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PICARD'S THEOREM, MITTAG-LEFFLER METHODS, AND CONTINUITY OF CHARACTERS ON FRÉCHET ALGEBRAS

BY J. ESTERLE

ABSTRACT. –The general question of continuity of characters on commutative Fréchet algebras can be reduced to the question of continuity of characters on some "test algebras". We discuss the algebraic structure of some quotients $\mathcal{U}/I$ where $\mathcal{U}$ is one of these test algebras. The notion of Picard-Borel algebra, related to the classical Picard theorem, plays an important role in these investigations. We also use the Mittag-Leffler theorem to exhibit large semigroups in quotients of the form A/I where A is a commutative unital Fréchet algebra and I a dense, countable union of closed prime ideals of A. We point out new algebraic obstructions to the construction of discontinuous characters on $\mathcal{U}$ related to the Picard theorem, and relate to extension properties of joint spectra of finite families of a quotient of $\mathcal{U}$ a question about iteration of Bieberbach mappings raised in 1986 by P. G. Dixon and the author.

1. Introduction

In this paper we consider Fréchet algebras, *i.e.* algebras A over $\mathbb{C}$ equipped with a nondecreasing family $(\|\cdot\|_n)_{n\geq 1}$ of submultiplicative seminorms, such that $\cap_{n\geq 1}\mathrm{Ker}\|\cdot\|_n = \{0\}$, with respect to which A is complete (when A possesses a unit e, elementary standard arguments show that it can be assumed without loss of generality that $\|e\|_n = 1\,(n \geq 1)$).

Whether characters are necessarily continuous on Fréchet algebras is still an open problem. This question, known as Michael's problem, was raised by Michael in 1952 in his memoir [36] where he established a structure theorem which shows that a Fréchet algebra A is isomorphic to a projective limit $\varprojlim(A_n, \theta_n)$ where A_n is a Banach algebra and $\theta_n : A_{n+1} \to A_n$ a norm-decreasing algebra homomorphism with dense range for every $n \geq 1$ (in fact A_n is the completion of the quotient algebra $A/\mathrm{Ker}\|\cdot\|_n$ with respect to the norm induced on $A/\mathrm{Ker}\|\cdot\|_n$ by the seminorm $\|\cdot\|_n$). It follows easily from this structure theorem that if A is commutative and unital, then the spectrum $\sigma_A(x)$ of any $x \in A$ is given by the formula $\sigma_A(x) = \{\chi(x)\}_{\chi\in\hat{A}}$ where we denote by $\hat{A}$ the set of continuous characters of A.

Despite a lot of effort by various mathematicians to solve Michael's problem, it seems that only four significant ideas appeared in the literature since 1952.

1) Arens [2] proved in 1958 that if $(a_1, \ldots, a_n)$ is a finite family of elements of a unital Fréchet algebra, A, and if $a_1.A \ldots + a_n.A$ is dense in A, then $a_1.A \ldots + a_n.A = A$. This shows in particular that if A is commutative and unital then the joint spectrum $\sigma_A(b)$ of any finite family $b = (b_1, \ldots, b_k)$ of elements of A is given by the formula

$$\sigma_A(b) = \{\chi(b_1), \ldots, \chi(b_k)\}_{\chi \in \hat{A}}.$$

It follows from this fact that if A is polynomially generated by a finite set $a = (a_1, \ldots, a_n)$ (which means that the continuous map $P \to P(a)$ from $\mathbb{C}[X_1, \ldots, X_n]$ into A has dense range) then all characters on A are continuous (this result extends to Fréchet algebras which are rationnally generated by a finite set). For example denote by $\mathcal{O}(M)$ the algebra of holomorphic functions on a Stein manifold M. It follows from the classical embedding theorem for Stein manifolds and from Cartan's theorems [31, p. 224 and p. 243] that $\mathcal{O}(M)$ is polynomially generated by a set of $2n+1$ elements, where n is the dimension of M, so that characters on $\mathcal{O}(M)$ are continuous (hence given by point evaluations $f \to f(z)$ for some $z \in M$).

2) Real-valued characters on real Fréchet algebras are continuous. This follows from results obtained in 1960 by Shah [47] concerning positive linear functionals on Fréchet algebras equipped with a continuous involution. In particular if Ω is a locally compact space such that $\Omega = \cup_{n \geq 1} K_n$, where $K_n \subset \overset{\circ}{K}_{n+1}$ is compact for every $n \geq 1$, then all characters on the Fréchet algebra $\mathcal{C}(\Omega)$ are continuous.

3) Clayton produced in 1975 [9] some "test algebras" for Michael's problem, *i.e.* commutative, unital Fréchet algebras A such that the existence of a discontinuous character on some commutative, unital Fréchet algebra would imply the existence of a discontinuous character on A. Craw [11] had produced in 1970 a locally multiplicatively convex complete algebra B such that the existence of a discontinous character on some commutative, unital Fréchet algebra would imply the existence of an unbounded character on B. Other examples of test algebras were obtained by Schottenloher [46] and Mujica [37], and a noncommutative test algebra was given by P. G. Dixon and the author in [16].

4) P. G. Dixon and the author proved in 1986 [16] that the existence of a discontinuous character on some Fréchet algebra would imply that $\varprojlim(\mathbb{C}^{p_n}, F_n) \neq \emptyset$ for every projective system $(\mathbb{C}^{p_n}, F_n)$ where $(p_n)_{n \geq 1}$ is a sequence of positive integers and $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ a holomorphic map for every $n \geq 1$. It follows immediately from Picard's theorem [41] that if $(f_n)_{n \geq 1}$ is any sequence of entire functions on $\mathbb{C}$ there exists a sequence $(z_n)_{n \geq 1}$ of complex numbers such that $z_n = f_n(z_{n+1})$ for every $n \geq 1$, so that $\varprojlim(\mathbb{C}^{p_n}, F_n)$ is indeed nonempty if $p_n = 1\,(n \geq 1)$ (in fact if f_n is nonconstant for every $n \geq 1$, the first term z_1 of the sequence $(z_n)_{n \geq 1}$ can be chosen arbitrarily in $f_1(\mathbb{C})$, and $\mathrm{Card}(\mathbb{C} \setminus f_1(\mathbb{C})) \leq 1$ by Picard's theorem). But the "Poincaré-Fatou-Bieberbach phenomenon" ([5], [26], [42]) shows that there exist entire, one-to-one maps $F : \mathbb{C}^2 \to \mathbb{C}^2$, of jacobian equal to 1, such that $F(\mathbb{C}^2)$ is not dense in $\mathbb{C}^2$. This gives some hope to construct a sequence $(F_n)_{n \geq 1}$ of entire mappings from $\mathbb{C}^2$ into itself such that $\cap_{n \geq 1}(F_1 \ldots \circ F_n)(\mathbb{C}^2) = \emptyset$, which would give a positive answer to Michael's problem.

There was some recent progress in the study of one-to-one (or nondegenerate) maps from $\mathbb{C}^p$ into $\mathbb{C}^p$ with nondense range ([10], [30], [39], [40], [43], [44]), and in the study of complex dynamics in several variables related to these map [27], [34], but the existence of a projective system $(\mathbb{C}^{p_n}, F_n)$, with $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ entire for every $n \geq 1$, such that $\varprojlim (\mathbb{C}^{p_n}, F_n) \neq \varnothing$ is still an open problem. It is even unknown whether $\cap_{n\geq 1} F^n(\mathbb{C}^2) \neq \varnothing$ (the power F^n being computed with respect to the composition of maps) for every entire map $F : \mathbb{C}^2 \to \mathbb{C}^2$.

The purpose of the paper is to propose a direct attack of Michael's problem (we will be only interested here in the commutative case, *i.e.* continuity of characters on commutative Fréchet algebras). We discuss in Section 2 the test algebra $\mathcal{U}$. This algebra is the second of the test algebras considered by Clayton in [9], and it is also the commutative version of the algebra of weighted power series in infinitely many variables considered by P. G. Dixon and the author in [16].

The algebra $\mathcal{U}$ is a subalgebra of $\mathbb{C}_{\mathbb{N}}[[X]]$, the algebra of formal power series in infinitely many (commuting) variables $X_1, \ldots, X_n, \ldots$, and it can also be interpreted, in a natural way, as the algebra of entire functions on ℓ^∞ (*see* the precise definitions in Section 2). Let $\mathcal{M}$ be the ideal of $\mathcal{U}$ consisting of elements of $\mathcal{U}$ with zero constant term, so that $\mathcal{M}$ is the kernel of a continuous character on $\mathcal{U}$. Let $\mathcal{J}_n = X_1.\mathcal{U} \ldots + X_n.\mathcal{U}$ $(n \geq 1)$, $\mathcal{J}_\infty = \cup_{n\geq 1}\mathcal{J}_n$. Then, as observed already by Clayton in [9], the existence of a discontinuous character on some commutative Fréchet algebra is equivalent to the existence of a (nonzero) character on the quotient algebra $\mathcal{V} = \mathcal{M}/\mathcal{J}_\infty$. It is also equivalent to the existence of a character on the quotient algebra $\mathcal{V}(f) = \mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$ where $f = \sum_{n=1}^\infty \lambda_n X_n$ and where $(\lambda_n)_{n\geq 1}$ is any element of $\ell^1 \backslash c_{00}$. These quotient algebras are described in Section 2. The ideals $\mathcal{J}_\infty$ and $\mathcal{J}_\infty + (f-1)\mathcal{U}$ are prime, and they are also "of countable type", in the sense of [24], which means that they are the union of a nondecreasing sequence of closed ideals of $\mathcal{U}$. This implies in particular that there exists a natural injection from the set of free ultrafilters on $\mathbb{N}$ into the set of maximal ideals of $\mathcal{V}(f)$ of infinite codimension [24].

The main new idea in Section 2 is the notion of Picard-Borel algebra. We say that a commutative, complex unital algebra A is a Picard-Borel algebra if any family $(x_\lambda)_{\lambda\in\Lambda}$ of invertible elements of A such $x_\lambda \notin \mathbb{C}.x_\mu$ for $\lambda \neq \mu$ is linearly independent over $\mathbb{C}$. Borel's extension [7], [38] of Picard's theorem shows $\mathcal{H}(\mathbb{C})$, the algebra of entire functions over $\mathbb{C}$, is a Picard-Borel algebra and we deduce from this fact that the algebras $\mathcal{U}$, $\mathcal{V}$, $\mathcal{V}(f)$ are also Picard-Borel algebras. A Picard-Borel algebra A is always semisimple, and if $x \in A \backslash \mathbb{C}\, e$ then the cardinal of $\mathbb{C}\backslash\sigma_A(x)$ (where we denote by $\sigma_A(x)$ the spectrum of x in A) is at most 1.

We introduce also in Section 2 the notion of Picard-Borel ideals: an ideal I of a commutative, unital complex algebra is a Picard-Borel ideal if the quotient algebra A/I is a Picard-Borel algebra. A routine application of Zorn's lemma shows that every Picard-Borel ideal is contained in a maximal Picard-Borel ideal. A maximal Picard-Borel ideal which is not the kernel of a character is of course infinite dimensional. Since the ideals $\mathcal{J}_\infty + (f-1)\,\mathbb{C}$ introduced above are Picard-Borel ideals of $\mathcal{U}$, the algebra $\mathcal{V}$ possesses a large quantity of maximal Picard-Borel ideals. The author believes that the study of maximal Picard-Borel ideals of $\mathcal{V}$ should play an important role in future research about Michael's problem (the existence of a discontinuous character on some commutative Fréchet algebra is equivalent to the existence of a maximal Picard-Borel ideal of $\mathcal{V}$ distinct from $\mathcal{M}/\mathcal{J}_\infty$ of codimension 1).

Some properties of the algebra $\mathcal{V}$ might depend on axioms of set theory. If the continuum hypothesis is assumed it follows from a general result of the author [20] that $\mathcal{V}$ possesses an algebra norm. It would be interesting to check whether there exist models of set theory, including the axiom of choice, for which the algebra $\mathcal{V}$ is not normable (for a discussion of the normability of the quotients $\ell^\infty/\mathcal{F}$, where $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$, *see* [14], [28], [51]).

The third section is devoted to a systematic use of the Mittag-Leffler theorem to investigate the structure of the quotient algebras A/I where A is a commutative Fréchet algebra and where I is a dense ideal of A. Recall that the Mittag-Leffler theorem shows that $\varprojlim (E_n, \theta_n) \neq \varnothing$ for every projective system (E_n, θ_n) where E_n is a complete metric space and $\theta_n : E_{n+1} \to E_n$ a continuous map with dense range (more precisely, the first projection of $\varprojlim (E_n, \theta_n)$ is dense in E_1). The Mittag-Leffler theorem was used explicitely by Arens, and by Dixon and the author to establish the results mentioned above [2], [16] (*see* the comments about the theorem of Arens at the end of the paper). The result of Shah [47] is also a consequence of the Mittag-Leffler theorem, *see* remark 3-8. We give a version of the Mittag-Leffler theorem suitable for "projective systems of quotients" (theorem 3-2) and then give various applications. For example, defining $F : B^p \to B^q$ in the natural way when $F : \mathbb{C}^p \to \mathbb{C}^q$ is an entire map and when B is any quotient of a commutative, unital Fréchet algebra A by an ideal of A, we show that $\varprojlim (B^{p_n}, F_n) \neq \varnothing$ when $B = A/I$, when I is a dense ideal of a commutative, unital Fréchet algebra A and when $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ is entire for every $n \geq 1$ (corollary 3-5). This result applies in particular to the quotient algebras $\mathcal{V}(f)$ discussed above, and the fact that $\varprojlim (\mathbb{C}^{p_n}, F_n) \neq \varnothing$ if there exists a discontinuous character on some commutative Fréchet algebra appears as a consequence of corollary 3-5 (*see* proposition 3-6 and corollary 3-7). But these "Mittag-Leffler methods" lead to some other new results about the quotient algebras $B = A/I$ where I is a dense ideal of a commutative Fréchet algebra. For exemple, if we denote by $\pi : B \to \cap_{n\geq 1} x^n.B$ the canonical surjection, then $\pi(x)$ belongs to the radical of $B/\cap_{n\geq 1} x^n.B$ for every $x \in B$ (*see* corollary 3-4 and remark 3-17).

Also if I is a prime, dense ideal of countable type, and if $B = A/I$ is nonunital, then B possesses nonzero rational semigroups (lemma 3-9). In particular the maximal ideal $\mathcal{M}/\mathcal{J}_\infty$ of $\mathcal{V}$ has nonzero rational semigroups. Using the algebraic methods introduced by the author [18] for his construction of discontinuous homomorphisms of $\mathcal{C}(K)$, we show that the quotient algebra $\mathbb{C}_\mathbb{N}[[X]]/\mathcal{J}_{\mathbb{N},\infty}$ contains a "big" algebra of formal power series in $\aleph_1$ variables in which every elements possesses roots of all orders (corollary 3-16). The structure of this very natural (in the author's opinion) quotient algebra certainly deserves more investigations.

In section 4 we investigate the multiplicative structure of the set $S(B)$ of noninvertible elements of a quotient algebra $B = A/I$, where A is a commutative, unital Fréchet algebra and where I is a dense, prime ideal of A of countable type. The author showed in a previous paper [25] that $S(B)$ is a universal multiplicative monoid, in the sense that, if the continuum hypothesis is assumed, every cancellative, non unital monoid without torsion can be embedded in $S(B)$. Using some technical results from [25] we establish in Section 4 a deeper result, via the "lifting lemma" (lemma 4-8): if b is any nonzero element of $S(B)$, then $\cap_{n\geq 1} b^n.S(B)$ is a universal multiplicative monoid. This result applies in particular to the quotient algebras $\mathcal{V}(f)$ introduced above. This multiplicative property of the algebras $\mathcal{V}(f)$ shows a sharp contrast between $\mathcal{V}(f)$ and $\mathcal{U}$, since $\cap_{n\geq 1} g^n.\mathcal{U} = \{0\}$ for every noninvertible element g of $\mathcal{U}$ (*see* remark 2-21).

The reader interested in Kaplansky's conjecture ([12], [19]) noticed some analogy between the above results and the fact that $\cap_{n\geq 1} a^n.\mathcal{R}$ is universal for every nonnilpotent element a of a commutative, radical Banach algebra $\mathcal{R}$ such that $a^n \in [a^{n+1}.\mathcal{R}]^-$ for some $n \geq 1$ ([21], [23]). In fact, the methods of Section 4 lead to a major simplification of the most technical part of the author's construction of discontinuous homomorphisms of $\mathcal{C}(K)$ ([19], [21], [23], [55]). This new approach is outlined in remark 4-11. The fact that $(\mathcal{E}/\mathcal{R}, +)$ embeds in $\cap_{n\geq 1} a^n.\mathcal{R}$ where $\mathcal{E}$ is the set of sequences of positive integers dominated by an arbitrary given sequence converging to infinity, and where $(u_n)_{n\geq 1}\, \mathcal{R}\, (v_n)_{n\geq 1}$ when u_n agrees eventually with v_n, could open the gate to the construction of new models of set theory where discontinuous homomorphisms of $\mathcal{C}(K)$ exist and where $2^{\aleph_0} > \aleph_1$ (the first models of this type were recently constructed by Woodin [51] and by Frankiewicz and Zbierski [28]. In these models, $2^{\aleph_0} = \aleph_2$).

In Section 5 we "test the test for the test algebra $\mathcal{U}$". The fact that the existence of a discontinuous character on some commutative unital Fréchet algebra implies the existence of a discontinuous character on $\mathcal{U}$, and more precisely the existence of a character χ on $\mathcal{V}$ such that $\chi(\sum_{n=1}^{\infty} \lambda_n X_n + \mathcal{J}_\infty) = \mu$, where $(\lambda_n)_{n\geq 1}$ in any element of $\ell^1\backslash c_{00}$ and μ any complex number, is based on the following simple property (*see* the proof of theorem 2-7): let I be a dense ideal of a commutative, unital Fréchet algebra A, and let I^∞ be the set of all bounded sequences of elements of I. Then if $f \in \ell^1\backslash c_{00}$, the map $a \to f(a)$ is a surjective map from I^∞ onto A (for a precise definition of $g(a)$ for $g \in \mathcal{U}$, *see* Section 2; all test algebras are based on a functional calculus operating on bounded sequences of elements of Fréchet algebras). We show in Section 5 that the map $a \to f(a)$ is also necessarily onto when $f = \sum_{n=1}^{\infty} \lambda_n X_n^p$, with $(\lambda_n)_{n\geq 1} \in \ell^1\backslash c_{00}$ and where p is any positive integer. This nontrivial fact is based on an elementary formula (formula 5-2) which shows that if A is any unital complex algebra and p any positive integer then every $x \in A$ can be written in the form $x = \sum_{j=1}^{p} x_i^p$, where $x_i \in A\,(i \leq p)$. It is then possible, given a dense ideal I in a commutative, unital Fréchet algebra A and an arbitrary element x of A, to find a sequence $(x_{1,m}, \ldots, x_{p,m})_{m\geq 1}$ of elements of I^p such that $\sum_{i=1}^{p} x_{i,m}^p \xrightarrow[m\to\infty]{} x$ with some control on $\|x_{i,m}\|_n$, and the result follows. We then produce "recalcitrant" elements of $\mathcal{M}$ (the terminology is borrowed from Thomas's proof of the Singer-Wermer conjecture [49]), *i.e.* elements g of $\mathcal{M}\backslash\mathcal{I}_\infty$ such that $g(a) \notin \operatorname{Inv} A$ for every bounded sequence of noninvertible elements of a Picard-Borel Fréchet algebra A. The basic idea of the construction is that the function z cannot be written as the sum of the cubes of two elements of $\mathcal{H}(\mathbb{C})$, and that if $u, v \in \mathcal{H}(\mathbb{C})$, and if $u^3 + v^3$ is invertible, then the family (u, v) has rank one, so that u and v are either invertible or identically equal to zero. These properties follow from Picard's theorem and have been known since the last century. The last property extends easily to Picard-Borel algebras. By using some version of normal families for the algebra $\mathcal{U}$, we construct a sequence (f_n) of elements of $\mathcal{M}$, where f_n depends only on the variables $(X_m)_{m\geq n}$, such that $f_n = \varepsilon_n.X_n^3 + f_{n+1}^3\,(n \geq 1)$, where $(\varepsilon_n)_{n\geq 1}$ is a suitable sequence of positive reals. Then each f_n is recalcitrant (theorem 5-9). We also observe that if f is a recalcitrant element of $\mathcal{M}$ and if $u = (u_n)_{n\geq 1}$ is any bounded sequence of elements of $\mathcal{U}$ such that $u_n + \mathcal{I}_\infty$ is noninvertible in $\mathcal{V}$ for each $n \geq 1$, then $f(u) + \mathcal{I}_\infty$ is noninvertible in $\mathcal{V}$ (a similar property holds for the quotient algebras $\mathcal{V}(f)$. It would certainly be interesting to get more information about the class of recalcitrant elements of $\mathcal{M}$. Also it would be interesting to check whether there exist elements $g \in \mathcal{M} \backslash \mathcal{I}_\infty$ such that the ideal of A generated by $g(I^\infty)$ is a proper ideal of A for some dense ideal I of some commutative, unital Fréchet algebra A (if g is such an element, there would be some hope to prove that $g + \mathcal{J}_\infty$ belongs to the intersection of the kernels of the characters of $\mathcal{V}$).

In Section 6 we discuss joint spectra. We define the joint spectrum $\sigma_A(M)$ (often denoted by $\sigma(M)$) of a subset M of a commutative, unital complex algebra A in the usual way. We first observe that, in general, $\sigma(M)$ is not given by the restrictions to M of elements of $\sigma(N)$ for $M \subset N$ (this situation occurs if and only if all maximal ideals of A are given by the kernels of the characters of A). We then discuss the countable (resp. finite, resp. p-extension) property for joint spectra. These properties are equivalent to the fact that if I is an ideal of A which is countably generated (resp. finitely generated, resp. generated by at most p elements of A) then the spectrum of $x+I$ in A/I is nonempty for every $x \in A$.

If A has the countable extension property, and if the continuum hypothesis is assumed, then A possesses a character (proposition 6-2). Unfortunately, if the quotient algebra A/I has the countable extension property for some ideal I of a commutative, unital Fréchet algebra A, then the spectrum of every element of A/I is a (nonempty) compact subset of $\mathbb{C}$ (proposition 6-3). So the algebras $\mathcal{V}$ and $\mathcal{V}(f)$ certainly do not possess the countable extension property. We then discuss a question which seems crucial in order to make progress on Michael's problem: does the algebra $\mathcal{V}$ have the finite extension property? If the answer was no, this would give a finite family of elements of $\mathcal{V}$ the joint spectrum of which is not given by characters of $\mathcal{V}$, a big step which could lead to a successful strategy to give a positive answer to Michael's problem. If the answer was yes, there would still be a long way to go to build a nontrivial character on $\mathcal{V}$ ($\mathcal{V}$ indeed does not have the countable extension property), but this would show that $\cap_{p\geq 1}(F_1 \ldots \circ F_n)(\mathbb{C}^p) \neq \varnothing$ for every $p \geq 2$ and every sequence $(F_n)_{n\geq 1}$ of entire, one-to-one maps from $\mathbb{C}^p$ into itself (theorem 6-6). Some sophisticated use of the theory of analytic functions of several complex variables is probably needed to make progress on this question.

The author hopes that the present work will encourage some people to invest some time and energy in order to make progress on the old standing question of continuity of characters on Fréchet algebras.

2. A test algebra for Michael's problem

First, we introduce some notations and recall standard facts about the algebra of formal power series in infinitely many variables. Let Δ be the set of all sequences $\alpha = (\alpha_n)_{n\geq 1}$ of elements of $\mathbb{Z}$ such that $\operatorname{Supp}\alpha = \{n \geq 1 | \alpha_n \neq 0\}$ is finite. Let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1}$ be two distinct elements of Δ, and let $m \geq 1$ be the largest integer such that $\alpha_m \neq \beta_m$. We define an order relation on Δ by setting $\alpha < \beta$ if $\alpha_m < \beta_m$. Clearly, $(\Delta, \leq)$ is a linearly ordered group.

Now set $S = \{\alpha = (\alpha_n)_{n\geq 1} \in \Delta | \alpha_n \geq 0\,(n \geq 1)\}$, and, for $k \geq 1$, $S_k = \{\alpha \in S | \operatorname{Supp} \alpha \subset \{1, \ldots, k\}\}$. Then $S = \cup_{n\geq 1}^{\infty} S_k$, and $(S_k, \leq)$ is isomorphic to $(\mathbb{Z}^+)^k$ equipped with reverse lexicographic order. If $\alpha \in S_k$, $\beta \leq \alpha$ then $\beta \in S_k$, and so $(S, \leq)$ is well-ordered.

We denote by $\mathbb{C}_{\mathbb{N}}[[X]]$ the algebra of formal power series in infinitely many variables: $\mathbb{C}_{\mathbb{N}}[[X]]$ is the linear space $\mathbb{C}^S$, equipped with the product topology and with the usual convolution product $(f_\alpha)(g_\alpha) = (\sum_{\beta+\gamma=\alpha} f_\beta\, g_\gamma)$.

For $\alpha = (\alpha_n)_{n\geq 1} \in S$ set $|\alpha| = \sum_{n=1}^{\infty} \alpha_n$, so that $|\alpha+\beta| = |\alpha|+|\beta|$ $(\alpha, \beta \in S)$, and for $f = (f_\alpha)_{\alpha\in S} \in \mathbb{C}_{\mathbb{N}}[[X]]$, $n \geq 1$, set $p_n(f) = \sum\limits_{\substack{\alpha\in S_n \\ |\alpha|\leq n}} |f_\alpha|$. The given topology on $\mathbb{C}_{\mathbb{N}}[[X]]$ is the topology defined by the family $(p_n)_{n\geq 1}$, and so $\mathbb{C}_{\mathbb{N}}[[X]]$ is a Fréchet algebra.

