \mathrm{Ind}_P^G\,(\sigma)$ *is admissible and supercuspidal.*

Proof. – The representation under consideration is smooth and irreducible; by Jacquet's theorem ([M] 4.10) it is admissible. On the other hand, it plainly has some compactly supported mod centre matrix coefficients (one produces them via the functions f_ν of [M] Sect. 4.5); by an irreducibility argument all its matrix coefficients must be compactly supported mod the centre, hence it is supercuspidal.

1.3. We now continue with the notation and conventions of [M] (*esp.* Sect. 3.12, 3.14). Let $P_J = P$ be a maximal parahoric subgroup with pro-unipotent radical $U_J = U$ and Levi component M; then M is a finite group of Lie type, and we let σ be a cuspidal representation of M.

Suppose that $(\pi,\, \mathcal{V})$ is an irreducible admissible representation of G such that $(\pi|P,\, \mathcal{V})$ contains a non zero U-fixed vector. The space $\mathcal{V}^U$ is then non zero and provides a representation of the group M; we shall suppose that this representation contains an isotypic part corresponding to σ. We shall abbreviate all this by saying that π *contains*

σ; from [M] 4.7 this implies that π corresponds to an isomorphism class of irreducible (unital) $\mathcal{H}(\sigma^t)$-modules, where σ^t is the contragredient to σ.

Next, let P^+ denote the normaliser of P; this is a compact mod centre, open subgroup of G. Moreover, P contains a neighbourhood base consisting of compact open characteristic (in P) subgroups (for this, *see* e.g. [P-R]), hence the remarks of 1.1-1.2 apply to P^+. In particular, let ρ be an irreducible admissible representation of P^+; we wish to study the smooth representation $c - \operatorname{Ind}_{P^+}^G(\rho)$.

1.4. The following fact generalises 5.3 of [M1].

PROPOSITION. – *Suppose that $\rho|\,U$ is trivial, and that $\rho|P$ is a sum of cuspidal representations. Then $c - \operatorname{Ind}_{P^+}^G(\rho)$ is an irreducible supercuspidal representation.*

Proof. – By Lemma 1.1 it is enough to show that the dimension of $\operatorname{Hom}_G(c-\operatorname{Ind}_{P^+}^G(\rho), c-\operatorname{Ind}_{P^+}^G(\rho))$ is equal to 1. Again by a variant [K] of the Mackey decomposition formula (1.1.1), it is enough to show that $\operatorname{Hom}_{P^+\cap{}^gP^+}({}^g\rho, \rho)$ is zero unless g is a representative for the coset P^+. Since there is an injection (of vector spaces) $\operatorname{Hom}_{P^+\cap{}^gP^+}({}^g\rho, \rho)$ to $\operatorname{Hom}_{P^+\cap{}^gP^+}({}^g\rho|P, \rho|P)$, it is enough to compute this last space which again is equal to

$$\operatorname{Hom}_{U\cap{}^gU\backslash P\cap{}^gP}({}^g\rho|P, \rho|P).$$

Thus we are reduced to an intertwining number calculation for finite groups. Let χ, χ' denote the characters of ρ, ${}^g\rho$ respectively. Then we find

$$\int_{U\cap{}^gU\backslash P\cap{}^gP} \chi(x)\,\overline{\chi'(x)} = \int_{P\cap{}^gU\backslash P\cap{}^gP} \overline{\chi'(x)}\left(\int_{U\cap{}^gU\backslash P\cap{}^gU} \chi(ux)\,du\right)dx$$

where the integrals are all finite sums. Now consider the inner sum on the right hand side. We show that the function $x \mapsto \displaystyle\int_{P\cap{}^gU} \chi_\rho(hx)\,dh$ is identically zero on $P\cap{}^gP$ when $g\notin P^+$.

We may assume that $g=n$ is a distinguished double coset representative projecting to w. By [M] 3.19-3.20, the image of $P\cap{}^gP$ in $U\backslash P$ is equal to the image of $P_{J\cap wJ}$ in $U\backslash P$, and $U\,.\,P\cap{}^gU = U_{J\cap wJ}$. By [B-T2] 4.6.33, the image of $P_{J\cap wJ}$ in $U\backslash P$ is a parabolic subgroup. On the other hand, up to a constant the integral/sum above is equal to the integral over

$$U\cap{}^gU\backslash P\cap{}^gU \simeq U\backslash(U\,.\,P\cap{}^gU).$$

Since $\rho|P$ is a sum of cuspidal representations we see that the integral in question is zero unless $U = U\,.\,(P\cap{}^gU)$ or $P_{J\cap wJ} = P = P_J$ (*loc. cit.*). This means that $w\,J = J$. It then follows as in [M] Appendix 1 that n actually belongs to the group P^+. This concludes the proof.

The aim of the following sections is to provide a converse to this result.

2. Supercuspidal representations (continued)

2.1. We now let P denote a maximal parahoric subgroup; $(\pi, \mathcal{W})$ will denote an irreducible admissible representation of G. Further, let (σ, V) denote an irreducible cuspidal representation of P contained in $(\pi, \mathcal{W})$. (*See* 1.3) We also write P^+ for the normaliser of P; this is a totally disconnected group, which is compact mod centre. If $\mathcal{U} = \mathcal{W}^U$ denotes the (non zero and finite dimensional) space of U fixed points, then $\mathcal{U}$ is stable by P^+ and it is a finite direct sum of spaces $\mathcal{U}_\chi$, where if χ is a quasicharacter of Z, $\mathcal{U}_\chi$ denotes the subspace of $\mathcal{U}$ transforming by χ. Applying e.g. [Ca, Corollary 1.1] we see that $\mathcal{U}$ is a finite direct sum of irreducible representations of P^+. It follows that the representation (σ, V) must occur in one of these, say ρ (and then that ρ is generated by V). Moreover, just as for finite groups, $\rho|P$ is a finite sum of conjugates of σ.

2.2. In particular, ρ is admissible and finite dimensional. Since $\rho|P$ is a sum of conjugates of σ, it is a sum of cuspidal representations. It follows from 1.4 that $c - \mathrm{Ind}_{P+}^G(\rho)$ is irreducible and supercuspidal. Again by construction there is a non zero intertwining map from $c - \mathrm{Ind}_{P+}^G(\rho)$ to $(\pi, \mathcal{W})$; since $(\pi, \mathcal{W})$ is irreducible we see that this map is an isomorphism. This proves the following result.

Proposition. – *Let $(\pi, \mathcal{W})$ be an irreducible smooth representation of G. If $(\pi, \mathcal{W})$ contains the cuspidal representation (σ, V) of P then there is an irreducible representation ρ of P^+ such that $(\pi, \mathcal{W})$ is isomorphic to $c - \mathrm{Ind}_{P+}^G(\rho)$. Furthermore, $(\pi, \mathcal{W})$ is supercuspidal.*

Remark. – In contrast to what happens for $\mathbb{GL}_n$ a maximal parahoric subgroup P in a reductive group may have a normaliser which is strictly larger than ZP. This already happens for $\mathbb{GS}\mathrm{p}_4$.

3. Principal representations

3.1. In this section, we shall prove that if P is not a maximal parahoric then an irreducible admissible representation which contains σ upon restriction to P can never be supercuspidal. The philosophy behind the proof is not new, and is based on a general result in Kutzko's paper [K1] which we now recall.

3.2. Let G be a locally compact totally disconnected unimodular group; we write Z for the centre of G as usual. Let K be a compact open subgroup of G, (σ, V) an irreducible admissible representation of K as in [M] 4.5. In what follows we shall write $\mathcal{H}(\sigma)$ for the intertwining algebra $\mathrm{Hom}_G(c - \mathrm{Ind}_K^G(\sigma), c - \mathrm{Ind}_K^G(\sigma))$ and $\mathcal{V}$ for (the space of) $c - \mathrm{Ind}_K^G(\sigma)$.

Let $(\pi, \mathcal{W})$ be an irreducible smooth representation of G; write $\mathcal{W}_\sigma$ for the (finite dimensional) space of vectors which transform according to σ. There is a canonical isomorphism $\mathcal{W}_\sigma \cong V \otimes \mathrm{Hom}_K(V, \mathcal{W})$ and by (1.1.2) this is canonically isomorphic to $V \otimes \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$. Since $\mathcal{V}$ is canonically a left unital $\mathcal{H}(\sigma)$-module, we see that $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ is canonically a right unital $\mathcal{H}(\sigma)$-module and then that $\mathcal{W}_\sigma$ is canonically a right unital $\mathcal{H}(\sigma)$-module.

3.3. The following is one of the main results in [K1].

PROPOSITION (*cf.* [K1], 1.5). – *Suppose that K, σ, V, are as above; let $(\pi, \mathcal{W})$ be an irreducible smooth representation of G, such that there is a non zero vector ν in $\mathcal{W}_\sigma$. Suppose that there are distinct elements $\Phi_j \in \mathcal{H}(\sigma)$ with support S_j $(j = 1, 2, ...)$ such that the sets ZS_j are disjoint and that $\nu\,\Phi_j \neq 0$ for each j. Then $(\pi, \mathcal{W})$ is not supercuspidal.*

3.4. *Remarks.* – (i) The proof of 3.3 proceeds by constructing matrix coefficients for $(\pi, \mathcal{W})$ which do not have compact support mod Z. It is based on an idea of Matsumoto.

(ii) In [K1] representations (σ, V) with the property:

$$\mathcal{W}_\sigma \neq \{0\} \text{ implies } \mathcal{W} \text{ cannot be supercuspidal},$$

are called (G, K)*-principal.*

3.5. We shall apply 3.3 to the case where $G = \mathbf{G}(F)$ is the group of rational points of a reductive group over a local non archimedean field, $K = P$ is a parahoric subgroup which is not maximal, and (σ, V) is a cuspidal representation of P/U. We show that we can produce many elements of $\mathcal{H}(\sigma)$ satisfying 3.3 in general. Indeed, consider theorem 7.12 of [M]. If *P* is not maximal, then we have $W(\sigma) = R(\sigma)\,.\,C(\sigma)$ by [M] proposition 7.3. If $R(\sigma) \neq 1$ then it is an infinite reflection group, and the elements T_ν are invertible, by [M] 7.12 (c), (d) (*cf.* [M] 6.8). It then follows from the definition in [M] 7.8 that the elements T_w are invertible in $\mathcal{H}(\sigma)$, for $w \in R(\sigma)$.

Suppose that $R(\sigma) = 1$, so that $W(\sigma) = C(\sigma)$. As in [M] 7.3 let $T(J)$ denote the (infinite) group of translations arising from the split centre $\mathbf{Z_M}$. Consider the elements T_d where $d \in T(J)$. By [M] 7.12 (a), (b) we see that these elements are invertible in $\mathcal{H}(\sigma)$. (In fact the cocycle of *loc. cit.* is trivial on $T(J)$ by the remark following the statement of [M] 7.12.)

In either case, we have an infinite family of elements Φ_j satisfying the conditions of 3.3, provided $\mathcal{W}_\sigma \neq \{0\}$.

