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THE KERNEL OF AN HOMOMORPHISM OF HARISH-CHANDRA

BY T. LEVASSEUR AND J. T. STAFFORD

ABSTRACT. – Let $\mathfrak{g}$ be a reductive, complex Lie algebra, with adjoint group G, let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ for the differential of this action. A classic result of Harish-Chandra shows that any invariant differential operator that kills $\mathcal{O}(\mathfrak{g})^G$, the algebra of invariant functions on $\mathfrak{g}$, also kills all invariant distributions on a real form of $\mathfrak{g}$. In this paper we generalize this result by showing that

$$\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) = \{\theta \in \mathcal{D}(\mathfrak{g}) : \theta(\mathcal{O}(\mathfrak{g})^G) = 0\}.$$

This answers a question raised by Dixmier, by Wallach and by Schwarz.

Key words and phrases. Invariant differential operators, invariant distributions, semi-simple Lie algebras.

1. Introduction

Let G be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ and let W be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$. Denote by $\mathcal{O}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$ the algebras of polynomial functions and differential operators on $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ by the adjoint action, and therefore has an induced action on $\mathcal{O}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. In [6], Harish-Chandra defines a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ and, by [14], δ is surjective. The significance of this result is illustrated by the fact that it allows one to give relatively easy proofs of important theorems of Harish-Chandra (*see* [21]).

One would also like to understand the kernel of δ and one aim of this paper is to study this ideal. Set

$$\mathcal{J} = \{d \in \mathcal{D}(\mathfrak{g}) \mid \forall f \in \mathcal{O}(\mathfrak{g})^G,\ d(f) = 0\}, \quad \text{and} \quad \mathcal{I} = \mathcal{J} \cap \mathcal{D}(\mathfrak{g})^G.$$

The differential of the adjoint action of G on $\mathfrak{g}$ will be denoted by $\tau_{\mathfrak{g}}$ (or simply τ); thus, $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ is a Lie algebra map. It is immediate that $\tau(\mathfrak{g}) \subset \mathcal{J}$ and the construction of δ ensures that $\mathcal{I} = \operatorname{Ker}\delta$. This leads to the natural question, raised in [4, 1.2], [21, Section 4] and [19, Section 3]:

$$(\dagger) \qquad \text{Does } \mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})?$$

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In [4, Theorem 2.1], Dixmier shows that this is true at the level of vector fields while, in [21, Lemma 4.1], Wallach proves that $\mathcal{I}/(\mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))$ is torsion with respect to the discriminant of $\mathfrak{g}$.

The first main result of this paper answers question (†) in the affirmative:

THEOREM 1.1. – *$\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Hence $\mathcal{I} = \mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.*

The analogue of this theorem for analytic differential operators also holds and follows routinely from the stated, algebraic result (*see* Corollary 5.6). One reason why Dixmier and Wallach raised Question (†) is that it has as an immediate corollary the following fundamental result of Harish-Chandra [8, Theorem 5]: Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and assume that Ω is an open, completely invariant subset of $\mathfrak{g}_0$. Write $\mathcal{D}'(\Omega)^{\mathcal{G}_0}$ for the set of distributions on Ω that are invariant under the action of the adjoint group G_0 of $\mathfrak{g}_0$. Then $\mathcal{I} = \{d \in \mathcal{D}(\mathfrak{g})^{\mathcal{G}} : d\mathcal{D}'(\Omega)^{\mathcal{G}_0} = 0\}$. The proof of Theorem 1.1, while very different from Harish-Chandra's proof of [8, Theorem 5], is no easier.

In proving Theorem 1.1 we also provide some detailed information about the structure of $\mathcal{N}_\mathfrak{g} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. In order to state the result, we need some notation. Identify $S(\mathfrak{g})$ with the constant coefficient differential operators on $\mathfrak{g}$. Recall that a finitely generated $\mathcal{D}(\mathfrak{g})$-module $\mathcal{M}$ is called Cohen-Macaulay if there exists $p \in \mathbb{N}$ such that $\operatorname{Ext}^i_{\mathcal{D}(\mathfrak{g})}(\mathcal{M}, \mathcal{D}(\mathfrak{g})) = (0)$ for $i \neq p$. In this case, p is the projective homological dimension of $\mathcal{M}$, and $\mathcal{M}$ is homogeneous of Gelfand-Kirillov dimension $2\dim\mathfrak{g} - p$ [1, Theorem II.7.1, Theorem II.7.8]. Observe that $\mathcal{N}_\mathfrak{g}$ is a right module over $\mathcal{D}(\mathfrak{g})^G$ and, hence, over $S(\mathfrak{g})^G$. Thus, the next result, which provides the Lie algebra version of [12, Theorem 3], makes sense.

THEOREM 1.2. – (i) *The $\mathcal{D}(\mathfrak{g})$-module $\mathcal{N}_\mathfrak{g} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ is Cohen-Macaulay of projective dimension $\dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$.*

(ii) *$\mathcal{N}_\mathfrak{g}$ is a flat right $S(\mathfrak{g})^G$-module.*

In outline, the proofs of the above theorems are as follows. In Section 2 we study certain $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$-bimodules. In Section 3 the results of Section 2 are combined with some easy, known facts about $\mathcal{N}_\mathfrak{g}$ to show that Theorem 1.2 holds for the module $N = \mathcal{D}(\mathfrak{g})/\mathcal{J}$ and that $\mathcal{L} = \operatorname{Ker}(\mathcal{N}_\mathfrak{g} \to N)$ is generated by its G-invariants. In Section 4 we provide an inductive argument that reduces the problem to the case where $\mathcal{L}$ is supported on $\mathbf{N}(\mathfrak{g})$, the nilpotent cone of $\mathfrak{g}$. As is shown in Section 5, it follows easily that $\mathcal{L}$ is of finite length. Moreover, if $\mathcal{L} \neq 0$, then some non-zero element of $\mathcal{L}$ is killed by a co-finite dimensional ideal of $S(\mathfrak{g})^G$ and by a power of the augmentation ideal of $S(\mathfrak{g}^*)^G$ as well as by $\tau(\mathfrak{g})$. At this stage one appeals to a result of Harish-Chandra (*see* Theorem 5.2 and the comments thereafter) to show that such an element must be zero.

2. $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$-bimodules

The aim of this section is to study $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$-bimodules satisfying the property of the following definition. The importance of this condition is that, as will be shown in the next section, it is satisfied by N. This will form the starting point of our proof of Theorems 1.1 and 1.2. The Gelfand-Kirillov dimension of a module M will be denoted by $\operatorname{GKdim} M$.

DEFINITION 2.1. – *Define a non-zero* $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$*-bimodule* M *to satisfy property* (*) *if* M *is a finitely generated left* $\mathcal{D}(\mathfrak{g})$*-module with* $\text{GKdim}_{\mathcal{D}(\mathfrak{g})} M \leq \text{rk}\,\mathfrak{g} + \dim \mathfrak{g}$.

Before stating the main results of this section, we need some definitions. The Krull dimension, in the sense of Rentschler and Gabriel, of a module M over a ring R will be denoted by $\text{Kdim}M$. The module M will be called *GK-homogeneous* (respectively *Krull-homogeneous*) if $\text{GKdim}M' = \text{GKdim}M$ (respectively $\text{Kdim}M' = \text{Kdim}M$) for all non-zero submodules M' of M. A module M with Gelfand-Kirillov dimension is called *critical* if $\text{GKdim}M' < \text{GKdim}M$, for all proper factor modules M' of M. More details about these concepts can be found in [15, Chapters 6 & 8].