For α, $\beta \in S$ denote by $\delta_{\alpha,\beta}$ the usual Kronecker symbol. Set $X^\beta = (\delta_{\alpha,\beta})_{\alpha\in S}$. Clearly, $X^{\beta+\gamma} = X^\beta . X^\gamma$ $(\beta, \gamma \in S)$, the family $(f_\alpha X^\alpha)_{\alpha\in S}$ is absolutely summable in $\mathbb{C}_{\mathbb{N}}[[X]]$ and $f = \sum_{\alpha\in S} f_\alpha X^\alpha$ for every $f = (f_\alpha)_{\alpha\in S} \in \mathbb{C}_{\mathbb{N}}[[X]]$. If we denote by $0 \in S$ the null sequence then $\mathbf{1} = X^0$ is the unit element of $\mathbb{C}_{\mathbb{N}}[[X]]$. The character $\chi_o = \sum_{\alpha\in S} f_\alpha X^\alpha \to f_0$ is the unique continuous character of $\mathbb{C}_{\mathbb{N}}[[X]]$, and so $\operatorname{Sp}(f) = \{\chi_o(f)\}$ for every $f \in \mathbb{C}_{\mathbb{N}}[[X]]$. Hence, as is well known, $\mathbb{C}_{\mathbb{N}}[[X]]$ is a local ring and its unique maximal ideal is $\mathcal{M}_{\mathbb{N}} = \operatorname{Ker} \chi_o$. For $f = \sum_{\alpha\in S} f_\alpha X^\alpha \in \mathbb{C}_{\mathbb{N}}[[X]]$, set $\operatorname{Supp} f = \{\alpha \in S | f_\alpha \neq 0\}$ and for $f \neq 0$ set $v(f) = \inf(\operatorname{Supp} f)$. Let f, g be two nonzero elements of $\mathbb{C}_{\mathbb{N}}[[X]]$ and let $\alpha = v(f)$, $\beta = v(g)$. Then $\gamma + \delta > \alpha + \beta$ for $\gamma \in \operatorname{Supp} f$, $\delta \in \operatorname{Supp} g$, $(\gamma, \delta) \neq (\alpha, \beta)$ and $fg \neq 0$, $v(fg) = \alpha + \beta = v(f) + v(g)$. In particular, $\mathbb{C}_{\mathbb{N}}[[X]]$ is an integral domain.

Set $\rho_m = (\delta_{m,n})_{n\geq 1}$, $X_m = X^{\rho_m}$. If $\alpha \in S_k$ then $X^\alpha = X_1^{\alpha_k} \ldots X_k^{\alpha_k}$. Now identify S_k with $(\mathbb{Z}^+)^k$. Then the algebra $\{f \in \mathbb{C}_{\mathbb{N}}[[X]] | \operatorname{Supp} f \subset S_k\}$ becomes $\mathbb{C}_k[[X]]$, the usual algebra of formal power series in k variables. The map $\pi_k : \sum_{\alpha\in S} f_\alpha X^\alpha \to \sum_{\alpha\in S_k} f_\alpha X^\alpha$ is a continuous homomorphism from $\mathbb{C}_{\mathbb{N}}[[X]]$ onto $\mathbb{C}_k[[X]]$, and this is also true for $\pi_{k,n} : \mathbb{C}_n[[X]] \to \mathbb{C}_k[[X]]$ defined in a similar way for $n \geq k$. It is easy to see that $\theta : f \to (\pi_k(f))_{k\geq 1}$ is a topological isomorphism from the algebra $\mathbb{C}_{\mathbb{N}}[[X]]$ onto the projective limit $\varprojlim(\mathbb{C}_k[[X]], \pi_{k,k+1}) = \{f_k)_{k\geq 1} \in \prod_{k\geq 1} \mathbb{C}_k[[X]] | f_k = \pi_{k,k+1}(f_{k+1})\,(k \geq 1)\}$.

The structure of closed ideals of $\mathbb{C}_k[[X]]$ is well-known (*see* for example [52, p. 260]): every ideal of $\mathbb{C}_k[[X]]$ is closed and finitely generated. This is not the case for $\mathbb{C}_{\mathbb{N}}[[X]]$: if we set $\mathcal{J}_{\mathbb{N},n} = X_1.\mathbb{C}_{\mathbb{N}}[[X]] \ldots + X_n.\mathbb{C}_{\mathbb{N}}[[X]]\,(n \geq 1)$, $\mathcal{J}_{\mathbb{N},\infty} = \cup_{n\geq 1}\mathcal{J}_{\mathbb{N},n}$ then $\mathcal{J}_{\mathbb{N},\infty}$ is dense in $\mathcal{M}_{\mathbb{N}}$, but $\mathcal{J}_{\mathbb{N},\infty} \subsetneq \mathcal{M}_{\mathbb{N}}$. Also if $g_1, \ldots, g_k \in \mathcal{J}_{\mathbb{N},\infty}$ then $g_1.\mathbb{C}_{\mathbb{N}}[[X]] \ldots + g_k.\mathbb{C}_{\mathbb{N}}[[X]] \in \mathcal{J}_{\mathbb{N},n} \subsetneq \mathcal{J}_{\mathbb{N},n+1}$ for some $n \geq 1$, and so $\mathcal{J}_{\mathbb{N},\infty}$ is not finitely generated. Now let $f \neq 0$ be an element of $\mathbb{C}_{\mathbb{N}}[[X]]$ and let $g \in [f.\mathbb{C}_{\mathbb{N}}[[X]]]^-$. These exists a sequence $(h_p)_{p\geq 1}$ in $\mathbb{C}_{\mathbb{N}}[[X]]$ such that $g = \lim_{p\to\infty} f.h_p$, and so $\pi_k(g) = \lim_{p\to\infty} \pi_k(f)\,\pi_k(h_p)$. Since all ideals are closed in $\mathbb{C}_k[[X]]$ there exists $u_k \in \mathbb{C}_k[[X]]$ such that $\pi_k(g) = \pi_k(f).u_k$. Let $r \geq 1$ such that $\pi_r(f) \neq 0$, so that $\pi_k(f) \neq 0$ for $k \geq r$. Then $\pi_{k,k+1}(u_{k+1}).\pi_k(f) = \pi_{k,k+1}[\pi_{k+1}(f).u_{k+1}] = \pi_{k,k+1}[\pi_{k+1}(g)] = \pi_k(g) = u_k.\pi_k(f)$ and so $u_k = \pi_{k,k+1}(u_{k+1})\,(k \geq r)$.

Set $v_k = u_k$ for $k \geq r$, $v_k = \pi_{k,r}(u_r)\,(k < r)$. Then $(v_k)_{k\geq 1} \in \varprojlim (\mathbb{C}_{\mathbb{N}}[[X]], \pi_{k,k+1})$, and so there exists $v \in \mathbb{C}_{\mathbb{N}}[[X]]$ such that $\pi_k(v) = v_k\,(k \geq 1)$. Since $\pi_k(fv) = \pi_k(f)\,v_k = \pi_k(g)$ for each k, $g = fv$ and we see that principal ideals of $\mathbb{C}_{\mathbb{N}}[[X]]$ are closed.

The author does not know whether finitely generated ideals of $\mathbb{C}_{\mathbb{N}}[[X]]$ are closed in general. Nevertheless the ideals $\mathcal{J}_{\mathbb{N},k}$ are indeed closed, since $\mathcal{J}_{\mathbb{N},k} = \{f \in \mathcal{M}_{\mathbb{N}} | \operatorname{Supp} f \cap V_k = \varnothing\}$, where $V_k = \{\alpha \in S | \operatorname{Supp}\alpha \cap \{1, \ldots, k\} = \varnothing\}$. Clearly, if $\alpha + \beta \in V_k$, then $\alpha \in V_k$ and $\beta \in V_k$. It follows from this observation that $\theta_k : \sum_{\alpha\in S} f_\alpha X^\alpha \to \sum_{\alpha\in V_k} f_\alpha X^\alpha$ is a homomorphism. We have $\operatorname{Ker}\theta_k = \mathcal{J}_{\mathbb{N},k}$, $\operatorname{Im}(\theta_k) = \{f \in \mathbb{C}_{\mathbb{N}}[[X]] | \operatorname{Supp} f \subset V_k\}$. For $\alpha = (\alpha_n)_{n\geq 1} \in S$, set $\tau_k(\alpha) = (\beta_n)_{n\geq 1}$ where $\beta_1 \ldots = \beta_k = 0$ and where $\beta_n = \alpha_{n-k}$ for $n \geq k+1$. The map $\sum_{\alpha\in S} f_\alpha X^\alpha \to \sum_{\alpha\in S} f_\alpha X^{\tau_k(\alpha)}$ is an isomorphism from $\mathbb{C}_{\mathbb{N}}[[X]]$ onto $\operatorname{Im}(\theta_k)$. Hence $\mathbb{C}_{\mathbb{N}}[[X]]/\mathcal{J}_{\mathbb{N},k}$ is isomorphic to $\mathbb{C}_{\mathbb{N}}[[X]]$, and $\mathcal{J}_{\mathbb{N},k}$ is a prime ideal for each $k \geq 1$. So $\mathcal{J}_{\mathbb{N},\infty} = \cup_{k\geq 1}\mathcal{J}_{\mathbb{N},k}$ is also prime.

The following notion was introduced in [24].

DEFINITION 2.1. – *An ideal I in a commutative Fréchet algebra A is of countable type if there exists a nondecreasing sequence $(I_n)_{n\geq 1}$ of closed ideals of A such that $I = \cup_{n\geq 1} I_n$.*

It follows then from the above considerations that we have the following result.

PROPOSITION 2.2. – *The ideal $\mathcal{J}_{\mathbb{N},\infty}$ is a prime ideal of countable type in $\mathbb{C}_{\mathbb{N}}[[X]]$, and $\mathcal{J}_{\mathbb{N},\infty}$ is dense in $\mathcal{M}_{\mathbb{N}}$.*

The structure of the quotient algebra $\mathbb{C}_{\mathbb{N}}[[X]]/\mathcal{J}_{\mathbb{N},\infty}$, which is of course a local ring, will be investigated later. We now turn to a test algebra for Michael's problem. This algebra is isomorphic to the second test algebra introduced by Clayton [9]. It is exactly the commutative algebra discussed by P. G. Dixon and the author in [16].

DEFINITION 2.3. – Set $\mathcal{U} = \{f = \sum_{\alpha\in S} f_\alpha X^\alpha \in \mathbb{C}_{\mathbb{N}}[[X]] |\, \|f\|_n = |\sum_{\alpha\in S} f_\alpha| n^{|\alpha|} < +\infty\,(n \geq 1)\}$, *$\mathcal{J}_\infty = \cup_{n\geq 1}\mathcal{J}_n$, where $\mathcal{J}_n = X_1\mathcal{U} \ldots + X_n\mathcal{U}$, and $\mathcal{M} = \{f = \sum_{\alpha\in S} f_\alpha X^\alpha \in U | f_0 = 0\}$.*

It follows from the definition of $\mathcal{U}$ that $(\mathcal{U}, (\|\cdot\|_n)_{n\geq 1})$ is a commutative, unital Fréchet algebra and that the injection $\mathcal{U} \to \mathbb{C}_{\mathbb{N}}[[X]]$ is continuous. Also if $X_i.f \in \mathcal{U}$, with $f \in \mathbb{C}_{\mathbb{N}}[[X]]$, then $f \in \mathcal{U}$, and it follows easily from this observation that $\mathcal{J}_n = \mathcal{J}_{\mathbb{N},n} \cap \mathcal{U}$. So each $\mathcal{J}_n$ is prime and closed, and $\mathcal{J}_\infty = \mathcal{J}_{\mathbb{N},\infty} \cap \mathcal{U}$ is a prime ideal of countable type in $\mathcal{U}$. This ideal is dense in $\mathcal{M}$ and distinct from $\mathcal{M}$, since $\sum_{n=1}^\infty \frac{X_n}{n^2} \in \mathcal{M}\backslash\mathcal{J}_\infty$.

Recall that a sequence $(a_n)_{n\geq 1}$ in a commutative Fréchet algebra $(A, (\|\cdot\|_m)_{m\geq 1})$ is said to be bounded if $\sup_{n\geq 1} \|a_n\|_m < +\infty$ for every $m \geq 1$.

If $a = (a_n)_{n\geq 1}$ is a bounded sequence in a commutative unital Fréchet algebra and if $\alpha = (\alpha_n)_{n\geq 1} \in S_k$ set $a^\alpha = a_1^{\alpha_1} \ldots a_k^{\alpha_k}$ (with the convention $x^0 = 1$ for every $x \in A$). This definition does not depend on the choice of k and if we define the product of bounded sequences in the obvious way we see that the map $(\alpha, a) \to a^\alpha$ obeys the usual rules for exponents. If $f = \sum_{\alpha\in S} f_\alpha X^\alpha \in \mathcal{U}$ then the family $(f_\alpha a^\alpha)_{\alpha\in S}$ is absolutely summable, hence summable, in A. The following result follows from the definitions.

PROPOSITION 2.4. – *Let A be a commutative, unital Fréchet algebra and for $V \subset A$ let V^∞ be the set of all bounded sequences of elements of V. For $f = \sum_{\alpha \in S} f_\alpha X^\alpha \in \mathcal{U}$, $a = (a_n)_{n \geq 1} \in A^\infty$ set $f(a) = \sum_{\alpha \in S} f_\alpha a^\alpha$. Then the map $\theta_a : f \to f(a)$ is continuous, and $\theta_a(X_n) = a_n (n \geq 1)$.*

For $z = (z_n)_{n \geq 1} \in \ell^\infty$ define $f(z)$ as above. The following easy proposition can be found in [9].

PROPOSITION 2.5. – *Set $\chi_z(f) = f(z)$ $(z \in \ell^\infty,\ f \in \mathcal{U})$. The map $z \to \chi_z$ is a bijection from ℓ^∞ onto the set $\hat{\mathcal{U}}$ of continuous characters of $\mathcal{U}$, which is a homeomorphism between the w^*-topology on ℓ^∞ and the Gelfand topology when restricted to bounded sets. Also, $\mathcal{U}$ is semisimple.*

The nontrivial fact in proposition 2.5 is the semi-simplicity of $\mathcal{U}$. An easy way to prove it is to remark that $\mathcal{U}$ is a subalgebra of the group algebra $\ell^1(\Delta)$. The dual group of Δ (we consider Δ as a discrete group) is the compact group $\Gamma^{\mathbb{N}}$, where we denote by Γ the unit circle. Hence if $f(z) = 0$ for every $z \in \Gamma^{\mathbb{N}}$, with $f \in \mathcal{U}$, then the Fourier transform of f vanishes, and so $f = 0$.

If A is a commutative, unital complex algebra and if $a = (a_\lambda)_{\lambda \in \Lambda}$ is a family of elements of A we will denote by $\sum_{\lambda \in \Lambda} a_\lambda . A$ the set of all finite sums $a_{\lambda_1}.u_1 \ldots + a_{\lambda_k}.u_k$ where $\lambda_1, \ldots, \lambda_k \in \Lambda$, $u_1 \ldots u_k \in A$. The joint spectrum of the family $(a_\lambda)_{\lambda \in \Lambda}$ is then the set $\sigma_A(a) = \{(z_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda | \sum_{\lambda \in \Lambda} (a_\lambda - z_\lambda.1) A \subsetneq A\}$. We will write $\sigma(a)$ instead of $\sigma_A(A)$ when there is no risk of confusion.

For $L \subset \mathcal{U}$ we set $Z(L) = \{z \in \ell^\infty | f(z) = 0 (f \in L)\}$. If $g = (g_1, \ldots, g_k) \in \mathcal{U}^k$, $z \in \ell^\infty$, we set $g(z) = (g_1(z), \ldots, g_k(z))$.

COROLLARY 2.6. – *For $1 \leq m \leq \infty$ let $\pi_m : \mathcal{U} \to \mathcal{U}/\mathcal{J}_m$ be the canonical map, and for $m \in \mathbb{N}$ set $\ell_m^\infty = \{z = (z_n)_{n \geq 1} \in \ell^\infty | z_1 \ldots = z_m = 0\}$. Then for every $g = (g_1, \ldots, g_k) \in \mathcal{U}^k$ we have $\sigma(\pi_\infty(g_1), \ldots, \pi_\infty(g_k)) = \cap_{m \geq 1} g(\ell_m^\infty)\,(k \geq 1)$.*

Proof. – Set $\pi_m(g) = (\pi_m(g_1), \ldots, \pi_m(g_k))\,(1 \leq m \leq \infty)$. Since $\mathcal{J}_\infty = \cup_{m \geq 1} \mathcal{J}_m$, we have $\sigma(\pi_\infty(g)) = \cap_{m \geq 1} \sigma(\pi_m(g))$. A basic result of Arens [2] shows that if $a = (a_1, \ldots, a_k)$ is a finite family in a commutative, unital Fréchet algebra A, we have $\sigma_A(a) = \{\chi(a_1), \ldots, \chi(a_k)\}_{\chi \in \hat{A}}$, where we denote by $\hat{A}$ the set of continuous characters on A. We have $Z(\mathcal{J}_m) = Z(\{X_1\}) \ldots \cap Z(\{X_m\}) = \ell_m^\infty$, and so continuous characters on $\mathcal{U}/\mathcal{J}_m$ have the form $\pi_m(h) \to h(z)$ for some $z \in \ell_m^\infty$. Hence $\sigma(\pi_m(g)) = g(\ell_m^\infty)$ $(m \geq 1)$ and the results follows.

The algebra $\mathcal{U}$ is a test algebra for Michael's problem, in the sense that if there exists a discontinuous character on some commutative Fréchet algebra, then there exists a discontinuous character on $\mathcal{U}$. A very short proof on this fact can be found in [16]. Clayton [9] proved more precisely that if there exists a discontinuous character on some commutative Fréchet algebra, there exists a nonzero character on the quotient $\mathcal{M}/\mathcal{J}_\infty$.

We summarize these facts, in a slightly more precise form, in the following statement (we denote by c_{00} the set of all $x = (x_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}}$ such that $\operatorname{Supp} x = \{n \geq 1 | x_n \neq 0\}$ is finite).

THEOREM 2.7. – *Let* $\lambda = (\lambda_n)_{n\geq 1} \in \ell^1 \backslash c_{00}$, *and let* $f = \sum_{n=1}^{\infty} \lambda_n X_n$. *The following properties are equivalent*

(1) *There exists a discontinuous character on some commutative Fréchet algebra* A.

(2) *There exists a discontinuous character on* $\mathcal{U}$.

(3) *There exists a (nonzero) character on the quotient algebra* $\mathcal{M}/\mathcal{J}_\infty$.

(4) *There exists a character on the quotient algebra* $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$.

Proof. – The result is essentially contained in [9], but we give a proof for the sake of completeness. Since $\mathcal{J}_\infty$ is dense in $\mathcal{M}$, and since discontinuous characters on A extend to $A \oplus \mathbb{C}\, e$, the algebra obtained by adjoining a unit to A if A is nonunital, it suffices to show that (1) implies (4).

Assume that (1) holds and let $(n_p)_{n\geq 1}$ be a strictly increasing sequence of integers such that $\lambda_{n_p} \neq 0\,(p \geq 1)$. Set $\mu_p = \lambda_{n_p}$, and let χ be a discontinuous character on A. Set $I = \operatorname{Ker}\chi$. Since I is dense in A, we can construct by induction a sequence $(e_p)_{p\geq 1}$ of elements of I, which satisfies the condition:

$$\|1 - \textstyle\sum_{n=1}^{p} e_n\|_p \leq \dfrac{\inf(|\mu_p|, |\mu_{p+1}|)}{2}\ (p \geq 1).$$

For $p \geq 1$ we have $\|1 - \sum_{n=1}^{p} e_n\|_m \leq \frac{|\mu_p|}{2}$, and so the series $\sum_{n=1}^{\infty} e_n$ converges in A, and $\sum_{n=1}^{\infty} e_n = 1$. Also $\|e_p\|_{p-1} \leq \|1 - \sum_{n=1}^{p-1} e_n\|_{p-1} + \|1 - \sum_{n=1}^{p} e_n\|_p \leq |\mu_p|\,(p \geq 2)$. So if we set $a_p = \mu_p^{-1}.e_p\,(p \geq 1)$, we see that $\|a_p\|_m \leq \|a_p\|_{p-1} \leq 1$ for $p \geq m+1$, and the sequence $(a_p)_{p\geq 1}$ is bounded in A. Now set $b_n = 0$ if $n \neq n_p$ for every $p \geq 1$, and $b_{n_p} = a_p\,(p \geq 1)$. Let $b = (b_n)_{n\geq 1}$. Set $\varphi = \chi \circ \theta_b$, where we denote as above by θ_b the map $g \to g(b)\,(g \in \mathcal{U})$. Then φ is a character on $\mathcal{U}$, $\varphi(X_n) = \chi(b_n) = 0\ (n \geq 1)$ and so $\mathcal{J}_\infty \subset \operatorname{Ker}\varphi$. We have $\varphi(f) = \chi(\sum_{n=1}^{\infty} \lambda_n b_n) = \chi(\sum_{p=1}^{\infty} e_p) = \chi(1) = 1$. Hence $\mathcal{J}_\infty + (1-f)\mathcal{U} \subset \operatorname{Ker}\chi$, and the quotient algebra $\mathcal{U}/\mathcal{J}_\infty + (1-f)\mathcal{U}$ possesses a character. This concludes the proof of the theorem.

We now wish to investigate the algebraic properties of the quotient algebras $\mathcal{U}/\mathcal{J}_\infty$ and $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$, the notations being as in theorem 2.7. Clearly, the algebra $\mathcal{U}$ is in some sense an algebra of entire functions on ℓ^∞. The following easy proposition gives a connection between $\mathcal{U}$ and analytic functions of several complex variables.

PROPOSITION 2.8. – *Let* $\Omega \subset \mathbb{C}^m$ *be open, and let*

$$g : \Omega \to \ell^\infty$$
$$u \to (g_n(u))_{n\geq 1}$$

be a map which satisfies the following conditions

(i) $g_n : \Omega \to \mathbb{C}$ *is analytic* $(n \geq 1)$.

(ii) *For every compact set* $K \subset \Omega$, *there exists* $M_K > 0$ *such that* $\sup_{u\in K} \|g(u)\|_\infty \leq M_K$.

Then $f \circ g : \Omega \to \mathbb{C}$

$u \to f(g(u))$ *is analytic for every* $f \in \mathcal{U}$.

Proof. – Let $f = \sum_{\alpha \in S}^{\infty} f_\alpha X^\alpha \in \mathcal{U}$. For $p \geq 1$ set $f_p = \sum_{\alpha \in S_p} f_\alpha X^\alpha$. Then $f_p \circ g$ is analytic for every $p \geq 1$, and $(f_p \circ g)(u) \to (f \circ g)(u)$, uniformly on compact subsets of Ω. So $f \circ g$ is analytic.

It is easy to deduce from corollary 2.6 and proposition 2.8 that the cardinal of $\mathbb{C} \backslash \sigma(f)$ (resp. $\mathbb{C} \backslash \sigma(\pi_\infty(f))$ is at most 1 if $f \notin \mathbb{C}.1$ (resp. $\pi_\infty(f) \notin \mathbb{C}.1$), by using Picard's theorem. In order to obtain some more precise information, we introduce the following notion:

DEFINITION 2.9 – Let A be a commutative, unital complex algebra and denote by $\operatorname{Inv} A$ the group of invertible elements of A. We shall say that A is a Picard-Borel algebra if every family $(x_\lambda)_{\lambda \in \Lambda}$ of elements of $\operatorname{Inv} A$ such that $x_\lambda \notin \mathbb{C}\, x_\mu$ for $\lambda,\ \mu \in \Lambda,\ \lambda \neq \mu$ is linearly independent over $\mathbb{C}$.

The following observation essentially goes back to Borel [7].

PROPOSITION 2.10. – *Let A be a Picard-Borel algebra, and let $x \in A$. If $x \notin \mathbb{C}.1$, then the cardinal of $\mathbb{C} \backslash \sigma_A(x)$ is at most 1. In particular, A is semisimple.*

Proof. – Let $x \in A$ be such that $\{a,\ b\} \cap \sigma_A(x) = \varnothing$, with $a \neq b$. We have

$$(x - a.1) + (b.1 - x) + (a - b).1 = 0.$$

Since $a - b \neq 0$, the family $(x - a.1,\ x - b.1, 1)$ is linearly dependent. Hence one of the pairs $(x - a.1,\ x - b.1)$, $(x - a.1,\ 1)$, $(x - b.1,\ 1)$ has rank 1. But this shows that $x \in \mathbb{C}.1$. Now if $x \in \operatorname{Rad}(A)$, the Jacobson radical of A, then certainly $\sigma_A(x) = 0$, and so $x = 0$.

The exponential map on a unital Fréchet algebra A is the usual map $x \to e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We will denote by $\exp A$ the range of the exponential map.

PROPOSITION 2.11. – *The algebra $\mathcal{U}$ is a Picard-Borel algebra, and* $\exp \mathcal{U} = \operatorname{Inv} \mathcal{U}$.

Proof. – Let $f = \sum_{\alpha \in S} f_\alpha X^\alpha \in \operatorname{Inv} \mathcal{U}$. For $t \in [0,\ 1]$ set $\gamma(t) = \frac{1}{f_0} (\sum_{\alpha \in S} |t|^\alpha . f_\alpha X^\alpha)$. Then $\gamma : [0,\ 1] \to \mathcal{U}$ is continuous. Also $\gamma(t)(z) = \frac{1}{f_0} f(tz) \neq 0$ for every $z \in \ell^\infty$, and so $\gamma(t) \in \operatorname{Inv} \mathcal{U}$ for every $t \in [0,\ 1]$. Since $\gamma(0) = 1$, it follows from [15, corollary 2.5] that $\gamma(1) \in \exp \mathcal{U}$, and so $f \in \exp(\mathcal{U})$.

