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DIETMAR BISCH
UFFE HAAGERUP

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COMPOSITION OF SUBFACTORS: NEW EXAMPLES OF INFINITE DEPTH SUBFACTORS

By Dietmar BISCH* and Uffe HAAGERUP

Abstract. – Let $N \subset P$ and $P \subset M$ be inclusions of II_1 factors with finite Jones index. We study the composed inclusion $N \subset P \subset M$ by computing the fusion of N-P and P-M bimodules and determine various properties of $N \subset M$ in terms of the "small" inclusions. A nice class of such subfactors arises in the following way: let H and K be two finite groups acting properly outerly on the hyperfinite II_1 factor M and consider the inclusion $M^H \subset M \rtimes K$. We show that properties like irreducibility, finite depth, amenability and strong amenability (in the sense of Popa) of $M^H \subset M \rtimes K$ can be expressed in terms of properties of the group G generated by H and K in Out$M$. In particular, the inclusion is amenable iff M is hyperfinite and the group G is amenable. We obtain many new examples of infinite depth subfactors (amenable and nonamenable ones), whose principal graphs have subexponential and/or exponential growth and can be determined explicitly. Furthermore, we construct irreducible, amenable subfactors of the hyperfinite II_1 factor which are not strongly amenable.

Key words and phrases. Jones' index, subfactors, principal graphs, amenability, entropy of groups.

1. Introduction

The *standard invariant* [Po4] or *paragroup* [Oc1] for an inclusion of II_1 factors $N \subset M$ with finite Jones index is a grouplike object which encodes combinatorially information about the *position* of the subfactor N in the ambient II_1 factor M. If this object has certain regularity properties, it will determine the subfactor uniquely up to conjugacy by an automorphism of M. Such properties were introduced in [Oc1], [Po4], [Po5], [Po6], *see* also [Po2], where it is shown that if the standard invariant of a subfactor of the hyperfinite II_1 factor R is "finite" ($N \subset M$ has *finite depth*) or more generally "amenable" ($N \subset M$ is *amenable* in the sense of Popa), then it is a complete invariant for this subfactor. However, direct computations of the standard invariant are practically impossible even for subfactors given by an explicit construction. To illustrate this statement, let us point out that if the subfactor is given by a crossed product with a finite group, then the standard invariant combines and encodes the group *and* its representation theory in one single object, the

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canonical commuting square [Po2]. Thus the theory of finite groups and their representation theory is part of the theory of "finite" standard invariants.

On the other hand one is in certain situations able to compute partial information contained in this invariant, such as the *principal graphs* associated to $N \subset M$ [GHJ], explicitly. A class of examples of subfactors where this calculation can be carried out is presented in this paper. The idea is simple: suppose we are given two inclusions of II_1 factors $N \subset P$ and $P \subset M$. We study then the *composed inclusion* $N \subset P \subset M$, *i.e.* the *relative position* of the two subfactors N, P in the II_1 factor M such that P is an intermediate subfactor of $N \subset M$. Let us recall that if $N \subset M$ has finite depth, then the intermediate inclusions $N \subset P$ and $P \subset M$ also do [Bi2] and a similar statement holds for amenable inclusions [Po5]. However, the converse does not hold, *i.e.* composing two amenable inclusions may result in a nonamenable one (*see* example 6.6 below). Furthermore, let us point out that intermediate subfactors of a given inclusion $N \subset M$ can be recognized abstractly by looking at appropriate projections in the first higher relative commutant of $N \subset M$ [Bi2]. - This composition of subfactors can be expressed in terms of bimodules associated to $N \subset P \subset M$ and is then described as combining the two tensor categories of bimodules associated to $N \subset P$ resp. $P \subset M$ in various ways according to the relative position of the subfactors N and P.

A particular class of such composed inclusions arises in the following way: let H and K be finite groups with a properly outer action on the hyperfinite II_1 factor R. Then we compose the subfactor obtained by taking the fixed point algebra under the H action with the one resulting from the crossed product by the K action, *i.e.* we study the *group type inclusion* $R^H \subset R \subset R \rtimes K$. The main theorem of this paper is the following

THEOREM. – *Let H and K be finite groups acting properly outerly on the hyperfinite II_1 factor R and let G be the group generated by H and K in the outer automorphism group* $\mathrm{Out}\, R = \mathrm{Aut}\, R/\mathrm{Int}\, R$. *Then*

a) $R^H \subset R \rtimes K$ has finite depth iff G is a finite group.

b) $R^H \subset R \rtimes K$ is amenable (in the sense of Popa) iff G is an amenable group.

c) Suppose the $H \times K$ acts freely on G and let $\mu = \frac{1}{|H||K|} \sum_{k \in K} \sum_{h \in H} \delta_{kh}$ *be a probability measure on G. Then $R^H \subset R \rtimes K$ is strongly amenable (in the sense of Popa) iff the entropy of G with respect to μ* ([Av], [KV]) *is zero.*

This theorem is a powerful tool to construct (strongly) amenable subfactors of the hyperfinite II_1 factor and it illustrates that the subfactor concepts of amenability and finite depth are very natural.

Let us now give a more detailed description of the different sections in this paper. In section 2 we fix the notation and recall the interpretation of the principal graphs associated to an inclusion of II_1 factors as fusion graphs for bimodules following Ocneanu [Oc2].

In section 3 we start the analysis of our composed inclusions $N \subset P \subset M$ and give a necessary and sufficient criterion for finite depth and irreduciblity of $N \subset M$ in terms of P-P bimodules.

The class of group type inclusions as defined above is studied in section 4. Let us point out that since any countable discrete group has an outer action on the hyperfinite II_1 factor

R, we can associate to any given $G = \langle H, K\rangle$ a group type inclusion $R^H \subset R \rtimes K$. We get therefore a great variety of examples of subfactors by performing this simple construction. – After giving a sufficient and necessary condition for the irreducibility of $R^H \subset R \rtimes K$, we proceed with describing how the principal graphs (Γ, Γ') of this inclusion can be calculated. If we denote as above by G the group generated by H and K in Out$R$, it turns out that the computation of the vertices of these principal graphs amounts to computing the H-H, H-K and K-K double cosets of G. The double cosets which have maximal cardinality represent then irreducible bimodules, *i.e.* vertices of the principal graphs, whereas the ones which have smaller cardinality represent reducible bimodules which one can then decompose further. The edges of (Γ, Γ') are determined by calculating the dimension of certain intertwiner spaces for bimodules, which can be done for our group type inclusions by performing some lenghty, but simple calculations in G. This gives a very explicit and efficient method to calculate the principal graphs of $R^H \subset R \rtimes K$.

We proceed with proving a formula, which allows us to compute the square norm of the principal graphs of a group type inclusion as the spectral radius of a natural operator associated to G. This theorem is based on some results describing how to compute norms (in the sense of [GHJ] of infinite graphs. In the case where G is the free product of H and K, we can give a general formula for the norm of Γ involving only $|H|$ and $|K|$. This formula for $||\Gamma||^2$ is then used to prove *b*) of the above theorem. We use here that an inclusion of hyperfinite II_1 factors is amenable iff the Jones index $[M : N]$ is equal to the square norm of the associated principal graph [Po5]. Next we study *strong amenability* of $R^H \subset R \rtimes K$, *i.e.* amenability of the inclusion and factoriality of the algebras obtained by taking the weak closure of the union of the higher relative commutants [Po4]. It turns out that this property of our group type inclusion is equivalent to the asymptotic ergodicity of the right random walk on G with respect to a natural probabitity measure μ on G (*see* [Av], [KV], [Bi1]). This can be made precise by using the notion of *entropy of* G with respect to μ or equivalently by studying *bounded* μ*-harmonic functions* on G. Our result can then be formulated in the following way: $R^H \subset R \rtimes K$ is strongly amenable iff all bounded μ-harmonic functions on G are constant.

We would like to use our main theorem to give new examples of finite and infinite depth subfactors. To this end we calculate in section 5 the entropy of random walks on groups which are quotients of the free product of two finite groups. Our calculations are based on the results in [Ka1], [Ka4], [KV]. In particular we find amenable groups G of this type such that the entropy of G with respect to any nondegenerate probability measure on G with finite first moment is positive. Furthermore, we show that a certain quotient of $PSL(2, \mathbb{Z})$, which has exponential growth, has zero entropy with respect to any symmetric measure with finite second moment.

In section 6 we use the results of the previous sections to give a wide variety of new examples of subfactors of the hyperfinite II_1factor. Let us summarize the type of examples we construct in the following list

a) irreducible, amenable subfactors, which are not strongly amenable and whose principal graphs have exponential growth;

b) an irreducible strongly amenable subfactor, whose principal graph has exponential growth;

c) new irreducible finite depth subfactors;

d) an irreducible, strongly amenable infinite depth subactor of small index (above 4) whose principal graph has polynomial growth;

e) an irreducible nonamenable subfactor corresponding to the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

In particular *a*) solves the problem [Po4, 5.4.2]. Note that our examples are the first "exotic" irreducible (strongly) amenable infinite depth subfactors of R whose principal graphs have all the possible growth properties. We compute the principal graphs of several subfactors in the above collection [*c*), *d*), *e*)] and list them graphically below. Our examples in case *d*) start at index 6. Prior to our work, Hiai constructed subfactors satisfying the conditions in *d*) of index 8, 9 and some higher values [Hi].

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2. Preliminaries

In this section we fix the notation and recall some facts about the bimodule calculus associated to an inclusion of II_1 factors. Let N be a subfactor of the II_1 factor M and denote by τ the unique normalized faithful trace on M. As usual we let $L^2(M)$ be the Hilbert space obtained by completing M in the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$, $x \in M$. Let $J : L^2(M) \to L^2(M)$ be the modular conjugation and denote by ρ the N-M Hilbert bimodule ${}_N L^2(M)_M$ with the action $x \cdot \xi \cdot y = xJy^*J(\xi)$, $x \in N$, $y \in M$ and $\xi \in L^2(M)$. The adjoint (or conjugate or contragredient) bimodule $\bar{\rho}$ is then just $L^2(M)$ as M-N Hilbert bimodule with the actions as above (exchanging N and M of course). As usual we work with unitary equivalence classes of A-B bimodules, A, $B \in \{N, M\}$ (*see* [Co], [Po1], [Oc1]).

Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index $[M : N]$ [Jo1] and let

$$(1) \qquad N \subset M \subset M_1 \subset M_2 \subset \ldots \subset \overline{\cup_{k\geq 1} M_k}^w = M_\infty$$

be the associated Jones tower of II_1 factors. The *principal graphs* [GHJ] (Γ, Γ') of the inclusion $N \subset M$ are the principal parts of the (infinite) Bratteli diagrams of the following inclusions of higher relative commutants:

$$(2) \qquad \mathbb{C} = N' \cap N \subset N' \cap M \subset N' \cap M \subset \ldots \qquad \text{resp.}$$

$$(3) \qquad \mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \ldots$$

Γ and Γ' are (possibly infinite) bipartite graphs with distinguished vertices $*$ and $*'$, corresponding to the copy of $\mathbb{C}$ in the above inclusion sequences (2) resp. (3). The vertices with even distance from $*$ (resp. $*'$) are denoted by Γ_{even} (resp. Γ'_{even}) and those with odd distance by Γ_{odd} (resp. Γ'_{odd}). If we denote by

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix} \quad \text{resp.} \tag{4}$$

$$\Delta_{\Gamma'} = \begin{pmatrix} 0 & G' \\ (G')^t & 0 \end{pmatrix} \tag{5}$$

the adjacency matrices of Γ, Γ', then

$$G = (G_{\gamma\delta})_{\gamma\in\Gamma_{\text{even}},\delta\in\Gamma_{\text{odd}}} \quad (\text{resp.} G' = (G'_{\gamma\delta})_{\gamma\in\Gamma'_{\text{even}},\delta\in\Gamma'_{\text{odd}}})$$

is a "$\Gamma_{\text{even}} \times \Gamma_{\text{odd}}$"-matrix (resp. "$\Gamma'_{\text{even}} \times \Gamma'_{\text{odd}}$"-matrix). Note that G, G' are called the *standard matrices* in [Po2].

Following Ocneanu's bimodule picture [Oc2], we can describe Γ_{even}, Γ_{odd}, Γ'_{even}, Γ'_{odd} as sets of irreducible A-B bimodules, A, $B \in \{N, M\}$. Before we do this, let us fix some notation:

DEFINITION 2.1. – Let A, B, C be II_1factors and let $\alpha =_A H_B$, $\beta =_A K_B$, $\gamma =_B L_C$ be A-B resp. B-C Hilbert bimodules.

a) We write $\alpha\gamma$ for the Hilbert bimodule ${}_AH_B{\otimes}_B{}_BL_C$.

b) We denote by $\langle\alpha, \beta\rangle$ the dimension of the space of A-B intertwiners from the bimodule ${}_AH_B$ to the bimodule ${}_AK_B$.

The following result holds

PROPOSITION (Frobenius reciprocity) 2.2. – *Let A, B, C be II_1 factors and let $\alpha =_A H_B$, $\beta =_B K_C$ and $\gamma =_A L_C$ be Hilbert bimodules. Then*

$$\langle\alpha\beta, \gamma\rangle = \langle\alpha, \gamma\bar{\beta}\rangle = \langle\beta, \bar{\alpha}\gamma\rangle.$$

For a proof *see* [Oc2], [Su], [Ya]. Note that it is trivial to see that

$$\langle\alpha\beta, \gamma\rangle = \langle\bar{\gamma}, \overline{\alpha\beta}\rangle = \langle\bar{\gamma}, \bar{\beta}\bar{\alpha}\rangle.$$

If we let $\rho =_N L^2(M)_M$ as above, then we get, due to the basic construction, that $(\rho\bar{\rho})^k =_N L^2(M_{k-1})_N$, $(\rho\bar{\rho})^k\rho =_N L^2(M_k)_M$, $(\bar{\rho}\rho)^k =_M L^2(M_k)_M$ and $(\bar{\rho}\rho)^k\bar{\rho} = {}_ML^2(M_k)_N$ (*see* for instance [Po1]). The higher relative commutants $M'\cap M_k$ and $N'\cap M_k$ become spaces of A-B intertwiners of the A-B bimodules $L^2(M_k)$, A, $B \in \{N, M\}$. More precisely, one has [Oc2]:

$$\begin{aligned} N' \cap M_{2k+1} &= \text{Hom}_{N-N}({}_NL^2(M_k)_N), \\ N' \cap M_{2k} &= \text{Hom}_{N-M}({}_NL^2(M_k)_M), \\ M' \cap M_{2k} &= \text{Hom}_{M-M}({}_ML^2(M_k)_M), \\ M' \cap M_{2k+1} &= \text{Hom}_{M-N}({}_ML^2(M_k)_N). \end{aligned}$$

This is due to the fact that $N \subset M_k \subset M_{2k+1}$, $M \subset M_k \subset M_{2k}$, etc. are basic constructions [Pi-Po1], which yields for instance that

$$\begin{aligned}\mathrm{Hom}_{N-N}({}_N L^2(M_k)_N) &= N' \cap (N^{\mathrm{opp}})' \cap B(L^2(M_k)) \\ &= N' \cap (J_k N J_k)' \cap B(L^2(M_k)) = N' \cap M_{2k+1},\end{aligned}$$

where J_k is the modular conjugation on $L^2(M_k)$. Since Γ_{even} (resp. Γ_{odd}) labels the simple summands of $N' \cap M_{2k+1}$ (resp. $N' \cap M_{2k}$), we get the following description of Γ_{even} and Γ_{odd}:

(6) $\Gamma_{\mathrm{even}} =$ set of (equivalence classes of) non-equivalent, irreducible N-N-bimodules appearing in ${}_N L^2(N)_N, (\rho\bar\rho)^k$, for some $k \geq 1$ and

(7) $\Gamma_{\mathrm{odd}} =$ set of (equivalence classes of) non-equivalent, irreducible N-M-bimodules appearing in $\rho = {}_N L^2(N)_M, (\rho\bar\rho)^k\rho$, for some $k \geq 1$.

Similarly, Γ'_{even} labels all nonequivalent irreducible M-M bimodules appearing in ${}_M L^2(M)_M$, $(\bar\rho\rho)^k$, for some $k \geq 1$ and Γ'_{odd} labels the nonequivalent irreducible M-N bimodules appearing in $\bar\rho$ and $(\bar\rho\rho)^k\bar\rho$, for some $k \geq 1$. From the definition of the principal graphs one has then

$$G_{\gamma\delta} = \langle \gamma\rho, \delta \rangle, \qquad \gamma \in \Gamma_{\mathrm{even}}, \quad \delta \in \Gamma_{\mathrm{odd}} \tag{8}$$

and

$$G'_{\gamma'\delta'} = \langle \gamma'\bar\rho, \delta' \rangle = \langle \gamma', \delta'\rho \rangle, \qquad \gamma' \in \Gamma'_{\mathrm{even}}, \quad \delta' \in \Gamma'_{\mathrm{odd}}. \tag{9}$$

The contragredient map (which is just $J \cdot J$) $\gamma \in \Gamma_{\mathrm{odd}} \to \bar\gamma \in \Gamma'_{\mathrm{odd}}$ gives a natural identification of Γ_{odd} and Γ'_{odd} and induces a permutation on the even levels (*see* for instance [Bi3] and [Ha] for more on this).

Recall that if Γ is a finite graph (equivalently Γ' is a finite graph), then we say that $N \subset M$ has *finite depth* [Oc1]. This is equivalent to the condition that there are only finitely many nonequivalent irreducible A-B bimodules, $A, B \in \{N, M\}$ appearing in the decomposition of $\oplus_{k\geq 0} L^2(M_k)$.

3. The relative position of two subfactors

Let $N \subset P$ and $P \subset M$ be inclusions of II_1 factors with finite Jones index. We study in this section the *relative position* of N and P in M, *i.e.* the *composed inclusion* $N \subset P \subset M$. In [Bi2] we showed, that if $N \subset M$ has finite depth, then the same is true for the two small inclusions. The same result holds if "finite depth" is replaced by "amenable" [Po5]. Furthermore, information on the graphs for $N \subset M$ can be obtained from those for $N \subset P$ and $P \subset M$ and vice versa [Bi2]. Suppose now that we are given two inclusions $N \subset P$ and $P \subset M$. Let $\alpha = {}_N L^2(P)_P$, $\beta = {}_P L^2(M)_M$ and

$\rho =_N L^2(M)_M = \alpha\beta =_N L^2(P)_P \otimes_P {}_P L^2(M)_M$. As we have seen in section 2, we need to decompose $(\rho\bar{\rho})^k$, $(\bar{\rho}\rho)^k$ and $(\rho\bar{\rho})^k\rho$ into irreducible bimodules to find the graphs of $N \subset M$. Let $N \subset M \subset M_1$, $N \subset P \subset P_1$ and $P \subset M \subset Q_1$ be the Jones basic constructions, then $\bar{\alpha}\alpha =_P L^2(P_1)_P$, $\alpha\bar{\alpha} =_N L^2(P)_N$, $\beta\bar{\beta} =_P L^2(M)_P$ and $\bar{\beta}\beta =_M L^2(Q_1)_M$. We calculate

$$\begin{aligned}
(\rho\bar{\rho})^k &= \alpha(\beta\bar{\beta}\bar{\alpha}\alpha)^{k-1}\beta\bar{\beta}\bar{\alpha} = \alpha\beta\bar{\beta}(\bar{\alpha}\alpha\beta\bar{\beta})^{k-1}\bar{\alpha},\\
(\bar{\rho}\rho)^k &= \bar{\beta}\bar{\alpha}\alpha(\beta\bar{\beta}\bar{\alpha}\alpha)^{k-1}\beta = \bar{\beta}(\bar{\alpha}\alpha\beta\bar{\beta})^{k-1}\bar{\alpha}\alpha\beta,\\
(\rho\bar{\rho})^k\rho &= \alpha(\beta\bar{\beta}\bar{\alpha}\alpha)^k\beta,\\
(\bar{\rho}\rho)^k\bar{\rho} &= \bar{\beta}(\bar{\alpha}\alpha\beta\bar{\beta})^k\bar{\alpha} = \overline{(\rho\bar{\rho})^k\rho)}.
\end{aligned}$$

This simple computation gives immediately the following result

PROPOSITION 3.1. – *Let $N \subset P$ and $P \subset M$ be II_1 factors with finite depth. Then $N \subset M$ has finite depth iff $\{(\beta\bar{\beta}\bar{\alpha}\alpha)^k = ({}_P L^2(M)_P \otimes_P {}_P L^2(P_1)_P)^{\otimes k}, \quad k \geq 1\}$ contains at most finitely many nonequivalent irreducible P-P bimodules.*

Remark 3.2. – The result in [Wi] is an immediate consequence of this proposition: if $N \subset P$ and $P \subset M$ have finite depth and K is a subfactor of M such that $N \subset P \subset M$ and $N \subset K \subset M$ form a nondegenerate commuting square [Po4], then $N \subset M$ has finite depth, since the hypothesis implies that ${}_P L^2(M)_P \otimes_P {}_P L^2(P_1)_P =_P L^2(P_1)_P \otimes_P {}_P L^2(M)_P$, i.e. $(\beta\bar{\beta}\bar{\alpha}\alpha)^k = (\beta\bar{\beta})^k(\bar{\alpha}\alpha)^k$, $k \geq 1$.

Note that $N \subset P$ and $P \subset M$ are *extremal* iff $N \subset M$ is extremal [Po4]. The *irreducibility* of $N \subset M$ is described in the following

PROPOSITION 3.3. – *Let $N \subset P$ and $P \subset M$ be irreducible II_1 factors with finite Jones index. Then $N \subset M$ is irreducible iff $\{H_i \,|\, H_i$ irreducible P-P subbimodule of $\bar{\alpha}\alpha\} \cap \{K_j \,|\, K_j$ irreducible P-P subbimodule of $\beta\bar{\beta}\} = \{{}_P L^2(P)_P\}$.*

Proof. – Let $\rho = \alpha\beta$ as above. Then $N' \cap M = \mathbb{C}$ iff $\mathrm{Hom}_{N-M}({}_N L^2(M)_M) = \mathbb{C}$ iff $\langle\rho,\rho\rangle = 1$. But $1 = \langle\rho,\rho\rangle = \langle\alpha\beta,\alpha\beta\rangle = \langle\bar{\alpha}\alpha,\beta\bar{\beta}\rangle = \sum_{i,j}\langle H_i, K_j\rangle$ by Frobenius reciprocity (Proposition 2.2). Note that ${}_P L^2(P)_P$ is an irreducible subbimodule of $\bar{\alpha}\alpha$ and $\beta\bar{\beta}$, which appears with multiplicity 1 since $N \subset P$ and $P \subset M$ are irreducible. The result follows. $\square$

Note that by definition $\langle\rho,\rho\rangle = \dim N' \cap M$ and hence $\dim N' \cap M = \langle\bar{\alpha}\alpha,\beta\bar{\beta}\rangle$.

4. Group type inclusions

We apply the bimodule techniques described in sections 2 and 3 to study inclusions of the form

$$N = P^H \subset P \subset P \rtimes K = M,$$

where H and K are finite groups acting properly outerly on the II_1 factor P. Observe that $P^H \subset P \rtimes K$ is an extremal inclusion (as composition of two finite depth, hence in particular extremal inclusions) and that its Jones index is $[P \rtimes K : P^H] = |K||H|$. Note also, that the inclusion $P^K \subset P \rtimes H$ is obtained from the basic construction for $P^H \subset P \rtimes K$, *i.e.* is just the dual inclusion to the latter one (*see* for instance [Bi2]). Recall that if θ is an automorphism of P, then we can identify $\theta \bmod \mathrm{Int}\, P$ with the (unitary) equivalence classes of the P-P bimodule $L^2(P)$ with the action $x \cdot \xi \cdot y = x\xi\theta(y) = xJ\theta(y^*)J(\xi)$, $x, y \in P$ and $\xi \in L^2(P)$. We denote this bimodule by θ or $L^2(\theta)$. Note that the contragredient bimodule $\overline{L^2(\theta)}$ is equal to $L^2(\theta^{-1})$ and $L^2(\theta_1\theta_2) = L^2(\theta_1) \otimes_P L^2(\theta_2)$ ([Co], [Oc2]). As in section 3 we let $\alpha = {}_{P^H} L^2(P)_P$, $\beta = {}_P L^2(P \rtimes K)_{P \rtimes K}$ such that $\rho = {}_{P^H} L^2(P \rtimes K)_{P \rtimes K} = \alpha\beta$. It is then well-known and easy to see that

$$\bar{\alpha}\alpha = {}_P L^2(P \rtimes H)_P = \bigoplus_{h \in H} L^2(h)$$

and

$$\beta\bar{\beta} = {}_P L^2(P \rtimes K)_P = \bigoplus_{k \in K} L^2(k).$$

To simplify the notation we write h instead of $L^2(h)$ etc., *i.e.* $\bar{\alpha}\alpha = \bigoplus_{i=0}^{r} h_i$, where $H = \{e = h_0, h_1, \ldots, h_r\}$ and $\beta\bar{\beta} = \bigoplus_{j=0}^{s} k_j$, where $K = \{e = k_0, k_1, \ldots, k_s\}$. The following corollary follows from propositions 3.1 and 3.3

COROLLARY 4.1. – (i) *$P^H \subset P \rtimes K$ is irreducible iff $H \cap K = \{e\}$ in* Out$P$.

(ii) *$P^H \subset P \rtimes K$ has finite depth iff the group generated by H and K in* Out$P$ *is a finite group.*

Proof. – (i) $\langle \bar{\alpha}\alpha, \beta\bar{\beta} \rangle = \sum_{i=0}^{r} \sum_{j=0}^{s} \langle h_i, k_j \rangle$. Since $\langle h_i, k_j \rangle = 1$ iff $h_i = k_j \bmod \mathrm{Int}\, P$, we get the result.

