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ON THE ZETA FUNCTION OF A COMPLETE INTERSECTION

BY ALAN ADOLPHSON* AND STEVEN SPERBER

ABSTRACT. – In this article, we compute the p-adic Dwork cohomology of a smooth complete intersection in $\mathbf{T}^m \times \mathbf{A}^n$ or $\mathbf{P}^N$ over a finite field (where $\mathbf{T}^m$ is the m-torus). As an application, we prove the "Katz Conjecture" (*i.e.*, the assertion that the Newton polygon lies over the Hodge polygon) for such varieties. This result is new in the case of $\mathbf{T}^m \times \mathbf{A}^n$. (The case of $\mathbf{P}^N$ is due to Mazur [14].)

1. Introduction

In [9], Dwork developed a p-adic cohomology theory for smooth projective hypersurfaces over finite fields. Given $f \in \mathbf{F}_q[x_0, x_1, \ldots, x_N]$, a form of degree d defined over the field of $q = p^a$ elements, Dwork constructed a complex $K^{\mathrm{Dw}}_{\cdot}(f)$ of p-adic Banach spaces. When the hypersurface $V(f)$ defined by the vanishing of f in $\mathbf{P}^N$ is nonsingular and has nonsingular intersection with every coordinate variety $H_A = \bigcap_{i \in A} \{x_i = 0\}$, where $A \subset S = \{0, \ldots, N\}$, $A \neq S$, then the complex $K^{\mathrm{Dw}}_{\cdot}(f)$ is acyclic except in degree 0. The characteristic polynomial of Frobenius acting on H_0 gives the primitive part of the middle-dimensional factor of the zeta function of $V(f)$. From this vantage point, there remained the problems of extending this work to varieties other than hypersurfaces, as well as to treat even in the hypersurface case open or singular varieties. Of course, the development of crystalline cohomology and rigid cohomology provided an excellent basis for these generalizations.

Our goal in the present paper is to use the approach of exponential modules or twisted de Rham theory pioneered by Dwork in the hypersurface case to treat complete intersections. In this we are continuing the early work of Ireland [11] and Barshay [4], who studied projective and multiprojective complete intersections from this point of view also. In their work, they constructed a complex of p-adic Banach spaces $K^{\mathrm{Dw}}_{\cdot}$ (related to the complex $K.(S, S)$ of section 6 below), proved the acyclicity except in degree 0 of this complex in the smooth case, and related the characteristic polynomial of Frobenius acting on H_0 to the zeta function of the complete intersection defined by the simultaneous vanishing of forms $f_1, \ldots, f_r \in \mathbf{F}_q[x_0, \ldots, x_N]$ in $\mathbf{P}^N$. Specifically, they showed that this characteristic

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polynomial equals a product of certain factors (which the Weil conjectures imply are polynomials), from which they concluded this polynomial has the correct degree. They were unable to show the factors themselves are polynomials. In particular, they were unable to construct a finite-dimensional p-adic vector space with action of Frobenius whose characteristic polynomial is the interesting factor of the zeta function of a smooth projective complete intersection. We construct such a theory here.

The main application of our work that we give here is a proof of the "Katz Conjecture" (*i.e.*, the assertion that the Newton polygon lies above the Hodge polygon: see [14]) for general smooth complete intersections in an affine space or a torus (as well as another proof in the projective case). Previously, such results were known only in the proper case. We note also that our approach eliminates the need to treat separately the case of hypersurfaces of degree divisible by p (compare [9], [10]). In a future article, we plan to describe the relation between the theory developed here and classical de Rham cohomology. In particular, we believe that the description we give here of middle-dimensional cohomology and of a procedure for finding a basis for it should be useful in calculations involving the Gauss-Manin connection.

We describe our results more precisely. In the present work we study both open smooth complete intersections (in $\mathbf{T}^m \times \mathbf{A}^n$) in sections 2-5 and projective smooth complete intersections in sections 6-7. In the open case, we let $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_{m+n}, (x_1 \cdots x_m)^{-1}]$ be Laurent polynomials and V be the variety in $\mathbf{T}^m \times \mathbf{A}^n$ defined by the simultaneous vanishing of the f_i's. If we set $g = \sum_{j=1}^{r} x_{m+n+j} f_j(x_1, \ldots, x_{m+n})$, then is is well-known that

$$L(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r, g; t) = Z(V/\mathbf{F}_q; q^r t),$$

where the right-hand side is the zeta function of V and the left-hand side is the L-function of the exponential sum associated to g. It is also known from our earlier work [1] that, with $S = \{1, \ldots, m+n+r\}$, $S_{\mathrm{af}} = \{m+1, \ldots, m+n+r\}$, there is a complex of p-adic Banach spaces $K.(S, S_{\mathrm{af}})$ which satisfies

$$(1.1) \qquad L(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r, g; t)^{(-1)^{m+n+r+1}} = \det(I - t\,\mathrm{Frob} \mid K.(S, S_{\mathrm{af}})),$$

where the right-hand side is shorthand for the alternating product of characteristic series of Frobenius acting on the complex $K.(S, S_{\mathrm{af}})$. While we have studied L-functions such as (1.1) in the past, our earlier results need further refinements here. Even if we make appropriate hypothesis on the f_i to ensure that g is nondegenerate (in the sense of Kouchnirenko [13]), the polynomial g is not commode (in the sense of [1]) with respect to the set $S_{\mathrm{du}} = \{m+n+1, \ldots, m+n+r\}$ (and therefore, *a fortiori*, not commode with respect to S_{af}). Setting $x_{m+n+1} = \cdots = x_{m+n+r} = 0$ in g gives the zero polynomial, *i.e.*, this substitution causes the dimension of its Newton polyhedron to drop by $m+n+r$, rather than simply r which is what the condition of being commode requires. We are nevertheless able to calculate $H_*(K.(S, S_{\mathrm{af}}))$ in the open case of a smooth complete intersection V in $\mathbf{T}^m \times \mathbf{A}^n$. Using this result, we are able to show that the Newton polygon of the primitive part of the middle-dimensional factor of the zeta function of V lies over its corresponding Hodge polygon.

In the projective case we change our enumeration and notation, letting $f_1, \ldots, f_r$ be forms in $\mathbf{F}_q[x_0, \ldots, x_N]$. Let $g = \sum_{j=1}^{r} x_{N+j} f_j(x_0, \ldots, x_N)$, $S = \{0, 1, \ldots, N+r\}$, $S_{\mathrm{sp}} = \{0, 1, \ldots, N\}$, $S_{\mathrm{du}} = \{N+1, \ldots, N+r\}$. The projective case is thus more complicated even when g is nondegenerate since the Newton polyhedron has dimension $N+r$ rather that $N+r+1$ $(= |S|)$ and since g vanishes when one specializes all the variables in either S_{sp} or S_{du} to be 0. As we noted above, Ireland [11] was able to compute the homology of $K_{\cdot}^{\mathrm{Dw}}$ in the smooth case. Here we compute the homology of $K_{\cdot}(S, S)$ and show that the characteristic polynomial of Frobenius acting on H_0 gives the primitive part of the middle-dimensional factor of the zeta function of the smooth projective complete intersection defined by the simultaneous vanishing of the f_i's in $\mathbf{P}^N$. Our main technical tools in this analysis are some properties of Koszul complexes which we specify explicitly in the appendix. It is interesting that these properties are, in the case of a hypersurface defined by the vanishing of a form f, sufficient to guarantee that

$$\det(I - t\,\mathrm{Frob} \mid K_{\cdot}^{\mathrm{Dw}}(f))^{(-1)^{N+1}} = Z(V(f)/\mathbf{F}_q; qt)(1-qt)\cdots(1-q^N t)$$

even in the case f is singular. In the projective case, we also compare Newton and Hodge polygons giving another proof of Mazur's theorem [14]. In the course of this, we specify a basis (valid for a Zariski open set of the moduli space of complete intersections with specified degrees) for the primitive middle-dimensional cohomology of a projective complete intersection, which may prove useful in explicit calculations.

The outline of the paper is as follows. In section 2, we compute the homology of the complex $K_{\cdot}(S, S_{\mathrm{af}})$, which gives the zeta function of a smooth complete intersection in $\mathbf{T}^m \times \mathbf{A}^n$ (Theorem 2.19). In order to estimate the Newton polygon of Frobenius acting on homology, we need a more precise description of a basis for these homology spaces. In section 3, we obtain such a description for $\bar{K}_{\cdot}'(S, S_{\mathrm{af}})$, the "reduction mod p" of $K_{\cdot}(S, S_{\mathrm{af}})$, by first obtaining such a description for its associated graded complex $\bar{K}_{\cdot}(S, S_{\mathrm{af}})$ (Theorems 3.26 and 3.37). We then explain in section 4 how this lifts to a basis for the homology of $K_{\cdot}(S, S_{\mathrm{af}})$, which in turn leads to a lower bound for the Newton polygon of Frobenius acting on homology (Theorem 4.13). In section 5, we identify this lower bound with a Hodge polygon by using the ideas of [6] to explicitly compute the Hodge polygon of a general complete intersection in $(\mathbf{T}^m \times \mathbf{A}^n)_{\mathbf{C}}$. In sections 6 and 7, we repeat the above procedure for smooth complete intersections in $\mathbf{P}^N$. Here some of our arguments are sketchier, because they are analogous to the case $\mathbf{T}^m \times \mathbf{A}^n$ and because the result in the projective case is already known [14]. In section 8, we collect some general results on complexes that are useful in sections 6 and 7.

2. Cohomology of toric and affine complete intersections

Let p be a prime number, $q = p^a$, and let $\mathbf{F}_q$ be the finite field of q elements. Let $\mathbf{T}^m$ be the m-torus over $\mathbf{F}_q$ (*i.e.*, $\mathbf{T}^m$ is the product over $\mathbf{F}_q$ of m copies of the multiplicative group) and let $\mathbf{A}^n$ be affine n-space over $\mathbf{F}_q$. Put $N = m + n$. Take $f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_N, (x_1 \cdots x_m)^{-1}]$ and let $V \subseteq \mathbf{T}^m \times \mathbf{A}^n$ be the variety

$f_1 = \cdots = f_r = 0$. We allow the possibility that m or n is zero. Let $V(\mathbf{F}_{q^s})$ be the set of $\mathbf{F}_{q^s}$-rational points of V and let $N_s(V)$ be its cardinality. The zeta function of V is defined to be

$$Z(V/\mathbf{F}_q;t) = \exp\biggl(\sum_{s=1}^{\infty} N_s(V)\frac{t^s}{s}\biggr).$$

We begin by re-expressing $Z(V/\mathbf{F}_q;t)$ in terms of L-functions of certain exponential sums.

Fix a nontrivial additive character $\Psi : \mathbf{F}_q \to \mathbf{C}^\times$ and let $\Psi_s = \Psi \circ \mathrm{Trace}_{\mathbf{F}_{q^s}/\mathbf{F}_q} : \mathbf{F}_{q^s} \to \mathbf{C}^\times$. For an $\mathbf{F}_q$-regular function h on an $\mathbf{F}_q$-variety X, define

$$S_s(X,h) = \sum_{x\in X(\mathbf{F}_{q^s})} \Psi_s(h(x)),$$

$$L(X,h;t) = \exp\biggl(\sum_{s=1}^{\infty} S_s(X,h)\frac{t^s}{s}\biggr).$$

We introduce dummy variables $x_{N+1},\dots,x_{N+r}$ and put

$$g = x_{N+1}f_1(x_1,\dots,x_N) + \cdots + x_{N+r}f_r(x_1,\dots,x_N) \in \mathbf{F}_q[x_1,\dots,x_{N+r},(x_1\cdots x_m)^{-1}].$$

It is easily seen that

$$q^{rs}N_s(V) = \sum_{x_1,\dots,x_m\in\mathbf{F}_{q^s}^\times}\ \sum_{x_{m+1},\dots,x_{N+r}\in\mathbf{F}_{q^s}} \Psi_s(g(x_1,\dots,x_{N+r})), \tag{2.1}$$

or equivalently,

$$Z(V/\mathbf{F}_q;q^rt) = L(\mathbf{T}^m\times\mathbf{A}^n\times\mathbf{A}^r,g;t). \tag{2.2}$$

Put $\mathbf{A}' = \mathbf{A}^r\setminus\{(0,\dots,0)\}$. Since g vanishes identically on $\mathbf{T}^m\times\mathbf{A}^n\times\{(0,\dots,0)\}$, an easy calculation gives

$$Z(V/\mathbf{F}_q;q^rt) = L(\mathbf{T}^m\times\mathbf{A}^n\times\mathbf{A}',g;t)\prod_{j=0}^{m}(1-q^{j+n}t)^{\binom{m}{j}(-1)^{m-j-1}}. \tag{2.3}$$

We regroup terms as follows. Define a rational function $P(t)$ by

$$P(t) = \biggl(L(\mathbf{T}^m\times\mathbf{A}^n\times\mathbf{A}',g;t)\prod_{j=0}^{r-n-1}(1-q^{j+n}t)^{\binom{m}{j}(-1)^{m-j-1}}\biggr)^{(-1)^{N-r-1}}, \tag{2.4}$$

so that (2.3) becomes

$$Z(V/\mathbf{F}_q;q^rt) = (P(t)(1-q^rt)^{\binom{m}{r-n}})^{(-1)^{N-r-1}}\prod_{j=1}^{N-r}(1-q^{j+r}t)^{\binom{m}{j+r-n}(-1)^{N-r-j-1}}, \tag{2.5}$$

where we understand $\binom{a}{b} = 0$ if $b<0$ or $b>a$. We shall identify the factors on the right-hand side with the action of Frobenius on p-adic cohomology spaces when V is sufficiently smooth. In particular, $P(t)$ will be the polynomial corresponding to the action of Frobenius on the primitive part of middle-dimensional cohomology. For purposes of induction on r, it will be convenient to allow $r = 0$. In this situation, we understand that $V = \mathbf{T}^m \times \mathbf{A}^n$ and $P(t) = 1$.

We apply the theory of [1] to the L-function $L(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r, g; t)$. For a Laurent polynomial h over any field in variables $x_1, x_1^{-1}, \dots, x_k, x_k^{-1}$, we denote by $\operatorname{supp}(h) \subset \mathbf{R}^k$ the set of exponents of the monomials appearing in h, thought of as lattice points in $\mathbf{R}^k$. Let $\Delta \subset \mathbf{R}^{N+r}$ be the convex hull of the origin and $\operatorname{supp}(g)$. Let $C(\Delta)$ be the real cone generated by Δ, *i.e.*, the collection of all nonnegative real multiples of points of Δ. Put $M = \mathbf{Z}^{N+r} \cap C(\Delta)$. For $u = (u_1, \dots, u_{N+r}) \in C(\Delta)$, define its *weight* $w(u)$ by $w(u) = u_{N+1} + \cdots + u_{N+r}$.

Let $\Omega_0 = \mathbf{Q}_p(\zeta_p, \zeta_{q-1})$, where $\mathbf{Q}_p$ denotes the p-adic numbers and ζ_p and ζ_{q-1} denote primitive p-th and $(q-1)$-st roots of unity, respectively. We normalize the p-adic valuation ord on Ω_0 by setting ord $p = 1$. Let $\mathcal{O}_0$ be the ring of integers of Ω_0 and let $\pi \in \mathcal{O}_0$ be a uniformizing parameter, so ord $\pi = 1/(p-1)$. A key role will be played by the p-adic Banach space B and its unit ball $B(\mathcal{O}_0)$:

$$B = \{\sum_{u \in M} A_u \pi^{w(u)} x^u \mid A_u \in \Omega_0,\ A_u \to 0 \text{ as } u \to \infty\},$$
$$B(\mathcal{O}_0) = \{\sum_{u \in M} A_u \pi^{w(u)} x^u \in B \mid A_u \in \mathcal{O}_0 \text{ for all } u\}.$$

The norm on B is defined by

$$\|\sum_{u \in M} A_u \pi^{w(u)} x^u\| = \sup_{u \in M} |A_u|.$$

It will be useful to consider some related spaces as well. For $b, c \in \mathbf{R}$, $b > 0$, define

$$L(b, c) = \{\sum_{u \in M} A_u x^u \mid A_u \in \Omega_0,\ \operatorname{ord} A_u \geq b w(u) + c\},$$
$$L(b) = \bigcup_{c \in \mathbf{R}} L(b, c).$$

Note that we have $L(b) \subset B \subset L(1/(p-1))$ for $b > 1/(p-1)$.

Define an operator ψ on formal power series over M by

$$\psi(\sum_{u \in M} A_u x^u) = \sum_{u \in M} A_{pu} x^u.$$

Observe that $\psi(L(b, c)) \subset L(pb, c)$. Associated to g is a series $F_0(x) \in L(p/q(p-1), 0)$ (see [1, equation (1.15)]) with the following property. Put $\alpha = \psi^a \circ F_0$, the composition of ψ^a with multiplication by $F_0(x)$, where $a = [\mathbf{F}_q : \mathbf{F}_p]$. Then α is a completely continuous Ω_0-linear endomorphism of B and of $L(b)$ for $0 < b \leq p/(p-1)$. Furthermore,

$$L(\mathbf{T}^{N+r}, g; t)^{(-1)^{N+r-1}} = \det(I - t\alpha)^{\delta^{N+r}}, \tag{2.6}$$

where $\det(I - t\alpha)$ is the Fredholm determinant of α as operator on B or any of the $L(b)$, $0 < b < p/(p-1)$, and δ is the operator on formal power series in t with constant term 1 defined by $h(t)^{\delta} = h(t)/h(qt)$.

We now explain how to modify this purely toric case to obtain $L(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r, g; t)$. Different variables will play different roles in the argument, and we index them accordingly. The set of all variables is indexed by $S = \{1, \ldots, N+r\}$. Toric variables are indexed by the set $S_{\text{to}} = \{1, \ldots, m\}$, affine variables by $S_{\text{af}} = \{m+1, \ldots, N+r\}$, space variables by $S_{\text{sp}} = \{1, \ldots, N\}$, dummy variables by $S_{\text{du}} = \{N+1, \ldots, N+r\}$. For any subset $I \subseteq S$, we use subscripts to denote its intersection with these subsets, e. g., $I_{\text{to}} = I \cap S_{\text{to}}$. For any finite set I, we let $|I|$ denote its cardinality.