Borel's theorem [7] shows that $\mathcal{H}(\mathbb{C})$, the algebra of entire functions on $\mathbb{C}$, satisfies definition 2-9. It is well-known that $\mathcal{H}(\mathbb{C}^p)$ is also a Picard-Borel algebra for every $p \geq 2$. To see this consider $f_1, \ldots, f_k \in \mathcal{H}(\mathbb{C}^p)$ and assume that $e^{f_1} + \ldots + e^{f_k} = 0$. Then for every $z,\ u \in \mathbb{C}^p$ there exist $i,\ j \in \{1, \ldots, k\}$, with $i < j$, such that the function $\lambda \to f_i(z + \lambda u) - f_j(z + \lambda u)$ is constant on $\mathbb{C}$, by Borel's theorem. Set $df = \frac{df}{\partial z_1} dz_1 \ldots + \frac{\partial f}{\partial z_p} dz_p$ for $f \in \mathcal{H}(\mathbb{C}^p)$. Then $df_i(z)(u) - df_j(z)(u) = 0$, and $\mathbb{C}^{2p} = \bigcup_{i<j} \Omega_{i,j}$ where $\Omega_{i,j} = \{(z,\ u) \in \mathbb{C}^{2p} \mid df_i(z)(u) = df_j(z)(u)\}$. But it follows from standard properties of analytic sets [31, p. 9] that we then have $\mathbb{C}^{2p} = \Omega_{i,j}$ for some pair $(i,\ j)$ with $i \neq j$; so $f_i - f_j$ is constant, and $\mathcal{H}(\mathbb{C}^p)$ is a Picard-Borel algebra.

Now let $g_1, \ldots, g_k \in \mathcal{U}$, with $k \geq 2$, and assume that $g_i - g_j \notin \mathbb{C} \cdot 1$ for $i \neq j$. Then there exists $u_{i,j}$ and $v_{i,j} \in \ell^\infty$ such that

$$g_i(u_{i,j}) - g_j(u_{i,j}) \neq g_i(v_{i,j}) - g_j(v_{i,j}) (i \leq k,\ j \leq k,\ i < j).$$

Set $p = \frac{k(k-1)}{2}$ and for $(\lambda,\ \mu) \in \mathbb{C}^{2p}$ with $\lambda = (\lambda_{i,j})_{i<j\leq k}$, $\mu = (\mu_{i,j})_{i<j\leq k}$, $s \leq k$ set

$$h_s(\lambda,\ \mu) = g_s(\sum_{i<j} (\lambda_{i,j} \cdot u_{i,j} + \mu_{i,j} \cdot v_{i,j})).$$

It follows from Proposition 2-8 that $h_s \in \mathcal{H}(\mathbb{C}^{2p})$, and it follows from the construction of the function h_s that $h_s - h_{s'}$ is not constant for $s \neq s'$. Hence $\sum_{s=1}^{k} e^{h_s} \neq 0$ in $\mathcal{H}(\mathbb{C}^{2p})$, and *a fortiori* $\sum_{s=1}^{k} e^{g_s} \neq 0$ in $\mathcal{U}$. This concludes the proof of the proposition.

We now wish to extend Proposition 2-11 to the quotient algebras $\mathcal{U}/\mathcal{J}_\infty$ and $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$, the notations being as in Theorem 2-7.

Notice that if I is an ideal of a commutative, unital Fréchet algebra, if $a \in A$ and if $h \in I$, then $e^{a+h} - e^a = e^a(e^h - 1) \in I$. So if we denote by $\pi : A \to A/I$ the canonical map, we can define the exponential map on A/I by the formula $e^{\pi(a)} = \pi(e^a)(a \in A)$, even if I is not closed.

DÉFINITION 2.12. – (1) *A Picard-Borel ideal of a commutative, unital complex algebra* A *is an ideal I of A such that the quotient algebra A/I is a Picard-Borel algebra.*

(2) *A logarithmic ideal of a commutative, unital Fréchet algebra A is an ideal I of A such that* $\exp(A/I) = \mathrm{Inv}(A/I)$.

LEMMA 2.13. – *The union of a chain of logarithmic ideals of a commutative, unital Fréchet algebra A is a logarithmic ideal of A.*

Proof. – Let $(I_\lambda)_{\lambda\in\Lambda}$ be a chain of logarithmic ideals of A, and let $I = \bigcup_{\lambda\in\Lambda} I_\lambda$. Denote by π (resp. π_λ) the canonical map $A \to A/I$ (resp. A/I_λ). Let $x \in A$ be such that $\pi(x) \in \mathrm{Inv}(A/I)$. Then there exists $y \in A$ such that $\pi(x) \cdot \pi(y) - 1 \in I$, and so $\pi_\lambda(x) \in \mathrm{Inv}(A/I_\lambda)$ for some $\lambda \in \Lambda$. Hence there exists $u \in A$ such that $\pi_\lambda(x) = e^{\pi_\lambda(u)} = \pi_\lambda(e^u)$, and so $\pi(x) = \pi(e^u) = e^{\pi(u)}$.

LEMMA 2.14. – *The union of a chain of Picard-Borel ideals of a commutative, unital complex algebra A is a Picard-Borel ideal of A.*

Proof. – Let $(I_\lambda)_{\lambda\in\Lambda}$ be a chain of Picard-Borel ideals of A, and set $I = \bigcup_{\lambda\in\Lambda} I_\lambda$. The notations for quotient maps being as above let $(x_t)_{t\in T}$ be a family of elements of A such that $\pi(x_t) \notin \mathbb{C}\cdot\pi(x_{t'})$ for $t \neq t'$. If $\gamma_1\cdot\pi(x_{t_1})\ldots+\gamma_k\cdot\pi(x_{t_k}) = 0$, where $\gamma_1, \ldots, \gamma_k \in \mathbb{C}$, $t_1, \ldots, t_k \in T$, $t_i \neq t_j$ for $i \neq j$, then $\gamma_1 \cdot \pi_\lambda(x_{t_1}) \ldots + \gamma_k \cdot \pi_\lambda(x_{t_k}) = 0$ for some $\lambda \in \Lambda$. Since $\pi(x_{t_i}) \notin \mathbb{C} \cdot \pi(x_{t_j})$ for $i \neq j$, we have *a fortiori* $\pi_\lambda(x_{t_i}) \notin \mathbb{C} \cdot \pi_\lambda(x_{t_j})$ for $i \neq j$, and so $\gamma_1 \ldots = \gamma_k = 0$, since the quotient algebra A/I_λ is a Picard-Borel algebra. Hence the family $(\pi(x_t))_{t\in T}$ is linearly independent, and A/I is a Picard-Borel algebra.

We deduce immediately from Zorn's lemma the following corollary.

COROLLARY 2.15. – *Every Picard-Borel ideal is contained in a maximal Picard-Borel ideal.*

By maximal Picard-Borel ideal, we mean of course a Picard-Borel ideal I in A which is not properly contained in any Picard-Borel ideal of A. If follows immediately from Proposition 2-10 that if J is a maximal ideal of A which is also a Picard-Borel ideal, then J is the kernel of a character of A.

We now turn to a version of Taylor's formula for the algebra $\mathcal{U}$. Unfortunately, the author was not able to find such a statement available in the literature. For $\alpha = (\alpha_n)_{n\geq 1} \in S$, set $\alpha! = \prod_{n=1}^{\infty} (\alpha_n !)$, with the usual convention $0! = 1$.

We will write $\alpha \geq \beta$ when $\alpha_n \geq \beta_n$ for every n, and in this case we set $\alpha - \beta = (\alpha_n - \beta_n)_{n\geq 1}$. If A is a Fréchet algebra, and if $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1}$ are bounded sequences of elements of A, we set $a + b = (a_n + b_n)_{n\geq 1}$.

LEMMA 2.16. – *For $\alpha, \beta \in S$ set*

$$\partial^\alpha (X^\beta) = 0 \qquad if \quad \beta \ngeq \alpha$$

$$\partial^\alpha (X^\beta) = \frac{\beta!}{(\beta-\alpha)!} X^{\beta-\alpha} \qquad if \quad \beta \geq \alpha;$$

Then the family $(f_\beta \, \partial^\alpha (X^\beta))_{\beta\in S}$ is absolutely summable in $\mathcal{U}$ for every $f = \sum_{\beta\in S} f_\beta X^\beta \in \mathcal{U}$ and every $\alpha \in S$. Moreover if we set $\partial^\alpha f = \sum_{\beta\in S} f_\beta \, \partial^\alpha (X^\beta)$ and if A is a commutative, unital Fréchet algebra, then the family $\left(\frac{(\partial^\alpha f)(a)}{\alpha!} b^\alpha\right)_{\alpha\in S}$ is absolutely summable in A for every $f \in \mathcal{U}$ and every $a, b \in A^\infty$, and we have

$$f(a+b) = \sum_{\alpha\in S} \frac{\partial^\alpha f(a)}{\alpha!} b^\alpha.$$

Proof. – For $\alpha, \beta \in S$, $\beta \geq \alpha$, $n \geq 1$ we have

$$\| \partial^\alpha (X^\beta) \|_n \leq n^{|\beta-\alpha|} \, |\beta|^{|\alpha|} \leq n^{|\beta|} \cdot |\beta|^{|\alpha|} \leq K(n, \alpha)(n+1)^{|\beta|}$$

for some $K(n, \alpha) > 0$, and so the family $(f_\beta \, \partial^\alpha (X^\beta))_{\beta\in S}$ is absolutely summable for every $f = \sum_{\beta\in S} f_\beta X^\beta \in \mathcal{U}$.

Now let $f = \sum_{\beta\in S} f_\beta X^\beta \in \mathcal{U}$, and let $a = (a_p)_{p\geq 1}$, $b = (b_p)_{p\geq 1} \in A^\infty$. Let $(\|\cdot\|_n)_{n\geq 1}$ be the sequence of seminorms which defines the topology of A, fix $n \geq 1$ and let $m \geq 1$ be such that $\| a_p \|_n \leq m$, $\| b_p \|_n \leq m$ for every $p \geq 1$.

Now consider the family

$$(u_{\beta,\gamma})_{\beta,\gamma\in S} = \left(\frac{(\beta+\gamma)!}{\beta!\gamma!} f_{\beta+\gamma} \, a^\beta \cdot b^\gamma \right)_{\beta,\gamma\in S}.$$

We have

$$\| u_{\beta,\gamma} \|_n = \left\| \frac{(\beta+\gamma)!}{\beta!\gamma!} f_{\beta+\gamma} \, a^\beta \cdot b^\gamma) \right\|_n \leq \frac{(\beta+\gamma)!}{\beta!\gamma!} \, | f_{\beta+\gamma} | \, m^{|\beta+\gamma|}.$$

For $\alpha \in S$, Supp $\alpha \subset \{1, \ldots, k\}$ we have

$$\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} = (1+1)^{\alpha_1} \ldots (1+1)^{\alpha_k} = 2^{|\alpha|}$$

and so if $L \subset S \times S$ is finite we have

$$\sum_{(\beta,\gamma)=L} \| u_{\beta,\gamma} \|_n \leq \sum_{\alpha\in S} | f_\alpha | (2m)^{|\alpha|} = \| f \|_{2m}.$$

This shows that the family $(u_{\beta,\gamma})_{\beta,\gamma\in S}$ is absolutely summable in $\mathcal{U}$.

We have

$$\sum_{\beta+\gamma=\alpha} u_{\beta,\gamma} = f_\alpha (a+b)^\alpha \quad \text{for} \quad \alpha \in S,$$

and so

$$\sum_{(\beta,\gamma)\in S} u_{\beta,\gamma} = \sum_{\alpha\in S} f_\alpha (a+b)^\alpha = f(a+b).$$

Also the family $(\sum_{\gamma\in S} u_{\beta,\gamma})_{\beta\in S}$ is absolutely summable, and we have

$$\sum_{\gamma\in S}(\sum_{\beta\in S} u_{\beta,\gamma}) = \sum_{\beta,\gamma\in S} u_{\beta,\gamma} = f(a+b).$$

But

$$\sum_{\beta\in S} u_{\beta,\gamma} = \frac{b^\gamma}{\gamma!}\left(\sum_{\beta\in S} f_{\beta+\gamma}\frac{(\beta+\gamma)!}{\beta!} a^\beta\right) = \frac{b^\gamma}{\gamma!}\left[\sum_{\delta\in S} f_\delta\cdot\partial^\gamma(X^\delta)\right](a) = \frac{\partial^\gamma(f)}{\gamma!}(a)\cdot b^\gamma,$$

which concludes the proof of the lemma.

Notice that it follows from the lemma that $f = \sum_{\alpha\in S}\frac{\partial^\alpha f(0)}{\alpha!}\cdot X^\alpha$ for every $f\in\mathcal{U}$ (here we denote by 0 the zero element of ℓ^∞).

LEMMA 2.17. – *Let* $(\lambda_n)_{n\geq 1}\in\ell^1\backslash c_{oo}$ *and let* $f=\sum_{n=1}^{\infty}\lambda_n X_n$. *For* $m\geq 1$ *set*

$$\mathcal{U}_m = \{g\in\mathcal{U}\,|\,\partial^\alpha g(0)=0 \quad for \quad \operatorname{Supp}\alpha\cap\{1,\ldots,m\}\neq\varnothing\}.$$

Then $\mathcal{U}_m$ *is isomorphic to* $\mathcal{U}$ *and we have the following properties*

1) *Set* $Y=(Y_n)_{n\geq 1}$ *where* $Y_n=0\,(n\leq m)$, $Y_n=X_n\,(n\geq m+1)$. *Then the map* $\theta_Y: g\to g(Y)$ *is a homomorphism from* $\mathcal{U}$ *onto* $\mathcal{U}_m$, *and* $\operatorname{Ker}\theta_Y=\mathcal{J}_m$.

2) *If* $\lambda_m\neq 0$ *set* $Z=(Z_n)_{n\geq 1}$ *where* $Z_n=0\,(n\leq m-1)$

$$Z_m = \frac{1}{\lambda_m} - \sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}X_n,\ Z_n = X_n \qquad (n\geq m+1).$$

Then $\theta_Z: g\to g(Z)$ *is a homomorphism from* $\mathcal{U}$ *onto* $\mathcal{U}_m$, *and* $\operatorname{Ker}\theta_Z = \mathcal{J}_{m-1}+(f-1)\mathcal{U}$.

Proof. – Set $T=\{\alpha=(\alpha_n)_{n\geq 1}\in S\,|\,\operatorname{Supp}\alpha\cap\{1,\ldots,m\}=\varnothing\}$, and set $U=(X_{n+m})_{n\geq 1}$ so that $U\in\mathcal{U}^\infty$. For $\alpha=(\alpha_n)_{n\geq 1}$, set $\sigma(\alpha)=(\beta_n)_{n\geq 1}$, where $\beta_n=0\,(n\leq m)$, $\beta_n=\alpha_{n-m}\,(n\geq m+1)$. Clearly, $\sigma: S\to T$ is a bijection. Also $\alpha!=\sigma(\alpha)!$, $|\sigma(\alpha)|=|\alpha|$ and $U^\alpha=X^{\sigma(\alpha)}\,(\alpha\in S)$.

For $g\in\mathcal{U}$, we have

$$\theta_U(g) = \sum_{\alpha\in S}\frac{\partial^\alpha g(0)}{\alpha!}\cdot X^{\sigma(\alpha)}.$$

So $\theta_U(g)\in\mathcal{U}_m$ and $\|\theta_U(g)\|_n=\|g\|_n\,(n\geq 1)$. If $h=\sum_{\alpha\in T}\frac{h^\alpha(0)}{\alpha!}\cdot X^\alpha\in\mathcal{U}_m$, then $g=\sum_{\alpha\in S}\frac{h^{\sigma(\alpha)}(0)}{\alpha!}\cdot X^\alpha\in\mathcal{U}$, and $\theta_U(g)=h$. Hence $\theta_U:\mathcal{U}\to\mathcal{U}_m$ is an isomorphism of Fréchet algebras. We have $Y^\alpha = 0$ if $\alpha \notin T$, $Y^\alpha = X^\alpha$ if $\alpha \in T$. Hence $\theta_Y(\mathcal{U}) \subset \mathcal{U}_m$, and $\theta_{Y|\mathcal{U}_m}$ is the identity map, so that $\theta_Y(\mathcal{U}) = \mathcal{U}_m$. Since $\theta_Y(X_n) = 0$ for $n \leq m$, $\mathcal{J}_m \subset \operatorname{Ker}\theta_Y$. Now set $X = (X_n)_{n\geq 1}$ and let $g \in \operatorname{Ker}\theta_Y$. It follows from lemma 2-16 that

$$g = g(X) = \sum_{\alpha\neq 0} \frac{\partial^\alpha g(Y)}{\alpha!}(X - Y)^\alpha.$$

But $(X - Y)^\alpha = 0$ if $\alpha \notin S_{m'}$ and $(X - Y)^\alpha = X^\alpha$ if $\alpha \in S_m$. Hence $g \in \mathcal{J}_m$, which concludes the proof of (2).

We have $Z_n \in \mathcal{U}_m\,(n \geq 1)$, and so $\theta_z(\mathcal{U}) \subset \mathcal{U}_m$. Again, $\theta_z\,|_{\mathcal{U}_m}$ is the identity map and so $\theta_z(\mathcal{U}) = \mathcal{U}_m$. We have $\theta_z(X_n) = 0\,(1 \leq n \leq m-1)$, and

$$\theta_z(f) = \sum_{n=1}^{\infty} \lambda_n Z_n = \lambda_m Z_m + \sum_{n=m+1}^{\infty} \lambda_n X_n = 1.$$

So $\mathcal{J}_{m-1} + (f-1)\mathcal{U} \subset \operatorname{Ker}\,\theta_z$. We see as above that if $g \in \operatorname{Ker}\,\theta_z$, then $g = \sum_{\substack{\alpha\in S_m\\ \alpha\neq 0}} \frac{\partial^\alpha g(Z)}{\alpha!}(X - Z)^\alpha$, by lemma 2-16. It also follows from lemma 2-16 that

$$\sum_{\alpha\in S} \left\| \frac{\partial^\alpha g(Z)}{\alpha!} \right\|_n \cdot n^{|\alpha|} = \sum_{\alpha\in S} \left\| \frac{\partial^\alpha g(Z)}{\alpha!} X^\alpha \right\|_n < +\infty$$

for every n. Set as before $\rho_k = (\delta_{k,n})_{n\geq 1}$ and set $\Lambda_k = \{\alpha = (\alpha_n)_{n\geq 1} \in S \,|\, \alpha_k \neq 0, \alpha_n = 0\,(1 \leq n \leq k-1\}$. If follows from the above inequalities that the family $\left(\frac{\partial^\alpha g(Z)}{\alpha!}(X - Z)^{\alpha-\rho_k}\right)_{\alpha\in\Lambda_k}$ is absolutely summable in $\mathcal{U}$ for $1 \leq k \leq m$ So there exist $h_1, \ldots, h_m \in \mathcal{U}$ such that

$$g = \sum_{k=1}^{m}(X_k - Z_k)\cdot h_k = \sum_{k=1}^{m-1} X_k.h_k + \frac{1}{\lambda_m} h_m \left[f - 1 - \sum_{k=1}^{m-1} \lambda_n X_n\right] \in \mathcal{J}_{m-1} + (f-1)\mathcal{U}.$$

This concludes the proof of the lemma.

Now for $\Omega \subset \ell^\infty$ set $\Omega^\perp = \{f \in \mathcal{U} \,|\, g(z) = 0\,(z \in \Omega)\}$.

COROLLARY 2.18. – *The notations being as in lemma 2-17, we have* $\mathcal{J}_m = [\ell^\infty_m]^\perp$,

$$\mathcal{J}_{m-1} + (f-1)\mathcal{U} = [\ell^\infty_{m-1} \cap Z(f-1)]^\perp\,(m \geq 1).$$

Proof. – Let p be the smallest integer such that $\lambda_p \neq 0$, $p \geq m$. Set $Y_n = X_n$ for $n \neq p$, $n \neq m$, $Y_p = X_m$, $Y_m = X_p$, and $Y = (Y_n)_{n\geq 1}$.

The map $\varphi : g \to g(Y)$ is clearly an isomorphism from $\mathcal{U}$ onto itself, and the quotient algebra $\mathcal{U}/\mathcal{J}_{m-1} + (f-1)\mathcal{U}$ is isomorphic to the quotient algebra $\mathcal{U}/\mathcal{J}_{m-1} + (\varphi(f) - 1)\mathcal{U}$, hence isomorphic to $\mathcal{U}$ by lemma 2-16. Now let $\mathcal{J}$ be one of the ideals considered in the corollary, and let $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{J}$ be the canonical map. Then $\mathcal{U}/\mathcal{J}$ is isomorphic to $\mathcal{U}$, and so $\mathcal{U}/\mathcal{J}$ is semisimple. So if $g \in Z(\mathcal{J})^\perp$, then $\sigma(\pi(g)) = g[Z(\mathcal{J})] = \{0\}, \pi(g) = 0$, and $g \in \mathcal{J}$. This proves the corollary.

THEOREM 2.19. – *Let* $(\lambda_n)_{n\geq 1} \in \ell^1\backslash c_{oo}$, *and let* $f = \sum_{n=1}^{\infty} \lambda_n X_n$. *Then* $\operatorname{Inv}(\mathcal{U}/\mathcal{J}_\infty) = \exp(\mathcal{U}/\mathcal{J}_\infty)$, $\operatorname{Inv}(\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}) = \exp(\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U})$, *the ideals* $\mathcal{J}_\infty$ *and* $\mathcal{J}_\infty + (f-1)\mathcal{U}$ *are Picard-Borel ideals and* $\mathcal{J}_\infty + (f-1)\mathcal{U}$ *is a dense, prime ideal of countable type of* $\mathcal{U}$.

Proof. – This follows from lemma 2-13, lemma 2-14 and lemma 2-17.

COROLLARY 2.20. – *Let f be as in theorem 2.19. If we assume the continuum hypothesis, then the following assertions imply each other*

(1) *There exists a discontinuous character on some commutative Fréchet algebra.*

(2) *There exists an algebra norm on the quotient algebra $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$.*

Proof. – A commutative unital normed algebra always has a character, and so it follows from theorem 2.7 that (2) implies (1). Now assume that (1) holds. By theorem 2.7, $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$ has a character. Since $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$ is an integral domain, by theorem 2.19, it follows from [20, Corollary 5.3] that there exists an embedding from $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$ into some Banach algebra if the continuum hypothesis is assumed.

Notice that if the continuum hypothesis is assumed then the quotient algebre $\mathcal{U}/\mathcal{J}_\infty$ has an algebra norm. It would be interesting to investigate the dependence of this property on the continuum hypothesis. There exist models of set theory for which if $\mathcal{F}$ is any free ultrafilter on $\mathbb{N}$, the quotient algebra $\ell^\infty/\mathcal{F}$ is not normable [14], [50]. Recently, models of set theory for which $2^{\aleph_0} = \aleph_2$ and for which $\ell^\infty/\mathcal{F}$ is normable for some free ultrafilter $\mathcal{F}$ were constructed [28], [51]. The author believes that there exist models of set theory in which $\mathcal{U}/\mathcal{J}_\infty$ is normable, but in which for any free ultrafilter $\mathcal{F}$ on $\mathbb{N}$, the quotient algebra $\ell^\infty/\mathcal{F}$ is not.

The multiplicative structure of the quotient algebras $\mathcal{U}/\mathcal{J}_\infty$ and $\mathcal{U}/\mathcal{J}_\infty + (f-1)\mathcal{U}$ will be investigated in the next sections. We conclude the present section by a few observations, which show in particular that the multiplicative structure of $\mathcal{U}/\mathcal{J}_m$ and $\mathcal{U}/\mathcal{J}_m + (f-1)\mathcal{U}$ (which are isomorphic to $\mathcal{U}$ for $m \geq 1$) is rather poor. Also the notion of logarithmic and Picard-Borel ideals are unrelated.

Remark 2.21. – 1) $\bigcap_{n\geq 1} g_1 \dots g_n \mathcal{M}_{\mathbb{N}} = (0)$ *for every sequence* $(g_n)_{n\geq 1}$ *of elements of* $\mathcal{M}_{\mathbb{N}}$.

2) $\bigcap_{n\geq 1} g^n \cdot \mathcal{U} = \{0\}$ *for every* $g \in \mathcal{U}\backslash \mathrm{Inv}\,\mathcal{U}$.

3) *There exists a commutative, unital Picard-Borel Fréchet algebra such that Inv A is connected and such that $exp\, A \subsetneq \mathrm{Inv}\, A$.*

Proof. – For $g \in \mathcal{M}_{\mathbb{N}}$, $g \neq 0$ set $V(g) = \{\alpha \in S \,|\, \partial^\alpha g(0) \neq 0\}$ and $k(g) = \inf_{\alpha \in V(g)} |\alpha|$. If $g_n \neq 0$ for every n, then $k(g) \geq n+1$ for every nonzero $g \in g_1 \dots g_n \mathcal{M}_{\mathbb{N}}$, which proves (1).

Now let $g \in \mathcal{U}\backslash \mathrm{Inv}\mathcal{U}$, and let $z_o \in Z(g)$. For $h \in \bigcap_{n\geq 1} g^n \cdot \mathcal{U}$, $z \in \ell^\infty$, the entire function $F : s \to h(z_o + s(z - z_o))$ has a zero of infinite order at z_o. Hence $F = 0$ and $h(z) = 0 \, (z \in \ell^\infty)$ so that $h = 0$.

