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SUBSTITUTION DYNAMICAL SYSTEMS : ALGEBRAIC CHARACTERIZATION OF EIGENVALUES

By Sébastien FERENCZI, Christian MAUDUIT and Arnaldo NOGUEIRA

ABSTRACT. – We give a necessary and sufficient condition allowing to compute explicitly the eigenvalues of the dynamical system associated to any primitive substitution; this yields a simple criterion to determine wether a substitution is weakly mixing; we apply these results to examples where the matrix has two expanding and two contracting eigenvalues.

Primitive substitutions, or morphisms of languages on a finite alphabet, form extensively studied examples of dynamical systems, *see* [QUE]. The computation of eigenvalues of the associated spectral operator (or dynamical system) is the first step towards the understanding of the geometrical structure of this system. They are well known in several classes of examples, such as substitutions of *constant length* ([DEK]), or the ones where the dominant eigenvalue of the *matrix of the substitution* is a Pisot number.

The computation for the general case has been the object of a number of papers: in [HOS], Host proved the continuity of the eigenfunctions, and gave a necessary and sufficient condition for a complex number to be an eigenvalue of the system; a similar condition was given simultaneously by Livshits ([LIV], *see* also [LIV-VER]), though he used the language of *adic systems* rather than the one of harmonic analysis. In both cases, the fundamental role played by the eigenvalues of the *matrix of the substitution* was apparent, but explicit conditions (in the sense that they are algorithmically computable for a given substitution) were given only for some limited classes of examples. Other examples were given by Solomyak ([SOL1], [SOL2]); in the particular case where the characteristic polynomial of the matrix of the substitution is irreducible, Solomyak gave in [SOL3] an explicit sufficient condition for a complex number to be an eigenvalue of the system, and an explicit necessary and sufficient condition for the system to be *weakly mixing*.

The aim of this work is to solve this problem for any primitive substitution with a non-periodical fixed point: in the present paper, we give an explicit necessary and sufficient condition allowing us to compute the eigenvalues of the system. This uses another reformulation of the condition of Host, in the language of *finite rank*, with a more geometrical proof, and the notion of *Pisot family*, with techniques developped by Mauduit in his study of normal sets associated to substitutive sequences of integers ([MAU]). It is easier to express and to apply when the expanding eigenvalues of the matrix are simple, which is a weaker condition than the irreducibility of the characteristic polynomial; it is more complicated in the general case, but takes a simple form if we want only to know the direction of the irrational eigenvalues, or to determine whether the system is weakly mixing (the last two criterions could be deducted from [MAU], but the proof we give here is more complete). We then proceed to use our condition on two examples, where the matrix has two expanding and two contracting eigenvalues (hence the dominant eigenvalue is not a Pisot number): one is proved to be weakly mixing, while for the other, the system has eigenvalues and we give them all explicitely (a very similar system has been studied in [SOL3], but without a complete description of the eigenvalues).

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1. Basic definitions and notations

Let $A = (a_1, ..., a_s)$ be a finite alphabet, and $A^\star$ the set of finite words on A. For a word w, we denote by $|w|$ its length. The concatenation of two words v and w is denoted by vw.

Let σ be a **substitution** on A, that is an application from A to $A^\star$, which extends into a morphism of $A^\star$ by the rule $\sigma(vw) = \sigma(v)\sigma(w)$.

In all what follows, we assume that σ has a fixed point, denoted by u (if σ has more than one fixed point, we chose one) ; T is the shift defined on A^N by $(Tx)_n = x_{n+1}$; X is the closed orbit of u under T. The substitution σ extends into a continuous map from X to X.

By the **eigenvalues** and **eigenvectors** of the dynamical system (X, T), we mean complex numbers λ and (borelian) measurable functions f such that

$$f \circ T = \lambda f.$$

If there are no eigenvectors except the constant functions, we say that the system is **weakly mixing**.

We denote by $[a]$ the cylinder of X defined by $(x_0 = a)$; for a word $w = w_0...w_r$, we denote by $[w]$ the cylinder $(x_0 = w_0, ..., x_r = w_r)$. Note that $\sigma[a] \subset [\sigma a]$, but this inclusion may be strict: for $\sigma 0 = 010$ et $\sigma 1 = 01$, we check that $\sigma[0]$ is the cylinder $[0100]$ and that $\sigma[1]$ is the cylinder $[0101]$.

σ is said to be **primitive** if there exists $k > 0$ such that $\sigma^k a$ contains b for each couple (a, b) of elements in A.

If σ is primitive, the dynamical system (X, T) is uniquely ergodic, that is admits only one invariant probability measure, which we denote by μ. Furthermore, there exists an at most countable set D, invariant under T and σ, such that T is a homeomorphism of X/D.