Fix $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and let $g_I \in \mathbf{F}_q[\{x_i\}_{i \in I}, (x_1, \ldots, x_m)^{-1}]$ be the polynomial obtained from g by setting $x_j = 0$ for $j \in S_{\text{af}} \setminus I$. For $j \in S_{\text{af}}$, let $\theta_j : B \to B$ be the map "set $x_j = 0$" and let θ_I be the composition $\theta_I = \prod_{j \in S_{\text{af}} \setminus I} \theta_j$. We define $B_I = \theta_I(B)$, $B_I(\mathcal{O}_0) = \theta_I(B(\mathcal{O}_0))$, $L_I(b,c) = \theta_I(L(b,c))$, $L_I(b) = \theta_I(L(b))$, and $\alpha_I = \psi^a \circ \theta_I(F_0)$. We observe that by [1]

$$L(\mathbf{T}^{|I|}, g_I; t)^{(-1)^{|I|-1}} = \det(I - t\alpha_I)^{\delta^{|I|}}, \tag{2.7}$$

where α_I may be regarded as acting on B_I or any of the $L_I(b)$, $0 < b \leq p/(p-1)$.

Equation (2.6) may be interpreted as follows. In [1, section 2] it is shown that there exist elements $H_i \in L(p/(p-1), -1)$ such that the (commuting) differential operators $D_i = x_i \partial/\partial x_i + H_i$, $i = 1, \ldots, N+r$, satisfy

$$\alpha \circ D_i = q D_i \circ \alpha \tag{2.8}$$

as operators on B or $L(b)$, $0 < b \leq p/(p-1)$. Let $K. = K.(B, \{D_i\}_{i=1}^{N+r})$ be the Koszul complex on B defined by $D_1, \ldots, D_{N+r}$. For $0 \leq l \leq N+r$, its component of degree l is

$$K_l = \bigoplus_{|A|=l} B\, e_A,$$

where the sum is over all subsets $A \subseteq S$ of cardinality l and e_A is a formal symbol. The boundary map $\partial_l : K_l \to K_{l-1}$ is given by: if $\xi \in B$ and $A = \{i_1, \ldots, i_l\}$ with $i_1 < \cdots < i_l$, then

$$\partial_l(\xi\, e_A) = \sum_{j=1}^{l} (-1)^{j-1} D_{i_j}(\xi)\, e_{A \setminus \{i_j\}}.$$

Define an endomorphism $\alpha_l : K_l \to K_l$ by

$$\alpha_l = \bigoplus_{|A|=l} q^l \alpha.$$

Then (2.8) implies that $\alpha.$ is a chain map on $K.$, hence by (2.6)

$$L(\mathbf{T}^{N+r}, g; t)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - t\alpha_l \mid K_l)^{(-1)^l}. \tag{2.9}$$

Passing to homology, we have

$$L(\mathbf{T}^{N+r}, g; t)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - t\bar{\alpha}_l \mid H_l(K.))^{(-1)^l}, \tag{2.10}$$

where $\bar{\alpha}_l$ is the endomorphism of $H_l(K.)$ induced by α_l on K_l.

For $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$, put $D_{I,i} = x_i\partial/\partial x_i + \theta_I(H_i)$. We denote by $K.(I)$ the Koszul complex on B_I formed by the operators $D_{I,i}$ for $i \in I$. In particular,

$$K_l(I) = \bigoplus_{A \subseteq I,\ |A|=l} B_I\, e_A.$$

We have in analogy with (2.10)

$$L(\mathbf{T}^{|I|}, g_I; t)^{(-1)^{|I|-1}} = \prod_{l=0}^{|I|} \det(I - t\bar{\alpha}_{I,l} \mid H_l(K.(I)))^{(-1)^l}. \tag{2.11}$$

Usually, no confusion will result if we denote $\bar{\alpha}_{I,l}$ simply by $\bar{\alpha}_l$ or even $\bar{\alpha}$.

The complexes $K.(I)$ can be tied together by introducing some subcomplexes $K.(I, I')$. For $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$ and $I' \subseteq I_{\mathrm{af}}$, let

$$B_I^{I'} = \bigcap_{i \in I'} \ker(\theta_i \mid B_I),$$

i.e., $B_I^{I'}$ consists of all elements of B_I that are divisible by x_i for all $i \in I'$. Let $K.(I, I')$ be the subcomplex of $K.(I)$ defined by

$$K_l(I, I') = \bigoplus_{A \subseteq I,\ |A|=l} B_I^{I' \setminus A}\, e_A.$$

Note that for $i \in I_{\mathrm{af}}$, $i \notin I'$, there is an exact sequence

$$0 \to B_I^{I' \cup \{i\}} \to B_I^{I'} \xrightarrow{\theta_i} B_{I\setminus\{i\}}^{I'} \to 0,$$

which induces an exact sequence of complexes

$$0 \to K.(I, I' \cup \{i\}) \to K.(I, I') \to K.(I \setminus \{i\}, I') \to 0. \tag{2.12}$$

Using this exact sequence, equation (2.11), and induction on $|I'|$, the natural toric decomposition of $\mathbf{A}^{|I'|}$ gives

$$L(\mathbf{T}^{|I|-|I'|} \times \mathbf{A}^{|I'|}, g_I; t)^{(-1)^{|I|-1}} = \prod_{l=0}^{|I|} \det(I - t\bar{\alpha} \mid H_l(K.(I, I')))^{(-1)^l}. \tag{2.13}$$

To be precise, the left-hand side is the L-function corresponding to the exponential sum $\sum \Psi(g_I(x))$, where x_i runs over $\mathbf{F}_q$ for $i \in I'$ and over $\mathbf{F}_q^\times$ for $i \in I$, $i \notin I'$. In particular, taking $I = S$, $I' = S_{\mathrm{af}}$, we have by (2.2) that

$$Z(V/\mathbf{F}_q; q^r t)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - t\bar{\alpha} \mid H_l(K.(S, S_{\mathrm{af}})))^{(-1)^l}. \tag{2.14}$$

Let $\Delta_I \subset \mathbf{R}^{|I|}$ be the convex hull of the origin and $\mathrm{supp}(g_I)$. In particular, $\Delta = \Delta_S$. We make two assumptions about g. We assume that for every subset $I \subseteq S$ such that $S_{\mathrm{to}} \subseteq I$ and $I_{\mathrm{du}} \neq \emptyset$ we have $\dim \Delta_I = |I|$. We call g *semi-convenient* when this condition is satisfied. (When g is semi-convenient, g will be "commode" in the sense of [1] with respect to any subset $I \subset S_{\mathrm{af}}$ such that $S_{\mathrm{du}} \not\subseteq I$.) When $n = 0$, this is equivalent to requiring that the convex hull of $\mathrm{supp}(f_i)$ have dimension m for $i = 1, \ldots, r$. When $m = 0$, it is equivalent to requiring that each f_i contain terms $a_{ij}x_j^{k_{ij}}$ for $j = 1, \ldots, n$, with $a_{ij} \neq 0$ and $k_{ij} > 0$, and that $f_i(0, \ldots, 0) \neq 0$ for $i = 1, \ldots, r$. We also assume that g is nondegenerate (*see* [1, section 2]), which implies that g_I is also nondegenerate for all $I \subseteq S$. Geometrically, this means that for all $I \subseteq S_{\mathrm{sp}}$ with $S_{\mathrm{to}} \subseteq I$ and $J \subseteq \{1, \ldots, r\}$, the equations $\theta_I(f_j) = 0$ for $j \in J$ define a smooth complete intersection X in the torus $\mathbf{T}^{|I|}$ and that there is a compactification Y of the torus in which its closure $\bar{X}$ is smooth and meets all orbits transversally. This condition is generically satisfied ([3], [12]).

By [1, Theorems 2.9 and 3.13] we have the following.

THEOREM 2.15. – *Suppose that g is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$ and $I' \subseteq I_{\mathrm{af}}$ with $I'_{\mathrm{du}} \neq I_{\mathrm{du}}$, we have $H_l(K.(I, I')) = 0$ for $l > 0$, $\bar{\alpha}$ is invertible on $H_0(K.(I, I'))$, and*

$$\dim_{\Omega_0} H_0(K.(I, I')) = \sum_{I \setminus I' \subseteq J \subseteq I} (-1)^{|I|-|J|}(|J|)!\,\mathrm{Vol}(\Delta_J), \tag{2.16}$$

where $\mathrm{Vol}(\Delta_J)$ denotes the volume of Δ_J relative to Lebesgue measure on $\mathbf{R}^{|J|}$. Thus by (2.13),

$$L(\mathbf{T}^{|I|-|I'|} \times \mathbf{A}^{|I'|}, g_I; t)^{(-1)^{|I|-1}} = \det(I - t\bar{\alpha} \mid H_0(K.(I, I')))$$

is a polynomial whose degree is given by (2.16).

Using this theorem and induction on r, we shall compute the homology of $K.(S, S_{\mathrm{af}})$, which by (2.14) is the complex that gives the zeta function $Z(V/\mathbf{F}_q; q^r t)$. First, for later application, we modify the formula (2.16) for $\dim_{\Omega_0} H_0(K.(I, I'))$. For $S_{\mathrm{to}} \subseteq I \subseteq S_{\mathrm{sp}}$, let $\Delta_i^I \subset \mathbf{R}^{|I|}$ denote the convex hull of $\mathrm{supp}(\theta_I(f_i))$, $i = 1, \ldots, r$. Regarding $\mathbf{R}^{|J|}$ as $\mathbf{R}^{|J_{\mathrm{sp}}|} \times \mathbf{R}^{|J_{\mathrm{du}}|}$, the projection of $\Delta_J \subseteq \mathbf{R}^{|J|}$ on $\mathbf{R}^{|J_{\mathrm{du}}|}$ is the simplex

$$\Lambda = \{(\lambda_j)_{j \in J_{\mathrm{du}}} \mid \sum_{j \in J_{\mathrm{du}}} \lambda_j \leq 1,\ \lambda_j \geq 0 \text{ for all } j\}.$$

The fiber of Δ_J over $(\lambda_j)_{j \in J_{\mathrm{du}}} \in \Lambda$ is the Minkowski sum $\sum_{j \in J_{\mathrm{du}}} \lambda_j \Delta_j^{J_{\mathrm{sp}}}$, thus

$$\mathrm{Vol}(\Delta_J) = \int_\Lambda \mathrm{Vol}\Big(\sum_{j \in J_{\mathrm{du}}} \lambda_j \Delta_j^{J_{\mathrm{sp}}}\Big) \bigwedge_{j \in J_{\mathrm{du}}} d\lambda_j,$$

where $\mathrm{Vol}(\sum_{j\in J_{\mathrm{du}}} \lambda_j \Delta_j^{J_{\mathrm{sp}}})$ denotes volume relative to Lebesgue measure on $\mathbf{R}^{|J_{\mathrm{sp}}|}$. It is a theorem of Minkowski that $\mathrm{Vol}(\sum_{j\in J_{\mathrm{du}}} \lambda_j \Delta_j^{J_{\mathrm{sp}}})$ is a homogeneous polynomial of degree $|J_{\mathrm{sp}}|$ in $\{\lambda_j\}_{j\in J_{\mathrm{du}}}$, specifically,

$$\mathrm{Vol}(\sum_{j\in J_{\mathrm{du}}} \lambda_j \Delta_j^{J_{\mathrm{sp}}}) = \sum_{\sum_{j\in J_{\mathrm{du}}} l_j = |J_{\mathrm{sp}}|} \frac{(|J_{\mathrm{sp}}|)!}{\prod_{j\in J_{\mathrm{du}}} l_j!} M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}}) \prod_{j\in J_{\mathrm{du}}} \lambda_j^{l_j},$$

where $M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}})$ denotes the Minkowski mixed volume of the collection of $|J_{\mathrm{sp}}|$ polytopes in $\mathbf{R}^{|J_{\mathrm{sp}}|}$ obtained by taking $\Delta_j^{J_{\mathrm{sp}}}$ with multiplicity l_j. Induction on $|J_{\mathrm{du}}|$ shows that

$$\int_\Lambda \prod_{j\in J_{\mathrm{du}}} \lambda_j^{l_j} \bigwedge_{j\in J_{\mathrm{du}}} d\lambda_j = \frac{\prod_{j\in J_{\mathrm{du}}} l_j!}{(|J_{\mathrm{du}}| + \sum_{j\in J_{\mathrm{du}}} l_j)!} = \frac{\prod_{j\in J_{\mathrm{du}}} l_j!}{(|J|)!},$$

hence evaluation of the above integral for $\mathrm{Vol}(\Delta_J)$ gives

$$(|J|)!\,\mathrm{Vol}(\Delta_J) = (|J_{\mathrm{sp}}|)! \sum_{\sum_{j\in J_{\mathrm{du}}} l_j = |J_{\mathrm{sp}}|} M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}}). \tag{2.17}$$

When $\Delta_j^{J_{\mathrm{sp}}}$ is a single point and $l_j = 0$ for all $j \in J_{\mathrm{du}}$, we define $M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}}) = 1$, so that (2.17) remains valid even when $J_{\mathrm{sp}} = \emptyset$. (This case will arise when we try to generalize (2.16) to the case where $I' = I_{\mathrm{af}}$ when $m = 0$.)

Substituting the right-hand side of (2.17) into the right-hand side of (2.16) gives

$$\dim_{\Omega_0} H_0(K.(I, I')) = \sum_{I\setminus I' \subseteq J \subseteq I} (-1)^{|I|-|J|} (|J_{\mathrm{sp}}|)! \sum_{\sum_{j\in J_{\mathrm{du}}} l_j = |J_{\mathrm{sp}}|} M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}}).$$

To simplify this expression, we define an equivalence relation on the pairs $(J, \{l_j\}_{j\in J_{\mathrm{du}}})$ with $I \setminus I' \subseteq J \subseteq I$ and $\sum_{j\in J_{\mathrm{du}}} l_j = |J_{\mathrm{sp}}|$ that appear in this sum. Let $J_+ = \{j \in J_{\mathrm{du}} \mid l_j > 0\}$. Define $(J, \{l_j\}_{j\in J_{\mathrm{du}}}) \sim (J', \{l'_j\}_{j\in J'_{\mathrm{du}}})$ if $J_{\mathrm{sp}} = J'_{\mathrm{sp}}$, $J_+ = J'_+$, and $l_j = l'_j$ for $j \in J_+$. When two pairs are equivalent, we have

$$M(\{\Delta_j^{J_{\mathrm{sp}}}, l_j\}_{j\in J_{\mathrm{du}}}) = M(\{\Delta_j^{J'_{\mathrm{sp}}}, l'_j\}_{j\in J'_{\mathrm{du}}})$$

since both collections consist of the same polytopes repeated with the same multiplicities. Note that each equivalence class contains exactly one representative of the form $(G \cup I_{\rm du}, \{l_j\}_{j\in I_{\rm du}})$, where $I_{\rm sp} \setminus I'_{\rm sp} \subseteq G \subseteq I_{\rm sp}$. It follows that

$$\begin{aligned}\dim_{\Omega_0} & H_0(K.(I,I')) \\ &= \sum_{I_{\rm sp}\setminus I'_{\rm sp}\subseteq G\subseteq I_{\rm sp}} (-1)^{|I_{\rm sp}|-|G|}(|G|)! \sum_{\sum_{j\in I_{\rm du}} l_j=|G|} M(\{\Delta_j^G, l_j\}_{j\in I_{\rm du}}){\sum}'(-1)^{|I_{\rm du}|-|J'_{\rm du}|},\end{aligned}$$

where $\sum'$ denotes a sum over pairs $(J', \{l'_j\}_{j\in J'_{\rm du}})$ in the equivalence class of the pair $(G \cup I_{\rm du}, \{l_j\}_{j\in I_{\rm du}})$. For every pair $(J', \{l'_j\}_{j\in J'_{\rm du}})$ in this equivalence class, the set $J'_{\rm du}$ can be represented in the form $J'_{\rm du} = I_{{\rm du},+} \cup F$ for a unique subset F, $I_{\rm du} \setminus (I'_{\rm du} \cup I_{{\rm du},+}) \subseteq F \subseteq I_{\rm du} \setminus I_{{\rm du},+}$ (namely, $j \in F$ if and only if $l'_j = 0$). In terms of F, the innermost sum in the previous equation becomes

$$\sum_{I_{\rm du}\setminus(I'_{\rm du}\cup I_{{\rm du},+})\subseteq F\subseteq I_{\rm du}\setminus I_{{\rm du},+}} (-1)^{|I_{\rm du}|-|I_{{\rm du},+}|-|F|}.$$

Putting $F' = I_{\rm du} \setminus (I_{{\rm du},+} \cup F)$, this sum becomes

$$\sum_{\emptyset\subseteq F'\subseteq I'_{\rm du}\setminus I_{{\rm du},+}} (-1)^{|F'|}.$$

But this clearly vanishes unless $I'_{\rm du} \subseteq I_{{\rm du},+}$ (*i.e.*, $l_j > 0$ for all $j \in I'_{\rm du}$), in which case it equals 1. We may thus restrict our sum to the classes of those pairs $(G \cup I_{\rm du}, \{l_j\}_{j\in I_{\rm du}})$ for which $l_j > 0$ for all $j \in I'_{\rm du}$ and take a single representative from each of these classes. With a slight change in notation, the formula becomes

$$\begin{aligned}(2.18)\qquad \dim_{\Omega_0} & H_0(K.(I,I')) \\ &= \sum_{I_{\rm sp}\setminus I'_{\rm sp}\subseteq J\subseteq I_{\rm sp}} (-1)^{|I_{\rm sp}|-|J|}(|J|)! \sum_{\substack{\sum_{i\in I_{\rm du}} l_i=|J| \\ l_i\geq 1 \text{ for } i\in I'_{\rm du}}} M(\{\Delta_i^J, l_i\}_{i\in I_{\rm du}}).\end{aligned}$$

Let $I \subseteq S$ with $S_{\rm to} \subseteq I$ and let $I' \subseteq I_{\rm af}$. When $H_l(K.(I,I'))$ is finite-dimensional for all l, we define

$$\chi(I,I') = \sum_{l=0}^{|I|}(-1)^l \dim_{\Omega_0} H_l(K.(I,I')).$$

THEOREM 2.19. – *Suppose that g is nondegenerate and semi-convenient. Then*

(i) $\dim_{\Omega_0} H_l(K.(S,S_{\rm af})) < \infty$ *for all* l.