Now let A be the algebra consisting of entire functions f on $\mathbb{C}$ such that the sequence $(f\,(2\,ip\,\pi))_{p\geq 1}$ is convergent. For $f \in A$, $n \geq 1$ set $\|f\|_n = \max [\sup_{|z|\leq n} |f(z)|, \sup_{p\geq 1} |f(2\,ip\,\pi)|]$. Routine vertifications show that $(A, (\|\cdot\|_n)_{n\geq 1})$ is a commutative, unital Fréchet algebra. Since $\mathrm{Inv}\, A \subset \mathrm{Inv}\,(\mathcal{H}(\mathbb{C}))$, A is also a Picard-Borel algebra.

Let $f \in \operatorname{Inv} A$, and fix $n \geq 1$. Set $\ell = \lim_{p \to \infty} f(2ip\pi)$. Since $f \in \operatorname{Inv} A$, $\ell \neq 0$ and by using a suitable branch of the logarithm on a neighbourhood of ℓ we can construct a convergent sequence $(\lambda_p)_{p \geq 1}$ such that $e^{\lambda_p} = f(2ip\pi)\,(p \geq 1)$.

It follows from the theory of Weierstrass products that there exists $g \in \mathcal{H}(\mathbb{C})$ such that $Z(g) = \{2ip\pi\}_{p \geq n+1}$, and by a standard interpolation result there exists $h \in \mathcal{H}(\mathbb{C})$ such that $h(2ip\pi) = \lambda_p\,(p \geq 1)$. Clearly, g and h are elements of A.

Let $u \in \mathcal{H}(\mathbb{C})$ such that $e^u = f$, and for $|z| < 2\pi(n+1)$ set $q(z) = \frac{u(z) - h(z)}{g(z)}$. By using the Taylor series of q at the origin we see that there exists a sequence $(p_m)_{m \geq 1}$ of polynomials such that $p_m(z) \underset{m \to \infty}{\longrightarrow} q(z)$ uniformly for $|z| \leq 2\pi n$. Set $v_m = gp_m + h\,(m \geq 1)$. We see that $v_m \in A$, and that $\|e^{v_m} - f\|_n \underset{n \to \infty}{\longrightarrow} 0$. Hence $\operatorname{Inv} A \subset [\exp A]^-$ is connected [15].

Now let $\varphi : z \to e^z$ be the usual exponential function. Then $\varphi \in \operatorname{Inv} A$, and it follows from the definition of A that $\varphi \neq e^u$ for every $u \in A$ (in fact $\varphi \neq v^k$ for every $v \in A$ and every $k \geq 2$).

Notice that if $w \in \mathcal{H}(\mathbb{C})$ satisfies $\omega(2ip\pi) = 2i(p!)\pi\,(p \geq 1)$ and if we set $v_t \in e^{t\omega}\,(t \in Q)$ then $(v_t)_{t \in Q}$ is a rational subgroup of $\operatorname{Inv} A$, but $v_t \notin \exp A$ for every $t \neq 0$.

The general link between invertible elements and exponentials in commutative Fréchet algebras was described by Davie in [15]. $\operatorname{Inv} A \cap [\exp A]^-$ is the component of the unit element in $\operatorname{Inv} A$, and $f \in A$ belongs to $\exp A$ if and only if there exists a continuous path $\gamma : [0, 1] \to \operatorname{Inv} A$ such that $\gamma(0) = 1$ and $\gamma(1) = f$ Davie's paper contains also an example where $\operatorname{Inv} A$ is connected and strictly contains $\exp A$, and gives extensions to Fréchet algebras of the Arens-Royden Theorem [4], [45].

3. Mittag-Leffler methods, action of entire mappings and structure of $\mathbb{C}_{\mathsf{N}}[[X]]/\mathcal{J}_{\mathsf{N},\infty}$

The Mittag-Leffler theorem shows that if (E_n, θ_n) is a projective system where E_n is a complete metric space and $\theta_n : E_{n+1} \to E_n$ a continuous map such that $\theta_n(E_{n+1})$ is dense in E_n for every $n \geq 1$, then $\pi_1(\varprojlim (E_n, \theta_n))$ is dense in E_1. Here

$$\varprojlim (E_n, \theta_n) = \{(x_n)_{n \geq 1} \in \prod_{n \geq 1} E_n \mid x_n = \theta_n(x_{n+1})(n \geq 1)\},$$

and $\pi_k : (x_n)_{n \geq 1} \to x_k$ is the k^{th} coordinate projection on the cartesian product $\prod_{n \geq 1} E_n$. This result, explicitely stated and proved by Arens [2], is fact an abstract version of the argument used by Mittag-Leffler to prove his classical theorem on the existence of meromorphic functions with prescribed singular parts on a given discrete set [22]. The Mittag-Leffler theorem was an essential ingredient in the proof of a basic result of Arens [2]: if A is a unital Fréchet algebra, and if $(x_1, \ldots, x_n)$ is a finite family of elements of A such that $x_1 A \ldots + x_n A \subsetneq A$, then $[x_1 A \ldots + x_n A]^- \subsetneq A$. In particular characters on commutative, unital Fréchet algebras which are polynomially (or rationally) generated by a finite set are necessarily continuous. This theorem was also the starting point of Allan's embedding of $\mathbb{C}[[X]]$ into some Banach algebra [1], and played an important role in the author's construction of discontinuous homomorphisms of $\mathcal{C}(K)$ [17], [18], [19].

We first give a version of the Mittag-Leffler theorem suitable for quotients. The proof uses an argument similar to the argument used by P. G. Dixon and the author [16] to establish a link between Michael's problem and iteration of entire mappings from $\mathbb{C}^p$ into $\mathbb{C}^p$ $(p \geq 2)$. We will see in fact that this version of the Mittag-Leffler theorem, implicit in [16], gives a general powerful tool to investigate the quotient of a commutative Fréchet algebra by a dense ideal.

DEFINITION 3.1. – *A projective system of quotients is a family* $(E_n, \theta_n, \mathcal{R}_n)_{n \geq 1}$ *where* $(E_n)_{n \geq 1}$ *is a family of sets, where* $\theta_n : E_{n+1} \to E_n$ *is a map and where* $\mathcal{R}_n$ *is an equivalence relation on* E_n *which satisfies the condition*

$$(1)\ \theta_n(x)\,\mathcal{R}_n\,\theta_n(y)\ for\ every\ (x, y) \in E_{n+1} \times E_{n+1}\ such\ that\ x\,\mathcal{R}_{n+1}\,y \qquad (n \geq 1).$$

If $(E_n, \theta_n, \mathcal{R}_n)_{n \geq 1}$ is a projective system of quotients, we will denote by $\tilde{\theta}_n : E_{n+1}/\mathcal{R}_{n+1} \to E_n/\mathcal{R}_n$ the map defined by the commutative diagram

$$\begin{array}{ccc} E_{n+1} & \xrightarrow{\rho_{n+1}} & E_{n+1}/\mathcal{R}_{n+1} \\ \theta_n \downarrow & & \downarrow \tilde{\theta}_n \\ E_n & \xrightarrow{\rho_n} & E_n/\mathcal{R}_n \end{array}$$

where $\rho_n : E_n \to E_n/\mathcal{R}_n$ is the canonical surjection $(n \geq 1)$.

THEOREM 3.2. – *Let* $(E_n, \theta_n, \mathcal{R}_n)_{n \geq 1}$ *be a projective system of quotients, where* E_n *is a complete metric space and* $\theta_n : E_{n+1} \to E_n$ *a continuous map* $(n \geq 1)$. *Assume that there exists for every* $n \geq 1$ *a family* W_n *of continuous functions from* E_n *into itself which satisfies the following conditions.*

$$(1) \qquad f(x)\,\mathcal{R}_n\,x \ (x \in E_n,\ f \in W_n,\ n \geq 1)$$

$$(2) \qquad \bigcup_{f \in W_n} (f \circ \theta_n)(E_{n+1})\ is\ dense\ in\ E_n\ (n \geq 1).$$

Then $\varprojlim (E_n/\mathcal{R}_n, \tilde{\theta}_n) \neq \emptyset$, *and* $\rho_1^{-1}(\pi_1(\varprojlim (E_n/\mathcal{R}_n, \tilde{\theta}_n)))$ *is dense in* E_1.

Proof. – Set $F_1 = E_1$, and $F_n = E_n \times W_1 \ldots \times W_{n-1}$ $(n \geq 2)$. We equip the sets W_n with the discrete topology, so that they are complete with respect to the distance $d(f, g) = \delta_{f,g}$, where $\delta_{f,g}$ is the usual Kronecker symbol. We equip E_n with the given topology, and F_n with the corresponding product topology, so that F_n is a complete metric space.

Define $\varphi_n : F_{n+1} \to F_n$ by the formula $\varphi_n(x, f_1, \ldots, f_n) = ((f_n \circ \theta_n)(x), f_1, \ldots, f_{n-1})$. An immediate verification shows that φ_n is continuous, and it follows from hypothesis (2) that $\varphi_n(F_{n+1}))$ is dense in F_n.

It follows then from the Mittag-Leffler Theorem (*see* [8, Chap. II, sect. 3, Theorem 1] or [22, Corollary 2.2]) that $\pi_1(\varprojlim (F_n, \varphi_n))$ is dense in $F_1 = E_1$.

Let $(u_n)_{n \geq 1} \in \varprojlim (F_n, \varphi_n)$. There exists $(x_n)_{n \geq 1} \in \prod_{n \geq 1} E_n$ and $(f_n)_{n \geq 1} \in \prod_{n \geq 1} W_n$ such that $u_n = (x_n, f_1, \ldots, f_{n-1})$ and we have $x_n = (f_n \circ \theta_n)(x_{n+1})(n \geq 1)$.

It follows from hypothesis (1) that $x_n \mathcal{R}_n \theta_n(x_{n+1})$. So, the notations being as in definition 3.1, we have

$$\rho_n(x_n) = (\rho_n \circ \theta_n)(x_{n+1}) = \tilde{\theta}_n(\rho_{n+1}(x_{n+1})),$$

and so $(\rho_n(x_n))_{n\geq 1} \in \varprojlim(E_n/\mathcal{R}_n, \tilde{\theta}_n)$. Hence $u_1 = x_1 \in \rho_1^{-1}(\pi_1(\varprojlim E_n/\mathcal{R}_n, \tilde{\theta}_n))$. So $\rho_1^{-1}(\pi_1(\varprojlim(E_n/\mathcal{R}_n, \tilde{\theta}_n)))$ is dense in E_1, which concludes the proof of the theorem.

Notice that if we define $\mathcal{R}_n$ to be the equality on E_n, and if we set $W_n = \{\mathbf{1}_{E_n}\}$, where $\mathbf{1}_{E_n}$ is the identity map on E_n, then $E_n/\mathcal{R}_n = E_n$, $\rho_n = \mathbf{1}_{E_n}$ and we obtain the usual version of the Mittag-Leffler theorem: if $(E_n)_{n\geq 1}$ is a family of complete metric spaces and if $\theta_n : E_{n+1} \to E_n$ is a continuous map such that $\theta_n(E_{n+1})$ is dense in E_n for every $n \geq 1$, then $\pi_1 \varprojlim(E_n, \theta_n))$ is dense in E_1.

We now deduce from theorem 3.2 the following corollary, which is a basic tool to investigate the structure of the quotient of a Fréchet algebra by a dense ideal.

COROLLARY 3.3. – *Let $(G_n)_{n\geq 1}$ be a family of complete, metrizable abelian groups. For each $n \geq 1$ let H_n be a subgroup of G_n and let $\theta_n : G_{n+1} \to G_n$ be a continuous map. Assume that $\theta_n(x) - \theta_n(y) \in H_n$ whenever $x, y \in G_{n+1}$, $x - y \in H_{n+1}$ and denote by $\tilde{\theta}_n : G_{n+1}/H_{n+1} \to G_n/H_n$ the map associated to θ_n. If H_n is dense in G_n for every $n \geq 1$, then $\varprojlim(G_n/H_n, \tilde{\theta}_n) \neq \varnothing$.*

Proof. – For $z \in G_n$ denote by $\tau_{n,z}$ the map $x \to x + z$. Set $W_n = \{\tau_{n,z}\}_{z\in H_n}$ and let $\mathcal{R}_n$ be the equivalence relation associated to H_n, *i.e.* $x \mathcal{R}_n y$ if and only if $x - y \in H_n$. Then $(G_n, \theta_n, \mathcal{R}_n, W_n)_{n\geq 1}$ satisfies the conditions of theorem 3-2, and the corollary follows.

COROLLARY 3.4. – *Let A be a commutative Fréchet algebra, let I be a dense ideal of A, and let (a_n) be a sequence of elements of A/I. Then there exists $a \in A/I$ such that*

$$a - \sum_{m=1}^{n} a_1 \ldots a_m \in (a_1 \ldots a_n) A/I \quad (n \geq 1).$$

Proof. – Set $G_n = A$, $H_n = I\,(n \geq 1)$. Let $\pi : A \to A/I$ be the canonical map, and let $(x_n)_{n\geq 1}$ be a sequence of elements of A such that $\pi(x_n) = a_n\,(n \geq 1)$. Set $\theta_n(u) = x_n \cdot u + x_n\,(u \in A, n \geq 1)$. Then $(G_n, H_n, \theta_n)_{n\geq 1}$ satisfies the conditions of corollary 3-3, and $\tilde{\theta}_n(b) = a_n \cdot b + a_n\,(b \in A/I)$. So there exists a sequence $(b_n)_{n\geq 1}$ of elements of A/I such that

$$b_n = \theta_n(b_{n+1}) = a_n \cdot b_{n+1} + a_n \qquad (n \geq 1).$$

An immediate induction shows that $b_1 = \sum_{m=1}^{n} a_1 \ldots a_m + a_1 \ldots a_n b_{n+1}\,(n \geq 1)$, which proves the corollary.

We now turn to the action of entire maps $F : \mathbb{C}^p \to \mathbb{C}^q$ on $(A/I)^p$, where I is a dense ideal of a commutative, unital Fréchet algebra A. Let

$$f : (z_1, \ldots, z_p) \to \sum_{\alpha\in\mathbb{N}^p} \lambda_\alpha z_1^{\alpha_1} \ldots z_p^{\alpha_p}$$

be an entire function from $\mathbb{C}^p$ into $\mathbb{C}$ (with the notation $\alpha = (\alpha_1 \dots \alpha_p)$ for $\alpha \in \mathbb{N}^p$). If $a = (a_1 \dots a_p)$ is a finite family in a commutative, unital Fréchet algebra A, we can define $f(a)$ by the formula

$$f(a) = \sum_{\alpha \in \mathbb{N}^p} \lambda_\alpha \, a_1^{\alpha_1} \dots a_p^{\alpha_p}$$

(*see* details in [16]); this is in fact a trivial case of the holomorphic functional calculus in several variables for commutative, unital, locally multiplicatively convex complete algebras, *see* Arens [3].

The notations being as above, set $S_k = \{\alpha \in S \mid \operatorname{Supp} \alpha \subset \{1, \dots, k\}\}$, as in section 2, and denote by $\sigma_p(\alpha)$ the sequence $(\beta_n)_{n \geq 1}$ where $\beta_n = \alpha_n \, (n \leq p) \, \beta_n = 0 \, (n \geq p)$ for $\alpha \in \mathbb{N}^p$. The map $f \to \sum_{\alpha \in \mathbb{N}^p} \lambda_\alpha X^{\sigma_p(\alpha)}$ defines an isomorphism from $\mathcal{H}(\mathbb{C}^p)$ onto $\mathcal{U}^{(p)} = \{f \in \mathcal{U} | \partial^\alpha f(o) = 0 \, (\alpha \notin S_p)\}$, and we can identity in this way $\mathcal{H}(\mathbb{C}^p)$ with a closed subalgebra of $\mathcal{U}$, identifying also $\mathbb{N}^p$ with S_p by using σ_p. Hence if $f \in \mathcal{H}(\mathbb{C}^p)$, $a, b \in A^p$ we have

$$f(a+b) = \sum_{\alpha \in \mathbb{N}^p} \frac{\partial^\alpha f(a)}{\alpha!} b^\alpha,$$

by formula 2-16 (this is of course well-known). In particular $f(a+b) - f(a) \in I$ if $b \in I^p$, where I is any ideal in the commutative, unital Fréchet algebra A.

Similarly if $F = (f_1, \dots, f_q)$ is a holomorphic map from $\mathbb{C}^p$ into $\mathbb{C}^q$ and if $a \in A^p$ we can set $F(a) = (f_1(a), \dots, f_q(a)) \in A^q$ and $F(a+b) - F(a) \in I^q$ if $b \in I^p$, where b is any ideal in A.

For $p \geq 1$, $a = (a_1, \dots, a_p) \in A^p$ set

$$\pi_p(A) = (\pi(a_1), \dots, \pi(a_p)) \in (A/I)^p.$$

Then

$$F : (A/I)^p \to (A/I)^q$$
$$\pi_p(a) \to \pi_q(F(a))$$

is well-defined. Since $F : A^p \to A^q$ is continuous, and since I^p can be identified in the trivial way with a dense ideal of A^p when I is a dense ideal of A, we obtain the following immediate consequence of Corollary 3.3.

COROLLARY 3.5. – *Let $(p_n)_{n \geq 1}$ be a sequence of integers, and for each $n \geq 1$ let $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ be holomorphic. If I is a dense ideal of a commutative, unital Fréchet algebra A, then $\varprojlim((A/I)^{p_n}, F_n) \neq \varnothing$.*

We now briefly discuss the spectral properties of the functional calculus on A/I introduced above. If χ is a character on A/I and if $b = (b_1, \dots, b_k) \in (A/I)^k$, we set $X_k(b) = (\chi(b_1), \dots, \chi(b_k))$, so that X_k maps $(A/I)^k$ into $\mathbb{C}^k$. Also if φ is any character on A, and if $f : \mathbb{C}^k \to \mathbb{C}$ is holomorphic, then $\varphi[f(a)] = f[\varphi(a_1), \dots, \varphi(a_k)]$ for $a = (a_1, \dots, a_k) \in A^k$. As observed in [16], this follows from Arens'theorem [2], since the restriction of φ to the closed unital subalgebra B of A generated by $(a_1, \dots, a_k)$ is continuous. A more direct argument is given by formula 2-16. If we set

$$\lambda = (\varphi(a_1), \dots, \varphi(a_k)),$$

we have

$$\varphi(f(a) - f(\lambda).1) = \varphi\left(\sum_{\substack{\alpha\in\mathbb{N}^k\\ \alpha\neq 0}} \frac{\partial^\alpha f}{\alpha!}(\lambda)(a-\lambda\cdot 1)^\alpha\right) = 0,$$

since

$$\sum_{\substack{\alpha\in\mathbb{N}^k\\ \alpha\neq 0}} \frac{\partial^\alpha f}{\alpha!}(\lambda)(a-\lambda\cdot 1)^\alpha \in (a_1 - \varphi(a_1)\cdot 1)A\ldots + (a_k - \varphi(a_k\cdot 1)A$$

(here we set $\lambda\cdot 1 = (\varphi(a_1).1, \ldots, \varphi(a_k)\cdot 1) \in A^k$).

PROPOSITION 3.6. – *Let I be an ideal of a commutative, unital Fréchet algebra A, let $F : \mathbb{C}^p \to \mathbb{C}^q$ be a holomorphic map, and let $a \in (A/I)^p$. Then the following properties hold*

1) $F(\sigma(a)) \subset \sigma(F(a))$

2) *If I is closed, or if $p = q$ and if F is one-to-one, then $F(\sigma(a)) = \sigma(F(a))$.*

3) *If χ is a character on A/I, then $\chi_q(F(a)) = F(\chi_p(a))$.*

Proof. – Let $\pi : A \to A/I$ be the canonical surjection. Set $a = (a_1, \ldots, a_p)$, $F = (f_1, \ldots, f_q)$ and for $i \leq p$ choose $x_i \in A$ such that $\pi(x_i) = a_i$. Set $x = (x_1, \ldots, x_p)$ and assume that $\lambda = (\lambda_1, \ldots, \lambda_p) \in \sigma(a)$.

For $i \leq q$, we have

$$f_i(x) - f_i(\lambda).1 = \sum_{\substack{\alpha\in\mathbb{N}^p\\ \alpha\neq 0}} \frac{\partial^\alpha f_i(\lambda)}{\alpha!}(x-\lambda.1)^\alpha$$

by formula 2.16, and so

$$f_i(x) - f_i(\lambda)\cdot 1 \in (x_1 - \lambda_1\cdot 1)A\ldots + (x_p - \lambda_p\cdot 1)\cdot A.$$

Hence

$$f_i(a) - f_i(\lambda)\cdot 1 = \pi[f_i(x) - f_i(\lambda)\cdot 1] \in (a_1 - \lambda_1\cdot 1)A/I\ldots + (a_p - \lambda_p\cdot 1)A/I \subsetneq A/I\ (i \leq q).$$

This shows that $(f_1(\lambda), \ldots, f_q(\lambda)) \in \sigma(f_1(a), \ldots, f_q(a)) = \sigma(F(a))$, which proves (1). Now assume that χ is a character on A/I. Then $\varphi = \chi\circ\pi$ is a character on A, and we have, for $i \leq q$,

$$\begin{aligned}\chi(f_i(a)) &= \varphi(f_i(x)) = f_i(\varphi(x_1), \ldots, \varphi(x_p))\\ &= f_i(\chi(a_1), \ldots, \chi(a_p)) = f_i(\chi_p(a)),\end{aligned}$$

and so $\chi_q(F(a)) = F(\chi_p(a))$, which proves (3).

Now assume that I is closed. It follows from the definitions that $F(a)$ is the element of $(A/I)^q$ obtained by applying directly the functional calculus to the Fréchet algebra A/I. Let H be the set of continuous characters on A/I. It follows from a standard result of Arens [2] that

$$\sigma(F(a)) = \{\chi_q(F(a))\}_{\chi\in H} = F[\{\chi_p(a)\}_{\chi\in H}] = F(\sigma(a)).$$

Now let I be any ideal of A and assume that $p = q$ and that F is one-to-one on $\mathbb{C}^p$. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \in \sigma(F(a))$, and let $y \in I$.

Since $\lambda \in \sigma(F(a))$, we have $(f_1(x) - \lambda_1 \cdot 1) A \ldots + (f_p(x) - \lambda_p \cdot 1).A + y \cdot A \subsetneq A$, and it follows from the theorem of Arens that there exists a continuous character φ on A such that

$$\varphi(f_1(x) - \lambda_1.1) \ldots = \varphi(f_p(x) - \lambda_p \cdot 1) = \varphi(y) = 0.$$

Hence

$$f_i[\varphi(x_1), \ldots, \varphi(x_p)] = \varphi(f_i(x)) = \lambda_i \, (i \leq p)$$

and $\lambda = F(\mu)$, where $\mu = (\varphi(x_1), \ldots, \varphi(x_p))$. We have

$$(x_1 - \varphi(x_1) \cdot 1) A \ldots + (x_p - \varphi(x_p) \cdot 1) \cdot A + y \cdot A \subsetneq A,$$

and μ is independent of y since F is one-to-one. So $\mu \in \sigma(a)$, which concludes the proof of the proposition.

The link established in [16] between Michael's problem and projective systems of entire mappings now appears as an immediate consequence of Corollary 3.5 and Proposition 3.6.

COROLLARY 3.7. – *Assume that there exists a discontinuous character on some commutative Fréchet algebra. Then* $\varprojlim(\mathbb{C}^{p_n}, F_n) \neq \emptyset$ *for every projective system* $(\mathbb{C}^{p_n}, F_n)$ *where* $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ *is entire for* $n \geq 1$.

Proof. – We can assume without loss of generality that there exists a discontinuous character φ on some commutative, unital Fréchet algebra A. Set $I = \mathrm{Ker}\varphi$, let $\pi : A \to A/I$ be the canonical map and let $\chi : A/I \to \mathbb{C}$ be the isomorphism satisfying $\chi \circ \pi = \varphi$. Then χ is a character on A/I, and it follows from proposition 3-6 that $\tilde{\chi} : (u_n)_{n\geq 1} \to (\chi_{p_n}(u_n))_{n\geq 1}$ is bijective from $\varprojlim((A/I)^{p_n}, F_n)$ onto $\varprojlim(\mathbb{C}^{p_n}, F_n)$. Since I is dense in A, it follows from Corollary 3.5 that $\varprojlim((A/I)^{p_n}, F_n) \neq \emptyset$, and the result follows.

Remark 3.8. – 1) The proof we give here of Corollary 3-7 is in fact very similar to the proof given in [16]. The advantage of the point of view presented here is that dense ideals exist in abundance in commutative, unital Fréchet algebras A for which Inv A is not open [24], so that Corollary 3-5 applies to a lot of situations whereas the existence of discontinuous characters is still unknown. The reader will check that Corollary 3-5 is in fact a rather obvious result when $A = \mathbb{C}^{\mathbb{N}}$ and $I = c_{oo}$, the set of complex sequences which vanish eventually.

2) Corollary 3-5 and Corollary 3-7 remain valid for real Fréchet algebras if we restrict attention to entire mappings with real Taylor coefficients at the origin. In particular if we set $p(x) = 1 + x^2$ we see that $\bigcap_{n\geq 1} p^n(\mathbb{R}) = \emptyset$ (where p^n is computed with respect to the composition of maps) and so real characters on commutative real Fréchet algebras are continuous (*see* [47]). Corollary 3-7 is also valid for noncommutative Fréchet algebras, see the details in [16].

3) It is not true in general that $\sigma(F(a)) = F(\sigma(a))$, the notations being as in Corollary 3-5. To see this let $A = \mathbb{C}^{\mathbb{N}}$, $I = c_{oo}$ and let $\pi : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}/c_{oo}$ be the canonical map. Set $u = (2\, ip\, \pi)_{p\geq 0}$. Then e^u is the unit element of $\mathbb{C}^{\mathbb{N}}$, so that $\pi(e^u) = 1$, $\sigma(e^{\pi(u)}) = \{1\}$. On the other hand we check easily that $\sigma(\pi(u)) = \emptyset$.