PROPOSITION 2.2. – *Let* M *be a* $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$*-bimodule that satisfies property* (*). *Then,* $\text{GKdim}_{\mathcal{D}(\mathfrak{g})} M = \text{rk}\,\mathfrak{g} + \dim \mathfrak{g}$ *and* $\text{Kdim}M = \text{rk}\,\mathfrak{g}$.

Proof. – Let $\text{Cdim}S$ denote the maximal Krull dimension of a commutative, finitely generated subring of a ring S. Clearly, $\text{Cdim}\mathcal{D}(\mathfrak{h})^W \geq \text{Cdim}S(\mathfrak{h})^W = \text{rk}\,\mathfrak{g}$. As $\mathcal{D}(\mathfrak{h})^W$ is simple [16, Theorem 2.15], the map $\mathcal{D}(\mathfrak{h})^W \to \text{End}_{\mathcal{D}(\mathfrak{g})}(M)$ induced by the right action of $\mathcal{D}(\mathfrak{h})^W$ on M is an injection. Thus, by [11, Proposition 1.1], one obtains

$$\text{rk}\,\mathfrak{g} \leq \text{Cdim}\mathcal{D}(\mathfrak{h})^W \leq \text{Cdim}\,\text{End}_{\mathcal{D}(\mathfrak{g})}(M) \leq \text{Kdim}M.$$

However, by [15, Corollary 8.5.6] and (*),

$$\text{Kdim}M \leq \text{GKdim}M - \dim \mathfrak{g} \leq \text{rk}\,\mathfrak{g},$$

as required. □

COROLLARY 2.3. – *Let* M *satisfy property* (*). *Then:*

(i) *As a left* $\mathcal{D}(\mathfrak{g})$*-module,* M *is Krull and GK-homogeneous. Moreover,* M *has finite length as a* $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$*-bimodule.*

(ii) *As a left* $\mathcal{D}(\mathfrak{g})$*-module,* M *is Cohen-Macaulay, with homological projective dimension* $\text{pd}_{\mathcal{D}(\mathfrak{g})} M = \dim \mathfrak{g} - \text{rk}\,\mathfrak{g}$.

Proof. – (i) If M is not GK-homogeneous, write T for the unique largest $\mathcal{D}(\mathfrak{g})$-submodule of M with $\text{GKdim}T < \text{GKdim}M$. Since T is mapped to itself by any $\mathcal{D}(\mathfrak{g})$-endomorphism of M, T is a $(\mathcal{D}(\mathfrak{g}),\, \mathcal{D}(\mathfrak{h})^W)$-bisubmodule of M. Thus, T satisfies (*). Thus, by Proposition 2.2, $\text{GKdim}T = \text{rk}\,\mathfrak{g} + \dim \mathfrak{g} = \text{GKdim}M$, a contradiction. Hence, M is GK-homogeneous and, similarly, M is Krull-homogeneous.

Next, let $M = M_0 \supset M_1 \supset \cdots$ be any proper, descending chain of bisubmodules of M. Then each M_i/M_{i+1} satisfies (*) and so Proposition 2.2 implies that

$$\text{Kdim}M_i/M_{i+1} = \text{rk}\,\mathfrak{g} = \text{Kdim}M \quad \text{for each } i.$$

By the definition of Krull dimension, this forces the chain to have finite length.

(ii) Let $j(M) = \min\{j : \text{Ext}^j_{\mathcal{D}(\mathfrak{g})}(M,\, \mathcal{D}(\mathfrak{g})) \neq 0\}$. By [1, Theorem II.5.15]

$$j(M) = \text{GKdim}\mathcal{D}(\mathfrak{g}) - \text{GKdim}_{\mathcal{D}(\mathfrak{g})} M = \dim \mathfrak{g} - \text{rk}\,\mathfrak{g}.$$

Also $\text{Ext}^{\text{pd}M}_{\mathcal{D}(\mathfrak{g})}(M,\, \mathcal{D}(\mathfrak{g})) \neq 0$. Thus, if either assertion of part (ii) of the corollary is false, there exists an integer $s > \dim \mathfrak{g} - \text{rk}\,\mathfrak{g}$ such that $E = \text{Ext}^s_{\mathcal{D}(\mathfrak{g})}(M,\, \mathcal{D}(\mathfrak{g})) \neq 0$. However,

E is naturally a $(\mathcal{D}(\mathfrak{h})^W, \mathcal{D}(\mathfrak{g}))$-bimodule, finitely generated as a right $\mathcal{D}(\mathfrak{g})$-module and, by [1, Proposition II.5.16],

$$\mathrm{GKdim}_{\mathcal{D}(\mathfrak{g})} E \leq \mathrm{GKdim}\mathcal{D}(\mathfrak{g}) - s < \dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g}.$$

This contradicts the left-hand analogue of Proposition 2.2. □

THEOREM 2.4. – *Let M be a $(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{h})^W)$-bimodule, finitely generated as a left $\mathcal{D}(\mathfrak{g})$-module. Then, M is a flat right $S(\mathfrak{h})^W$-module.*

Before proving this result we need some notation and a subsidiary lemma. Let $C_k = \mathbb{C}[x_1, \ldots, x_k]$ denote a polynomial ring in k variables and, for any non-zero linear polynomial $f \in C_k$, identify $C_k/(f) = C_{k-1}$. Define, inductively, $\mathcal{F}_k$ to be the family of torsion-free right C_k-modules L such that $L/Lf \in \mathcal{F}_{k-1}$, for all linear polynomials $0 \neq f \in C_k$.

LEMMA 2.5. – *If $L \in \mathcal{F}_k$, then L is a flat right C_k-module.*

Proof. – Suppose that L has weak homological dimension r. By [15, Proposition 7.1.13], there exists a simple C_k-module S such that $\mathrm{Tor}_r^{C_k}(L, S) \neq 0$. Let $f \in \mathrm{Ann}_{C_k}(S)$ be a non-zero linear form. By [3, p. 347], there is a spectral sequence associated to the change of rings $\Lambda = C_k \to \Gamma = C_k/fC_k$:

$$E^2_{p,q} = \mathrm{Tor}_q^\Gamma(\mathrm{Tor}_p^\Lambda(L, \Gamma), S) \Longrightarrow \mathrm{Tor}_m^\Lambda(L, S).$$

Since f is a non zero divisor in both L and Λ it is easy to see that $\mathrm{Tor}_p^\Lambda(L, \Gamma) = 0$ for $p \geq 1$. Since $\mathrm{Tor}_0^\Lambda(L, \Gamma) = L/fL$, the spectral sequence therefore collapses to isomorphisms

$$\mathrm{Tor}_m^\Gamma(L/fL, S) \cong \mathrm{Tor}_m^\Lambda(L, S).$$

By induction $\mathrm{Tor}_m^\Gamma(L/fL, S) = (0)$ for $m \neq 0$, hence the result. □

Proof of Theorem 2.4. – Set $R = \mathcal{D}(\mathfrak{h})^W \subset A = \mathcal{D}(\mathfrak{h})$ and identify A with $\mathcal{D}(\mathbb{C}^\ell)$ in such a way that $S(\mathfrak{h})$ is identified with C_ℓ. Let $L = M \otimes_R A$, which we regard as a $(\mathcal{D}(\mathfrak{g}), A)$-bimodule. Since ${}_R A$ is finitely generated, ${}_{\mathcal{D}(\mathfrak{g})}L$ is finitely generated. Since A is simple, this implies that L_A and L_{C_ℓ} are torsion-free (use [5, Lemma 7.1]). If $f \in C_\ell$ is a non-zero linear polynomial, we may choose the generators of C_ℓ such that $f = x_\ell$. Hence, f centralizes the natural copy of $\mathcal{D}(\mathbb{C}^{\ell-1})$ in $\mathcal{D}(\mathbb{C}^\ell)$. Consequently, L/Lf is a finitely generated left $\mathcal{D}(\mathfrak{g})$-module that is also a $(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathbb{C}^{\ell-1}))$-bimodule. Thus, L/Lf is also torsion-free as a right module over $\mathcal{D}(\mathbb{C}^{\ell-1})$ and $C_{\ell-1}$. By induction, therefore, $L \in \mathcal{F}_\ell$.