(ii) By Proposition 3.1 we need to calculate $(\bar{\alpha}\alpha\beta\bar{\beta})^n = (\sum_{i=0}^{r} \sum_{j=0}^{s} h_i k_j)^n$. But all words in the h_i's and k_j's of length $\leq n$ appear in this expression, which implies the result. $\square$

Observe that it follows from the above proof that $(P^H)' \cap P \rtimes K \cong \mathbb{C}(H \cap K)$ if H and K act properly outerly on P.

The situation becomes even more interesting when $G = \langle H, K \rangle$, the group generated by H and K in Out$P$, is an infinite group. From the computations given below, it follows that in this case $P^H \subset P \rtimes K$ has infinite principal graphs, *i.e.* the inclusion has *infinite depth*. We will show below that in the case $P = R$, the hyperfinite II_1 factor, the inclusion $P^H \subset P \rtimes K$ is amenable (in the sense of Popa [Po4]) iff G is amenable, and is strongly amenable (in the sense of Popa) iff the entropy $h(G, \mu)$ of G with respect to a certain distinguished probability measure μ on G vanishes. These results enable us then to construct a variety of examples of irreducible (strongly) amenable subfactors of the hyperfinite II_1 factor R whose principal graphs have all possible growth properties. Let us also point out, that any group $G = \langle H, K \rangle$, which is of course a quotient of the free product $H * K$, can be realized in Out$R$. In other words, given any $G = H * K/_{\sim}$, we can construct an index $|H||K|$-inclusion of hyperfinite factors $R^H \subset R \rtimes K$, such that $\langle H, K\rangle = G$ in Out$R$. All our computations below will not use hyperfiniteness, *i.e.* they are valid for an arbitrary II_1 factor P, which carries appropriate outer H and K actions.

We will determine the principal graphs (Γ, Γ') of our group type inclusion and compute their l^2-norm. Let $\rho = \alpha\beta$ as above, then (*see* section 3)

$$\begin{array}{rll} (\rho\bar\rho)^n & \text{decomposes as} & \alpha g\bar\alpha, \quad g \in G, \\ (\rho\bar\rho)^n\rho & \text{decomposes as} & \alpha g\beta, \quad g \in G, \\ (\bar\rho\rho)^n & \text{decomposes as} & \bar\beta g\beta, \quad g \in G, \\ (\bar\rho\rho)^n\bar\rho & \text{decomposes as} & \bar\beta g\bar\alpha, \quad g \in G. \end{array}$$

Clearly, all the bimodules on the right hand side appear for n big enough. If $g,\ g' \in G$, let

$$\begin{aligned} m^H_{g,g'} &= \text{number of times g can be written as } h_i g' h_j, \quad 0 \le i,\ j \le r, \\ m^K_{g,g'} &= \text{number of times g can be written as } k_i g' k_j, \quad 0 \le i,\ j \le s, \\ m^{H,K}_{g,g'} &= \text{number of times g can be written as } h_i g' k_j, \quad 0 \le i \le r, \quad 0 \le j \le s. \end{aligned}$$

Note that $m^H_{g,g'} = m^H_{g',g}$, $m^K_{g,g'} = m^K_{g',g}$. These multiplicities are obtained as

$$\langle \alpha g\bar\alpha, \alpha g'\bar\alpha\rangle = \langle g, \bar\alpha\alpha g'\bar\alpha\alpha\rangle = \sum_{i,j=0}^{r} \langle g, h_i g' h_j\rangle = \sum_{i,j=0}^{r} \delta_{g, h_i g' h_j} = m^H_{g,g'},$$

and similarly $\langle \bar\beta g\beta, \bar\beta g'\beta\rangle = m^K_{g,g'}$, $\langle \alpha g\beta, \alpha g'\beta\rangle = m^{H,K}_{g,g'}$, $\langle \bar\beta g\bar\alpha, \bar\beta g'\bar\alpha\rangle = m^{K,H}_{g,g'}$. We determine now which of the bimodules $\alpha g\bar\alpha$, $g \in G$ (resp. $\alpha g\beta$, $g \in G$, resp. $\bar\beta g\beta$, $g \in G$) are distinct (*i.e.* have no common subbimodules). Clearly, H and K act naturally by left resp. right multiplication on $H * K$, which induces an action on $G = H * K/_{\sim}$. We show that the orbits under this action provide a labeling of all the distinct bimodules of the above form. More precisely, we have

PROPOSITION 4.2. – *Consider the group type inclusion $P^H \subset P \rtimes K$ and let G be the group generated by H and K in* Out$P$. *Let H and K act by left resp. right multiplication on G. Then we have:*

(i) *The double cosets ${}_H\backslash G/_H$* (i.e. *the orbits of the $H \times H$ action $(h, h') \cdot g = hgh'$ on G) label the distinct P^H-P^H bimodules of the form $\alpha g\bar\alpha$, $g \in G$.*

(ii) *The double cosets ${}_H\backslash G/_K$ label the distinct P^H-$P \rtimes K$ bimodules of the form $\alpha g\beta$, $g \in G$.*

(iii) *The double cosets ${}_K\backslash G/_K$ label the distinct $P \rtimes K$-$P \rtimes K$ bimodules of the form $\bar\beta g\beta$, $g \in G$.*

Proof. – (i): Let $g,\ g' \in G$ such that $g' \notin HgH$. Then

$$\langle \alpha g\bar\alpha, \alpha g'\bar\alpha\rangle = \sum_{i,j} \langle h_i g h_j, g'\rangle = m^H_{g,g'} = 0,$$

i.e. $\alpha g\bar{\alpha} \cap \alpha g'\bar{\alpha} = \emptyset$ (which means that they have no common subbimodules). Observe that $\alpha h = \alpha$ for all $h \in H$. This holds since α and αh are irreducible P^H-P bimodules (since $\langle \alpha h, \alpha h\rangle = \langle \alpha, \alpha\rangle = 1$ (Proposition 2.2)) and $\langle \alpha h, \alpha\rangle = \langle h, \bar{\alpha}\alpha\rangle = \sum_{h' \in H} \langle h, h'\rangle = \sum_{h' \in H} \delta_{h,h'} = 1$. Hence, if $g' \in HgH$, *i.e.* $g' = hgh'$, then $\alpha g'\bar{\alpha} = \alpha hgh'\bar{\alpha} = \alpha g\bar{\alpha}$. If $\alpha g\bar{\alpha} \cap \alpha g'\bar{\alpha} \neq \emptyset$ for some g, $g' \in G$, then

$$\begin{aligned} 1 \leq \langle \alpha g\bar{\alpha}, \alpha g'\bar{\alpha}\rangle &= \langle g, \bar{\alpha}\alpha g'\bar{\alpha}\alpha\rangle \\ &= \sum_{i,j=0}^{r} \langle g, h_i g' h_j\rangle = m^H_{g,g'} \end{aligned}$$

and therefore $g' \in HgH$. This completes the proof of (i).

(ii) and (iii) : The proof is as in (i) using $k\beta = \beta$ for all $k \in K$, which holds since β, $k\beta$ are irreducible P-$P \rtimes K$ bimodules and $\langle k\beta, \beta\rangle = \langle k, \beta\bar{\beta}\rangle = \sum_{k' \in K} \langle k, k'\rangle = \sum_{k' \in K} \delta_{k,k'} = 1$. □

Let us emphasize that the bimodules appearing in the above proposition are not necessarily irreducible. However, if $H \times K$ acts *freely* on G, *i.e.* if

$$h_i g k_j = g, \quad \text{some} \quad g \in G, \quad \text{implies} \quad h_i = k_j = e, \tag{10}$$

then $\langle \alpha g\beta, \alpha g\beta\rangle = \sum_{i=0}^{r}\sum_{j=0}^{s} \langle g, h_i g k_j\rangle = 1$ and $\alpha g\beta = \alpha g'\beta$ iff $g' \in HgK$. Thus Γ_{odd} (resp. Γ'_{odd}) is labeled by the double cosets ${}_H\backslash G/{}_K$ (resp. ${}_K\backslash G/{}_H$) in this case. Note that $H \times H$ resp. $K \times K$ never act freely on G. In general, Γ_{even} and Γ_{odd} (resp. Γ'_{even} and Γ'_{odd}) depend on the H and K actions on G, *i.e.* the multiplicities $m^H_{g,g'}$, $m^K_{g,g'}$ and $m^{H,K}_{g,g'}$, and the representation theory of H and K ($\alpha\bar{\alpha}$ and $\bar{\beta}\beta$ appear as P^H-P^H and $P \rtimes K$-$P \rtimes K$ bimodules for instance). Let us also remark, that in the special case when $G = H \cdot K$, we get with the above observation from Proposition 4.2 (ii) that Γ_{odd} consists of one single point, which means that the associated inclusion has depth two and is therefore by a well-known result obtained as a crossed product with a finite dimensional Kac algebra (*see* for instance [Sz]).

A few further things about the principal graphs can be said in general. Since α and β are irreducible bimodules, the bimodules αg, $g\beta$, $\bar{\beta}g$ and $g\bar{\alpha}$ are also irreducible for every $g \in G$. The computation

$$\langle \alpha g, \alpha g'\rangle = \sum_{i=0}^{r} \langle g, h_i g'\rangle = \begin{Bmatrix} 0, & \text{if} \quad g \notin Hg' \\ 1, & \text{if} \quad g \in Hg', \end{Bmatrix}$$

shows that ${}_H\backslash G$ labels the irreducible P^H-P bimodules. Similarly, $G/{}_K$ labels the irreducible P-$P \rtimes K$ bimodules, ${}_K\backslash G$ the irreducible $P \rtimes K$-P bimodules and $G/{}_H$ labels the irreducible P-P^H bimodules. Using this interpretation of these cosets we have

LEMMA 4.3. – *Consider the formal $\mathbb{Z}$-linear combinations of the sets G, ${}_H\backslash G$, $G/_H$, ${}_K\backslash G$, $G/_K$, Γ_{even}, Γ'_{even}, Γ_{odd} and Γ'_{odd}. Then the following diagrams commute:*

(i)

$$\begin{array}{ccc} \mathbb{Z}(G) & \xrightarrow{R_\beta} & \mathbb{Z}(G/_K) \\ {\scriptstyle L_\alpha}\downarrow & & \downarrow{\scriptstyle L_\alpha} \\ \mathbb{Z}({}_H\backslash G) & \xrightarrow{R_\beta} & \mathbb{Z}(\Gamma_{\text{odd}}) \end{array} \tag{11}$$

(i')

$$\begin{array}{ccc} \mathbb{Z}(G) & \xrightarrow{R_\beta} & \mathbb{Z}(G/_K) \\ {\scriptstyle L_{\bar\beta}}\downarrow & & \downarrow{\scriptstyle L_{\bar\beta}} \\ \mathbb{Z}({}_K\backslash G) & \xrightarrow{R_\beta} & \mathbb{Z}(\Gamma'_{\text{even}}) \end{array} \tag{12}$$

(ii)

$$\begin{array}{ccc} \mathbb{Z}(G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(G/_H) \\ {\scriptstyle L_\alpha}\downarrow & & \downarrow{\scriptstyle L_\alpha} \\ \mathbb{Z}({}_H\backslash G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(\Gamma_{\text{even}}) \end{array} \tag{13}$$

(ii')

$$\begin{array}{ccc} \mathbb{Z}(G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(G/_H) \\ {\scriptstyle L_{\bar\beta}}\downarrow & & \downarrow{\scriptstyle L_{\bar\beta}} \\ \mathbb{Z}({}_K\backslash G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(\Gamma'_{\text{odd}}) \end{array} \tag{14}$$

where $R_{\bar\alpha}$ (resp. R_β) denotes right multiplication by $\bar\alpha$ (resp. β) and L_α (resp. $L_{\bar\beta}$) denotes left multiplication by α (resp. $\bar\beta$).

Proof. – (i) $L_\alpha : \mathbb{Z}(G/_K) \to \mathbb{Z}(\Gamma_{\text{odd}})$ is defined as the linear extension of the map which assigns gK to $\alpha g\beta$, where the image is decomposed into irreducible P^H-$P \rtimes K$ bimodules. Since $G/_K$ labels the irreducible P-$P \rtimes K$ bimodules, this map is well-defined. Similarly, $R_\beta : \mathbb{Z}({}_H\backslash G) \to \mathbb{Z}(\Gamma_{\text{odd}})$ is defined as the linear extension of the map which assigns Hg to $\alpha g\beta$, where the image is again decomposed into irreducible P^H-$P \rtimes K$ bimodules. The map is again well-defined since ${}_H\backslash G$ labels the irreducible P^H-P bimodules. The commutativity of the diagram in now trivial: on the basis of $\mathbb{Z}(G)$ we have $L_\alpha R_\beta(g) = \alpha g\beta = R_\beta L_\alpha(g)$.

(ii) A similar argument as in (i) shows (ii). (i') and (ii') are then proven in the same way. $\square$

We prove next a formula that allows us to compute the norm of the principal graphs of $P^H \subset P \rtimes K$. We have

THEOREM 4.4. – *Let (Γ, Γ') be the principal graphs of the group type inclusion $R^H \subset R \rtimes K$ and let $G = \langle H, K\rangle$ be the group generated by H and K in* Out$R$. *Set $x = \sum_{h\in H} h$, $y = \sum_{k\in K} k$ and consider x and y in the left (or right) regular representation* *of G. Then*

$$\|\Gamma\|^2 = \|\Gamma'\|^2 = r(xy) = \frac{1}{|H||K|}\|xy\|^2. \tag{15}$$

In order to prove this theorem we need some results that allow us to compute the norms of infinite graphs. Recall that a (possibly infinite) matrix $A = (a_{kl})_{k,l\in I}$ with nonnegative entries $a_{kl} \in \mathbb{R}^+$ is called *irreducible* if for all $i,\ j \in I$, there is an $n \in \mathbb{N}$, $n = n(i,j)$, such that $(A^n)_{ij} > 0$. The following lemma is probably well-known to specialists

LEMMA 4.5. – *Let $A = (a_{kl})_{k,l\in I}$ be a (possibly infinite) self-adjoint square matrix with nonnegative entries such that A is irreducible and defines a bounded operator on $l^2(I)$. Then for all $x = (x_i)_{i\in I} \in l^2(I)^+\backslash\{0\}$, i.e. $x \neq 0$, $x_i \geq 0$, we have*

$$\|A\| = \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}}.$$

Proof. – Let $A = \int_{sp(A)} \lambda dE(\lambda)$ be the spectral resolution of A. Note that each $y \in l^2(I)^+$ defines a Borel measure on the spectrum $sp(A)$ of A via $\mu_y(S) = (E(S)y|y)$, $S \subset sp(A)$ a Borel set. First we show that $\lim\limits_{n\to\infty} \|A^n x\|^{\frac{1}{n}}$ always exists. For this consideration, we may assume $\|x\| = 1$. Then using the Minkowski inequality we get

$$\begin{aligned}
\|A^n x\|^{\frac{1}{n}} &= \left(\int_{sp(A)} \lambda^{2n} d\mu_x(\lambda)\right)^{\frac{1}{2n}} \\
&\leq \left[\left(\int_{sp(A)} \lambda^{\frac{2n n+1}{n}} d\mu_x(\lambda)\right)^{\frac{n}{n+1}} \left(\int_{sp(A)} 1 d\mu_x(\lambda)\right)^{\frac{1}{n+1}}\right]^{\frac{1}{2n}} \\
&= \|A^{n+1}x\|^{\frac{1}{n+1}} \cdot 1.
\end{aligned}$$

Hence, the above limit exists. Next set

$$\rho(x) = \max|\text{supp}(\mu_x)| = \max\{|t|,\ t \in \text{supp}(\mu_x)\}.$$

Clearly

$$\|A^n x\| = \left(\int_{sp(A)} \lambda^{2n} d\mu_x(\lambda)\right)^{\frac{1}{2}} \leq \rho(x)^n \|x\|$$

and if $\rho(x) > 0$, then we have for all $c \in (0, \rho(x))$

$$\|A^n x\| \geq \left(\int_{|\lambda|\geq c} \lambda^{2n} d\mu_x(\lambda)\right)^{\frac{1}{2}} \geq c^n \mu_x(\{\lambda \in sp(A) \,|\, |\lambda| \geq c\}).$$

Note that by the definition of $\rho(x)$, $\{\lambda \in sp(A) \,|\, |\lambda| \geq c\}$ is not a μ_x-nullset. Hence

$$c \leq \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}} \leq \rho(x)$$

and since this holds for all $c \in (0, \rho(x))$, we have

$$\rho(x) = \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}}. \tag{16}$$

This formula holds also if $\rho(x) = 0$.

Let now δ_i, $i \in I$, be the standard basis for $l^2(I)$ and fix $i, j \in I$. By irreducibility of A there is an $m \in \mathbb{N}$ and a constant $c > 0$ such that $A^m \delta_i \geq c\delta_j$ (entrywise inequality). Hence

$$A^{n+m}\delta_i \geq cA^n\delta_j, \qquad \text{for all } n \in \mathbb{N}$$

and using that the inner product on $l^2(I)$ is increasing in both variables, we get that

$$\|A^n\delta_j\| \leq \frac{1}{c}\|A^{n+m}\delta_i\| \leq \frac{\|A\|^m}{c}\|A^n\delta_i\|, \qquad n \in \mathbb{N}.$$

Thus $\rho(\delta_j) \leq \rho(\delta_i)$. Hence, by symmetry, $\rho(\delta_i)$ is independent of $i \in I$. Let ρ denote their common value. Clearly $\rho \leq \|A\|$. Moreover, the supports of all the measures μ_{δ_i} are contained in $[-\rho, \rho]$. Since $(\delta_i)_{i\in I}$ is a total set in $l^2(I)$, the nullsets of the spectral measure E are precisely the common nullsets for all the measures μ_{δ_i}, $i \in I$. Hence

$$\mathrm{sp}(A) = \mathrm{supp}(E) \subset [-\rho, \rho],$$

which proves that $\rho \geq r(A) = \|A\|$. Altogether, we have shown that $\rho(\delta_i) = \|A\|$ for all $i \in I$. Finally, let x be a general element in $l^2(I)^+\backslash\{0\}$. Then there exists an $i \in I$ and a constant $c > 0$ such that $x \geq c\delta_i$. Hence

$$A^n x \geq cA^n\delta_i, \qquad \text{for all } n \in \mathbb{N}$$

and therefore

$$\lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}} \geq \lim_{n\to\infty} \|A^n\delta_i\|^{\frac{1}{n}} = \|A\|.$$

The converse inequality is trivial. $\square$

PROPOSITION 4.6. – *Let I_1, I_2, I_3 and I_4 be four (possibly infinite) index sets, and let G, H, K and L be matrices with nonnegative entries indexed by $I_1 \times I_2$, $I_3 \times I_4$, $I_1 \times I_3$ and $I_2 \times I_4$ respectively. Assume that G, H, K and L define bounded operators between the $l^2(I_k)$-spaces and that $GL = KH$ and $LH^t = G^tK$. If moreover the bipartite graphs Γ_G and Γ_H associated to G and H are connected and $K \neq 0$ (equivalently $L \neq 0$), then $\|G\| = \|H\|$.*

Proof. – Set $A = G^tG$ and $B = HH^t$. By connectedness of Γ_G and Γ_H, A and B are irreducible matrices. Moreover $AK = KB$. Since K has a nonzero column, there is an index k such that $K\delta_k \geq 0$ (entrywise), $K\delta_k \neq 0$. Thus by Lemma 4.5 we have

$$\begin{aligned}\|G\|^2 = \|A\| &= \lim_{n\to\infty} \|A^nK\delta_k\|^{\frac{1}{n}} = \lim_{n\to\infty} \|KB^n\delta_k\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \|B^n\delta_k\|^{\frac{1}{n}} = \|B\| = \|H\|^2,\end{aligned}$$

i.e. $\|G\| \leq \|H\|$. The converse inequality is obtained in the same way, using $K^tA = BK^t$. This completes the proof. $\square$

Remark 4.7. – Note that we do not assume in the above proposition that K and L define connected graphs. Note also that if all the involved matrices above were finite, a Perron-Frobenius argument would give the result.

We can now proceed with the proof of Theorem 4.4.

Proof of Theorem 4.4. – We have the following commutative diagrams

$$\begin{array}{ccccc} \mathbb{Z}(G/_H) & \xrightarrow{R_\alpha} & \mathbb{Z}(G) & \xrightarrow{R_\beta} & \mathbb{Z}(G/_K) \\ \downarrow L_\alpha & & \downarrow L_\alpha & & \downarrow L_\alpha \\ \mathbb{Z}(\Gamma_{\text{even}}) & \xrightarrow{R_\alpha} & \mathbb{Z}(_H\backslash G) & \xrightarrow{R_\beta} & \mathbb{Z}(\Gamma_{\text{odd}}) \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{Z}(G/_K) & \xrightarrow{R_{\bar\beta}} & \mathbb{Z}(G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(G/_H) \\ \downarrow L_\alpha & & \downarrow L_\alpha & & \downarrow L_\alpha \\ \mathbb{Z}(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar\beta}} & \mathbb{Z}(_H\backslash G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{Z}(\Gamma_{\text{even}}) \end{array}$$

by transposing the diagram in Lemma 4.3 (ii) (resp (i)) and combining it with the one in Lemma 4.3 (i) (resp. (ii)). Let A_1, A_2, B_1, B_2, C_1, C_2 and C_3 be the matrices corresponding to the linear maps of the first diagram, *i.e.*

$$\begin{array}{ccccc} \mathbb{Z}(G/_H) & \xrightarrow{A_1} & \mathbb{Z}(G) & \xrightarrow{B_1} & \mathbb{Z}(G/_H) \\ \downarrow C_1 & & \downarrow C_2 & & \downarrow C_3 \\ \mathbb{Z}(\Gamma_{\text{even}}) & \xrightarrow{A_2} & \mathbb{Z}(_H\backslash G) & \xrightarrow{B_2} & \mathbb{Z}(\Gamma_{\text{odd}}). \end{array}$$

By Frobenius reciprocity (Proposition 2.2), the matrices corresponding to the second diagram are given by

$$\begin{array}{ccccc} \mathbb{Z}(G/_K) & \xrightarrow{B_1^t} & \mathbb{Z}(G) & \xrightarrow{A_1^t} & \mathbb{Z}(G/_H) \\ \downarrow C_3 & & \downarrow C_2 & & \downarrow C_1 \\ \mathbb{Z}(\Gamma_{\text{odd}}) & \xrightarrow{B_2^t} & \mathbb{Z}(_H\backslash G) & \xrightarrow{A_2^t} & \mathbb{Z}(\Gamma_{\text{even}}). \end{array}$$

Therefore

$$(B_2A_2)C_1 = C_3(B_1A_1) \qquad \text{and} \qquad (B_2A_2)^tC_3 = C_1(B_1A_1)^t,$$

so that by Proposition 4.6 $\|B_1A_1\| = \|B_2A_2\|$. Clearly, the adjacency matrix Δ_Γ of the graph Γ is

$$\Delta_\Gamma = \begin{pmatrix} 0 & B_2A_2 \\ (B_2A_2)^t & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} \|\Gamma\|^2 &= \|B_2A_2\|^2 = \|B_1A_1\|^2 \\ &= \|A_1^tB_1^tB_1A_1\| \\ &= r(A_1^tB_1^tB_1A_1) = r(A_1A_1^tB_1^tB_1). \end{aligned}$$

But the matrix $A_1A_1^t$ is the matrix of the composed map from $\mathbb{Z}(G)$ into $\mathbb{Z}(G)$ given by right multiplication by $\bar\alpha\alpha$. Hence $A_1A_1^t$ is the matrix of right convolution by $x = \sum_{h\in H} h$.

Similarly, $B_1^t B_1$ is the matrix of right convolution by the measure $y = \sum_{k \in K} k$. Hence $A_1 A_1^t B_1^t B_1$ is the matrix of right convolution by yx, and therefore unitary equivalent to the matrix given by left convolution by $(yx)^\vee = xy$. Hence $||\Gamma||^2 = r(xy)$. Note that $||\Gamma|| = ||\Gamma'||$ holds always ([Po4], but it follows also from Proposition 4.6). □

Before we proceed with studying the amenability of $R^H \subset R \rtimes K$, let us give a corollary of Theorem 4.4.

COROLLARY 4.8. – *Let $R^H \subset R \rtimes K$ be a group type inclusion as above and let* (Γ, Γ') *be the principal graphs. If the group G generated by H and K in* Out$R$ *is the free product,* i.e. *$G = H * K$, then*

$$||\Gamma||^2 = ||\Gamma'||^2 = (\sqrt{|H| - 1} + \sqrt{|K| - 1})^2.$$

Proof. – Let λ be the left regular representation of G and set $p = \frac{1}{|H|} \sum_{h \in H} \lambda(h)$, $q = \frac{1}{|K|} \sum_{k \in K} \lambda(k)$. By Theorem 4.4 we have that

$$||\Gamma||^2 = ||\Gamma'||^2 = |H||K| r(pq).$$

If the groups H and K are interchanged, then the principal graphs will be interchanged, so it is sufficient to treat the case where $|H| \leq |K|$. Since the projections p and q are free in the sense of Voiculescu [Voi], the arguments of [ABH, pp. 10-18] go through: Set

$$\alpha = r(p) = \frac{1}{|H|}, \qquad \beta = r(q) = \frac{1}{|K|}$$

and

$$t_1 = \left(\sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta}\right)^2, \qquad t_2 = \left(\sqrt{(-\alpha)(1-\beta)} + \sqrt{\alpha\beta}\right)^2.$$

Denote by p_0, q_0 the projections in $M_2(C([t_1, t_2]))$ given by $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q_0 = \begin{pmatrix} 1-t & \sqrt{t-t^2} \\ \sqrt{t-t^2} & t \end{pmatrix}$.