Through all this paper, σ will be a primitive substitution, and u a non-periodical fixed point. The **matrix** of σ is defined by $M = ((m_{i,j}))$ where $m_{i,j}$ is the number of times the letter a_j appears in the word σa_i: then the vector $|\sigma^n a_1|, ...|\sigma^n a_s|$ is computed simply by applying the matrix M^n to the vector $e = (1, ...1)$. Let P be the characteristic polynomial of M, and $\theta_1, ...\theta_t$ its eigenvalues (not to be confused with the eigenvalues of the dynamical system) ; let d_i be the multiplicity of θ_i. We will denote by I the $s \times s$-identity matrix.

The property of **bilateral recognizability** is proved in [MOS] :

LEMMA 1. – *Let σ be primitive and u a non-periodical fixed point; let E be the subset of* $\mathbf{N}$ *defined by*

$$E = (0) \cup (|\sigma(u_0...u_{p-1})|,\ p \in \mathbf{N}).$$

Then there exists $l > 0$ *such that,* if $n \in E$ *and* $u_{n-l}...u_{n+l-1} = u_{m-l}...u_{m+l-1}$, *then* $m \in E$.

We call l the **index of recognizability** of σ and E the **set of bars** of σ.

2. Host's criterion revisited

2.1. The following lemma is proved in [QUE] under a stronger hypothesis, but the proof remains the same.

LEMMA 2. – *Let σ be primitive and u a non-periodical fixed point; for integers n, p and q and letters a and b in A, if $0 \leq p < |\sigma^n a|$ and $0 \leq q < |\sigma^n b|$, then*

$$T^p\sigma^n[a] \cap T^q\sigma^n[b] \cap X/D = \emptyset$$

except if $p = q$ and $\sigma^n a = \sigma^n b$.

Proof. – Suppose that $T^p\sigma^n[a] \cap T^q\sigma^n[b] \cap X/D \neq \emptyset$, with $q > p$; then $\sigma^n x = T^{q-p}\sigma^n y$ for a point x in $[a]$ and a point y in $[b]$; as all these points are in X/D, we have still $T^{-l}\sigma^n x = T^{-l+q-p}\sigma^n y$, if l is the index of recognizability of σ^n. Let E be the set of bars of σ^n.

We have $y = \lim T^{m_i}u$ and $x = \lim T^{p_i}u$; let $s_i = |\sigma^n(u_0...u_{m_i-1})|$ and $t_i = |\sigma^n(u_0...u_{p_i-1})|$; s_i and t_i are in E, and $u_{-l+s_i}...u_{l+s_i-1} = u_{-l+t_i+q-p}...u_{l+t_i+q-p-1}$, hence the $t_i + q - p$ are still in E. But the hypotheses on q and p and the definition of E force $q = p$.

If $p = q$ and $\sigma^n a \neq \sigma^n b$ while the considered intersection is nonempty, there cannot exist any j such that $(\sigma^n a)_j \neq (\sigma^n b)_j$, and then for example $\sigma^n a$ is a strict prefix of $\sigma^n b$, and a new application of the recognizability property gives the conclusion. Q.E.D.

We need now to define some particular sequences of integers, associated to σ and explicitly computable for each given σ, which shall play the key role in the characterization of eigenvalues of the dynamical system ; the following lemma, whose proof is straightforward from the definitions, the decomposition of the matrix M under Jordan form, and the expression of the projectors on the eigenspaces, sums up what we need to know about them :

LEMMA 3. – *Let σ be a primitive substitution, u a non-periodical fixed point of σ and (X, T) the associated dynamical system; then there exist an integer $1 < r \leq s$ and an integer N such that, for every $n \geq N$, the set $\sigma^n A$ has exactly r elements.*

We call a **return word** *any word $b_1...b_{z-1}$ appearing in u and satisfying*

$$\forall n \geq N, \quad \sigma^n b_z = \sigma^n b_1, \quad \sigma^n b_j \neq \sigma^n b_1 \qquad \forall 1 < j < z.$$

For a given return word $C = b_1...b_{z-1}$, we define the associated **return time sequence** *by*

$$r_n(C) = |\sigma^n b_1| + ... + |\sigma^n b_{z-1}|$$

for all $n \geq 1$.

Then there exists only a finite collection of return words $C_1,..,C_q$, which all appear inside the words of the set $(\sigma^k a \sigma^k b,\ a \in A,\ b \in A)$, where k is given in the definition of primitivity. And there exists an $s \times q$-matrix L, explicitly computable for any given σ, such that, if R_n denotes the vector $(r_n(C_1),...,r_n(C_q))$, we have

$$R_n = LM^n e.$$

Hence

$$r_n(C_j) = \sum_{i=1}^{t} \sum_{h=0}^{d_i-1} W_{i,h,C_j}(\theta_i) V_h(n) \theta_i^n,$$

for some polynomials $W_{i,h,C_j} \in \mathbb{Q}[X]$ and $V_h(n) = n(n-1)...(n-h+1)$, $V_0(n) = 1$. Whenever θ_i and θ_k are algebraically conjugate, then $d_i = d_k$, and $W_{i,h,C} = W_{k,h,C}$ for any $0 \leq h \leq d_i - 1$ and any return word C. If θ_i is a simple eigenvalue, $d_i = 1$ and $W_{i,0,C_j}(\theta_i)$ is the j-th coordinate of the vector $\frac{1}{P'(\theta_i)} L \prod_{v \neq i} (M - \theta_v I)e$.