By Lemma 2.5, L is a flat right $S(\mathfrak{h})$-module. Since $S(\mathfrak{h})$ is a free $S(\mathfrak{h})^W$-module, by [15, Proposition 7.2.2], L is a flat right $S(\mathfrak{h})^W$-module.

As a $\mathbb{C}[W]$-module, $\mathcal{D}(\mathfrak{h})$ decomposes as $\mathcal{D}(\mathfrak{h}) = \mathcal{D}(\mathfrak{h})^W \oplus D$, where D is the direct sum of the non-trivial W-modules. This is clearly a $\mathcal{D}(\mathfrak{h})^W$-bimodule decomposition and so, under the natural embedding, $R = \mathcal{D}(\mathfrak{h})^W$ is an R-bimodule summand of $A = \mathcal{D}(\mathfrak{h})$. Thus, as a right R-module, and therefore as a right $S(\mathfrak{h})^W$-module, M is a summand of $L = M \otimes_R A$. Hence, by the last paragraph, M is a flat right $S(\mathfrak{h})^W$-module; as required. □

3. Preliminary results on $\mathcal{N}_\mathfrak{g}$

Let G, $\mathfrak{g}$, $\mathfrak{h}$ and W be as in Section 1 and write $n = \dim \mathfrak{g}$ and $\ell = \dim \mathfrak{h}$. In this section we make some preliminary observations, mostly well-known, about the $\mathcal{D}(\mathfrak{g})$-module $\mathcal{N}_\mathfrak{g} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Combined with the results of the last section, these show that $N = \mathcal{D}(\mathfrak{g})/\mathcal{J}$ satisfies condition (*) of Definition 2.1 and hence that Theorem 1.2 holds for N.

Let κ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. As usual, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through κ. If $T^*\mathfrak{g} \xrightarrow{\pi_\mathfrak{g}} \mathfrak{g}$ denotes the cotangent bundle, then $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \equiv \mathfrak{g} \times \mathfrak{g}$ is a G-variety under the adjoint action. Throughout, $\mathcal{D}(\mathfrak{g})$ and its subspaces will be filtered by degree of differential operators. The principal symbol $\mathrm{gr}(\tau(v)) \in \mathrm{gr}(\mathcal{D}(\mathfrak{g})) = \mathcal{O}(T^*\mathfrak{g})$ of the vector field $\tau(v)$, for $v \in \mathfrak{g}$, will be denoted by $\sigma(v)$; hence $\sigma(v)(x,\xi) = \kappa(v, [\xi, x])$ for $(x,\xi) \in \mathfrak{g} \times \mathfrak{g}$. Given a finitely generated $\mathcal{D}(\mathfrak{g})$-module M, we denote its *characteristic variety* by $\mathrm{Ch}M \subset T^*\mathfrak{g}$, its *characteristic cycle* by $\mathbf{Ch}M$ and its *support* by $\mathrm{Supp}M = \pi_\mathfrak{g}(\mathrm{Ch}M) \subseteq \mathfrak{g}$, as defined in [2]. Recall from [2, VI.1.17] that $\mathrm{Supp}M$ is Zariski closed.

We denote the discriminant of $\mathfrak{g}$ by d_ℓ, and write $\mathfrak{g}' = \{x \in \mathfrak{g} \mid d_\ell(x) \neq 0\}$ for the set of generic elements. Let $\mathfrak{g}^x$ denote the centralizer of $x \in \mathfrak{g}$ and write $\mathbf{N}(\mathfrak{g})$ for the nilpotent cone of $\mathfrak{g}$. Recall that the *commuting variety* of $\mathfrak{g}$ is

$$\mathcal{C}(\mathfrak{g}) = \{(x,\xi) \in T^*\mathfrak{g} \mid \mathrm{ad}\, x.\xi \equiv [x,\xi] = 0\}.$$

It follows from [17] that $\mathcal{C}(\mathfrak{g})$ is irreducible of dimension $n + \ell$. Note that $\mathcal{C}(\mathfrak{g})$ is the set of zeroes of the ideal $\mathbf{a} = (\sigma(x); x \in \mathfrak{g}) \subset \mathcal{O}(T^*\mathfrak{g}) = \mathrm{gr}(\mathcal{D}(\mathfrak{g}))$. Set $\mathbf{b} = \mathrm{gr}(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))$ and denote by $\mathbf{p}$ the prime ideal defining $\mathcal{C}(\mathfrak{g})$; hence $\mathbf{a} \subseteq \mathbf{b}$ and $\sqrt{\mathbf{a}} = \mathbf{p}$. Therefore

$$\mathrm{Ch}\mathcal{N}_\mathfrak{g} \subseteq \mathcal{C}(\mathfrak{g}) \tag{3.1}$$

In fact, it is known that one has equality in (3.1). Since we have been unable to find an appropriate reference for this assertion, we will give a proof using the results of the last section:

LEMMA 3.1. – (i) *Both N and $N' = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})\mathcal{I})$ satisfy condition* (*) *of Definition 2.1. In particular,* $\mathrm{GKdim} N = \mathrm{GKdim} N' = \mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{g}$.

(ii) $\mathrm{GKdim}\mathcal{N}_\mathfrak{g} = \dim \mathcal{C}(\mathfrak{g})$ *and hence* $\mathrm{Ch}\mathcal{N}_\mathfrak{g} = \mathcal{C}(\mathfrak{g})$.

Proof. – (i) By construction N and hence N' are non-zero. Recall [14, Theorem 1] that there exists an isomorphism $\delta : \mathcal{D}(\mathfrak{g})^G/\mathcal{I} \to \mathcal{D}(\mathfrak{h})^W$. Under the embedding $\mathcal{D}(\mathfrak{g})^G \hookrightarrow \mathcal{D}(\mathfrak{g})$, the left ideal $\mathcal{J}$ of $\mathcal{D}(\mathfrak{g})$ becomes a right $\mathcal{D}(\mathfrak{g})^G$-module. Hence, both N and N' are right modules over $\mathcal{D}(\mathfrak{g})^G/\mathcal{I} \cong \mathcal{D}(\mathfrak{h})^W$. Since both modules are factors of $\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$, it follows from (3.1) that

$$\mathrm{GKdim}N \leq \mathrm{GKdim}N' \leq \dim \mathcal{C}(\mathfrak{g}) = \mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{g}.$$

Thus, both modules satisfy (*) and so part (i) follows from Proposition 2.2.