(ii) $H_l(K.(S,S_{\rm af})) = 0$ *for* $l > N - r$.

(iii) *For* $l = 1, \dots, N-r$, $\dim_{\Omega_0} H_l(K.(S, S_{\mathrm{af}})) = \binom{m}{l+r-n}$ *and Frobenius operates as multiplication by* q^{l+r}. *In particular,* $\det(I - t\bar{\alpha} \mid H_l(K.(S, S_{\mathrm{af}}))) = (1 - q^{l+r}t)^{\binom{m}{l+r-n}}$ *for* $l = 1, \dots, N-r$.

(iv) $\det(I - t\bar{\alpha} \mid H_0(K.(S, S_{\mathrm{af}}))) = P(t)(1 - q^r t)^{\binom{m}{r-n}}$.

(v) *There is a subspace* $H' \subseteq H_0(K.(S, S_{\mathrm{af}}))$ *of dimension* $\binom{m}{r-n}$ *on which Frobenius operates as multiplication by* q^r. *In particular,* $\det(I - t\bar{\alpha} \mid H') = (1 - q^r t)^{\binom{m}{r-n}}$ *and* $P(t)$ *is a polynomial.*

(vi) *For* $r \geq 1$,

$$\chi(S, S_{\mathrm{af}}) = \sum_{S_{\mathrm{to}} \subseteq J \subseteq S_{\mathrm{sp}}} (-1)^{N-|J|}(|J|)! \sum_{\substack{i_1 + \cdots + i_r = |J| \\ i_j \geq 1 \text{ for all } j}} M(\Delta_1^J, i_1; \dots; \Delta_r^J, i_r).$$

Proof. – Note that assertion (iv) follows immediately from equations (2.5) and (2.14) and assertions (ii) and (iii). We prove statements (i), (ii), (iii), (v), and (vi) by induction on r. To simplify notation, we write $H_l(I, I')$ in place of $H_l(K.(I, I'))$.

Suppose $r = 0$, so that $S_{\mathrm{af}} = \{m+1, \dots, m+n\}$. In this case $g = 0$, so $\Delta = (0, \dots, 0) \in \mathbf{R}^N$, $M = (0, \dots, 0) \in \mathbf{R}^N$, $B = \Omega_0$, and the differential operators are $D_i = x_i \partial/\partial x_i$, which act trivially on Ω_0. It follows that $K.(S, S_{\mathrm{af}})$ is the complex with $K_l(S, S_{\mathrm{af}}) = (\Omega_0)^{\binom{m}{l-n}}$ and all boundary maps trivial. Thus when $r = 0$, $H_l(S, S_{\mathrm{af}})$ is a space of dimension $\binom{m}{l-n}$ with Frobenius acting as multiplication by q^l. This is exactly the assertion of the theorem. We observe that for $r = 0$,

$$\chi(S, S_{\mathrm{af}}) = \sum_{l=0}^{N} (-1)^l \binom{m}{l-n} = \begin{cases} 0 & \text{for } m > 0, \\ (-1)^n & \text{for } m = 0. \end{cases} \tag{2.20}$$

Suppose the theorem true for $r - 1$. For notational convenience, put $S' = \{1, \dots, N + r - 1\}$. From (2.12) we get a short exact sequence

$$0 \to K.(S, S_{\mathrm{af}}) \to K.(S, S'_{\mathrm{af}}) \to K.(S', S'_{\mathrm{af}}) \to 0, \tag{2.21}$$

and by Theorem 2.15, $H_l(S, S'_{\mathrm{af}}) = 0$ for $l > 0$. The associated long exact homology sequence then gives an exact sequence

$$0 \to H_1(S', S'_{\mathrm{af}}) \to H_0(S, S_{\mathrm{af}}) \to H_0(S, S'_{\mathrm{af}}) \to H_0(S', S'_{\mathrm{af}}) \to 0 \tag{2.22}$$

and isomorphisms for $l \geq 1$

$$H_l(S, S_{\mathrm{af}}) \simeq H_{l+1}(S', S'_{\mathrm{af}}). \tag{2.23}$$

Assertion (i) is now immediate from Theorem 2.15 and the induction hypothesis. We apply the induction hypothesis to compute the homology of $K.(S', S'_{\mathrm{af}})$. We have $H_l(S', S'_{\mathrm{af}}) = 0$ for $l > N - r + 1$, so by (2.23) we have $H_l(S, S_{\mathrm{af}}) = 0$ for $l > \max\{0, N - r\}$. This establishes (ii) when $N - r \geq 0$. (The proof of (ii) when $N - r < 0$, *i.e.*, the proof that $H_l(S, S_{\mathrm{af}}) = 0$ for all l when $N - r < 0$, is given below.) For $1 \leq l \leq N - r + 1$,

$H_l(S', S'_{\mathrm{af}})$ is $\binom{m}{l+r-1-n}$-dimensional with Frobenius acting as multiplication by q^{l+r-1}, hence for $1 \leq l \leq N-r$, it follows from (2.23) that $H_l(S, S_{\mathrm{af}})$ is $\binom{m}{l+r-n}$-dimensional with Frobenius acting as multiplication by q^{l+r}. This establishes assertion (iii). The space H' of assertion (v) is the image of $H_1(S', S'_{\mathrm{af}})$ under the injection $H_1(S', S'_{\mathrm{af}}) \hookrightarrow H_0(S, S_{\mathrm{af}})$ of (2.22).

To prove (vi), we observe that the short exact sequence (2.21) gives

$$\chi(S, S_{\mathrm{af}}) = \chi(S, S'_{\mathrm{af}}) - \chi(S', S'_{\mathrm{af}}).$$

From (2.18) and Theorem 2.15 we have

$$\chi(S, S'_{\mathrm{af}}) = \sum_{S_{\mathrm{to}} \subseteq J \subseteq S_{\mathrm{sp}}} (-1)^{N-|J|}(|J|)! \sum_{\substack{i_1+\cdots+i_r=|J| \\ i_1,\ldots,i_{r-1}\geq 1}} M(\Delta_1^J, i_1; \ldots; \Delta_r^J, i_r). \tag{2.24}$$

When $r \geq 2$, the induction hypothesis gives

$$\chi(S', S'_{\mathrm{af}}) = \sum_{S_{\mathrm{to}} \subseteq J \subseteq S_{\mathrm{sp}}} (-1)^{N-|J|}(|J|)! \sum_{\substack{i_1+\cdots+i_{r-1}=|J| \\ i_1,\ldots,i_{r-1}\geq 1}} M(\Delta_1^J, i_1; \ldots; \Delta_{r-1}^J, i_{r-1}). \tag{2.25}$$

Subtracting (2.25) from (2.24) gives assertion (vi) when $r \geq 2$. When $r = 1$, we follow the same argument except that $\chi(S', S'_{\mathrm{af}})$ is no longer given by the induction hypothesis but rather by the right-hand side of (2.20).

It remains to prove (ii) and (vi) when $N-r < 0$. Since the expression for $\chi(S, S_{\mathrm{af}})$ in (vi) vanishes for $N-r < 0$, we see that (vi) follows from (ii) in this case. Suppose $N-r = -1$. Then $\dim_{\Omega_0} H_0(S, S'_{\mathrm{af}}) = \chi(S, S'_{\mathrm{af}})$ by Theorem 2.15 and $\dim_{\Omega_0} H_0(S', S'_{\mathrm{af}}) = \chi(S', S'_{\mathrm{af}})$ since $H_l(S', S'_{\mathrm{af}}) = 0$ for $l > 0$ by (ii) (in the case $N-r = 0$, which was already proved). But $\chi(S, S'_{\mathrm{af}}) = \chi(S', S'_{\mathrm{af}})$ by evaluating (2.24) and (2.25) with $N-r = -1$, so the map $H_0(S, S'_{\mathrm{af}}) \to H_0(S', S'_{\mathrm{af}})$ in (2.22) is an isomorphism. Since $H_1(S', S'_{\mathrm{af}}) = 0$, it follows that $H_0(S, S_{\mathrm{af}}) = 0$, thus (ii) and (vi) are established for $N-r = -1$. For $N-r < -1$, we have $H_0(S, S'_{\mathrm{af}}) = 0$ by (2.24), hence $H_0(S, S_{\mathrm{af}}) = 0$ by (2.22).

3. Complexes in characteristic p

From Theorem 2.19 we have

$$P(t) = \det(I - t\bar{\alpha} \mid H_0(K.(S, S_{\mathrm{af}})))/(1 - q^r t)^{\binom{m}{r-n}}. \tag{3.1}$$

Our goal (Corollary 4.14 below) is to use this formula to give a lower bound for the Newton polygon of $P(t)$. Later, we shall identify this lower bound with the Hodge polygon of the primitive part of middle-dimensional cohomology of the complete intersection $F_1 = \cdots = F_r = 0$ in $(\mathbf{T}^m \times \mathbf{A}^n)_{\mathbf{C}}$, where $F_i \in \mathbf{C}[x_1, \ldots, x_N, (x_1 \cdots x_m)^{-1}]$ is the generic polynomial with the property that the convex hull of $\mathrm{supp}(F_i)$ coincides with the convex hull of $\mathrm{supp}(f_i)$.

The first step is to describe a basis for $H_0(K.(S, S_{\mathrm{af}}))$. In fact, it is no more difficult to do this for all $H_l(K.(S, S_{\mathrm{af}}))$. We begin by considering some related complexes in characteristic p. Let R be the ring $R = \mathbf{F}_q[x^u \mid u \in M]$. This ring is graded by the weight function defined earlier, namely, let $R^{(k)}$, the homogeneous part of degree k, be the span of all monomials x^u with $w(u) = k$. For $i = 1, \ldots, N + r$, put

$$g_i = x_i \frac{\partial g}{\partial x_i} \in R^{(1)}. \tag{3.2}$$

Let $\bar{K}. = K.(R, \{g_i\}_{i=1}^{N+r})$ be the Koszul complex on R defined by $g_1, \ldots, g_{N+r}$. Grade $\bar{K}.$ so that the boundary maps ∂_l are homogeneous of degree 0, *i.e.*,

$$\bar{K}_l^{(k)} = \bigoplus_{A \subseteq S,\, |A| = l} R^{(k-l)}\, e_A.$$

For $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$, we may regard θ_I as an endomorphism of R, homogeneous of degree 0, and define $R_I = \theta_I(R)$. We also put

$$g_{I,i} = x_i \frac{\partial g_I}{\partial x_i} \in R_I^{(1)} \tag{3.3}$$

for $i \in I$ and let $\bar{K}.(I) = K.(R_I, \{g_{I,i}\}_{i \in I})$ be the Koszul complex on R_I defined by the $g_{I,i}$ for $i \in I$. For $I' \subseteq I_{\mathrm{af}}$, let

$$R_I^{I'} = \bigcap_{i \in I'} \ker(\theta_i \mid R_I),$$

the elements of R_I divisible by x_i for all $i \in I'$. Let $\bar{K}.(I, I')$ be the subcomplex of $\bar{K}.(I)$ defined by

$$\bar{K}_l(I, I') = \bigoplus_{A \subseteq I,\, |A| = l} R_I^{I' \setminus A}\, e_A. \tag{3.4}$$

For $i \in I_{\mathrm{af}}$, $i \notin I'$, we have, as in (2.12), an exact sequence of (graded) complexes

$$0 \to \bar{K}.(I, I' \cup \{i\}) \to \bar{K}.(I, I') \to \bar{K}.(I \setminus \{i\}, I') \to 0. \tag{3.5}$$

We shall compute the dimension (over $\mathbf{F}_q$) of $H_l(\bar{K}.(S, S_{\mathrm{af}}))^{(k)}$, the homogeneous component of $H_l(\bar{K}.(S, S_{\mathrm{af}}))$ of degree k, for all l and k and use this information to describe a basis for $H_l(K.(S, S_{\mathrm{af}}))$. The answer will be expressed in terms of certain invariants of certain polyhedra. As before, let $\Delta_i^{I_{\mathrm{sp}}}$ be the convex hull of $\mathrm{supp}(\theta_{I_{\mathrm{sp}}}(f_i))$, $i = 1, \ldots, r$. When $m = 0$ and $I_{\mathrm{sp}} = \emptyset$, we assume $f_i(0, \ldots, 0) \neq 0$ for $i = 1, \ldots, r$, so that $\Delta_i^{\emptyset} \neq \emptyset$. As before, Δ_I will denote the convex hull of the origin and $\mathrm{supp}(\theta_I(g))$.

For any set $Y \subseteq \mathbf{R}^N$, let $\ell(Y)$ denote the cardinality of $Y \cap \mathbf{Z}^N$. For any subset $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$, set $\delta(I) = \{j \in \{1, \ldots, r\} \mid N + j \in I_{\mathrm{du}}\}$ and define a power series in the variables t_j for $j \in \delta(I)$ by

$$P_I(\{t_j\}_{j \in \delta(I)}) = \sum_{\substack{k_j = 0 \\ \text{for } j \in \delta(I)}}^{\infty} \ell\Big(\sum_{j \in \delta(I)} k_j \Delta_j^{I_{\mathrm{sp}}}\Big) \prod_{j \in \delta(I)} t_j^{k_j} \in \mathbf{Z}[[\{t_j\}_{j \in \delta(I)}]]. \tag{3.6}$$

It is easily seen that

$$P_I = P_{I_{\rm sp}\cup S_{\rm du}}|_{t_j=0 \text{ for } j\notin\delta(I)}. \tag{3.7}$$

It is well-known that $\ell(k_1\Delta_1^{I_{\rm sp}}+\cdots+k_r\Delta_r^{I_{\rm sp}})$ is a rational polynomial of degree $\leq |I_{\rm sp}|$ in $k_1,\ldots,k_r$, say,

$$\ell(k_1\Delta_1^{I_{\rm sp}}+\cdots+k_r\Delta_r^{I_{\rm sp}}) = \sum_{e_1+\cdots+e_r\leq|I_{\rm sp}|} a^{I_{\rm sp}}_{e_1\cdots e_r} k_1^{e_1}\cdots k_r^{e_r} \in \mathbf{Q}[k_1,\ldots,k_r].$$

Thus

$$\begin{aligned}
P_{I_{\rm sp}\cup S_{\rm du}}(t_1,\ldots,t_r) &= \sum_{k_1,\ldots,k_r=0}^{\infty}\ \sum_{e_1+\cdots+e_r\leq|I_{\rm sp}|} a^{I_{\rm sp}}_{e_1\cdots e_r}\prod_{j=1}^{r}\left(t_j\frac{\partial}{\partial t_j}\right)^{e_j} t_1^{k_1}\cdots t_r^{k_r}\\
&= \sum_{e_1+\cdots+e_r\leq|I_{\rm sp}|} a^{I_{\rm sp}}_{e_1\cdots e_r}\prod_{j=1}^{r}\left(t_j\frac{\partial}{\partial t_j}\right)^{e_j}\left(\prod_{k=1}^{r}\frac{1}{1-t_k}\right)\\
&= \sum_{e_1+\cdots+e_r\leq|I_{\rm sp}|}\frac{p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)}{\prod_{j=1}^{r}(1-t_j)^{e_j+1}},
\end{aligned}$$

for some polynomial $p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)\in\mathbf{Q}[t_1,\ldots,t_r]$. Note that if $e_j>0$, then $p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)$ is divisible by t_j, and if $e_j=0$, then t_j does not appear in $p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)$. Furthermore, $\deg p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)=e_1+\cdots+e_r$ unless $a^{I_{\rm sp}}_{e_1\cdots e_r}=0$, in which case this polynomial vanishes. From (3.7) we have

$$P_I(\{t_j\}_{j\in\delta(I)}) = \sum_{\substack{e_1+\cdots+e_r\leq|I_{\rm sp}|\\ e_j=0 \text{ for } j\notin\delta(I)}}\frac{p^{I_{\rm sp}}_{e_1\cdots e_r}(t_1,\ldots,t_r)}{\prod_{j\in\delta(I)}(1-t_j)^{e_j+1}}. \tag{3.8}$$

It is easily seen from the definitions of Δ_I and the weight function w that

$$\dim_{\mathbf{F}_q} R_I^{(k)} = \ell(k\Delta_I)-\ell((k-1)\Delta_I),$$

i.e.,

$$\sum_{k=0}^{\infty}(\dim_{\mathbf{F}_q} R_I^{(k)})t^k = (1-t)\sum_{k=0}^{\infty}\ell(k\Delta_I)t^k. \tag{3.9}$$

We regard $\mathbf{R}^{|I|}$ as being fibered over $\mathbf{R}^{|I_{\rm du}|}$. The fiber of $C(\Delta_I)$ over a point $(k_j)_{j\in\delta(I)}\in\mathbf{N}^{|I_{\rm du}|}$ is $\sum_{j\in\delta(I)} k_j\Delta_j^{I_{\rm sp}}$. Hence

$$\ell(k\Delta_I) = \sum_{\sum_{j\in\delta(I)} k_j\leq k}\ell\left(\sum_{j\in\delta(I)}k_j\Delta_j^{I_{\rm sp}}\right),$$

i.e., by (3.6)

$$(1-t)\sum_{k=0}^{\infty}\ell(k\Delta_I)t^k = P_I(\{t_j\}_{j\in\delta(I)})|_{t_j=t \text{ for } j\in\delta(I)}. \tag{3.10}$$

It now follows from (3.8) and (3.9) that

$$\sum_{k=0}^{\infty}(\dim_{\mathbf{F}_q} R_I^{(k)})t^k = \sum_{\substack{e_1+\cdots+e_r\le|I_{\mathrm{sp}}|\\ e_j=0 \text{ for } j\notin\delta(I)}} \frac{p^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t,\ldots,t)}{(1-t)^{|I_{\mathrm{du}}|+\sum_{j=1}^r e_j}}.$$

Define

$$q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t) = p^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t,\ldots,t)(1-t)^{|I_{\mathrm{sp}}|-\sum_{j=1}^r e_j} \in \mathbf{Q}[t]. \tag{3.11}$$

Then

$$\sum_{k=0}^{\infty}(\dim_{\mathbf{F}_q} R_I^{(k)})t^k = (1-t)^{-|I|}\sum_{\substack{e_1+\cdots+e_r\le|I_{\mathrm{sp}}|\\ e_j=0 \text{ for } j\notin\delta(I)}} q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t). \tag{3.12}$$

Note that if $A = \operatorname{card}\{j \mid e_j \ge 1\}$, then $q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t)$ is divisible by t^A. Furthermore, $\deg q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t) \le |I_{\mathrm{sp}}|$. It is clear from the definitions that $p^{I_{\mathrm{sp}}}_{0\cdots 0}(t_1,\ldots,t_r) = 1$, hence $q^{I_{\mathrm{sp}}}_{0\cdots 0}(t) = (1-t)^{|I_{\mathrm{sp}}|}$.