2.2. We are now ready to give a new version of Host's criterion:

PROPOSITION 1. – *The complex number λ of modulus 1 is an eigenvalue of the dynamical system (X, T) if and only if*

$$\lambda^{r_n(C)} \to 1 \tag{1}$$

when $n \to +\infty$, for every return word C.

Proof. – To simplify notations, we suppose first of all that $N = 1$ and $r = s$, that is σ and its iterates are injective on letters.

Stacks

We build a sequence of *Rokhlin stacks* generating the system (X, T). At stage n, there are s stacks, of bases $F_{n,a} = \sigma^n[a]$ and heights $h_{n,a} = |\sigma^n a|$, for every element a in A. The sets $T^i F_{n,a}$, $a \in A$, $0 \leq i \leq h_{n,a} - 1$, form, because of Lemma 2, a partition Q_n of X (up to a set of measure zero). By Lemma 6 of [HOS], these partitions increase towards the σ-algebra of X; the system (X, T) is of **finite rank**, and the sets

$$\tau_{n,a} = \cup_{i=0}^{h_{n,a}-1} T^i F_{n,a}$$

are the **Rokhlin stacks** of the system, the $\tau_{n,a}$ being referred to as the n-stacks; the $F_{n,a}$ are the **bases** and the $T^i F_{n,a}$ the **levels** of the stacks.

The Rokhlin stacks can be naturally built by recurrence in the following way : if

$$\sigma a_i = a(i,1)...a(i,c(i))$$

for $1 \leq i \leq r$ and if $d(j)$ is the number of couples (i,k), $1 \leq i \leq r$, $1 \leq k \leq c(i)$ such that $a(i,k) = a_j$, we cut each base F_{n,a_j} into $d(j)$ pieces $(D_{n,a_j,z},\ z \in U(j))$. The stack

τ_{n+1,a_i} has as a base some $D_{n,a(i,1),z(i,1)}$, and its successive levels are: its $h_{n,a(i,1)}-1$ first iterates, some $D_{n,a(i,2),z(i,2)}$, its $h_{n,a(i,2)}-1$ iterates, ..., some $D_{n,a(i,c(i)),z(i,c(i))}$, its $h_{n,a(i,c(i))}-1$ iterates, the $z(i,j)$, $1 \leq j \leq r$, taking exactly $d(i)$ different values. At the first stage, each stack $\tau_{0,a}$ is made with only one level, the cylinder $[a]$. The constraints

$$\mu(D_{n,a(i,k),z(i,k)}) = \mu(D_{n,a(i,l),z(i,l)})$$

for all $1 \leq k \leq c(i)$, $1 \leq l \leq c(i)$, and

$$\sum_{z \in U(j)} \mu(D_{n,a_j,z}) = \mu(F_{n,a_j})$$

for all j, ensure that the measures of the $D_{n,a_j,z}$, and hence of the new stacks, are determined recursively.

The set $\bigcup_{k=0}^{h_{n,a_j}} T^k D_{n,a_j,z}$, for any fixed z, is called a **column** of the stack τ_{n,a_j}. Note that the measures of the stacks, and hence of the columns (provided they are always numbered in the same order) are independent of n.

Necessary condition

Let f be an eigenvector of T for the eigenvalue λ; for fixed ϵ and for n large enough, there exists a function f_n, constant on each atom of the partition Q_n, such that

$$\| f_n - f \|_2 < \epsilon.$$

Let $C = b_1 \ldots b_{z-1}$ be a return word, appearing in one $\sigma^p a$. When we built geometrically the stack $\tau_{n+p,a}$ it includes a column of τ_{n,b_1}, topped with a column of τ_{n,b_2},..., etc, until we finish with a column of τ_{n,b_z}, which is the same as τ_{n,b_1}. And this pattern will appear, by primitivity, in all the m-stacks for m large enough.

Hence we found a full column D_n of τ_{n,b_1}, of fixed measure, $\mu(D_n) = \gamma$, such that, for each point x of D_n, $T^{r_n(C)}x$ is again in D_n, and on the same level.

But then

$$\begin{aligned} &\int_{D_n} |T^{r_n(C)} f_n - \lambda^{r_n(C)} f_n|^2 \\ &\quad < \int_{D_n} |T^{r_n(C)} f_n - T^{r_n(C)} f|^2 + \int_{D_n} |\lambda^{r_n(C)} f - \lambda^{r_n(C)} f_n|^2 < 2\epsilon. \end{aligned}$$

And, on D_n, $T^{r_n(C)} f_n = f_n$ as f_n is constant on the levels of τ_{n,b_1}, so we get

$$|\lambda^{r_n(C)} - 1| < \frac{4\epsilon}{\gamma \| f \|_2}$$

for all n large enough, hence the criterion is necessary.