(ii) This follows from part (i) and (3.1). □

LEMMA 3.2. – (i) *Let* $(x,\xi) \in \mathcal{C}(\mathfrak{g})$. *Set* $k = \dim \mathfrak{g}^x \cap \mathfrak{g}^\xi$ *and denote by* $\mathbf{p}_{(x,\xi)}$ *the localization of* $\mathbf{p}$ *at* (x,ξ). *Then* $\mathbf{p}_{(x,\xi)}$ *contains a regular sequence of the form* $\{\sigma(v_1),\ldots,\sigma(v_{n-k})\}$, *for some* $v_j \in \mathfrak{g}$.

(ii) $\mathbf{a}_{d_\ell} = \mathbf{b}_{d_\ell} = \mathbf{p}_{d_\ell}$.

Proof. – (i) Let $(A,\mathbf{m})$ be the local ring of $T^*\mathfrak{g}$ at (x,ξ). If $v \in \mathfrak{g}$ it is easily seen that the differential, $d\sigma(v)(x,\xi)$, of $\sigma(v)$ at (x,ξ) is given by $d\sigma(v)(x,\xi)(a,b) = \kappa(v,[b,x]+[\xi,a])$. It follows that the subvector space of $\mathbf{m}/\mathbf{m^2} = T^*_{(x,\xi)}(T^*\mathfrak{g})$ generated by the $d\sigma(v)(x,\xi)$, $v \in \mathfrak{g}$, is of dimension $n-k$. Hence (i) follows from the fact that A is a regular local ring.

(ii) By Lemma 3.1 (ii), $\mathbf{a} \subseteq \mathbf{b} \subseteq \mathbf{p}$. Assume that $(x,\xi) \in \mathcal{C}(\mathfrak{g})$ with $x \in \mathfrak{g}'$. Since $\mathfrak{g}^x$ is a Cartan subalgebra of $\mathfrak{g}$ we obtain that $\mathfrak{g}^x = \mathfrak{g}^x \cap \mathfrak{g}^\xi$ is of dimension ℓ. By the proof of (i), the ideal generated by the $\sigma(v_i)$, $i = 1,\ldots,n-\ell$, is prime in A. Since $\mathbf{p}$ is prime of height $n-\ell$, we can deduce $\mathbf{a}_{(x,\xi)} = \mathbf{b}_{(x,\xi)} = \mathbf{p}_{(x,\xi)}$. Hence the result. $\square$.

COROLLARY 3.3. – $\mathrm{Ch}\mathcal{N}_\mathfrak{g} = [\mathcal{C}(\mathfrak{g})]$. *Moreover,* $\mathcal{N}_\mathfrak{g}$ *has a unique submodule* T *with* GKdim$T$ $< \mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{g}$ *such that* $\mathcal{N}_\mathfrak{g}/T$ *is a GK-critical module of Gelfand-Kirillov dimension* $\mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{g}$.

Proof. – Set $R = \mathcal{O}(T^*\mathfrak{g})$. From Lemma 3.2 we know that $\mathbf{p}_{d_\ell} = \mathbf{b}_{d_\ell} = \mathbf{a}_{d_\ell}$. Since $d_\ell \notin \mathbf{p}$ it follows that $\mathbf{p}\mathrm{R}_\mathbf{p} = \mathbf{b}\mathrm{R}_\mathbf{p}$. Recall that the multiplicity of $\mathcal{N}_\mathfrak{g}$ along $\mathcal{C}(\mathfrak{g}) = \mathcal{V}(\mathbf{p})$ is defined to be

$$\mathrm{mult}_{\mathcal{V}(\mathbf{p})}\mathcal{N}_\mathfrak{g} = \mathrm{length}_{R_\mathbf{p}}(\mathrm{gr}(\mathcal{N}_\mathfrak{g}))_\mathbf{p}.$$

Since $(\mathrm{gr}(\mathcal{N}_\mathfrak{g}))_\mathbf{p} = R_\mathbf{p}/\mathbf{p}\mathrm{R}_\mathbf{p}$ we have $\mathrm{mult}_{\mathcal{V}(\mathbf{p})}\mathcal{N}_\mathfrak{g} = 1$. Therefore, by definition of the characteristic cycle,

$$\mathbf{Ch}\mathcal{N}_\mathfrak{g} = \mathrm{mult}_{\mathcal{V}(\mathbf{p})}\mathcal{N}_\mathfrak{g}.[\mathcal{C}(\mathfrak{g})] = [\mathcal{C}(\mathfrak{g})].$$

In order to prove the second claim, note that $\mathcal{N}_\mathfrak{g}$ does have a unique submodule T, maximal with respect to GKdim$T < n+\ell$. Moreover, $M = \mathcal{N}_\mathfrak{g}/T$ is GK-homogeneous with GKdim$M = n+\ell$. For any $\mathcal{D}(\mathfrak{g})$-module M we set $\mathbf{Ch}_{n+\ell}M = \sum_\mathcal{V} \mathrm{mult}_\mathcal{V} M.[\mathcal{V}]$, the sum being taken over the irreducible components of ChM of dimension $n+\ell$. Consequently if M' is a nonzero submodule of M then, by additivity of $\mathbf{Ch}_{n+\ell}(\ )$, one obtains $\mathbf{Ch}_{n+\ell}M/M' = 0$. Hence M is critical. $\square$

Summarizing the results of this and the last section for the module $N = \mathcal{D}(\mathfrak{g})/\mathcal{J}$, we obtain the following result. As in Corollary 3.3, we write T for the largest submodule T of $\mathcal{N}_\mathfrak{g}$ with GKdim$T <$ GKdim$\mathcal{N}_\mathfrak{g}$ and we write $\mathcal{L} = \mathrm{Ker}(\mathcal{N}_\mathfrak{g} \to N)$.

COROLLARY 3.4. – (i) *If* $N' = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})\mathcal{I})$, *then* $N' = N = \mathcal{N}_\mathfrak{g}/T$. *Thus,* $\mathcal{L} = T$.

(ii) $\mathcal{N}_\mathfrak{g} = N \iff \mathcal{N}_\mathfrak{g}$ *is GK-critical* $\iff \mathcal{N}_\mathfrak{g}$ *is GK-homogeneous.*

(iii) *As a left* $\mathcal{D}(\mathfrak{g})$*-module,* N *is Cohen-Macaulay with* $\mathrm{pd}_{\mathcal{D}(\mathfrak{g})}N = \dim \mathfrak{g} - \mathrm{rk}\,\mathfrak{g}$.

(iv) *As a right* $S(\mathfrak{g})^G$*-module* N *is flat.*

(v) N *is a simple* $(\mathcal{D}(\mathfrak{g}),\ \mathcal{D}(\mathfrak{h})^W)$*-bimodule.*

Proof. – Lemma 3.1 and Corollary 2.3 imply that N and N' are homogeneous left $\mathcal{D}(\mathfrak{g})$-modules of Gelfand-Kirillov dimension $\mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{g}$. Hence, by Corollary 3.3, $N = N' = \mathcal{N}_\mathfrak{g}/T$ is GK-critical. Now apply Corollary 2.3 and Theorem 2.4. $\square$

4. Restriction to a reductive subalgebra

Recall that Theorem 1.1 asserts that $\mathcal{L} = \mathrm{Ker}(\mathcal{N}_{\mathfrak{g}} \to N)$ is zero. The aim of this section is to give an inductive result that shows that, in order to prove this, one may assume that $\mathcal{L}$ has finite length. Specifically, we prove that, if Theorem 1.1 holds for every proper, reductive subalgebra of a semisimple Lie algebra $\mathfrak{g}$, then $\mathcal{L}$ is supported on the nilpotent cone $\mathbf{N}(\mathfrak{g})$. It follows easily that $\mathcal{L}$ has finite length.