LEMMA 3.13. – *Suppose that g is nondegenerate and semi-convenient. Then for $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$ and $I_{\mathrm{du}} \neq \emptyset$, we have $H_l(\bar{K}.(I)) = 0$ for $l > 0$ and $\dim_{\mathbf{F}_q} H_0(\bar{K}.(I))^{(k)}$ is the coefficient of t^k in*

$$\sum_{\substack{e_1+\cdots+e_r\le|I_{\mathrm{sp}}|\\ e_j=0 \text{ for } j\notin\delta(I)}} q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t).$$

Proof. – The complex $\bar{K}.(I)$ is acyclic in positive dimension by [13] (*see* also [1, Theorem 2.17]). The formula for $\dim_{\mathbf{F}_q} H_0(\bar{K}.(I))^{(k)}$ follows from this acyclicity and (3.12).

Since $\bar{K}_l(I,I')^{(k)}$ and $H_l(\bar{K}.(I,I'))^{(k)}$ are always finite-dimensional, we may define

$$\begin{aligned}\bar{\chi}^{(k)}(I,I') &= \sum_{l=0}^{|I|}(-1)^l \dim_{\mathbf{F}_q} \bar{K}_l(I,I')^{(k)}\\ &= \sum_{l=0}^{|I|}(-1)^l \dim_{\mathbf{F}_q} H_l(\bar{K}.(I,I'))^{(k)}.\end{aligned}$$

LEMMA 3.14. – *Suppose that g is nondegenerate and semi-convenient. Let $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and let $I' \subseteq I_{\text{af}}$. Then $\bar{\chi}^{(k)}(I, I')$ is the coefficient of t^k in*

$$\sum_{I_{\text{sp}} \setminus I'_{\text{sp}} \subseteq J \subseteq I_{\text{sp}}} (-1)^{|I_{\text{sp}}|-|J|} \sum_{\substack{e_1+\cdots+e_r \leq |J| \\ e_j=0 \text{ for } j \notin \delta(I) \\ e_j \geq 1 \text{ for } j \in \delta(I')}} q^J_{e_1\cdots e_r}(t). \tag{3.15}$$

If in addition $I'_{\text{du}} \neq I_{\text{du}}$, then $H_l(\bar{K}.(I, I')) = 0$ for $l > 0$, hence $\dim_{\mathbf{F}_q} H_0(\bar{K}.(I, I'))^{(k)}$ is the coefficient of t^k in (3.15).

Proof. – When $I'_{\text{du}} \neq I_{\text{du}}$, g_I is nondegenerate and commode (in the sense of [1]) with respect to I', so the complex $\bar{K}.(I, I')$ is acyclic in positive dimension by [1, Theorem 2.17], and we need to prove only the formula (3.15) for $\bar{\chi}^{(k)}(I, I')$.

From the definitions,

$$\bar{K}_l(I, I')^{(k)} = \bigoplus_{A \subseteq I,\, |A|=l} R_I^{I' \setminus A, (k-l)} e_A.$$

We fix a set $T \subseteq I'$ and ask for which $A \subseteq I$, $|A| = l$, we have $I' \setminus A = T$. Clearly, we must have $A = (I' \setminus T) \cup \tilde{T}$, where $\tilde{T} \subseteq I \setminus I'$ has cardinality $l + |T| - |I'|$. Thus

$$\bar{\chi}^{(k)}(I, I') = \sum_{l=0}^{|I|} (-1)^l \sum_{T \subseteq I'} \binom{|I| - |I'|}{l + |T| - |I'|} \dim_{\mathbf{F}_q} R_I^{T,(k-l)}. \tag{3.16}$$

A formula for $\dim_{\mathbf{F}_q} R_I^{T,(k-l)}$ can be obtained from (3.12). A standard inclusion-exclusion argument shows that

$$\sum_{k=0}^{\infty} (\dim_{\mathbf{F}_q} R_I^{T,(k)}) t^k = \sum_{I \setminus T \subseteq J \subseteq I} (-1)^{|I|-|J|} \sum_{k=0}^{\infty} (\dim_{\mathbf{F}_q} R_J^{(k)}) t^k.$$

Substituting from (3.12) into this formula and using (3.16) we see that $\bar{\chi}^{(k)}(I, I')$ is the coefficient of t^k in

$$\begin{aligned} &\sum_{l=0}^{|I|} (-1)^l \sum_{T \subseteq I'} \binom{|I| - |I'|}{l + |T| - |I'|} t^l \\ &\quad \times \sum_{I \setminus T \subseteq J \subseteq I} (-1)^{|I|-|J|} (1-t)^{-|J|} \sum_{\substack{e_1+\cdots+e_r \leq |J_{\text{sp}}| \\ e_j=0 \text{ for } j \notin \delta(J)}} q^{J_{\text{sp}}}_{e_1\cdots e_r}(t). \end{aligned} \tag{3.17}$$

Fix a subset $K \subseteq I_{\text{sp}}$ and fix $e_1, \ldots, e_r$ with $e_1 + \cdots + e_r \leq |K|$. We ask for the coefficient of $q^K_{e_1\cdots e_r}(t)$ in (3.17). Let $A \subseteq S_{\text{du}}$ be the set of all indices $N + j$ such that $e_j \neq 0$. The equality of (3.15) and (3.17) is equivalent to the assertion that the coefficient of $q^K_{e_1\cdots e_r}(t)$ in (3.17) is

$$\begin{cases} (-1)^{|I_{\text{sp}}|-|K|} & \text{if } I_{\text{sp}} \setminus I'_{\text{sp}} \subseteq K \subseteq I_{\text{sp}} \text{ and } I'_{\text{du}} \subseteq A \subseteq I_{\text{du}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

It is clear that $q^K_{e_1\cdots e_r}(t)$ does not appear in (3.17) unless $I_{\mathrm{sp}} \setminus I'_{\mathrm{sp}} \subseteq K \subseteq I_{\mathrm{sp}}$ and $A \subseteq I_{\mathrm{du}}$. Assume from now on that these conditions are satisfied. Then $q^K_{e_1\cdots e_r}(t)$ will appear in (3.17) for each J, $I \setminus T \subseteq J \subseteq I$, such that $J_{\mathrm{sp}} = K$ and $J_{\mathrm{du}} \supseteq A$, thus the coefficient of $q^K_{e_1\cdots e_r}(t)$ in (3.17) is

$$\sum_{l=0}^{|I|}(-1)^l \sum_{T\subseteq I'} \binom{|I|-|I'|}{l+|T|-|I'|} t^l \sum_{\substack{I\setminus T\subseteq J\subseteq I\\ J_{\mathrm{sp}}=K \text{ and } J_{\mathrm{du}}\supseteq A}} (-1)^{|I|-|J|}(1-t)^{-|J|}. \tag{3.19}$$

We show this expression equals $(-1)^{|I_{\mathrm{sp}}|-|K|}$ if $I'_{\mathrm{du}} \subseteq A$ and equals 0 otherwise.

We evaluate the innermost sum in (3.19) first. Note that it is the empty sum unless $I_{\mathrm{sp}} \setminus T_{\mathrm{sp}} \subseteq K$. When this condition is satisfied, it equals

$$\begin{aligned}&\frac{(-1)^{|I_{\mathrm{sp}}|-|K|}}{(1-t)^{|K|}} \sum_{A\cup(I_{\mathrm{du}}\setminus T_{\mathrm{du}})\subseteq L\subseteq I_{\mathrm{du}}} (-1)^{|I_{\mathrm{du}}|-|L|}(1-t)^{-|L|}\\ &= \frac{(-1)^{|I_{\mathrm{sp}}|-|K|}}{(1-t)^{|K|}} \sum_{j=|A\cup(I_{\mathrm{du}}\setminus T_{\mathrm{du}})|}^{|I_{\mathrm{du}}|} \binom{|I_{\mathrm{du}}|-|A\cup(I_{\mathrm{du}}\setminus T_{\mathrm{du}})|}{j-|A\cup(I_{\mathrm{du}}\setminus T_{\mathrm{du}})|}(-1)^{|I_{\mathrm{du}}|-j}(1-t)^{-j},\end{aligned}$$

where we have set $j = |L|$ in the second expression. Replacing j by $j + |A \cup (I_{\mathrm{du}} \setminus T_{\mathrm{du}})|$ and using the fact that $|I_{\mathrm{du}}| - |A \cup (I_{\mathrm{du}} \setminus T_{\mathrm{du}})| = |T_{\mathrm{du}} \setminus A|$, this simplifies to

$$\frac{(-1)^{|I_{\mathrm{sp}}|-|K|}t^{|T_{\mathrm{du}}\setminus A|}}{(1-t)^{|K|+|I_{\mathrm{du}}|}}.$$

Substitution into (3.19) transforms that expression into

$$\sum_{l=0}^{|I|}(-1)^l \sum_{T\subseteq I'} \binom{|I|-|I'|}{l+|T|-|I'|} t^l \cdot \begin{cases} \dfrac{(-1)^{|I_{\mathrm{sp}}|-|K|}t^{|T_{\mathrm{du}}\setminus A|}}{(1-t)^{|K|+|I_{\mathrm{du}}|}} & \text{if } I_{\mathrm{sp}}\setminus T_{\mathrm{sp}}\subseteq K,\\ 0 & \text{otherwise.}\end{cases} \tag{3.20}$$

Interchanging the order of summation transforms this into

$$\frac{(-1)^{|I_{\mathrm{sp}}|-|K|}}{(1-t)^{|K|+|I_{\mathrm{du}}|}} \sum_{\substack{T\subseteq I'\\ I_{\mathrm{sp}}\setminus T_{\mathrm{sp}}\subseteq K}} t^{|T_{\mathrm{du}}\setminus A|} \sum_{l=0}^{|I|} \binom{|I|-|I'|}{l+|T|-|I'|}(-t)^l. \tag{3.21}$$

The inner sum in (3.21) equals

$$\begin{aligned}\sum_{l=|I'|-|T|}^{|I|-|T|} \binom{|I|-|I'|}{l+|T|-|I'|}(-t)^l &= \sum_{l=0}^{|I|-|I'|}\binom{|I|-|I'|}{l}(-t)^{l+|I'|-|T|}\\ &= (-t)^{|I'|-|T|}(1-t)^{|I|-|I'|}.\end{aligned}$$

Substituting this last expression into (3.21) gives

$$\frac{(-1)^{|I_{\mathrm{sp}}|-|K|}}{(1-t)^{|I'|+|K|-|I_{\mathrm{sp}}|}} \sum_{\substack{T\subseteq I'\\ I_{\mathrm{sp}}\setminus T_{\mathrm{sp}}\subseteq K}} t^{|T_{\mathrm{du}}\setminus A|}(-t)^{|I'|-|T|}. \tag{3.22}$$

The decomposition $T = T_{\rm sp} \cup T_{\rm du}$ induces a decomposition of the sum appearing in (3.22) as a product of two other sums, *i.e.*, (3.22) equals

$$\frac{(-1)^{|I_{\rm sp}|-|K|}}{(1-t)^{|I'|+|K|-|I_{\rm sp}|}} \sum_{I_{\rm sp}\setminus K\subseteq T_1\subseteq I'_{\rm sp}} (-t)^{|I'_{\rm sp}|-|T_1|} \sum_{T_2\subseteq I'_{\rm du}} t^{|T_2\setminus A|}(-t)^{|I'_{\rm du}|-|T_2|}, \tag{3.23}$$

where we are implicitly using the assumption made earlier that $I_{\rm sp} \setminus I'_{\rm sp} \subseteq K$, otherwise the first of these sums is empty. Evaluating the first sum we get

$$\begin{aligned}\sum_{I_{\rm sp}\setminus K\subseteq T_1\subseteq I'_{\rm sp}} (-t)^{|I'_{\rm sp}|-|T_1|} &= \sum_{j=|I_{\rm sp}|-|K|}^{|I'_{\rm sp}|} \binom{|I'_{\rm sp}|-|I_{\rm sp}|+|K|}{j-|I_{\rm sp}|+|K|}(-t)^{|I'_{\rm sp}|-j}\\ &= \sum_{j=0}^{|I'_{\rm sp}|-|I_{\rm sp}|+|K|} \binom{|I'_{\rm sp}|-|I_{\rm sp}|+|K|}{j}(-t)^{|I'_{\rm sp}|-|I_{\rm sp}|+|K|-j}\\ &= (1-t)^{|I'_{\rm sp}|-|I_{\rm sp}|+|K|},\end{aligned}$$

hence (3.23) becomes

$$\frac{(-1)^{|I_{\rm sp}|-|K|}}{(1-t)^{|I'_{\rm du}|}} \sum_{T_2\subseteq I'_{\rm du}} t^{|T_2\setminus A|}(-t)^{|I'_{\rm du}|-|T_2|}. \tag{3.24}$$

To complete the proof of Lemma 3.14, we need to show that the sum in (3.24) equals $(1-t)^{|I'_{\rm du}|}$ if $I'_{\rm du} \subseteq A$ and is 0 otherwise. Putting $B = T_2 \setminus A$ we have

$$\sum_{T_2\subseteq I'_{\rm du}} t^{|T_2\setminus A|}(-t)^{|I'_{\rm du}|-|T_2|} = \sum_{B\subseteq I'_{\rm du}\setminus A} \sum_{\substack{T_2\subseteq I'_{\rm du}\\ T_2\setminus A=B}} (-1)^{|I'_{\rm du}|-|T_2|} t^{|I'_{\rm du}|-|T_2|+|B|}. \tag{3.25}$$

Putting $j = |T_2|$ and writing $T_2 = B \cup (T_2 \cap A)$, a disjoint union with $T_2 \cap A \subseteq I'_{\rm du} \cap A$, we see that the inner sum equals

$$\begin{aligned}\sum_{j=|B|}^{|B|+|I'_{\rm du}\cap A|} \binom{|I'_{\rm du}\cap A|}{j-|B|}(-1)^{|I'_{\rm du}|-j}t^{|I'_{\rm du}|-j+|B|} &= \sum_{j=0}^{|I'_{\rm du}\cap A|} \binom{|I'_{\rm du}\cap A|}{j}(-1)^{|I'_{\rm du}|-j+|B|}t^{|I'_{\rm du}|-j}\\ &= (-1)^{|B|}(-t)^{|I'_{\rm du}\setminus A|}(1-t)^{|I'_{\rm du}\cap A|}.\end{aligned}$$

Substituting this result into the right-hand side of (3.25), that expression becomes

$$(-t)^{|I'_{\rm du}\setminus A|}(1-t)^{|I'_{\rm du}\cap A|} \sum_{B\subseteq I'_{\rm du}\setminus A} (-1)^{|B|}.$$

But this sum is clearly 0 unless $I'_{\rm du} \subseteq A$, in which case the entire expression clearly equals $(1-t)^{|I'_{\rm du}|}$.

THEOREM 3.26. – *Suppose that g is nondegenerate and semi-convenient. Then*

(i) $H_l(\bar{K}.(S,S_{\mathrm{af}})) = 0$ *for* $l > N-r$.

(ii) *For* $l = 1,\dots,N-r$, $\dim_{\mathbf{F}_q} H_l(\bar{K}.(S,S_{\mathrm{af}}))^{(k)} = \begin{cases} \binom{m}{l+r-n} & \textit{if } k = l+r, \\ 0 & \textit{otherwise}. \end{cases}$

(iii) $\dim_{\mathbf{F}_q} H_0(\bar{K}.(S,S_{\mathrm{af}}))^{(k)}$ *is the coefficient of* t^k *in*

$$\sum_{j=r+1}^{N} (-1)^{j-r-1}\binom{m}{j-n} t^j + \sum_{S_{\mathrm{to}}\subseteq J\subseteq S_{\mathrm{sp}}} (-1)^{N-|J|} \sum_{\substack{e_1+\cdots+e_r\le |J| \\ e_j\ge 1 \text{ for all } j}} q^J_{e_1\cdots e_r}(t).$$

In particular, $H_0(\bar{K}.(S,S_{\mathrm{af}}))^{(k)} = 0$ *for* $k < r$ *or* $k > N$.