By [ABH, Theorem 13] we have:

a) If $|K| > |H| \geq 2$, then up to isomorphism

$$C^*(p, q, 1) = \mathbb{C} \oplus M_2(C([t_1, t_2])) \oplus \mathbb{C},$$

where $p = 0 \oplus p_0 \oplus 1$ and $q = 0 \oplus q_0 \oplus 0$.

b) If $|K| = |H| > 2$, then $t_2 = 1$ and up to isomorphism

$$C^*(p, q, 1) = \mathbb{C} \oplus M_2(C([t_1, t_2])),$$

where $p = 0 \oplus p_0$ and $q = 0 \oplus q_0$.

c) If $|K| = |H| = 2$, then $t_1 = 0$, $t_2 = 1$ and up to isomorphism

$$C^*(p, q, 1) = M_2(C([t_1, t_2])),$$

where $p = p_0$ and $q = q_0$.

In all three cases we get

$$\begin{aligned} r(pq) = \|pqp\| = 1 - t_1 &= 1 - (\sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta})^2 \\ &= \alpha + \beta - 2\alpha\beta + 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \end{aligned}$$

Thus

$$\begin{aligned} \|\Gamma\|^2 = \frac{1}{\alpha\beta} r(pq) &= \frac{1}{\alpha\beta}(\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)})^2 \\ &= \left(\sqrt{\frac{1-\beta}{\beta}} + \sqrt{\frac{1-\alpha}{\alpha}}\right)^2 \\ &= (\sqrt{|K|-1} + \sqrt{|H|-1})^2. \quad \square \end{aligned}$$

We prove next that amenability of the *inclusion* $R^H \subset R \rtimes K$ is equivalent to the amenabitity of the *group* $G = \langle H, K \rangle$.

THEOREM 4.9. – *Let P be a II_1 factor carrying an outer action of the finite groups H and K. Denote the principal graphs of the inclusion $P^H \subset P \rtimes K$ by (Γ, Γ'). Then*

$$|H||K| = [P \rtimes K : P^H] = \|\Gamma\|^2 \quad \textit{iff } G = \langle H, K \rangle \subset \mathrm{Out}P \textit{ is an amenable group.}$$

In particular $R^H \subset R \rtimes K$ is amenable (in the sense of Popa) iff G is an amenable group in Out$R$.

Proof. – Note that the second statement follows from the first by Popa's characterization of amenability ([Po4, 5.4], [Po5, Theorem 2.1]).
Define a measure μ on G with $\mathrm{supp}\,\mu = \{kh \,|\, k \in K,\ h \in H\}$ and weights $\mu(kh) = \frac{1}{|H||K|}$. Then μ is a non-symmetric probability measure on G whose support generates all of G. As usual, μ defines a Markov operator $P_\mu : l^2(G) \to l^2(G)$ via

$$(P_\mu f)(t) = \sum_{s \in G} f(ts)\mu(s), \qquad f \in l^2(G).$$

From the generalizations of Kesten's characterization of amenability of a group [Ke] given by Derriennic-Guivarc'h ([DG], Theorem 1) and Berg-Christensen ([BC], Theorem 5), we get that G is amenable iff the spectral radius $r(P_\mu)$ is equal to 1. Let $\rho : G \to l^2(G)$ be the right regular representation of G and let $x = \sum_{i=0}^{r} \rho(h_i)$, $y = \sum_{j=0}^{s} \rho(k_j)$, then we have for all $f \in l^2(G)$

$$(xy)f = \frac{1}{|H||K|} P_\mu f.$$

Thus

$$\begin{aligned} G \text{ is amenable} \quad &\text{iff} \quad r(P_\mu) = 1 \\ &\text{iff} \quad r(xy) = |H||K| \\ &\text{iff} \quad \|\Gamma\|^2 = |H||K| \end{aligned}$$

where the last equivalence follows from Theorem 4.4. □

Remark 4.10. – *a*) Note that $R^H \subset R \rtimes K$ will be in particular always amenable when G has subexponential growth [KV].

b) Using Theorem 4.9 and Corollary 4.8 we also obtain the well-known result that a group $G = H * K$ is an amenable group if and only if $|H||K| = (\sqrt{|H|-1} + \sqrt{|K|-1})^2$, which happens iff $|H| = |K| = 2$, *i.e.* $H = K = \mathbb{Z}_2$. In other words, the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ is the only amenable group which is a free product of two finite groups.

It is now natural to ask if strong amenability [Po4] of $R^H \subset R \rtimes K$ can also be translated into an appropriate growth condition on the group G. As in [Bi1], [Po3] we will see that it is the *entropy* of G with respect to a certain distinguished measure, which will describe the ergodicity behaviour of the standard model [Po4] associated to the inclusion $R^H \subset R \rtimes K$.

Recall that if Γ is a principal graph of $N \subset M$ and $\gamma \in \Gamma$ denotes an irreducible A-B-bimodule, A, $B \in \{N, M\}$, we let

$$\xi(\gamma) = [R_\gamma(B)' : L_\gamma(A)]^{\frac{1}{2}},$$

where $L_\gamma(A) \subset R_\gamma(B)'$ is the inclusion defined by the bimodule $\gamma = {}_A H_B$. L_γ (resp. R_γ) denotes the left (resp. right) action of A (resp. B) on the Hilbert space H. Equivalently, $\xi(\gamma)$ is the square root of the index of a reduced subfactor corresponding to γ (*see* (6), (7)). If $N \subset M$ is extremal, we have that $\Delta_\Gamma \xi = [M : N]^{\frac{1}{2}} \xi$ and similarly for the other principal graph Γ' (*see* for instance [Po4]). If $N = P^H \subset M = P \rtimes K$, then $\xi(\alpha g \bar{\alpha}) = |H|$, $\xi(\alpha g \beta) = |H|^{\frac{1}{2}} |K|^{\frac{1}{2}} = \xi(\bar{\beta} g \bar{\alpha})$ and $\xi(\bar{\beta} g \beta) = |K|$. As we have seen above, the bimodules $\alpha g \bar{\alpha}$, $\alpha g \beta$, $\bar{\beta} g \beta$ and $\bar{\beta} g \bar{\alpha}$ may not be irreducible, but since the square root of the indices of the irreducible components add up to $\xi(\alpha g \bar{\alpha})$ (resp. $\xi(\alpha g \beta)$, etc.), we see that ξ is certainly a bounded eigenvector of Δ_Γ to the eigenvalue $(|H||K|)^{\frac{1}{2}}$ (and similarly for $\Delta_{\Gamma'}$ and its eigenvector ξ'). By ([Po4], Cor. 5.3.7]) we have that the strong amenability of $R^H \subset R \rtimes K$ is equivalent to the ergodicity of the principal graph Γ (which in this case is equivalent to the ergodicity of Γ'). This in turn means by definition ([Po4, 1.4.2]) that ξ is up to a scalar multiple the unique ξ-bounded eigenvector of Δ_Γ to the eigenvalue $(|H||K|)^{\frac{1}{2}}$. Before we come to the theorem relating strong amenability of $R^H \subset R \rtimes K$ to certain growth properties of G, let us state for clarity of exposition the following immediate

LEMMA 4.11. – *Let $N \subset M$ be an extremal inclusion of II_1 factors with principal graph Γ. Denote by $\Delta_\Gamma = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$ the adjacency matrix of Γ and let ξ be the standard eigenvector of local indices satisfying $\Delta_\Gamma \xi = [M : N]^{\frac{1}{2}} \xi$. Then the following are equivalent*

(i) *Up to scalar multiples ξ is the unique ξ-bounded eigenvector of Δ_Γ to the eigenvalue $[M : N]^{\frac{1}{2}}$ (*i.e. *Γ is ergodic).*

(ii) *Up to scalar multiples* $\xi|_{\Gamma_{\text{even}}}$ *is the unique* $\xi|_{\Gamma_{\text{even}}}$*-bounded eigenvector of* AA^t *to the eigenvalue* $[M : N]$.

(iii) *Up to scalar multiples* $\xi|_{\Gamma_{\text{odd}}}$ *is the unique* $\xi|_{\Gamma_{\text{odd}}}$*-bounded eigenvector of* A^tA *to the eigenvalue* $[M : N]$.

Proof. – Easy exercise. □

In order to state our main theorem on *strong amenability* of the inclusion $R^H \subset R \rtimes K$ we need to recall some facts on harmonic functions on groups.

Let G be a countable discrete group and ν a probability measure on G. A function $f \in l^\infty(G)$ is called *ν-harmonic*, if

$$f(s) = \sum_{g \in G} f(sg)\nu(g), \qquad \text{for all } s \in G$$

(*see* for instance [Fu], [KV] or [Wo] for a review). As usual we denote by $\check{\nu}$ the probability measure defined by $\check{\nu}(g) = \nu(g^{-1})$, $g \in G$. Ergodicity properties of the *right random walk* on G defined by the measure ν are then expressed in terms of the *entropy* $h(G, \nu)$ of G with respect to ν, which is defined as

$$h(G, \nu) = \lim_{n \to \infty} \frac{1}{n} H(\nu^{*n}),$$

where $H(\nu) = -\sum_{g \in G} \nu(g) \ln \nu(g)$ is the *entropy of* ν and ν^{*n} denotes the n-th convolution power of the measure ν (*see* [Av], [KV]). It is shown there that if $H(\nu) < \infty$, then the vanishing of $h(G, \nu)$ is equivalent to the triviality of the Poisson boundary $\Gamma(G, \nu)$, which means that all bounded ν-harmonic functions on G are constant.

Let now $G = \langle H, K \rangle$ be as in Theorem 4.9 and consider the measure μ on G with support $\{kh \,|\, k \in K, h \in H\}$ and weigths $\mu(kh) = \dfrac{1}{|H||K|}$, $k \in K$, $h \in H$ (*see* proof of Theorem 4.4). μ is then a finitely supported, non-symmetric (*i.e.* $\check{\mu} \neq \mu$) probability measure on G whose support generates all of G (such a measure is called *nondegenerate*). The following lemma shows that we can replace μ by a *symmetric* measure for entropy considerations.

LEMMA 4.12. – *Let* $\mu = \dfrac{1}{|H||K|} \sum_{k \in K} \sum_{h \in H} \delta_{kh}$ *be the above probability measure on* $G = \langle H, K \rangle$. *Then*

$$h(G, \mu) = h(G, \check{\mu}) = h(G, \mu * \check{\mu}) = h(G, \check{\mu} * \mu).$$

Proof. – Set $\mu_1 = \dfrac{1}{|K|} \sum_{k \in K} \delta_k$ and $\mu_2 = \dfrac{1}{|H|} \sum_{h \in H} \delta_h$, then $\mu = \mu_1 * \mu_2$. Note that $\mu_i * \mu_i = \mu_i$, $i = 1, 2$ and therefore $\mu * \check{\mu} = \mu_1 * \mu_2 * \mu_1$, $\check{\mu} * \mu = \mu_2 * \mu_1 * \mu_2$. Thus

$$\mu^{*n} = (\mu_1 * \mu_2)^{*n} = (\mu * \check{\mu})^{*(n-1)} = \mu_1 * (\check{\mu} * \mu)^{*(n-1)},$$

which implies $H(\mu^{*n}) \leq H((\mu * \check{\mu})^{*(n-1)}) + H(\mu_2)$ ([KV, Proposition 1.1]) and hence $h(G,\mu) \leq h(G, \mu * \check{\mu})$ by definition of the entropy. Similarly, $h(G,\mu) \leq h(G, \check{\mu} * \mu)$. Furthermore we have $(\mu * \check{\mu})^{*n} = \mu^{*n} * \mu_1$ and $(\check{\mu} * \mu)^{*n} = \mu_2 * \mu^{*n}$. Thus

$$H((\mu * \check{\mu})^{*n}) \leq H(\mu^{*n}) + H(\mu_1)$$
$$H((\check{\mu} * \mu)^{*n}) \leq H(\mu^{*n}) + H(\mu_2)$$

which gives $h(G, \mu * \check{\mu}) \leq h(G,\mu)$ and $h(G, \check{\mu} * \mu) \leq h(G,\mu)$, from which the lemma follows. □

We can now state our main theorem on strong amenability of the inclusion $R^H \subset R \rtimes K$.

THEOREM 4.13. – *Let H and K be finite groups acting properly outerly on the hyperfinite II_1 factor R and let G be the group generated by H and K in* Out$R$. *Suppose that the action $(h,k) \cdot g = hgk$ of $H \times K$ on G is free ((10)) and let μ be the probability measure given by $\mu = \frac{1}{|H||K|} \sum_{k \in K} \sum_{h \in H} \delta_{kh}$. Then the following are equivalent*

(i) $R^H \subset R \rtimes K$ *is strongly amenable (in the sense* [Po4]).

(ii) *All bounded μ-harmonic functions on G are constant.*

(iii) *The entropy $h(G,\mu)$ of G with respect to μ is zero.*

(i') $R^K \subset R \rtimes H$ *is strongly amenable.*

(ii') *All bounded $\check{\mu}$-harmonic functions on G are constant.*

(iii') $h(G,\check{\mu}) = 0$.

(iv) *All bounded $\mu * \check{\mu}$ (resp. $\check{\mu} * \mu$)-harmonic functions on G are constant.*

(iv') $h(G, \mu * \check{\mu}) = 0$ (resp. $h(G, \check{\mu} * \mu) = 0$).

Proof. – Let (Γ, Γ') be the principal graphs of $N = R^H \subset R \rtimes K = M$. Since $R^K \subset R \rtimes H$ is just $M \subset M_1$ from the basic construction $N \subset M \subset M_1$, we get (i) ⇔ (i') from [Po4]. Observe that this also follows by repeating the proof below for Γ' instead of Γ. Note that strong amenability of $R^H \subset R \rtimes K$ is equivalent to amenability and ergodicity of Γ as mentioned above. The equivalences (ii) ⇔ (iii), (ii') ⇔ (iii') and (iv) ⇔ (iv') are classical results ([Av], [KV]). The equivalence (iii) ⇔ (iii') ⇔ (iv') is shown in Lemma 4.12. Hence we are left with showing (i) ⇔ (ii).

Let $\Delta_\Gamma = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$ be the adjacency matrix of Γ, *i.e.* A is the standard matrix of $R^H \subset R \rtimes K$. From Lemma 4.3 we get the following commutative diagram (17)

$$\begin{array}{ccccccccc}
\mathbb{R}(G/_K) & \xrightarrow{\tilde{R}_{\bar{\beta}}} & \mathbb{R}(G) & \xrightarrow{\tilde{R}_{\bar{\alpha}}} & \mathbb{R}(G/_H) & \xrightarrow{\tilde{R}_{\alpha}} & \mathbb{R}(G) & \xrightarrow{\tilde{R}_{\beta}} & \cdots \\
\downarrow L_\alpha & & \downarrow L_\alpha & & \downarrow L_\alpha & & \downarrow L_\alpha & & \\
\mathbb{R}(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar{\beta}}} & \mathbb{R}(_H\backslash G) & \xrightarrow{R_{\bar{\alpha}}} & \mathbb{R}(\Gamma_{\text{even}}) & \xrightarrow{R_{\alpha}} & \mathbb{R}(_H\backslash G) & \xrightarrow{R_{\beta}} & \cdots \\
\downarrow L_{\bar{\alpha}} & & \downarrow L_{\bar{\alpha}} & & \downarrow L_{\bar{\alpha}} & & \downarrow L_{\bar{\alpha}} & & \\
\mathbb{R}(G/_K) & \xrightarrow{\tilde{R}_{\bar{\beta}}} & \mathbb{R}(G) & \xrightarrow{\tilde{R}_{\bar{\alpha}}} & \mathbb{R}(G/_H) & \xrightarrow{\tilde{R}_{\alpha}} & \mathbb{R}(G) & \xrightarrow{\tilde{R}_{\beta}} & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots \xrightarrow{\tilde{R}_\beta} & \mathbb{R}(G/_K) & \xrightarrow{\tilde{R}_{\bar{\beta}}} & \mathbb{R}(G) \\
\downarrow & \downarrow L_\alpha & & \downarrow L_\alpha \\
\cdots \xrightarrow{R_\beta} & \mathbb{R}(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar{\beta}}} & \mathbb{R}(_H\backslash G) \\
\downarrow & \downarrow L_{\bar{\alpha}} & & \downarrow L_{\bar{\alpha}} \\
\cdots \xrightarrow{\tilde{R}_\beta} & \mathbb{R}(G/_K) & \xrightarrow{\tilde{R}_{\bar{\beta}}} & \mathbb{R}(G)
\end{array}$$

where as usual R_γ, $\tilde{R}_\gamma$, L_γ, $\gamma \in \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$ denotes right resp. left multiplication by γ. As in the proof of Theorem 4.4 we regard these operators as matrices acting on the appropriate spaces according to the commutative diagram (17).

(ii) $\Rightarrow$ (i): Let η be a $\xi|_{\Gamma_{\text{odd}}}$-bounded (and hence bounded) vector on Γ_{odd} satisfying

$$A^t A\eta = |H||K|\eta \quad \text{or equivalently} \quad R_{\alpha\beta}R_{\overline{\alpha\beta}}\eta = |H||K|\eta. \tag{18}$$

Define $f = L_{\bar{\alpha}} R_{\bar{\beta}}\eta \in l^\infty(G)$ and compute

$$\begin{aligned}
\tilde{R}_{\bar{\alpha}\alpha\beta\bar{\beta}} f &= \tilde{R}_{\bar{\beta}}\tilde{R}_\beta\tilde{R}_\alpha\tilde{R}_{\bar{\alpha}} f \\
&= \tilde{R}_{\bar{\beta}}\tilde{R}_\beta\tilde{R}_\alpha\tilde{R}_{\bar{\alpha}} L_{\bar{\alpha}} R_{\bar{\beta}}\eta \\
&= L_{\bar{\alpha}} R_{\bar{\beta}} R_\beta R_\alpha R_{\bar{\alpha}} R_{\bar{\beta}}\eta \qquad ((17)) \\
&= |H||K| L_{\bar{\alpha}} R_{\bar{\beta}}\eta = |H||K| f.
\end{aligned}$$

Thus f satisfies

$$f(s) = \frac{1}{|H||K|}\sum_{k\in K}\sum_{h\in H} f(skh),$$

i.e. f is a μ-harmonic function on G. Observe that since $H \times K$ acts freely on G, we have that $R_\beta L_\alpha : \mathbb{R}(G) \to \mathbb{R}(\Gamma_{\text{odd}})$ is surjective (*see* considerations before Lemma 4.3), and hence $L_{\bar{\alpha}} R_{\bar{\beta}} = (R_\beta L_\alpha)^t$ is injective. Thus, if Γ is not ergodic, *i.e.* $A^t Ax = |H||K|x$ has at least two linearly independent $\xi|_{\Gamma_{\text{odd}}}$-bounded solutions, icithen by lifting them up to G via $L_{\bar{\alpha}} R_{\bar{\beta}}$, we get two linearly independent μ-harmonic functions on G, *i.e.* $h(G, \mu) > 0$. This proves (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Denote the space of bounded μ-harmonic functions on G by $H^\infty(G, \mu)$ and suppose that there are nonconstant bounded μ-harmoncic functions on G. Then by ([Ka2], 3.3.1, Corollary 1) it follows that $\dim H^\infty(G, \mu) = \infty$. Let $f \in H^\infty(G, \mu)$ and average f by H from the left, *i.e.* let $\tilde{f}(g) = \dfrac{1}{|H|}\sum_{h\in H} f(hg)$, $g \in G$. Then it is easy to check that $\tilde{f} \in H^\infty(G, \mu)$, *i.e.* $\tilde{f}$ satisfies $\tilde{R}_{\bar{\alpha}\alpha\beta\bar{\beta}}\tilde{f} = |H||K|\tilde{f}$ (since f does) or equivalently

$$\tilde{f}(g) = \frac{1}{|H||K|}\sum_{k\in K}\sum_{h\in H}\tilde{f}(gkh), \qquad g \in G. \tag{19}$$

Observe that $\tilde{f}(hg) = \tilde{f}(gk) = \tilde{f}(hgk)$, for all $h \in H$, $k \in K$ and $g \in G$. Since $\dim H^\infty(G, \mu) = \infty$, and H is a finite group, we get that the space of bounded μ-harmonic functions on G, which are constant on H-K double cosets, is still infinite dimensional. Let f be such a function, *i.e.* $f \in l^\infty(G)$ with $\tilde{R}_{\bar{\alpha}\alpha\beta\bar{\beta}} f = |H||K| f$ and $L_{\bar{\alpha}\alpha} f = |H| f$ (the latter follows from $f(hg) = f(g)$, $h \in H$, $g \in G$). Set

$$\eta = R_\beta R_\alpha R_{\bar{\alpha}} L_\alpha f = R_\beta L_\alpha \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f,$$

which defines a bounded function on Γ_{odd} (*see* (17)). Then using (17) we get

$$\begin{aligned} R_{\alpha\beta} R_{\overline{\alpha\beta}} \eta &= R_\beta R_\alpha R_{\bar{\alpha}} R_{\bar{\beta}} R_\beta R_\alpha R_{\bar{\alpha}} L_\alpha f \\ &= R_\beta R_\alpha R_{\bar{\alpha}} L_\alpha \tilde{R}_{\bar{\beta}} \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f \qquad ((17)) \\ &= |H||K| R_\beta R_\alpha R_{\bar{\alpha}} L_\alpha f = |H||K| \eta, \end{aligned}$$

i.e. $A^t A \eta = |H||K| \eta$. Suppose f_1 and f_2 are two such functions and assume $\eta_1 \stackrel{\text{def}}{=} R_\beta L_\alpha \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_1 = R_\beta L_\alpha \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_2 \stackrel{\text{def}}{=} \eta_2$. Then, since we get from (17) that

$$L_{\bar{\alpha}} R_\beta L_\alpha \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} = \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} L_{\bar{\alpha}} L_\alpha,$$

we have

$$\begin{aligned} L_{\bar{\alpha}} \eta_1 &= \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} L_{\bar{\alpha}} L_\alpha f_1 \\ &= |H| \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_1 \\ &= L_{\bar{\alpha}} \eta_2 \\ &= |H| \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_2 \end{aligned}$$

Thus

$$f_1 = \frac{1}{|H||K|} \tilde{R}_{\bar{\beta}} \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_1 = \frac{1}{|H||K|} \tilde{R}_{\bar{\beta}} \tilde{R}_\beta \tilde{R}_\alpha \tilde{R}_{\bar{\alpha}} f_2 = f_2.$$

Hence, if $h(G, \mu) > 0$, then the equation $A^t A x = |H||K| x$ has infinitely many linear independent bounded (and hence $\xi|_{\Gamma_{\text{odd}}}$-bounded) solutions, which means that Γ is not ergodic (Lemma 4.12). This completes the proof of $i) \Rightarrow ii)$ and hence the proof of the theorem. □

Remark 4.14. – *a*) If P is an arbitrary II_1 factor with H and K actions as in the theorem, then we actually prove above that the principal graphs (Γ, Γ') of $P^H \subset P \rtimes K$ are ergodic (*i.e.* the associated hyperfinite standard inclusion [Po4] consists of factors) if and only if all bounded μ (resp. $\check{\mu}$, resp. $\mu * \check{\mu}$, resp. $\check{\mu} * \mu$) - harmonic functions on G are constant.

b) The proof of (i) $\Rightarrow$ (ii) does not use the hypothesis that $H \times K$ acts freely on G. Instead we use that the space of bounded μ-harmonic functions on G is either one dimensional or infinite dimensional ([Ka2], [Fu]).

c) If $h(G, \mu) = 0$, then G is automatically amenable ([Av], [KV]). The statement $h(G, \mu) = 0 \Leftrightarrow h(G, \check{\mu}) = 0$, is a translation of the fact that for amenable inclusions N^{st} is a factor iff M^{st} is a factor (*see* [Po4] for the notation), where $N = R^H$ and $M = R \rtimes K$. This happens for the inclusion $R^H \subset R \rtimes K$ precisely in the strongly amenable case.

d) It is shown in ([Po4], 1.4.2, Corollary) that if $N \subset M$ is extremal, then $Z(N^{st}) \cap Z(M^{st}) = \mathbb{C}$. If we let $N = R^H$, $M = R \rtimes K$ be as above, then this results is a special case of ([Ka3], Theorem 3 and Theorem 4), where it is shown that the (μ, μ) jointly harmonic functions on G, *i.e.* $f \in l^\infty(G)$ with $\mu f \mu = f$, are just the constants. This is obtained from the following commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{R}(\Gamma'_{\text{even}}) & \xrightarrow{R_{\bar\beta}} & \mathbb{R}({}_K\backslash G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{R}(\Gamma'_{\text{odd}}) & \xrightarrow{R_{\alpha}} & \mathbb{R}({}_K\backslash G) & \xrightarrow{R_{\beta}} & \cdots \\
{\scriptstyle L_{\bar\beta}}\uparrow & & {\scriptstyle L_{\bar\beta}}\uparrow & & {\scriptstyle L_{\bar\beta}}\uparrow & & {\scriptstyle L_{\bar\beta}}\uparrow & & \\
\mathbb{R}(G/_K) & \xrightarrow{R_{\bar\beta}} & \mathbb{R}(G) & \xrightarrow{\tilde R_{\bar\alpha}} & \mathbb{R}(G/_H) & \xrightarrow{\tilde R_{\alpha}} & \mathbb{R}(G) & \xrightarrow{\tilde R_{\beta}} & \cdots \\
\downarrow{\scriptstyle L_\alpha} & & \downarrow{\scriptstyle L_\alpha} & & \downarrow{\scriptstyle L_\alpha} & & \downarrow{\scriptstyle L_\alpha} & & \\
\mathbb{R}(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar\beta}} & \mathbb{R}({}_H\backslash G) & \xrightarrow{R_{\bar\alpha}} & \mathbb{R}(\Gamma_{\text{even}}) & \xrightarrow{R_{\alpha}} & \mathbb{R}({}_H\backslash G) & \xrightarrow{R_{\beta}} & \cdots
\end{array}$$

$$\begin{array}{ccc}
\cdots\mathbb{R}(\Gamma'_{\text{even}}) & \xrightarrow{R_{\bar\beta}} & \mathbb{R}({}_K\backslash G) \\
{\scriptstyle L_{\bar\beta}}\uparrow & & {\scriptstyle L_{\bar\beta}}\uparrow \\
\cdots\mathbb{R}(G\backslash_K) & \xrightarrow{\tilde R_{\bar\beta}} & \mathbb{R}(G) \\
\downarrow{\scriptstyle L_\alpha} & & \downarrow{\scriptstyle L_\alpha} \\
\cdots\mathbb{R}(\Gamma_{\text{odd}}) & \xrightarrow{R_{\bar\beta}} & \mathbb{R}({}_H\backslash G)
\end{array}$$

Problem. – The entropy $h(G, \mu)$ measures how far N^{st} resp. M^{st} are from being factors (*see* Remark 4.13 c)). Is there a formula relating the relative Connes-Størmer entropy $H(M^{st}|N^{st})$ to $h(G, \mu)$ as in ([Bi1], Theorem 1.9)?