COROLLARY. – *Let $(\pi, \mathcal{W})$ be an irreducible smooth representation of G, containing the cuspidal representation (σ, V) of P. If P is not maximal, then $(\pi, \mathcal{W})$ cannot be supercuspidal.*

Remark. – In [M2] another proof of this is given which uses Jacquet functors, and a "Casselman type lemma".

4. The unipotent cuspidal case

4.1. In this section we suppose that the group G is simple. We begin with a couple of results that are analogues of results in [C] 10.10. For this we return to the notation of [M] section 2; thus we take a subset $J \subseteq \Pi$. We suppose that $|\Pi - J| \geq 2$ as usual. In addition, we shall assume that for each $a \in \Pi - J$, the longest element w_K in the spherical Weyl group corresponding to the root system arising from $K = J \cup \{a\}$ satisfies $w_K\, J = -J$. We shall abbreviate this by saying that J is *self opposed.* (This is adapted from the definition for spherical systems in [C].) Now recall the groups S_J, R_J of [M] 2.2, 2.6 respectively. The next few results are analogues of those in [C] pp. 351-352.

LEMMA. – *Assume that J is self opposed. Then $S_J = \Omega_J R_J$.*

Proof. – We have $S_J = R_J X$, where X is defined as in [M] 2.8. Let $w \in S_J$. By [M] Lemma 2.5 we may write $w = \rho\nu[a_r, K_r]...\nu[a_1, K_1]$ where each $K_i \subseteq \Pi$, $\nu[a_i, K_i]K_i = K_{i+1}$, for $1 \leq i \leq r$, $\rho K_{r+1} = J$ and $\rho \in \Omega$ (where Ω is the stabiliser of the chamber corresponding to Π). Since J is self opposed, it follows immediately by induction that $K_i = J$ for each i, and then from [M] 2.6 that each $\nu[a_i, K_i] \in R_J$. The result follows.

4.2. The same argument shows that if J is self opposed then each element of $\Pi - J$ provides an element of the set Q defined in [M] 2.7. Let $B = A'/A'_\Theta$ be the space defined in [M] 2.7. By definition, this space is spanned by the images of the elements of $\Pi - J$, and these images are linearly independent; on the other hand each such element provides an element in the quotient system Q'. From [M] 2.7 we then deduce that the elements a themselves provide a basis for the affine root system Q' in [M] Theorem 2.7. We summarise all this in the following result.

LEMMA. – *If J is self opposed, then the elements $a \in \Pi - J$ provide a basis for the quotient root system Q' defined in [M] Theorem 2.7.*

4.3. Now suppose that J is self opposed and that (σ, V) is a cuspidal representation which is stable under all automorphisms of the group $\mathcal{M}_J$ which arise from conjugation by representatives of $W(\sigma)$.

LEMMA. – *With these assumptions, we have $W(\sigma) = \Omega_J R_J$.*

Proof. – This follows from the preceding results and the definition of $W(\sigma)$.

4.4. We now show that the preceding results apply whenever σ is unipotent. This will follow from results of Lusztig [L4] on finite groups of the Lie type, and will enable us to describe the algebra $\mathcal{H}(\sigma)$ somewhat more explicitly in this case.

In the first place, if we limit our attention to finite groups of Lie type, the following is known ([L4] p. 33): if σ is unipotent cuspidal and occurs in the Levi component of a standard parabolic P then the subset J corresponding to P is the unique F-stable subgraph of its type occurring in the (absolute) Dynkin diagram of G. (Here, F is some chosen Frobenius element for the algebaic closure of the appropriate finite field.) In particular, if G is split, J must be the unique subdiagram of its type.

Returning to the situation at hand, and applying the above facts to the groups $U_K \backslash P_J \subseteq U_K \backslash P_K$ we see that our set J will indeed be self opposed.

To apply 4.3 we note that any unipotent cuspidal representation is invariant under all automorphisms of M_J which arise from algebraic automorphisms of the underlying algebraic group. (*See* [DM] proposition 13.20.) If g represents an element of $R(\sigma)$ then it normalises $\mathcal{M}_J$, and then it must normalise the group scheme underlying $\mathcal{M}_J$. This implies that it acts as an algebraic automorphism of M_J. From 4.3 we can compute the group $R(\sigma)$. Applying the recipes in [M] Section 6, together with the tables on p. 35 of [L4], we find that the numbers p_a are never 1 so that $R(\sigma) = R_J$. From 4.3, the complement $C(\sigma) = \Omega_J$. This proves the following result.

PROPOSITION. – *If σ is unipotent then $W(\sigma) = R_J \Omega_J$.*

From 4.2 we see that the root system Γ of [M] 7.3 is described by an extended Dynkin diagram whose vertices correspond to the complement of J. (Further, the Dynkin diagram of the system Γ can be determined completely with the help of 10.10.3 of [C] if $|\Pi - J| > 2$. Otherwise, it must be of type $\widetilde{A}_1$.)

In other words the algebra $\mathcal{H}(\sigma)$ is described by elements $T_w\,(w \in R(J))$, $T_\omega\,(\omega \in \Omega_J(\sigma))$ subject to the relations 7.12 (a)-(d).

4.5. *Remark.* – (i) The groups $\Omega_J(\sigma)$ can be computed explicitly from the extended Dynkin diagram, and the tables on p. 35 of [L4]. For example, if G is split of type E_6 one finds that the parahoric P with subdiagram D_4 has a (unique) unipotent cuspidal representation. The group $\Omega = \mathbb{Z}_3$ is a homomorphic image of the normaliser of P and it follows by uniqueness that $\Omega_J(\sigma) = \Omega$. A similar situation occurs for G split of type E_7 with $\Omega = \mathbb{Z}_2$. Similar case by case arguments show that in general, $\Omega_J(\sigma) = \Omega$, or $\{1\}$, unless G is of type D_n.

The group $\Omega_J(\sigma)$ may be non trivial, even if P is maximal. We refer the reader to sections 5 and 6 below for examples of this nature.

(ii) Let ${}^L G$ denote the (complex) dual group of G; it is simply connected if G is split adjoint. In [L2], Lusztig has suggested that there should be a natural bijection between irreducible smooth representations of G containing a unipotent cuspidal σ as above, and triples (up to ${}^L G$ conjugacy) (s, N, ρ), where $s \in {}^L G$ is semisimple, N is a nilpotent element in Lie $({}^L G)$ with $\mathrm{Ad}(s)\,N = q\,N$, and ρ is an irreducible representation of the group of components of the simultaneous centraliser $Z_{{}^L G}(s, N)$ on which the (finite) centre of ${}^L G$ acts trivially. We shall consider this problem when P is maximal in the next section.

(iii) It is easily seen that the algebra H constructed in [L2] is the part of the algebra $\mathcal{H}(\sigma)$ arising from the group R_J above; this follows from the description of H in [L2] and our description above.

4.6 We now specialise the above to the case where the parahoric is maximal, and G is an inner form of a split adjoint simple group. In this situation we can make the results of sections 1-3 more precise. The group P^+/P is always finite abelian since G is adjoint, and $W(\sigma) = \Omega_J = P^+/P$. This last assertion follows from 4.4 and [M], Appendix. Thus the Hecke algebra in this case is a group algebra possibly twisted by a 2-cocycle μ.

Furthermore we know from Section 2 that any supercuspidal representation of G which contains σ must be of the form $c - \mathrm{Ind}_{P^+}^{G}(\rho)$ where ρ is an irreducible admissible representation of P^+ which contains σ.

PROPOSITION. – (a) *The cocycle μ is trivial.*

(b) *The representation σ extends to a representation of P^+.*

(c) *There are $|P^+/P|$ distinct irreducible representations of P^+ which contain σ.*

(d) *There are $|P^+/P|$ distinct irreducible supercuspidal representations of G which contain σ.*

Proof. – Suppose that (b) is true. From [M] 6.1-6.2 there is a projective representation extending σ on the group denoted $N(J, \sigma)$ there which defines the 2-cocycle μ. In fact this projective representation is inflated from one on P^+/U; indeed, there is an obvious projection of $N(J, \sigma)$ onto P^+/U. (In the split case this last group is just a semidirect product of $M = M_J$ by Ω_J since Ω_J is a subgroup of the group of outer automorphisms of M.) The projective representation is just that arising from the intertwining operators induced from elements of P^+/U. Thus if (b) holds then the projective representation is trivial and (a) also holds. Moreover, (c) follows from (b) by a well known result; *see* [CR] Corollary 11.7 for example. Finally (d) follows from (c), 1.2, 2.2, and 3.5.

It remains to prove (b). Write $M^+ = P^+/U$. In all cases of interest to us the quotient M^+/M is either finite cyclic or the non cyclic group of order-4. In the former case the representation σ will extend by standard results; *see* [CR] 11.47 for example. In the latter situation there are three cases of interest, and they are all of absolute type D_n. (*See* Sect. 6.2 below). We treat each of these cases in the following three subsections.

4.7. We begin with the split adjoint form of D_n. Then $n = 2t^2$ where t is even; the maximal parahoric of concern is that which corresponds to omitting the middle node in the local Dynkin diagram. It then has index 4 in its normaliser, and the quotient is the 2×2 group.

Let P (resp. P^+) denote this parahoric (resp. its normaliser), and U its pro-unipotent radical. Passing to $M = P/U$, we have the exact sequence

$$0 \to M \to M^+ \to \Omega \to 0$$

where Ω denotes the component group (non cyclic of order 4). In fact M is a central direct product, isogenous to $SO_{2l} \times SO_{2l}$ (and the sequence above splits as a semidirect product since all groups being considered are split). One of the generators τ of Ω can be taken to be an element τ in M^+ which acts as an outer automorphism on each factor SO_{2l} simultaneously. In fact τ can be taken to be the projection of an element τ which lies in PSO_{2n} such that $\tau = \tau_1 \times \tau_2$ with $\tau_1 = \tau_2$. The other generator ν can be taken to be an element which interchanges the 2 factors.

Let $\sigma \otimes \sigma$ denote the unique (up to isomorphism) unipotent cuspidal representation of M. (Note that unipotent representations are trivial on central elements, so that the central product has an action on the tensor product.) The descriptions of τ, ν imply that this representation can be extended to M^+. Indeed as an intertwining operator for ν we can choose the operator S which switches the factors in the tensor product. On the other hand we may choose an intertwining operator $T = T_1 = T_2$ for τ_i which extends σ; then we take $T \otimes T$ for the intertwining operator for τ. We evidently have $S \circ (T \otimes T) = (T \otimes T) \circ S$. It follows that $\sigma \otimes \sigma$ extends.

4.8. For non split inner forms of a simple split adjoint group we shall also need to know that the analogous cocycles (maximal parahoric) are trivial. From the tables and recipes in [T] one sees that the cases of interest are ${}^2D'_{2n}$ and ${}^2D''_{2n}$ (notation of loc. cit.; *see* also Sect. 6 below). In this first case the maximal parahoric in question corresponds to omitting the middle node of the relative local Dynkin diagram; in the second case each maximal parahoric subgroup has index 4 in its normaliser, unless one omits a special vertex. In either case the appropriate parahoric has index 4 in its normaliser and the quotient group is the non cyclic group of order 4.