Remark. – When $r=0$, *i.e.*, $S_{\mathrm{af}} = \{m+1,\dots,m+n\}$, $S_{\mathrm{du}} = \emptyset$, we understand the second sum in Theorem 3.26 (iii) to be simply $\sum_{S_{\mathrm{to}}\subseteq J\subseteq S_{\mathrm{sp}}} (-1)^{N-|J|} q^J_{0\cdots 0}(t) = (-t)^n(1-t)^m$. The whole expression in (iii) is then just equal to 1 if $n=0$ and equal to 0 if $n>0$.

Proof. – To simplify notation, we write $H_l(I,I')$ in place of $H_l(\bar{K}.(I,I'))$. We note first that by Lemma 3.14, (iii) follows from (i) and (ii).

The proof of (i) and (ii) is by induction on r. Suppose $r=0$. Then $\bar{K}.(S,S_{\mathrm{af}})$ is the complex $\bar{K}_l(S,S_{\mathrm{af}}) = (\mathbf{F}_q)^{\binom{m}{l-n}}$ with all boundary maps trivial. Hence $H_l(S,S_{\mathrm{af}}) = 0$ for $l > N$, and for $0 \le l \le N$ we have

$$\dim_{\mathbf{F}_q} H_l(S,S_{\mathrm{af}})^{(k)} = \begin{cases} \binom{m}{l-n} & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

This is exactly the assertion of the theorem when $r=0$.

Suppose the theorem true for $r-1$. For notational convenience, put $S' = \{1,\dots,N+r-1\}$. From (3.5) we get an exact sequence

$$0 \to \bar{K}.(S,S_{\mathrm{af}}) \to \bar{K}.(S,S'_{\mathrm{af}}) \to \bar{K}.(S',S'_{\mathrm{af}}) \to 0,$$

and by Lemma 3.14, $H_l(S,S'_{\mathrm{af}}) = 0$ for $l > 0$. The associated long exact homology sequence then gives an exact sequence

$$0 \to H_1(S',S'_{\mathrm{af}}) \to H_0(S,S_{\mathrm{af}}) \to H_0(S,S'_{\mathrm{af}}) \to H_0(S',S'_{\mathrm{af}}) \to 0 \tag{3.28}$$

and isomorphisms for $l \ge 1$

$$H_l(S,S_{\mathrm{af}}) \simeq H_{l+1}(S',S'_{\mathrm{af}}). \tag{3.29}$$

By induction we have $H_l(S',S'_{\mathrm{af}}) = 0$ for $l > N-r+1$, and for $1 \le l \le N-r+1$

$$\dim_{\mathbf{F}_q} H_l(S',S'_{\mathrm{af}})^{(k)} = \begin{cases} \binom{m}{l+r-1-n} & \text{if } k = l+r-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

Parts (i) and (ii) of the theorem now follow from (3.29).

Let the $g_{I,i}$ be as in (3.3) and put

$$D'_{I,i} = x_i \frac{\partial}{\partial x_i} + g_{I,i},$$

an operator on R_I. Let $\bar{K}.'(I) = K.(R_I, \{D'_{I,i}\}_{i\in I})$ be the Koszul complex on R_I defined by the $D'_{I,i}$ for $i \in I$. Let $\bar{K}.'(I,I')$ be the subcomplex of $\bar{K}.'(I)$ defined by taking $\bar{K}'_l(I,I') = \bar{K}_l(I,I')$. The rings $R_I^{I'}$ have an increasing filtration $F.R_I^{I'}$ defined by taking $F_k R_I^{I'}$ to be the span of monomials x^u, $u \in M$, with $w(u) \le k$. This induces a filtration $F.(\bar{K}.'(I,I'))$ on the complexes $\bar{K}.'(I,I')$, namely,

$$F_k \bar{K}'_l(I,I') = \bigoplus_{A\subseteq I,\, |A|=l} F_{k-l} R_I^{I'\setminus A} e_A,$$

which in turn induces a filtration $F.(H.(\bar{K}.'(I,I')))$ on the homology spaces $H.(\bar{K}.'(I,I'))$. We shall compute $\mathrm{gr}^{\cdot}(H.(\bar{K}.'(I,I')))$, the associated graded of this filtration. It is clear from the definitions that $\mathrm{gr}^{\cdot}(\bar{K}.'(I,I'))$, the graded complex associated to the filtered complex $\bar{K}.'(I,I')$, is identified with the complex $\bar{K}.(I,I')$ considered previously.

Associated to the filtered complex $\bar{K}.'(I,I')$ is a convergent E^1 spectral sequence [17, Chapter 9] with

$$E^1_{k,l} = H_{k+l}(\bar{K}.(I,I'))^{(k)},$$
$$E^\infty_{k,l} = \mathrm{gr}^{(k)} H_{k+l}(\bar{K}.'(I,I')).$$

Suppose we are in the situation of Lemma 3.14. Then $E^1_{k,l} = 0$ for $k+l>0$ or $k+l<0$, from which it is easily seen that all the differentials $d^s_{k,l} : E^s_{k,l} \to E^s_{k-s,l+s-1}$ of the spectral sequence are 0. Hence $E^1_{k,l} \simeq E^2_{k,l} \simeq \cdots \simeq E^\infty_{k,l}$ for all k,l. By Lemma 3.14 we have the following.

LEMMA 3.31. – *Suppose that g is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{\mathrm{to}} \subseteq I$ and $I' \subseteq I_{\mathrm{af}}$ with $I'_{\mathrm{du}} \neq I_{\mathrm{du}}$, we have $H_l(\bar{K}.'(I,I')) = 0$ for $l > 0$ and $\dim_{\mathbf{F}_q} \mathrm{gr}^{(k)} H_0(\bar{K}.'(I,I'))$ is the coefficient of t^k in*

$$\sum_{I_{\mathrm{sp}}\setminus I'_{\mathrm{sp}} \subseteq J \subseteq I_{\mathrm{sp}}} (-1)^{|I_{\mathrm{sp}}|-|J|} \sum_{\substack{e_1+\cdots+e_r \le |J| \\ e_j = 0 \text{ for } j \notin \delta(I) \\ e_j \ge 1 \text{ for } j \in \delta(I')}} q^J_{e_1\cdots e_r}(t).$$

Now consider the case $I = S$, $I' = S_{\mathrm{af}}$, so that

$$E^1_{k,l} \simeq H_{k+l}(\bar{K}.(S,S_{\mathrm{af}}))^{(k)} \tag{3.32}$$
$$E^\infty_{k,l} \simeq \mathrm{gr}^{(k)} H_{k+l}(\bar{K}.'(S,S_{\mathrm{af}})). \tag{3.33}$$

We shall again show that $E^1_{k,l} \simeq E^\infty_{k,l}$ for all k,l. We begin by computing $\dim_{\mathbf{F}_q} H_l(\bar{K}.'(S,S_{\mathrm{af}}))$ for all l.

LEMMA 3.34. – *Suppose that g is nondegenerate and semi-convenient. Then for all l,*

$$\dim_{\mathbf{F}_q} H_l(\bar{K}.'(S, S_{\mathrm{af}})) = \dim_{\mathbf{F}_q} H_l(\bar{K}.(S, S_{\mathrm{af}})),$$

hence $\dim_{\mathbf{F}_q} H_l(\bar{K}.'(S, S_{\mathrm{af}}))$ *can be computed from Theorem* 3.26.

Proof. – The existence of the spectral sequence implies that

$$\sum_{l=0}^{N+r}(-1)^l \dim_{\mathbf{F}_q} H_l(\bar{K}.'(S, S'_{\mathrm{af}})) = \sum_{l=0}^{N+r}(-1)^l \dim_{\mathbf{F}_q} H_l(\bar{K}.(S, S_{\mathrm{af}})),$$

hence it suffices to prove the stated equality for $l \geq 1$. The proof is by induction on r. The case $r = 0$ is trivial since $\bar{K}.'(S, S_{\mathrm{af}}) = \bar{K}.(S, S_{\mathrm{af}})$ in that case. Put $S' = \{1, \dots, N + r - 1\}$. As in (3.5), we have an exact sequence of complexes

$$0 \to \bar{K}.'(S, S_{\mathrm{af}}) \to \bar{K}.'(S, S'_{\mathrm{af}}) \to \bar{K}.'(S', S'_{\mathrm{af}}) \to 0. \tag{3.35}$$

By Lemma 3.31, $H_l(\bar{K}.'(S, S'_{\mathrm{af}})) = 0$ for $l > 0$, hence the long exact homology sequence gives isomorphisms

$$H_l(\bar{K}.'(S, S_{\mathrm{af}})) \simeq H_{l+1}(\bar{K}.'(S', S'_{\mathrm{af}}))$$

for $l \geq 1$. By induction we have $\dim_{\mathbf{F}_q} H_l(\bar{K}.'(S', S'_{\mathrm{af}})) = \dim_{\mathbf{F}_q} H_l(\bar{K}.(S', S'_{\mathrm{af}}))$ and by (3.29) we have $H_l(\bar{K}.(S, S_{\mathrm{af}})) \simeq H_{l+1}(\bar{K}.(S', S'_{\mathrm{af}}))$ for $l \geq 1$. It follows that $\dim_{\mathbf{F}_q} H_l(\bar{K}.'(S, S_{\mathrm{af}})) = \dim_{\mathbf{F}_q} H_l(\bar{K}.(S, S_{\mathrm{af}}))$ for $l \geq 1$. This proves Lemma 3.34.

From (3.32) and Theorem 3.26, we see that $E^1_{k,l} = 0$ for all k, l except possibly $E^1_{l+r,-r}$ for $l = 1, \dots, N - r$ and $E^1_{k,-k}$ for $k = r, \dots, N$. Therefore $\mathrm{gr}^{(k)} H_{k+l}(\bar{K}.'(S, S_{\mathrm{af}})) = 0$ except possibly for $\mathrm{gr}^{(l+r)} H_l(\bar{K}.'(S, S_{\mathrm{af}}))$, $l = 1, \dots, N - r$, and $\mathrm{gr}^{(k)} H_0(\bar{K}.'(S, S_{\mathrm{af}}))$, $k = r, \dots, N$. If $l \in \{1, \dots, N - r\}$, it now follows from Lemma 3.34 and Theorem 3.26 that

$$\dim_{\mathbf{F}_q} \mathrm{gr}^{(l+r)} H_l(\bar{K}.'(S, S_{\mathrm{af}})) = \dim_{\mathbf{F}_q} H_l(\bar{K}.(S, S_{\mathrm{af}}))^{(l+r)}.$$

Examining the differentials of the spectral sequence shows that $E^1_{k,-k} \simeq E^2_{k,-k} \simeq \cdots \simeq E^\infty_{k,-k}$ if $k \neq r$, *i.e.*,

$$\mathrm{gr}^{(k)} H_0(\bar{K}.'(S, S_{\mathrm{af}})) \simeq H_0(\bar{K}.(S, S_{\mathrm{af}}))^{(k)} \tag{3.36}$$

for $k = r + 1, \dots, N$. For $k = r$, one sees that $E^\infty_{r,-r}$ is a quotient of $E^1_{r,-r}$ (in fact, $E^\infty_{r,-r} \simeq E^2_{r,-r}$), hence $\dim_{\mathbf{F}_q} E^\infty_{r,-r} \leq \dim_{\mathbf{F}_q} E^1_{r,-r}$. But by Lemma 3.34 and equation (3.36), we see that we must have $\dim_{\mathbf{F}_q} E^\infty_{r,-r} = \dim_{\mathbf{F}_q} E^1_{r,-r}$, also. By Theorem 3.26 we now have the following.

THEOREM 3.37. – *Suppose that g is nondegenerate and semi-convenient. Then*

(i) $H_l(\bar{K}.'(S, S_{\mathrm{af}})) = 0$ *for* $l > N - r$.

(ii) *For* $l = 1, \dots, N - r$,

$$\dim_{\mathbf{F}_q} \mathrm{gr}^{(k)} H_l(\bar{K}.'(S, S_{\mathrm{af}})) = \begin{cases} \dbinom{m}{l + r - n} & \text{if } k = l + r, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) $\dim_{\mathbf{F}_q} \mathrm{gr}^{(k)} H_0(\bar{K}.'(S, S_{\mathrm{af}}))$ *is the coefficient of* t^k *in*

$$\sum_{j=r+1}^{N} (-1)^{j-r-1} \binom{m}{j-n} t^j + \sum_{S_{\mathrm{to}} \subseteq J \subseteq S_{\mathrm{sp}}} (-1)^{N-|J|} \sum_{\substack{e_1+\cdots+e_r \leq |J| \\ e_j \geq 1 \text{ for all } j}} q^J_{e_1\cdots e_r}(t).$$

In particular, $\mathrm{gr}^{(k)} H_0(\bar{K}.'(S, S_{\mathrm{af}})) = 0$ *for* $k < r$ *or* $k > N$.

4. p-Adic estimates

We now prove a lemma on lifting homology from characteristic p to characteristic 0, which will allow us to use the results of section 3 to obtain information about the complexes of section 2. Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer π and let K be an $\mathcal{O}$-module. We call K *flat* if multiplication by π is injective and *separated* if $\bigcap_{j=1}^{\infty} \pi^j K = 0$. A separated $\mathcal{O}$-module K has an obvious metric space structure with the $\{\pi^j K\}$ forming a fundamental system of neighborhoods of 0. We call K $\mathcal{O}$*-complete* if it is complete in this metric.

LEMMA 4.1. – *Let* $K. = \{\cdots \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} 0\}$ *be a complex of flat, separated,* $\mathcal{O}$*-complete* $\mathcal{O}$*-modules with* $\mathcal{O}$*-linear boundary maps. Let* $\bar{K}.$ *be the complex obtained by reducing* $K.$ *modulo* π. *If* $H_l(\bar{K}.)$ *has dimension* d *over* $\mathcal{O}/(\pi)$ *and multiplication by* π *is injective on* $H_l(K.)$ *and* $H_{l-1}(K.)$, *then* $H_l(K.)$ *is a finite, free* $\mathcal{O}$*-module of rank* d. *Furthermore, any lifting of any basis for* $H_l(\bar{K}.)$ *is a basis for* $H_l(K.)$.

Proof. – Consider $H_l(K.) = \ker \partial_l / \mathrm{im}\ \partial_{l+1}$. We claim $\mathrm{im}\ \partial_{l+1}$ is complete. Let $\{\partial_{l+1} z_s\}_{s=1}^{\infty}$ be a Cauchy sequence in $\mathrm{im}\ \partial_{l+1}$, say,

$$\partial_{l+1} z_{s+1} - \partial_{l+1} z_s = \pi^{A(s)} w_s,$$

where $A(s) \to \infty$ as $s \to \infty$. Suppose we have found $\tilde{z}_1, \ldots, \tilde{z}_s \in K_{l+1}$ such that $\partial_{l+1} \tilde{z}_j = \partial_{l+1} z_j$ and $\tilde{z}_{j+1} \equiv \tilde{z}_j \pmod{\pi^{A(j)}}$. Then $\partial_{l+1} z_{s+1} - \partial_{l+1} \tilde{z}_s = \pi^{A(s)} w_s$, so $\pi^{A(s)} w_s = 0$ in $H_l(K.)$. By the injectivity of π on $H_l(K.)$, $w_s = 0$ in $H_l(K.)$, *i.e.*, $w_s = \partial_{l+1} y_s$ for some $y_s \in K_{l+1}$. Hence $\partial_{l+1}(z_{s+1} - \tilde{z}_s - \pi^{A(s)} y_s) = 0$. Put $\tilde{z}_{s+1} = \tilde{z}_s + \pi^{A(s)} y_s$. Then $\tilde{z}_{s+1} \equiv \tilde{z}_s \pmod{\pi^{A(s)}}$ and $\partial_{l+1} \tilde{z}_{s+1} = \partial_{l+1} z_{s+1}$. Since K_{l+1} is complete, the Cauchy sequence $\{\tilde{z}_s\}$ has a limit $\tilde{z}$. It is then clear that $\{\partial_{l+1} z_s\}$ converges to $\partial_{l+1} \tilde{z}$.

There is a short exact sequence of complexes

$$0 \to K. \xrightarrow{\pi} K. \to \bar{K}. \to 0,$$

where the second arrow means multiplication by π. The associated long exact homology sequence and the hypothesis that multiplication by π is injective on $H_{l-1}(K.)$ imply that we have an isomorphism $H_l(K.)/\pi H_l(K.) \simeq H_l(\bar{K}.)$. The completeness of $\mathrm{im}\ \partial_{l+1}$ implies that $H_l(K.)$ is separated. It is then straightforward to check that if $\bar{\xi}_1, \ldots, \bar{\xi}_d$ is a basis for $H_l(\bar{K}.)$, then $\xi_1, \ldots, \xi_d$ is an $\mathcal{O}$-basis for $H_l(K.)$, where ξ_i is any lifting of $\bar{\xi}_i$ to $H_l(K.)$.

We shall apply this result in the following setting. In the definitions of our complexes in section 2, we may replace B by its unit ball $B(\mathcal{O}_0)$, thus obtaining complexes $K.(I, I'; \mathcal{O}_0)$ of $\mathcal{O}_0$-modules. We define a reduction map $\rho : B(\mathcal{O}_0) \to R$ as follows. If $\xi = \sum_{u \in M} A_u \pi^{w(u)} x^u \in B(\mathcal{O}_0)$, then $\rho(\xi) = \sum_{u \in M} \bar{A}_u x^u \in R$, where $\bar{A}_u \in \mathbf{F}_q$ is the reduction of A_u modulo π. There are induced maps $\rho : B_I^{I'}(\mathcal{O}_0) \to R_I^{I'}$ for all $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and $I' \subseteq I_{\text{af}}$. By [1, section 2], the image of the complex $K.(I, I'; \mathcal{O}_0)$ under ρ is the complex $\bar{K}.'(I, I')$. Thus we have short exact sequences of complexes

$$0 \to K.(I, I'; \mathcal{O}_0) \xrightarrow{\pi} K.(I, I'; \mathcal{O}_0) \xrightarrow{\rho} \bar{K}.'(I, I') \to 0. \tag{4.2}$$

Before we can apply Lemma 4.1, we must check that multiplication by π is injective on all $H_l(K.(I, I'; \mathcal{O}_0))$.