Sufficient condition

Suppose that λ is a complex number of modulus one satisfying (1). Then, by Lemma 1 of [HOS], for each return time sequence $r_n(C)$, we have

$$|\lambda^{r_n(C)} - 1| < K\rho^n$$

for some $\rho < 1$ and some constant K.

We can then define f_n in the following way :

- $f_n = 1$ on the basis of τ_{n,a_1},
- for each $b \neq a_1$, we chose a word $a_1 b_1 ... b_q b$ appearing in u, independent of n, and we put $f_n = \lambda^{|\sigma^n a_1| + |\sigma^n b_1| + ... |\sigma^n b_i|}$ on the basis of $\tau_{n,b}$,
- f_n is defined on the other levels of the n-stacks by $f(Tx) = \lambda f(x)$ for every x except those on the top levels.

Then we have

$$|f_{n+1}(x) - f_n(x)| < |\lambda^{s_n(x)} - 1|,$$

where $s_n(x)$ is a finite sum of return time sequences. Hence

$$\| f_{n+1} - f_n \|_\infty < C\rho^n,$$

and the f_n converge uniformly to a function which will be an eigenfunction for λ; hence the criterion is sufficient.

General case for N and r

In that case, let $(A_1, ... A_r)$ be the partition of A according to the different values of $\sigma^n a$ for all $n \geq N$; this partition is independent of n. Then the n-stacks will have as bases $F'_{n,i} = \bigcup_{a \in A_i} \sigma^n[a]$. The geometrical construction has to be made in two steps: the stacks $\tau_{n,a}$ are built recursively in exactly the same way as before, but the stacks generating the system are the $\tau'_{n,i} = \bigcup_{a \in A_i} \tau_{n,a}$; the columns D_n used in the proof are of course columns of the $\tau'_{n,i}$. Everything else in the proof is unchanged. Q.E.D.

2.3. We would like to mention that the technics developed here may be applied to **self-inducing interval exchange transformations**.

Let π be a permutation on m letters and $\xi = (\xi_1, ..., \xi_m)$ be a vector in the positive cone $\mathbb{R}^{+m}$. Set $\xi^\pi = (\xi_{\pi^{-1}1}, ..., \xi_{\pi^{-1}m})$, $|\xi| = \sum_{i=1}^{m} \xi_j$, $a_0(\xi) = 0$, $a_i(\xi) = \sum_{j=1}^{i} \xi_j$, $I_i(\xi) = [a_{i-1}, a_i[$ for $1 \leq i \leq m$. Let $T = T_{(\pi,\xi)}$ be the map defined by

$$Tx = x + a_{\pi i - 1}(\xi^\pi) - a_{i-1}(\xi)$$

for $x \in I_i$.

T is called an **interval exchange transformation**, as it translates the interval $I_i(\xi)$ to the interval $I_{\pi i}(\xi^\pi)$ for each $1 \leq i \leq m$.

Let $T = T_{(\pi,\xi)}$, and $I = [0, b[$ be a subinterval of $[0, |\xi|[$. Let S be the Poincaré first return map induced by T on I, namely

$$Sx = T^{n(x)}x$$

where $n(x) = \min(n > 0; T^n x \in I)$. For suitably chosen I, S will be an exchange of the same number of intervals as T; in this case, we write $T = T_{(\pi',\xi')}$, with $\xi' = B\xi$ where B is a positive integer-valued $m \times m$ matrix.

We say that T is a **self-inducing interval exchange transformation** if there exists a subinterval I such that

$$\pi' = \pi \quad \text{and} \quad \xi' = \beta\xi$$

where $0 < \beta < 1$.

For basic references on the subject, we mention the works of Keane [KEA] and Veech [VEE]. The weak mixing property of interval exchange transformations is still conjectural, although Nogueira and Rudolph [NOG-RUD] have proved that, for $m \geq 3$, when π is an irreducible permutation, then for Lebesgue-almost all ξ in $\mathbb{R}^{+m}$ the interval exchange transformation $T_{(\pi,\xi)}$ is topologically weakly mixing, which means that all continuous eigenfunctions of T are constant.

As for self-inducing interval exchange transformations, they are metrically isomorphic to substitutions, and hence all the results in this paper may be applied to them; but also, it follows straight from [NOG-RUD] that, if T is a self-inducing interval-exchange transformation and if

$$\lambda = e^{2\pi i\alpha}, \quad \text{for } \alpha \in \mathbb{R}$$

is an eigenvalue for $T_{(\pi,\xi)}$, then there exists an integer-valued vector (depending on α) $v = (v_1, ..., v_m) \in \mathbb{Z}^m$ such that

$$\lim_{n\to+\infty} ({}^tB)^n(\alpha e - v) = 0$$

where B is the matrix in the definition of the induced map, and satisfies $\xi = B(\beta\xi)$. This implies that

$$\alpha = \frac{v_1\xi_1 + ... + v_m\xi_m}{|\xi|}.$$

3. Algebraic characterization

3.1. Back to substitutions, we begin by giving an easy characterization of the rational eigenvalues of the dynamical system.