5. Entropy of random walks on groups

We would like to use Theorems 4.9 and 4.13 to construct examples of strongly amenable and of amenable, but *not* strongly amenable subfactors of the hyperfinite II_1 factor. According to these we need to find groups G which are of the form $H * K/_\sim$ for two finite groups H and K such that G is amenable, but $h(G, \mu) > 0$ (then $R^H \subset R \rtimes K$ is amenable, but not strongly amenable) and/or such that G is amenable and $h(G, \mu) = 0$ (then $R^H \subset R \rtimes K$ is strongly amenable), where μ is as in Theorem 4.13. We are therefore lead to analyze ergodicity properties of the right random walk, defined by a probability measure on G, on various classes of groups G of the above form.

Let us recall some elementary facts about groups and fix some notation. Given a group G, we let $\{G, G\}$ denote its *commutator subgroup*, *i.e.* the subgroup generated by $\{xyx^{-1}y^{-1} \,|\, x, y \in G\}$. Note that the sequence $N_1(G)$, $N_2(G), \ldots$, inductively defined by

$$N_1(G) = \{G, G\}, \quad N_{n+1}(G) = \{N_n(G), N_n(G)\}$$

is decreasing sequence of normal subgroups of G. $N_n(G)$ is called the *n-th commutator subgroup* of G and G is called *solvable* if $N_n(G) = \{e\}$ for some $n \in \mathbb{N}$. Note that solvable groups are amenable (*see* for instance [G], Theorem 2.3.3).

Let G be a finitely generated group and S a symmetric set of generators of G. Using S, one can define a *length function* on G via

$$|g|_S = \min\{n \mid g = s_1 s_2 \cdots s_n,\ s_i \in S\}, \tag{20}$$

if $g \neq e$ and $|e|_S = 0$ (e denotes the identity in G). If S' is another finite symmetric set of generators of G, then it is easy to see that there are positive constants a and b such that

$$a|g|_S \leq |g|_{S'} \leq b|g|_S, \qquad g \in G. \tag{21}$$

We say that the probability measure μ on G has *finite p-th moment*, $p \in \mathbb{R}^+$, if

$$\sum_{g \in G} |g|_s^p \mu(g) < \infty \tag{22}$$

for some finite symmetric set of generators S (equivalently for all finite symmetric set of generators by (21)). Recall also that a probability measure μ on G is called *nondegenerate* if $G = \bigcup_{n=1}^{\infty} (\mathrm{supp}\mu)^n$.

For our entropy calculations below the following groups play an important role ([KV], [Ka1], [Ka2], [Ka4]): let G_k denote the semi-direct product

$$G_k = F_0(\mathbb{Z}^k, \mathbb{Z}_2) \rtimes \mathbb{Z}^k \tag{23}$$

where $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ is the abelian group of finitely supported functions from $\mathbb{Z}^k \to \mathbb{Z}_2$ and where the action of $\mathbb{Z}^k$ on $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ is given by translation T_x, *i.e.*

$$(T_x f)(y) = f(y - x), \quad f \in F_0(\mathbb{Z}^k, \mathbb{Z}_2), \quad x, y \in \mathbb{Z}^k.$$

Observe that these groups are amenable, since they are solvable of lenght 2. We will need the following result from ([Ka1], Theorem 3.3, *see* also [KV], Proposition 6.4).

LEMMA 5.1. – *Let G_k be the groups defined in* (23). *If $k \geq 3$ and if μ is a probability measure on G_k with finite first moment for which the induced measure $\bar{\mu}$ on $\mathbb{Z}^k = G_k/F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ is nondegenerate and for which the subgroup generated by the support of μ is non-abelian, then the entropy $h(G_k, \mu)$ is strictly positive.*

Proof. – The induced random walk $(\mathbb{Z}^k, \bar{\mu})$ is transient since $k \geq 3$ (*see* for instance [Sp], Theorems 8.1, 8.2). Then ([Ka1], Theorem 3.3) yields the result. □

Using this result we can prove

LEMMA 5.2. – *Let F_k, $3 \leq k \leq \infty$, be the free group on k generators and let $H_k = F_k/N_2(F_k)$ be the quotient of F_k by its second commutator subgroup. If μ is a nondegenerate probability measure on H_k with finite first moment, then $h(H_k, \mu) > 0$.*

Proof. – Let $z_1, \dots z_k$ be the standard generators of $\mathbb{Z}^k$ and define $f_0 \in F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ by

$$f_0(y) = \left\{ \begin{array}{lll} 1, & \text{if} & y = 0 \\ 0, & \text{if} & y \notin \mathbb{Z}^k \backslash \{0\}. \end{array} \right\}$$

It is easily seen that $z_1, \cdots, z_k$, f_0 form a set of $k+1$ generators of G_k. Let $x_1, \dots, x_k$ be the generators of F_k and let $y_1, \dots, y_k$ be their images in $H_k = F_k/N_2(F_k)$ by the quotient map. Note that there is a unique homomorphismus ϕ from F_k onto a subgroup G'_k of G_k such that

$$\phi(x_1) = f_0 z_1 \quad \text{and} \quad \phi(x_i) = z_i, \qquad 2 \le i \le k.$$

Observe that for $i = 2, \dots, k$ we have

$$\begin{aligned}(f_0 z_1) z_i (f_0 z_1)^{-1} z_i^{-1} &= f_0 z_i f_0^{-1} z_i^{-1} \\ &= f_0 - T_{z_i}(f_0) \neq 0,\end{aligned}$$

where we use additive notation for the group multiplication in $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$. In particular, G'_k is not abelian. Since the second commutator subgroup of G_k is trivial, the same holds for G'_k. Therefore ϕ can be factored through $H_k = F_k/N_2(F_k)$, *i.e.* there is a homomorphism ϕ_0 from H_k onto G'_k such that

$$\phi_0(y_1) = f_0 z_1 \quad \text{and} \quad \phi_0(y_i) = z_i, \qquad 2 \le i \le k. \tag{24}$$

Let $|\cdot|_{H_k}$ (resp. $|\cdot|_{G_k}$ be the length function on H_k (resp. G_k) given by the set of generators $\{y_1, \dots, y_k\}$ (resp. $\{f_0, z_1, \dots, z_k\}$) and their inverses. Then by (24)

$$|\phi_0(h)|_{G_k} \le 2|h|_{H_k}, \qquad h \in H_k.$$

Therefore, since μ has finite first moment on H_k, the image μ' of μ on G_k given by ϕ_0 has also finite first moment. Since μ is nondegenerate, we have $\bigcup_{n=1}^{\infty} (\mathrm{supp}\mu')^n = G'_k$, which is a non-abelian subgroup of G_k as we have seen above. Moreover, the map $G_k \to G_k/F_0(\mathbb{Z}^k, \mathbb{Z}_2) \cong \mathbb{Z}^k$ is nondegenerate, because the image of $f_0 z_1, z_2, \dots, z_k$ under the quotient map gives obviously the standard generators of $\mathbb{Z}^k$. Hence, by Lemma 5.1, we have that $h(G_k, \mu') > 0$. Since $(\mu')^n$ is the image of μ^n, the n-th convolution power of μ, by the map ϕ_0, we have

$$(\mu')^n(\phi_0(h)) = \sum_{k \in hKer(\phi_0)} \mu^n(k), \quad \text{for all } h \in H_k.$$

Hence we can calculate the entropy

$$\begin{aligned} H((\mu')^n) &= -\sum_{g \in G'_k} (\mu')^n(g) \ln((\mu')^n(g)) \\ &= -\sum_{h \in H_k} \mu^n(h) \ln((\mu')^n(\phi_0(h))) \\ &\le -\sum_{h \in H_k} \mu^n(h) \ln \mu^n(h) \\ &= H(\mu^n) \end{aligned}$$

since $(\mu')^n(\phi_0(h)) = \sum_{k \in hKer(\phi_0)} \mu^n(k) \ge \mu^n(h)$. Thus

$$h(H_k, \mu) \ge h(G_k, \mu') > 0. \quad \square$$

In the next proposition we determine a class of groups, which are quotients of the free product of two finite groups, and have positive entropy with respect to a large class of probability measures on these groups.

THEOREM 5.3. – *Let $G = H * K$ be the free product of two nontrivial finite groups H and K and let $N_3(G)$ be the third commutator subgroup of G. If $|H||K| > 6$, then every nondegenerate probability measure μ on $G' = G/N_3(G)$ with finite first moment satisfies $h(G', \mu) > 0$.*

Proof. – Let $N_1(G)$ be the first commutator subgroup of G. Then $N_1(G)$ is freely generated by $\{hkh^{-1}k^{-1} \,|\, h \in H\backslash\{e\},\ k \in K\backslash\{e\}\}$. Hence $N_1(G) \cong F_m$ for $m = (|H| - 1)(|K| - 1)$. Since $|H||K| > 6$, we have either ($|H| \geq 2$ and $|K| \geq 4$) or ($|H| \geq 3$ and $|K| \geq 3$) or ($|H| \geq 4$ and $|K| \geq 2$). In all three cases we get $m \geq 3$. Set $G' = G/N_3(G)$. Since $N_1(G)$ has finite index in G (namely index $|H||K|$), the image G'_0 of $N_1(G)$ under the quotient map $\pi : G \to G'$ is a subgroup of G' of finite index and satisfies $G'_0 \cong N_1(G)/N_3(G) \cong F_m/N_2(F_m)$. Let μ be a nondegenerate probability measure on G' with finite first moment. By ([Ka1], Lemmas 2.2, 2.3, *see* also Lemma 5.5 below) we can find a nondegenerate probability measure μ_0 on G'_0 with finite first moment, such that the Poisson boundaries $\Gamma(G', \mu)$ and $\Gamma(G'_0, \mu_0)$ are isomorphic. Since $H(\mu)$, $H(\mu_0) < \infty$ (both measures have finite first moment), we have in particular ([KV], Theorem 1.1)

$$h(G', \mu) > 0 \text{ iff } h(G'_0, \mu_0) > 0.$$

But by Lemma 5.2 we know that $h(G'_0, \mu_0) > 0$. Thus the proof is complete. □

In the following lemmas and propositions we treat the case when $|H||K| = 6$. By symmetry we may assume $|H| = 2$ and $|K| = 3$.

LEMMA 5.4. – *Let F_2 be the free group on two generators and let $H_2 = F_2/N_2(F_2)$ be the quotient of F_2 by its second commutator subgroup. If μ is a symmetric probability measure on H_2 with finite second moment, then $h(H_2, \mu) = 0$.*

Proof. – Let $N_1(F_2)$ denote the first commutator subgroup of F_2. Since $N_1(F_2) \supset N_2(F_2)$, we have a surjective homomorphism $\rho : H_2 \to F_2/N_1(F_2) = \mathbb{Z}^2$. Moreover, $\mathrm{Ker}(\rho) = N_1(F_2)/N_2(F_2)$ is abelian. Let x_1, x_2 be the generators of F_2 and let y_1, y_2 be the generators of H_2 obtained as the images of x_1, x_2 by the quotient map $F_2 \to H_2$. Furthermore, let z_1, z_2 denote the generators of $\mathbb{Z}^2$. Then clearly $\rho(y_i) = z_i$, $i = 1, 2$. Thus, if $|\cdot|_{H_2}$ (resp. $|\cdot|_{\mathbb{Z}^2}$) denote the length function on H_2 (resp. $\mathbb{Z}^2$) associated to the generators $\{y_1, y_2\}$ (resp. $\{z_1, z_2\}$) and their inverses, then

$$|\rho(h)|_{\mathbb{Z}^2} \leq |h|_{H_2}.$$

Hence, if μ is a symmetric probability measure on H_2 with finite second moment, then the image μ' of μ under ρ is a symmetric probability measure on $\mathbb{Z}^2$ with finite second moment and zero mean displacement since μ' is symmetric. Thus μ' is recurrent (*see* for instance [Sp], Theorem 8.1 or [Wo], Theorem 4.4) and hence the subgroup $\mathrm{Ker}(\rho)$ is a recurrent subgroup for μ. By ([Ka], Lemma 2.2) there is then a probability measure μ_0 on $\mathrm{Ker}(\rho)$ which has the same Poisson boundary as μ, *i.e.* $\Gamma(H_2, \mu) \cong \Gamma(\mathrm{Ker}\rho, \mu_0)$. But since $\mathrm{Ker}\rho$ is an abelian group, its Poisson boundary is trivial by the classical Choquet-Deny theorem

[CD] and hence $\Gamma(H_2, \mu)$ is trivial, which implies that $h(H_2, \mu) = 0$ ([KV], Theorem 1.1, note that $H(\mu) < \infty$ since μ has finite second moment). $\square$
We will need the following extension of ([Ka1], Lemma 2.3):

LEMMA 5.5. – *Let G be a finitely generated group and let G_0 be a normal subgroup of G with finite index. Then G_0 is finitely generated. Moreover, if μ is a probability measure on G with finite p-th moment ($p \in \mathbb{R}^+$) and μ_0 is the measure on G_0 defined as the distribution of the point where the μ-random walk starting at e returns to G_0 for the first time, then μ_0 has also finite p-th moment.*

Proof. – By ([Ka1], Lemma 2.1), G_0 is finitely generated. For $n \in \mathbb{N}$, let θ_n be the probability that the μ-random walk on G starting at e returns to G_0 after exactly n steps. Denote $\bar{\mu}$ be the image of μ on the quotient group $A = G/G_0$. Then θ_n is the probability that the $\bar{\mu}$-random walk starting at e_A returns to e_A after precisely n steps. Since $|G/G_0| < \infty$, $\bar{\mu}$ is recurrent, *i.e.*

$$\sum_{n=1}^{\infty} \theta_n = 1. \tag{25}$$

Moreover, from the theory of Markov chains on finite sets it follows (*see* Appendix), that the probability distribution on $\mathbb{N}$ given by $(\theta_n)_{n=1}^{\infty}$ has finite q-th moment for any $q \in \mathbb{R}^+$, *i.e.*

$$\sum_{n=1}^{\infty} n^q \theta_n < \infty \quad \text{for all } q \in \mathbb{R}^+. \tag{26}$$

Let $|\cdot|$ and $|\cdot|_0$ be the lenght functions on G resp. G_0 based on given symmetric finite sets of generators of G resp. G_0. Then by the proof of ([Ka1], Lemma 2.1) there are positive constants c and d such that

$$c|h| \leq |h|_0 \leq d|h|, \qquad h \in G_0. \tag{27}$$

For $h \in G_0$ denote by $\mu_0(h, n)$ the probability that the μ-random walk starting at e returns to G_0 for the first time at $h \in G_0$ and that this happens after precisely n steps. Then

$$\sum_{n=1}^{\infty} \mu_0(h, n) = \mu_0(h) \quad \text{and} \quad \sum_{h \in G_0} \mu_0(h, n) = \theta_n.$$

Using (27) we have

$$\sum_{h \in G_0} |h|_0^p \mu_0(h) \leq d^p \sum_{n=1}^{\infty} \left(\sum_{h \in G_0} |h|^p \mu_0(h, n) \right) \tag{28}$$

Let $\pi : G \to G/G_0 = A$ be the quotient map. For each $a \in A$, let H_a be the corresponding G_0-coset in G_0, *i.e.* $H_a = \pi^{-1}(a)$. Since μ has finite p-th moment and since A is finite, there is a constant $c_p > 0$, such that

$$\sum_{g \in H_a} |g|^p \mu(g) \leq c_p \sum_{g \in H_a} \mu(g) \tag{29}$$

for all $a \in A$. Next set

$$P_n = \{\tilde{g} = (g_1, \dots, g_n) \in G^n \mid g_1 g_2 \cdot \ldots \cdot g_i \notin G_0 \text{ for } i = 1, \dots, n-1 \text{ and } g_1 g_2 \cdot \ldots \cdot g_n \in G_0\}$$

and

$$Q_n = \{\tilde{a} = (a_1, \dots, a_n) \in A^n \mid a_1 a_2 \cdot \ldots \cdot a_i \notin A \text{ for } i = 1, \dots, n-1 \text{ and } a_1 a_2 \cdot \ldots \cdot a_n = e_A\}.$$

Then we have clearly

$$P_n = \{\tilde{g} = (g_1, \dots, g_n) \in G^n \mid (\pi(g_1), \dots, \pi(g_n)) \in Q_n\}. \tag{30}$$

Observe that

$$\theta_n = \sum_{h \in G_0} \mu_0(h, n) = \sum_{g \in P_n} \mu(g_1) \cdot \ldots \cdot \mu(g_n) \tag{31}$$

and

$$\begin{aligned} \sum_{h \in G_0} |h|^p \mu_0(h, n) &= \sum_{\tilde{g} \in P_n} |g_1 \cdot \ldots \cdot g_n|^p \mu(g_1) \cdot \ldots \cdot \mu(g_n) \\ &\le \sum_{g \in P_n} (|g_1| + \ldots + |g_n|)^p \mu(g_1) \cdot \ldots \cdot \mu(g_n) \end{aligned} \tag{32}$$

Assume first that $p \geq 1$. Then

$$(|g_1| + \ldots + |g_n|)^p \le n^{p-1}(|g_1|^p + \ldots + |g_n|^p). \tag{33}$$

Moreover, by (29) and (30) we have for any fixed i that

$$\begin{aligned} \sum_{\tilde{g} \in P_n} |g_i|^p \mu(g_1) \ldots \mu(g_n) &= \sum_{\tilde{a} \in Q_n} \left(\sum_{g_i \in H_{a_i}} |g_i|^p \mu(g_i) \right) \prod_{j \neq i} \left(\sum_{g_j \in H_{a_j}} \mu(g_j) \right) \\ &\le c_p \sum_{\tilde{a} \in Q_n} \left(\sum_{\tilde{g} = (g_1, \dots, g_n), g_j \in H_{a_j}} \mu(g_1) \ldots \mu(g_n) \right) \\ &= c_p \sum_{\tilde{g} \in P_n} \mu(g_1) \ldots \mu(g_n) \end{aligned}$$

Hence, we get from (31), (32) and (33)

$$\sum_{h \in G_0} |h|^p \mu_0(h, n) \le c_p n^p \theta_n,$$

so that by (28) and (26)

$$\sum_{h \in G_0} |h|_0^p \mu_0(h) \le c_p d^p \sum_{n=1}^{\infty} n^p \theta_n < \infty.$$

Hence μ_0 has finite p-th moment on G_0.

If $0 < p < 1$, we must replace (33) by

$$(|g_1| + \ldots + |g_n|)^p \leq |g_1|^p + \ldots |g_n|^p,$$

which gives as above

$$\sum_{h \in G_0} |h|_0^p \mu_0(h) \leq c_p d^p \sum_{n=1}^{\infty} n\theta_n < \infty.$$

This completes the proof of the lemma. □

Note that if μ is symmetric, then μ_0 is automatically symmetric too.

We can now determine the ergodicity behaviour of some classes of random walks on certain amenable quotients of $PSL(2, \mathbb{Z})$.

THEOREM 5.6. – *Let $G = PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$. Then for every symmetric measure μ on $G' = G/N_3(G)$ with finite second moment one has that $h(G', \mu) = 0$.*

Proof. – The first commutator subgroup $N_1(G)$ of G is isomorphic to F_2. Let G'_0 be the image of $N_1(G)$ in G' under the quotient map $G \to G'$. Then G'_0 has finite index in G' (namely 6) and

$$G'_0 \cong N_1(G)/N_3(G) \cong F_2/N_2(F_2).$$

Let μ_0 be the symmetric measure on G'_0 associated to μ as in ([Ka1, Lemma 2.2]). By Lemma 5.5, we know that μ_0 has finite second moment, and hence we get by Lemma 5.4 that $h(G'_0, \mu_0) = 0$ and thus also $h(G', \mu) = 0$ again by ([Ka1, Lemma 2.2]). □

THEOREM 5.7. – *Let $G = PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$. Then for every nondegenerate probability measure μ on $G'' = G/N_4(G)$ with finite first moment, one has that $h(G'', \mu) > 0$.*

Proof. – Let G''_0 be the image of $N_1(G)$ under the quotient map $G \to G''$. Then G''_0 has finite index in G'' and $G''_0 \cong N_1(G)/N_4(G) \cong F_2/N_3(F_2)$. Let $H = \mathbb{Z}_3 * \mathbb{Z}_3/N_3(\mathbb{Z}_3 * \mathbb{Z}_3)$. Since H is generated by two elements, there is a surjective homomorphism from F_2 to H and since H has trivial third commutator subgroup, this homomorphism factors through $N_3(F_2)$. Hence, there is a surjective homomorphism $\rho : G''_0 \to H$. Let μ now be a nondegenerate probability measure on G'' with finite first moment and let μ_0 be the corresponding probability measure on G''_0 as in ([Ka1, Lemma 2.2], *see* also Lemma 5.5). Then μ_0 is still nondegenerate and has also finite first moment ([Ka1] or Lemma 5.5). If $|\cdot|_{G''_0}$ denotes the length function on G''_0 associated to a given finite symmetric set of generators S and $|\cdot|_H$ is the length function on H associated to the set of generators $\rho(S)$, then we have clearly

$$|\rho(g)|_H \leq |g|_{G''_0} \quad \text{for } g \in G''_0.$$

Therefore, the image $\bar{\mu}_0$ of μ_0 by ρ is a nondegenerate probability measure on H with finite first moment and hence Theorem 5.3 implies that $h(H, \bar{\mu}_0) > 0$. However, as in the proof of Lemma 5.2, one has that $h(G''_0, \mu_0) \geq h(H, \bar{\mu}_0)$. Hence $h(G''_0, \mu_0) > 0$, which implies that $h(G'', \mu) > 0$ (again by [Ka1, Lemma 2.2]). □

The following proposition shows that the groups studied in Theorems 5.3 and 5.6 have exponential growth.

PROPOSITION 5.8. – *Let H and K be two nontrivial finite groups such that $|H||K| \geq 6$ (i.e. H and K are not both equal to $\mathbb{Z}_2$) and set $G = H * K$. Then the quotient group $G' = G/N_3(G)$ of G by its third commutator subgroup has exponential growth.*

Proof. – Note first that $N_1(G)$ is isomorphic to the free group F_l on $l = (|H|-1)(|K|-1)$ generators and that $l \geq 2$. Hence the subgroup $G'_0 = N_1(G)/N_3(G)$ of G' is isomorphic to $F_l/N_2(F_l)$. By [KV, 6.1], the group $G_k = F_0(\mathbb{Z}^k, \mathbb{Z}_2) \rtimes \mathbb{Z}^k$ has exponential growth for all $k \geq 1$. Set $k = l - 1 \geq 1$. Then with the notation of the proof of Lemma 5.2, we have that $z_1, \ldots, z_k$, f_0 is a set of $l = k + 1$ generators for G_k. Hence there is a surjective homomorphism $\psi : F_l \to G_k$, which factors through $F_l/N_2(F_l)$ since the second commutator subgroup of G_k is trivial. Note that $F_l/N_2(F_l)$ is isomorphic to G'_0. Hence G_k is a quotient of G'_0, and thus the fact that G_k has exponential growth implies that G'_0 has exponential growth. Since G'_0 is a subgroup of G', we are done. $\square$

Remark 5.9. – Let $G = H * K$ be as in the Proposition 5.8. Then the group $G/N_2(G)$ has polynomial growth because $N_1(G)/N_2(G)$ is a normal subgroup of $G/N_2(G)$ with finite index. But this subgroup has polynomial growth since $N_1(G)/N_2(G) \cong F_l/N_1(F_l) \cong \mathbb{Z}^l$ for $l = (|H| - 1)(|K| - 1)$ as above.

Let us also point out that the groups $G/N_k(G)$ which we have studied in the above propositions are all amenable since they are solvable.

6. Examples

We construct explicitly various new examples of finite and infinite depth subfactors of the hyperfinite II_1 factor R with finite Jones index using the group type inclusions studied in chapter 4. In particular, we obtain the first examples of irreducible amenable subfactors of R, which are *not* strongly amenable, thus solving Problem 5.4.2 in [Po4]. The "smallest" of our examples appear already in index 6. Furthermore, we construct (strongly) amenable, irreducible subfactors of R whose principal graphs have subexponential (or polynomial) and/or exponential growth and compute some of these principal graphs.

Since every group $G = \mathbb{Z}_2 * \mathbb{Z}_3/_\sim$ (*i.e.* a quotient of $PSL(2, \mathbb{Z})$) produces an irreducible group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$, we get a large variety of distinct irreducible index 6 inclusions, whose principal graphs can be (in principle) determined explicitly according to our results in section 4. However, since these principal graphs will be rather big in size, it will be difficult to list them all graphically. In particular, a complete list of principal graphs as in [Ha] seems out of reach for indices ≥ 6.

Note that the "smallest" group type inclusions are obtained as $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_2$, which produces strongly amenable, irreducible index 4 inclusions with principal graphs $D_n^{(1)}$ (in the case $G = \langle a, b \,|\, a^2 = b^2 = (ab)^{n-2} = 1\rangle$, the dihedral groups) and D_∞ (case $G = \langle a, b \,|\, a^2 = b^2 = 1\rangle$, the infinite dihedral group) (*see* [GHJ], [Po4]).