Our first proof of this result used analytic $\mathcal{D}$-modules. We would like to thank G. Schwarz and M. Van den Bergh for pointing out that one could obtain a direct, algebraic proof using Luna's slice theorem. It is the latter proof that we present here. We should emphasize that the key result, Proposition 4.4, is implicit in [18] and its proof is merely part of the proof of [18 Theorem 4.9]. Unexplained terminology can be found in [18].

Assume that G is an arbitrary reductive algebraic group and that X is an affine G-variety. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\tau_X : \mathfrak{g} \to \mathcal{D}(X)$ be the Lie algebra homomorphism induced by the G-action. Set

$$\mathcal{K}(X) = \{d \in \mathcal{D}(X) \,:\, \forall f \in \mathcal{O}(X)^G,\, d(f) = 0\}.$$

Then $\mathcal{D}(X)\tau_X(\mathfrak{g}) \subseteq \mathcal{K}(X)$. Note that $\mathcal{K}(\mathfrak{g}) = \mathcal{J}$. Given a reductive subgroup $M \subseteq G$ and an affine M-variety Y, define $G \times^M Y = (G \times Y)/M$, under the M-action $m.(g, y) = (gm^{-1}, my)$. Recall Luna's slice theorem, as stated in [18, Theorem 1.14].

THEOREM 4.1. – *Let X be a smooth affine algebraic G-variety. Let $G.b \subset X$ be a closed orbit, and denote by $M = G^b$ the centralizer of b. Then M is reductive and $T_bX = N \oplus T_b(G.b)$ for an M-module N. Thus (N, M) is the slice representation at the point b.*

There is a locally closed smooth affine M-stable subvariety $S \subset X$ containing b such that $U = G.S$ is an affine open subset of X which satisfies:

(i) *There exists an excellent surjective G-morphism $\varphi : G \times^M S \twoheadrightarrow U$.*

(ii) *There exists $f \in \mathcal{O}(N)^M$ with $f(0) \neq 0$, and an excellent surjective morphism $\psi : S \twoheadrightarrow N_f$, such that $\psi(b) = 0$ and the induced G-morphism $\phi : G \times^M S \twoheadrightarrow G \times^M N_f$ is excellent.*

LEMMA 4.2. – *Let $\varphi : Z \to U$ be an excellent surjective G-morphism of smooth affine G-varieties. Then, $\mathcal{K}(Z) = \mathcal{D}(Z)\tau_Z(\mathfrak{g})$ if and only if $\mathcal{K}(U) = \mathcal{D}(U)\tau_U(\mathfrak{g})$.*

Proof. – Set $A = \mathcal{O}(U)$ and $B = \mathcal{O}(Z)$. Then B^G is faithfully flat over A^G. It follows from the proof of [18, Corollary 4.4] that there is a natural identification $B^G \otimes_{A^G} \mathcal{D}(U) = \mathcal{D}(Z)$, which induces identities $1 \otimes_{A^G} \tau_U(\mathfrak{g}) = \tau_Z(\mathfrak{g})$ and $\mathcal{K}(Z) = B^G \otimes_{A^G} \mathcal{K}(U)$.

If $\mathcal{K}(Z) = \mathcal{D}(Z)\tau_Z(\mathfrak{g})$, then combining these observations gives:

$$B^G \otimes_{A^G} \mathcal{D}(U)\tau_U(\mathfrak{g}) = \mathcal{D}(Z)\tau_Z(\mathfrak{g}) = \mathcal{K}(Z) = B^G \otimes_{A^G} \mathcal{K}(U).$$

Therefore $\mathcal{D}(U)\tau_U(\mathfrak{g}) = \mathcal{K}(U)$ by faithful flatness. The other implication is similar. □

LEMMA 4.3. – *Let M be a reductive subgroup of G and Y a smooth affine M-variety and set $Z = G \times^M Y$. Assume that $\mathcal{K}(Y) = \mathcal{D}(Y)\tau_Y(\mathfrak{g})$. Then $\mathcal{K}(Z) = \mathcal{D}(Z)\tau_Z(\mathfrak{g})$.*

Proof. – Set $L = G \times M$ and $V = G \times Y$. Let L act on V by $(g_1, h).(g_2, y) = (g_1 g_2 h^{-1}, hy)$ for $g_1, g_2 \in G$, $h \in M$ and $y \in Y$. This induces actions of G and M such that $Z = V/M$, and $\mathcal{O}(V)^L = \mathcal{O}(G \times^M Y)^G = \mathcal{O}(Y)^M$. Set $\mathfrak{m} = \mathrm{Lie}(M)$ and $\mathfrak{g} = \mathrm{Lie}(G)$. Under the natural identification $\mathcal{D}(V) = \mathcal{D}(G) \otimes_{\mathbb{C}} \mathcal{D}(Y)$, note that $\tau_Y(\mathfrak{m})$ and $\tau_V(\mathfrak{m})$ differ only by elements of $\mathrm{Der}\,\mathcal{O}(G) = \mathcal{O}(G)\tau_G(\mathfrak{g})$. We may write

$$\mathcal{D}(V) = \mathcal{D}(V)\tau_V(\mathfrak{g}) \oplus \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}(Y).$$

Therefore $\mathcal{K}(V) = \{d \in \mathcal{D}(V) : d(\mathcal{O}(V)^L) = 0\} = \mathcal{D}(V)\tau_G(\mathfrak{g}) + \mathcal{K}'$, where

$$\mathcal{K}' = \{\theta \in \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}(Y) : \theta(\mathcal{O}(Y)^M) = 0\}.$$

Clearly $\mathcal{K}' = \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{K}(Y) = (\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}(Y))\tau_Y(\mathfrak{m})$. Hence,

$$\mathcal{K}(V) = \mathcal{D}(V)(\tau_G(\mathfrak{g}) + \tau_Y(\mathfrak{m})) = \mathcal{D}(V)(\tau_G(\mathfrak{g}) + \tau_V(\mathfrak{m})).$$

Now let $Q \in \mathcal{K}(Z)$. As M acts freely on V, [18, Corollary 4.5(i)] provides a short exact sequence:

$$0 \longrightarrow (\mathcal{D}(V)\tau_V(\mathfrak{m}))^M \longrightarrow \mathcal{D}(V)^M \xrightarrow{\gamma} \mathcal{D}(Z) \longrightarrow 0$$

Set $Q = \gamma(Q_1)$ for some $Q_1 \in \mathcal{D}(V)^M$. Since $\tau_V(\mathfrak{m})$ kills $\mathcal{O}(Z)$, the action of Q_1 on $\mathcal{O}(Z)$ is the same as that of $\gamma(Q_1) = Q$. Thus Q_1 kills $\mathcal{O}(Z)^G = \mathcal{O}(V)^L$. By the first paragraph, and since M is reductive, this implies that

$$Q_1 \in \mathcal{K}(V)^M = (\mathcal{D}(V)\tau_G(\mathfrak{g}) + \mathcal{D}(V)\tau_V(\mathfrak{m}))^M = (\mathcal{D}(V)\tau_G(\mathfrak{g}))^M + (\mathcal{D}(V)\tau_V(\mathfrak{m}))^M.$$