Consider the case of a group G of type ${}^2D''_{2n}$; then G is isogenous to a quaternionic orthogonal group (with involution). If we omit any non special vertex from the relative local

Dynkin diagram (corresponding to an orbit of the Galois group in the local index) we obtain a maximal parahoric subgroup whose Levi component is a central direct product, isogenous to $SU_{2\,l} \times R(SO_{2\,k})$. Here $SU_{2\,l}$ denotes the unitary group in $2\,l$ variables, R denotes restriction of scalars from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$, and $SO_{2\,k}$ denotes the special orthogonal group.

Now write $0 \to M \to M^+ \to \Omega \to 0$ as before. One of the elements in Ω corresponds to an element in the group of similitudes which can be taken to be $\nu = \pi_D\, I$ in the standard matrix representation. (Here D denotes the quaternion algebra defining G.) Under reduction mod $\mathfrak{p}$ it amounts to the action of a Frobenius element ν. The other generator τ can be taken to be an element which is trivial on the unitary factor and which provides the non trivial diagram automorphism on the other factor.

Now suppose that k, l are such that $SU_{2\,l}$, $R(SO_{2\,k})$ each admit a unipotent cuspidal representation. (There is only one for each group if it exists.) We then proceed to imitate the argument in 4.7; we shall use analogous notation. One must exercise a little more care however, in the present situation, in showing that the representation $\sigma = \sigma_1 \otimes \sigma_2$ (they are not the same!) extends to the group Ω. Let V_i be the space of σ_i and let T_ν be the operator such that

$$T_\nu\, \sigma\,({}^{v}(a,\, b))\,(v_1 \otimes v_2) = \sigma\,(a,\, b)\, T_\nu\,(v_1 \otimes v_2).$$

There is also an operator $T_\tau : V_2 \to V_2$ such that

$$(1 \otimes T_\tau)\, \sigma\,({}^{\tau}(a,\, b))\,(v_1 \otimes v_2) = \sigma\,(a,\, b)\,(v_1 \otimes T_\tau\, v_2).$$

Now, we can choose $T_\nu = T_1 \otimes T_2$ where T_i is an intertwining operator on V_i such that

$$T_i\, \sigma_i\,({}^{\nu_i}(a)\,(w_i) = \sigma_i\,(a)\, T_i\,(w_i)$$

where w_i is in V_i and ν_i corresponds to Galois action on the coordinates on the appropriate factor.

An easy computation shows that everything reduces to showing that we can arrange $T_2\, T_\tau = T_\tau\, T_2$ but this obvious from our choice for these operators.

4.9. The remaining case to consider is that of groups of type ${}^2D'_{2\,n}$. The group G in question can be represented by the adjoint group of similitudes of a quadratic form q which has an anisotropic part of dimension 4. For this case one can argue in a similar fashion to that for the split orthogonal groups in Section 4.7.

The relevant parahoric subgroup can only be that obtained by omitting the middle node of the relative local Dynkin diagram. The Levi component is a central direct product isogenous to 2 copies of the non split special orthogonal group over the residue field, and we suppose that this group admits a unipotent cuspidal representation σ. (There is at most one.) We then get a unipotent cuspidal representation $\sigma \otimes \sigma$ on the Levi component just as in Section 4.6. The group Ω can be described by 2 generators τ, ν where ν acts on each component simultaneously. The element τ can be taken to be a pure diagram automorphism, interchanging the two factors.

One may now imitate the argument in 4.7. The essential point is that we may extend τ by the switch operator S and that we may choose $T_\nu = T \otimes T$ where T intertwines σ. Thus $S \circ (T \otimes T) = (T \otimes T) \circ S$.

This concludes the proof of Proposition 4.6.

5. Unramified supercuspidal representations

5.1. Let G be a split adjoint group defined over F. Let P be a maximal parahoric subgroup of G with prounipotent radical U. From Sections 1-3 we have seen that if σ is a cuspidal representation of P/U then $c - \mathrm{Ind}_P^G(\sigma)$ is a finite sum of irreducible supercuspidal representations.

From Langlands' philosophy one expects that these induced representations will correspond to certain admissible homomorphisms of the Weil-Deligne group. In this Section we shall examine this in case σ is *unipotent*, following a conjecture of Lusztig. For this we use Lusztig's classiciation of irreducible admissible cuspidal complexes in [L3]; much of what we need can be found in [L5] sections 20, 21, and 23.

5.2. Let $\mathbb{G}$ a complex semisimple group. Recall that a conjugacy class $\mathcal{C}$ in $\mathbb{G}$ is *isolated* if the centraliser of the semisimple part of any element in it has semisimple rank equal to that of G. (The number of such classes is finite.) Now let $\mathcal{P}$ be a conjugacy class of parabolic subgroups in $\mathbb{G}$; let $P \in \mathcal{P}$ with Levi component L and unipotent radical U. If $l \in L$ and $\mathcal{C}$ is an isolated conjugacy class in $\mathbb{G}$, let

$$\delta = \delta(l, P) = \dim(\mathcal{C}) - \dim(L) + \dim(Z_L(l)).$$

We recall that an *irreducible cuspidal local system* on $\mathbb{G}$ consists of a pair $(\mathcal{C}, \mathcal{E})$ where $\mathcal{E}$ is an isolated conjugacy class in $\mathbb{G}$ and $\mathcal{E}$ is a $\mathbb{G}$-equivariant irreducible local system on $\mathcal{C}$ with the following properties.

(i) $\mathcal{E}$ admits a central character.

(ii) For all P, L, l as above $H_c^{\delta}(l\,U_P \cap \mathcal{C}, \mathcal{E}) = 0$.

Such systems have been classified implicitly in [L3]; we shall make use of the results freely. (In (ii) the cohomology is with compact supports, with coefficients in $\mathcal{E}$.)

For the *classical groups* we shall proceed as follows. First we enumerate the set $\mathcal{LT}'$ of conjugacy classes of irreducible cuspidal systems on ${}^LG$. In doing so we shall also describe another (combinatoric) set $\mathcal{LT}$ which is closely related to $\mathcal{LT}'$-essentially it describes the support of elements in $\mathcal{LT}'$-and (implicitly) a finite-to-one map $\varphi : \mathcal{LT}' \to \mathcal{LT}$. In this way we obtain all irreducible cuspidal systems on ${}^LG$ which have trivial central character.

Put another way, we are able to describe all triples (s', u, ρ) where s' is semi simple and isolated, u is unipotent and centralised by s', and ρ is an irreducible representation of $Z_{{}^LG}(s', u)/Z_{{}^LG}\, Z^0_{{}^LG}(s', u)$ such that the condition (ii) above is satisfied. Proposition 2.8 of [L3] guarantees that $Z^0_{{}^LG}(s', u)$ contains no non trivial torus, and then Proposition 2.5 of [L0] guarantees that the pair (s', u) corresponds to a unique pair (s, N) with the properties that s is semi-simple, N is a nilpotent element in the Lie algebra of G, $\mathrm{Ad}(s)\,N = q\,N$, and there is no non trivial torus centralising both s and N.

Corollary 2.6 of [L0] says that $Z_{{}^LG}(s, N)$ is finite and the discussion there also implies that

$$Z_{{}^LG}(s, N) \simeq Z_{{}^LG}(s', u)/Z^0_{{}^LG}(s', u)$$

whence an isomorphism

$$Z_{{}^LG}(s, N)/Z_{{}^LG} \simeq Z_{{}^LG}(s', u)/Z_{{}^LG}\, Z^0_{{}^LG}(s', u)$$

as well. In this way we can identify irreducible cuspidal complexes on ${}^LG$ with trivial central character, with a subset of those triples (s, N, ρ) where (s, N) is an L^2-pair in ${}^LG$ in the sense of [L0], and ρ is an irreducible representation of $Z_{{}^LG}(s, N)/Z_{{}^LG}$. Such a triple describes an admissible homomorphism from the Weil-Deligne group to ${}^LG$.

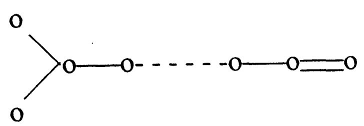
On the local side we describe the set $\mathcal{T}$ of conjugacy classes of pairs (P, σ) where P is a parahoric subgroup of G, and σ is a unipotent cuspidal representation of the Levi component of P. We remind the reader that the Levi component is the group of rational points of a reductive group defined over the residue field $\mathbb{F}_q$ of F. *For each group of classical type we exhibit an explicit bijection* $\mathcal{T} \to \mathcal{LT}$. We remark that almost identical bijections to these, and those in Section 6 below, have been used by Lusztig in [L5] to establish that the set of cuspidal character sheaves is the same as the set of irreducible cuspidal complexes.

Given an element $(P, \sigma) \in \mathcal{T}$ we form the smooth representation $c - \mathrm{Ind}_P^G(\sigma)$. Proposition 4.6 implies that this always splits into a finite sum of *distinct* irreducible supercuspidal representations. Let $\lambda \in \mathcal{LT}$ correspond to (P, σ) under the bijection above; *for each group of classical type we find that* $\varphi^{-1}(\lambda)$ *contains as many elements as there are supercuspidal pieces in the decomposition of* $c - \mathrm{Ind}_P^G(\sigma)$. (In many cases this number is 1.)

For the *exceptional groups* we shall describe the irreducible cuspidal complexes more or less explicitly, as well as the supercuspidal representations, and match them. (The analogues of the underlined statements above do not hold for these groups.)

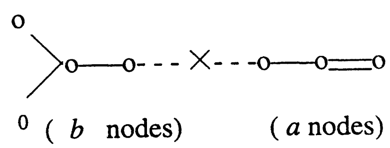
In what follows we shall use repeatedly the fact that there is a bijection between the set of irreducible unipotent cuspidal representations on a finite group of Lie type and the corresponding set on the group of adjoint type. (*See* [C], 12.1, p. 380.)

5.3. To begin we consider the case where G is split adjoint of type B_n, with local diagram



(n + 1 vertices)

The two left end nodes are hyperspecial. Let P be a (standard) maximal parahoric subgroup of type



where "$\times$" means that the corresponding vertex and nodes are omitted, and $n = a + b$. The Levi component of P has a (unique) unipotent cuspidal representation precisely when $a = t(t+1)$, and $b = s^2$ where s is even.

Let $\mathcal{T}$ denote the set of ordered pairs

$$\{(s^2, t(t+1)) | s, t \in \mathbb{N} \cup \{0\}, s \text{ even}, n = s^2 + t(t+1)\}.$$

Let $\mathcal{LT}$ denote the set of *un*ordered pairs of triangular numbers $\{\nu, w\}$ such that $n = \nu + w$. We have a bijection $\mathcal{T} \to \mathcal{LT}$ given by the rule

$$(s^2, t(t+1)) \mapsto \{(s+t)(s+t+1)/2, (s-t)(s-t-1)/2).$$

The inverse map is given by the rule

$$\begin{aligned}&\{l(l+1)/2, k(k+1)/2\}\\ &\quad\mapsto \begin{cases} ((l+k+1)/2)^2, (l-k-1/2)(l-k+1/2)), \text{just one of } l, k \text{ even}\\ ((l-k)/2)^2, (l+k)(l-k+2)/4), \text{otherwise}\end{cases}\end{aligned}$$

Here we take $l \geq k$, as we may.