The next "bigger" group type inclusions are of the form $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$, coming from an outer action of $G = \langle \mathbb{Z}_2, \mathbb{Z}_3 \rangle$, a quotient of $PSL(2, \mathbb{Z})$, on the hyperfinite II_1 factor R. Various such quotients are discussed in detail below.

We recall for the convenience of the reader the various notions of growth for the principal graphs (Γ, Γ') of a group type inclusion $N = R^H \subset M = R \rtimes K$. To this end let k_n be the number of simple summands of the higher relative commutant $N' \cap M_{2n-1}$ and recall that the standard eigenvectors associated to (Γ, Γ') are bounded (*see* considerations before Lemma 4.11). We say that Γ has *polynomial growth* if $k_n \sim O(n^r)$, some $r \geq 1$, or equivalently if $k_n \leq cn^r$, for some $c \in \mathbb{R}^+$ and some $r \geq 1$. If $r = 1$, then Γ has *linear growth*. Γ has *subexponential growth* (*see* [KV, Definition 1.2] or [Po4, 5.3.8]) if $\lim_{n\to\infty} k_n^{\frac{1}{n}} = 1$ and *exponential growth* if this limit is > 1.

Let us now give the announced examples of (strongly) amenable irreducible subfactors of R.

Examples 6.1. – *(irreducible, amenable subfactors of R, which are not strongly amenable and whose principal graphs have exponential growth).*

Consider the amenable group $G = \mathbb{Z}_2 * \mathbb{Z}_3 / N_4(\mathbb{Z}_2 * \mathbb{Z}_3)$ and let it act properly outerly on the hyperfinite II_1 factor R. The index 6 group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ is irreducible (Corollary 4.1), amenable (Theorem 4.9), but not strongly amenable by Theorem 5.7 and Theorem 4.13. In particular the principal graph Γ has exponential growth by [Po4]. The fact that Γ has exponential growth can also be obtained in the following way: the group $\mathbb{Z}_2 * \mathbb{Z}_3 / N_3(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a quotient of G, and since this quotient has exponential growth by Proposition 5.8, G itself must have exponential growth. Hence by Proposition 4.2 and the considerations after Proposition 4.2, the graph Γ has exponential growth.

More examples are obtained in the following way: let $G = H * K / N_3(H * K)$ with nontrivial finite groups H and K, $|H||K| > 6$. Since G is amenable (solvable), $R^H \subset R \rtimes K$ is an amenable inclusion (Theorem 4.9), but not strongly amenable by Theorem 5.3 and Theorem 4.13. For instance $G = \mathbb{Z}_2 * \mathbb{Z}_4 / N_3(\mathbb{Z}_2 * \mathbb{Z}_4)$ resp. $G = \mathbb{Z}_3 * \mathbb{Z}_3 / N_3(\mathbb{Z}_3 * \mathbb{Z}_3)$ give amenable, but not strongly amenable inclusions with Jones index 8 resp. 9. In this way we get countably many irreducible, amenable subfactors of R, which are not strongly amenable and whose principal graphs have exponential growth by Propositions 5.8 and 4.2.

Example 6.2. – *(irreducible strongly amenable subfactor of R, whose principal graph has exponential growth).*

Consider the amenable group $G = \mathbb{Z}_2 * \mathbb{Z}_3 / N_3(\mathbb{Z}_2 * \mathbb{Z}_3)$ and the associated irreducible, index 6 group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$. Note that this inclusion is strongly amenable by Theorem 4.13, using Theorem 5.6 and Lemma 4.12. The principal graphs have exponential growth by Propositions 5.8 and 4.2 (*see* also considerations after Proposition 4.2).

Example 6.3. – *(irreducible strongly amenable infinite depth subfactor of R whose principal graphs have polynomial growth).*

Let H and K be finite groups and set $G = H * K / N_2(H * K)$. By remark 5.9, G is infinite and has polynomial growth. Hence the associated index $|H||K|$-inclusion $R^H \subset R \rtimes K$ has infinite depth, and its principal graphs have polynomial growth, so in particular the inclusion is strongly amenable. In example 6.5 we will compute the principal graphs Γ and Γ' in the special case $G = \mathbb{Z}_2 * \mathbb{Z}_3 / N_2(\mathbb{Z}_2 * \mathbb{Z}_3)$.

Next we will work out the principal graphs of group type inclusions associated to some special quotients of $PSL(2, \mathbb{Z})$. Consider the *polyhedral groups*

$$G_{l,m,n} = \langle a, b \,|\, a^l = b^m = (ab)^n = 1 \rangle$$

(*see* [Bu, p. 408], [CM, Chapter 6.4]). It is shown in ([CM, Chapters 4.3, 4.4 and 5.3]) that these groups are finite iff the number $k = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right)$ is a positive number, which happens for instance in the cases $(l, m, n) \in \{(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5), n \in \mathbb{N}\}$. Observe that $G_{2,2,n}$ are the dihedral groups. The groups $G_{l,m,n}$ with $2 \leq l \leq m \leq n$, $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ are Fuchsian groups and usually called the (l, m, n)-triangle groups. They have nice geometrical presentations in terms of rotations of hyperbolic triangles with angles $\frac{\pi}{l}$ and $\frac{\pi}{m}$ [K] and have been studied extensively in the literature (*see* for instance [M], [R] and references there).

Examples 6.4. – *(irreducible finite depth subfactors of R based on the groups $G_{2,3,n}$, $2 \leq n \leq 5$).*

We determine the principal graphs Γ and Γ' of the group type inclusions $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ corresponding to the finite groups $G_{2,3,2}$, $G_{2,3,3}$, $G_{2,3,4}$ and $G_{2,3,5}$. Observe that $G_{2,3,1} = \{1\}$.

a) Let $G = G_{2,3,2} = \langle a, b \,|\, a^2 = b^3 = (ab)^2 = 1 \rangle$ and let $H = \{e, a\}$ and $K = \{e, b, b^2\}$. Then $G \cong S_3$ via $a \to (12)$ and $b \to (123)$. Moreover $H \cap K = \{e\}$ and $G = HK$, so the associated inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ is irreducible and has depth 2, because Γ_{odd} consists of only one vertex by Proposition 4.2 ii). Hence the inclusion is isomorphic to $R \subset R \rtimes L$ for some Kac algebra L of dimension 6 (*see* for instance [Sz]). Since Kac algebras of dimension less than 8 come from either groups or group duals, the inclusion must be isomorphic to one of the inclusions $R \subset R \rtimes \mathbb{Z}_6$, $R \subset R \rtimes S_3$ or $R^{S_3} \subset R$.

We compute next Γ_{even} and Γ'_{even}. The $H \times H$ cosets of G are $\{e, a\}$ and $\{b, b^2, ab, ab^2\}$. By Proposition 4.2 we have

$$\alpha\bar{\alpha} = \alpha a \bar{\alpha}, \quad \alpha b \bar{\alpha} = \alpha b^2 \bar{\alpha} = \alpha a b \bar{\alpha} = \alpha a b^2 \bar{\alpha}.$$

Moreover

$$\langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle = m^H_{e,e} = 2, \quad \langle \alpha b \bar{\alpha}, \alpha b \bar{\alpha} \rangle = m^H_{b,b} = 1.$$

Hence $\alpha b \bar{\alpha}$ is irreducible and $\alpha\bar{\alpha}$ is the sum of two inequivalent bimodules (if $\alpha\bar{\alpha}$ contained two equivalent bimodules, then $\langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$ would be at least 4). In fact, the components of $\alpha\bar{\alpha}$ can be identified with the elements of the dual group $\widehat{\mathbb{Z}_2} = \{1, a'\}$. Hence by the considerations after Proposition 4.2, we get that

$$\Gamma_{\text{even}} = \{1, a', \alpha b \bar{\alpha}\}$$

and since $\alpha\beta\overline{(\alpha\beta)} = \alpha(1+b+b^2)\bar{\alpha} = 1 + a' + 2\alpha b\bar{\alpha}$, the graph Γ is given by

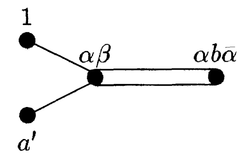

Fig. 1

The $K \times K$ cosets of G are $\{1, b, b^2\}$ and $\{a, ab, ab^2\}$. Hence the vertices of Γ'_{even} consist of the irreducible bimodules contained in $\bar{\beta}\beta$ and $\bar{\beta}a\beta$. Since the numbers

$$\langle\bar{\beta}\beta, \bar{\beta}\beta\rangle = m^K_{e,e} = 3, \qquad \langle\bar{\beta}a\beta, \bar{\beta}a\beta\rangle = m^K_{a,a} = 3$$

are strictly less than 4, neither $\bar{\beta}\beta$ nor $\bar{\beta}a\beta$ can contain two equivalent irreducible bimodules. Thus $\bar{\beta}\beta$ and $\bar{\beta}a\beta$ each split into three inequivalent irreducible bimodules, which in the case of $\bar{\beta}\beta$ can be identified with the elements fo the dual group $\widehat{\mathbb{Z}_3} = \{1, b', (b')^2\}$, *i.e.*

$$\bar{\beta}\beta = 1 + b' + (b')^2, \qquad \bar{\beta}a\beta = (\bar{\beta}a\beta)_1 + (\bar{\beta}a\beta)_2 + (\bar{\beta}a\beta)_3.$$

Therefore

$$\Gamma'_{\text{even}} = \{1, b', (b')^2, (\bar{\beta}a\beta)_1, (\bar{\beta}a\beta)_2, (\bar{\beta}a\beta)_3\}.$$

Since

$$\overline{(\alpha\beta)}\alpha\beta = \bar{\beta}(1+a)\beta = 1 + b' + (b')^2 + (\bar{\beta}a\beta)_1 + (\bar{\beta}a\beta)_2 + (\bar{\beta}a\beta)_3,$$

we get that Γ' is given by

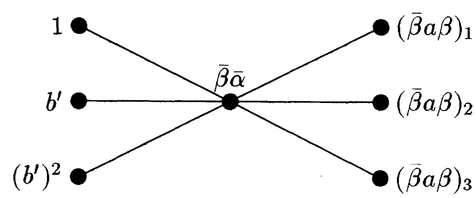

Fig. 2

It follows from the form of Γ and Γ' that $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ is isomorphic to the crossed product of R by the dual of a non-commutative group of order 6. Hence the inclusion must be isomorphic to $R^{S_3} \subset R$.

b) Let $G = G_{2,3,3} = \langle a, b \,|\, a^2 = b^3 = (ab)^3 = 1\rangle$. Then $G \cong A_4$, the alternating group, via the isomorphism $a \longrightarrow (12)(34)$ (a product of two transpositions) and $b \longrightarrow (234)$ (a cyclic permutation of order 3). One checks that $G = \{1, a, b, b^2, ab, ba, b^2a, ab^2, bab, b^2ab, bab^2, b^2ab^2\}$ and $b^2ab^2 = aba$. We compute the principal graphs (Γ, Γ') of the associated group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$, *i.e.* the group generated by $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in Out$R$ is just G. The first step consists of computing the vertices of Γ (resp. Γ'). According to Proposition 4.2 we need to calculate the double cosets ${}_H\backslash G/_H$, ${}_H\backslash G/_K$ and ${}_K\backslash G/_K$ with $H = \mathbb{Z}_2$ and $K = \mathbb{Z}_3$. We compute first ${}_H\backslash G/_H$. We have that $H \cdot 1 \cdot H = \{1, a\}$, $HbH = \{b, ab, ba, aba\}$, $Hb^2H = \{b^2, ab^2, b^2a, ab^2a = bab\}$ and $Hbab^2H = \{bab^2, abab^2 = b^2ab, bab^2a = b^2ab, abab^2a = bab^2\}$, which determines ${}_H\backslash G/_H$. Applying Proposition 4.2 and using the notation of chapter 4 we obtain that $\alpha\bar{\alpha}$, $\alpha b\bar{\alpha}$, $\alpha b^2\bar{\alpha}$ and $\alpha bab^2\bar{\alpha}$ are all the distinct $R^{\mathbb{Z}_2}$-$R^{\mathbb{Z}_2}$ bimodules appearing in $(\rho\bar{\rho})^n$, $n \in \mathbb{N}$, which are of the form $\alpha g\bar{\alpha}$, $g \in G$. To calculate Γ_{even}, we need to decompose these bimodules into irreducibles, *i.e.* we need to calculate the dimension of certain intertwiner spaces, which is done using Frobenius reciprocity (Proposition 2.2):

$$\begin{aligned}
\langle \alpha\bar{\alpha}, \alpha\bar{\alpha}\rangle &= \langle \bar{\alpha}\alpha, \bar{\alpha}\alpha\rangle = \langle 1 + a, 1 + a\rangle = 2,\\
\langle \alpha b\bar{\alpha}, \alpha b\bar{\alpha}\rangle &= \langle b, \bar{\alpha}\alpha b\bar{\alpha}\alpha\rangle = \langle b, (1 + a)b(1 + b + b^2)\rangle\\
&= \langle b, b\rangle = \langle b, b + ab + ba + aba\rangle = 1,\\
\langle \alpha b^2\bar{\alpha}, \alpha b^2\bar{\alpha}\rangle &= \langle b^2, b^2 + ab^2 + b^2a + bab\rangle = \langle b^2, b^2\rangle = 1,\\
\langle \alpha bab^2\bar{\alpha}, \alpha bab^2\bar{\alpha}\rangle &= \langle bab^2, bab^2 + b^2ab + b^2ab + bab^2\rangle = 2.
\end{aligned}$$

Note that we have in general that $\alpha g\bar{\alpha}$ is irreducible iff the double coset HgH has maximal cardinality (namely $|H|^2$). If we denote the dual group of $\mathbb{Z}_2$ by $\widehat{\mathbb{Z}_2} = \{1, a'\}$, then $\alpha\bar{\alpha} = 1 + a'$. $\alpha bab^2\bar{\alpha} = b_1 + b_2$ decomposes into two inequivalent irreducible bimodules b_1, b_2 (argue as in *a*)) and hence

$$\Gamma_{\text{even}} = \{1, a', \alpha b\bar{\alpha}, \alpha b^2\bar{\alpha}, b_1, b_2\}.$$

Next we compute ${}_H\backslash G/_K$. We have that all H-K double cosets of G are given by $H \cdot 1 \cdot K = \{1, a, b, ab, b^2, ab^2\}$, $HbaK = \{ba, aba, bab, (ab)^2 = b^2a, bab^2, abab^2 = b^2ab\}$. Hence $\alpha\beta$ and $\alpha ba\beta$ are all the distinct $R^{\mathbb{Z}_2}$-$R \rtimes \mathbb{Z}_3$ bimodules appearing in $(\rho\bar{\rho})^n\rho$, $n \in \mathbb{N}$, which are of the form $\alpha g\beta$, $g \in G$ (Proposition 4.2). A computation as above (or again maximality of the cardinality of the double cosets) shows that these two bimodules are irreducible and thus

$$\Gamma_{\text{odd}} = \{\alpha\beta, \alpha ba\beta\}.$$

In the second step we compute the edges of Γ according to (8). First we calculate the edges of $\gamma \in \Gamma_{\text{even}}$ to $\alpha\beta \in \Gamma_{\text{odd}}$ (recall that $\rho = \alpha\beta$ in (8)). Then

$$\begin{aligned}
\langle 1 \cdot \alpha\beta, \alpha\beta\rangle &= 1,\\
\langle \alpha\bar{\alpha}\alpha\beta, \alpha\beta\rangle &= \langle \bar{\alpha}\alpha, \bar{\alpha}\alpha\beta\bar{\beta}\rangle = \langle 1 + a, (1 + a)(1 + b + b^2)\rangle = 2,\\
\langle \alpha b\bar{\alpha}\alpha\beta, \alpha\beta\rangle &= \langle b(1 + a), (1 + a)(1 + b + b^2)\rangle\\
&= \langle b + ba, 1 + a + b + ab + b^2 + ab^2\rangle = 1
\end{aligned}$$

and similarly $\langle \alpha b^2\bar{\alpha}\alpha\beta, \alpha\beta\rangle = 1$. Note that the first two identities imply that

$$\langle a'\alpha\beta, \alpha\beta\rangle = 1.$$

Finally one has

$$\langle \alpha bab^2\bar{\alpha}\alpha\beta, \alpha\beta\rangle = \langle bab^2(1 + a), (1 + a)(1 + b + b^2)\rangle = 0.$$

Hence we have precisely one single edge from 1, a', $\alpha b\bar{\alpha}$ and $\alpha b^2\bar{\alpha}$ to $\alpha\beta$ and no edges from b_1 or b_2 to $\alpha\beta$. Next we compute the edges from $\gamma \in \Gamma_{\text{even}}$ to $\alpha ba\beta$. Let $g = (1+a)ba(1+b+b^2) = ba + aba + bab + b^2a + bab^2 + b^2ab$, then

$$\begin{aligned}
\langle \alpha\bar{\alpha}\alpha\beta, \alpha ba\beta\rangle &= \langle \bar{\alpha}\alpha, \bar{\alpha}\alpha ba\beta\bar{\beta}\rangle = \langle 1+a, (1+a)ba(1+b+b^2)\rangle \\
&= \langle 1+a, g\rangle = 0, \\
\langle \alpha b\bar{\alpha}\alpha\beta, \alpha ba\beta\rangle &= \langle b+ba, g\rangle = 1, \\
\langle \alpha b^2\bar{\alpha}\alpha\beta, \alpha ba\beta\rangle &= \langle b^2+b^2a, g\rangle = 1, \\
\langle \alpha bab^2\bar{\alpha}\alpha\beta, \alpha ba\beta\rangle &= \langle bab^2+b^2ab, g\rangle = 2.
\end{aligned}$$

Hence, since Γ is connected, we have precisely one single edge from $\alpha b\bar{\alpha}$, $\alpha b^2\bar{\alpha}$, b_1 and b_2 to $\alpha ba\beta$ and no edges from 1 or a' to $\alpha ab\beta$. Thus Γ is given by the following graph

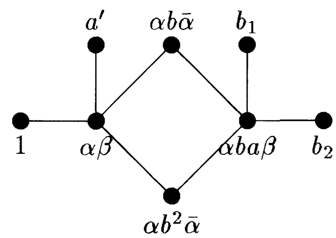

Fig. 3

Next we compute the principal graph Γ'. Since $\Gamma'_{\text{odd}} = \overline{\Gamma_{\text{odd}}}$, where $^{-}$ denotes the contragredient map, we get that

$$\Gamma'_{\text{odd}} = \{\bar{\beta}\bar{\alpha}, \bar{\beta}(ba)^{-1}\bar{\alpha}\} = \{\bar{\beta}\bar{\alpha}, \bar{\beta}ab^2\bar{\alpha}\}.$$

To obtain the even vertices of Γ' we need to compute first ${}_K\backslash G/_K$. These double cosets are given as $K\cdot 1\cdot K = \{1, b, b^2\}$ and $KaK = \{a,\ ba,\ b^2a,\ ab,\ bab,\ b^2ab,\ ab^2,\ bab^2,\ b^2ab^2\}$. Hence $\bar{\beta}\beta$ and $\bar{\beta}a\beta$ are the distinct $R \rtimes \mathbb{Z}_3$-$R \rtimes \mathbb{Z}_3$ bimodules appearing in $(\bar{\rho}\rho)^n$, $n \in \mathbb{N}$, which are of the form $\bar{\beta}g\beta$, $g \in G$. We check irreducibility: denote by $\widehat{\mathbb{Z}_3} = \{1, b', (b')^2\}$ the dual group of $\mathbb{Z}_3$, then $\bar{\beta}\beta = 1 + b' + (b')^2$. Since $\langle \bar{\beta}a\beta, \bar{\beta}a\beta\rangle = \langle a, (1+b+b^2)a(1+b+b^2)\rangle = 1$, we have

$$\Gamma'_{\text{even}} = \{1, b', (b')^2, \bar{\beta}a\beta\}.$$

Next we compute the edges of Γ' according to (9). First we calculate the edges of $\gamma \in \Gamma'_{\text{even}}$ to $\bar{\beta}\bar{\alpha}$. We have that

$$\begin{aligned}
\langle \bar{\beta}\beta\bar{\beta}\bar{\alpha}, \bar{\beta}\bar{\alpha}\rangle &= \langle \beta\bar{\beta}, \beta\bar{\beta}\bar{\alpha}\alpha\rangle = \langle 1+b+b^2, (1+b+b^2)(1+a)\rangle = 3, \\
\langle \bar{\beta}a\beta\bar{\beta}\bar{\alpha}, \bar{\beta}\bar{\alpha}\rangle &= \langle a(1+b+b^2), (1+b+b^2)(1+a)\rangle = 1.
\end{aligned}$$

Hence we have precisely one single edge from $\bar{\beta}a\beta$ to $\bar{\beta}\bar{\alpha}$ and three edges from $\bar{\beta}\beta$ to $\bar{\beta}\bar{\alpha}$. Now we compute the edges from the even vertices of Γ' to $\bar{\beta}ab^2\bar{\alpha}$. We have

$$\begin{aligned}
\langle \bar{\beta}\beta\bar{\beta}\bar{\alpha}, \bar{\beta}ab^2\bar{\alpha}\rangle &= 0, \\
\langle \bar{\beta}a\beta\bar{\beta}\bar{\alpha}, \bar{\beta}ab^2\bar{\alpha}\rangle &= \langle a(1+b+b^2), (1+b+b^2)ab^2(1+a)\rangle \\
&= \langle a+ab+ab^2, ab^2+bab^2+b^2ab^2+bab+b^2ab+ab\rangle = 2.
\end{aligned}$$

Since Γ' is connected, we have therefore precisely one edge from 1, b' and $(b')^2$ to $\bar{\beta}\bar{\alpha}$ and no edges from 1, b' and $(b')^2$ to $\bar{\beta}ab^2\bar{\alpha}$. Furthermore we have a double edge from $\bar{\beta}a\beta$ to $\bar{\beta}ab^2\bar{\alpha}$. Thus the principal graph Γ' is given by

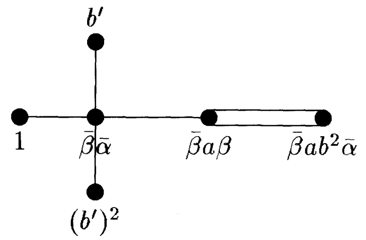

Fig. 4

c) Let $G = G_{2,3,4} = \langle a, b \,|\, a^2 = b^3 = (ab)^4 = 1\rangle$ and consider the associated group type inclusion as in *a*). Then $G \cong S_4$, the symmetric group, via the isomorphism which maps $a \to (12)$ and $b \to (243)$. We have that $G = \{$ 1, a, b, b^2, ab, ba, bab, aba, ab^2, b^2a, $(ab)^2$, ab^2a, b^2ab, bab^2, $(ba)^2$, $abab^2$, bab^2a, ab^2ab, b^2aba, b^2ab^2, $abab^2a$, bab^2ab, ab^2aba, $abab^2ab\}$. Let $H = \mathbb{Z}_2$ and $K = \mathbb{Z}_3$ and let us compute the principal graphs of the associated irreducible group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$. One checks that

$$\begin{aligned} {}_H\backslash G/_H &= \{H, HbH, Hb^2H, HbabH, Hb^2abH, Hbab^2H, Hbab^2abH\}, \\ {}_H\backslash G/_K &= \{HK, HbaK, Hb^2aK, Hbab^2aK\}, \\ {}_K\backslash G/_K &= \{K, KaK, KabaK, Kab^2abaK\}. \end{aligned}$$

Hence $\alpha\bar{\alpha}$, $\alpha b\bar{\alpha}$, $\alpha b^2\bar{\alpha}$, $\alpha bab\bar{\alpha}$, $\alpha b^2ab\bar{\alpha}$, $\alpha bab^2\bar{\alpha}$ and $\alpha bab^2ab\bar{\alpha}$ are all the distinct $R^{\mathbb{Z}_2}$-$R^{\mathbb{Z}_2}$ bimodules of the form $\alpha g\bar{\alpha}$, $g \in G$ (Proposition 4.2). To determine irreducibility one computes

$$\begin{aligned} \langle \alpha b\bar{\alpha}, \alpha b\bar{\alpha}\rangle = \langle \alpha b^2\bar{\alpha}, \alpha b^2\bar{\alpha}\rangle = \langle \alpha bab\bar{\alpha}, \alpha bab\bar{\alpha}\rangle &= \langle \alpha b^2ab\bar{\alpha}, \alpha b^2ab\bar{\alpha}\rangle \\ &= \langle \alpha bab^2\bar{\alpha}, \alpha bab^2\bar{\alpha}\rangle = 1 \end{aligned}$$

and $\alpha\bar{\alpha} = 1 + a'$, where $\widehat{\mathbb{Z}_2} = \{1, a'\}$, $\langle \alpha bab^2ab\bar{\alpha}, \alpha bab^2ab\bar{\alpha}\rangle = 2$, hence $\alpha bab^2ab\bar{\alpha} = a_1 + a_2$ with $a_1 \neq a_2$ irreducible bimodules (if $a_1 = a_2$ we would get again a graph with square norm > 6). Thus

$$\Gamma_{\text{even}} = \{1, a', \alpha b\bar{\alpha}, \alpha b^2\bar{\alpha}, \alpha bab\bar{\alpha}, \alpha b^2ab\bar{\alpha}, \alpha bab^2\bar{\alpha}, a_1, a_2\}.$$

By the above double coset computation we obtain next that $\alpha\beta$, $\alpha ba\beta$, $\alpha b^2a\beta$ and $\alpha bab^2a\beta$ are all the distinct $R^{\mathbb{Z}_2}$-$R \rtimes \mathbb{Z}_3$ bimodules of the form $\alpha g\beta$, $g \in G$, which are all irreducible since the action of $H \times K$ on G is free. Thus

$$\Gamma_{\text{odd}} = \{\alpha\beta, \alpha ba\beta, \alpha b^2a\beta, \alpha bab^2a\beta\}.$$