Since the actions of G and M commute, $(\mathcal{D}(V)\tau_G(\mathfrak{g}))^M = \mathcal{D}(V)^M\tau_G(\mathfrak{g})$. Hence $Q = \gamma(Q_1) \in \gamma(\mathcal{D}(V)^M\tau_G(\mathfrak{g})) = \mathcal{D}(Z)\gamma(\tau_G(\mathfrak{g})) = \mathcal{D}(Z)\tau_Z(\mathfrak{g})$, as required. □

PROPOSITION 4.4. (Schwarz). – *Let X be a smooth affine G-variety and $G.b$ be a closed orbit in X. Set $M = G^b$ and let (N, M) be the slice representation at the point b. If $\mathcal{K}(N) = \mathcal{D}(N)\tau_N(\mathfrak{m})$ on an M-neighbourhood of* 0 *in N, then $\mathcal{K}(X) = \mathcal{D}(X)\tau_X(\mathfrak{g})$ on a G-neighbourhood of b.*

Proof. – The result follows from Lemma 4.3, Lemma 4.2 and Theorem 4.1. □

We now return to the situation where the *semisimple* group G acts on $\mathfrak{g}$ by the adjoint representation. Recall the well known result (*see*, for example, [13, Section 3, Theorem 3]):

LEMMA 4.5. – *Let S be a G-stable Zariski closed subset of* $\mathfrak{g}$. *Assume that* 0 *is the unique semisimple element of* $\mathfrak{g}$ *contained in S. Then, $S \subseteq \mathbf{N}(\mathfrak{g})$.*

THEOREM 4.6. – *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Assume that $\mathcal{K}(\mathfrak{m}) = \mathcal{D}(\mathfrak{m})\tau_{\mathfrak{m}}(\mathfrak{m})$ for all proper reductive Lie subalgebras* $\mathfrak{m}$ *of* $\mathfrak{g}$. *Then* $\mathrm{Supp}\mathcal{L} \subseteq \mathbf{N}(\mathfrak{g})$ *and so $\mathcal{N}_{\mathfrak{g}} = N$ outside* $\mathbf{N}(\mathfrak{g})$.

Proof. – Let $0 \neq b \in \mathfrak{g}$ be a semisimple element, and set $\mathfrak{q} = [b, \mathfrak{g}]$, $M = G^b$ and $\mathfrak{m} = \mathrm{Lie}(M) = \mathfrak{g}^b$. Thus M is reductive and $\mathfrak{q}$ identifies with the tangent space $T_b(G.b) \subset T_b\mathfrak{g} \equiv \mathfrak{g}$. Furthermore, $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$ and $(\mathfrak{m}, M)$ is the slice representation at the point b.

Therefore it follows from Proposition 4.4 that $\mathcal{L} = 0$ on a neighbourhood of each non-zero semisimple element of $\mathfrak{g}$. Since $\mathcal{L}$ is a rational G-module, $\mathrm{Supp}\mathcal{L}$ is a Zariski closed G-stable subset of $\mathfrak{g}$. Thus, the result follows from Lemma 4.5. □

5. Proof of Theorem 1.1

In this section we combine the earlier results of this paper to prove Theorem 1.1 for a reductive Lie algebra $\mathfrak{g}$. Together with Corollary 3.4, this also completes the proof of Theorem 1.2. We begin with some preliminary results.

LEMMA 5.1. – *Assume that $\mathfrak{g}$ is semisimple. Let $\mathcal{M}$ be a finitely generated $\mathcal{D}(\mathfrak{g})$-module such that*

$$\mathrm{Ch}\mathcal{M} \subset (\mathbf{N}(\mathfrak{g}) \times \mathfrak{g}^*) \cap \mathcal{C}(\mathfrak{g}).$$

Then $\mathcal{M}$ is holonomic, and so, in particular, has finite length. Moreover, each element of $\mathcal{M}$ is killed by a power of $S_+(\mathfrak{g}^*)^G$.

Proof. – A proof is given in [12, Lemma 3.1] although, for sake of completeness, we sketch it here. Denote by $\pi : \mathfrak{g} \times \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ the projection onto the first factor. Let Z be an irreducible component of $(\mathbf{N}(\mathfrak{g}) \times \mathfrak{g}) \cap \mathcal{C}(\mathfrak{g})$. Since G is connected, Z is G-stable and therefore $Y = \overline{\pi(Z)}$ is a closed irreducible G-subvariety of $\mathbf{N}(\mathfrak{g})$. Thus Y is the closure of a single nilpotent orbit, say $G.u$. Then $\dim(\pi^{-1}(u) \cap Z) \leq \dim \mathfrak{g}^u$, which yields $\dim Z \leq \dim G.u + \dim \mathfrak{g}^u = \dim \mathfrak{g}$. Since $\dim(\mathbf{N}(\mathfrak{g}) \times \mathfrak{g}) \cap \mathcal{C}(\mathfrak{g}) \geq \dim \mathfrak{g}$ is obvious, we obtain that $\dim(\mathbf{N}(\mathfrak{g}) \times \mathfrak{g}) \cap \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g}$. Hence $\mathcal{M}$ is holonomic, and therefore has finite length.

By [13, Proposition 16], $\mathbf{N}(\mathfrak{g})$ is the set of zeros of $S_+(\mathfrak{g}^*)^G$ while, by hypothesis, $\mathrm{Supp}\mathcal{M}$ is a closed subset of $\mathbf{N}(\mathfrak{g})$. Thus, each element of $\mathcal{M}$ is killed by some power of $S_+(\mathfrak{g}^*)^G$. □

The proof of the next result uses analytic $\mathcal{D}$-modules about which we need to make some remarks. Let X be a smooth algebraic complex variety or a complex analytic manifold and write $\mathcal{O}_X$ for its structure sheaf. We denote by $\mathcal{D}_X$ the sheaf of differential operators on X. The basic definitions and facts concerning $\mathcal{D}_X$-modules can be found in [1], [2]. When X is affine algebraic, the global section functor provides an equivalence of categories between $\mathcal{D}_X$-modules and $\mathcal{D}(X)$-modules [2, Proposition VII.9.1]. One can associate to X a complex analytic manifold X^{an}, and if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module one defines a quasi-coherent $\mathcal{O}_{X^{an}}$-module by $\mathcal{F}^{an} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{F}$. The functor $\mathcal{F} \longrightarrow \mathcal{F}^{an}$ is exact and faithful. If $\mathcal{F}$ is a coherent $\mathcal{D}_X$-module, the sheaf $\mathcal{F}^{an}$ is naturally endowed with the structure of a coherent $\mathcal{D}_{X^{an}}$-module.