For each *ordered* pair $(v, w) = (l(l+1)/2, k(k+1)/2)$ with $w \neq \nu$ there is a unique cuspidal local system on $\mathbb{S}\mathrm{p}_{2n}(\mathbb{C})$ with trivial central character. This can be obtained as follows. Since v, w are triangular, [L3] tells us that there are unique cuspidal local systems $\mathcal{E}_v$, $\mathcal{E}_w$ on $\mathbb{S}\mathrm{p}_{2v}(\mathbb{C})$, $\mathbb{S}\mathrm{p}_{2w}(\mathbb{C})$ respectively. Since we are interested in complexes which have trivial central character on the centre of $\mathbb{S}\mathrm{p}_{2n}(\mathbb{C})$ (the diagonal part of the centre of $\mathbb{S}\mathrm{p}_{2n}(\mathbb{C}) \times \mathbb{S}\mathrm{p}_{2w}(\mathbb{C})$), we require these to both have trivial central character, or non trivial central character. The table in [L3] then implies that v, w are both odd or both even. The complex $\mathcal{E}_{v,w} = \mathcal{E}_v \otimes \mathcal{E}_w$ corresponds to the desired complex. (Compare [L3] 2.10). It is parametrised by a triple (s, u, ρ) where

$$Z(s) = \mathbb{S}\mathrm{p}_{2v}(\mathbb{C}) \times \mathbb{S}\mathrm{p}_{2w}(\mathbb{C})$$

and u is a certain unipotent element in $Z(s)$ which can be explicitly described: on each factor it will have distinct Jordan block widths of even block width. Furthermore [L3] 2.10 implies that all irreducible cuspidal local systems with trivial central character are obtained in this fashion. Thus to each *un*ordered pair of triangular numbers $\{v, w\}$ as above we obtain *two* irreducible cuspidal local systems.

It also follows from this discussion that there is a "forgetful" map

$$\varphi : \mathcal{LT}' \to \mathcal{LT}.$$

On the other hand an element $(s^2, t(t+1))$ of $\mathcal{T}$ corresponds to a unique unipotent cuspidal representation σ of the Levi component of a uniquely specified parahoric subgroup P. Since G is split adjoint, the normaliser of P has index 2 over P unless $s = 0$, $n = t(t+1)$. From section 4.6 we see that when we inflate σ to P and compactly induce to G the resulting (supercuspidal) representation π splits into the sum of two irreducible supercuspidal representations, unless $s = 0$, $n = t(t+1)$ (in which case it remains irreducible).

(It is not clear to the author how one matches the two cuspidal systems with the two corresponding supercuspidal representations.)

5.4. Next consider split adjoint groups of type C_n. In this case conjugacy classes of pairs (P, σ) as above are parametrised by *un*ordered pairs $\{s(s+1), t(t+1)\}$ such that

$n = s(s+1) + t(t+1)$; let $\mathcal{T}$ denote this set. Let $\mathcal{LT}$ denote the set of unordered pairs $\{v^2, w^2\}$ such that $2n+1 = v^2 + w^2$. There is a bijection $\mathcal{T} \to \mathcal{LT}$ given by the rule

$$\{s(s+1),\, t(t+1)\} \mapsto \{(s+t+1)^2,\, (s-t)^2\}.$$

Given $\{v^2, w^2\}$ one can associate

(a) one irreducible cuspidal system on $\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})$ if $v \,.\, w \neq 0$;

(b) two irreducible cuspidal systems on $\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})$ otherwise.

Consider case (a). We shall argue as in the previous paragraph. Omitting a non special node from the completed Dynkin diagram corresponds to an isolated semi-simple conjugacy class in $\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})$. The centraliser of this element is $H = \mathbb{S}\mathrm{pin}_{2k}(\mathbb{C}) \times \mathbb{S}\mathrm{pin}_{2l+1}(\mathbb{C})/\langle(\varepsilon, \varepsilon')\rangle$. Here ϵ denotes the kernel of the map $\mathbb{S}\mathrm{pin}_{2k}(\mathbb{C}) \to \mathbb{S}\mathbb{O}_{2k}(\mathbb{C})$ and ε' denotes the analogous element for $\mathbb{S}\mathrm{pin}_{2l+1}(\mathbb{C})$, while $(\varepsilon, \varepsilon')$ denotes the element in $Z(\mathbb{S}\mathrm{pin}_{2k}(\mathbb{C}) \times \mathbb{S}\mathrm{pin}_{2l+1}(\mathbb{C}))$ whose components are given by $\varepsilon, \varepsilon'$ respectively. In this identification the centre of $\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})$ is generated by the image of ε. We seek complexes in H which are trivial on this last group. Let u be any unipotent element in $\mathbb{S}\mathrm{pin}_{2k} \times \mathbb{S}\mathrm{pin}_{2l+1}$, with image $\overline{u}$ in *H*; then $Z(u) \mapsto Z(\overline{u})$ is onto and it follows that

$$Z(u)/Z^0(u) \to Z(\overline{u})/Z^0(\overline{u})$$

is also onto with kernel either $\langle(\varepsilon, \varepsilon')\rangle$ or $\{1\}$. From this we see that we only have to construct complexes on $\mathbb{S}\mathrm{pin}_{2k}(\mathbb{C}) \times \mathbb{S}\mathrm{pin}_{2l+1}(\mathbb{C})$ whose central characters take the same values on $\varepsilon, \varepsilon'$, and then that they must be trivial on the subgroups generated by these two elements. (Just consider what happens as one passes to the group generated by the image of ϵ in H.) The Table in [L3] implies that the only possibility is that $2k$, $2l+1$ are each squares and the resulting admissible complex on H then has the form $\pi_*\,(\mathcal{E}_{2k} \otimes \mathcal{E}_{2l+1})$, where π denotes the quotient map to H. (*See* [L3] 2.10.) The support of the system is the closure of a class su where $Z_{\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})}(s)$ doubly covers $\mathbb{S}\mathbb{O}_{v^2} \times \mathbb{S}\mathbb{O}_{w^2}$ and u is an explicitly describable unipotent element in this group. The arguments for case (b) are the same except that there are now two conjugacy classes of semisimple elements provided by the two elements of the centre of $\mathbb{S}\mathrm{pin}_{2n+1}(\mathbb{C})$.

This matches the behaviour on the local side: if $\{s(s+1),\, t(t+1)\}$ corresponds to (P, σ) then the compactly induced representation splits into two irreducible supercuspidal pieces precisely when $s = t$, which corresponds to $v \,.\, w = 0$; otherwise we obtain a unique irreducible supercuspidal representation.

5.5. We may argue in a similar fashion for split adjoint groups of type D_n. Passing first to the group $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$ we consider unordered pairs $\{v^2, w^2\}$ such that $v^2 + w^2 = 2n$; we suppose that $4|n$.

The technique for obtaining the possible irreducible cuspidal local systems is similar to what has gone before. One first describes the possibilities via the simply connected (double covering of a group of the form $Z(s)$ where s represents a semi-simple conjugacy class in $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$; using the results in [L3] one then obtains the actual cases that can occur. All such systems are obtained this way. We omit the details.

Given $\{v^2, w^2\}$ one can associate

(a) four irreducible cuspidal systems on $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$ if $v \,.\, w = 0$;

(b) two irreducible cuspidal systems on $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$ if $v \,.\, w \neq 0$, and $v \neq w$;

(c) one irreducible cuspidal system on $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$ if $v \,.\, w \neq 0$, and $v = w$.

The complexes in case (a) are supported on the closure of the conjugacy class of an element su where s runs through the centre and u is a certain (fixed) unipotent element. (The representation ρ is the same in each case).

Next consider case (b); there are two conjugacy classes of semisimple elements with centraliser of type $D_{v^2} \times D_{w^2}$; each gives rise to one complex. (Compare case (b), 5.4.)

In case (c) the complex is supported on the closure of a conjugacy class su where s has centraliser of type $D_{v^2} \times D_{v^2}$; it is analogous to case (a) of 5.4.

Let $\mathcal{LT}$ denote the set of unordered pairs $\{v^2, w^2\}$ such that $v^2 + w^2 = 2n\,(4|n)$. Let $\mathcal{T}$ denote the set of unordered pairs $\{s^2, t^2\}$ such that $s^2 + t^2 = n$, where s, t are both even. There is a bijection $\mathcal{T} \to \mathcal{LT}$ given by the rule

$$\{s^2, t^2\} \mapsto \{(s+t)^2, (s-t)^2\}$$

Each element of $\mathcal{LT}$ gives rise to one, two, or four (explicitly describable) irreducible cuspidal complexes on $\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})$ corresponding to cases (c), (b), or (a) above.

Note that since n is even $Z_{\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C})} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \pi_1(G)$. It is useful to note how the latter acts on the completed Dynkin diagram. In this context there are three non trivial elements ρ_0, ρ_1, ρ_n. Here ρ_1 interchanges the two left hand nodes, interchanges the two right hand nodes, and fixes the rest; ρ_n interchanges the extreme bottom nodes, interchanges the extreme top nodes and swaps node i with node $n-i$ for the rest (numbered left to right); finally $\rho_0 = \rho_1\,\rho_n$.

Returning to the local group $G = \mathbb{PSO}_{2n}(F)$, we know that for each unordered pair $\{s^2, t^2\} \in \mathcal{T}$ there is a unique (up to conjugacy) pair (P, σ) just as before. We form $c - \mathrm{Ind}_P^G(\sigma)$ as before and consider how it splits.

(c) If s or $t = 0$ (omit an end node) the induced representation does not split. In fact P is hyperspecial and is self normalising. This corresponds to case (c) for complexes on the dual side.

(b) If $s \neq t$, P has index 2 in its normaliser (corresponding to the element ρ_1 above). This implies that the representation splits into two irreducible pieces. This corresponds to case (b) for complexes on the dual side.

(a) If $s = t$, P has index 4 in its normaliser (corresponding to the full group $\mathbb{Z}_2 \times \mathbb{Z}_2$ above). This implies that the representation splits into irreducible pieces. This corresponds to case (a) for complexes on the dual side.

Again there is an ambiguity with respect to matching individual representations with the individual complexes (triples).

5.6. We pass to exceptional split groups of adjoint type; we shall treat the groups of type G_2, F_4, E_8 first since they are also simply connected. In any case it is worthwhile to note that the only maximal parahoric subgroups that can play a role here are hyperspecial; this follows by inspection of the possible Levi components. In particular any unipotent cuspidal representation will (compactly) induce irreducibly to a supercuspidal representation on G; this follows from Sections 1-2 after noting that a hyperspecial parahoric subgroup is self normalising in an adjoint group.