LEMMA 4.3. – *Suppose that g is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and $I' \subseteq I_{\text{af}}$ with $I'_{\text{du}} \neq I_{\text{du}}$, multiplication by π is injective on $H_l(K.(I, I'; \mathcal{O}_0))$ for all l.*

Proof. – The long exact homology sequence associated to (4.2) is

$$\begin{aligned} \cdots \to H_{l+1}(\bar{K}.'(I, I')) &\to H_l(K.(I, I'; \mathcal{O}_0)) \\ \xrightarrow{\pi} H_l(K.(I, I'; \mathcal{O}_0)) &\to H_l(\bar{K}.'(I, I')) \to \cdots. \end{aligned} \tag{4.4}$$

The result now follows immediately from Lemma 3.31.

Applying Lemma 4.1 and Lemma 3.31 gives the following.

COROLLARY 4.5. – *Suppose that g is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and $I' \subseteq I_{\text{af}}$ with $I'_{\text{du}} \neq I_{\text{du}}$, one has $H_l(K.(I, I'; \mathcal{O}_0)) = 0$ for $l > 0$ and $H_0(K.(I, I'; \mathcal{O}_0))$ is a free $\mathcal{O}_0$-module of finite rank.*

LEMMA 4.6. – *Suppose that g is nondegenerate and semi-convenient. Then for all $I \subseteq S$ with $S_{\text{to}} \subseteq I$ and for all l, multiplication by π is injective on $H_l(K.(I, I_{\text{af}}; \mathcal{O}_0))$.*

Proof. – The proof is by induction on $|I_{\text{du}}|$. When $I_{\text{du}} = \emptyset$, $K.(I, I_{\text{af}}; \mathcal{O}_0)$ is the complex with $K_l(I, I_{\text{af}}; \mathcal{O}_0) = (\mathcal{O}_0)^{\binom{m}{l-|I_{\text{sp}}|}}$ and all boundary maps trivial. Thus $H_l(K.(I, I_{\text{af}}; \mathcal{O}_0))$ is a free $\mathcal{O}_0$-module of rank $\binom{m}{l-|I_{\text{sp}}|}$, so multiplication by π is injective for all l. Now suppose we know the result for all subsets $I \subseteq S$ with I_{du} of a given cardinality. Let $j \in S_{\text{du}}$, $j \notin I_{\text{du}}$, and put $J = I \cup \{j\}$. There is a short exact sequence of complexes

$$0 \to K.(J, J_{\text{af}}; \mathcal{O}_0) \to K.(J, I_{\text{af}}; \mathcal{O}_0) \xrightarrow{\theta_j} K.(I, I_{\text{af}}; \mathcal{O}_0) \to 0.$$

By Corollary 4.5, the associated long exact homology sequence gives an exact sequence

$$0 \to H_1(I, I_{\text{af}}) \to H_0(J, J_{\text{af}}) \to H_0(J, I_{\text{af}}) \to H_0(I, I_{\text{af}}) \to 0 \tag{4.7}$$

and isomorphisms for $l \geq 1$

$$H_l(J, J_{\mathrm{af}}) \simeq H_{l+1}(I, I_{\mathrm{af}}).$$

The induction hypothesis then implies that multiplication by π is injective on $H_l(J, J_{\mathrm{af}})$ for $l \geq 1$. By the induction hypothesis (resp. Lemma 4.3), multiplication by π is injective on $H_1(I, I_{\mathrm{af}})$ (resp. $H_0(J, I_{\mathrm{af}})$). The injectivity on $H_0(J, J_{\mathrm{af}})$ then follows from the exact sequence (4.7).

By Lemma 4.6, we may apply Lemma 4.1 to obtain a basis for $H_0(K.(S, S_{\mathrm{af}}; \mathcal{O}_0))$ by lifting a basis for $H_0(\bar{K}.'(S, S_{\mathrm{af}}))$. We choose a basis for $H_0(\bar{K}.'(S, S_{\mathrm{af}}))$ as follows. For $k = r, \ldots, N$, let $\mathcal{B}^{(k)} = \{\xi_1^{(k)}, \ldots, \xi_{d(k)}^{(k)}\}$ be a set of monomials of weight k in $\{x^u \mid u \in M\}$ such that their images in $\mathrm{gr}^{(k)} H_0(\bar{K}.'(S, S_{\mathrm{af}}))$ form a basis. By Theorem 3.37,

$$d(k) = \text{coefficient of } t^k \text{ in } \sum_{j=r+1}^{N} (-1)^{j-r-1} \binom{m}{j-n} t^j + \sum_{S_{\mathrm{to}} \subseteq J \subseteq S_{\mathrm{sp}}} (-1)^{N-|J|} \sum_{\substack{e_1+\cdots+e_r \leq |J| \\ e_j \geq 1 \text{ for all } j}} q^J_{e_1 \cdots e_r}(t). \tag{4.8}$$

Clearly then, the set $\mathcal{B} = \bigcup_{k=r}^{N} \mathcal{B}^{(k)}$ is a basis for $H_0(\bar{K}.'(S, S_{\mathrm{af}}))$, so by Lemma 4.1, $\mathcal{B}$ is also a basis for $H_0(K.(S, S_{\mathrm{af}}; \mathcal{O}_0))$.

Let W be the Ω_0-span of $\mathcal{B}$ and set $W(b, c) = W \cap L(b, c)$. From the above remarks, we have

$$B_S^{S_{\mathrm{af}}}(\mathcal{O}_0) = W\left(\frac{1}{p-1}, 0\right) \oplus \sum_{i=1}^{N+r} D_i B_S^{S_{\mathrm{af}} \setminus \{i\}}(\mathcal{O}_0), \tag{4.9}$$

$$B_S^{S_{\mathrm{af}}} = W \oplus \sum_{i=1}^{N+r} D_i B_S^{S_{\mathrm{af}} \setminus \{i\}}. \tag{4.10}$$

Let $L_I^{I'}(b, c) = \bigcap_{i \in I'} \ker(\theta_i \mid L_I(b, c))$. Set $e = b - 1/(p-1)$. Repeating the proofs of [1, Proposition 3.6 and Theorem 3.8] (for a somewhat improved version of these arguments, see [2, Lemma 3.3 and Theorem 3.14]), we get the following.

THEOREM 4.11. – *For $b, c \in \mathbf{R}$ with $1/(p-1) \leq b \leq p/(p-1)$,*

$$L_S^{S_{\mathrm{af}}}(b, c) = W(b, c) + \sum_{i=1}^{N+r} D_i L_S^{S_{\mathrm{af}} \setminus \{i\}}(b, c+e),$$

$$L_S^{S_{\mathrm{af}}}(b) = W \oplus \sum_{i=1}^{N+r} D_i L_S^{S_{\mathrm{af}} \setminus \{i\}}(b).$$

We now follow the method of [10, section 7]. Put $\Omega_1 = \mathbf{Q}_p(\zeta_p)$ and let $\tau \in \mathrm{Gal}(\Omega_0/\Omega_1)$ be a lifting of the Frobenius automorphism of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$. By [1, section 1], there exists

$F(x) \in L(1/(p-1), 0)$ such that the operator $\beta = \tau^{-1} \circ \psi \circ F$, an Ω_1-linear endomorphism of B and of $L(b)$ for $0 < b \leq p/(p-1)$, satisfies $\beta^a = \alpha$. Let $\{\sigma_l\}_{l=1}^a$ be an integral basis for Ω_0 over Ω_1. For $\xi_{j'}^{(k')} \in \mathcal{B}^{(k')}$, write

$$\beta(\pi^{k'} \sigma_{l'} \xi_{j'}^{(k')}) = \sum_{l=1}^{a} \sum_{k=r}^{N} \sum_{j=1}^{d(k)} A(j,k,l;j',k',l') \pi^k \sigma_l \xi_j^{(k)} \quad \left(\bmod \sum_{i=1}^{N+r} D_i B_S^{S_{\mathrm{af}} \setminus \{i\}} \right).$$

By (4.10), the $A(j,k,l;j',k',l')$ are uniquely determined by this relation.

LEMMA 4.12. – ord $A(j,k,l;j',k',l') \geq k$.

Proof. – Since $\pi^{k'} \sigma_{l'} \xi_{j'}^{(k')} \in L(1/(p-1), 0)$, it follows that $\beta(\pi^{k'} \sigma_{l'} \xi_{j'}^{(k')}) \in L(p/(p-1), 0)$. The lemma is then an immediate consequence of Theorem 4.11.

Let ord_q be the p-adic valuation normalized by requiring $\mathrm{ord}_q\, q = 1$. We compute all Newton polygons with respect to this valuation. By the argument that establishes [10, equation (7.7)], we now have the following. (*See* the two paragraphs preceding [10, Theorem 7.1] for the assertion about the endpoints.)

THEOREM 4.13. – *Suppose that g is nondegenerate and semi-convenient. Then the Newton polygon of the polynomial* $\det \left(I - t\bar{\alpha} \mid B_S^{S_{\mathrm{af}}} / \sum_{i=1}^{N+r} D_i B_S^{S_{\mathrm{af}} \setminus \{i\}} \right)$ *lies on or above the Newton polygon of the polynomial* $\prod_{k=r}^{N} (1 - q^k t)^{d(k)}$ *and their endpoints coincide.*

By Theorem 2.19 (iv), we finally have:

COROLLARY 4.14. – *The Newton polygon of the polynomial $P(t)$ lies on or above the Newton polygon of the polynomial* $(1 - q^r t)^{-\binom{m}{r-n}} \prod_{k=r}^{N} (1 - q^k t)^{d(k)}$ *and their endpoints coincide.*

Remark. – Note that, in view of (2.5), the corresponding factor of $Z(V/\mathbf{F}_q; t)$ is $P(q^{-r}t)$, which has lower bound determined by the polynomial $(1-t)^{-\binom{m}{r-n}} \prod_{k=0}^{N-r} (1 - q^k t)^{d(k+r)}$.

5. Hodge polygon

The purpose of this section is to identify the lower bounds of Theorem 4.13 and Corollary 4.14 with certain Hodge polygons of toric and affine complete intersections. This is a rather straightforward calculation, based on the results of [6]. We begin by recalling the definition of Hodge polygon.

Let X be a complex variety and let $H_c^l(X, \mathbf{Q})$ denote rational cohomology with compact supports. By Deligne [7, 8], there is a mixed Hodge structure on $H_c^l(X, \mathbf{Q})$, in particular, there is a decreasing (Hodge) filtration $F^{\cdot}$ on $H_c^l(X, \mathbf{C})$. The *Hodge numbers* $h^k(H_c^l(X, \mathbf{C}))$ are defined by

$$h^k(H_c^l(X, \mathbf{C})) = \dim_{\mathbf{C}} F^k H_c^l(X, \mathbf{C}) / F^{k+1} H_c^l(X, \mathbf{C}),$$

and the *Hodge polygon* of $H^l_c(X,\mathbf{C})$ is defined to be the Newton polygon of the polynomial $\prod_{k\geq 0}(1-q^k t)^{h^k(H^l_c(X,\mathbf{C}))}$. Alternatively, there is also an increasing (weight) filtration $W.$ on $H^l_c(X,\mathbf{Q})$ such that the Hodge filtration $F^\cdot$ induces a pure Hodge structure of weight s on $W_sH^l_c(X,\mathbf{C})/W_{s-1}H^l_c(X,\mathbf{C})$, *i.e.*,

$$W_sH^l_c(X,\mathbf{C})/W_{s-1}H^l_c(X,\mathbf{C}) \simeq \bigoplus_{a+b=s} H^{a,b},$$

where $H^{b,a}=\bar{H}^{a,b}$ and $\bar{H}^{a,b}$ is the complex conjugate of $H^{a,b}$. If we put $h^{a,b}(H^l_c(X,\mathbf{C}))=\dim_{\mathbf{C}} H^{a,b}$, then

$$h^k(H^l_c(X,\mathbf{C}))=\sum_{b\geq 0} h^{k,b}(H^l_c(X,\mathbf{C})).$$

Consider the case where $F_i\in\mathbf{C}[x_1,\dots,x_N,(x_1\cdots x_m)^{-1}]$, $i=1,\dots,r$, is the generic polynomial with the property that the convex hull of $\operatorname{supp}(F_i)$ equals $\Delta_i^{S_{\mathrm{sp}}}$ and let $X\subseteq(\mathbf{T}^m\times\mathbf{A}^n)_{\mathbf{C}}$ be the smooth complete intersection $F_1=\cdots=F_r=0$. Since X is affine, of complex dimension $N-r$, we have $H^l_c(X,\mathbf{C})=0$ for $l<N-r$ or $l>2N-2r$. Furthermore, the Gysin map $H^{N-r+i}_c(X,\mathbf{C})\to H^{N+r+i}_c((\mathbf{T}^m\times\mathbf{A}^n)_{\mathbf{C}},\mathbf{C})$ is an isomorphism for $i=1,\dots,N-r$ and is surjective for $i=0$. It is a morphism of mixed Hodge structures sending $H^{a,b}(X)$ to $H^{a+r,b+r}((\mathbf{T}^m\times\mathbf{A}^n)_{\mathbf{C}})$, and by [6] we have for $i=0,\dots,N$

$$h^{a,b}(H^{N+i}_c((\mathbf{T}^m\times\mathbf{A}^n)_{\mathbf{C}},\mathbf{C}))=\begin{cases}\dbinom{m}{i-n} & \text{if } a=b=i,\\ 0 & \text{otherwise.}\end{cases} \tag{5.1}$$

Hence for $i=1,\dots,N-r$

$$h^{a,b}(H^{N-r+i}_c(X,\mathbf{C}))=\begin{cases}\dbinom{m}{r+i-n} & \text{if } a=b=i,\\ 0 & \text{otherwise,}\end{cases} \tag{5.2}$$

in particular,

$$h^{k}(H^{N-r+i}_c(X,\mathbf{C}))=\begin{cases}\dbinom{m}{r+i-n} & \text{if } k=i,\\ 0 & \text{otherwise.}\end{cases} \tag{5.3}$$

Thus the only nontrivial case is the middle-dimensional cohomology $H^{N-r}_c(X,\mathbf{C})$. The *primitive part* $PH^{N-r}_c(X,\mathbf{C})$ of middle-dimensional cohomology is defined to be the kernel of the (surjective) Gysin map $H^{N-r}_c(X,\mathbf{C})\to H^{N+r}_c((\mathbf{T}^m\times\mathbf{A}^n)_{\mathbf{C}},\mathbf{C})$. By (5.1), $PH^{N-r}_c(X,\mathbf{C})$ has codimension $\dbinom{m}{r-n}$ in $H^{N-r}_c(X,\mathbf{C})$. More precisely,

$$h^k(PH^{N-r}_c(X,\mathbf{C}))=\begin{cases}h^k(H^{N-r}_c(X,\mathbf{C})) & \text{if } k\neq 0,\\ h^0(H^{N-r}_c(X,\mathbf{C}))-\dbinom{m}{r-n} & \text{if } k=0.\end{cases} \tag{5.4}$$

In [6] is described a procedure for computing the Hodge-Deligne numbers of X in terms of invariants of the polytopes Δ_i^I, $S_{\rm to} \subseteq I \subseteq S_{\rm sp}$, $i = 1, \ldots, r$. We carry out part of this procedure in order to verify the following.

THEOREM 5.5. – *The lower bound for the Newton polygon of the polynomial* $P(q^{-r}t)$ *given by Corollary* 4.14 *is the Hodge polygon of* $PH_c^{N-r}(X, \mathbf{C})$.

Proof. – In view of (5.4) and the remark following Corollary 4.14, the theorem is equivalent to the assertion that

$$h^k(H_c^{N-r}(X, \mathbf{C})) = d(k+r) \tag{5.6}$$

for all k, where $d(k)$ is given by (4.8).

Following [6], we set

$$e^k(X) = \sum_{l \geq 0} (-1)^l h^k(H_c^l(X, \mathbf{C})). \tag{5.7}$$

Consider first the case $r = 1$, $n = 0$, *i.e.*, X is the hypersurface $F_1 = 0$ in $\mathbf{T}_{\mathbf{C}}^m$. From section 3, we have

$$\sum_{k=0}^{\infty} \ell(k\Delta_1^{S_{\rm to}}) t^k = (1-t)^{-m-1} \sum_{e_1=0}^{m} q_{e_1 0 \cdots 0}^{S_{\rm to}}(t).$$

One of the basic results of [6] is the following.

THEOREM 5.8 ([6, Remark 4.6]). – *If* X *is the hypersurface* $F_1 = 0$ *in* $\mathbf{T}_{\mathbf{C}}^m$, *then*

$$(-1)^{m-1} e^k(X) = (-1)^k \binom{m}{k+1} + \textit{coefficient of } t^{k+1} \textit{ in } \sum_{e_1=0}^{m} q_{e_1 0 \cdots 0}^{S_{\rm to}}(t).$$

This result may be generalized to complete intersections.

PROPOSITION 5.9. – *If* X *is the complete intersection* $F_1 = \cdots = F_r = 0$ *in* $(\mathbf{T}^m \times \mathbf{A}^n)_{\mathbf{C}}$, *then*

$$(-1)^{N-r} e^k(X) = \textit{coefficient of } t^{k+r} \textit{ in } \sum_{S_{\rm to} \subseteq J \subseteq S_{\rm sp}} (-1)^{N-|J|} \sum_{\substack{e_1 + \cdots + e_r \leq |J| \\ e_j \geq 1 \text{ for all } j}} q_{e_1 \cdots e_r}^J(t).$$

Remark. – This proposition implies Theorem 5.5, because it is now straightforward to prove (5.6) using (5.3) and (4.8).