We now investigate the multiplicative structure of A/I where I is a dense, prime ideal of countable type in a commutative Fréchet algebra A. Here we do not assume that A is unital, and so the result applies to some Banach algebras.

LEMMA 3.9. – *Let I be a dense, prime ideal of countable type in a commutative Fréchet algebra, and denote by B the set of nonzero elements of A/I. Then for every sequence $(a_n)_{n\geq 1}$ of elements of B there exists a sequence $(u_n)_{n\geq 1}$ of elements of B*

$$\textit{such that}\ u_n = a_1^n \dots a_{n-1}^n \cdot a_n^{n+1} \cdot u_{n+1}^{n+1}\ (n \geq 1).$$

Proof. – Set $D = A\backslash I$, so that D is stable under products. There exists a sequence $(I_n)_{n\geq 1}$ of closed ideals of A such that $I = \bigcup_{n\geq 1} I_n$, and so $D = \bigcap_{n\geq 1} (A\backslash I_n)$ is a G_δ set. So D is homeomorphic to a complete metric space. Let $\mathcal{R}$ be the restriction to D of the equivalence relation defined by I. Let $(x_n)_{n\geq 1}$ be a sequence of elements of D such that $\pi(x_n) = a_n\ (n \geq 1)$ where $\pi : A \to A/I$ is the canonical map. For $n \geq 1$, $y \in D$ set $\theta_n(y) = (x_1 \dots x_{n-1})^n (x_n y)^{n+1}$. For $z \in I$, $y \in D$, we have $y + z \in D$, since I is an ideal, and so $\tau_z : y \to y + z$ is a continuous map from D into itself. Set $W = (\tau_z)_{z\in I}$ and for $n \geq 1$ set $E_n = D$, $\mathcal{R}_n = \mathcal{R}$, $W_n = W$. Then the family $(E_n, \theta_n, \mathcal{R}_n, W_n)_{n\geq 1}$ satisfies the conditions of theorem 3-2, and there exists a sequence (u_n) of elements of B such that

$$u_n = \tilde{\theta}_n(u_{n+1}) = \pi(x_1^n \dots x_{n-1}^n x_n^{n+1})\, u_{n+1}^{n+1} = a_1^n \dots a_{n-1}^n \cdot a_n^{n+1} \cdot u_{n+1}^{n+1}.$$

This concludes the proof of the lemma.

We now introduce some notations. We will denote by K_{ω_1} the rational linear space defined as follows

$$K_{\omega_1} = \{r = (r_\xi)_{\xi<\omega_1} \in Q^{\omega_1} \,|\, \{\xi \,|\, r_\xi \neq 0\} \text{ is finite}\} \tag{3.10}$$

and we equip K_{ω_1} with the reverse lexicographic order on K_{ω_1}, *i.e.*

(3.11) $(r_\xi)_{\xi<\omega_1} < (r'_\xi)_{\xi<\omega_1}$ if there exists $\eta < \omega_1$ such that $r_\eta < r'_\eta$ and $r_\xi = r'_\xi\ (\xi > \eta)$.

Then K_{ω_1} is a linearly ordered rational linear space. Set

$$L_{\omega_1} = \{r \in K_{\omega_1} \,|\, r \geq 0\},\ L'_{\omega_1} = \{r \in K_{\omega_1} \,|\, r > 0\}. \tag{3.12}$$

If $(G, +)$ is a linearly ordered, abelian group, we will denote by $\mathcal{F}_{(1)}(G, \mathbb{C})$ the set of all formal power series $f = \sum_{\alpha\in G} \lambda_\alpha Z^\alpha$ such that $\operatorname{Supp} f = \{\alpha \in G \,|\, \lambda_\alpha \neq 0\}$ is well-ordered and at most countable. It follows from old results of Hahn [32] (*see* references in [17], [18]) that $\mathcal{F}_{(1)}(G, \mathbb{C})$ is a field. Also $\mathcal{F}_{(1)}(G, \mathbb{C})$ is algebraically closed if G is a divisible group, by a result of Mac Lane [35]. Set.

$$\begin{aligned} \mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C}) &= \{f \in \mathcal{F}_{(1)}(K_{\omega_1}, \mathbb{C}) \,|\, \operatorname{Supp} f \subset L_{\omega_1}\}, \\ \mathcal{F}_{(1)}(L'_{\omega_1}, \mathbb{C}) &= \{f \in \mathcal{F}_{(1)}(K_{\omega_1}, \mathbb{C}) \,|\, \operatorname{Supp} f \subset L'_{\omega_1}\}. \end{aligned} \tag{3.13}$$

Then $\mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C})$ is the valuation ring associated to the standard valuation

$$f \to v(f) = \inf(\operatorname{Supp} f)(f \in \mathcal{F}_{(1)}(K_{\omega_1}, \mathbb{C}),\ f \neq 0)$$

and the unique maximal ideal of $\mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C})$ is $\mathcal{F}_{(1)}(L'_{\omega_1}, \mathbb{C})$, which is the kernel of the (unique) character

$$\sum_{\alpha \in L_{\omega_1}} \lambda_\alpha Z^\alpha \to \lambda_0 \quad \text{on} \quad \mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C}).$$

We now deduce from lemma 3-9 the following result (recall that an ideal I of an algebra A is modular if A/I is unital).

THEOREM 3.14. – *Let A be a commutative, non unital Fréchet algebra and let I be a dense prime, nonmodular ideal of A of countable type. Then there exists for every nonzero $a \in A/I$ a one-to-one map*

$$\psi : L'_{\omega_1} \to \bigcap_{n \geq 1} a^n (A/I) \text{ such that } \psi(r + r') = \psi(r) \cdot \psi(r')(r,\ r' \in L'_{\omega_1}).$$

Proof. – Notice that the above condition implies that ψ maps L'_{ω_1} into the set of nonzero elements of A/I.

Denote by V the set of all nonzero elements of $\bigcap_{n\geq 1} a^n(A/I)$, and denote by W the set of sequences $u = (u_n)_{n\geq 1}$ of elements of V such that $u_n = u_{n+1}^{n+1}\ (n \geq 1)$. For $u = (u_n)_{n\geq 1}$, $v = (v_n)_{n\geq 1} \in W$ set $u.v = (u_n.v_n)_{n\geq 1}$. Also for $p \in \mathbb{N}$, $q \in \mathbb{N}$, $u = (u_n)_{n\geq 1} \in W$ set $u^{p/q} = (u_{nq}^{p\frac{(nq)!}{q(n!)}})_{n\geq 1}$. Routine verifications show that $u^{p/q}$ is a well-defined element of W, and that $u^{p'/q'} = u^{p/q}$ if $qp' - pq' = 0$.

So we can define this way u^s for $u \in W$, $s \in Q^{+*} = \{s \in Q | s > 0\}$, and it follows from the definitions that the usual rules $u^{s+s'} = u^s.u^{s'}$, $(u^s)^{s'} = u^{ss'}$ are satisfied for $u \in W$, s, $s' \in Q^{+*}$.

Apply Lemma 3-9 with $a_n = a(n \geq 1)$. We obtain a sequence $(v_n)_{n\geq 1}$ of nonzero elements of A/I such that $a^{n-1}.v_n = (a^n.v_{n+1})^{n+1}\,(n \geq 1)$. So if we set $u_n = a^{n-1}.v_n\,(n \geq 1)$, we see that $(u_n)_{n\geq 1} \in W$, so that $W \neq \emptyset$.

For $\eta < \omega_1$ set $e_\eta = (\delta_{\eta,\xi})_{\xi<\omega_1}$ where we denote by $\delta_{\eta,\xi}$ the Kronecker symbol, and set $L'_\eta = \{(r_\xi)_{\xi<\omega_1} \in L'_{\omega_1} | r_\xi = 0\ \eta < \xi < \omega_1\}$. We now define by transfinite induction a family $(\varphi_\eta)_{\eta<\omega_1}$, where φ_η maps L'_η into W, such that $\varphi_\eta(r^s) = \varphi_\eta(r)^s$; $\varphi_\eta(r + r') = \varphi_\eta(r).\varphi_\eta(r')$, and $\varphi_{\eta|L'_\xi} = \varphi_\xi$ for $\xi < \eta$ $(r,\ r' \in L'_\eta,\ s \in Q^{+*},\ \eta < \omega_1)$.

Since $W \neq \emptyset$, we can set $\varphi_0(se_0) = u_0^s(s \in Q^{+*})$, where u_0 is the element of W constructed above. Now assume that a family $(\varphi_\xi)_{\xi<\eta}$ which has the required properties has been constructed for some $\eta \in (0,\ \omega_1)$. Let $\rho : (u_n)_{n\geq 1} \to u_1$ be the first projection map from W into V. Set $u_\xi = \varphi_\xi(e_\xi)\,(\xi < \eta)$, $L' = \bigcup_{\xi<\eta} L'_\xi$ (so that $L' = L'_\sigma$ if $\eta = \sigma + 1$ is a sucessor ordinal), and define $\varphi : L' \to W$ by the condition $\varphi|_{L'_\xi} = \varphi_\xi\,(\xi < \eta)$. If η is a limit ordinal, let (ξ_n) be a strictly increasing, cofinal sequence in $[0,\ \eta)$. In this case set $r^{(n)} = e_{\xi_n}$. If $\eta = \sigma + 1$ is a sucessor ordinal, set $r^{(n)} = ne_\sigma\,(n \geq 1)$. In both cases the sequence $(r^{(n)})_{n\geq 1}$ is cofinal in L'. Set $\varphi(r^{(m)}) = (a_{m,n})_{n\geq 1}$. It follows from Lemma 3-9 that there exists a sequence $(v_n)_{n\geq 1}$ of elements of D such that

$a_{1,1} \cdots a_{n-1,1} \cdot v_n = (a_{1,1} \cdots a_{n,1} \cdot v_{n+1})^{n+1}$ $(n \geq 1)$. Set $u_n = a_{1,1} \cdots a_{n-1,1} \cdot v_n$ $(n \geq 1)$ so that $u = (u_n)_{n \geq 1} \in W$.

Let $c \in L'$. Then there exists $m \geq 1$ such that $r^{(m)} > c$, so that $d = r^{(m)} - c \in L'$. Let $s \in Q^{+*}$ and set $u^s = (w_n)_{n \geq 1}$. It follows from the definition of u^s that if we fix $n \geq 1$ there exists $k \geq 1$ such that $w_n \in u_k.V$ for $k \geq p$. So for $k = \max(p, m+1)$ we obtain $w_n \in a_{m,1}.V \subset a_{m,n}.V$. Hence there exists a sequence $(y_n)_{n \geq 1}$ of elements of V such that $w_m = a_{m,n}.y_n$ $(n \geq 1)$. We obtain $a_{m,n}.y_n = a_{m,n}.y_{n+1}^{n+1}$, and so $y_n = y_{n+1}^{n+1}$ for $n \geq 1$ since I is not modular. So $(y_n)_{n \geq 1} \in W$, and $u^s \in \varphi(r^{(m)}).W$. Hence $u^s \in \varphi(c).W$ $(c \in L')$. Using again the fact that I is not modular, we see in fact that the equation $u^s = \varphi(c).x$ has a unique solution in W, which we can denote by $u^s/\varphi(c)$. Now let $r = (r_\xi)_{\xi < \omega_1} \in L'_\eta$. If $r \in L'$ set $\varphi_\eta(r) = \varphi(r)$. If $r \in L'_\eta \backslash L'$ then $r_\eta > 0$ and we can write $r = r_\eta.e_\eta + b - c$, where $b, c \in L'$. In this case set $\varphi_\eta(r) = [u^{r_\eta}/\varphi(c)].\varphi(b)$. It follows from the above discussion that this definition does not depend on the choice of b and c, and immediate verifications show that φ_η has the required properties. So we can construct the family $(\varphi_\eta)_{\eta < \omega_1}$ by transfinite induction.

Now define $\varphi_{\omega_1} : L'_{\omega_1} \to W$ by the condition $\varphi_{\omega_1 | L'_\eta} = \varphi_\eta$ $(\eta < \omega_1)$ and set $\psi = \rho \circ \varphi_{\omega_1}$. Clearly, ψ satisfies the conditions of the theorem.

COROLLARY 3-15. – *Let A be a commutative, unital Banach algebra, let M be a maximal ideal of A and let I be a prime ideal of countable type of A which is dense in M. Then for every nonzero $a \in M/I$ there exists a one-to-one homomorphism $\theta : \mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C})$ into A such that $\theta(\mathcal{F}_{(1)}(L'_{\omega_1}, \mathbb{C})) \subset \bigcap_{n \geq 1} a^n(A/I)$.*

Proof. – Since I is dense in M, I is not modular. It follows from the theorem that there exists $\psi : L'_{\omega_1} \to \bigcap_{n \geq 1} a^n(A/I)$ such that $\psi(r + r') = \psi(r).\psi(r')$ $(r, r' \in L'_{\omega_1})$ which is one-to-one.

Let $a_0 \in M/I$, let $a_2, \ldots, a_n \in A/I$, and let $\pi : A \to A/I$ be the canonical map. For $k \geq 2$, let $b_k \in M$ such that $\pi(b_k) = a_k$. Set $f(x) = x + \sum_{k=2}^n b_k.x^k (x \in M)$. Then $f : M \to M$ is continuously differentiable on M and $Df(0)$ is the identity map. If follows then from the standard inversion theorem that there exists $\rho > 0$ such that the equation $f(x) = u$ has a solution in M for every $u \in M$ such that $||u|| < \rho$.

Since I is dense in M, there exists $u \in M$ such that $\pi(u) = -a_0$ and such that $||u|| < \rho$. Let x be a solution in M of the equation $f(x) = u$, and set $y = \pi(x)$. Then $a_0 + y + a_2 y^2 \cdots + a_n y^n = 0$.

Since M/I is radical, and since for every sequence $(u_n)_{n \geq 1}$ of elements of M/I there exists $\rho \in M/I$ such that $\rho - \sum_{k=2}^n u_1 \ldots u_k \in u_1 \ldots u_{k+1}(A/I)$, by Corollary 3-4, we can apply directly the method used by the author in [18] to obtain an embedding $\theta : \mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C}) \to A/I$ such that $\theta(Z^r) = \psi(r)$ $(r \in L'_{\omega_1})$, which proves the corollary.

COROLLARY 3-16. – *For every nonzero $a \in \mathcal{M}_{\mathbb{N}}/\mathcal{I}_{\mathbb{N},\infty}$ there exists a one-to-one homomorphism $\theta : \mathcal{F}_{(1)}(L_{\omega_1}, \mathbb{C}) \to \mathbb{C}_{\mathbb{N}}[[X]]/\mathcal{I}_{\mathbb{N},\infty}$ such that*

$$\theta(\mathcal{F}_{(1)}(L'_{\omega_1}, \mathbb{C})) \subset \bigcap_{n \geq 1} a^n[\mathcal{M}_{\mathbb{N}}/\mathcal{I}_{\mathbb{N},\infty}].$$

Proof. – Let $a_0 \in \mathcal{M}_{\mathbb{N},k} = \mathcal{M}_{\mathbb{N}} \cap \mathbb{C}_k[[X]]$, and let $a_2 \ldots a_n \in \mathbb{C}_k[[X]]$, Consider the polynomial $P(t) = t^n + t^{n-1} + a_2 a_0 t^{n-2} \ldots + a_{n-1} a_0^{n-2} t + a_n.a_0^{n-1}$.

In $\mathbb{C}_k[[X]]/\mathcal{M}_{\mathbb{N},k}$, $P(t) \equiv t^{n-1}(t+1)$. It follows then from Hensel's Lemma [52, p. 279] that there exists a monic polynomial $Q(t)$ of degree $n-1$, and an element u of $\mathcal{M}_{\mathbb{N},k}$ such that $P(t) = (t+1+u)\,Q(t)$. In particular, $P(-1-u) = 0$. Since $\mathbb{C}_k[[X]]$ is a local ring, $-1-u$ is invertible, and if we set $x = (-1-u)^{-1}a_0$, we obtain $a_0 + x + a_2 x^2 \ldots + a_n x^n = 0$, with $x \in \mathcal{M}_{\mathbb{N},k}$. Also x is the only solution of this equation in $\mathcal{M}_{\mathbb{N},k}$, for if y is any other solution we have $(x-y) + a_2(x^2-y^2)\ldots + a_n(x_n - y^n) = 0$, hence $(x-y)(1+w) = 0$, where $w \in \mathcal{M}_{\mathbb{N},k}$. Since $\mathbb{C}_{\mathbb{N}}[[X]] \simeq \varprojlim (\mathbb{C}_k[[X]], \pi_{k,k+1})$ (*see* section 2), we deduce immediately from these facts that if $b_0 \in \mathcal{M}_{\mathbb{N}}$, $b_2, \ldots, b_n \in \mathbb{C}_{\mathbb{N}}[[X]]$ then the equation $b_0 + x + b_2 x^2 \ldots + b_n x^n = 0$ has a solution in $\mathcal{M}_{\mathbb{N}}$. *A fortiori* the same property holds for similar equations with coefficients in $\mathbb{C}_{\mathbb{N}}[[X]]/\mathcal{I}_{\mathbb{N},\infty}$.

Since $\mathcal{M}_{\mathbb{N}}/\mathcal{I}_{\mathbb{N},\infty}$ is radical, we can use the same argument as in Corollary 3-15 to conclude the proof.

Remark 3-17. – 1) Corollary 3-15 obviously does not extend to Fréchet algebras, because the algebra $\mathcal{F}_{(1)}(L'_{\omega_1}, \mathbb{C})$ is radical and, for example, the quotient algebra $\mathcal{M}/\mathcal{I}_\infty$ is semisimple, as observed in section 2. Also the equation $a_0 + x + x^2 = 0$ has no solution in $\mathcal{M}/\mathcal{I}_\infty$ for some $a_0 \in \mathcal{M}/\mathcal{I}_\infty$. To see this it suffices to take $a_0 \in \mathcal{M}/\mathcal{I}_\infty$ such that 1-4 a_0 has no square root in $\mathcal{U}/\mathcal{I}_\infty$, which is the case for example if $a_0 = \pi(\sum_{n=1}^{\infty} \lambda_n X_n)$, where $(\lambda_n)_{n\geq 1} \in \ell^1 \backslash c_{00}$. The details are left to the reader.

2) Lemma 3-9 is no longer true if we omit the hypothesis that the prime ideal I is of countable type: Dales and McClure construted in [13] a prime ideal I of a commutative, unital Banach algebra A, which is dense in the kernel of a character φ of A, such that the quotient algebra A/I is isomorphic to $\mathbb{C}[[X]]$, the algebra of formal power series in one variable.

In the other direction it follows easily from Corollary 3-4 that if I is dense in a commutative Fréchet algebra A then there exists for every $a \in A/I$ and every sequence $(\lambda_n)_{n\geq 1}$ of complex numbers some $\rho \in A/I$ such that $\rho - \sum_{k=1}^{n} \lambda_k a^k \in a^n.\ A/I\ (n \geq 1)$. Now let $\delta = A/I \to (A/I)/\bigcap_{n\geq 1} a^n \cdot A/I$ be the canonical map. If I is prime and nonmodular, and if $a \neq 0$, it is easy to see that the map $\theta : \sum_{n=1}^{\infty} \lambda_n X^n \to \delta(\rho)$ is a well-defined one-to-one homomorphism form $\mathbb{C}_0[[X]]$ into $(A/I)/\bigcap_{n\geq 1} a^n \cdot A/I$ such that $\theta(X) = \delta(a)$, where $\mathbb{C}_0[[X]] = \{\sum_{n=1}^{\infty} \lambda_n X^n \in \mathbb{C}[[X]] | \lambda_0 = 0\}$. Similarly if I is a dense, prime, modular ideal of a commutative Fréchet algebra A, and if $a \neq 0$ is a singular element of A/I, there exists an embedding $\theta : \mathbb{C}_0[[X]] \to (A/I)\bigcap_{n\geq 1} a^n \cdot A/I$ such that $\theta(X) = \delta(a)$, the notations being as above. If we omit the condition that I is prime, then there still exists a unique homomorphism θ from $\mathbb{C}_0[[X]]$ (resp. $\mathbb{C}[[X]]$) in the modular case) into the above quotient algebra such that $\theta(X) = \delta(a)$, but θ is not necessarily one-to-one. Since $\mathbb{C}_0[[X]]$ is a radical algebra, we see in both cases that $\delta(a) \in \mathrm{Rad}((A/I)/\bigcap_{n\geq 1} a^n \cdot A/I)$. Now let $\pi : A \to A/I$ be the canonical map. Since $(A/I)/\bigcap_{n\geq 1} \pi(b)^n\, A/I$ is isomorphic to $A/\bigcap_{n\geq 1} I + b^n \cdot A$ for $b \in A$, we see that $\pi(b) \in \mathrm{Rad}(A/\bigcap_{n\geq 1} I + b^n \cdot A)$ for every $b \in A$ if I is a dense ideal of A (the quotient algebra $A/\bigcap_{n\geq 1} I + b^n \cdot A$ reduces to $\{0\}$ in the trivial case where I is modular and where $\pi(b) \in \mathrm{Inv}(A/I)$). In particular, if I is a dense ideal of A and if χ is a (discontinuous) character of A such that $\bigcap_{n\geq 1} I + b^n \cdot A \subset \mathrm{Ker}\,\chi$, then $\chi(b) = 0$.

4. Universal multiplicative properties of the quotient of a commutative, unital Fréchet algebra by a dense, prime ideal of countable type

We showed in the previous section that if I is a prime, dense, nonmodular ideal of a commutative nonunital Fréchet algebra then there exists for every nonzero $a \in A/I$ a one-to-one map $\psi : L'_{\omega_1} \to \bigcap_{n\geq 1} a^n\, A/I$ such that $\psi\,(r + r') = \psi\,(r)\,\psi\,(r')\,(r,\ r' \in L'_{\omega_1})$. In this section we prove a stronger result for $\bigcap_{n\geq 1} a^n\,(A/I)$ where $a \neq 0$ is a singular element of A/I and where I is a dense, modular prime ideal of countable type in a commutative Fréchet algebra A. In fact, if the continuum hypothesis is assumed, there exists for any cancellative, abelian, non unital monoid $(V,\ +)$ without torsion such that Card $V \leq 2^{\aleph_0}$ a one-to-one map $\psi : V \to \bigcap_{n\geq 1} a^n \cdot A/I$ such that $\psi\,(r + r') = \psi\,(r)\,\psi\,(r')\,(r,\ r' \in V)$.

We first introduce some notations used in [17], [18].

(4.1) We denote by $\mathcal{S}_{\omega_1}$ the set of all dyadic sequences $(\varepsilon_\xi)_\xi < \omega_1$, with $\varepsilon_\xi \in \{0, 1\}$ $(\xi < \omega_1)$ such that the set $\{\xi < \omega_1 | \varepsilon_\xi = 1\}$ is nonempty and possesses a largest element. We equip $\mathcal{S}_{\omega_1}$ with lexicographic order.

(4.2) We denote by $G^{(1)}_{\omega_1}$ the real linear space consisting of all functions $\rho : \mathcal{S}_{\omega_1} \to \mathbb{R}$ such that Supp $\rho = \{r \in \mathcal{S}_{\omega_1} | \rho(r) \neq 0\}$ is a well-ordered, at most countable set.

For $\rho,\ \rho' \in G^{(1)}_{\omega_1},\ \rho \neq \rho'$ set $\rho > \rho'$ if $\rho\,(s) > \rho'\,(s)$ where $s = \inf\,\{r \in \mathcal{S}_{\omega_1} | \rho\,(r) \neq \rho'\,(r)\}$ so that $(G^{(1)}_{\omega_1},\ \leq)$ is a totally ordered real linear space.

(4.3) We set $T'_{\omega_1} = \{\rho \in G^{(1)}_{\omega_1} | \rho > 0\}$, $T_{\omega_1} = T'_{\omega_1} \cup \{0\}$.

An abelian additive monoid is a nonempty set V equipped with an associative, abelian law $(x,\ y) \to x + y$. We will say that $(V,\ +)$ is cancellative if the map $x \to x + y$ is one-to-one for every $y \in V$, and we will say that $(V,\ +)$ is without torsion if $n.d. \neq nd'$ for $n \geq 2$, $d,\ d' \in V$, $d \neq d'$. Also we will say that the positive rationals operate on $(V,\ +)$ if there exists a map $(r,\ d) \to r.d.$ from $Q^{+*} \times V$ into V such that $1.d = 1$, $r\,(d + d') = r.d + r.d'$, $(r + r').d = r.d + r'.d$, $r\,(r'\,d) = (rr').d$ $(r,\ r' \in Q^{+*},\ d,\ d' \in V)$. Clearly, $(V,\ +)$ is without torsion if the positive rationals operate on $(V,\ +)$. We define in a similar way multiplicative abelian monoids, cancellative multiplicative abelian monoids, etc.

DEFINITION 4.4. – 1) *A partially ordered set* $(\mathcal{E},\ <)$ *is universal if there exists a map* $\theta : \mathcal{S}_{\omega_1} \to \mathcal{E}$ *such that* $\theta\,(u) > \theta\,(v)$ *for* $u,\ v \in \mathcal{S}_{\omega_1},\ u > v$.

2) *An additive (resp. multiplicative) monoid* $(V,\ +)$ *(resp.* $(V,\ .)$*) is universal if there exists a one-to-one map* $\psi : T'_{\omega_1} \to V$ *such that* $\psi(\rho + \rho') = \psi(\rho) + \psi(\rho')$ *(resp.* $\psi(\rho + \rho') = \psi(\rho).\psi(\rho')$*)* $(\rho,\ \rho' \in T'_{\omega_1})$.