Consider a group G of type G_2 with hyperspecial parahoric subgroup P. The Levi component of P is of type G_2 as well. There are four unipotent cuspidal representations in this case denoted by $G_2\,[-1]$, $G_2\,[\theta]$, $G_2\,[\theta^2]$, $G_2\,[1]$ in [C] 13.8. Each one of these inflates to a representation of P and then compactly induces to an irreducible supercuspidal representation of G. According to *op. cit.* each representation $G_2\,[\,.\,]$ corresponds to a certain pair $(g_i,\, \rho)$ where g_i is an element of order i in $\mathfrak{S}_3$, the symmetric group on three letters, and ρ is an irreducible representation of $Z_{\mathfrak{S}_3}(g_i)\,(=\langle g_i\rangle$ if $i \neq 1)$. Write ε for the non trivial character of $\mathbb{Z}_2 \simeq \mathfrak{S}_3/\mathbb{Z}_3$ and let θ denote the complex number $e^{2\,\pi i/3}$; we also write θ, θ^2 for the character of $\mathbb{Z}_3$ whose restriction to g_3 is θ, θ^2 respectively. We can then describe this correspondence by the table below.

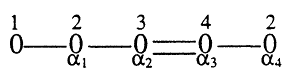
Representation	**Pair** $(g_i,\, \theta)$
$G_2\,[1]$	$(1,\, \varepsilon)$
$G_2\,[-1]$	$(g_2,\, \varepsilon)$
$G_2\,[\theta]$	$(g_3,\, \theta)$
$G_2\,[\theta^2]$	$(g_3,\, \theta^2)$

The pairs in the right column can be interpreted in another way. Indeed the triples $(s',\, N,\, \rho)$ that correspond to irreducible cuspidal complexes in $\mathbb{G}_2\,(\mathbb{C})$ (*cf.* 5.2, where we take s' centralising N) can be described by the following table.

$Z\,(s')$	$N \in H = Z\,(s')$	$Z_H\,(N)/Z_H^0\,(N)$	ρ
$\mathbb{G}_2$	subregular	$\mathfrak{S}_3$	ε
$\mathbb{SO}_4$	regular	$\mathbb{Z}_2$	ε
$\mathbb{SL}_3$	regular	$\mathbb{Z}_3$	$\theta,\, \theta^2$

In fact the N in question is always a subregular element of $\mathbb{G}_2$. We see that there are four such triples and then that there is a correspondence between the two tables: $G_2\,[1]$ corresponds to the first row of the second table, $G_2\,[-1]$ corresponds to the second row, and $G_2\,[\theta]$, $G_2\,[\theta^2]$ correspond to the last row. (This depends on the choice of generator for $\mathbb{Z}_3$.)

5.7. Consider the case of a split simple group of type F_4. The completed Dynkin diagram for this group over F or $\mathbb{C}$ has the form



The number above each node refers to the order of a semisimple element s whose conjugacy class is isolated (over $\mathbb{C}$). Then $Z\,(s)$ is a semisimple group whose type is obtained by omitting the corresponding vertex and connecting edges. There are seven irreducible cuspidal local systems on $\mathbb{F}_4\,(\mathbb{C})$; we shall list them as we did for $\mathbb{G}_2$. Some preliminary comments are in order.

If $\mathbb{Z}_n$ is a cyclic group of order n we shall describe a character by the value it takes on a generator. (In particular, i has its customary meaning as a primitive 4th root of unity.) We write $\mathbb{D}_8$ for the dihedral group of order 8. The representation ϵ on $\mathbb{Z}_2 \times \mathbb{Z}_2$ is simply the tensor product of the non trivial representation on each copy of $\mathbb{Z}_2$. Finally if $\mathfrak{S}_n$ denotes the symmetric group on n letters, we write λ^j for the j-th exterior power of the reflection representation.

The left column below denotes the isogeny class of $Z(s')$. (For the first and last entries, "isogeny" can be replaced by "isomorphism".) Under the column marked "N" we have described the nilpotent element in $Z(s')$ using Jordan block size for the classical groups where appropriate. "$F_4(a_3)$" refers to a particular conjugacy class, using notation that is standard. (*See* [C] chapter 13, for example.)

$Z(s')$	$N \in H = Z(s')$	$Z_H(N)/Z_H^0(N)$	ρ
$\mathbb{F}_4$	$\mathbb{F}_4(a_3)$	$\mathfrak{S}_4$	λ^3
$\mathbb{SL}_2 \times \mathbb{S}\mathrm{p}_6$	regular $\times$ (2, 4)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\varepsilon = \varepsilon' \otimes \varepsilon''$
$\mathbb{SL}_3 \times \mathbb{SL}_3$	regular	$\mathbb{Z}_3$	$\theta,\ \theta^2$
$\mathbb{SL}_4 \times \mathbb{SL}_2$	regular	$\mathbb{Z}_4$	$i,\ -i$
$\mathbb{S}\mathrm{pin}_9$	(1, 3, 5)	$\mathbb{D}_8$	$\varepsilon = \varepsilon' \otimes \varepsilon''$

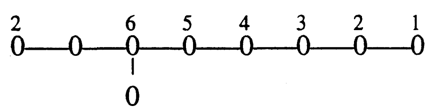
(The last entry in the right hand column is via the map $\mathbb{D}_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$.)

On the other hand, up to conjugacy there is a single hyperspecial parahoric subgroup P of the local group $G = \mathbb{F}_4(F)$ whose Levi component has type $\mathbb{F}_4$ over the residue field. The split group of type $\mathbb{F}_4$ over a finite field has seven unipotent cuspidal representations. We can inflate these to P and compactly induce to obtain seven distinct irreducible supercuspidal representations. We may describe these representations by means of pairs (g_n, ρ) where g_n is a certain element of order n in the symmetric group $\mathfrak{S}_4$ (*cf.* G_2). In this case the appropriate elements of $\mathfrak{S}_4$ are 1, g_2 (transposition), g_2' (cycle type 22), g_3, g_4. (These are representatives of the conjugacy classes in $\mathfrak{S}_4$.) The centralisers in $\mathfrak{S}_4$ of these elements are respectively $\mathfrak{S}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{D}_8$, $\mathbb{Z}_3$, $\mathbb{Z}_4$. As above we write λ^j for the j-th exterior power of the reflection representation of $\mathfrak{S}_4$. The labels for the other representations follow previous conventions.

Representation	**Pair** (g_n, θ)
$F_4^{II}[1]$	$(1, \lambda^3)$
$F_4[-1]$	(g_2, ε)
$F_4[\theta],\ F_4[\theta^2]$	$(g_3, \theta),\ (g_3, \theta^2)$
$F_4[i],\ F_4[-i]$	$(g_4, i),\ (g_4, -i)$
$F_4^{I}[1]$	(g_2', ε)

Comparing the rows of each table we see that indeed we have a correspondence between the irreducible (unipotent) supercuspidal representations in the second table and the triples in the first table.

5.8. The completed Dynkin diagram for a split group of type E_8 over F or $\mathbb{C}$ is given as follows



Again the number above a given node is the order of an element in an isolated semisimple conjugacy class over $\mathbb{C}$; the isogeny type of the centraliser of such an element is given by omitting the node and connecting edges.

There are thirteen irreducible cuspidal local systems on $\mathbb{E}_8(\mathbb{C})$; we shall list them below.

Remarks. – We employ conventions similar to those used in Section 5.6 and 5.7; in particular we write λ^4 for the 4-th power of the reflection representation on the symmetric group $\mathfrak{S}_5$, θ has its previous meaning, and ζ will denote a primitve 5-th root of unity (or the corresponding character on $\mathbb{Z}_5$ following our previous convention).

Under the column "$\mathbf{Z_H(N)/Z_H^0(N)}$" we have written a group which is a quotient of $Z_H(N)/Z_H^0(N)$; in most cases this is an isomorphism. To the right of each such group "ρ" denotes a representation of that group. In row 2, "ϵ" denotes the sign representation of $\mathfrak{S}_3$; in row 3 it denotes the non trivial representation of $\mathbb{Z}_2$; in the last row it is the tensor product of the non trivial representation of each of the factors.

$Z(s')$	$N \in H{=}Z(s')$	$Z_H(N)/Z_H^0(N)$	ρ
E_8	$2A_4 = E_8(a_7)$	$\mathfrak{S}_5$	λ^4
$E_7 \times A_1$	$E_7(a_5) \times$ regular	$\mathfrak{S}_3 \times \mathbb{Z}_2$	$-\varepsilon$
$E_6 \times A_2$	$E_6(a_3) \times$ regular	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\theta\varepsilon,\ \theta^2\varepsilon$
$D_5 \times A_3$	$(3, 7) \times$ regular	$\mathbb{Z}_4$	$i,\ -i$
$A_4 \times A_4$	regular	$\mathbb{Z}_5$	$\zeta^m,\ 1 \leq m \leq 4$
$A_2 \times A_1 \times A_5$	regular	$\mathbb{Z}_6$	$-\theta,\ -\theta^2$
D_8	$(1, 3, 5, 7)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	ε

It is perhaps worth describing briefly how this table is obtained. Consider for example the row marked "$D_5 \times A_3$". There is a semisimple element of order four in E_8 whose centraliser is isogenous to $\mathbb{S}\mathrm{pin}_{10} \times \mathbb{SL}_4$. (It is the central product of this group.) According to [L3] there are two cuspidal systems on $\mathbb{S}\mathrm{pin}_{10}$ supported on the closure of the unipotent class indicated and the centre $\mathbb{Z}_4$ acts via a faithful character in each case. Similarly there are two cuspidal systems supported on the closure of a regular unipotent class in $\mathbb{SL}_4$ and the centre also acts via a faithful character in each case. Using [L3] 14.2 one finds $Z_H(N)/Z_H^0(N) \simeq \mathbb{Z}_4$. Using [L3] 2.10.1 it follows that we must produce a (tensor) representation on $\mathbb{Z}_4 \times \mathbb{Z}_4$ (each of whose pieces corresponds to the appropriate cuspidal complex) which is trivial on (the central) $\mathbb{Z}_4$ part so that it factors through $\mathbb{Z}_4$. One finds easily that the only possibility is the representation described in the right hand column. The other entries can be obtained by similar arguments. (*See* preceding paragraphs, or section 6 below, for more computations of this nature.)

Again, the nilpotent elements not described by block sizes are labelled according to standard conventions.

There are also thirteen unipotent cuspidal representations on a finite group of Lie type E_8. They can be listed by pairs (g_n, ρ) where g_n is a certain element of order n in the symmetric group $\mathfrak{S}_5$ (*cf.* F_4). In this case the appropriate elements of $\mathfrak{S}_5$ are 1, g_2 (transposition), g_2' (cycle type 22), g_3, g_4, g_5, g_6. (These are representatives of the conjugacy classes in $\mathfrak{S}_5$.) The centralisers in $\mathfrak{S}_5$ of these elements are respectively $\mathfrak{S}_5$, $\mathfrak{S}_3 \times \mathbb{Z}_2$, $\mathbb{D}_8$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$. As usual we write λ^j for the j-th exterior power of the reflection representation of $\mathfrak{S}_5$. The labels for the other representations follow previous conventions.