PROPOSITION 2. – *A number of the form* $e^{2\pi i\frac{p}{q}}$ *is an eigenvalue of the dynamical system* (X, T) *if and only if* q *divides* $r_n(C)$ *for any return word* C *and any* n *large enough.*

Proof. – It follows immediately from Proposition 1. Q.E.D.

3.2. We now give the characterization of the eigenvalues of the dynamical system in its fullest generality. For this, we need the following lemma about Pisot families, first introduced and studied in [MAU], using the arguments developped by Pisot in [PIS], *see* also [SAL]:

LEMMA 4. – *Let* $(\eta_1, ... \eta_k)$ *be algebraic numbers of modulus greater or equal to one, which are roots of multiplicity* d_i *of the polynomial*

$$P(X) = X^d + r_1 X^{d-1} + ... + r_d = \prod_{i=1}^{l} (X - \eta_i)^{d_i} \in \mathbb{Z}[X].$$

Let

$$R(X) = X^d P\left(\frac{1}{X}\right),$$

and $d' = \sup_{1 \leq i \leq k} d_i$. *Then the two following properties are equivalent :*

$$\text{(i)} \qquad \sum_{i=1}^{k} \sum_{h=0}^{d_i - 1} \beta_{i,h} V_h(n) \eta_i^n \to 0 \mod 1$$

when $n \to +\infty$, *for* $\beta_{i,h} \in \mathbb{C}$, $1 \leq i \leq k$, $0 \leq h \leq d_i - 1$, *with* $\beta_{i,d_i} \neq 0$ *for each* i, *and* $V_h \in \mathbb{Z}[X]^\star$, *of degree* h.

(ii) *a) whenever* η *is conjugate to some* η_i *and* $\eta \neq \eta_1, ..., \eta \neq \eta_k$, *then* $|\eta| < 1$, $((\eta_1, ... \eta_k)$ are then said to form a **Pisot family**),
and

b) there exists Q *in* $\mathbb{Z}[X]$ *such that , for any* $1 \leq i \leq k$, *the rational fractions*

$$\frac{Q(X)}{R(X)^{d'}}$$

and

$$\sum_{h=0}^{d_i - 1} \beta_{i,h} \frac{\Pi_h(\eta_i X)}{(1 - \eta_i X)^{h+1}}$$

have the same simple elements of denominator $\left(X - \frac{1}{\eta_i}\right)^{h+1}$ *for any* $h \geq 0$, *where* Π_h *is defined by*

$$\sum_{n=0}^{+\infty} V_h(n) z^n = \frac{\Pi_h(z)}{(1 - z)^{h+1}}.$$

Proof. – By Lemma 1 of [MAU], Π_h is in $\mathbb{Z}[X]$, has degree h and satisfies $\Pi_h(1) \neq 0$.
(i) implies (ii) :
We have

$$\sum_{i=1}^{k} \sum_{h=0}^{d_i - 1} \beta_{i,h} V_h(n) \eta_i^n = a_n + e_n,$$

where $a_n \in \mathbb{Z}$ and $e_n \to 0$ when $n \to +\infty$.

Hence

$$f(z) = \sum_{n=0}^{+\infty} e_n z^n$$

is defined and analytic for $|z| < 1$, and has no poles of modulus one (here in fact it is defined on $|z| \leq 1$ as the convergence is geometric).

Also, for all i, $P(\eta_i) = 0$, hence, if γ_j, $0 \leq j \leq d''$, are the coefficients of the polynomial $P^{d'}$, we have

$$\sum_{j=0}^{d''} \gamma_j \eta_i^j = \sum_{j=0}^{d''} j\gamma_j \eta_i^j = \ldots = \sum_{j=0}^{d''} j^{d'-1} \gamma_j \eta_i^j = 0.$$

Hence

$$\sum_{i=1}^{k} \sum_{h=0}^{d_i-1} \sum_{j=0}^{d''} \gamma_j \beta_{i,h} V_h(n+j) \eta_i^{n+j} = 0.$$

Then, for all n big enough,

$$\sum_{j=0}^{d''} \gamma_{d''-j} a_{n+j} = 0,$$

and hence

$$\sum_{n=0}^{+\infty} a_n z^n = \frac{Q(z)}{R(z)^{d'}},$$

where Q is a polynomial with integer coefficients and R is as defined above.