A slightly lengthy, but simple computation of the edges as in *a*) yields the following principal graph Γ

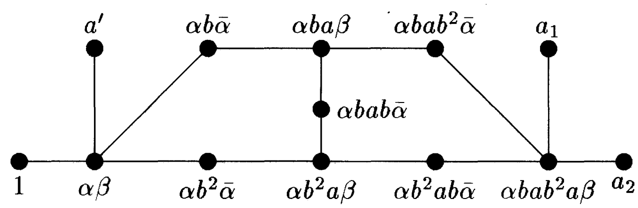

Fig. 5

Next we compute the principal graph Γ'. We know that

$$\Gamma'_{\mathrm{odd}} = \{\bar{\beta}\bar{\alpha}, \bar{\beta}(ba)^{-1}\bar{\alpha}, \bar{\beta}(b^2a)^{-1}\bar{\alpha}, \bar{\beta}(bab^2a)^{-1}\bar{\alpha}\} = \{\bar{\beta}\bar{\alpha}, \bar{\beta}ab^2\bar{\alpha}, \bar{\beta}ab\bar{\alpha}, \bar{\beta}abab^2\bar{\alpha}\}.$$

Checking irreducibility of $\bar{\beta}\beta$, $\bar{\beta}a\beta$, $\bar{\beta}aba\beta$ and $\bar{\beta}ab^2aba\beta$, which are all the distinct $R \rtimes \mathbb{Z}_3$-$R \rtimes \mathbb{Z}_3$ bimodules of the form $\bar{\beta}g\beta$, $g \in G$, we get that

$$\Gamma'_{\mathrm{even}} = \{1, b', (b')^2, \bar{\beta}a\beta, \bar{\beta}aba\beta, a_1, a_2, a_3\},$$

where $\widehat{\mathbb{Z}_3} = \{1, b', (b')^2\}$ denotes the dual group of $\mathbb{Z}_3$ and a_1, a_2 and a_3 denote the irreducible components of $\bar{\beta}ab^2ab\beta$. Computing again the edges according to (9) as in a), we obtain the following graph for Γ'

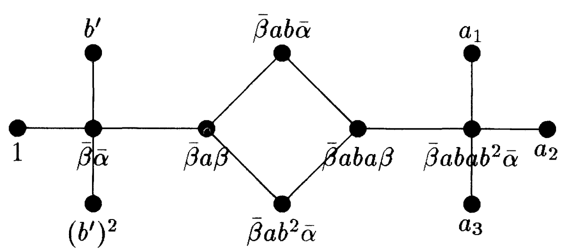

Fig. 6

d) Let $G = G_{2,3,5} = \langle a, b \,|\, a^2 = b^3 = (ab)^5 = 1\rangle$ and consider the associated irreducible index 6 group type inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$. Observe that $G \cong A_5$, the alternating group, via $a \to (12)(45)$ and $b \to (134)$ (this presentation of A_5 goes back to Hamilton, 1856, *see* [CM, page 137]). A rather lengthy and tedious computation as in a), b), which will be omitted here (and can be done as an exercise by the reader) yields the following principal graphs:

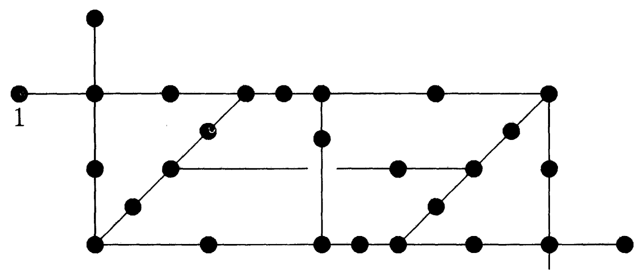

Fig. 7

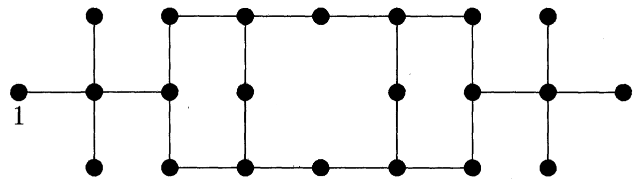

Fig. 8

Example 6.5. – *(calculation of the principal graphs* Γ *and* Γ' *in the case* $G = G_{2,3,6} = \mathbb{Z}_2 * \mathbb{Z}_3/N_2(\mathbb{Z}_2 * \mathbb{Z}_3)$).

We show first that $G_{2,3,6} = \langle a, b \,|\, a^2 = b^3 = (ab)^6 = 1\rangle$ is equal to $\mathbb{Z}_2 * \mathbb{Z}_3/N_2(\mathbb{Z}_2 * \mathbb{Z}_3)$. Set $\tilde{G} = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle \tilde{a}, \tilde{b} \,|\, \tilde{a}^2 = \tilde{b}^3 = 1\rangle$ and let $N_1(\tilde{G})$, $N_2(\tilde{G})$ be the first and the second commutator subgroups of $\tilde{G}$. Then $N_1(\tilde{G})$ is the subgroup of $\tilde{G}$ generated by $\tilde{x} = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^2$ and $\tilde{y} = \tilde{a}\tilde{b}^2\tilde{a}\tilde{b}$. Hence $N_2(\tilde{G})$ is the smallest normal subgroup of $\tilde{G}$, which contains

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = (\tilde{b}\tilde{a})^{-1}(\tilde{a}\tilde{b})^{-6}(\tilde{b}\tilde{a}),$$

i.e. $N_2(\tilde{G})$ is also the smallest subgroup of $\tilde{G}$ containing $(\tilde{a}\tilde{b})^6$. Therefore $\tilde{G}/N_2(\tilde{G})$ can be expressed in terms of generators and relations as

$$\tilde{G}/N_2(\tilde{G}) = \{a, b \,|\, a^2 = b^3 = (ab)^6 = 1\} = G_{2,3,6}.$$

Set now $G = \tilde{G}/N_2(\tilde{G}) = G_{2,3,6}$ and set $a = \phi(\tilde{a})$, $b = \phi(\tilde{b})$, where $\phi : \tilde{G} \to \tilde{G}/N_2(\tilde{G})$ denotes the quotient homomorphism. Then $x = abab^2$ and $y = ab^2ab$ generate a normal subgroup of G, namely the group

$$N = \phi(N_1(\tilde{G})) = N_1(\tilde{G})/N_2(\tilde{G}) \cong F_2/N_1(F_2) \cong \mathbb{Z}^2,$$

with x and y as free abelian generators. It is well-known that $\tilde{G} = \coprod_{n=0}^{5} (\tilde{a}\tilde{b})^n N_1(\tilde{G})$ (*see* for instance [Ne]) and therefore $G = \coprod_{n=0}^{5} (ab)^n N$ and hence $[G : N] = 6$. Note that since N is abelian, this implies that G has polynomial growth. Furthermore, G has the structure of a (twisted) crossed product of N by $\mathbb{Z}_6$. Let us determine this structure: we have that $\mathrm{Ad}a(x) = axa^{-1} = bab^2a = x^{-1}$, $\mathrm{Ad}a(y) = aya^{-1} = b^2aba = y^{-1}$, $\mathrm{Ad}b(x) = babab = x^{-1}y$ and $\mathrm{Ad}b(y) = bab^2a = x^{-1}$. Thus $\mathrm{Ad}ab(x) = xy^{-1}$ and $\mathrm{Ad}ab(y) = x$. Using $G = \coprod_{n=0}^{5} N(ab)^n$, we see that $G \cong N \rtimes \mathbb{Z}_6$, where $\mathbb{Z}_6 = \langle ab \rangle$ acts via $\mathrm{Ad}ab$ on N. Let us give a geometrical realization of this crossed product. Since $N = \langle x, y \rangle \cong \mathbb{Z}^2$, we can realize the lattice N in $\mathbb{C}$ as $\mathbb{Z} + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\mathbb{Z}$ with $x = 1$ and $y = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. $\mathrm{Ad}a$ is then a counterclockwise rotation by $180°$ and $\mathrm{Ad}b$ one by $120°$ around 0. $\mathrm{Ad}ab$ becomes a clockwise rotation by $60°$ around 0. This geometrical realization of N will be useful for the subsequent computations. - Next we will simplify the right N-coset decomposition of G. Since $(ab)^2y^{-1}x = b^2$, $(ab)^3y^{-1}x = a$, $(ab)^4y^{-1}xy = b$ and $(ab)^5y^{-1}xy = ab^2$, we get that G can be written as a disjoint union

$$G = N \cup aN \cup bN \cup abN \cup b^2N \cup ab^2N \tag{34}$$

Let us compute now the vertices of the principal graphs (Γ, Γ'). Clearly, $\mathbb{Z}_2 \times \mathbb{Z}_3$ (resp. $\mathbb{Z}_3 \times \mathbb{Z}_2$) acts freely on G, so that Γ_{odd} (resp. Γ'_{odd}) is labeled by the double cosets ${}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_3}$ (resp. ${}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_2}$) (Proposition 4.2). If we write each element in G as a word in a and b with *positive* powers, then it follows from (34) that the elememts in N are precisely those words whose sum of powers in a (resp. b) equals $0 \bmod 2$ (resp. $0 \bmod 3$). Now let $g \in G$ and consider the double coset $\mathbb{Z}_2 g \mathbb{Z}_3$, which all have cardinality 6 (since $\mathbb{Z}_2 \times \mathbb{Z}_3$ acts freely). Clearly, there is precisely one element $n \in \mathbb{Z}_2 g \mathbb{Z}_3$ whose sum of powers in a (and b) is equal $0 \bmod 2$ ($0 \bmod 3$). Hence N forms a set of representatives for ${}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_3}$ and we have therefore

$$\Gamma_{\mathrm{odd}} = \{\alpha n \beta, \quad n \in N\}, \qquad \Gamma'_{\mathrm{odd}} = \{\bar{\beta} n \bar{\alpha}, \quad n \in N\}. \tag{35}$$

Next, we determine the even vertices of Γ resp. Γ'. According to Proposition 4.2 we need to analyze the double cosets ${}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_2}$ and ${}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_3}$. A double coset $\mathbb{Z}_2 g \mathbb{Z}_2$ (resp. $\mathbb{Z}_3 g \mathbb{Z}_3$) will represent an irreducible bimodule iff $|\mathbb{Z}_2 g \mathbb{Z}_2| = 4$ (resp. $|\mathbb{Z}_3 g \mathbb{Z}_3| = 9$). Clearly, if $g \in \{1, a\}$ (resp. $g \in \{1, b, b^2\}$), then $|\mathbb{Z}_2 g \mathbb{Z}_2| = 2$ (resp. $|\mathbb{Z}_3 g \mathbb{Z}_3| = 3$). The corresponding bimodule decomposes therefore into 2 (resp. 3) irreducible bimodules. We show that the double cosets have maximal cardinality in all other cases.

LEMMA 6.5.1. – *Let* $g \in G$, $\mathbb{Z}_2 g \mathbb{Z}_2 = \{g, ag, ga, aga\}$ *and suppose* $|\mathbb{Z}_2 g \mathbb{Z}_2| < 4$. *Then* $g \in \{1, a\}$.

Proof. – $|\mathbb{Z}_2 g \mathbb{Z}_2| < 4$ implies that either $g = aga$ or $ag = ga$, which are identical conditions. Let us use (34) and the geometrical realization of N to see that this condition

$$g = aga = \mathrm{Ad}a(g) \tag{36}$$

implies indeed $g \in \{1, a\}$. This is done by a case-by-case analysis. If $g \in N$, then (36) implies that g is invariant under a 180° rotation, which is only possible if $g = 0$ (note that the identity of G is written additively as 0 when regarded in the abelian group N). If $g \in aN$, then $g = ag'$, for some $g' \in N$. Then (36) implies that $g = g'a$, *i.e.* $ag' = g'a$ and hence $g' = 0$ and therefore $g = a$ by the above argument. If $g \in bN$, then $g = bg'$ for some $g' \in N$ and (36) implies $bg' = abg'a$, *i.e.* $g' = b^2abg'a = y^{-1}\mathrm{Ad}a(g')$ (this is a 180° rotation and a translation by y^{-1}). Clearly, no $g' \in N$ satisfies this relation. If $g \in abN$, then $g = abg'$ for some $g' \in N$ and (36) implies $abg' = bg'a$, *i.e.* $g' = b^2abg'a$, which is impossible by the previous argument. If $g \in b^2N$, then $g = b^2g'$ for some $g' \in N$ and (36) implies $g' = bab^2g'a = x^{-1}\mathrm{Ad}a(g')$, which is again impossible. Finally, if $g \in ab^2N$, then $g = ab^2g'$ for some $g' \in N$ and (36) implies that $ab^2g' = b^2g'a$ which holds iff $g' = bab^2aag'a = x^{-1}\mathrm{Ad}a(g')$, which is impossible as we have just seen. $\square$

Thus, if $\{1, g_i, i \in I\}$ is a set of $\mathbb{Z}_2$ double coset representatives of G, then

$$\Gamma_{\text{even}} = \{1, a', \alpha g_i \bar{\alpha},\ i \in I\},$$

where $\alpha\bar{\alpha} = 1 + a'$, $\widehat{\mathbb{Z}_2} = \{1, a'\}$. A similar analysis has to be carried out for ${}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_3}$.

LEMMA 6.5.2. – *Let* $g \in G$, $\mathbb{Z}_3 g \mathbb{Z}_3 = \{g, bg, b^2g, gb, gb^2, bgb, bgb^2, b^2gb, b^2gb^2\}$ *and suppose* $|\mathbb{Z}_3 g \mathbb{Z}_3| < 9$. *Then* $g \in \{1, b, b^2\}$.

Proof. – If $|\mathbb{Z}_3 g \mathbb{Z}_3| < 9$, then $g = bgb$ or $g = bgb^2$ or $g = b^2gb$ or $g = b^2gb^2$, which reduces to $g = bgb^{-1}$ (iff $g = bgb^2$ iff $g = b^2gb$) or $g = bgb$ (iff $g = b^2gb^2$). In particular $|\mathbb{Z}_3 g \mathbb{Z}_3| = 3$. We sketch the arguments, which are analagous to the ones used in the proof of Lemma 6.5.1. Suppose first

$$g = bgb^{-1} = \mathrm{Ad}b(g). \tag{37}$$

If $g \in N$, then $g = 0$, since $\mathrm{Ad}b$ is a 120° rotation. If $g \in aN$, $g = ag'$, $g' \in N$, then (37) implies $g' = abab^2bg'b^{-1} = x\mathrm{Ad}b(g')$, which is a rotation by 120° and a translation by x. Observe however that this transformation has no fixed point in N. If $g \in bN$, $g = bg'$, $g' \in N$, then (37) implies that $g' = \mathrm{Ad}b(g')$, and hence $g' = 0$ and thus $g = b$. If $g \in abN$, $g = abg'$, $g' \in N$, then (37) implies that $g' = y^{-1}\mathrm{Ad}b(g')$, which cannot be satisfied by any element in N. If $g \in b^2N$, $g = b^2g'$, $g' \in N$, then (37) implies that $g' = \mathrm{Ad}b(g')$, *i.e.* $g' = 0$ and hence $g = b^2$. Finally, if $g \in ab^2N$, $g = ab^2g'$, $g' \in N$, then (37) implies $g' = x^{-1}y\mathrm{Ad}b(g')$, which is again impossible in N. Now suppose

$$g = bgb = b^2\mathrm{Ad}b(g). \tag{38}$$

If $g \in N$, then (38) implies $g \in b^2N$, *i.e.* $g \in N \cap b^2N = \varnothing$, which is impossible ((34)). If $g \in aN$, $g = ag'$, $g' \in N$, then (38) implies $g' = (ab)^2(b^{-1}g'b) \in (ab)^2N = b^2N$, which is impossible. If $g \in bN$, $g = bg'$, $g' \in N$, then (38) implies $g' = bg'b$, which is impossible by the first argument. If $g \in abN$, $g = abg'$, $g' \in N$, then (38) implies $g' = b^2x(b^{-1}g'b) \in b^2N$, which is impossible. If $g \in b^2N$, $g = b^2g'$, $g' \in N$, then (38) implies that $g' = bg'b$, which cannot occur as seen above. Finally, if $g \in ab^2N$, $g = ab^2g'$, $g' \in N$, then (38) implies $g' = b^2y^{-1}(b^{-1}g'b) \in b^2N$, which is again impossible. Thus $g = bgb$ cannot occur for any $g \in G$ and the lemma is proven. $\square$

If we let $\{1, g'_j,\ j \in J\}$ be a set of representatives for the $\mathbb{Z}_3$ double cosets of G, then

$$\Gamma'_{\text{even}} = \{1, b', (b')^2, \bar{\beta}g'_j\beta,\ j \in J\},$$

where $\bar{\beta}\beta = 1 + b' + (b')^2$ as in section 6.3. This determines (not very explicitly) the vertices of (Γ, Γ').

Next, we compute the edges for Γ. As usual we use the notation of chapter 4 and recall that the computation of edges amounts to computing the dimensions of various intertwiner spaces. Let $\alpha n\beta \in \Gamma_{\text{odd}}$ and consider

$$\langle \alpha g\bar{\alpha}\alpha\beta, \alpha n\beta\rangle = \langle \alpha g\beta, \alpha n\beta\rangle + \langle \alpha ga\beta, \alpha n\beta\rangle. \tag{39}$$

Note that $\alpha g\beta$ and $\alpha ga\beta$ are irreducible bimodules (since $\mathbb{Z}_2 \times \mathbb{Z}_3$ acts freely on G) and therefore $\langle \alpha g\bar{\alpha}\alpha\beta, \alpha n\beta\rangle = 0$, 1 or 2. If $g \in \{g_i,\ i \in I\}$ (hence $g \neq 1, a$), then $\langle \alpha g\bar{\alpha}\alpha\beta, \alpha n\beta\rangle = 0$ or 1, since g and ga cannot be contained in the same double coset $\mathbb{Z}_2 n\mathbb{Z}_3$ (easy exercise, as proof of Lemma 6.5.1). If $g = 1$, then $\langle \alpha\bar{\alpha}\alpha\beta, \alpha n\beta\rangle = 2$ if $n = e$ (the identity of G) and 0 otherwise, since N labels all distinct double cosets. Hence 1 resp. a' are connected by a single edge precisely to $\alpha\beta$. Furthermore, $\alpha g_i\bar{\alpha}$ and $\alpha n\beta$ are connected iff $g_i \in \mathbb{Z}_2 n\mathbb{Z}_3$ or $g_i a \in \mathbb{Z}_2 n\mathbb{Z}_3$. In this case they are connected by a single edge. This determines Γ. However, in order to represent Γ graphically in an explicit way, we need to do some more work.

The odd vertices of Γ are labeled by $N = \mathbb{Z} + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\mathbb{Z}$ as shown above. Furthermore, we have just seen that each even vertex $\alpha g_i\bar{\alpha}$ is connected by a single edge to precisely two odd ones $\alpha n_1\beta$ and $\alpha n_2\beta$, where g_i or $g_i a \in \mathbb{Z}_2 n_j\mathbb{Z}_3$, $j = 1, 2((39))$. The odd vertex $\alpha\beta$ is connected by a single edge precisely to the even vertices 1, a', $\alpha b\bar{\alpha}$ and $\alpha b^2\bar{\alpha}$ since $\langle \alpha g_i\bar{\alpha}\alpha\beta, \alpha\beta\rangle = \langle (1+a)g_i(1+a), 1 + b + b^2\rangle = 2\delta_{g_i,e} + \delta_{g_i,b} + \delta_{g_i,b^2}$. All other odd vertices $\alpha n\beta$, $n \in N\backslash\{e\}$, are connected to precisely three even vertices by a single edge. To see this, consider the composite map ${}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_2} \to {}_{\mathbb{Z}_2}\backslash G \to {}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_3}$, which maps $\mathbb{Z}_2 g\mathbb{Z}_2 \to \mathbb{Z}_2 g \to \mathbb{Z}_2 g\mathbb{Z}_3$. This map is clearly 3:1 and hence we have that $\alpha n\beta$ is connected (by a single edge as shown above) to those three vertices $\alpha g_{i_j}\bar{\alpha}$, $i_j \in I$, $j = 1, 2, 3$, for which $\mathbb{Z}_2 g_{i_j}\mathbb{Z}_3 = \mathbb{Z}_2 n\mathbb{Z}_3$, $1 \le j \le 3$. With this notation we have the following local picture of Γ

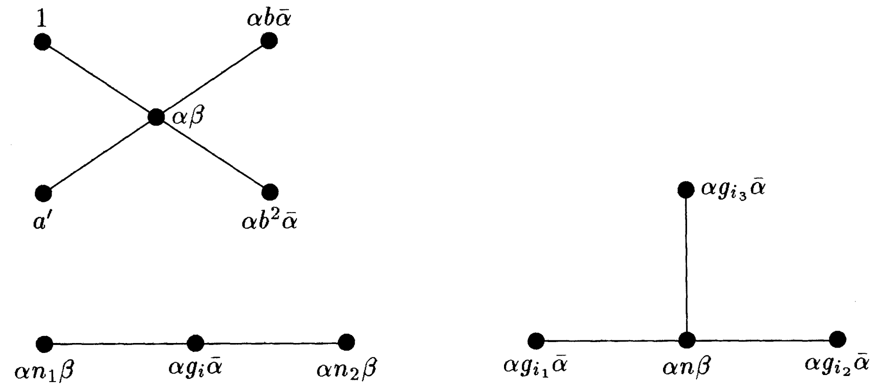

Fig. 9

To get a global picture of Γ, we use the labeling of Γ_{odd} by $N = \mathbb{Z} + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\mathbb{Z}$. First we will find the structure of $\Gamma^t\Gamma$, *i.e.* we determine to which odd vertices a given odd vertex $\alpha n\beta$ is connected by going two steps on Γ. We compute

$$\begin{aligned}\langle \alpha n\beta\overline{\alpha\beta}\alpha\beta, \alpha n'\beta\rangle &= \langle (1+a)n(1+b+b^2), n'(1+b+b^2)(1+a)\rangle\\ &= \langle n + nb + nb^2 + an + anb + anb^2, n' + n'b + n'b^2 + n'a\\ &\quad + n'ba + n'b^2a\rangle\\ &= 3\delta_{n,n'} + \langle an, n'a\rangle + \langle anb, n'ba\rangle + \langle anb^2, n'b^2a\rangle\\ &= 3\delta_{n,n'} + \langle ana, n'\rangle + \langle ana, n'bab^2a\rangle + \langle ana, n'b^2aba\rangle\\ &= 3\delta_{n,n'} + \delta_{ana^{-1},n'} + \delta_{ana^{-1},n'x^{-1}} + \delta_{ana^{-1},n'y^{-1}}\end{aligned}$$

where the third equality follows from (34). Thus $\alpha n\beta$ is connected on $\Gamma^t\Gamma$ three times to itself and by a single edge to the vertices $\alpha ana^{-1}\beta$, $\alpha ana^{-1}x\beta$ and $\alpha ana^{-1}y\beta$. The position of these bimodules in the lattice N is indicated in the following figure (Fig. 10). In order to realize Γ as a planar graph, where adjacent vertices are connected by edges, we label each odd vertex *twice*, namely by the corresponding lattice point $n \in N$ and the midpoint of the triangle with vertice ana^{-1}, $ana^{-1}x$ and $ana^{-1}y$. Note that this midpoint is obtained by reflecting n at the point $\frac{1}{4} + i\frac{\sqrt{3}}{8}$. Using this labeling the graph $\Gamma^t\Gamma$ is given by the following figure (omitting self-connecting lines and modulo the identification of points resulting from the doubling) (*see* Fig. 11).

Thus, by our calculations above (summarized in Figure 9), we can picture the principal graph Γ (modulo the above identification) in the following way (*see* Fig. 12).