THEOREM 5.2. – *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Let F be a left ideal of $\mathcal{D}(\mathfrak{g})$ such that $F \supseteq \mathcal{D}(\mathfrak{g})K + \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})K'$, where K is an ideal of $S(\mathfrak{g}^*)^G$ containing a power of $S_+(\mathfrak{g}^*)^G$ and K' is an ideal of finite codimension in $S(\mathfrak{g})^G$. Then, $\mathcal{D}(\mathfrak{g}) = F$.*

Proof. – If $\mathcal{D}(\mathfrak{g})/F \neq 0$, pick $s \in S(\mathfrak{g})^G$ such that $[s + F] \neq 0$, but $Ps = 0$, for some maximal ideal P of $S(\mathfrak{g})^G$. By its definition, $\tau(\mathfrak{g})$ commutes with $\mathcal{D}(\mathfrak{g})^G$, while $S(\mathfrak{g}^*)^G$ acts ad-nilpotently on $\mathcal{D}(\mathfrak{g})$. Thus, $[s+F]$ is still killed by $\tau(\mathfrak{g})$ and by a power of $S_+(\mathfrak{g}^*)^G$. In other words, replacing $\mathcal{D}(\mathfrak{g})/F$ by $\mathcal{D}(\mathfrak{g})s$, we may assume that $K' = P$ is a maximal ideal of $S(\mathfrak{g})^G$. As such, $\mathcal{D}(\mathfrak{g})/F$ is a homomorphic image of $\mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})P)$ supported on $\mathbf{N}(\mathfrak{g})$.

By the remarks before the statement of the theorem, in order to prove that $\mathcal{D}(\mathfrak{g})/F = 0$, we may work with sheaves of analytic $\mathcal{D}$-modules. The theorem is now a special case of [10, Theorem 6.7.2]. $\square$

As is observed in [10], [10, Theorem 6.7.2] is an interpretation in terms of $\mathcal{D}$-modules of Harish-Chandra's famous result on the regularity of invariant eigendistributions [8, Theorem 1]. Similarly, Theorem 5.2 is the interpretation of Harish-Chandra's theorem on eigendistributions with nilpotent support [6, Theorem 5]. It is not difficult to prove Theorem 5.2 by modifying the proof of [6, Theorem 5] as given, for example, in [20, Chapter 5, Section 6].

LEMMA 5.3. – *Set $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_1$ is semisimple. Then, $\mathcal{N}_\mathfrak{g}$ is GK-homogeneous if and only if $\mathcal{N}_{\mathfrak{g}_1}$ is GK-homogeneous.*

Proof. – We identify $\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}_1) \otimes_\mathbb{C} \mathcal{D}(\mathfrak{z})$. Thus, given any $\mathcal{D}(\mathfrak{g}_1)$-module M, then $\text{GKdim}_{\mathcal{D}(\mathfrak{g})}\mathcal{D}(\mathfrak{z}) \otimes_\mathbb{C} M = \text{GKdim}_{\mathcal{D}(\mathfrak{g}_1)}M + \text{GKdim}\mathcal{D}(\mathfrak{z})$. Since tensoring over $\mathbb{C}$ is exact, one concludes that the $\mathcal{D}(\mathfrak{g}_1)$-module M is GK-homogeneous if and only if the same is true for the $\mathcal{D}(\mathfrak{g})$-module $\mathcal{D}(\mathfrak{z}) \otimes_\mathbb{C} M = \mathcal{D}(\mathfrak{g}) \otimes_{\mathcal{D}(\mathfrak{g}_1)} M$.

Since $\tau(\mathfrak{z}) = 0$, clearly $\tau(\mathfrak{g}) = \tau(\mathfrak{g}_1)$ and so $\mathcal{N}_\mathfrak{g} = \mathcal{D}(\mathfrak{g}) \otimes_{\mathcal{D}(\mathfrak{g}_1)} \mathcal{N}_{\mathfrak{g}_1}$. Now apply the observations of the last paragraph. $\square$

LEMMA 5.4. – *Let $x \in \mathcal{D}(\mathfrak{g})$ and $p \in S(\mathfrak{g})^G$ be such that $xp \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Then $p^m x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$, for some $m \geq 1$.*

Proof. – If $v \in \mathcal{D}(\mathfrak{g})$, write $\operatorname{ad} p(v) = pv - vp$ and set $x_m = (\operatorname{ad} p)^m(x)$ for $m \geq 0$. Since $\tau(\mathfrak{g})$ commutes elementwise with $\mathcal{D}(\mathfrak{g})^G$, certainly $\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})p \subseteq \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Hence, for any $m \geq 0$,

$$x_m p \in \sum_{i=0}^{m} \mathcal{D}(\mathfrak{g})xp^{i+1} \subseteq \sum_{i=0}^{m} \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})p^i = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}).$$

In particular, for any $i, j \geq 1$,

$$p^i x_{j-1} = p^{i-1}x_j + p^{i-1}x_{j-1}p \in p^{i-1}x_j + \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}).$$

Finally, as $S(\mathfrak{g})$ acts locally ad-nilpotently on $\mathcal{D}(\mathfrak{g})$, certainly $x_m = 0 \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$, for some $m \geq 1$. By the last displayed equation, and induction, this implies that $p^m x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. $\square$

We can now prove Theorem 1.1. By Corollary 3.4, this also completes the proof of Theorem 1.2.

THEOREM 5.5. – *Let $\mathfrak{g}$ be a reductive Lie algebra and set $N = \mathcal{D}(\mathfrak{g})/\mathcal{J}$, as in the introduction. Then $\mathcal{N}_\mathfrak{g} = N$.*

Proof. – By Corollary 3.4 it suffices to prove that $\mathcal{N}_\mathfrak{g}$ is GK-critical. We prove this result by induction on $\dim \mathfrak{g}$, with the case $\mathfrak{g} = 0$ being trivial. If $\mathfrak{g}$ is not semisimple, and in particular if $\dim \mathfrak{g} \leq 2$, the result follows immediately from Lemma 5.3. Thus, we may assume that $\mathfrak{g}$ is semisimple and that the result holds for every proper, reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$.

Set $\mathcal{L} = \operatorname{Ker}(\mathcal{N}_{\mathfrak{g}} \to N)$. Then $\operatorname{Ch}\mathcal{L} \subseteq (\mathbf{N}(\mathfrak{g}) \times \mathfrak{g}^*) \cap \mathcal{C}(\mathfrak{g})$, by Lemma 3.1 (ii) and Theorem 4.6. Hence, by Lemma 5.1, $\mathcal{L}$ has finite length and so, by [15, Corollary 9.1.8], $\operatorname{End}_{\mathcal{D}(\mathfrak{g})}(\mathcal{L})$ is a finite dimensional $\mathbb{C}$-vector space. Now, $\mathcal{L}$ is a right $\mathcal{D}(\mathfrak{g})^G$-module and hence is a right $S(\mathfrak{g})^G$-module. Since the image of $S(\mathfrak{g})^G$ in $\operatorname{End}_{\mathcal{D}(\mathfrak{g})}(\mathcal{L})$ is necessarily finite dimensional, $\mathcal{L}$ is killed on the right by an ideal K_1 of finite codimension of $S(\mathfrak{g})^G$.

Recall from Corollary 3.4 (i) that $\mathcal{J} = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})\mathcal{I}$. Thus, if $\mathcal{L} \neq 0$, we may pick $x \in \mathcal{I}$ such that $x \notin \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Thus, for any $p \in K_1$, one has $xp \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. By Lemma 5.4, $p^n x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$, for some n, and hence $K'x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ for some ideal K' of finite codimension in $S(\mathfrak{g})^G$. Lemma 5.1, applied to the $\mathcal{D}(\mathfrak{g})$-submodule $(\mathcal{D}(\mathfrak{g})x + \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ of $\mathcal{L}$, shows that $Kx \subset \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$, where K is a power of $S_+(\mathfrak{g}^*)^G$. Finally, as $x \in \mathcal{I}$ and $\mathcal{I}$ commutes with $\tau(\mathfrak{g})$, certainly $\tau(\mathfrak{g})x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.