Representation	**Pair** $(g_n,\, \theta)$
$E_8^{II}[1]$	$(1,\, \lambda^4)$
$E_8[-1]$	$(g_2,\, -\varepsilon)$
$E_8[\theta],\, E_8[\theta^2]$	$(g_3,\, \varepsilon\theta),\, (g_3,\, \varepsilon\theta^2)$
$E_8[i],\, E_8[-i]$	$(g_4,\, i),\, (g_4,\, -i)$
$E_8[\zeta^n],\, 1 \leq n \leq 4$	$(g_5,\, \zeta^n)$
$E_8[-\theta],\, E_8[-\theta^2]$	$(g_6,\, -\theta),\, (g_6,\, -\theta^2)$
$E_8^I[1]$	$(g_2',\, \varepsilon)$

Again we have arranged matters so that row k in the two tables correspond.

5.9. Finally we treat the cases E_6 and E_7. Matters are slightly more subtle on the dual group side because simply connected groups of this type have centres of orders 3 and 2 respectively, and we only want complexes which have trivial central character. Apart from this there are fewer cases to discuss.

The simply connected group $\tilde{\mathbb{E}}_6(\mathbb{C})$ has fourteen irreducible cuspidal local systems; only two of these have trivial central character. They can be described as follows. There is an isolated semisimple conjugacy class in the adjoint group whose elements have order 3; their connected centraliser is isogenous to $\mathbb{SL}_3 \times \mathbb{SL}_3 \times \mathbb{SL}_3$ corresponding to the branch node of the completed Dynkin diagram for E_6. In the group $\tilde{\mathbb{E}}_6(\mathbb{C})$ there is a corresponding group H isogenous to $\mathbb{SL}_3 \times \mathbb{SL}_3 \times \mathbb{SL}_3$; this group has centre isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and if $x,\, y$, represent generators for the two copies of $\mathbb{Z}_3$, the centre of $\tilde{\mathbb{E}}_6(\mathbb{C})$ is the cyclic group given by $\langle (x,\, y) \rangle$. It follows from this and the description of cuspidal complexes on $\mathbb{SL}_n$ that there are just two complexes on $\tilde{\mathbb{E}}_6(\mathbb{C})$ which have trivial character on the centre and they can be labelled by the two faithful characters of $\mathbb{Z}_3$; their support is the closure of the conjugacy classes of xu or yu where u is a regular unipotent element in H. (Since u is regular in H we have $Z_H(u)/Z_{\tilde{\mathbb{E}}_6(\mathbb{C})}\, Z_H^0(u) \simeq \mathbb{Z}_3$.)

Now pass to the adjoint group $\mathbb{E}_6(F)$ and the Levi component of the (unique up to conjugacy) hyperspecial parahoric subgroup. There are two unipotent cuspidal representations for the split group of type E_6 over a finite field, labelled $E_6[\theta]$, $E_6[\theta^2]$ in [C]. We can match these to the above complexes in the same way as before (and subject to the same caveat).

There is a similar discussion for the group of type E_7. This time there is an element of order 4 in the adjoint group over $\mathbb{C}$ corresponding to the branch node of the completed Dynkin diagram. Arguing as before one finds two irreducible cuspidal complexes; the relevant unipotent element u is again regular in a group H isogenous to $\mathbb{SL}_4 \times \mathbb{SL}_4 \times \mathbb{SL}_2$ and one has $Z_H(u)/Z_{\tilde{\mathbb{E}}_7(\mathbb{C})}\, Z_H^0(u) \simeq \mathbb{Z}_4$. The cuspidal complexes are parametrised by the faithful characters of $\mathbb{Z}_4$. Correspondingly there are two unipotent cuspidal representations on the finite group of Lie type E_7; these give rise to two irreducible supercuspidal representations. We can match them to the complexes just as before.

5.10. *Remark.* – The preceding analysis for the exceptional groups makes it easy to predict the size of the L-packet (*cf.* [Ko]) containing any one of the supercuspidal representations we have exhibited. For example in the case of E_7 just discussed, the philosophy of L-packets predicts that the two supercupisdal representations we have exhibited will be contained in one L-packet which in addition should contain two further (square integrable representations). Similarly for the split adjoint E_6 there should be an L-packet containing both the supercuspidal representations and one further (square integrable) representation. For G_2, one can write down the L-packets explicitly; this is essentially contained in [L0]. For this case (and some others) M. Reeder shows in [R] that the formal degrees of the representations in each packet are all integer multiples of a "generic" representation (which is never supercuspidal). In each case the multiple is the degree of the representation given by the matching triple on the dual side. Reeder's methods apply more generally but for the present are difficult to use.

6. Non split inner forms

6.1. In this section we shall sketch some analogues of the results of section 5 for inner forms of split simple adjoint groups. We shall emphasise the new phenomena that can occur; we remind the reader that $\mathbb{F}_q$ denotes the residue field of the local field F. The results in this section were suggested by a remark made by Lusztig during a lecture at the Institute in the Fall of 1988.

6.2. To start we remark that the recipes in [T] 3.51.-2 enable us to determine the type/index of M from the table in [T] if a standard P is given. Furthermore [T] 2.5 enables us to determine that part of P^+ arising from diagram automorphisms, and [T] 3.5.3 enables us to determine the rest of P^+.

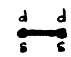
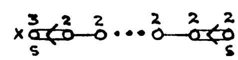
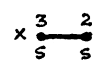
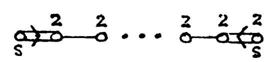
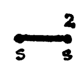
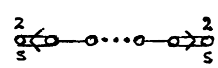
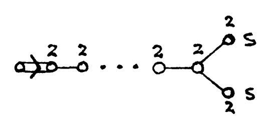
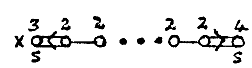
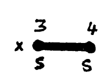
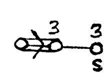
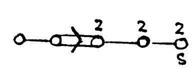
We shall suppose our group $\mathbf{G}$ is simple, split, adjoint. We are interested in $H^1(\Gamma, \mathbf{G})$; this set can be identified with the set of "$\overline{F}$-forms" $f : \mathbf{G} \to \mathbf{G}'$ over $\overline{F}$ such that for each element $\sigma \in \mathrm{Gal}(\overline{F}/F)$, the automorphism $f^{-1} \circ {}^{\sigma} f$ is inner, modulo the equivalence relation obtained by conjugating by elements of $\mathbf{G}(\overline{F})$. In the discussion that follows, "equivalent" will mean equivalent in $H^1(\Gamma, \mathbf{G})$.

The diagrams of equivalence classes of inner forms of $\mathbf{G}$ can easily be extracted from the table in [T]. We shall begin by reproducing them below in Table 6.3 by type, together with their local Dynkin diagrams; we ignore the split case. In section 6.4 we shall differentiate among equivalence classes of forms which have the same type, name, and (local) index.

► **Warning.** Table 6.3 below omits the description of the local index: the reader who wishes to verify our later assertions concerning the type of a particular M will have to consult the table in [T]. ◄

In what follows we write the *index* of $\mathbf{G}$ (defined as in [T2]); the *name* (as in [T]), and under the name, the affine root system [T]; and the relative local Dynkin diagram (as in [T]). We remind the reader that the index corresponds to an absolute Dynkin diagram [T2] together with a Galois action (the **-action* of [T2]), and a set of *distinguished vertices*: in case the form is inner, the *-action is trivial. The label for the échelonnage includes the type of the split affine root system: when it is hyphenated, the left label gives the type, otherwise it is given by the label (omit any superscripts). In particular, the local Dynkin diagram encodes the parameters describing the standard Iwahori-Hecke algebra attached to G; forgetting the arrows tells the type of the affine Weyl group involved, while the integer $d(a)$ above each node (corresponding to a simple root) defines the parameter q_a by $q_a = q^{d(a)}$. (The rest of the Hecke algebra comes from diagram automorphisms.)

6.3. Table

Index	**Name**	**Local Dynkin diagram**
${}^1\mathrm{A}^{(d)}_{md-1,\, m-1}$ $d \geq 2,\ m \geq 3$	${}^d\mathrm{A}_{md-1}$ A_{m-1}	cycle of length m, all nodes special; $d\,(a) = d$
${}^1\mathrm{A}^{(d)}_{d-1,\, 0}$	$\varnothing$	–
${}^1\mathrm{A}^{(d)}_{2d-1,\, 1}$	${}^d\mathrm{A}_{2d-1}$ A_1	
$\mathrm{B}_{n,\, n-1}$ $n \geq 3$	${}^2\mathrm{B}_n$ $\mathrm{C}-\mathrm{B}_{n-1}$	
$\mathrm{C}^{(2)}_{2m-1, m-1}$ $m \geq 3$	${}^2\mathrm{C}_{2m-1}$ $\mathrm{C}-\mathrm{BC}^{\mathrm{IV}}_{n-1}$	
$\mathrm{C}^{(2)}_{3,1}$	${}^2\mathrm{C}_3$ $\mathrm{C}-\mathrm{BC}^{\mathrm{IV}}_1$	
$\mathrm{C}^{(2)}_{2m, m}$ $m \geq 2$	${}^2\,\mathrm{C}_{2m}$ C_m	
$\mathrm{C}^{(2)}_{2,1}$	${}^2\mathrm{C}_2$ A_1	
${}^1\mathrm{D}^{(1)}_{n, n-2}$ $n \geq 4$	${}^2\mathrm{D}'_n$ $\mathrm{C}-\mathrm{B}_{n-2}$	
${}^1\mathrm{D}^{(2)}_{2m, m}$ $m \geq 3$	${}^2\mathrm{D}''_{2m}$ $\mathrm{B}-\mathrm{C}_m$	
${}^1\mathrm{D}^{(2)}_{2m+1, m-1}$ $m \geq 3$	${}^4\mathrm{D}_{2m+1}$ $\mathrm{C}-\mathrm{BC}^{\mathrm{I}}_{m-1}$	
${}^1\mathrm{D}^{(2)}_{5,1}$	${}^4\mathrm{D}_5$ $\mathrm{C}-\mathrm{BC}^{\mathrm{I}}_1$	
${}^1\mathrm{E}^{16}_{6,2}$	${}^3\mathrm{E}_6$ $\mathrm{G}^{\mathrm{I}}_2$	
$\mathrm{E}^{9}_{7,4}$	${}^2\mathrm{E}_7$ $\mathrm{F}^{\mathrm{I}}_4$	

6.4. Some remarks about the preceding table are in order. First note that the equivalence class of (adjoint) groups corresponding to the name in 6.3 is unique except in the cases of type ${}^{d}A_{md-1}$, ${}^{4}D_{2m+1}$, ${}^{2}\mathrm{D}''_{2m}$, and ${}^{3}E_6$. In the first case fix a positive integer n and let $d|n$. Central algebras of degree n are classified by $H^2(F, \mu_n) \simeq Br_n(F) \simeq \mathbb{Z}/n\mathbb{Z}$. If $\bar{r}$ is invertible in $\mathbb{Z}/n\mathbb{Z}$ the corresponding central algebra is a division algebra of degree n; otherwise, selecting $0 \leq r \leq n-1$ and writing $r = ld$, $n = md$, where d $= (r, n)$ we obtain $1 \leq l \leq d-1$, whence an invertible element of $\mathbb{Z}/d\mathbb{Z}$ which corresponds to a division algebra of degree d. Thus fixing n, we see that there are $\varphi(n)$ equivalence classes of anisotropic forms of A_{n-1} and the remaining inner forms are of type $\mathbb{M}_m(D)$ where D is a division algebra of degree d and $md = n$.