Next we determine the principal graph Γ'. Let $\bar\beta n\bar\alpha \in \Gamma'_{\text{odd}}$, $g \in G$, and consider

$$\langle \bar\beta g\beta\bar\beta\bar\alpha, \bar\beta n\bar\alpha\rangle = \langle \bar\beta g\bar\alpha, \bar\beta n\bar\alpha\rangle + \langle \bar\beta gb\bar\alpha, \bar\beta n\bar\alpha\rangle + \langle \bar\beta gb^2\bar\alpha, \bar\beta n\bar\alpha\rangle. \tag{40}$$

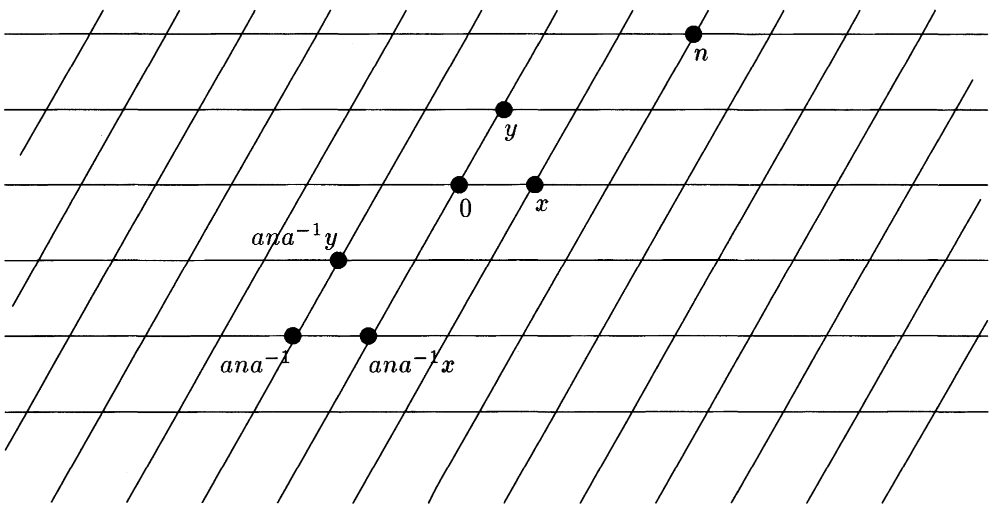

Fig. 10

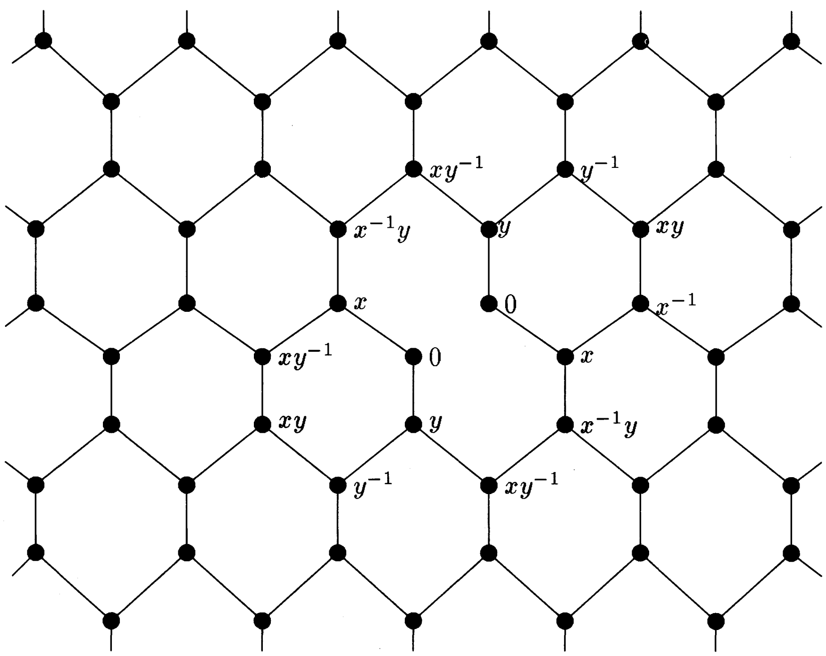

Fig. 11

Since $\bar{\beta}g\bar{\alpha}$, $\bar{\beta}gb\bar{\alpha}$ and $\bar{\beta}gb^2\bar{\alpha}$ are irreducible bimodules, we have $\langle\bar{\beta}g\beta\bar{\beta}\bar{\alpha}, \bar{\beta}n\bar{\alpha}\rangle = 0$, 1, 2 or 3. If $g \in \{g'_j,\ j \in J\}$ (hence $g \neq 1, b, b^2$), then one shows easily that g, gb and gb^2 are contained in three distinct double cosets of the form $\mathbb{Z}_3 n \mathbb{Z}_2$,

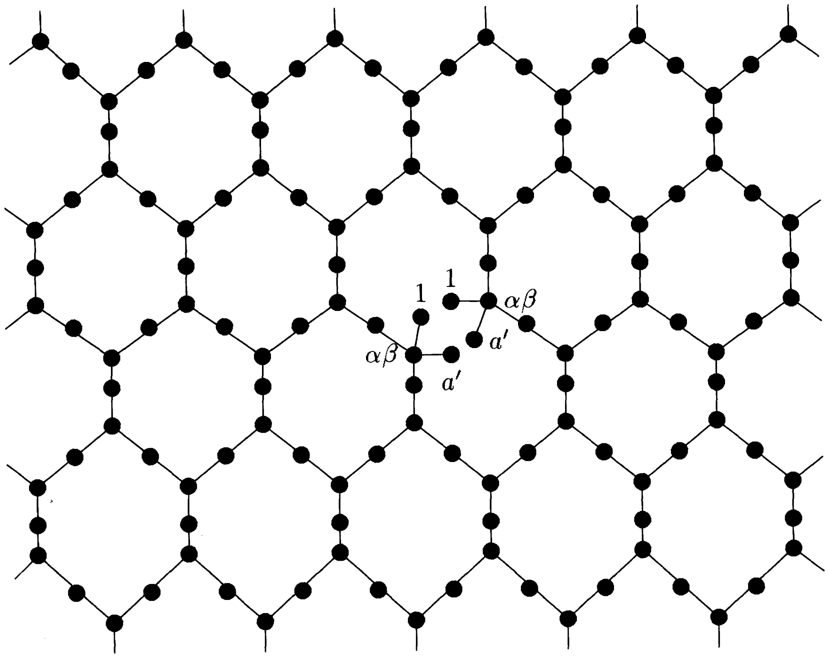

Fig. 12

$n \in N$. Hence $\langle \bar\beta g\beta\bar\beta\bar\alpha, \bar\beta n\bar\alpha \rangle = 0$ or 1 ((40)). If $g = 1$, then $\langle \bar\beta\beta\bar\beta\bar\alpha, \bar\beta n\bar\alpha \rangle = \langle 1 + b + b^2, (1 + b + b^2)n(1 + a) \rangle = 3$ if $n = e$ and 0 otherwise. Hence 1 resp. b' resp. $(b')^2$ are connected by a single edge precisely to $\bar\beta\bar\alpha$. Furthermore, $\bar\beta g_j'\beta$ and $\bar\beta n\bar\alpha$ are connected iff g_j' or $g_j'b$ or $g_j'b^2 \in \mathbb{Z}_3 n\mathbb{Z}_2$ and in this case they are connected by a single edge. This determines the principal graph Γ', but to draw it explicitly some more information is needed.

(40) shows that each even vertex is connected by a single edge to precisely three odd vertices. The odd vertex $\bar\beta\bar\alpha$ is connected by a single edge precisely to 1, b', $(b')^2$ and $\bar\beta a\beta$, since

$$\langle \bar\beta g_j'\beta\bar\beta\bar\alpha, \bar\beta\bar\alpha \rangle = \langle (1 + b + b^2)g_j'(1 + b + b^2), (1 + a) \rangle = 3\delta_{g_j',e} + \delta_{g_j',a}.$$

All other odd vertices $\bar\beta n\bar\alpha$, $n \in N\backslash\{e\}$, are connected by a single edge to precisely two even vertices. To see this, consider as above the map ${}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_3} \to {}_{\mathbb{Z}_3}\backslash G \to {}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_2}$, which maps $\mathbb{Z}_3 g\mathbb{Z}_3 \to \mathbb{Z}_3 g \to \mathbb{Z}_3 g\mathbb{Z}_2$. This map is clearly 2:1 and hence we have that $\bar\beta n\bar\alpha$ is connected (by a single edge as shown above) to those two vertices $\bar\beta g_{j_i}'\beta$, $i = 1, 2$, $j_i \in J$, for which $\mathbb{Z}_3 g_{j_i}\mathbb{Z}_2 = \mathbb{Z}_3 n\mathbb{Z}_2$, $i = 1, 2$. Thus we have the following local picture of Γ'

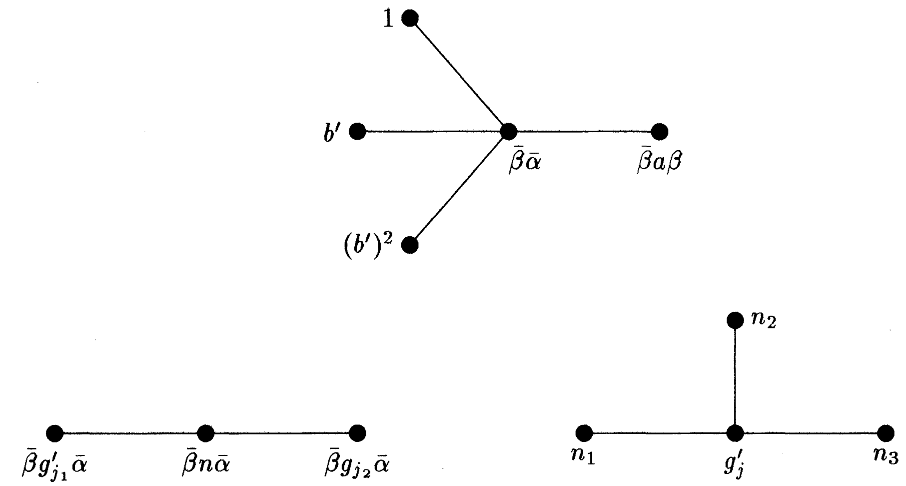

Fig. 13

As in the calculation of Γ, we will determine first the structure of $(\Gamma')^t\Gamma'$. To this end we compute

$$\begin{aligned}
\langle \bar\beta n\bar\alpha\overline{\bar\beta\bar\alpha}\bar\beta\bar\alpha, \bar\beta n'\bar\alpha\rangle &= \langle (1+b+b^2)n(1+a), n'(1+a)(1+b+b^2)\rangle \\
&= \langle n+na+bn+bna+b^2n+b^2na, n'+n'b+n'b^2+n'a \\
&\quad + n'ab+n'ab^2\rangle \\
&= 2\delta_{n,n'} + \langle bn, n'b\rangle + \langle bna, n'ab\rangle + \langle b^2n, n'b^2\rangle + \langle b^2na, n'ab^2\rangle \\
&= 2\delta_{n,n'} + \langle bnb^2, n'\rangle + \langle bnb^2, n'abab^2\rangle + \langle b^2nb, n'\rangle \\
&\quad + \langle b^2nb, n'ab^2ab\rangle \\
&= 2\delta_{n,n'} + \delta_{bnb^{-1},n'} + \delta_{bnb^{-1},n'x} + \delta_{b^{-1}nb,n'} + \delta_{b^{-1}nb,n'y},
\end{aligned}$$

where the third equality follows again from (34) (by taking inverses). Thus $\bar\beta n\bar\alpha$, $n\in N\backslash\{e\}$, is connected on $(\Gamma')^t\Gamma'$ twice to itself and by a single edge to $\bar\beta bnb^{-1}\bar\alpha$, $\bar\beta bnb^{-1}x^{-1}\bar\alpha$, $\bar\beta b^{-1}nb\bar\alpha$ and $\bar\beta b^{-1}nby^{-1}\bar\alpha$. Observe that $\bar\beta\bar\alpha$ is connected on $(\Gamma')^t\Gamma'$ four times to itself and by a single edge to $\bar\beta x^{-1}\bar\alpha$ and $\bar\beta y^{-1}\bar\alpha$. The position of these vertices on the lattice N is pictured in the following figure (*see* Fig. 14).

Thus, in order to realize Γ' as a planar graph where adjacent vertices are connected by edges, we label each odd vertex *three times*, namely by the corresponding lattice point $n\in N$ and the midpoints of the segments $[bnb^{-1}x^{-1}, bnb^{-1}]$ and $[b^{-1}nby^{-1}, b^{-1}nb]$ as indicated in figure 14. Note that these midpoints are obtained by rotating $n\in N$ by 120° and 240° around the point $-\frac{1}{4}-i\frac{\sqrt{3}}{8}$. Modulo the identification by this 120° rotation (twice),

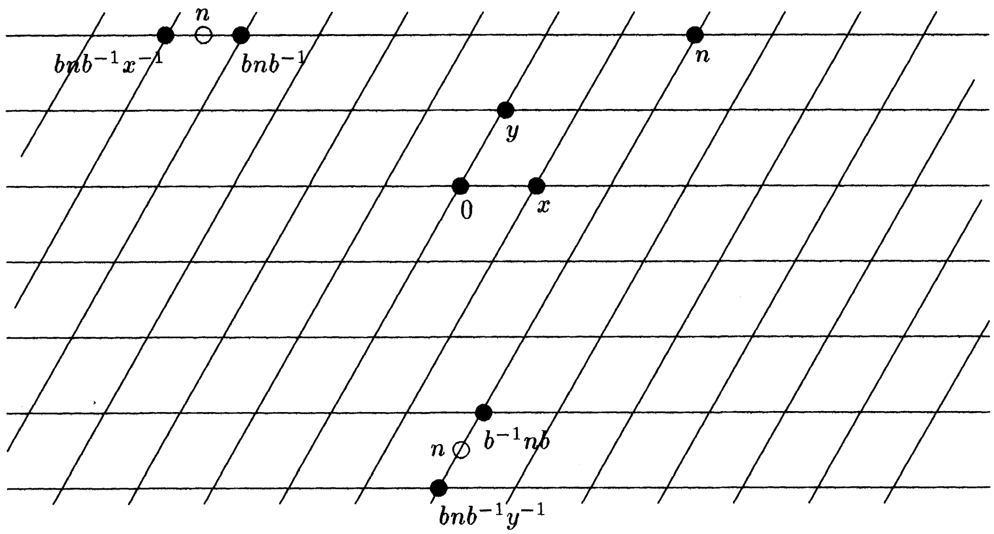
n
bnb^{-1}x^{-1}
bnb^{-1}
n
y
0
x
b^{-1}nb
n
bnb^{-1}y^{-1}

Fig. 14

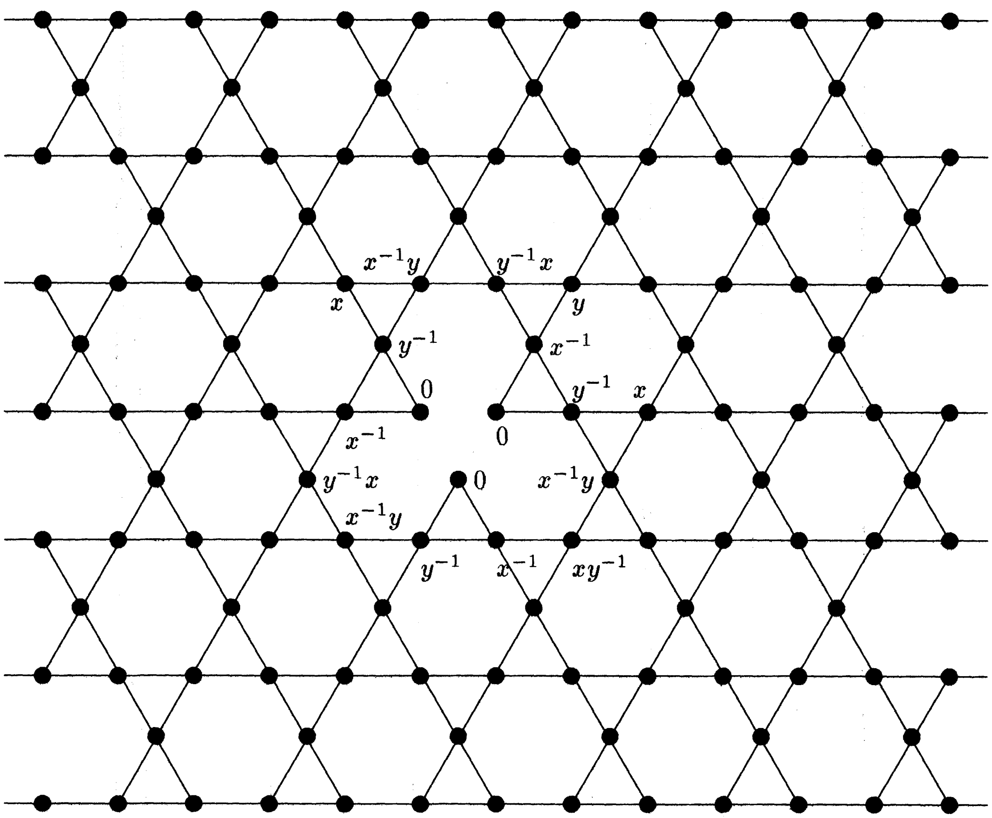
x^{-1}y
y^{-1}x
x
y
y^{-1}
x^{-1}
0
y^{-1}
x
x^{-1}
0
y^{-1}x
0
x^{-1}y
x^{-1}y
y^{-1}
x^{-1}
xy^{-1}

Fig. 15

we get the following graph for $(\Gamma')^t\Gamma'$ (omitting the self-connecting lines) (*see* Fig. 15) where a generic part is given by

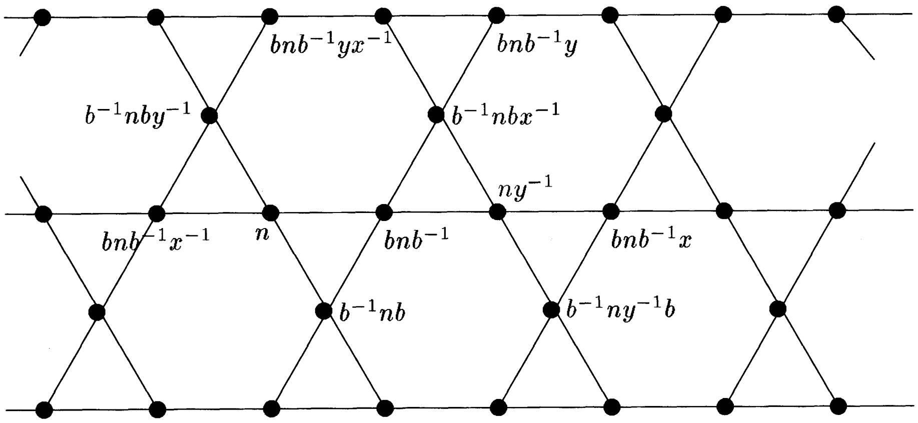

Fig. 16

Next we construct Γ' from $(\Gamma')^t\Gamma'$ and our calculations above (summarized in figure 13). Let $\bar{\beta}g_1\beta$ and $\bar{\beta}g_2\beta$ be the two even vertices connected (by a single edge necessarily) to $\bar{\beta}n\bar{\alpha}$ (on Γ'). Since the edge from $\bar{\beta}n\bar{\alpha}$ to $\bar{\beta}b^{-1}nb\bar{\alpha}$ on $(\Gamma')^t\Gamma'$ has to go through precisely one even vertex, one of the two even vertices above, call it $\bar{\beta}g_1\beta$, must be connected to $\bar{\beta}b^{-1}nb\bar{\alpha}$. Similarly, there must be precisely one even vertex through which the edge from $\bar{\beta}n\bar{\alpha}$ to $\bar{\beta}bnb^{-1}\bar{\alpha}$ (on $(\Gamma')^t\Gamma'$) has to pass. Since $\bar{\beta}n\bar{\alpha}$ is connected to precisely the two even vertices $\bar{\beta}g_i\beta$, $i = 1, 2$, this even vertex must be either $\bar{\beta}g_1\beta$ or $\bar{\beta}g_2\beta$. We show that it is necessarily $\bar{\beta}g_1\beta$.

LEMMA 6.5.3. – *Let $\bar{\beta}n\bar{\alpha} \in \Gamma'_{\mathrm{odd}}$, $n \neq e$, and denote by $\bar{\beta}g_i\beta$, $i = 1, 2$, the two even vertices connected to $\bar{\beta}n\bar{\alpha}$. Suppose that $\bar{\beta}g_1\beta$ is also connected to $\bar{\beta}b^{-1}nb\bar{\alpha}$, then it is necessarily connected to $\bar{\beta}bnb^{-1}\bar{\alpha}$.*

Proof. – We have seen above, that the multiplicities of all involved edges here is 1. Thus our assumption reads

$$\langle \bar{\beta}g_1\beta\bar{\beta}\bar{\alpha}, \bar{\beta}n\bar{\alpha}\rangle = \langle \bar{\beta}g_1\beta\bar{\beta}\bar{\alpha}, \bar{\beta}b^{-1}nb\bar{\alpha}\rangle = 1.$$

But

$$\langle \bar{\beta}g_1\beta\bar{\beta}\bar{\alpha}, \bar{\beta}n\bar{\alpha}\rangle = \langle \bar{\beta}g_1\beta, \bar{\beta}n\beta\rangle + \langle \bar{\beta}g_1\beta, \bar{\beta}na\beta\rangle$$

and similarly

$$\langle \bar{\beta} g_1 \beta \bar{\beta} \bar{\alpha}, \bar{\beta} b^{-1} n b \bar{\alpha} \rangle = \langle \bar{\beta} g_1 \beta, \bar{\beta} b^{-1} n b \beta \rangle + \langle \bar{\beta} g_1 \beta, \bar{\beta} b^{-1} n b a \beta \rangle.$$

Note that

$$\begin{aligned} \langle \bar{\beta} g_1 \beta, \bar{\beta} b n b^{-1} \beta \rangle &= \langle g_1, (1 + b + b^2) n (1 + b + b^2) \rangle \\ &= \langle \bar{\beta} g_1 \beta, \bar{\beta} n \beta \rangle = \langle \bar{\beta} g_1 \beta, \bar{\beta} b^{-1} n b \beta \rangle. \end{aligned}$$

Thus, if $\langle \bar{\beta} g_1 \beta, \bar{\beta} n \beta \rangle = 1$ or $\langle \bar{\beta} g_1 \beta, \bar{\beta} b^{-1} n b \beta \rangle = 1$, then $\langle \bar{\beta} g_1 \beta \bar{\beta} \bar{\alpha}, \bar{\beta} b n b^{-1} \bar{\alpha} \rangle = \langle \bar{\beta} g_1 \beta, \bar{\beta} b n b^{-1} \beta \rangle + \langle \bar{\beta} g_1 \beta, \bar{\beta} b n b^{-1} a \beta \rangle \geq 1$. In the other case, *i.e.* $\langle \bar{\beta} g_1 \beta, \bar{\beta} n a \beta \rangle = 1$ and $\langle \bar{\beta} g_1 \beta, \bar{\beta} b^{-1} n b a \beta \rangle = 1$, we get that $\mathbb{Z}_3 g_1 \mathbb{Z}_3 = \mathbb{Z}_3 n a \mathbb{Z}_3 = \mathbb{Z}_3 b^{-1} n b \mathbb{Z}_3$, and hence $na \in \mathbb{Z}_3 n b a \mathbb{Z}_3$. A simple case-by-case analysis shows that this is impossible. This proves the lemma. $\square$

A similar analysis for g_2 shows that Γ' looks locally like

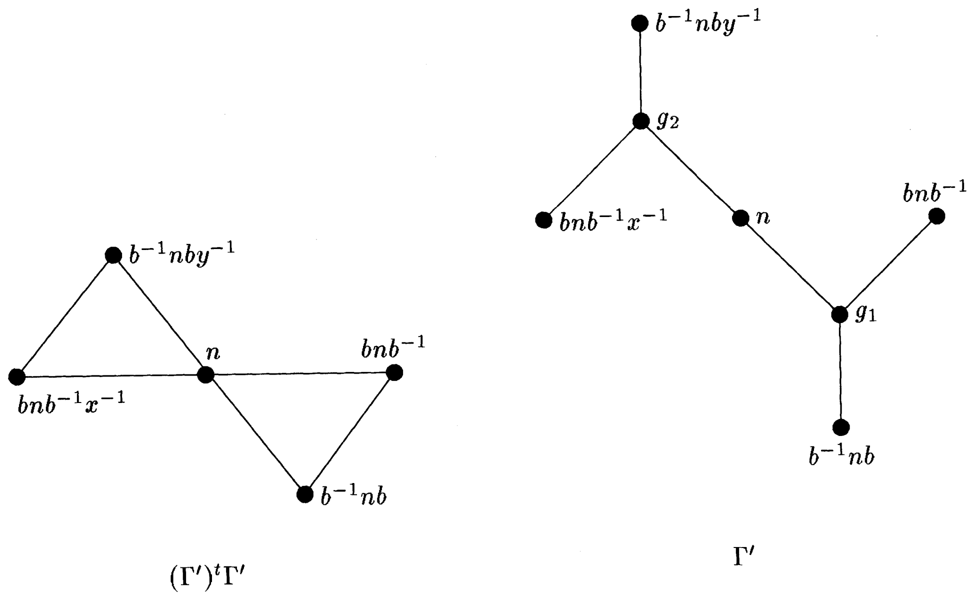

Fig. 17

Finally, taking into account the structure of Γ' around the vertex $\bar{\beta}\bar{\alpha}$ (Fig. 13), we can picture Γ' in the following way (modulo the identification by the above 120° rotation of course) (*see* Fig. 18).

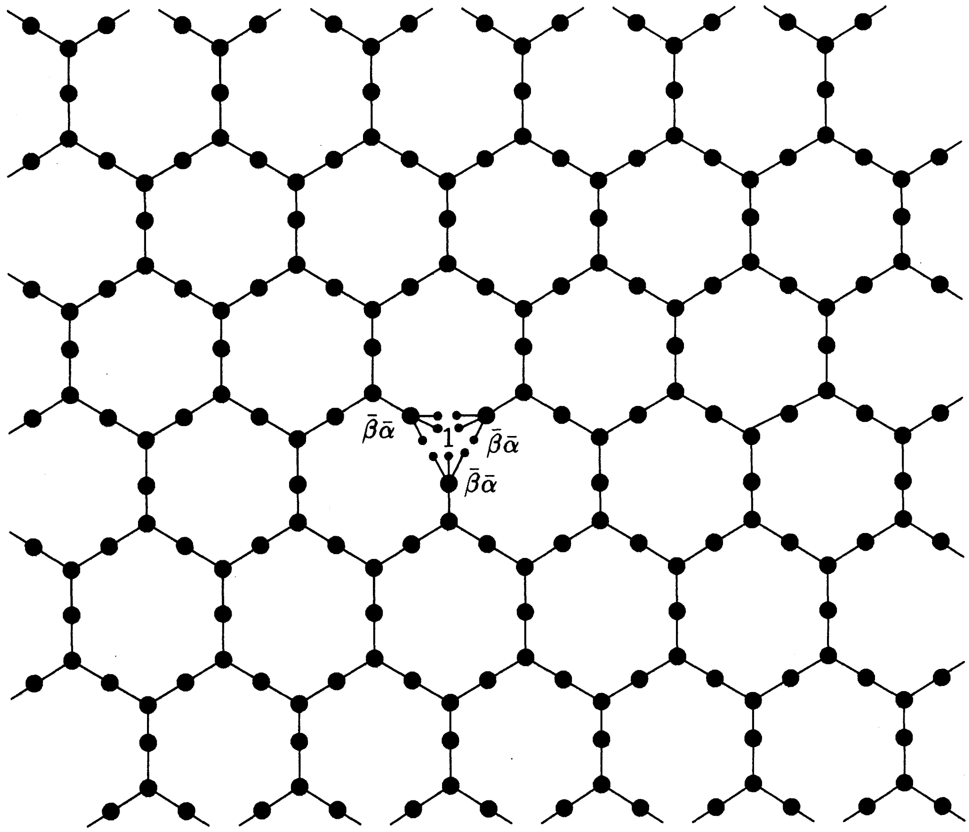

Fig. 18

Note that (Γ, Γ') have trivially polynomial growth (since $N \cong \mathbb{Z}^2$ labels the odd vertices) and that the here discussed group type inclusion is therefore strongly amenable (Theorem 4.13) (*see* also Remark 5.9).