Proof of Proposition 5.9. – We follow the outline given in [6, section 6]. Put

$$G = x_{N+1} F_1 + \cdots + x_{N+r} F_r - 1$$

and let Y be the hypersurface $G = 0$ in $(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r)_{\mathbf{C}}$. By [6, section 6.2],

$$e^k(X) = e^k((\mathbf{T}^m \times \mathbf{A}^n)_{\mathbf{C}}) - e^{k+r-1}(Y), \tag{5.10}$$

and by (5.1),

$$e^k((\mathbf{T}^m \times \mathbf{A}^n)_{\mathbf{C}}) = (-1)^{N-k}\binom{m}{k-n}. \tag{5.11}$$

We cannot apply Theorem 5.8 directly to compute $e^{k+r-1}(Y)$ because Y is not a hypersurface in a torus but rather a hypersurface in $(\mathbf{T}^m \times \mathbf{A}^n \times \mathbf{A}^r)_{\mathbf{C}}$. So for $I \subseteq S$ with $S_{\text{to}} \subseteq I$ we put $G_I = \sum_{i \in I_{\text{du}}} x_i \theta_{I_{\text{sp}}}(F_{i-N}) - 1$ and let Y_I be the hypersurface $G_I = 0$ in $\mathbf{T}_{\mathbf{C}}^{|I|}$. (Note that $Y_I = \emptyset$ if $I_{\text{du}} = \emptyset$.) Then $Y = \bigcup_{S_{\text{to}} \subseteq I \subseteq S} Y_I$ (a disjoint union) and by [6, Proposition 1.6],

$$e^k(Y) = \sum_{S_{\text{to}} \subseteq I \subseteq S} e^k(Y_I). \tag{5.12}$$

We shall apply Theorem 5.8 to each Y_I to compute $e^k(Y_I)$, then substitute into (5.12) to find $e^k(Y)$ and substitute into (5.10) to find $e^k(X)$.

The convex hull of $\operatorname{supp}(G_I)$ is Δ_I. By (3.9) and (3.12),

$$\sum_{k=0}^{\infty} \ell(k\Delta_I)t^k = (1-t)^{-|I|-1} \sum_{\substack{e_1+\cdots+e_r \leq |I_{\text{sp}}| \\ e_j = 0 \text{ for } j \notin \delta(I)}} q^{I_{\text{sp}}}_{e_1\cdots e_r}(t).$$

So by Theorem 5.8,

$$(-1)^{|I|-1} e^k(Y_I) = (-1)^k \binom{|I|}{k+1} + \text{coefficient of } t^{k+1} \text{ in } \sum_{\substack{e_1+\cdots+e_r \leq |I_{\text{sp}}| \\ e_j = 0 \text{ for } j \notin \delta(I)}} q^{I_{\text{sp}}}_{e_1\cdots e_r}(t).$$

From (5.12),

$$\begin{aligned} e^k(Y) = &\sum_{S_{\text{to}} \subseteq I \subseteq S} (-1)^{|I|+k-1} \binom{|I|}{k+1} \\ &+ \text{coefficient of } t^{k+1} \text{ in } \sum_{S_{\text{to}} \subseteq I \subseteq S} (-1)^{|I|-1} \sum_{\substack{e_1+\cdots+e_r \leq |I_{\text{sp}}| \\ e_j = 0 \text{ for } j \notin \delta(I)}} q^{I_{\text{sp}}}_{e_1\cdots e_r}(t). \end{aligned} \tag{5.13}$$

Using the decomposition $S = S_{\text{sp}} \cup S_{\text{du}}$ we have

$$\begin{aligned} &\sum_{S_{\text{to}} \subseteq I \subseteq S} (-1)^{|I|-1} \sum_{\substack{e_1+\cdots+e_r \leq |I_{\text{sp}}| \\ e_j = 0 \text{ for } j \notin \delta(I)}} q^{I_{\text{sp}}}_{e_1\cdots e_r}(t) \\ &= \sum_{S_{\text{to}} \subseteq I_1 \subseteq S_{\text{sp}}} \sum_{I_2 \subseteq S_{\text{du}}} (-1)^{|I_1|+|I_2|-1} \sum_{\substack{e_1+\cdots+e_r \leq |I_1| \\ e_j = 0 \text{ for } j \notin \delta(I_2)}} q^{I_1}_{e_1\cdots e_r}(t) \\ &= \sum_{S_{\text{to}} \subseteq I_1 \subseteq S_{\text{sp}}} (-1)^{|I_1|+r-1} \sum_{\substack{e_1+\cdots+e_r \leq |I_1| \\ e_j \geq 1 \text{ for all } j}} q^{I_1}_{e_1\cdots e_r}(t) \end{aligned}$$

by a standard inclusion-exclusion argument. We also have

$$\sum_{S_{\mathrm{to}}\subseteq I\subseteq S}(-1)^{|I|+k-1}\binom{|I|}{k+1}=\sum_{j=m}^{N+r}(-1)^{j+k-1}\binom{n+r}{j-m}\binom{j}{k+1}$$
$$=(-1)^{N-r+k-1}\binom{m}{k+1-r-n}$$

by a straightforward combinatorial argument. Thus (5.13) becomes

$$e^k(Y)=(-1)^{N-r+k+1}\binom{m}{k+1-r-n}$$
$$+\text{ coefficient of } t^{k+1} \text{ in } \sum_{S_{\mathrm{to}}\subseteq I_1\subseteq S_{\mathrm{sp}}}(-1)^{|I_1|+r-1}\sum_{\substack{e_1+\cdots+e_r\leq|I_1|\\ e_j\geq 1 \text{ for all } j}} q^{I_1}_{e_1\cdots e_r}(t).$$

Combining this with (5.10) and (5.11) yields Proposition 5.9.

6. Cohomology of projective complete intersections

In this section we change notation slightly. Let $f_1,\dots,f_r\in\mathbf{F}_q[x_0,\dots,x_N]$ be homogeneous polynomials with $\deg f_i=d_i$ for $i=1,\dots,r$. Let $V\subseteq\mathbf{P}^N$ be the projective variety $f_1=\cdots=f_r=0$. Put $g=x_{N+1}f_1+\cdots+x_{N+r}f_r\in\mathbf{F}_q[x_0,\dots,x_{N+r}]$. Let $V^*\subseteq\mathbf{A}^{N+1}$ be the affine variety $f_1=\cdots=f_r=0$. As in (2.2),

$$Z(V^*/\mathbf{F}_q;q^rt)=L(\mathbf{A}^{N+1}\times\mathbf{A}^r,g;t). \tag{6.1}$$

On the other hand, $N_s(V^*)=(q^s-1)N_s(V)+1$, hence

$$Z(V^*/\mathbf{F}_q;t)=Z(V/\mathbf{F}_q;qt)Z(V/\mathbf{F}_q;t)^{-1}(1-t)^{-1}. \tag{6.2}$$

Write

$$Z(V/\mathbf{F}_q;t)=\frac{P(t)^{(-1)^{N-r-1}}}{(1-t)(1-qt)\cdots(1-q^{N-r}t)}.$$

Then (6.1) and (6.2) imply that

$$L(\mathbf{A}^{N+1}\times\mathbf{A}^r,g;t)^{(-1)^{N-r}}=\frac{P(q^rt)(1-q^{N+1}t)^{(-1)^{N-r-1}}}{P(q^{r+1}t)}. \tag{6.3}$$

Let $S=\{0,1,\dots,N+r\}$, $S_{\mathrm{sp}}=\{0,\dots,N\}$, $S_{\mathrm{du}}=\{N+1,\dots,N+r\}$. By (2.13),

$$L(\mathbf{A}^{N+1}\times\mathbf{A}^r,g;t)^{(-1)^{N-r}}=\prod_{l=0}^{N+1+r}\det(I-t\bar{\alpha}_l\mid H_l(K.(S,S)))^{(-1)^l}. \tag{6.4}$$

For purposes of induction, we shall have occasion to consider cases where $S_{\mathrm{sp}}=\emptyset$ or $S_{\mathrm{du}}=\emptyset$. Although such cases have no geometric interpretation in terms of complete intersections, the L-function $L(\mathbf{A}^{N+1} \times \mathbf{A}^r, g; t)$ is still defined. Recall [1] that we say f_i is *commode* if it contains each of the monomials $x_0^{d_i}, \dots, x_N^{d_i}$ with nonzero coefficient.

THEOREM 6.5. – *Suppose that $r \leq N$, g is nondegenerate, and f_i is commode for $i = 1, \dots, r$. Then*

(i) $\dim_{\Omega_0} H_l(K.(S,S)) < \infty$ *for all* l.

(ii) $H_l(K.(S,S)) = 0$ *for* $l \neq 0, 1, N+1-r$.

(iii) *Suppose* $N - r > 0$. *Then* $\dim_{\Omega_0} H_{N+1-r}(K.(S,S)) = 1$ *and Frobenius acts on* $H_{N+1-r}(K.(S,S))$ *as multiplication by* q^{N+1}. *The space* $H_1(K.(S,S))$ *with Frobenius* $\bar{\alpha}_1$ *is isomorphic (as Frobenius module) to* $H_0(K.(S,S))$ *with Frobenius* $q\bar{\alpha}_0$.

(iv) *Suppose* $N - r = 0$. *Then* $H_1(K.(S,S))$ *with Frobenius* $\bar{\alpha}_1$ *is isomorphic to a direct sum* $H_0(K.(S,S)) \oplus H'$, $\dim_{\Omega_0} H' = 1$, *with Frobenius acting on* $H_0(K.(S,S))$ *by* $q\bar{\alpha}_0$ *and acting on* H' *by multiplication by* q^{N+1}.

(v) *The Frobenius endomorphism* $\bar{\alpha}_0$ *is invertible on* $H_0(K.(S,S))$ *and*

$$\dim_{\Omega_0} H_0(K.(S,S)) = (-1)^{N-r+1}(N-r+1) + (-1)^{N+1} \sum_{l=r+1}^{N+1} (-1)^l \binom{N+1}{l} \sum_{\substack{i_1+\cdots+i_r=l-1 \\ i_j \geq 1 \text{ for all } j}} d_1^{i_1} \cdots d_r^{i_r}.$$

Comparing the right-hand sides of (6.3) and (6.4), we have immediately the following.

COROLLARY 6.6. – *Suppose that $r \leq N$, g is nondegenerate, and f_i is commode for $i = 1, \dots, r$. Then*

$$P(q^r t) = \det(I - t\bar{\alpha}_0 \mid H_0(K.(S,S))),$$

in particular, $P(t)$ is a polynomial. Its degree is given by Theorem 6.5 (v).

Remark. – If $f_i = 0$, $i = 1, \dots, r$, are smooth hypersurfaces in $\mathbf{P}^N$ in general position, then one can always make a coordinate change on $\mathbf{P}^N$ (defined over a finite extension of $\mathbf{F}_q$) so that g is nondegenerate and all f_i's are commode. (*See* [3])

The proof of Theorem 6.5 will require several steps. First observe that the homogeneity condition on the f_i's implies that every monomial $x_0^{a_0} \cdots x_N^{a_N} x_{N+1}^{b_1} \cdots x_{N+r}^{b_r}$ appearing in g satisfies

$$a_0 + \cdots + a_N = d_1 b_1 + \cdots + d_r b_r.$$

This in turn implies that every $u \in M = \mathbf{Z}^{N+1+r} \cap C(\Delta)$ satisfies the same condition. It follows that

$$D_0 + \cdots + D_N - d_1 D_{N+1} - \cdots - d_r D_{N+r} = 0 \tag{6.7}$$

as operator on B or $L(b)$. More generally, for $I \subseteq S$ we have

$$\sum_{i \in I_{\mathrm{sp}}} D_{I,i} - \sum_{i \in I_{\mathrm{du}}} d_{i-N} D_{I,i} = 0 \tag{6.8}$$

as operator on B_I or $L_I(b)$. This relation implies that for $I' \subseteq I$, the map from $K_0(I,I') = B_I^{I'}\, e_\emptyset$ to $K_1(I,I') = \bigoplus_{i\in I} B_I^{I'\setminus\{i\}}\, e_{\{i\}}$ defined by

$$\xi\, e_\emptyset \mapsto \sum_{i\in I_{\mathrm{sp}}} \xi\, e_{\{i\}} + \sum_{i\in I_{\mathrm{du}}} (-d_{i-N})\xi\, e_{\{i\}} \tag{6.9}$$

has image lying in the set of 1-cycles. Suppose $\xi\, e_\emptyset$ is a 0-boundary, *i.e.*, there exist $\xi_i \in B_I^{I'\setminus\{i\}}$ such that $\xi = \sum_{i\in I} D_{I,i}(\xi_i)$. Put

$$\epsilon_i = \begin{cases} 1 & \text{if } i \in I_{\mathrm{sp}}, \\ -d_{i-N} & \text{if } i \in I_{\mathrm{du}}, \end{cases}$$

and define $\eta_{ij} = \epsilon_i\xi_j - \epsilon_j\xi_i \in B_I^{I'\setminus\{i,j\}}$ for $i,j \in I$. Then $\{\eta_{ij}\}_{i,j\in I}$ is a skew-symmetric set and $\sum_{j\in J} D_{I,j}(\eta_{ij}) = \epsilon_i\xi$, *i.e.*, the image of $\xi\, e_\emptyset$ under the map (6.9) is a 1-boundary. Thus (6.9) induces a map

$$\phi_I^{I'} : H_0(K.(I,I')) \to H_1(K.(I,I')).$$

It is clear that $\phi_I^{I'} \circ q\bar\alpha_0 = \bar\alpha_1 \circ \phi_I^{I'}$, *i.e.*, $\phi_I^{I'}$ respects the Frobenius structure.

PROPOSITION 6.10. – *Suppose that g is nondegenerate and f_i is commode for $i = 1,\dots,r$. Then for $I' \subseteq I \subseteq S$ with $I'_{\mathrm{sp}} \neq I_{\mathrm{sp}}$ and $I'_{\mathrm{du}} \neq I_{\mathrm{du}}$, we have $H_l(K.(I,I')) = 0$ for $l > 1$,*

$$\dim_{\Omega_0} H_0(K.(I,I')) = \sum_{I_{\mathrm{sp}}\setminus I'_{\mathrm{sp}} \subseteq J \subseteq I_{\mathrm{sp}}} (-1)^{|I_{\mathrm{sp}}|-|J|} \sum_{\substack{\sum_{i\in I_{\mathrm{du}}} l_i = |J|-1 \\ l_i \geq 1 \text{ for } i\in I'_{\mathrm{du}}}} \Big(\prod_{i\in I_{\mathrm{du}}} d_i^{l_i}\Big), \tag{6.11}$$

$\bar\alpha_0$ is invertible on $H_0(K.(I,I'))$, and $\phi_I^{I'}$ is an isomorphism.

Proof. – The proof is by induction on $|I'|$. When $I' = \emptyset$, Proposition 8.3 of the appendix says that $H_l(K.(I,I')) = 0$ for $l > 1$ and $\phi_I^{I'}$ is an isomorphism. The invertibility of $\bar\alpha_0$ is given by [1, Theorem 3.13] and the formula for $\dim_{\Omega_0} H_0(K.(I,I'))$ follows from [1, Theorem 2.9] and a calculation as in the derivation of (2.18).

Suppose $|I'| > 0$ and let $i \in I'$. Then there is a short exact sequence of complexes

$$0 \to K.(I,I') \to K.(I, I'\setminus\{i\}) \to K.(I\setminus\{i\}, I'\setminus\{i\}) \to 0.$$

The vanishing of $H_l(K.(I,I'))$ for $l > 1$ follows immediately from the associated long exact homology sequence and application of the induction hypothesis to $K.(I, I'\setminus\{i\})$ and $K.(I\setminus\{i\}, I'\setminus\{i\})$. The maps $\phi_I^{I'\setminus\{i\}}$ and $\phi_{I\setminus\{i\}}^{I'\setminus\{i\}}$ are isomorphisms by the induction hypothesis, so Proposition 8.6 of the appendix implies that $\phi_I^{I'}$ is an isomorphism and that the sequence

$$0 \to H_0(K.(I,I')) \to H_0(K.(I,I'\setminus\{i\})) \to H_0(K.(I\setminus\{i\}, I'\setminus\{i\})) \to 0$$

is exact. By induction, $\bar{\alpha}_0$ on $H_0(K.(I, I' \setminus \{i\}))$ and on $H_0(K.(I \setminus \{i\}, I' \setminus \{i\}))$ is an isomorphism, hence $\bar{\alpha}_0$ on $H_0(K.(I, I'))$ is also an isomorphism. We also get

$$\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_0(K.(I, I' \setminus \{i\})) - \dim_{\Omega_0} H_0(K.(I \setminus \{i\}, I' \setminus \{i\})),$$

so the desired formula for $\dim_{\Omega_0} H_0(K.(I, I'))$ follows from the induction hypothesis.

Remark. – Note that in the case $|I'_{\mathrm{du}}| \geq |I_{\mathrm{sp}}|$, the proposition implies all homology vanishes since the index set for the inner sum on the right-hand side of (6.11) is empty.

PROPOSITION 6.12. – *Suppose that g is nondegenerate and f_i is commode for $i = 1, \ldots, r$. Let $I' \subseteq I \subseteq S$ with $I'_{\mathrm{sp}} = I_{\mathrm{sp}}$ but $I'_{\mathrm{du}} \neq I_{\mathrm{du}}$. For notational convenience, set $|I'_{\mathrm{du}}| - |I_{\mathrm{sp}}| = k$ and $|I_{\mathrm{du}}| - |I_{\mathrm{sp}}| = \tilde{k}$. Then $H_l(K.(I, I')) = 0$ if*

$$l \notin \{0, 1\} \cup \{k, k+1, \ldots, \tilde{k}\}.$$

If in addition $k \geq 0$, then

$$H_l(K.(I, I')) = 0 \qquad \text{if } l \notin \{k, k+1, \ldots, \tilde{k}\}, \tag{6.13}$$

$$H_l(K.(I, I')) \simeq (\Omega_0)^{\binom{\tilde{k}-k}{l-k}} \qquad \text{if } l \in \{k, k+1, \ldots, \tilde{k}\}. \tag{6.14}$$

Proof. – The proof is by induction on $|I_{\mathrm{sp}}|$. When $I_{\mathrm{sp}} = \emptyset$, one has

$$B_I^{I' \setminus A} = \begin{cases} \Omega_0 & \text{if } I' \subseteq A, \\ 0 & \text{if } I' \not\subseteq A, \end{cases}$$

hence

$$K_l(I, I') = \bigoplus_{I' \subseteq A \subseteq I,\ |A| = l} \Omega_0 \, e_A$$

and all boundary maps of the complex $K.(I, I')$ are zero. Thus

$$H_l(K.(I, I')) \simeq (\Omega_0)^{\binom{|I| - |I'|}{l - |I'|}}, \tag{6.15}$$

which implies the proposition when $|I_{\mathrm{sp}}| = 0$.