It is well-known that for every partially ordered set $(\mathcal{F},\ <)$ such that card $\mathcal{F} \leq \aleph_1$ there exists a one-to-one, order preserving map $\mathcal{F} \to \mathcal{S}_{\omega_1}$. Similarly for every cancellative, non unital additive (resp. multiplicative) monoid $(V,\ +)$ (resp. $(V,\ .)$) without torsion such that card $V \leq \aleph_1$ there exists a one-to-one map $\varphi : V \to T'_{\omega_1}$ such that $\varphi\,(x + x') = \varphi\,(x) + \varphi\,(x')$ (resp. $\varphi\,(x + x') = \varphi\,(x).\varphi\,(x')$) $(x,\ x' \in V)$. So if the continuum hypothesis is assumed, and if $\mathcal{E}$ is a universal set, then every partially ordered set $\mathcal{F}$ such that card $\mathcal{F} \leq 2^{\aleph_0}$ is order-isomorphic to a subset of $\mathcal{E}$. Similarly if V is a universal abelian monoid, and if W is a cancellative, non unital abelian monoid without torsion such that card $W \leq 2^{\aleph_0}$, then W is isomorphic as an abelian monoid to a sub-monoid of V. All these results are well-known, and the details can be found in [17], [25] (the corresponding result for algebras shows that if the continuum hypothesis is assumed then any commutative, non unital complex algebra B which is an integral domain is isomorphic to some subalgebra of the algebra $\mathbb{C}'_{\omega_1}$ introduced in [18], *see* [20]). The perhaps simplest example of a universal set is given by a natural quotient of the set $\mathbb{N}^{\mathbb{N}}$ of sequences $(p_n)_{n\geq 1}$ of positive integers.

We equip $\mathbb{N}^{\mathbb{N}}$ with coordinatewise addition, and with the following binary relations, defined for $p = (p_n)_{n\geq 1}$, $q = (q_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ as follows

(4.5) $p > q$ if $p \in q + \mathbb{N}^{\mathbb{N}}$.

(4.6) $p \mathcal{R} q$ if $\{n \geq 1 | p_n \neq q_n\}$ is finite or empty.

The equivalence relation $\mathcal{R}$ is compatible with addition and order on $\mathbb{N}^{\mathbb{N}}$ and so the quotient $\mathbb{N}^{\mathbb{N}}/\mathcal{R}$ is a partially ordered, abelian, cancellative, non unital abelian monoid.

It is a standard fact that $(\mathbb{N}^{\mathbb{N}}/\mathcal{R}, \leq)$ is a universal set. It is also true that $(\mathbb{N}^{\mathbb{N}}/\mathcal{R}, +)$ is a universal additive monoid. More precisely we have the following result, proved in detail in [25] (there should be much earlier references for analogous results).

LEMMA 4.7. – *Let $(p_n)_{n\geq 1}$ be a sequence of positive integers such that $p_n \xrightarrow[n\to\infty]{} \infty$. Set $\mathcal{E} = \left((m_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}} | \frac{m_n}{p_n} \xrightarrow[n\to\infty]{} 0\right)$. Then $(\mathcal{E}/\mathcal{R}, +)$ is a universal additive abelian monoid.*

The author showed in [25] that if I is a dense, prime modular ideal of countable type in a commutative Fréchet algebra A then the set of nonzero, singular elements of A/I is a universal multiplicative abelian monoid. We now wish to extend this result to the set of nonzero elements of $\bigcap_{n\geq 1} a^n A/I$ where a is a nonzero, singular element of A/I. The method of [25], based on natural infinite products in A, cannot be applied directly here, because remark 2-21 shows that $\bigcap_{n\geq 1} a^n \cdot A$ may reduce to $\{0\}$ for every singular element a of A. We will need the following technical result.

LEMMA 4.8. – *Let $(G, +)$ be a totally ordered, divisible group, set $V = \{d \in G | d > 0\}$, and let $(B, .)$ be a cancellative, unital abelian monoid. Let a be a noninvertible element of B, and assume that there exists a sequence $(a_n)_{n\geq 1}$ of elements of B such that $a_1 = a$ and $a_n = a_{n+1}^{n+1} (n \geq 1)$. For $\alpha_1, \alpha_2 \in B$ set $\alpha_1 \mathcal{R}_a \alpha_2$ if and only if $\{a^p.\alpha_1\}_{p\geq 1} \cap \{a^p.\alpha_2\}_{p\geq 1} \neq \emptyset$. Let $\varphi : V \to \bigcap_{n\geq 1} a^n B$ be a map such that $\varphi(d_1 + d_2) \mathcal{R}_a \varphi(d_1).\varphi(d_2) (d_1, d_2 \in V)$. Then there exists $\psi : V \to \bigcap_{n\geq 1} a^n B$ which satisfies the two following conditions*

(1) $\psi(d_1 + d_2) = \psi(d_1).\psi(d_2) (d_1, d_2 \in V)$.

(2) *For every $d \in V$, there exists $n \geq 1$ such that $\psi(nd) \mathcal{R}_a \varphi(nd)$.*

Proof. – Since G is linearly ordered and divisible, the equation $nx = d$ has a unique solution in G for every $n \geq 1$, and routine, well-known verifications [29] show that G possesses a structure of totally ordered, rational linear space which extends the map $(n, d) \to n.d.$ $(n \in \mathbb{Z}, d \in G)$. For $p \geq 1$, $q \geq 1$ set $a^{p/q} = a_q^{p(q-1)!}$. As in the proof of theorem 3-10 we see that the map $r \to a^r$ is well-defined on $Q^{+*} = \{r \in Q | r > 0\}$ and that $a^{r+r'} = a^r.a^{r'}$ $(r, r' \in Q^{+*})$. Clearly, $a^1 = a$, and $\bigcap_{n\geq 1} a^n B = \bigcap_{r\in Q^{+*}} a^r B$.

Let $u \in \bigcap_{n\geq 1} a^n B$. Since B is cancellative, the equation $a^r.x = u$ has a unique solution in B for $r \in Q^{+*}$, and $x \in \bigcap_{n\geq 1} a^n B$. So we can define $a^r \cdot u \in \bigcap_{n\geq 1} a^n B$ for $r \in Q$, $u \in \bigcap_{n\geq 1} a^n B$, and we have $a^{r_1}.(a^{r_2}.u) = a^{r_1+r_2}.u$, $a^{r_1}(uv) = (a^{r_1}u).v$ ($u, v \in \bigcap_{n\geq 1} a^n B$, $r_1, r_2 \in Q$).

Now let $d \in V$, and let $r \in Q$, $n \geq 1$. There exists $p \in \mathbb{Z}$ such that $a^p \cdot \varphi\left(\frac{d}{n}\right)^n = \varphi(d)$. Hence $[a^{p/n+r/n}.\varphi(d/n)]^n = a^r\varphi(d)$, and we see that we can construct by induction a sequence $(r_n)_{n\geq 1}$ of rational numbers, with $r_1 = 0$, such that $a^{r_n}.\varphi(d/n) = [a^{r_{n+1}}.\varphi(d/n+1)]^{n+1}$ $(n \geq 1)$.

Let $(d_\lambda)_{\lambda\in\Lambda} \subset V$ be a Hamel basis of G over Q. For every $\lambda \in \Lambda$ there exists a sequence $(r_{n,\lambda})_{n\geq 1}$ of rational numbers, with $r_{1,\lambda} = 0$, such that if we set $d_{n,\lambda} = a^{r_n}.\varphi(d_\lambda/n)$, we have $d_{n,\lambda} = (d_{n+1,\lambda})^{n+1}$ $(n \geq 1)$.

For $p \geq 1$, $q \geq 1$ set $\varphi(d_\lambda)^{p/q} = d_{q,\lambda}^{p.(q-1)!}$. We see as above that we obtain a well-defined map $r \to \varphi(d_\lambda)^r$ from Q^{+*} into $\bigcap_{n\geq 1} a^n B$ such that $\varphi(d_\lambda)^{r_1+r_2} = \varphi(d_\lambda)^{r_1}.\varphi(d_\lambda)^{r_2}$ for $r_1, r_2 \in Q^{+*}$, and such that for every $r \in Q^{+*}$ there exists $s \in Q$ satisfying $\varphi(d_\lambda)^r = a^s.\varphi(d_\lambda^r)$.

Denote by W the set of elements of V with nonnegative coordinates with respect to the basis $d_\lambda)_{\lambda\in\Lambda}$. For $\lambda_1, \ldots, \lambda_k \in \Lambda$, $r_1, \ldots, r_k \in Q^{+*}$ set $\psi\left(\sum_{i=1}^k r_i.d_{\lambda_1}\right) = \prod_{i=1}^k \varphi(d_{\lambda_1})^{r_i}$. Then $\psi : W \to \bigcap_{n\geq 1} a^n B$ is a well-defined map such that $\psi(d_1 + d_2) = \psi(d_1).\psi(d_2)$ $(d_1, d_2 \in W)$. Also for every $d \in W$ there exists $s \in Q$ such that $\psi(d) = a^s.\varphi(d)$.

Now let $d \in V$, and let $d_1, d_2 \in W$ such that $d = d_1 - d_2$. Since $\varphi(d_1) = \varphi(d + d_2)\,\mathcal{R}_a\,\varphi(d).\varphi(d_2)$, there exists $s \in Q$ such that $\psi(d_1) = a^s.\varphi(d).\psi(d_2)$. Set $\psi(d) = a^s.\varphi(d)$. Since B is a cancellative monoid, an easy verification shows that this definition of $\psi(d)$ does not depend on the choice of d_1 and d_2. Now let $d, d' \in V$, and let $d_1, d_2, d_1', d_2' \in W$ be such that $d = d_1 - d_2$, $d' = d_1' - d_2'$. We have $\psi(d_1 + d_1') = \psi(d_1).\psi(d_1') = \psi(d)\psi(d')\,\psi(d_2 + d_2')$, and so $\psi(d + d') = \psi(d).\psi(d')$. Also, if $d \in V$, there exists $s \in Q$ such that $\psi(d) = a^s.\varphi(d)$. Set $s = p/q$, where $p \in \mathbb{Z}$, $q \geq 1$. We have $\psi(q.d) = a^p.\varphi(d)^q\,\mathcal{R}_a\,\varphi(q.d)$, which concludes the proof of the lemma.

THEOREM 4.9. – *Let A be a commutative, unital Fréchet algebra, and let I be a dense, prime modular ideal of A of countable type. Let B be the set of nonzero, singular elements of A/I. Then $B \neq \emptyset$, and $\bigcap_{n\geq 1} a^n B$ is a universal multiplicative abelian monoid for every $a \in B$.*

Proof. – If A is not unital, let $A^\#$ be the Fréchet algebra obtained by adjoining a unit element to A, and let $f \in A$ be such that $x - xf \in I\,(x \in A)$. Set $J = A \oplus \mathbb{C}\,(f - 1)$. Easy verifications (*see* [25]) show that J is a prime, dense ideal of $A^\#$ of countable type, and that the quotient algebras A/I and $A^\#/J$ are isomorphic. So we may assume without loss of generality that A is unital. Since I is a dense ideal of A of countable type, there exists a continuous, onto map $\varphi : A \to \mathbb{C}^{\mathbb{N}}$ such that $\varphi(I) = c_{oo}$ [24, Theorem 3-1]. So I certainly not maximal, and $B \neq \emptyset$ (of course, the fact that $B \neq \emptyset$ also follows from [25, Theorem 2-10]).

Let $a \in B$. It follows from Lemma 3-9 that there exists a sequence $(b_n)_{n\geq 1}$ of elements of $\bigcap_{n\geq 1} a^n B$ such that $b_n = b_{n+1}^{n+1}\,(n \geq 1)$. Let $(p_n)_{n\geq 1}$ be a sequence of positive integers such that $p_n \underset{n\to\infty}{\longrightarrow} \infty$, let $\pi : A \to A/I$ be the canonical surjection and let $b \in A\backslash I$ such that $\pi(b) = b_1$.

Since I is of countable type, $A\backslash I$ is a G_δ subset of A and we can define the given topology on $A\backslash I$ by a distance d with respect to which $A\backslash I$ is a complete set. Since I is dense in A, there exists a sequence $(y_n)_{n\geq 1}$ of elements of I such that $b + y_n \underset{n\to\infty}{\longrightarrow} 1$. Hence $(b + y_n)^p \underset{n\to\infty}{\longrightarrow} 1$ for every $p \geq 1$.

So we can define by induction a sequence $(z_n)_{n\geq 1}$ of elements of $A\backslash I$, with $z_1 = 0$, which satisfies the following condition

$$d\left[\prod_{k=1}^{n-1}(b+z_k)^{m_k},\ \prod_{k=1}^{n}(b+z_k)^{m_k}\right] < 2^{-n} \tag{1}$$

$$(0 \leq m_1 \leq p_1,\ \dots,\ 0 \leq m_n \leq p_n,\ n \geq 2).$$

Set $\mathcal{E} = \left\{m = (m_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}} \mid \frac{m_n}{p_n} \underset{n\to\infty}{\longrightarrow} 0\right\}$. It follows from condition (1) that for every $p \geq 1$, the sequence $\left(\prod_{k=p}^{n}(b+z_k)^{m_k}\right)_{n\geq p}$ converges in $A\backslash I$ for $m = (m_n)_{n\geq 1} \in \mathcal{E}$. Set $\varphi_p(m) = \lim_{n\to\infty} \prod_{k=p}^{n}(b+z_k)^{m_k}$ $(m \in \mathcal{E})$. Then $\varphi_p(m+m') = \varphi_p(m)\,\varphi_p(m)$ $(m,\ m' \in \mathcal{E})$.

Since $\varphi_p(m) = \prod_{k=p}^{p+q}(b+z_k)^{m_k}\,\varphi_{p+q+1}(m)$ $(p \geq 1,\ q \geq 0)$ we have $(\pi \circ \varphi_p)(m) \in \bigcap_{n\geq 1} b_1^n . A/I \subset \bigcap_{n\geq 1} a^n . A/I$ and $(\pi \circ \varphi_p)(m)\,\mathbb{R}_{b_1}\,(\pi \circ \varphi_1)(m)$ $(m \in \mathcal{E},\ p \geq 1)$, the notations being as in Lemma 4-8. Now if $m,\ m' \in \mathcal{E}$ and if $m\ \mathcal{R}\ m'$ in the sense of definition 4-6, we have $\varphi_p(m) = \varphi_p(m')$ for some $n \geq 1$ and so $(\pi \circ \varphi_1)(m)\,\mathcal{R}_{b_1}\,(\pi \circ \varphi_1)(m')$.

Let $\omega : \mathcal{E} \to \mathcal{E}/\mathcal{R}$ be the canonical surjection, and let $\delta : \mathcal{E}/\mathcal{R} \to \mathcal{E}$ be a map such that $(\omega \circ \delta)(u) = u\,(u \in \mathcal{E}/\mathcal{R})$.

Since $\mathcal{E}/\mathcal{R}$ is universal, by Lemma 4-7, there exists $\theta : T'_{\omega_1} \to \mathcal{E}/\mathcal{R}$ such that $\theta(d_1 + d_2) = \theta(d_1) + \theta(d_2)$ $(d_1,\ d_2 \in T'_{\omega_1})$. Set $\varphi = \pi \circ \varphi_1 \circ \delta \circ \theta$. For $d_1,\ d_2 \in T'_{\omega_1}$, we have $(\omega \circ \delta \circ \theta)\ (d_1 + d_2) = \theta(d_1) + \theta(d_2) = \omega\ [(\delta \circ \theta)(d_1) + (\delta \circ \theta)(d_2)]$. Hence $(\delta \circ \theta)(d_1 + d_2)\,\mathcal{R}\,[(\delta \circ \theta)(d_1) + (\delta \circ \theta)(d_2)]$. So

$$\varphi(d_1 + d_2) = (\pi \circ \varphi_1)((\delta \circ \theta)(d_1 + d_2))\,\mathcal{R}_{b_1}\,(\varphi(d_1) + \varphi(d_2)).$$

It follows then from Lemma 4-8 that there exists $\psi : T'_{\omega_1} \to \bigcap_{n\geq 1} b_1^n . A/I \subset \bigcap_{n\geq 1} a^n . A/I = \bigcap_{n\geq 1} a^n . B$ such that $\psi(d_1 + d_2) = \psi(d_1).\psi(d_2)$ $(d_1,\ d_2 \in T'_{\omega_1})$. Hence $\bigcap_{n\geq 1} a^n . B$ is a universal monoid, which concludes the proof of the theorem.

Remark 4-11. – 1) The reader has certainly noticed that Corollary 3-4, Lemma 3-9 and Theorem 3-14 remain valid for general commutative, metrizable complete algebras. Theorem 4-10 is also valid for quotients A/I where I is a dense, prime, modular ideal of a commutative, metrizable complete algebra A if I is not maximal. The author did not investigate whether the hypothesis that I is not maximal is redundant in this context.

2) The most technical part of the authors's solution of Kaplansky's problem consists in embedding T'_{ω_1} in the multiplicative monoid $\Omega = \{a \in \mathcal{R} \mid [a\mathcal{R}]^- = \mathcal{R}\}$ where $\mathcal{R}$ is

a (nonzero) commutative radical Banach algebra which possesses dense principal ideals (the case of a commutative, radical Banach algebra which possesses elements of finite closed descent in the sense of Allan [1] reduces easily to this situation, *see* [23], [55]). The strategy used in this section suggests a much simpler approach. Denote by $\mathcal{R}^{\#}$ the algebra obtained by adjoining a unit element to $\mathcal{R}$, and set $\mathcal{G} = \exp(\mathcal{R}^{\#}) = \mathrm{Inv}(\mathcal{R}^{\#})$. Since Ω is a G_δ-subset of $\mathcal{R}$, Ω is homeomorphic to a complete, metrizable set. Also $a.\mathcal{G}$ is dense in Ω for every $a \in \Omega$ since $\mathcal{R}$ is radical.

A direct application of the Mittag-Leffler theorem shows then that for every $a \in \Omega$ there exists a sequence $(b_n)_{n\geq 1}$ of elements of $\bigcap_{n\geq 1} a^n.\Omega$ such that $b_n = b_{n+1}^{n+1}\,(n \geq 1)$. Fix $p \geq 1$, and set $b = b_1$. Since $b.b_p.\mathcal{G}$ is dense in Ω, there exists a sequence $(e_n)_{n\geq 1}$ of elements of $\mathcal{G}$ such that $b_p = \lim_{n\to\infty} b.\,b_p.e_n$. Hence for $1 \leq k \leq p$, we obtain $b_p^{\frac{p!}{k}} = \lim_{n\to\infty} b_p^{\frac{p!}{k}}\, b.e_n$. So $b = [b_p^{\frac{p!}{k}}]^k = \lim_{n\to\infty} b.\,b^k.e_n^k$, and more generally $u = \lim_{n\to\infty} u.\,b^k.e_n^k\,(1 \leq k \leq p)$ for every $u \in b.\mathcal{R}^{\#}$.

The given topology on Ω can be defined by a distance d such that $(\Omega,\ d)$ is a complete set. Let $(p_n)_{n\geq 1}$ be a sequence of positive integers such that $p_n \underset{n\to\infty}{\longrightarrow} \infty$. It follows from the above discussion that we can construct by induction a sequence $(u_n)_{n\geq 1}$ of elements of $\mathcal{G}$ such that the following condition holds

$$(2)\qquad d\left[\prod_{k=1}^{n-1} b^{m_k}.u_k^{m_k},\ \prod_{k=1}^{n} b^{m_k}.u_k^{m_k}\right] < 2^{-n}$$
$$(0 \leq m_1 \leq p_1,\ \ldots,\ 0 \leq m_n \leq p_n,\ m_1 \cdots + m_{n-1} > 0,\ n \geq 2).$$

Define $\mathcal{E}$ as in the proof of Theorem 4-10. Clearly, for every $p \geq 1$ and every $m = (m_n)_{n\geq 1} \in \mathcal{E}$ the sequence $\left(\prod_{k=p}^{n} b^{m_k}.u_k^{m_k}\right)_{n\geq p}$ has a limit $\varphi_p(m)$ in $\mathcal{R}$, and $\varphi_1(m) = (\prod_{k=1}^{p} b^{m_k}.u_k^{m_k}).\varphi_{p+1}(m)$. Now for $\alpha_1,\ \alpha_2 \in \Omega$ set $\alpha_1\ \tilde{\mathcal{R}}_b\ \alpha_2$ if there exists $p \geq 1$, $q \geq 1$ and $u \in \mathcal{G}$ such that $b^p.c = b^q.u.d$. We obtain an equivalence relation on the cancellative monoid $(\Omega,\ .)$, and since $\mathcal{E}/\mathcal{R}$ is universal we can construct as in the proof of theorem 4-10 a map $\varphi : T'_{\omega_1} \to \bigcap_{n\geq 1} b^n.\Omega$ such that $\varphi(d_1+d_2)\,\tilde{\mathcal{R}}_b\,\varphi(d_1).\varphi(d_2)$ $(d_1,\ d_2 \in T'_{\omega_1})$. But $\mathcal{G}$ is a divisible abelian group, and a slight modification of the proof of lemma 4-9 gives a map $\psi : T'_{\omega_1} \to \bigcap_{n\geq 1} b^n.\Omega$ such that $\psi(d_1+d_2) = \psi(d_1).\psi(d_2)$ $(d_1,\ d_2 \in T'_{\omega_1})$ (and such that for every $d \in T'_{\omega_1}$ there exists $n \geq 1$ satisfying $\varphi(nd)$ $\tilde{\mathcal{R}}_b\ \psi(nd)$).

The situation is even simpler if we restrict attention to the case where the radical Banach algebra $\mathcal{R}$ possesses a sequential, bounded approximate identity. We can assume without loss of generality that $\mathcal{R}$ possesses a sequential bounded approximate identity (e_n) such that $||e_n|| = 1\,(n \geq 1)$, *see* [48].

Choose $\lambda \in (0,1)$. We can construct by induction a sequence $(f_n)_{n\geq 1}$ of elements of $\mathcal{R}$ such that for every $m = (m_n)_{n\geq 1} \in \mathcal{E}$ and every $p \geq 1$ the sequence $\left(\prod_{k=p}^{n}(1-\lambda(e-e_k))^{m_k}\right)_{n\geq p}$ has a limit $\varphi_p(m)$ in Ω, so that $\varphi_1(m)\ \tilde{\mathcal{R}}\ \varphi_1(m')$ if $m\mathcal{R}m'$ $(m,\ m' \in \mathcal{E})$. Here we define $\tilde{\mathcal{R}}$ to be the relation $\alpha_1\ \tilde{\mathcal{R}}\ \alpha_2$ if and only if $\alpha_1 \in \alpha_2\,\mathcal{G}$ on Ω. Since $\mathcal{G}$ is a divisible group, a modification of the proof of lemma 4-9 gives again a map $\psi : T'_{\omega_1} \to \Omega$ such that $\psi(d_1 + d_2) = \psi(d_1).\psi(d_2)$ $(d_1, d_2 \in T'_{\omega_1})$ (and such that $\psi(d)\,\tilde{\mathcal{R}}\,\varphi_1(d)$ for every $d \in T'_{\omega_1}$). By using standard factorization methods [48] it is also possible to arrange, $a \in \mathcal{R}$ being given, that $a \in \psi(d).\mathcal{R}$ for every $d \in T'_{\omega_1}$ (in fact, the construction of the map φ_1 is sufficient to perform the embedding: $\mathbb{C}_{\omega_1} \to \mathcal{R}^{\#}$ as in [18], and so the construction of [19] and [54] can be replaced by a few lines to obtain a quick solution of Kaplansky's problem, assuming the continuum hypothesis).

The details of the above computations will be published elsewhere. The proofs we outlined here are clearly much simpler than the original proofs by the author [19], [21], and also significantly simpler than the improvements of these proofs given in [23], [54], [55].

Elements of $\mathcal{E}/\mathcal{R}$ such that the equation $m.x = d$ has a (unique) solution in $(\mathcal{E}/\mathcal{R}, +)$ for every $m \geq 1$ can be constructed very easily. Construct a strictly increasing sequence $(k_i)_{i\geq 1}$ of integers such that $p_n \geq (i+1)!$ for every $n \geq k_i$, set $k_0 = 0$ and for $k_i \leq n < k_{i+1}$ set $u_n = i!$. Then $u = (u_n)_{n\geq 1} \in \mathcal{E}$, and the equivalence class d of u satisfies the required condition. The proof of lemma 4-7 (*see* the details in [25]) is then based on the standard arguments, which go back to the last century, used to prove the universality of the set $(\mathbb{N}^{\mathbb{N}}, <)$. Also the arguments given here show that if the set of positive elements V of a totally ordered group G can be embedded in $(\mathcal{E}/\mathcal{R}, +)$, then it can be embedded in the multiplicative monoid $(\Omega, .)$ where $\Omega = \{x \in \mathcal{R} | [x\mathcal{R}]^- = \mathcal{R}\}$ for every commutative radical Banach algebra $\mathcal{R}$ which possesses dense principal ideals. Conversely, if there is an embedding of V into $(U, .)$, where we denote by U the set of nonzero elements of $\mathcal{R}$, then V is order isomorphic to a subset of $(\mathbb{N}^{\mathbb{N}}/\mathcal{R}, <)$ (*see* [50]). Since $(\mathcal{E}/\mathcal{R}, +)$ is an object familiar to logicians, the new approach presented here of the constructions of [18], [21] should be helpful in the discussion of the dependence of existence of discontinuous homomorphisms of $\mathcal{C}(K)$ on axioms of set theory (there was recent progress in this area, *see* [28], [51]). We refer the reader interested in this matter to [14], [50].