Thus

$$\sum_{n=0}^{+\infty} \sum_{i=1}^{k} \sum_{h=0}^{d_i-1} \beta_{i,h} V_h(n) \eta_i^n z^n = \frac{Q(z)}{R(z)^{d'}} + f(z)$$

for all z such that $|z| \leq 1$, which is equivalent to

$$\sum_{i=1}^{k} \sum_{h=0}^{d_i-1} \beta_{i,h} \frac{\Pi_h(\eta_i z)}{(1-\eta_i z)^{h+1}} = \frac{Q(z)}{R(z)^{d'}} + f(z).$$

As the poles of $\dfrac{1}{R(z)}$ of modulus smaller or equal to one are all the $\left(\dfrac{1}{\eta}, \eta \in H, |\eta| \geq 1\right)$, we get that each of these must be equal to some $\dfrac{1}{\eta_i}$, which is (ii) a).

And, for each $1 \leq i \leq k$, by multiplying the equality above by $(1-\eta_i z)^{d_i}$, and making $z \to \dfrac{1}{\eta_i}$, we must get the same limit l_i; then we substract $\dfrac{l_i}{(1-\eta_i z)^{d_i}}$, multiply by $(1-\eta_i z)^{d_i - 1}$ and so on, and finally we get (ii) b)

(ii) implies (i):

We put

$$f(z)=\sum_{i=1}^{k}\sum_{h=0}^{d_i-1}\beta_{i,h}\frac{\Pi_h(\eta_i z)}{(1-\eta_i z)^{h+1}}-\frac{Q(z)}{R(z)^{d'}}.$$

Because of (ii) b), f has no pole at $\frac{1}{\eta_i}$ for $1\le i\le k$; hence (ii) a) ensures that every pole of f has modulus strictly greater than one. Hence

$$f(z)=\sum_{n=0}^{+\infty}e_n z^n$$

for $|z|\le 1$, where $e_n\to 0$ if $n\to+\infty$.

Now, if H is the set of all roots of P, the multiplicity of each root η being denoted by $d(\eta)$,

$$\frac{1}{R(z)}=\frac{1}{\prod_{\eta\in H}(1-\eta z)^{d(\eta)}}=\prod_{\eta\in H}\left(\sum_{n=0}^{+\infty}\eta^n z^n\right)^{d(\eta)},$$

and, as $d(\eta)$ is the same for all elements of a given class of algebraic conjugacy, we have,

$$\frac{Q(z)}{R(z)^{d'}}=\sum_{n=0}^{+\infty}a_n z^n$$

for $|z|\le 1$, with $a_n\in\mathbb{Z}$.

And, by identifying the coefficients of z^n in (2), we get

$$\sum_{i=1}^{k}\sum_{h=0}^{d_i-1}\beta_{i,h}V_h(n)\eta_i^n=a_n+e_n.$$

Q.E.D.

Our main result follows now immediately from Lemmas 3 and 4 and Proposition 1 :

PROPOSITION 3. – *Let σ be a primitive substitution, u a nonperiodical fixed point of σ, $\theta_1,\ldots\theta_t$ the eigenvalues of its matrix, P its characteristic polynomial, D the (finite) set of its return words, the $(r_n(C),\ C\in D)$ its return time sequences.*

For $C\in D$, we define $A(C)$ to be the set of $1\le i\le s$ such that $|\theta_i|\ge 1$ and that $W_{i,h,C}(\theta_i)\neq 0$ for at least one $0\le h\le d_i-1$.

Let $B(C)$ be the set of i such that $(\theta_i, i\in B(C))$ is the closure under algebraic conjugacy of $(\theta_i, i\in A(C))$.

For $i\in A(C)$, let $d_{i,C}$ be the biggest $h\le d_i$ such that $W_{i,h-1,C}(\theta_i)\neq 0$. Let $d'(C)=\sup_{i\in A(C)} d_{i,C}$.

Let

$$R_C(X)=\prod_{i\in B(C)}(1-\theta_i X)^{d_{i,C}}.$$

Let V_h and $W_{i,h,C}$ be as defined in Lemma 3, Π_h as defined in Lemma 4. Let $v_{h,i,C} = U_{h,i,C}(\theta_i)$, $U_{h,i,C} \in \mathbb{Q}[X]$, be the numerator of the simple element of the rational fraction

$$\sum_{h=0}^{d_{i,C}-1} W_{h,i,C}(\theta_i)\frac{\Pi_h(\theta_i X)}{(1-\theta_i X)^{h+1}}$$

of denominator $\left(X - \dfrac{1}{\theta_i}\right)^{h+1}$.

Then $\lambda = e^{2\pi i\alpha}$ is an eigenvalue of the dynamical system associated to σ if and only if, for every $C \in D$, there exists a polynomial $Q_C \in \mathbb{Z}[X]$ such that, for every $i \in A(C)$, the numerator of the simple element of the rational fraction $\dfrac{Q_C(X)}{R_C(X)^{d'(C)}}$ of denominator $\left(X - \dfrac{1}{\theta_i}\right)^{h+1}$ is $\alpha U_{i,h,C}(\theta_i)$ for $0 \leq h \leq d_{i,C} - 1$, and 0 for $h \geq d_{i,C}$.

Remark. – Each $(\theta_i, i \in A(C))$ is the intersection of the set of expanding eigenvalues of M with a set closed under algebraic conjugacy, and contains the Perron-Frobenius eigenvalue.