For the case ${}^{4}D_{2m+1}$ we proceed as follows. Let Z denote the centre of ${}^{L}\mathbf{G}$. There is a natural isomorphism of functors $Z^{\wedge} \simeq H^1(\Gamma, \mathbf{G})$. (*See* [Ko] 6.4-6.5) Applying this to $\mathbb{S}\mathrm{pin}_{4m+2}(\mathbb{C})$ we see that there are four equivalence classes of inner forms of $\mathbb{PSO}_{4m+2}$ (including the split form) corresponding to the four four characters of $Z \simeq \mathbb{Z}_4$ in this case. The trivial character corresponds to the split form; the character of order two corresponds to the name/type ${}^{2}\mathrm{D}'_{2m+1}$. To see this note that the tables in [T] tell us that this latter type is isogenous to $\mathbb{SO}(q)$ where q is a quadratic form in $2r' + 4$ variables (r' being the Witt index) with discriminant one. Let $\mathbb{SO}_{4m+2}$ denote the special orthogonal group of the quadratic form with maximal index; then q provides an element in $H^1(\Gamma, \mathbb{SO}_{4m+2})$. The anisotropic part of the form q can be taken to be the norm form of the non trivial quaternion algebra of degree 2. If we consider the bijection $H^1(\Gamma, \mathbb{SO}_{4m+2}) \to Br_2$ induced by the map from the simply connected cover to $\mathbb{SO}_{4m+2}$ this means that the isomorphism class of q maps to the non trivial element. On the other hand the character of order two vanishes on the kernel of the map

$$\mathbb{S}\mathrm{pin}_{4m+2}(\mathbb{C}) \to \mathbb{SO}_{4m+2}(\mathbb{C})$$

and is the non trivial element of $H^1(\Gamma, \mathbb{SO}_{4m+2})$ under the natural isomorphism $Z^{\wedge} \simeq H^1(\Gamma, \mathbb{SO}_{4m+2})$; it must indeed correspond to the name/type ${}^{2}\mathrm{D}'_{2m+1}$. It also follows that there must be two inner forms of name/type ${}^{4}D_{2m+1}$ corresponding to the two faithful characters of $\mathbb{Z}_4$.

It will also be necessary for later purposes to distinguish between the various inner forms of $\mathbb{PSO}_{4m}$. In this case $Z \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, so there are three characters of order two. One of them vanishes on the subgroup $\langle\varepsilon\rangle = \ker(\mathbb{S}\mathrm{pin}_{4m} \to \mathbb{SO}_{4m})$; an argument similar to that above implies that it must correspond to the non split inner form of type ${}^{2}\mathrm{D}'_{2m}$. The other two vanish on the two remaining subgroups of order two of $\mathbb{Z}_2 \times \mathbb{Z}_2$; they must correspond to forms of type ${}^{2}\mathrm{D}''_{2m}$. (In fact they are isogenous to quaternionic special orthogonal groups.)

There are two non-split inner forms of type E_6 corresponding to the two division algebras of degree 3.

6.5. Consider the anisotropic case of type A. First observe that

$$\mathbb{Z}/n\mathbb{Z} \simeq (\mu_n)^{\wedge} \simeq (Z_{SL_n(\mathbb{C})})^{\wedge} \simeq H^2(F, \mu_n) \simeq H^1(F, \mathbb{PGL}_n)$$

In any case an invertible element of the left most group above corresponds naturally to both a faithful character of the centre of the simply connected group $\mathbb{SL}_n(\mathbb{C})$ and to a division algebra of degree n over F. According to [L3] 10.3.2, 2.10, there are n cuspidal local systems on $\mathbb{SL}_n(\mathbb{C})$ transforming by a fixed faithful character; they are described by triples (z, u, χ) where u is a regular unipotent element, z is an element of the centre of $\mathbb{SL}_n(\mathbb{C})$ and χ is the character in question. The support of the system corresponding to (z, u, χ) is the closure of the conjugacy class of zu. All cuspidal local systems are obtained in this fashion.

On the other side the local division algebra D corresponding to a given χ has a unique maximal order $\mathfrak{O}$; passing to the adjoint group and taking reductive quotients we obtain an anisotropic torus over the residue field $\mathbb{F}_q$ whose group of rational points is a quotient of $(\mathbb{F}_{q^n})^{\times}$. The trivial representation of this latter group is the unique unipotent cuspidal representation; it inflates to a representation of $\mathfrak{O}^{\times} \pmod{F^{\times}}$. The index of this latter group in $D^{\times} \pmod{F^{\times}}$ is n; in fact the quotient is cyclic and generated by any generator of the maximal prime ideal in $\mathfrak{O}$. Thus if we induce the trivial representation to the full group it breaks into a sum of n distinct irreducible representations.

The cuspidal systems above are determined by a triple (z, N, χ) where N is a regular nilpotent element and χ is the faithful character of $Z_{\mathrm{SL}_n(\mathbb{C})} \simeq Z_{LG}(N)/Z^0_{LG}(N)$ in question. Given N there is a unique regular semisimple element s such that $\mathrm{Ad}(s)N = qN$, and then $Z_{LG}(s, N)/Z^0_{LG}(s, N) \simeq Z_{LG}(N)/Z^0_{LG}(N) = Z_{LG}$.

6.6. Next, consider adjoint inner forms of type ${}^2\mathrm{B}_n$. Using the methods just sketched together with those employed previously, one readily finds that the type of a Levi component for a standard maximal parahoric P is ${}^2\mathrm{D}_r \times \mathrm{B}_{n-r}$ where ${}^2\mathrm{D}_r$ denotes the quasisplit form of type D_r over $\mathbb{F}_q$. It then follows that pairs (P, σ) are indexed by ordered pairs $(s^2, t(t+1))$ where s is odd and $n = s^2 + t(t+1)$. In particular, n is odd.

Following Section 5 we denote this set by $\mathcal{T}$.

On the dual side one knows from [L3] (*see* [L5] Sect. 23) that the irreducible cuspidal local systems on $\mathbb{S}\mathrm{p}_{2n}(\mathbb{C})$ with non trivial central character are parametrised by ordered pairs (v, w) of triangular numbers with $n = v + w$ (n odd). Let $\mathcal{LT}$ be the set given in 5.3 but with n odd. There is a bijection $\mathcal{T} \to \mathcal{LT}$ just as in 5.3. Each element of $\mathcal{LT}$ corresponds to a pair of irreducible cuspidal local systems on $\mathbb{S}\mathrm{p}_{2n}(\mathbb{C})$ with non trivial central character. On the other hand each element of $\mathcal{T}$ gives rise to a pair of irreducible supercuspidal representations via the usual compact induction argument, and an argument as in 6.2 for computing the normaliser. Thus we obtain a correspondence subject to the usual ambiguity about elements within corresponding pairs.

6.7. Next consider inner forms of type ${}^2\mathrm{C}_{2m-1}$. In this case one finds first that the type of M (over the residue field) for a standard maximal P is ${}^2\mathrm{A}_r \times \mathrm{C}_{2m-1-r}$. It follows from this and [T] 2.6, and then [L4] that the set $\mathcal{T}$ of pairs (P, σ) is indexed by ordered pairs $(s(s+1)/2, 2t(t+1))$ whose sum is $2m-1$. Indeed one first observes that [T] 3.5.2. implies that the Levi component of a standard maximal parahoric is isogenous to the group $\mathbb{SU}_{l+1} \times \mathrm{R}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{S}\mathrm{p}_{2r}$ where "R" denotes restriction of scalars, $\mathbb{SU}$ denotes the special unitary group with respect to the quadratic extension $\mathbb{F}_{q^2}/\mathbb{F}_q$, and $l + 2r = 2(m-1)$. The assertion then follows by applying the criteria for the existence of unipotent cuspidal representations for such a group, after noting that such representations occur if and only if they exist on the corresponding adjoint group which is a direct product of its simple factors. (We have used this observation repeatedly in Section 5.) One then observes that [T] 3.5.3 implies that such a parahoric will *always* have index 2 in its normaliser; this implies that the compactly induced representation will split into 2 irreducible supercuspidal representations on G.

On the dual side [L3] (*see* also [L5] Sect. 23) one knows that each element of the set $\mathcal{LT}$ of ordered pairs (v, w) of triangular numbers with $4m-1 = v+w$ (v even) gives rise to two irreducible cuspidal local systems on $\mathbb{S}\mathrm{pin}_{4m-1}(\mathbb{C})$ with non trivial central character. All such cuspidal systems are obtained in this manner. There is a bijection $\mathcal{T} \to \mathcal{LT}$ via the rule

$$(s(s+1)/2,\, 2t(t+1)) \mapsto \{(s+2t+1)(s+2t+2)/2,\, (s-2t)(s-2t-1)/2\}.$$

(Note that exactly one of the two terms on the right must be even.)

From this we see that there is a correspondence, subject to the usual caveat.

6.8. The argument for ${}^2\mathrm{C}_{2m}$ is entirely similar.

6.9. Consider groups of type ${}^2\mathrm{D}'_n$. We argue as in 6.7 to see that the Levi components of standard maximal parahorics are isogenous to products ${}^2D_l \times {}^2\mathrm{D}_k$ where $k+l=n$, and the superscript indicates the quasisplit form over a finite field. It follows that the pairs (P, σ) are parametrised by unordered pairs of odd squares $\{v^2, w^2\}$ such that $n = v^2 + w^2$. (In particular $n \equiv 2 \pmod 4$.) We also find that P has index 2 in its normaliser (case (a)) unless it corresponds to omitting the middle vertex in either the local index or the relative local Dynkin diagram, and then the index is 4 (case (b)). In case (a) the compactly induced representation splits into two irreducible pieces; in case (b) it splits into four irreducible pieces.

Now consider the dual side. We want to consider complexes whose central character is non trivial and factors through the kernel of the map

$$\mathbb{S}\mathrm{pin}_{2n}(\mathbb{C}) \to \mathbb{S}\mathbb{O}_{2n}(\mathbb{C}).$$

The argument for describing these is just like those given in Section 5. We shall state the results and omit the details. First, $n \equiv 2 \pmod 4$ and then to each unordered pair of even squares (a, b) with $2n = a+b$ one can associate

(a) two complexes if $ab > 0$;

(b) four complexes if $ab = 0$.

Let $\mathcal{T}$ denote the set of unordered pairs $\{v^2, w^2\}$ of odd squares such that $n = v^2 + w^2$; let $\mathcal{LT}$ denote the set of unordered pairs of even squares as above. There is a bijection $f : \mathcal{T} \to \mathcal{LT}$ given by the rule

$$\{a^2, b^2\} \mapsto \{(a+b)^2, (a-b)^2\}$$

and the discussion above tells us that there are as many complexes in $\varphi^{-1}(\lambda)$ (φ as in Sect. 5) for an element λ of $\mathcal{LT}$ as there are supercuspidal representations corresponding to $f^{-1}(\lambda)$.