5. Testing the test for the test algebra $\mathcal{U}$

The notations being as in section 2, set $\mathcal{V} = \mathcal{U}/\mathcal{I}_\infty$ and denote by $\pi_\infty : \mathcal{U} \to \mathcal{V}$ the canonical surjection. Denote by $\widehat{\mathcal{V}}$ the set of all characters of $\mathcal{V}$. Then $\widehat{\mathcal{V}}$ contains at least one element, the trivial character $\chi_0 : \pi_\infty(\sum_{\alpha\in S} \lambda_\alpha X^\alpha) \to \lambda_0$. Of course, if all characters on commutative Fréchet algebras were continuous, then $\widehat{\mathcal{V}}$ would reduce to χ_o. In the opposite direction it follows from Theorem 2-7 that if f is a linear element of $\mathcal{M}$, *i.e.* $f = \sum_{n=1}^\infty \lambda_n X_n$, where $(\lambda_n)_{n\geq 1} \in \ell^1$, and if we set $v = \pi_\infty(f)$, then assuming that there exists a discontinuous character on some commutative Fréchet algebra we would have $\sigma(v) = \{\chi(v)\}_{v\in\widehat{\mathcal{V}}}$ (the case where $(\lambda_n)_{n\geq 1} \in c_{oo}$ is trivial; notice that $\sigma(v) = \mathbb{C}$ when $(\lambda_n)_{n\geq 1} \in \ell^1 \backslash c_{oo}$).

Let $M \subset A$, where A is a commutative, unital Fréchet algebra. As in section 2 we denote by M^∞ the set of all bounded sequences of elements of M. For $f \in \mathcal{U}$, $a \in A^\infty$, $f(a) \in A$ is defined as in Proposition 2-4.

A special case of the following easy observation was used in the proof of Theorem 2-7 (4).

PROPOSITION 5-1. – *Let $v \in \mathcal{V}$, and let $f \in \pi_\infty^{-1}(\{v\})$. Denote by $\Lambda(v)$ the set of all $\lambda \in \mathbb{C}$ such that, for every dense ideal I of an arbitrary commutative unital Fréchet algebra A, there exists $a \in I^\infty$ such that $f(a) - \lambda e \in I$. Then the definition of $\Lambda(v)$ does not depend on the choice of f, $\Lambda(v) \subset \sigma(v)$ and if there exists a discontinuous character on some commutative Fréchet algebra A, $\Lambda(v) \subset \{\chi(v)\}_{\chi \in \widehat{\mathcal{V}}}$.*

Proof. – If $g \in \mathcal{I}_\infty$, then $g(a) \in I$ for every $a \in I^\infty$, and so the choice of $\Lambda(v)$ does not depend on the choice of f. Let I be a dense ideal in a commutative unital, Fréchet algebra A and let $\pi : A \to A/I$ be the canonical map. The map $\theta : \pi_\infty(g) \to \pi(g(a))$ is a homomorphism from $\mathcal{V}$ into A/I for every $a \in I^\infty$. If $f(a) - \lambda e \in I$ then $\theta(v) = \lambda.e$ and $\lambda \in \sigma(\theta(v)) \subset \sigma(v)$. Now assume that $\lambda \in \Lambda(v)$, and that there exists a discontinuous character χ on some commutative Fréchet algebra A. We can assume without loss of generality that A is unital. Set $I = \mathrm{Ker}\,\chi$ and let $a \in I^\infty$ such that $f(a) - \lambda.e \in I$. The map $\varphi : \pi_\infty(g) \to \pi(g(a))$ is well-defined, and $\varphi \in \widehat{\mathcal{V}}$. We have $\varphi(v) = \chi(f(a)) = \lambda$, which concludes the proof of the proposition.

The proof of theorem 2-7 (4) was based upon the observation that if $(\lambda_n)_{n\geq 1} \in \ell^1 \backslash c_{00}$, and if we set $f = \sum_{n=1}^\infty \lambda_n X_n$, then the map $a \to f(a)$ is a surjective map from I^∞ onto A if I is a dense ideal of a commutative Fréchet Algebra A. So in this case $\Lambda(\pi_\infty(f)) = \mathbb{C}$. In this section, we produce examples of less trivial elements f of $\mathcal{M}$ for which $\Lambda(\pi_\infty(f)) = \mathbb{C}$. In the opposite direction, we will construct some $g \in \mathcal{M} \backslash \mathcal{I}_\infty$ for which $\Lambda(\pi_\infty(g)) = \{0\}$. So for these elements g, the existence of a discontinuous character on some commutative Fréchet algebra would not give any direct information about the set $\{\chi(\pi_\infty(g))\}_{\chi \in \widehat{\mathcal{V}}}$.

We will use the following elementary identity, which is valid for $p \geq 2$ for any element x of a complex algebra with unit e.

(5.2) $p^2.x = \sum_{j=1}^p \rho_j (x + \rho_j.e)^p$ where $\rho_1 \dots \rho_p$ are the p$^{\text{th}}$ roots of the unit in $\mathbb{C}$.

Identity 5-2 follows immediately from the fact if $k \in \mathbb{Z}$, $k \notin p\,\mathbb{Z}$ then $\sum_{j=1}^p p_j^k = 0$.

LEMMA 5.3. – *Let $(A, (||.||_n)_{n\geq 1}$ be a commutative, unital Fréchet algebra, let $x \in A$, let I be a dense ideal of A, and let $n \geq 1$, $p \geq 2$. Then there exists a sequence $(a_{1,m}, \dots, a_{p,m})_{m\geq 1}$ of elements of I^p such that $\overline{\lim_{m\to\infty}} ||a_{j,m}||_n \leq \frac{2\,||x||_n^{1/p}}{p^{2/p}}$ $(j \leq p, m \geq 1)$, and such that $x = \lim_{m\to\infty} \sum_{j=1}^p a_{j,m}^p$.*

Proof. – Let $(e_m)_{m\geq 1}$ be a sequence of elements of I such that $e_m \xrightarrow[m\to\infty]{} 1$, so that $||e_m||_n \xrightarrow[m\to\infty]{} 1$. Then $e_m^p \xrightarrow[m\to\infty]{} 1$. The notations being as in formula 5-2, choose $\delta_j \in \mathbb{C}$ such that $\delta_j^p = \rho_j\,(j \leq p)$, and set $\varepsilon_m = ||x||_n + \frac{1}{m}$, $a_{j,m} = \delta_j.p^{-2/p}.(\varepsilon_m^{-1}.x + \rho_j.e).(\varepsilon_m^{1/p}.e_m)\,(m \geq 1)$. Then $\overline{\lim_{m\to\infty}} ||a_{j,m}||_n \leq 2.p^{-2/p}.||x||_n^{1/p}$, and it follows from formula 5-2 that $x = \lim_{m\to\infty} x.e_m^p = \lim_{m\to\infty} \varepsilon_m^{-1}.x\,(\varepsilon_m^{1/p}.e_m)^p = \lim_{m\to\infty} \sum_{j=1}^p a_{j,m}^p$, which proves the lemma.

THEOREM 5.4. – *Let* $(\lambda_n)_{n\geq 1} \in \ell^1\backslash c_{00}$, $p \geq 1$, *let* $f = \sum_{n=1}^{\infty} \lambda_n X_n^p$, $v = \pi_\infty(f)$, *and let* I *be a dense ideal of a commutative, unital Fréchet algebra* A. *Then* $f(I^\infty) = A$. *In particular,* $\Lambda(v) = \mathbb{C}$.

Proof. – The result has been proved in section 2 when $p = 1$, and so we can assume that $p \geq 2$. Also we can assume without loss of generality that $\lambda_n \neq 0\,(n \geq 1)$, since $(\sum_{k=1}^{\infty} \lambda_{n_k} X_k^p)(I^\infty) \subset f(I^\infty)$ for every strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers. Let $x \in A$. We construct by induction a sequence $V_n = (v_{n,1}, \ldots v_{n,p})_{n\geq 1}$ of elements of I^p such that if we set $w_n = \sum_{j=1}^{p} \lambda_{(n-1)p+j}\, v_{n,j}^p$, $\varepsilon_n = p^{-2}.2^{-p-1} \inf\{|\lambda_j|\}_{(n-1)p+1\leq j\leq np}$, the following properties hold

$$(1) \qquad \left\| x - \sum_{m=1}^{n} w_m \right\|_{n+1} \leq \varepsilon_{n+1} \quad (n \geq 1)$$

$$(2) \qquad \|v_{n,j}\|_n \leq 1 \quad (1 \leq j \leq p,\ n \geq 2)$$

Choose a sequence $(\mu_j)_{j\geq 1}$ of complex numbers such that $\mu_j^p = \lambda_j\,(p \geq 1)$. It follows from lemma 5-3 that there exists a sequence $(a_{1,k}, \ldots a_{p,k})_{k\geq 1}$ of elements of I^p such that $x = \lim_{k\to\infty} \sum_{j=1}^{p} a_{j,k}$. If we set $V_1 = (\mu_1^{-1} a_{1,k}, \ldots, \mu_p^{-1} a_{p,k})$ with k sufficiently large, we see that condition (1) is satisfied by V_1.

Now assume that have constructed $V_1, \ldots V_n$ so that (1) holds for $1 \leq m \leq n$ and (2) holds for $2 \leq m \leq n$. Then $\|x - \sum_{m=1}^{n} w_m\|_{n+1} \leq \varepsilon_{n+1}$. It follows then from lemma 5-3 that there exists a sequence $(b_{1,k}, \ldots, b_{p,k})$ of elements of I^p such that $x - \sum_{m=1}^{n} w_m = \lim_{k\to\infty} \sum_{j=1}^{p} b_{j,k}^p$, and such that $\overline{\lim_{k\to\infty}} \|b_{j,k}\|_{n+1} \leq 2.p^{-2/p}.\varepsilon_{n+1}^{1/p}\,(j \leq p)$. Hence $\overline{\lim_{k\to\infty}} \|\mu_{np+j}^{-1} b_{j,k}\|_{n+1} < 1$, and we see that if we set $V_{n+1} = (\mu_{np+1}^{-1} b_{1,k}, \ldots, \mu_{np+p}^{-1} b_{p,k})$ with k sufficiently large then (1) and (2) are satisfied. So we can construct the required sequence $(V_n)_{n\geq 1} = (v_{n,1}, \ldots, v_{n,p})_{n\geq 1}$ by induction. Set $u_{(n-1)p+j} = v_{n,j}\,(n \geq 1,\ 1 \leq j \leq p)$, $u = (u_n)_{n\geq 1}$. It follows from (2) that $u \in I^\infty$. We have $f(u) = \lim_{m\to\infty} \sum_{n=1}^{mp} \lambda_n u_n^p = \lim_{m\to\infty} \sum_{k=1}^{m} w_n = x$, and the theorem is proved.

We now deduce immediately from proposition 5-1 the following result (proved in section 2 for $p = 1$).

COROLLARY 5-5. – *Let* $(\lambda_n)_{n\geq 1} \in \ell^1\backslash c_{00'}$, *and let* $p \geq 1$. *Let* $v = \pi_\infty(\sum_{n=1}^{\infty} \lambda_n X_n^p)$. *If there exists a discontinuous character on some commutative Fréchet algebra, then* $\{\chi(v)\}_{\chi\in\widehat{\mathcal{V}}} = \mathbb{C}$.

Theorem 5-4 and corollary 5-5 extend, with a similar proof, to elements of $\mathcal{M}\backslash\mathcal{I}_\infty$ of the form $f = \sum_{n=1}^{\infty} \left(\sum_{j=p_n}^{p_{n+1}-1} \lambda_j X_j^{q_n}\right)$ where $(p_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers and where $(q_n)_{n\geq 1}$ is a sequence of positive integers such that $q_n \leq p_{n+1} - p_n$ when n is sufficiently large, with suitable growth conditions on $(\lambda_j)_{j\geq 1}$ to guarantee that $f \in \mathcal{U}$. More generally it is possible to extend along the same lines theorem 5-4 and corollary 5-5 to elements of $\mathcal{U}\backslash\mathcal{I}_\infty$ of the form $f = \sum_{n=1}^{\infty} f_n(X_n)$, where $(f_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{H}(\mathbb{C})$, satisfying $\sum_{n=1}^{\infty}[\sup_{|z|\leq m} |f_n(z)|] < +\infty$ for every $m \geq 1$ and vanishing at the origin, with some suitable conditions on the sequence $(\gamma_n)_{n\geq 1}$ where γ_n is the order of the zero of f_n at the origin. We leave the details to the reader, and some new ideas would be anyway necessary to remove the restrictions on the sequence $(\gamma_n)_{n\geq 1}$ (the method used to prove theorem 5-4 does not apply for example to $f = \sum_{n=1}^{\infty}\left(\frac{X_n^n}{n!}\right)$.

Formula 5-2 shows that if A is a commutative, unital complex algebra then for every $x \in A$ there exists $x_1, \ldots, x_p \in A$ such that $x = x_1^p \cdots + x_p^p$. In particular, any $x \in A$ can be written in the form $x = x_1^3 + x_2^3 + x_3^3$.

It is not possible in general to obtain a decomposition of the form $x = y_1^3 + y_2^3$. To see this, consider the Fréchet algebra $\mathcal{H}(\mathbb{C})$. For $u, v \in \mathcal{H}(\mathbb{C})$, we have, with $j = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $u^3 + v^3 = (u+v)\,(u+jv)\,(u+\bar{j}\cdot v) = (u+v)\,((1-j)u+j(u+v))\,((1-\bar{j})\,u+\bar{j}(u+v))$. If $u^3(z) + v^3(z) = z$ for every $z \in \mathbb{C}$, we can assume that $u+v$ never vanishes on $\mathbb{C}$. Set $\varphi = \frac{u}{u+v}$. Then $\varphi^{-1}(\{\frac{j}{j-1}, \frac{\bar{j}}{\bar{j}-1},\}) \subset \{0\}$. It would follow then from Picard's theorem that φ is a nonsurjective polynomial function, hence a constant. But then (u, v) has rank one, and so, say, $u = \lambda v$ where $\lambda \in \mathbb{C}$. We would obtain $(1+\lambda^3)\,v^3(z) = z\,(z \in \mathbb{C})$, a contradiction. A similar argument shows that if $u, v \in \mathcal{H}(\mathbb{C})$, and if $u^3(z) + v^3(z) = 1$ for every $z \in \mathbb{C}$, then u and v are constant. More generally, we have the following property, which is the starting point of the construction of some nonzero $v \in \mathcal{V}$ such that $\Lambda(v) = \{0\}$ which will follow.

LEMMA 5.6. – *Let A be a Picard-Borel algebra, and let $u, v \in A$. If $u^m + v^m \in \operatorname{Inv} A$ for some $m \geq 3$, then (u, v) has rank one. In particular, $\{u, v\} \subset \operatorname{Inv} A \cup \{0\}$.*

Proof. – Let $\gamma_1, \ldots, \gamma_m$ be the solutions in $\mathbb{C}$ of the equation $z^m + 1 = 0$. Since $u^m + v^m = (u + \gamma_1 v) \ldots (u + \gamma_m v)$, we have $u + \gamma_k v \in \operatorname{Inv} A$ for every $k \leq m$. Since $m \geq 3$, the family $(u + \gamma_1 v, \ldots, u + \gamma_m v)$ is linearly dependent, and so there exists $k_1 \neq k_2$ such that $(u + \gamma_{k_1} v, u + \gamma_{k_2} v)$ has rank 1. So (u, v) has rank 1. If $u \neq 0$, there exists $\lambda \in \mathbb{C}$ such that $v = \lambda u$, and so $(1+\lambda^m)\,u^m \in \operatorname{Inv} A$. Hence $1 + \lambda^m \neq 0$, $u^m \in \operatorname{Inv} A$ and $u \in \operatorname{Inv} A$. Similarly $v \in \operatorname{Inv} A$ if $v \neq 0$.

We will also need the following technical result.

LEMMA 5.7. – *Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive real numbers and let $(p_n)_{n\geq 1}$ be a sequence of positive integers. If $\overline{\lim}_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} < \frac{1}{2}$, then there exists a sequence $(\beta_n)_{n\geq 1}$ of positive real numbers such that $\beta_n \geq (n.\varepsilon_n)^{p_n} + \beta_{n+1}^{p_n}\,(n \geq 1)$*

Proof. – There exists $n_0 \geq 1$ such that $2\,\varepsilon_{n+1}.(n+1) < n.\varepsilon_n < 1$ for $n \geq n_0$. For $n \geq n_0$ set $\beta_n = 2.\varepsilon_n^{p_n}.n^{p_n}$. We have $\beta_{n+1}^{p_n} + (n.\varepsilon_n)^{p_n} = 2^{p_n}\,[(n+1).\varepsilon_{n+1}]^{p_n \cdot p_{n+1}} + (n.\varepsilon_n)^{p_n} \leq [(2\,n+2).\varepsilon_{n+1}]^{p_n} + (n.\varepsilon_n)^{p_n} \leq 2.(n.\varepsilon_n)^{p_n} = \beta_n\,(n \geq n_0)$. We then define $\beta_{n_0-1}, \ldots, \beta_1$ by the formula $\beta_{i-1} = [\varepsilon_{i-1}\,(i-1)]^{p_{i-1}} + \beta_i^{p_{i-1}}\,(2 \leq i \leq n_0)$ to obtain the required sequence.

If A is a commutative, unital complex algebra, we will denote by $\operatorname{Sg}(A) = A \backslash \operatorname{Inv} A$ the set of singular elements of A.

DEFINITION 5.8. – *A recalcitrant element of* $\mathcal{M}$ *is an element* $f \in \mathcal{M}\backslash\mathcal{J}_\infty$ *such that* $f(u) \in \operatorname{Sg} A$ *for every commutative Picard-Borel Fréchet algebra* A *and for every* $u \in [\operatorname{Sg} A]^\infty$.

We now produce recalcitrant elements of $\mathcal{M}$. We set $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_n = \mathcal{U}_n \cap \mathcal{M}$ $(n \geq 1)$, the notations being as in Lemma 2.17.

THEOREM 5.9. – *Let* $(\varepsilon_n)_{n\geq 1}$ *be a sequence of positive real numbers such that* $\overline{\lim}_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} < \frac{1}{2}$. *Then for every sequence* $(p_n)_{n\geq 1}$ *of positive integers there exists a sequence* $(f_n)_{n\geq 1}$ *of elements of* $\mathcal{M}\backslash\mathcal{J}_\infty$ *such that* $f_n \in \mathcal{M}_{n-1}$ *and* $f_n = (\varepsilon_n X_n)^{p_n} + f_{n+1}^{p_n}$ $(n \geq 1)$. *If, further,* $p_n \geq 3$ *for every* $n \geq 1$, *and if* $(f_n)_{n\geq 1}$ *satisfies the above conditions, then* f_n *is recalcitrant for every* $n \geq 1$.

Proof. – For $p \geq 1$ set $W_p = \{f = \sum_{\alpha\in S} f_\alpha . X^\alpha \in \mathbb{C}_{\mathbb{N}}[[X]] \mid \|f\|_p = \sum_{\alpha\in S} |f_\alpha . |p^{|\alpha|} < +\infty\}$. Then W_p, as a Banach space, is isomorphic to $\ell^1(\mathbb{N})$. Since $\ell^1(\mathbb{N})$ is the dual space of c_0, this isomorphism induces on W_p a w^*-topology which we denote by w_p^*. Clearly, the w_p^*-topology agrees on bounded subsets of W_p with the topology of pointwise convergence, *i.e.* the restriction to W_p of the natural topology of $\mathbb{C}_{\mathbb{N}}[[X]]$. Let $(\beta_n)_{n\geq 1}$ be a sequence of positive integers which satisfies the conditions of Lemma 5.7 with respect to $(\varepsilon_n)_{n\geq 1}$ and $(p_n)_{n\geq 1}$. Set

$$K_1 = \{f = \sum_{\alpha\in S} f_\alpha . X^\alpha \in W_1 | f_0 = 0,\ \|f\|_1 \leq \beta_1\}, \text{ and for } n \geq 2 \text{ set}$$

$$K_n = \{f = \sum_{\alpha\in S} f_\alpha . X^\alpha \in W_n | \ \|f\|_n \leq \beta_n,\ f_0 = 0,$$
$$f_\alpha = 0 \text{ for } \operatorname{Supp}\alpha \cap \{1, \ldots, n-1\} \neq \varnothing\}.$$

Clearly, K_n is w_n^*-compact for every $n \geq 1$. We now define

$$\theta_n : K_{n+1} \to W_{n+1}$$
$$f \to (\varepsilon_n X_n)^{p_n} + f^{p_n}$$

We have $\theta_n(K_{n+1}) \subset W_{n+1} \subset W_n$, and if we set $\theta_n(f) = g = \sum_{\alpha\in S} g_\alpha . X^\alpha$ for $f \in K_{n+1}$, we have $g_\alpha = 0$ if $\operatorname{Supp}\alpha \cap \{1, \ldots, n-1\} \neq \varnothing$. Also $\|\theta_n(f)\|_n \leq (\varepsilon_n . n)^{p_n} + \|f\|_{n+1}^{p_n} \leq (\varepsilon_n . n)^{p_n} + \beta_{n+1}^{p_n} \leq \beta_n$, and so $\theta_n(K_{n+1}) \subset K_n$ $(n \geq 1)$.

For $\alpha \in S$, the set $\{(\beta_1, \ldots, \beta_m) \in S^m | \beta_1 \ldots + \beta_m = \alpha\}$ is finite for every $m \geq 2$. It follows immediately from this observation that $\theta_n : (K_{n+1}, w_{n+1}^*) \to (K_n, w_n^*)$ is continuous.

Since (K_n, w_n^*) is compact for every $n \geq 1$, a standard consequence of Tychonoff's theorem shows that $\underleftarrow{\lim}(K_n, \theta_n) \neq \varnothing$. So there exists $(f_n)_{n\geq 1} \in \prod_{n\geq 1} K_n$ such that $f_n = \theta_n(f_{n+1}) = (\varepsilon_n X_n)^{p_n} + f_{n+1}^{p_n}$ $(n \geq 1)$.

An immediate induction shows that $f_n \in \bigcap_{p\geq 1} W_p = \mathcal{U}$, and it follows from the definition of K_n that $f_n \in \mathcal{M}_{n-1}$ for every $n \geq 1$.

We have $f_{n+1}^{p_n} \in \mathcal{M}_n$, $\mathcal{M}_n \cap \mathcal{I}_n = \{0\}$ and $f_n = (\varepsilon_n X_n)^{p_n} + f_{n+1}^{p_n}$, and so $f_n \neq 0$, $f_{n+1}^{p_n} \notin \mathcal{I}_n$, and $f_n \notin \mathcal{I}_n$ $(n \geq 1)$.

We see by induction that $f_1 - f_{n+1}^{p_1 \ldots + p_n} \in \mathcal{I}_n$ $(n \geq 1)$. Since $\mathcal{I}_n$ is prime, $f_{n+1}^{p_1 \ldots + p_n} \notin \mathcal{I}_n$ and $f_1 \notin \mathcal{I}_\infty = \cup_{n\geq 1}\mathcal{I}_n$. But $\pi_\infty(f_1) = \pi_\infty(f_{n+1})^{p_1 \ldots + p_n}$ and so $\pi_\infty(f_{n+1}) \neq 0$, $f_{n+1} \notin \mathcal{I}_\infty$ for every $n \geq 1$. Hence $f_n \notin \mathcal{I}_\infty$ $(n \geq 1)$. This proves the first assertion of the theorem. Now assume that $p_n \geq 3\,(n \geq 1)$, and let $u = (u_n)_{n\geq 1}$ be a bounded sequence of singular elements of a Picard-Borel Fréchet algebra A. Let $m \geq 1$, and assume that $f_m(u) \in \mathrm{Inv}\,A$. We have $(\varepsilon_m\, u_m)^{p_m} + f_{m+1}^{p_m}(u) \in \mathrm{Inv}\,A$, and it follows from Lemma 5.6 that $u_m = 0$. Hence $f_{m+1}(u) \in \mathrm{Inv}\,A$.

Assume that $u_p = 0$, and that $f_{p+1}(u) \in \mathrm{Inv}\,A$ for $m \leq p \leq n$. Then $(\varepsilon_{n+1}.u_{n+1})^{p_{n+1}} + f_{n+2}^{p_{n+1}}(u) \in \mathrm{Inv}\,A$, and it follows again from Lemma 5.6 that $u_{n+1} = 0$, so that $f_{n+2}(u) \in \mathrm{Inv}\,A$. We thus see that $u_n = 0\,(n \geq m)$. Since $f_m \in \mathcal{M}_{m-1}$, we obtain $f_m(u) = 0$, a contradiction. Hence f_m is recalcitrant for every $m \geq 1$.

Of course, if $f \in \mathcal{M}$ is recalcitrant, then $\Lambda(\pi_\infty(f)) = \{0\}$, the notations being as in Proposition 5.1. Notice that if we set $v_n = \pi_\infty(f_n)\,(n \geq 1)$ where (f_n) is the sequence of recalcitrant elements constructed in Theorem 5.9, then $v_n = v_{n+1}^{p_n}$, with $p_n \geq 3$, and so $\sigma(v_n) = \{\lambda^{p_n}\}_{\lambda \in \sigma(v_{n+1})}$. Since $\mathbb{C}\backslash\sigma(v_{n+1})$ contains at most one element, $\sigma(v_n) = \mathbb{C}$ for every $n \geq 1$. Notice also that Theorem 5.9 gives a new construction of rational semigroups in the set of nonzero elements of $\mathcal{M}/\mathcal{I}_\infty$. We do not know whether the fact that $\pi_\infty(f)$ has roots of all orders in $\mathcal{M}/\mathcal{I}_\infty$, with $f \in \mathcal{M}\backslash\mathcal{I}_\infty$, implies that f is recalcitrant.

Concerning action of recalcitrant elements of $\mathcal{M}$ on the quotients of Fréchet algebras by Picard-Borel ideals, we have the following observation.