Thus, we have shown that $F = \{r \in \mathcal{D}(\mathfrak{g}) : rx \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})\}$ satisfies the hypotheses of Theorem 5.2. Thus, $F = \mathcal{D}(\mathfrak{g})$ and $x \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Hence, $N = \mathcal{N}_{\mathfrak{g}}$, as required. □

We end the paper with several comments on extensions and applications of Theorem 5.5.

An interesting question is whether Theorem 5.5 holds at the graded level: Does $\mathcal{J} \cap \mathcal{D}(\mathfrak{g})_m = \mathcal{D}(\mathfrak{g})_{m-1}\tau(\mathfrak{g})$ for all m, where $\mathcal{D}(\mathfrak{g})_m$ denotes the differential operators of order $\leq m$? Both Dixmier [4] and Schwarz [19] raise their questions in this generality. This is slightly weaker than another well-known problem: is the ideal of zeros of the commuting variety generated by the obvious functions $\{\sigma(x) : x \in \mathfrak{g}\}$? Equivalently, does $\mathbf{a} = \mathbf{p}$ in the notation of Section 3?

If V is a finite dimensional linear representation of G, one can ask again whether $\operatorname{Ker}(\mathcal{D}(V)^G \to \mathcal{D}(V/G))$ equals $\mathcal{D}(V)^G \cap \mathcal{D}(V)\tau_V(\mathfrak{g})$ (*see* [19, Section 0]). For some positive results, complementary to Theorem 5.5, the reader is referred to [18, Theorem 8.9].

Finally we make some comments on $\mathcal{D}(\mathfrak{g}^{an})$, the ring of differential operators on $\mathfrak{g}$ with analytic coefficients. Recall from the introduction that we claimed that Theorem 5.5 had, as an immediate consequence, Harish-Chandra's theorem [8, Theorem 5] that asserts that any invariant differential operator that kills all invariant functions on $\mathfrak{g}$ also kills all invariant distributions on $\mathfrak{g}_0$. Since Harish-Chandra's result is concerned with analytic differential operators, this is only immediate if one has an analogue of Theorem 5.5 for $\mathcal{D}(\mathfrak{g}^{an})$. As the next corollary shows, this follows easily from Theorem 5.5. We remark, however, that the stated algebraic version of Theorem 5.5 is useful for applications to the analytic theory since it allows one to regard invariant distributions as modules over $\mathcal{D}(\mathfrak{g})^G/\mathcal{I} \cong \mathcal{D}(\mathfrak{h})^{\mathcal{W}}$ and hence to relate them to Weyl group representations. For example, by copying the proof that [21, Theorem 5.4] implies [21, Theorem 5.3], one obtains another proof of [7, Theorem 3]. We would like to thank Nolan Wallach for this observation.

In the next result we show how to extend Theorem 5.5 to the analytic case. We would like to thank the referee for a significant simplification in the proof.

COROLLARY 5.6. – *Let $\mathfrak{g}$ be a reductive Lie algebra and define a sheaf $\mathcal{F}$ of left ideals of $\mathcal{D}_{\mathfrak{g}^{an}}$ by $\mathcal{F}(U) = \{\theta \in \mathcal{D}_{\mathfrak{g}^{an}}(U) \mid \theta(\mathcal{O}(\mathfrak{g}^{an})^G) = 0\}$ for any open set $U \subseteq \mathfrak{g}^{an}$. Then:*

(i) $\mathcal{F} = \mathcal{D}_{\mathfrak{g}^{an}}\tau(\mathfrak{g})$;

(ii) *In particular, if U is a Stein open connected subset of $\mathfrak{g}^{an}$, then $\mathcal{F}(U) = \mathcal{D}_{\mathfrak{g}^{an}}(U)\tau(\mathfrak{g})$.*

Proof. – (i) Let $\tilde{\mathcal{F}}$ be the sheaf defined by $\tilde{\mathcal{F}}(U) = \{\theta \in \mathcal{D}_{\mathfrak{g}^{an}}(U) \mid \theta(\mathcal{O}(\mathfrak{g})^G) = 0\}$. Since $\mathcal{D}_{\mathfrak{g}^{an}}\tau(\mathfrak{g}) \subseteq \mathcal{F} \subseteq \tilde{\mathcal{F}}$, it suffices to prove that $\tilde{\mathcal{F}} = \mathcal{D}_{\mathfrak{g}^{an}}\tau(\mathfrak{g})$ and, in turn, it is enough to prove this locally. Fix $p \in \mathfrak{g}$ and write $R = \mathcal{D}_{\mathfrak{g},p} \subset S = \mathcal{D}_{\mathfrak{g}^{an},p}$ and $F = \tilde{\mathcal{F}}_p$. Pick coordinates $x_1, \dots, x_n$ on $\mathfrak{g}$ and set $\partial_i = \frac{\partial}{\partial x_i}$ as usual.

Fix $m \in \mathbb{N}$. Let $d \in S$ be a differential operator of order $\leq m$ and write $d = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$, where $a_\alpha \in B = \mathcal{O}_{\mathfrak{g}^{an},p}$ and $\partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$. Let $f_1, \dots, f_\ell$ be algebra generators of $\mathcal{O}(\mathfrak{g})^G$. Then $d \in F$ if and only if $d(f) = 0$ for all monomials $f = f_{i_1} \cdots f_{i_t}$ with $t \leq m$. For any such f, $\partial^\alpha(f) \in A = \mathcal{O}_{\mathfrak{g},p}$. Thus we obtain a system of linear equations $\{\sum_\alpha y_\alpha \lambda_{\alpha,i} = 0\}$ with $\lambda_{\alpha,i} \in A$ such that $d \in F$ if and only if $\sum_\alpha a_\alpha \lambda_{\alpha,i} = 0$ for all i.

Now, B is a flat A-module and so this system of equations has a solution in B if and only if it is soluble in A. In other words, if $d \in F$, then there exist $\beta_{\alpha,j} \in A$ and $v_j \in B$ such that $\sum_\alpha \beta_{\alpha,j} \lambda_{\alpha,i} = 0$, for all j, i, and $a_\alpha = \sum_j v_j \beta_{\alpha,j}$, for all α. By Theorem 5.5, this implies that

$$\beta_j = \sum_{|\alpha| \leq m} \beta_{\alpha,j} \partial^\alpha \in \{\theta \in R \mid \theta(\mathcal{O}(\mathfrak{g})^G) = 0\} = R\tau(\mathfrak{g}).$$

Moreover, $d = \sum_j v_j \beta_j$ and so $d \in S\tau(\mathfrak{g})$, as required.

(ii) By part (i), $\mathcal{F}$ is a coherent $\mathcal{D}_{\mathfrak{g}^{an}}$-module. Since U is connected and Stein, Cartan's Theorem B implies that $\mathcal{F}(U) = \mathrm{H}^0(U, \mathcal{F}) = \mathrm{H}^0(U, \mathcal{D}_{\mathfrak{g}^{an}}\tau(\mathfrak{g})) = \mathcal{D}_{\mathfrak{g}^{an}}(U)\tau(\mathfrak{g})$ (*see* [9, Theorem 7.4.3] and [1, Section V.2.9]). $\square$

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