Example 6.6. – *(irreducible nonamenable infinite depth subfactor of R corresponding to the free product of two finite groups).*

We compute the principal graphs of $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ in the case where the group G generated by $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in Out$R$ is just the free product, *i.e.* $G = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \,|\, a^2 = b^3 = 1\rangle \cong PSL(2, \mathbb{Z})$. Let W_a (resp. W_b) be all the nontrivial reduced words in the letters a and b, which begin and end with a (resp. b or b^2) and let $W_{a,b}$ (resp. $W_{b,a}$) be all the nontrivial reduced words in a, b which begin with a (resp. begin with b or b^2) and end with b or b^2 (resp. end with a). Clearly, the $\mathbb{Z}_2$-$\mathbb{Z}_2$ double cosets of G are given by

$$_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_2} = \{\mathbb{Z}_2 g \mathbb{Z}_2,\ g \in W_b \cup \{1\}\}$$

and similarly

$$\begin{aligned}
{}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_3} &= \{\mathbb{Z}_3 g \mathbb{Z}_3,\ g \in W_a \cup \{1\}\},\\
{}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_3} &= \{\mathbb{Z}_2 g \mathbb{Z}_3,\ g \in W_{b,a} \cup \{1\}\},\\
{}_{\mathbb{Z}_3}\backslash G/_{\mathbb{Z}_2} &= \{\mathbb{Z}_3 g \mathbb{Z}_2,\ g \in W_{a,b} \cup \{1\}\}.
\end{aligned}$$

Let $\widehat{\mathbb{Z}_2} = \{1, a'\}$ and $\widehat{\mathbb{Z}_3} = \{1, b', (b')^2\}$ be the dual groups and use the notation of chapter 4. We compute first the vertices of the principal graphs Γ resp. Γ', *i.e.* we check which double cosets label irreducible bimodules. Let $b^{\epsilon_1} w b^{\epsilon_2} \in W_b$, ϵ_1, $\epsilon_2 \in \{1,2\}$ and calculate $\langle \alpha b^{\epsilon_1} w b^{\epsilon_2} \bar{\alpha}, \alpha b^{\epsilon_1} w b^{\epsilon_2} \bar{\alpha}\rangle = \langle b^{\epsilon_1} w b^{\epsilon_2}, (1+a) b^{\epsilon_1} w b^{\epsilon_2} (1+a)\rangle = 1$. Since $\alpha\bar{\alpha} = 1 + a'$, we have therefore

$$\Gamma_{\text{even}} = \{1, a', \alpha g \bar{\alpha},\ g \in W_b\}.$$

Since $\mathbb{Z}_2 \times \mathbb{Z}_3$ acts freely on G, we have that Γ_{odd} is labeled by ${}_{\mathbb{Z}_2}\backslash G/_{\mathbb{Z}_3}$, *i.e.*

$$\Gamma_{\text{odd}} = \{\alpha g \beta,\ g \in W_{b,a} \cup \{1\}\}.$$

Similarly, we obtain

$$\begin{aligned}
\Gamma'_{\text{even}} &= \{1, b', (b')^2, \bar{\beta} g \beta,\ g \in W_a\},\\
\Gamma'_{\text{odd}} &= \{\bar{\beta} g \bar{\alpha},\ g \in W_{a,b} \cup \{1\}\}.
\end{aligned}$$

Next we compute the edges of the principal graph Γ. If $\alpha g_1 \bar{\alpha} \in \Gamma_{\text{even}}$ and $\alpha g_2 \beta \in \Gamma_{\text{odd}}$, then the number of edges between these two vertices is given by

$$\langle \alpha g_1 \bar{\alpha} \alpha \beta, \alpha g_2 \beta\rangle = \langle g_1(1+a), (1+a) g_2 (1 + b + b^2)\rangle. \tag{41}$$

In particular, if $g_1 = 1$, this formula shows that $\alpha\bar{\alpha} = 1 + a'$ is connected only to $\alpha\beta$ with multiplicity 2. Since Γ is connected we must therefore have precisely one single edge from 1 resp. a' to $\alpha\beta$. Similarly, $\alpha\beta$ is connected precisely to 1, a', $\alpha b \bar{\alpha}$ and $\alpha b^2 \bar{\alpha}$ ((41)) with a single edge. Next we show that $\alpha b \bar{\alpha}$ resp. $\alpha b^2 \bar{\alpha}$ are connected precisely to $\alpha b a \bar{\alpha}$ resp. $\alpha b^2 a \bar{\alpha}$ with a single edge. To this end, let $g_1 = b^{\epsilon}$, $\epsilon \in \{1,2\}$, $g_2 \in W_{b,a}$ (in particular $g_2 \neq 1$), then $\langle \alpha g_1 \bar{\alpha} \alpha \beta, \alpha g_2 \beta\rangle = 1$ iff $g_2 = b^{\epsilon} a$ since $\langle \alpha g_1 \bar{\alpha} \alpha \beta, \alpha g_2 \beta\rangle = \langle b^{\epsilon} + b^{\epsilon} a, g_2 + g_2 b + g_2 b^2\rangle = \langle b^{\epsilon} a, g_2\rangle$. This shows the claim. Now let $g_1 \in W_b \backslash \{b, b^2\}$, $g_2 \in W_{b,a}$, then (34) implies

$$\begin{aligned}
\langle \alpha g_1 \bar{\alpha} \alpha \beta, \alpha g_2 \beta\rangle &= \langle g_1 + g_1 a, g_2 + g_2 b + g_2 b^2\rangle\\
&= \langle g_1 a, g_2\rangle + \langle g_1, g_2 b\rangle + \langle g_1, g_2 b^2\rangle\\
&= \delta_{g_1, g_2 a} + \delta_{g_1, g_2 b} + \delta_{g_1, g_2 b^2} \qquad (42)\\
&= \delta_{g_1 a, g_2} + \delta_{g_1 b, g_2} + \delta_{g_1 b^2, g_2}. \qquad (43)
\end{aligned}$$

Thus $\alpha g_1 \bar{\alpha}$, $g_1 \in W_b \backslash \{b, b^2\}$ is connected with a single edge precisely to $\alpha g_1 a \beta$ and $\alpha g_1 b^{\epsilon} \beta$ with $\epsilon \in \{1,2\}$ such that $g_1 b^{\epsilon} \in W_{b,a}$ ((42)). Similarly, $\alpha g_2 \beta$, $g_2 \in W_{b,a}$, is connected with a single edge precisely to $\alpha g_2 a\bar{\alpha}$, $\alpha g_2 b\bar{\alpha}$ and $\alpha g_2 b^2\bar{\alpha}$ ((43)). This gives the following principal graph Γ

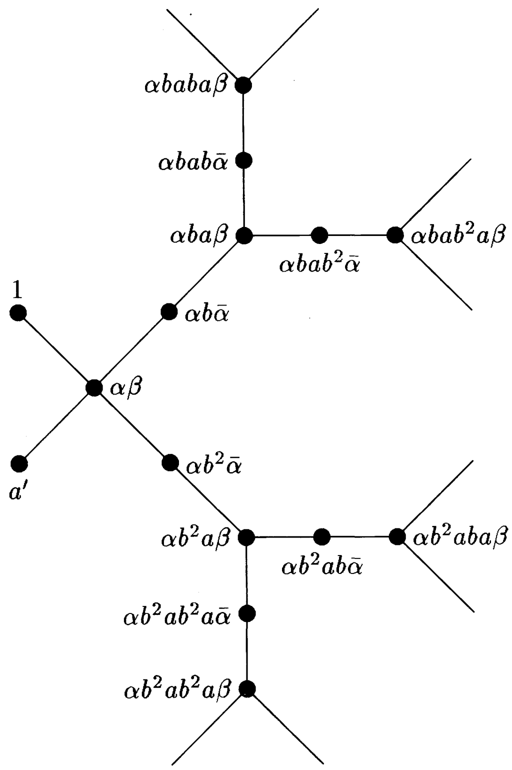

Fig. 19

We compute now the edges of Γ' in a similar manner. The number of edges from $\bar{\beta}g_1\beta \in \Gamma'_{\mathrm{even}}$ to $\bar{\beta}g_2\bar{\alpha} \in \Gamma'_{\mathrm{odd}}$ is obtained as

$$\langle \bar{\beta}g_1\beta\bar{\beta}\bar{\alpha}, \bar{\beta}g_2\bar{\alpha}\rangle = \langle g_1(1+b+b^2), (1+b+b^2)g_2(1+a)\rangle. \tag{44}$$

If $g_1 = 1$, then $\bar{\beta}\beta = 1 + b' + (b')^2$ is connected precisely to $\bar{\beta}\bar{\alpha}$ with multiplicity 3, hence, by connectedness of Γ', we get that there is a single edge from 1 resp. b' resp. $(b')^2$ to $\bar{\beta}\bar{\alpha}$. Similarly, $\bar{\beta}\bar{\alpha}$ is connected precisely to 1, b', $(b')^2$ and $\bar{\beta}a\beta$ by a single edge ((44)). Using (44), we compute that $\langle \bar{\beta}a\beta\bar{\beta}\bar{\alpha}, \bar{\beta}g_2\bar{\alpha}\rangle = \langle a + ab + ab^2, g_2 + g_2 a\rangle = 1$ iff $g_2 = 1$ or $g_2 = ab$ or $g_2 = ab^2$. In all other cases, we have $\langle \bar{\beta}a\beta\bar{\beta}\bar{\alpha}, \bar{\beta}g_2\bar{\alpha}\rangle = 0$. Thus $\bar{\beta}a\beta$ is connected by a single edge precisely to $\bar{\beta}\bar{\alpha}$, $\bar{\beta}ab\bar{\alpha}$ and $\bar{\beta}ab^2\bar{\alpha}$. Now let $g_1 \in W_a\backslash\{a\}$ and $g_2 \in W_{a,b}$, then (44) implies

$$\begin{aligned}\langle \bar{\beta}g_1\beta\bar{\beta}\bar{\alpha}, \bar{\beta}g_2\bar{\alpha}\rangle &= \langle g_1 + g_1 b + g_1 b^2, g_2 + g_2 a\rangle\\ &= \langle g_1, g_2 a\rangle + \langle g_1, g_2 b\rangle + \langle g_1, g_2 b^2\rangle\\ &= \delta_{g_1, g_2 a} + \delta_{g_1, g_2 b} + \delta_{g_1, g_2 b^2}\end{aligned} \tag{45}$$

$$= \delta_{g_1 a, g_2} + \delta_{g_1 b, g_2} + \delta_{g_1 b^2, g_2}. \tag{46}$$

Thus $\bar{\beta} g_1 \beta$, $g_1 \in W_a \backslash \{a\}$, is connected by a single edge precisely to $\bar{\beta} g_1 a \bar{\alpha}$, $\bar{\beta} g_1 b \bar{\alpha}$ and $\bar{\beta} g_1 b^2 \bar{\alpha}$ ((45)) and $\bar{\beta} g_2 \bar{\alpha}$, $g_2 \in W_{a,b}$, is connected by a single edge precisely to $\bar{\beta} g_2 a \beta$ and $\bar{\beta} g_2 b^\epsilon \bar{\alpha}$, where $\epsilon \in \{1,2\}$ such that $g_2 b^\epsilon \in W_a$ ((46)). Therefore Γ' is given as

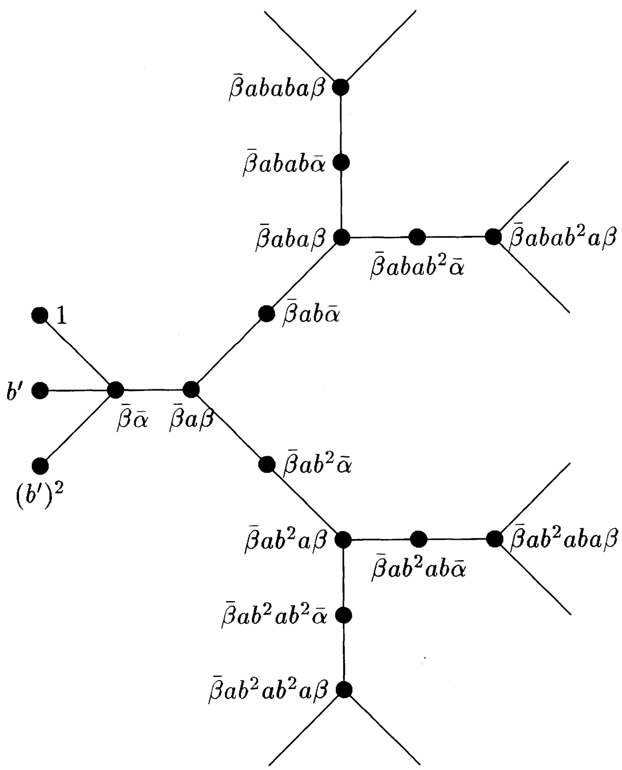

Fig. 20

Note that $\|\Gamma\|^2 = \|\Gamma'\|^2 = 3 + 2\sqrt{2} < 6$ by Corollary 4.8. In particular, $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ is an irreducible, nonamenable index 6 inclusion, whose principal graphs have exponential growth.

Examples 6.7. – *(Nonamenable inclusions based on the Fuchsian groups $G_{2,3,n}$, $n \geq 7$).*

By the remarks prior to example 6.4, the Fuchsian groups $G_{l,m,n}$, $\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1\right)$ can be realized as transformation groups of the hyperbolic plane $\mathfrak{H}$ such that $\mathfrak{H}/G_{l,m,n}$ are compact. In particular, $G_{l,m,n}$ becomes a lattice in the group $SO(1,2)$ of orientation preserving isometries of $\mathfrak{H}$. Therefore $G_{l,m,n}$ is nonamenable. Hence the inclusion $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ based on $G = G_{2,3,n}$, is nonamenable for every $n \geq 7$. The computation of the corresponding principal graphs Γ and Γ' is lengthy, so we will only state the results here. In analogy with Figure 12, Γ can be constructed from a tiling of the hyperbolic plane $\mathfrak{H}$ with regular n-gons all having angles $\frac{2\pi}{3}$, so that three n-gons will meet at each vertex. One identifies the points in the hyperbolic plane modulo rotation of 180° around the midpoint of a specified edge. Now Γ_{odd} consists of the vertices of the resulting tiling of $\mathfrak{H}/\mathbb{Z}_2$ and Γ_{even} consists of the midpoints of all the edges in this tiling with the exception of the rotation point, which corresponds to two vertices of Γ_{even} (namely 1 and a') as in the case $n = 6$. The description of Γ' is analogous to Figure 18. One identifies the points in the hyperbolic plane modulo a rotation of 120° around a specified vertex of the original tiling of $\mathfrak{H}$. Then Γ'_{odd} consists of all the midpoints if the edges of the resulting tiling of $\mathfrak{H}/\mathbb{Z}_3$ and Γ'_{even} consists of the vertices of this tiling with the exception of the rotation point, which corresponds to the three vertices of Γ'_{even} (namely 1, b' and $(b')^2$) as in the case $n = 6$.

7. Appendix

In the proof of Lemma 5.5 we used the following fact about Markov chains of finite sets. Since we have been unable to find a specific reference in the literature, we include here a proof.

PROPOSITION 7.1. – *Let (S, P) be an irreducible Markov chain on a finite set of states S with transition matrix $P = (P(x, y))_{x,y\in S}$. For $x \in S$, let $\theta_n(x)$ denote the probability that a path starting at x returns to x for the first time after exactly n steps. Then*

$$\sum_{n=1}^{\infty} n^p \theta_n(x) < \infty \tag{47}$$

for all $x \in S$ and all $p \in \mathbb{R}^+$.

Proof. – Since irreducible Markov chains on finite sets are positively recurrent (*see* [HPS, Section 2.4] and [Se, Definition 5.1]), we have

$$\sum_{n=1}^{\infty} \theta_n(x) = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} n\theta_n(x) < \infty.$$

In particular, (47) holds for $0 < p \leq 1$. To prove (47) for a general $p \in \mathbb{R}^+$, we will show that the radius of convergence of the power series $\sum_{n=1}^{\infty} s^n \theta_n(x)$ is strictly greater than 1. This immediately implies (47), because $\frac{n^p}{s^n} \to 0$ for $n \to \infty$ when $s > 1$. Set

$$L(s, x) = \sum_{n=1}^{\infty} s^n \theta_n(x), \quad |s| < 1, \quad s \in S$$

and

$$P(s,x,y) = \sum_{n=0}^{\infty} s^n P^n(x,y), \quad |s| < 1, \quad x,y \in S,$$

where P^n denotes the n-th power of the transition matrix P. Since for $|s| < 1$ we have

$$\sum_{n=0}^{\infty} s^n P^n = (1 - sP)^{-1} = \frac{1}{\det(1 - sP)} \operatorname{cof}(1 - sP)^t$$

by the cofactor formula for the inverse of a matrix, it follows that $P(s,x,y)$ can be extended to a meromorphic function on the complex plane for all $x,y \in S$. By ([Se, Lemma 5.3]) we have

$$P(s,x,x) = \frac{1}{1 - L(s,x)}, \qquad |s| < 1.$$

Since $\sum_{n=1}^{\infty} \theta_n(x) = 1$, we have $\lim_{s \to 1} -L(s,x) = 1$. Hence $P(s,x,x)$ has a pole at $s = 1$. Therefore

$$L(s,x) = 1 - \frac{1}{P(s,x,x)}$$

defines a meromorphic extension of $L(s,x)$ with a removable singularity at $s = 1$. Let $\rho(x) \in [1, \infty]$ denote the radius of convergence of the power series defining $L(s,x)$. Since $\theta_n(x) \geq 0$ for all $n \in \mathbb{N}$, we have that either $\rho(x) = \infty$ or $\rho(x)$ is a singular point for the meromorphic extension of $L(s,x)$. This proves that $\rho(x) > 1$ for $x \in S$. $\square$

REFERENCES

[ABH] J. Anderson, B. Blackadar and U. Haagerup, *Minimal projections in the reduced groups C^*-algebra of $\mathbb{Z}_n * \mathbb{Z}_M$*, (*J. of Operator Theory*, Vol. 26, 1991, pp. 3-23).

[Av] Avez, *Entropy des groupes de type fini* (*C. R. Acad. Sci. Paris*, Vol. 275A, 1972, pp. 1363-1366).

[BC] Ch. Berg and J. P. R. Christensen, *Sur la norme des opérateurs de convolution* (*Invent. math.*, Vol. 23, 1974, pp. 173-178).

[Bi1] D. Bisch, *Entropy of groups and subfactors* (*Journal of Funct. Analysis*, Vol. 103, 1992, pp. 190-208).

[Bi2] D. Bisch, *A note on intermediate subfactors* (*Pacific Journal of Math.*, Vol. 163, 1994, pp. 201-216).

[Bi3] D. Bisch, *On the structure of finite depth subfactors* (in *Algebraic Methods in Operator Theory, Birkhäuser*, 1994, pp. 175-194).

[Bu] W. Burnside, *Theory of groups of finite order*, Cambridge University Press, 1911.

[CD] G. Choquet, J. Deny, *Sur l'équation de convolution $\mu = \mu * \sigma$* (*C. R. Acad. Sci. Paris*, Vol. 250A, 1960, pp. 799-801).

[Co] A. Connes, *Notes on correspondences*, unpublished manuscript.

[CM] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups* (*Springer-Verlag*, New York, Berlin, Heidelberg, 2nd edition, 1972).

[DG] Y. DERRIENNIC and Y. GUIVARC'H, *Théorème de renouvellement pour les groups non-moyennables* (*C. R. Acad. Sci. Paris*, Série A, Vol. 277, 1973, pp. 613-615).

[Fu] H. FURSTENBERG, *Random walks and discrete subgroups of Lie groups* (in *Adv. Prob. Related Topics*, Dekker, New York, Vol. 1, 1971, pp. 1-63).

[GHJ] F. GOODMAN, P. de la HARPE, and V. JONES, *Coxeter graphs and towers of algebras, Springer-Verlag*, MSRI publications, 1989.

[G] F. P. GREENLEAF, *Invariant means on topological groups and their applications*, Van Nostrand, New York, 1969.

[Ha] U. HAAGERUP, *Principal graphs of subfactors in the index range* $4 < [M : N] < 3 + \sqrt{2}$ (in "Subfactors", *Proceedings of the Taniguchi Symposium on Operator Algebras*, 1994, pp. 1-38).

[Hi] F. HIAI, *Entropy and growth for derived towers of subfactors* (in "Subfactors", *Proceedings of the Taniguchi Symposium on Operator Algebras*, 1994, pp. 206-232).

[HPS] G. HOLE, S. C. PORT and C. J. STONE, *Introduction to stochastical processes*, Houghton Mifflin Co., 1972.

[Jo] V. F. R. JONES, *Index for subfactors* (*Invent. Math.*, Vol. 72, 1983, pp. 1-25).

[KV] V. A. KAIMANOVICH and A. M. VERSHIK, *Random walks on discrete groups: Boundary and entropy* (*Ann. Probab.*, Vol. 11, No. 3, 1983, pp. 457-490).

[Ka1] V. A. KAIMANOVICH, *Poisson boundaries of random walks on discrete solvable groups* (Proc. 10th Oberwolfach Conf. on *Prob. Measures on groups*, ed. H. Heyer, Plenum, New York, 1991, pp. 205-238).

[Ka2] V. A. KAIMANOVICH, *The Poisson boundary of covering Markov operators*, to appear in *Israel Journal of Math.*

[Ka3] V. A. KAIMANOVICH, *Bi-harmonic functions on groups* (*C. R. Acad. Sci. Paris*, Vol. 314, 1992, pp. 259-262).

[Ka4] V. A. KAIMANOVICH, *Examples of non-commutative groups with non-trivial exit boundary* (*J. Soviet Math.*, Vol. 28, 1985, pp. 579-591).

[Ke] H. KESTEN, *Symmetric random walks on groups* (*Trans. Amer. Math. Soc.*, Vol. 92, 1959, pp. 336-354).

[K] A. W. KNAPP, *Doubly generated Fuchsian groups* (*Michigan Math. Journal*, Vol. 15, 1968, pp. 289-304).

[M] J. P. MATELSKI, *The classification of discrete 2-generator subgroups of* $PSL(2, \mathbb{R})$ (*Israel J. Math.*, Vol. 42, 1982, pp. 309-317).

[Me] J. L. MENNICKE, *Finite factor groups of the unimodular group* (*Ann. of Math.*, Vol. 81, 1965, pp. 31-37).

[Ne] M. NEWMAN, *The structure of some subgroups of the modular group* (*Ill. Jour. of Math.*, Vol. 6, 1962, pp. 480-487).

[Oc1] A. OCNEANU, *Quantized group string algebras and Galois theory for operator algebras, in Operator Algebras and Applications 2* (*London Math. Soc. Lect.*, Notes Series, Vol. 136, 1988, pp. 119-172).

[Oc2] A. OCNEANU, *Quantum symmetry, differential geometry of finite graphs and classification of subfactors* University of Tokyo Seminary Notes (notes recorded by Y. Kawahigashi), Vol. 45, 1991.

[Po1] S. POPA, *Correspondences*, Increst preprint, 1986.

[Po2] S. POPA, *Classification of subfactors: reduction to commuting squares* (*Invent. Math.*, Vol. 101, 1990, pp. 19-43).

[Po3] S. POPA, *Sous-facteurs, actions des groupes et cohomologie* (*C. R. Acad. Sci. Paris*, Vol. 309, 1989, pp. 771-776).

[Po4] S. POPA, *Classification of amenable subfactors of type II* (*Acta Math.*, Vol. 172, 1994, pp. 352-445).

[Po5] S. POPA, *Symmetric enveloping algebras, amenability and AFD properties for subfactors* (*Mathematical Research Letters*, Vol. 1, 1994, pp. 409-425).

[Po6] S. POPA, *Approximate innerness and central freeness for subfactors: A classification result*, preprint 1993.

[R] G. ROSENBERGER, *All generating pairs of all two-generator Fuchsian groups* (*Arch. Math.*, Vol 46, 1986, pp. 198-204).

[Se] E. SENETA, *Non-negative matrices and Markov chains*, Springer Verlag, 1981.

[Sp] F. SPITZER, *Principles of Random walks*, 2nd ed. Springer Verlag, New York, Heidelberg, Berlin, 1976.

[Su] V. S. SUNDER, II_1 *factors, their bimodules and hypergroups* (*Trans. Amer. Math. Soc.*, Vol. 330, 1992, pp. 227-256).

[Sz] W. SZYMANSKI, *Finite index subfactors and Hopf algebra crossed products* (*Proc. of the Amer. Math. Soc*, Vol. 120, 1994, pp. 519-528).

[Voi] D. VOICULESCU, *Symmetries of some reduced free product* C^**-algebras* (in Operator Algebras and Their Connections with Topology and Ergodic Theory, *Lecture Notes in Mathematics*, Springer Verlag, Vol. 1132, 1985, pp. 556-588).

[Wi] J. Wierzbicki, *The estimation of the depth from an intermediate subfactor*, preprint 1993.

[Wo] W. Woess, *Random walks on infinite graphs and groups - a survey on selected topics* (*Bull. London Math. Soc.*, Vol. 26, 1994, pp. 1-60).

[Ya] S. Yamagami, *A note on Ocneanu's approach to Jones' index theory* (*Internat. J. of Math.*, Vol. 4, 1993, pp. 859-871).

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D. Bisch, U. Haagerup
UC Berkeley,
Department of Mathematics Berkeley,
CA 94720, USA
and
Odense Universitet,
Department of Mathematics,
DK-5230 Odense M,
Denmark.