3.3. An important particular case is the one where every expanding eigenvalue of the matrix is simple; then each $d'(C)$ is equal to 1, and, because of Lemma 3 and the expression of the simple elements related to simple poles, we have

PROPOSITION 4. – *Let σ be a primitive substitution such that every eigenvalue of modulus greater or equal to one of its matrix M is a simple eigenvalue, u a nonperiodical fixed point of σ, $\theta_1, \ldots\theta_t$ the eigenvalues of its matrix, D the (finite) set of its return words, the $(r_n(C),\ C \in D)$ its return time sequences.*

For $C_j \in D$, let $h(i, C_j)$ be the j-th coordinate of the vector $L\prod_{v\neq i}(M - \theta_v I)e$, and let

$$A(C_j) = (i \in (1, \ldots t);\ |\theta_i| \geq 1 \text{ and } h(i, C_j) \neq 0).$$

Then $\lambda = e^{2\pi i\alpha}$ is an eigenvalue of the dynamical system associated to σ if and only if, for every $C \in D$, there exists a polynomial $Q_C \in \mathbb{Z}[X]$ such that

$$\alpha = \frac{1}{h(i,C)}\theta_i^{s-1}Q_C\left(\frac{1}{\theta_i}\right)$$

for every $i \in A(C)$.

Remarks. – When P is irreducible over $\mathbb{Q}[X]$, each set $(\theta_i, i \in A(C))$ is just the set of expanding eigenvalues of M, for any C.

Note also that $\mathbb{Z}(\theta_i) \subset \mathbb{Z}\left(\dfrac{1}{\theta_i}\right) \subset \mathbb{Q}(\theta_i)$, the inclusions being strict in general.

3.4. The criterion in Proposition 3 has to be a little complicated (though still explicitely computable if we know σ) as we want the exact values of α ; however, if we are satisfied by knowing only the direction $\mathbb{Z}\alpha$, it takes a much simpler form :

PROPOSITION 5. – *Under the hypotheses and with the notations of Proposition 3, let* $A = \bigcup_{C \in D} A(C)$, *and* $F = (E_1, ...E_z)$ *be the set of algebraic conjugacy classes of the* θ_i, *where* $E_k = (\theta_i, i \in G_k)$.

If $e^{2\pi i\alpha}$ *is an eigenvalue of the dynamical system associated to* σ, *then, for every* $E = (\theta_i, i \in G) \in F$, *there exists a polynomial* $S_E \in \mathbb{Q}[X]$ *such that*

$$\alpha = S_E(\theta_i)$$

for every $i \in A \cap G$.

If, for every $E = (\theta_i, i \in G) \in F$, *there exists a polynomial* $S_E \in \mathbb{Q}[X]$ *such that*

$$\alpha = S_E(\theta_i)$$

for every $i \in A \cap G$, *then there exists* $b \in \mathbb{Z}$ *such that* $e^{2\pi ib\alpha}$ *is an eigenvalue of the dynamical system associated to* σ.

Proof. – All the elements in the formulas giving α in Proposition 3 are rational polynomials in θ_i, and, if θ_i and θ_j are in the same algebraic conjugacy class, all these elements involve the same rational polynomials in θ_i and θ_j; reciprocally, if each $bW_{i,h,C}(\theta_i)\alpha$ is an integer polynomial in θ_i, which remains the same if we replace θ_i by one of its algebraic conjugates, then (1) is satisfied. Q.E.D.

COROLLARY 1. – *The dynamical system has irrational eigenvalues if and only if for every algebraic conjugacy class* $E = (\theta_i, i \in G)$ *such that* $A \cap G \neq \emptyset$, *there exists a polynomial* $S_E \in \mathbb{Q}[X]$ *such that* $S_E(\theta)$ *takes the same irrational value for every* $\theta \in E$.

Corollary 1, together with Proposition 2, gives an easy necessary and sufficient condition for a substitution to be weakly mixing.

4. Examples

4.1.

$$\begin{array}{ccc} a & \rightarrow & abbbccccccccccdddddddd \\ b & \rightarrow & bccc \\ c & \rightarrow & d \\ d & \rightarrow & a \end{array}$$

$$M = \begin{pmatrix} 1 & 3 & 10 & 8 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P(X) = X^4 - 2X^3 - 7X^2 - 2X + 1 = (X^2 - (1 + \sqrt{10})X + 1)(X^2 - (1 - \sqrt{10})X + 1)$$

$$\theta_1 = \frac{1 + \sqrt{10} + \sqrt{7 + 2\sqrt{10}}}{2}$$

$$\theta_2 = \frac{1 + \sqrt{10} - \sqrt{7 + 2\sqrt{10}}}{2}$$

$$\theta_3 = \frac{1 - \sqrt{10} + \sqrt{7 - 2\sqrt{10}}}{2}$$

$$\theta_4 = \frac{1 - \sqrt{10} - \sqrt{7 - 2\sqrt{10}}}{2}.$$

We have $\theta_1 > 1$, $\theta_2 < 1$, $-1 < \theta_3 < 0$, and $\theta_4 < -1$, M has two expanding and two contracting eigenvalues.