PROPOSITION 5.10. – *Let A be a commutative, unital Fréchet algebra, let $(I_n)_{n\geq 1}$ be a nondecreasing sequence of Picard-Borel closed ideals of A, let $I = \bigcup_{n\geq 1} I_n$ and let $\pi : A \to A/I$ be the canonical map. Then $\pi(f(u))$ is a singular element of A/I for every recalcitrant element f of $\mathcal{M}$ and for every bounded sequence $u = (u_n)_{n\geq 1}$ of elements of A such that $\pi(u_n)$ is singular for every $n \geq 1$.*

Proof. – Assume that $u \in A^\infty$ and that $\pi(f(u))$ is invertible in A/I. Then there exists $m \geq 1$ such that $\pi_m(f(u))$ is invertible in A/I_m where $\pi_m : A \to A/I_m$ is the canonical map. We have $\pi_m(f(u)) = f(\pi_m(u))$, where $\pi_m(u) = (\pi_m(u_n))_{n\geq 1}$. Since f is recalcitrant, and since A/I_n is a Picard-Borel Fréchet algebra, $\pi_m(u_n) \in \mathrm{Inv}\,A/I_m$ for some $n \geq 1$, and so $\pi(u_n) \in \mathrm{Inv}\,A/I$.

Notice that Proposition 5-10 applies in particular to the ideals $I = \mathcal{I}_\infty + (f-1)\mathcal{U}$ discussed in Theorem 2.7.

The fact that the function z cannot be written as the sum of the cubes of two entire functions was observed in the last century. *See* Halphen [33, Chap. II] for a discussion of more general equations involving entire functions, and Bloch [6] for a discussion of the existence of meromorphic curves on "irregular" surfaces. Of course, is I is a dense ideal of a commutative, unital Fréchet algebra, the trivial formula $x = \lim_{t\to 0} \frac{e^{tx}-1}{t}$ shows that for every $p \geq 2$ and every $x \in A$ there exist two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of elements of I such that $x = \lim_{n\to\infty} a_n^p + b_n^p$, but the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are in general unbounded.

It follow from the definition of the set $\mathcal{R}$ of recalcitrant elements of $\mathcal{M}$ that $fg \in \mathcal{R}\cup\mathcal{I}_\infty$ for every $f \in \mathcal{R}\cup\mathcal{I}_\infty$ and every $g \in \mathcal{V}$. The author did not investigate whether $\mathcal{R}\cup\mathcal{I}_\infty$ contains some ideal J of $\mathcal{M}$ such that $\mathcal{I}_\infty \subsetneq J$.

6. Joint spectra in $\mathcal{V}$ and Michael's problem

Let A be a commutative, unital complex algebra. If $M \subset A$, $M \neq \emptyset$, the joint spectrum $\sigma_A(M)$ (denoted by $\sigma(M)$ if there is no risk of confusion) is the set of all maps $\lambda : M \to \mathbb{C}$ such that the set $\{x - \lambda(x).e\}_{x \in M}$ is contained in some proper ideal of A. One can also define the joint spectrum $\sigma_A((x_t)_{t \in T})$, where $(x_t)_{t \in T}$ is an arbitrary family of elements of A, to be the set of elements $(\lambda_t)_{t \in T}$ of $\mathbb{C}^T$ such that the family $(x_t - \lambda_t.e)_{t \in T}$ is contained in some proper ideal of A. We will occasionnally use this point of view (*see* below). There is of course no danger of confusion, because if $(\lambda_t)_{t \in T} \in \sigma_A((x_t)_{t \in T})$ and if $x_t = x_{t'}$, then certainly $\lambda_t = \lambda_{t'}$, so that the map $x_t \to \lambda_t$ is well-defined on the set $\{x_t\}_{t \in T}$.

Denote by $\hat{A}$ the set of all characters on A. Then $\{\chi_{|M}\}_{\chi \in \hat{A}} \subset \sigma(M)$ for every $M \subset A$. Now if $a, b \in A$, $\mu \in \mathbb{C}$ and if $(z_1, z_2, z_3) \in \sigma(a, b, a+b)$, $(z_1, z_4) \in \sigma(a, \mu a)$ then $z_3 = z_1 + z_2$, $z_4 = \mu z_1$. Also if $(z_1, z_2, z_3) \in \sigma(a, b, ab)$ then $z_3 = z_1 z_2$, since $ab - z_3 e = a.(b - z_2 e) + z_2(a - z_1 e) + (z_1.z_2 - z_3).e$.

So, with the above notations, we have $\sigma(A) = \hat{A}$. It follows then form the results of section 2 that the existence of a discontinuous character on some commutative Fréchet algebra is equivalent to the fact that $\sigma(\mathcal{V})$ does not reduce to $\{\chi_o\}$, the notations being as in section 5.

Now if A is a complex, commutative unital algebra, and if $N \neq \emptyset$, $N \subset M \subset A$ we will denote by $\mathcal{P}_{N, M} : \lambda \to \lambda_{|N}$ the restriction map from $\sigma(M)$ into $\sigma(N)$.

When every maximal ideal of A is the kernel of a character of A, then $\sigma(M) = \{\chi_{|M}\}_{M \in \hat{A}}$ and so $\mathcal{P}_{N, M} : \sigma(M) \to \sigma(N)$ is always onto. Conversely assume that $\mathcal{P}_{N, N \cup \{x\}}$ is onto for every nonempty subset N of A and every $x \in A$. If J is a maximal ideal of A, let $x \in A \backslash J$. Since the map $u \to 0$ belongs to $\sigma(J)$, there exists $\lambda \in \mathbb{C}$ such that $J \cup \{x - \lambda e\} \subset J$. Hence $x - \lambda e \in J$ and J has codimension 1 in A, which means that J is the kernel of a character of A. So the following conditions are equivalent

(i) Every maximal ideal of A is the kernel of a character of A

(ii) $\sigma(M) = \{\chi_{|M}\}_{\chi \in \hat{A}}$ $(M \subset A,\ M \neq \emptyset)$

(iii) $\mathcal{P}_{N, N \cup \{x\}} : \sigma(N \cup \{x\}) \to \sigma(N)$ is onto $(x \in A,\ N \subset A,\ N \neq \emptyset)$

(iv) $\mathcal{P}_{N, M} : \sigma(M) \to \sigma(N)$ is onto $(N \subset M \subset A,\ N \neq \emptyset)$.

This situation occurs if A is a commutative, unital Banach algebra, or if $A = B/I$ where B is a commutative, unital Banach algebra and I an ideal of B, closed or not. The above property never holds if A is a commutative, unital Fréchet algebra such that $\operatorname{Inv} A$ is not open, for in this case A has infinite – codimensional maximal ideals [24], [53]. This suggests the following weaker notions.

DEFINITION 6.1. – Let A be a commutative, unital complex algebra. Then A has the countable (resp. finite) extension property if $\mathcal{P}_{N, N \cup \{x\}} : \sigma(N \cup \{x\}) \to \sigma(N)$ is onto for every at most countable (resp. finite) nonempty subset N of A and every $x \in A$. Similarly A has the p-extension property if $\mathcal{P}_{N, N \cup \{x\}} : \sigma(N \cup \{x\}) \to \sigma(N)$ is onto for every subset N of A such that $\operatorname{Card} N \leq p$ and every $x \in A$.

Clearly, A has the finite extension property if and only if A has the p-extension property for every $p \geq 1$. Notice also that if $M = \bigcup_{t \in T} M_T$, where $(M_t)_{t \in T}$ is linearly ordered by inclusion, and if $(\lambda_t)_{t\in T} \in \Pi_{t\in T}\,\sigma(M_t)$, with $\lambda_{t|M_{t'}} = \lambda_{t'}$ for $M_{t'} \subset M_t$, then $\lambda : M \to \mathbb{C}$ defined by the relations $\lambda_{|M_t} = \lambda_t\,(t \in T)$ is an element of $\sigma(M)$. It follows immediately from this observation that if A has the countable extension property then $\mathcal{P}_{N,M} : \sigma(M) \to \sigma(N)$ is onto if $N \neq \varnothing$, $N \subset M \subset A$, and if M is at most countable. Similarly if A has the finite extension property then $\mathcal{P}_{N,M} : \sigma(M) \to \sigma(N)$ is onto if $N \neq \varnothing$, $N \subset M \subset A$, and if M is a finite set.

It follows from the theorem of Arens [2] that if A is a commutative, unital Fréchet algebra and if $M \subset A$ is finite, then $\sigma(M) = \{\chi_{|M}\}_{\chi\in\tilde{A}}$ where we denote by $\tilde{A}$ the set of continuous characters of A. Hence a commutative, unital Fréchet algebra has the finite extension property. Of course, this property does not extend to quotients of Fréchet algebras (consider a quotient by a maximal ideal of infinite codimension). More precisely it follows from [24, Theorem 3.1] that if A is a commutative, unital Fréchet algebra and if $\operatorname{Inv} A$ is not open, then A has a dense ideal I of countable type such that $\sigma(x) = \varnothing$ for some $x \in A/I$, so that A/I does not even have the 1-extension property.

The following observation is probably known by everyone who seriously tried to construct discontinuous characters one some Fréchet algebra. Since the author did not find it in the literature, a proof is included for the sake of completeness.

PROPOSITION 6.2. – *Let A be a commutative, unital complex algebra such that* $\operatorname{Card} A = 2^{\aleph_0}$. *If A possesses the countable extension property, and if the continuum hypothesis is assumed, then $\hat{A} \neq \varnothing$ and $\sigma(M) = \{\chi_{|M}\}_{\chi\in\hat{A}}$ for every nonempty $M \subset A$ such that* $\operatorname{Card} M \leq \aleph_0$.

Proof. – We can assume without loss of generality that $1 \in M$ and that $\operatorname{Card} M = \aleph_0$. Since the continuum hypothesis is assumed we can write $A = \{a_\xi\}_{\xi<\omega_1}$, where we denote by ω_1 the first uncountable ordinal, and assume that $M = \{a_\xi\}_{\xi<\omega_0}$, with $a_0 = 1$. For $0 < \xi < \omega_1$ set $N_\xi = \{a_\eta\}_{\eta<\xi}$. Since A indeed has the finite extension property we can construct by induction a sequence $(\lambda_n)_{n\geq 1}$, with $\lambda_n \in \sigma(N_n)$ $(n \geq 1)$ and with $\lambda_{n|N_m} = \lambda_m$ for $1 \leq m < n$. Define $\lambda_{\omega_0} : N_{\omega_0} \to \mathbb{C}$ by the formulae $\lambda_{\omega_0|N_n} = \lambda_n$ $(n \geq 1)$. Then $\lambda_{\omega_0} \in \sigma(N_{\omega_0}) = \sigma(M)$, and so $\sigma(M) \neq \varnothing$.

Now let $\mu \in \sigma(M)$. Set $\lambda_\xi = \mu_{|N_\xi}$ for $0 < \xi \leq \omega_0$. We construct by transfinite induction an extension $(\lambda_\xi)_{\xi<\omega_1}$ of the family $(\lambda_\xi)_{\xi\leq\omega_0}$ defined above, with $\lambda_\xi \in \sigma(N_\xi)$ and $\lambda_{\xi|N_\eta} = \lambda_\eta$ for $0 < \eta < \xi$, $\xi < \omega_1$.

Assume that a family (λ_ξ) with the required properties has been constructed for $\xi < \eta$, where $\eta > \omega_0$. If $\eta = \sigma + 1$ is a successor ordinal, then $N_\eta = N_\sigma \cup \{a_\sigma\}$. Since A has the countable extension property there exists $\lambda_\eta \in \sigma(N_\eta)$ such that $\lambda_{\eta|N_\sigma} = \lambda_\sigma$, and the family $(\lambda_\xi)_{\xi\leq\eta}$ satisfies the required conditions. If η is a limit ordinal, then $N_\eta = \bigcup_{\xi<\eta} N_\xi$ and we define $\lambda_\eta \in \sigma(N_\eta)$ by the condition $\lambda_{\eta|N_\xi} = \lambda_\xi$ $(\xi < \eta)$. Again, the family $(\lambda_\xi)_{\xi\leq\eta}$ satisfies the required condition. So we can construct the desired family $(\lambda_\xi)_{\xi<\omega_1}$ by transfinite induction.

Now define $\lambda : A \to \mathbb{C}$ by the conditions $\lambda_{|N_\xi} = \lambda_\xi$ $(\xi < \omega_1)$. Then $\lambda \in \sigma(A) = \hat{A}$, and $\mu = \lambda_{|M}$. This concludes the proof of the proposition.

Unfortunately, the algebra $\mathcal{V}$ does not have the countable extension property. More generally we have the following result (we adopt the convention that a compact set is nonempty).

PROPOSITION 6.3. – *Let A be a commutative, unital Fréchet algebra, and let I be an ideal of A. If the quotient algebra A/I has the countable extension property, then $\sigma(u)$ is compact for every $u \in A/I$.*

Proof. – Set $B = A/I$. If $\sigma(u) = \emptyset$ for some $u \in B$, then B does not have the 1-extension property. If $\sigma(u) \neq \emptyset$, and if $\sigma(u)$ is not compact for some $u \in A/I$, then $\sigma(u)$ is unbounded or $\sigma(u)$ is not closed. In the second case let $\delta \in \overline{\sigma(u)} \backslash \sigma(u)$ and set $v = (u - \delta e)^{-1}$. Let $(\delta_n)_{n \geq 1}$ be a sequence of elements of $\sigma(u)$ such that $\delta_n \xrightarrow[n \to \infty]{} \delta$. Then $\frac{1}{\delta_n - \delta} \in \sigma(v)$ $(n \geq 1)$ and so we can assume without loss of generality that $\sigma(u)$ is unbounded for some $u \in B$. Let $(z_n)_{n \geq 1}$ be a sequence of distinct elements of $\sigma(u)$ such that $|z_n| \xrightarrow[n \to \infty]{} \infty$. It follows from the theory of Weierstrass products that there exists for each $p \geq 1$ an entire function f_p on $\mathbb{C}$ such that $f_p^{-1}(\{0\}) = \{z_n\}_{n \geq p}$.

For $f \in \mathcal{H}(\mathbb{C})$, $a \in A$ we set as in section 3 $f(\pi(a)) = \pi(f(a))$ where $\pi : A \to A/I$ is the canonical surjection. Then $f(b) - f(z).e \in (b - ze).B$ $(b \in B,\ z \in \mathbb{C})$.

For $p \geq 1$, set $v_p = f_p(u)$. Since $f_n(z_p) = 0$ for $n \leq p$, we have $v_1.B \ldots + v_p B \subset (u - z_p.e).B \subsetneq B$. So if we set $M = \{v_p\}_{p \geq 1}$ we see that the zero map $0_M : v_p \to 0$ $(p \geq 1)$ belongs to $\sigma(M)$. Let $z \in \mathbb{C}$. Since $\cap_{p \geq 1} \{z_n\}_{n \geq p} = \emptyset$ there exists $p \geq 1$ such that $f_p(z) \neq 0$. Since $v_p - f_p(z).e \in (u - ze).B$, we have $e \in v_p.B + (u - z.e)B$. In particular $\lambda_{|M} \neq O_M$ for every $\lambda \in \sigma(M \cup \{u\})$, and B does not have the countable extension property.

It is well-known that if the group of invertible elements of a commutative, unital Fréchet algebra A is not open, then A has elements u such that $\sigma(u)$ is unbounded (this follows for example from the fact that there exists a surjective homomorphism from A onto $\mathbb{C}^{\mathbb{N}}$, *see* [24]). So in this case it follows from Proposition 6.3 that A does not have the countable extension property, despite the fact that A has the finite extension property by the theorem of Arens [2] mentioned above.

More generally if I is a dense ideal of countable type in a commutative unital Fréchet algebra A, there exists a surjective homomorphism from A/I onto $\mathbb{C}^{\mathbb{N}}/c_{00}$ [24] and so $\sigma(u)$ is unbounded for some $u \in A/I$, since a similar property holds for $\mathbb{C}^{\mathbb{N}}/c_{00}$. This shows that A/I does not have the countable extension property. Of course, it follows also from Proposition 6.3 that if I is a Picard-Borel ideal of a commutative, unital Fréchet algebra A, and if I is not the kernel of a character of A, then A/I does not have the countable extension property. In particular we cannot apply Proposition 6.2 to $\mathcal{V} = \mathcal{U}/\mathcal{I}_\infty$ or to the quotient algebras $\mathcal{U}/\mathcal{I}_\infty + (f - 1)\mathcal{U}$ introduced in section 2. It seems that a crucial step to make progress towards a solution to Michael's problem would consist in answering the following question.

PROBLEM 6.5. – *Does the algebra $\mathcal{V} = \mathcal{U}/\mathcal{I}_\infty$ possess the finite extension property?*

A negative answer to Problem 6.5 would give a finite family $(u_1, \ldots, u_p)$ of elements of $\mathcal{V}$ such that $\sigma(u_1, \ldots, u_p) \neq \{\chi(u_1), \ldots, \chi(u_p)\}_{\chi \in \hat{\mathcal{V}}}$, a very important piece of information to build any strategy to construct some $\chi \in \hat{\mathcal{V}} \backslash \{\chi_o\}$ (or to prove that $\hat{\mathcal{V}} = \{\chi_o\}$). In the other direction, a positive answer to Problem 6.5 would give an important information about the "Poincaré-Fatou-Bieberbach phenomenon", given by the following result.

THEOREM 6.6. – *Let $p \geq 2$, and assume that $\mathcal{V}$ possesses the p-extension property. Then $\bigcap_{n \geq 1}(F_1 \ldots \circ F_n)(\mathbb{C}^p) \neq \emptyset$ for every sequence $(F_n)_{n \geq 1}$ of entire, one-to-one maps from $\mathbb{C}^p$ into $\mathbb{C}^p$.*

Proof. – Let $f = \sum_{n=1}^{\infty} \lambda_n X_n$, where $(\lambda_n)_{n \geq 1} \in \ell^1 \backslash c_{00}$, let $I = \mathcal{I}_\infty + (f-1)\mathcal{U}$ and let $\pi : \mathcal{U} \to \mathcal{U}/I$ be the canonical map. Then I is dense in $\mathcal{U}$, and it follows then from the results of section 3 that $\bigcap_{n \geq 1}(F_1 \ldots \circ F_n)\ [(\mathcal{U}/I)^p] \neq \emptyset$. So there exists a sequence $(U_n)_{n \geq 1} = (u_{1,n}, \ldots, u_{p,n})_{n \geq 1}$ of elements of $(\mathcal{U}/I)^p$ such that $U_n = F_n\ (U_{n+1})\,(n \geq 1)$. For $1 \leq j \leq p$, let $f_j \in \mathcal{U}$ such that $\pi(f_j) = u_{j,1}$. Since $1 \in \sigma(\pi_\infty(f))$, and since we assumed that $\mathcal{V}$ possesses the p-extension property, there exists $(\lambda_0, \lambda_1, \ldots, \lambda_p) \in \sigma(\pi_\infty(f), \pi_\infty(f_1), \ldots, \pi_\infty(f_p))$ such that $\lambda_0 = 1$. But this shows that $\mathcal{I}_\infty + (f-1)\mathcal{U} + (f_1 - \lambda_1)\mathcal{U} \ldots + (f_p - \lambda_p)\mathcal{U}$ is contained in some proper ideal of $\mathcal{U}$, and so $\mu_1 = (\lambda_1, \ldots, \lambda_p) \in \sigma(U_1)$.

Since $F_n : \mathbb{C}^p \to \mathbb{C}^p$ is one-to-one, it follows from the results of section 3 that $\sigma(U_n) = F_n(\sigma(U_{n+1}))\ (n \geq 1)$. Hence, μ_1 being defined as above, there exists a sequence $(\mu_n)_{n \geq 1}$ of elements of $\mathbb{C}^p$ such that $\mu_n = F_n(\mu_{n+1})\ (n \geq 1)$, which concludes the proof of the theorem.

We see at this point which role Corollary 3.7 plays with respect to Michael's problem: if $\varprojlim(\mathbb{C}^{p_n}, F_n) = \emptyset$ for some family $(F_n)_{n \geq 1}$ of entire maps and some family $(p_n)_{n \geq 1}$ of positive integers, then $\mathcal{U}/\mathcal{I}_\infty + (f-1)\mathcal{U}$ could not have any character, because some necessary condition on the values of characters on the terms of a sequence $(U_n)_{n \geq 1}$, where $U_n = F_n(U_{n+1}) \in [(\mathcal{U}/\mathcal{I}_\infty + (f-1)\,U)]^{p_n}$, could not be satisfied.

If we knew that $\varprojlim(\mathbb{C}^{p_n}, F_n)$ is indeed nonempty, this would just mean that some obstruction related to joint spectra of special finite families of elements of $\mathcal{U}/\mathcal{I}_\infty + (f-1)\mathcal{U}$ disappears, but constructing a character on $\mathcal{U}/\mathcal{I}_\infty + (f-1)\mathcal{U}$ means dealing with joint spectra of all subsets of $\mathcal{U}/\mathcal{I}_\infty + (f-1)\mathcal{U}$. So there would still be a long way to go before getting a character on $\mathcal{U}/\mathcal{I}_\infty + (f-1)\mathcal{U}$, hence a discontinuous character on $\mathcal{U}$.

7. Appendix : a remark on the theorem of Arens

The basic theorem of Arens [2], mentioned repeatedly in the present paper, shows that if A is a (nonnecessarily commutative) unital Fréchet algebra and if $(a_1, \ldots, a_p) \in A^p$ is such that $a_1 A \ldots + a_p A$ is dense in A, then $a_1.A \ldots + a_p.A = A$.

Since A is isomorphic to a projective limit $\varprojlim(A_n, \theta_n)$ [36], where A_n is a unital Banach algebra and $\theta_n : A_{n+1} \to A_n$ *a* continuous homomorphism such that $\theta_n(A_{n+1})$ is dense in A_n for every $n \geq 1$, we can write $a_j = (a_{j,n})_{n \geq 1}$ where $a_{j,n} \in A_n$, $a_{j,n} = \theta_n(a_{j,n+1})\,(1 \leq j \leq p,\ n \geq 1)$. If we set $\tilde{\theta}_n(b_1, \ldots, b_p) = (\theta_n(b_1), \ldots, \theta_n(b_p))$ for $(b_1, \ldots, b_p) \in A_{n+1}^p$, $V_n = \{(b_1, \ldots, b_p) \in A_n^p | a_{1,n} b_1 \ldots + a_{p,n} b_p = 1\}$, we have $\tilde{\theta}_n(V_{n+1}) \subset V_n$, and $V_n \neq \emptyset$ since $a_{1,n}.A_n \ldots + a_{p.n}.A_n$, being dense in the Banach algebra A_n, must equal A_n.

The proof of Arens consists in showing that $\tilde{\theta}_n(V_{n+1})$ is dense in V_n, and then in applying the Mittag-Leffler theorem (*see* section 3) to show that $\varprojlim(V_n, \tilde{\theta}_n) \neq \emptyset$. If

$(v_{1,n}, \ldots, v_{p,n})_{n\geq 1} \in \varprojlim (V_n, \tilde{\theta}_n)$ and if we set $v_j = (v_{j,n})_{n\geq 1}$, then $v_j \in \varprojlim (A_n, \theta_n)$, $(j \leq p)$ and $a_1 v_1 \ldots + a_p v_p = 1$, the desired result.

The simplification of the original proof of Arens we propose here concerns the proof that $\tilde{\theta}_n (V_{n+1})$ is dense in V_n. If F is a Banach algebra and if $u = (u_1, \ldots, u_p) \in F^p$, $v = (v_1, \ldots, v_p) \in F^p$, $a \in F$ we set $a.u = (au_1, \ldots, au_p)$, $u.a = (u_1.a, \ldots, u_p.a)$, $\langle u, v\rangle = u_1 v_1 \ldots + u_p v_p$ so that $a.\langle u, v\rangle = \langle au, v\rangle$, $\langle u, v\rangle.a = \langle u, v.a\rangle$.

Now let F, G be two unital Banach algebras, and let $\theta : F \to G$ be a continuous homomorphism with dense range. Let $u \in F^p$ and set $L = \{v \in F^p | \langle u, v\rangle = 1\}$, $M = \{w \in G^p | \langle \theta (u), w\rangle = 1\}$, so that $\theta (L) \subset M$. For $v = (v_1, \ldots, v_p) \in F^p$, set $\tilde{\theta} (v) = (\theta (v_1), \ldots, \theta(v_p)) \in G^p$. Assume that $L \neq \varnothing$. Let $w \in M$. There exists a sequence $(\omega_n)_{n\geq 1}$ of elements of F^p such that $\tilde{\theta} (\omega_n) \xrightarrow[n\to\infty]{} w$.

Let $v \in L$. We have $\langle u, v + \omega_n - v \langle u, \omega_n\rangle\rangle = \langle u, v\rangle + \langle u, \omega_n\rangle - \langle u, v\rangle \langle u, \omega_n\rangle = 1$, and so $v + \omega_n - v \langle u, \omega_n\rangle \in L$. Also $\tilde{\theta} (v + \omega_n - v \langle u, \omega_n\rangle) = \tilde{\theta} (v) + \tilde{\theta} (\omega_n) - \tilde{\theta} (v).\theta (\langle u, \omega_n\rangle) = \tilde{\theta} (v) + \tilde{\theta} (\omega_n) - \tilde{\theta} (v).\langle \tilde{\theta} (u), \tilde{\theta} (\omega_n)\rangle \xrightarrow[n\to\infty]{} \tilde{\theta} (v) + w - \tilde{\theta} (v).\ \langle \theta\, u), w\rangle = w$. This shows that $\tilde{\theta} (L)$ is dense in M and so, with the above notations, $\tilde{\theta} (V_{n+1})$ is dense in V_n (this approach seems technically much simpler than the original computations of [2]).

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