6.10. We pass to inner forms of type ${}^2\mathrm{D}''_{2m}$. In this case one finds that the Levi component of a standard maximal parahoric is isogenous to $\mathbb{S}\mathbb{U}_{l+1} \times \mathrm{R}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{S}\mathbb{O}_{2r}$ and then that the pairs (P, σ) are parametrised by ordered pairs $(s(s+1)/2, t^2)$ such that $s(s+1)/2 + 2t^2 = 2m$. The index of P in its normaliser is 4 unless P corresponds to omitting a special point; in this case the index is 2. From this we see that if P is not special, one obtains four irreducible supercuspidal representations from the pair (P, σ); in the special case one obtains two irreducible supercuspidal representations. (This group splits over an unramified extension and the Galois group acts as a cyclic group of order two.)

Passing to the dual, we wish to consider those complexes with non trivial central character which factor through the kernel of $\mathbb{S}\mathrm{pin}_{4m} \to \frac{1}{2}\,\mathbb{S}\mathrm{pin}_{4m}$. Here $\frac{1}{2}\,\mathbb{S}\mathrm{pin}_{4m}$ denotes the quotient of $\mathbb{S}\mathrm{pin}_{4m}$ by a subgroup of order two which is *not* $\ker(\mathbb{S}\mathrm{pin}_{4m} \to \mathbb{S}\mathbb{O}_{4m})$. Let $\mathcal{LT}$ denote the set of unordered pairs $\{a, b\}$ of even triangular numbers whose sum is $4m$.

We shall explain briefly how one obtains the relevant complexes in this case. Let a, b be two positive integers whose sum is $4m$; then a, b must be even. There is a subgroup H in $\mathbb{S}\mathrm{pin}_{4m}$ which is isogenous to $\mathbb{S}\mathrm{pin}_{2a} \times \mathbb{S}\mathrm{pin}_{2b}$ and whose centre is an elementary abelian 2-group. Indeed if $\langle \varepsilon \rangle$ (respectively $\langle \varepsilon' \rangle$) denotes $\ker(\mathbb{S}\mathrm{pin}_{2a} \to \mathbb{S}\mathbb{O}_{2a})$ (respectively $\ker(\mathbb{S}\mathrm{pin}_{2b} \to \mathbb{S}\mathbb{O}_{2b})$ then H is $\mathbb{S}\mathrm{pin}_{2a} \times \mathbb{S}\mathrm{pin}_{2b}/\langle(\varepsilon, \varepsilon')\rangle$. This group doubly covers the identity component of the corresponding subgroup in $\frac{1}{2}\,\mathbb{S}\mathrm{pin}_{4m}$. (If $Z(\mathbb{S}\mathrm{pin}_{2a}) = \langle \varepsilon \rangle \times \langle \omega \rangle$, $Z(\mathbb{S}\mathrm{pin}_{2b}) = \langle \varepsilon' \rangle \times \langle \omega' \rangle$, one forms $H/\langle(\omega, \omega')\rangle$.) We want complexes whose central characters are equal, but not trivial, on ε, ε', and which are both trivial or both non trivial on ω, ω'. The table in [L3] tells us that this will occur if and only if $2a$, $2b$ are triangular numbers, and there are exactly two such complexes. Thus there are exactly two irreducible cuspidal complexes supported on the closure of a conjugacy class su where $Z(s)$ is isogenous to $\mathbb{S}\mathrm{pin}_a \times \mathbb{S}\mathrm{pin}_b$ with central character of the desired type. Similarly we obtain two irreducible cuspidal complexes corresponding to (b, a).

If $ab = 0$ then we find by a familiar argument that there are four complexes corresponding to the four elements of the centre. Thus to each unordered pair $\{a, b\}$ of even triangular numbers whose sum is $4m$ and $a \neq b$ we obtain four irreducible cuspidal local systems on $\mathbb{S}\mathrm{pin}_{4m}$. If $a = b$ we find two irreducible cuspidal local systems on $\mathbb{S}\mathrm{pin}_{4m}$ of the desired type. All such cuspidal local systems are obtained in this manner.

There is a bijection $f : \mathcal{T} \to \mathcal{LT}$ given by

$$(s(s+1)/2, t^2) \mapsto \{(2t+s)(2t+s+1)/2, (2t-s)(2t-s-1)/2\}$$

and using these results, we obtain a correspondence with similar properties to that obtained in 6.9.

6.11. To finish inner forms of classical type we must consider type ${}^4\mathrm{D}_{2m+1}$. Here the Levi component of a standard parahoric is isogenous to $\mathbb{S}\mathbb{U}_{l+1} \times \mathrm{R}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\, {}^2\mathbb{S}\mathbb{O}_{2r}$, where the superscript denotes the quasisplit form over $\mathbb{F}_{q^2}$. Then pairs (P, σ) are parametrised by ordered pairs $(s(s+1)/2, t^2)$ where t is odd and $s(s+1)/2 + 2t^2 = 2m+1$; we denote this set by $\mathcal{T}$ as usual. In all cases P has index 4 in its normaliser. (There are no relative diagram automorphisms for this case.) This implies that each member of $\mathcal{T}$ gives rise to four irreducible supercuspidal representations of the local group (isogenous to the orthogonal group of a certain quaternionic quadratic form), by the usual compact induction argument.

Passing to the dual side, we must consider the group $\mathbb{S}\mathrm{pin}_{4m+2}(\mathbb{C})$, and irreducible cuspidal complexes with faithful central character. Let $\mathcal{LT}$ denote the set of unordered pairs (a, b) of even triangular numbers whose sum is $4m+2$; there is a bijection $\mathcal{T} \to \mathcal{LT}$ given by

$$(s(s+1)/2,\, t^2) \mapsto \{(2t+s)(2t+s+1)/2,\, (2t-s)(2t-s-1)/2\}.$$

Each element (a, b) of $\mathcal{LT}$ gives rise to four irreducible cuspidal local systems: for example if $a, b > 1$ then two of these have support on the closure of a conjugacy class su where $Z(s)$ is doubly covered by $\mathbb{S}\mathrm{pin}_a \times \mathbb{S}\mathrm{pin}_b$ and $u \in Z(s)$ can be explicitly described; the other two have support on the closure of a conjugacy class su where $Z(s)$ is doubly covered by $\mathbb{S}\mathrm{pin}_b \times \mathbb{S}\mathrm{pin}_a$ and $u \in Z(s)$. (The $Z(s)$ correspond to omitting different nodes of the completed Dynkin diagram; they are not conjugate, since we are considering the simply connected group.)

6.12. Consider the case of a group of type ${}^3\mathrm{E}_6$. Applying the recipes in [T] 3.51-2 as usual allows us to determine the type of M, and then we may determine from [L4] (say) whether there is a possible unipotent cuspidal representation. The only possibility is when P is special; this corresponds to removing the right-most vertex in the relative local Dynkin diagram. In this case by applying [T] 3.5.2 we see immediately that M is of type ${}^3\mathrm{D}_4$ over the residue field; such a group has two unipotent cuspidal representations. On the other hand *op. cit.* 3.5.3 implies that P has index 3 in its normaliser, which is the pointwise stabiliser of the vertex attached to P in the affine building. We note in passing that there are no (relative) diagram automorphisms in this case. Each of these cuspidal representations is fixed under the action of the normaliser from the results in 4.6 (or since we have a group of order three acting on a set with two elements) and when one compactly induces to G one obtains a direct sum of three (non isomorphic) irreducible supercuspidal representations. Thus one obtains a total of six irreducible supercuspidal representations for each non split group of type ${}^3\mathrm{E}_6$; in addition there are two previously obtained irreducible supercuspidal representations arising from the split form of E_6. Thus one obtains a total of fourteen irreducible supercuspidal representations arising from such forms.

One the dual side [L3] Section 15 tells us that there are also fourteen irreducible cuspidal local systems for a simply connected $\mathbb{E}_6(\mathbb{C})$. Two of them have already been used in Section 5; they have support on the closure of the conjugacy class of su where s is a semisimple element whose centraliser is isogenous to $\mathbb{SL}_3 \times \mathbb{SL}_3 \times \mathbb{SL}_3$ and where u is regular unipotent in $\mathbb{SL}_3 \times \mathbb{SL}_3 \times \mathbb{SL}_3$. The others can be described as follows.

There are two unipotent cuspidal complexes whose support is the closure of a certain unipotent conjugacy class $\{u\}$ in $\mathbb{E}_6(\mathbb{C})$; each corresponds to a choice of faithful central character. Replacing $\{u\}$ by $\{zu\}$ where z runs through the centre of $\mathbb{E}_6(\mathbb{C})$ we obtain six cuspidal complexes.

Again, in the completed Dynkin diagram for E_6 we can omit any non extremal, non branch node. This corresponds to a subgroup H in $\mathbb{E}_6(\mathbb{C})$ which is the quotient

$$\mathbb{SL}_2 \times \mathbb{SL}_6/\langle \varepsilon, \rho^3 \rangle$$

where $\langle \varepsilon \rangle$ (respectively $\langle \rho \rangle$) denotes the centre of $\mathbb{SL}_2$ (respectively $\mathbb{SL}_6$). The centre of $\mathbb{E}_6(\mathbb{C})$ in H is generated by the image of ρ^2. Using [L3] 2.10.1 as we have done in previous paragraphs we obtain two cuspidal complexes on $\mathbb{E}_6(\mathbb{C})$ which correspond to two unipotent cuspidal complexes on H of the form $\mathcal{E} \otimes \mathcal{E}_\chi$ where $\mathcal{E}$ is the unique cuspidal complex on $\mathbb{SL}_2$ and $\mathcal{E}_\chi$ is a complex on $\mathbb{SL}_6$ transforming by a (faithful) central character χ. There are three ways to omit such a node, so this gives six such complexes.

All such complexes are obtained in this way.

6.13. A similar phenomenon occurs for the non split inner form ${}^2\mathrm{E}_7$. Again one need only consider the special point; the corresponding parahoric has a Levi component of type ${}^2\mathrm{E}_6$ over the residue field and is of index 2 in its normaliser. The group of type ${}^2\mathrm{E}_6$ has three unipotent cuspidal representations; in P. Compactly inducing and applying 4.6 we obtain six irreducible supercuspidal representations; adding these to the two previously obtained for the split E_7 we obtain a total of eight. On the dual side [L3] provides eight irreducible cuspidal local systems for a simply connected $\mathbb{E}_7(\mathbb{C})$; their description is similar to that given above for $\mathbb{E}_6$.

6.14. *Remark.* – Of course the bijections obtained in 6.12 and 6.13 are even less satisfactory than those obtained previously. Once one has made choices for central characters one can lump the appropriate set of supercuspidal representations with the appropriate set of triples; beyond that, things are murky. Consider once again the case of ${}^3\mathrm{E}_6$. Having fixed central characters, one has two sets consisting of three irreducible supercuspidal representations each, for each form. On the dual side there are also two sets of three irreducible cuspidal local systems each: one set comes from E_6 and the other set comes from a group isogenous to $\mathbb{SL}_2 \times \mathbb{SL}_6$. It is not clear to the author how these sets are paired with each other, or even if they should be paired.

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