Among the return words, we find b, c, d, and a, hence the convergence of every $r_n(C)\alpha$ to zero modulo 1 is equivalent to the convergence of every $l(\sigma^n a_j)\alpha$ to zero modulo 1, $j = 1, \ldots 4$; hence we may take $L = I$.

The four groups of conditions in Proposition 3 are equivalent; the simplest to write is for $j = 3$, which gives :

$$\alpha = \frac{\theta_1^3}{(1+\sqrt{10})\theta_1 + 11 - \sqrt{10}} Q\left(\frac{1}{\theta_1}\right) = \frac{\theta_4^3}{(1-\sqrt{10})\theta_4 + 11 + \sqrt{10}} Q\left(\frac{1}{\theta_4}\right),$$

and computations prove that this gives only integer values for α, or, equivalently, that there are no rational eigenvalues by Proposition 2 and that the hypotheses of Corollary 1 are satisfied : **the dynamical system associated to this substitution is weakly mixing**.

4.2.

$$\begin{array}{ccc} a & \rightarrow & abdd \\ b & \rightarrow & bc \\ c & \rightarrow & d \\ d & \rightarrow & a \end{array}$$

$$M = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P(X) = X^4 - 2X^3 - X^2 + 2X - 1 = (X^2 - X - 1 - \sqrt{2})(X^2 - X - 1 + \sqrt{2})$$

$$\theta_1 = \frac{1 + \sqrt{5 + 4\sqrt{2}}}{2}$$

$$\theta_2 = \frac{1 - \sqrt{5 + 4\sqrt{2}}}{2}$$

$$\theta_3 = \frac{1 + i\sqrt{-5 + 4\sqrt{2}}}{2}$$

$$\theta_4 = \frac{1 - i\sqrt{-5 + 4\sqrt{2}}}{2}$$

We have $\theta_1 > 1$, $\theta_2 < -1$, $|\theta_3| = |\theta_4| = \sqrt{2} - 1 < 1$, M has two expanding and two contracting eigenvalues.

Among the return words, we find d, a, dab, and $bcaa$, hence the convergence of every $r_n(C)\alpha$ to zero modulo 1 is equivalent to the convergence of every $l(\sigma^n a_j)\alpha$ to zero modulo 1, $j = 1, \ldots 4$; hence we may take $L = I$.

The four groups of conditions in Proposition 3 are equivalent; the simplest to write is for $j = 4$, which gives :

$$\alpha = \frac{1}{\sqrt{2}\theta_1 + 3}\theta_1^3 Q\left(\frac{1}{\theta_1}\right) = \frac{1}{\sqrt{2}\theta_2 + 3}\theta_2^3 Q\left(\frac{1}{\theta_2}\right),$$

and the second equation is satisfied if and only if

$$\left(\theta_1 - \frac{1}{2} + \frac{\sqrt{2}}{4}\right) Q\left(\frac{1}{\theta_1}\right) = \left(\theta_2 - \frac{1}{2} + \frac{\sqrt{2}}{4}\right) Q\left(\frac{1}{\theta_2}\right);$$

taking into account the expression of θ_i and the facts that, for $x = \theta_1$ or $x = \theta_2$, we have $\sqrt{2} = x^2 - x - 1$ and $x = 2 + \frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}$, we check that this is nontrivially satisfied if and only if

$$Q\left(\frac{1}{\theta_i}\right) = k\left(\theta_i - \frac{1}{2} - \frac{\sqrt{2}}{4}\right) + \frac{l}{\left(\theta_i - \frac{1}{2} + \frac{\sqrt{2}}{4}\right)}$$

for $i = 1, 2$, and (k, l) such that the above expression is an integer polynomial in $\frac{1}{\theta_i}$, or, equivalently, in θ_i; computations show that this is the case if and only if $l \in \frac{47}{4}\mathbb{Z}$ and $k \in 12l + 4\mathbb{Z}$; we get

$$\alpha = \frac{k}{4}(4 + 3\sqrt{2}) + \frac{l}{47}(12 + 5\sqrt{2}),$$

and we conclude that **the eigenvalues of the dynamical system associated to this substitution are the $e^{ik\pi\sqrt{2}}$ for $k \in \mathbb{Z}$.** Note that we do have $S(\theta_1) = S(\theta_2) = 1 + \sqrt{2}$, with the polynomial $S(X) = X^2 - X \in \mathbb{Z}[X]$ playing a key role in the equalities above; $\alpha = (S(\theta_i))^k$, for $k \in \mathbb{Z}$, is a sufficient condition for $e^{2\pi i\alpha}$ to be an eigenvalue of the system (this appears in [SOL3], with a slightly different substitution, where the matrix has the same eigenvalues), but the necessary condition is strictly weaker.

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