If $|I_{\mathrm{sp}}| > 0$, let $i \in I_{\mathrm{sp}}$ and consider the short exact sequence

$$0 \to K.(I, I') \to K.(I, I' \setminus \{i\}) \to K.(I \setminus \{i\}, I' \setminus \{i\}) \to 0. \tag{6.16}$$

The complex $K.(I, I' \setminus \{i\})$ satisfies the hypotheses of Proposition 6.10, so $H_l(K.(I, I' \setminus \{i\})) = 0$ if $l \neq 0, 1$. Thus the long exact homology sequence associated to (6.16) gives isomorphisms

$$H_{l+1}(K.(I \setminus \{i\}, I' \setminus \{i\})) \simeq H_l(K.(I, I')) \tag{6.17}$$

for $l > 1$. The assertion for $H_l(K.(I, I'))$ now follows by applying the induction hypothesis to $H_{l+1}(K.(I \setminus \{i\}, I' \setminus \{i\}))$. Suppose in addition that $|I'_{\mathrm{du}}| \geq |I_{\mathrm{sp}}|$. Then

$H_l(K.(I, I' \setminus \{i\})) = 0$ for all l by equation (6.11), so (6.17) holds for all $l \geq 0$. Induction on $|I_{\text{sp}}|$ now reduces (6.13) and (6.14) to the case $I_{\text{sp}} = \emptyset$, where the result follows from (6.15).

When $I_{\text{sp}} = I'_{\text{sp}}$ and $|I_{\text{du}}| = |I'_{\text{du}}| + 1$, there is a simple way to check when $\phi_I^{I'}$ is an isomorphism. To fix ideas, suppose $I_{\text{du}} = I'_{\text{du}} \cup \{N+r\}$ and let $\Xi = \sum_{i\in I} \xi_i \, e_{\{i\}} \in K_1(I, I')$ be a 1-cycle. Then $\xi_i \in B_I^{I'\setminus\{i\}}$, in particular, $\xi_{N+r} \in B_I^{I'}$. If the homology class represented by this 1-cycle lies in the image of $\phi_I^{I'}$, then there exists $\xi \in B_I^{I'}$ such that Ξ is homologous to the 1-cycle $\displaystyle\sum_{i\in I_{\text{sp}}} \xi \, e_{\{i\}} + \sum_{i\in I_{\text{du}}} (-d_{i-N})\xi \, e_{\{i\}}$. Thus there is a skew-symmetric set $\eta_{ij} \in B_I^{I'\setminus\{i,j\}}$ such that

$$\sum_{j\in I} D_j(\eta_{ij}) = \xi_i - \begin{cases} \xi & \text{if } i \in I_{\text{sp}}, \\ (-d_{i-N})\xi & \text{if } i \in I_{\text{du}}. \end{cases}$$

Taking $i = N + r$ we get $\eta_{N+r,j} \in B_I^{I'\setminus\{j\}}$ such that

$$\xi_{N+r} + d_r\xi = \sum_{j\in I} D_j(\eta_{N+r,j}).$$

But this equation says that ξ_{N+r} and $-d_r\xi$ are homologous 0-cycles in $H_0(K.(I, I'))$. In other words, the map defined by sending Ξ to $(-1/d_r)\xi_{N+r}$ is, by the argument above, defined on homology classes in the image of $\phi_I^{I'}$ and is a left inverse to $\phi_I^{I'}$:

$$\psi_I^{I'} \circ \phi_I^{I'} = \text{identity} : H_0(K.(I, I')) \to H_0(K.(I, I')).$$

Thus $\phi_I^{I'}$ is an isomorphism whenever

$$\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_1(K.(I, I')) < \infty.$$

COROLLARY 6.18. – *Suppose that g is nondegenerate and f_i is commode for $i = 1, \ldots, r$. Let $I' \subseteq I \subseteq S$ with $I'_{\text{sp}} = I_{\text{sp}}$ and $|I_{\text{du}}| = |I'_{\text{du}}| + 1 = |I_{\text{sp}}| + 1$. Then $H_l(K.(I, I')) = 0$ for $l > 1$, $\phi_I^{I'}$ is an isomorphism, and* $\dim_{\Omega_0} H_0(K.(I, I')) = 1$.

Proof. – It follows from (6.13) that $H_l(K.(I, I')) = 0$ for $l > 1$. To prove the remaining assertions, we see by the above discussion that it suffices to check that

$$\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_1(K.(I, I')) = 1.$$

But this follows from (6.14).

We now analyze the case $I = S$, $I' = S \setminus \{N+r\}$ in more detail.

COROLLARY 6.19. – *Suppose that $1 \leq r \leq N+2$, g is nondegenerate, and f_i is commode for $i = 1, \ldots, r$. Then $H_l(K.(S, S \setminus \{N+r\})) = 0$ if $l > 1$ and $\phi_S^{S\setminus\{N+r\}}$ is an isomorphism. Furthermore,*

$$(6.20) \quad \dim_{\Omega_0} H_0(K.(S, S \setminus \{N+r\})) \\ = (-1)^{N+r} + (-1)^{N+1} \sum_{l=r}^{N+1} (-1)^l \binom{N+1}{l} \sum_{\substack{i_1+\cdots+i_r=l-1 \\ i_1,\ldots,i_{r-1}\geq 1}} d_1^{i_1} \cdots d_r^{i_r}.$$

Proof. – For notational convenience, we put $S' = S \setminus \{N+r\}$. The vanishing of $H_l(K.(S,S'))$ for $l > 1$ is an immediate consequence of Proposition 6.12 and the hypothesis that $r \leq N+2$. We prove that $\phi_S^{S'}$ is an isomorphism by induction on $N+2-r$. Suppose that $r < N+2$ and consider the short exact sequence

$$0 \to K.(S,S') \to K.(S,S' \setminus \{N\}) \to K.(S \setminus \{N\}, S' \setminus \{N\}) \to 0. \tag{6.21}$$

By Proposition 6.10, $\phi_S^{S'\setminus\{N\}}$ is an isomorphism, so by Proposition 8.6 of the appendix, $\phi_S^{S'}$ will be an isomorphism provided $\phi_{S\setminus\{N\}}^{S'\setminus\{N\}}$ is. Thus by induction it suffices to establish the result when $r = N+2$. But $I = S$, $I' = S'$, $r = N+2$, is exactly the case covered by Corollary 6.18. Note that Corollary 6.18 also establishes (6.20) when $r = N+2$. We also prove (6.20) by induction on $N+2-r$. Since $\phi_{S\setminus\{N\}}^{S'\setminus\{N\}}$ is an isomorphism, Proposition 8.6 of the appendix implies that the connecting homomorphism in the exact homology sequence associated to (6.21) is zero, hence

$$\dim_{\Omega_0} H_0(K.(S,S')) = \dim_{\Omega_0} H_0(K.(S,S'\setminus\{N\})) - \dim_{\Omega_0} H_0(K.(S\setminus\{N\},S'\setminus\{N\})).$$

The value of the first term on the right-hand side is given by (6.11). The desired expression for the left-hand side then follows by applying the induction hypothesis to the second term on the right-hand side.

Proof of Theorem 6.5. – Note that Theorem 6.5 (i) is a consequence of Theorem 6.5 (ii)-(v), so we need only prove parts (ii)-(v). We proceed by induction on r, proving additionally that when $N - r > 0$, ϕ_S^S is an isomorphism. When $r = 0$, *i.e.*, $S = \{0, 1, \ldots, N\}$, it is straightforward to check that $K.(S,S)$ is the complex with $K_{N+1}(S,S) = \Omega_0$ and $K_l(S,S) = 0$ for $l \neq N+1$, where all boundary maps are zero and Frobenius acts on $K_{N+1}(S,S)$ as multiplication by q^{N+1}. This establishes Theorem 6.5 in that case (when one interprets appropriately the sum appearing in Theorem 6.5 (v)).

Suppose $r \geq 1$. We again put $S' = S \setminus \{N+r\}$. There is a short exact sequence of complexes

$$0 \to K.(S,S) \to K.(S,S') \to K.(S',S') \to 0. \tag{6.22}$$

By Corollary 6.19, $H_l(K.(S,S')) = 0$ if $l \neq 0, 1$, and by the induction hypothesis, $H_l(K.(S',S')) = 0$ if $l \neq 0, 1, N+2-r$. It follows from the associated long exact homology sequence that $H_l(K.(S,S)) = 0$ if $l \neq 0, 1, N+1-r$, establishing Theorem 6.5 (ii).

If $N - r > 0$, then the long exact homology sequence gives an isomorphism

$$H_{N+2-r}(K.(S',S')) \simeq H_{N+1-r}(K.(S,S)),$$

hence, by the induction hypothesis, $\dim_{\Omega_0} H_{N+1-r}(K.(S,S)) = 1$ and Frobenius acts as multiplication by q^{N+1}. When $N - r > 0$, we also have by the induction hypothesis that $H_2(K.(S',S')) = 0$ and $\phi_{S'}^{S'}$ is an isomorphism. By Corollary 6.19, $\phi_S^{S'}$ is also an isomorphism, hence ϕ_S^S is an isomorphism by Proposition 8.6 of the appendix. This establishes Theorem 6.5(iii).

Now suppose $N - r = 0$. By the induction hypothesis, $\phi_{S'}^{S'}$ is an isomorphism, and by Corollary 6.19, $\phi_S^{S'}$ is an isomorphism. Hence by Proposition 8.6 of the appendix, the connecting homomorphism $H_1(K.(S',S')) \to H_0(K.(S,S))$ of the long exact homology sequence associated to (6.22) is zero. Thus the nontrivial terms of the long exact homology sequence give us a commutative diagram with exact rows (For typographical convenience, in this diagram we denote $H_l(K.(I,I'))$ by $H_l(K_I^{I'})$.):

$$\begin{array}{ccccccccc} 0 \to & H_2(K_{S'}^{S'}) & \xrightarrow{\tau} & H_1(K_S^S) & \to & H_1(K_S^{S'}) & \to & H_1(K_{S'}^{S'}) & \to 0 \\ & & & \phi_S^S \uparrow & & \phi_S^{S'} \uparrow & & \phi_{S'}^{S'} \uparrow & \\ 0 & & \to & H_0(K_S^S) & \to & H_0(K_S^{S'}) & \to & H_0(K_{S'}^{S'}) & \to 0. \end{array} \tag{6.23}$$

By the induction hypothesis, $\dim_{\Omega_0} H_2(K.(S',S')) = 1$ and Frobenius acts on it as multiplication by q^{N+1}. Thus $\tau(H_2(K.(S',S')))$ is a one-dimensional subspace of $H_1(K.(S,S))$ with Frobenius acting as multiplication by q^{N+1}. A diagram chase, using the above observation that the two right-most vertical arrows in (6.23) are isomorphisms, now shows that

$$H_1(K.(S,S)) \simeq \tau(H_2(K.(S',S'))) \oplus \phi_S^S(H_0(K.(S,S))), \tag{6.24}$$

which establishes Theorem 6.5(iv).

What we have done so far implies that

$$\dim_{\Omega_0} H_0(K.(S,S)) = \dim_{\Omega_0} H_0(K.(S,S')) - \dim_{\Omega_0} H_0(K.(S',S')).$$

Evaluating the first term on the right-hand side by Corollary 6.19 and the second term on the right-hand side by the induction hypothesis establishes Theorem 6.5 (v).

7. Newton and Hodge polygons in the projective case

The calculation of a lower bound for the Newton polygon of $P(t)$ proceeds exactly as in sections 3 and 4, therefore we just summarize results while pointing out some differences in the projective case.

Let the ring R be defined as in section 3 and let $\bar{K}. = K.(R, \{g_i\}_{i=0}^{N+r})$ be the Koszul complex on R defined by $g_0, \ldots, g_{N+r}$. As in section 3, one can define the related complexes $\bar{K}.(I)$, $\bar{K}.(I,I')$ for $I' \subseteq I \subseteq S$. All these complexes are graded as in section 3.

Let $I \subseteq S$ with $I_{\rm sp} \neq \emptyset$. The homogeneity condition on $f_1, \ldots, f_r$ implies that the polytopes $\Delta_1^{I_{\rm sp}}, \ldots, \Delta_r^{I_{\rm sp}}$ are $(|I_{\rm sp}|-1)$-dimensional (instead of $|I_{\rm sp}|$-dimensional). Thus $\ell(k_1\Delta_1^{I_{\rm sp}} + \cdots + k_r\Delta_r^{I_{\rm sp}})$ is a rational polynomial of degree $\leq |I_{\rm sp}|-1$ in $k_1, \ldots, k_r$, say,

$$\ell(k_1\Delta_1^{I_{\rm sp}} + \cdots + k_r\Delta_r^{I_{\rm sp}}) = \sum_{e_1+\cdots+e_r \leq |I_{\rm sp}|-1} a_{e_1\cdots e_r}^{I_{\rm sp}} k_1^{e_1} \cdots k_r^{e_r} \in \mathbf{Q}[k_1, \ldots, k_r].$$

The hypothesis that g is commode implies that for each j, $\Delta_j^{I_{\rm sp}}$ is the $(|I_{\rm sp}|-1)$-simplex in $\mathbf{R}^{|I_{\rm sp}|}$ with vertices $(d_j, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_j)$, so, in fact,

$$\ell(k_1\Delta_1^{I_{\rm sp}} + \cdots + k_r\Delta_r^{I_{\rm sp}}) = \binom{k_1d_1 + \cdots + k_rd_r + |I_{\rm sp}| - 1}{|I_{\rm sp}| - 1}.$$

This formula shows that the lattice point function on the left-hand side depends only on $|I_{\mathrm{sp}}|$, not on I_{sp} itself, thus the same is true for the formulas that follow.

The polynomials $p^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t_1,\dots,t_r)$ can be defined as before and we have

$$\sum_{k=0}^{\infty}(\dim_{\mathbf{F}_q} R_I^{(k)})t^k = \sum_{\substack{e_1+\cdots+e_r\leq |I_{\mathrm{sp}}|-1\\ e_j=0 \text{ for } j\notin\delta(I)}} \frac{p^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t,\dots,t)}{(1-t)^{|I_{\mathrm{du}}|+\sum_{j=1}^r e_j}}.$$

Define

$$q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t) = p^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t,\dots,t)(1-t)^{|I_{\mathrm{sp}}|-1-\sum_{j=1}^r e_j} \in \mathbf{Q}[t].$$

Then

$$\sum_{k=0}^{\infty}(\dim_{\mathbf{F}_q} R_I^{(k)})t^k = (1-t)^{-|I|+1} \sum_{\substack{e_1+\cdots+e_r\leq |I_{\mathrm{sp}}|-1\\ e_j=0 \text{ for } j\notin\delta(I)}} q^{I_{\mathrm{sp}}}_{e_1\cdots e_r}(t). \tag{7.1}$$

Using induction on $|I'|$, as in the proof of Proposition 6.10, gives the following. (The case $|I'|=0$ is Proposition 8.4 of the appendix.)

PROPOSITION 7.2. – *Suppose that g is nondegenerate and f_i is commode for $i=1,\dots,r$. Then for $I'\subseteq I\subseteq S$ with $I'_{\mathrm{sp}}\neq I_{\mathrm{sp}}$ and $I'_{\mathrm{du}}\neq I_{\mathrm{du}}$ we have $H_l(\bar{K}.(I,I'))=0$ for $l>1$,*

$$\dim_{\mathbf{F}_q} H_1(\bar{K}.(I,I'))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(I,I'))^{(k-1)},$$

and $\dim_{\mathbf{F}_q} H_0(\bar{K}.(I,I'))^{(k)}$ is the coefficient of t^k in

$$\sum_{I_{\mathrm{sp}}\setminus I'_{\mathrm{sp}}\subseteq J\subseteq I_{\mathrm{sp}}} (-1)^{|I_{\mathrm{sp}}|-|J|} \sum_{\substack{e_1+\cdots+e_r=|J|-1\\ e_i=0 \text{ if } i\notin\delta(I)\\ e_i\geq 1 \text{ if } i\in\delta(I')}} q^{J}_{e_1\cdots e_r}(t).$$

Arguing as in the proof of Proposition 6.12 gives the following.

PROPOSITION 7.3. – *Suppose that g is nondegenerate and f_i is commode for $i=1,\dots,r$. Let $I'\subseteq I\subseteq S$ with $I'_{\mathrm{sp}}=I_{\mathrm{sp}}$ but $I'_{\mathrm{du}}\neq I_{\mathrm{du}}$. Then $H_l(\bar{K}.(I,I'))=0$ if*

$$l\notin\{0,1\}\cup\{|I'_{\mathrm{du}}|-|I_{\mathrm{sp}}|, |I'_{\mathrm{du}}|-|I_{\mathrm{sp}}|+1,\dots,|I_{\mathrm{du}}|-|I_{\mathrm{sp}}|\}.$$

If in addition $|I'_{\mathrm{du}}|\geq|I_{\mathrm{sp}}|$, then $H_l(\bar{K}.(I,I'))=0$ for $l\notin\{|I'_{\mathrm{du}}|-|I_{\mathrm{sp}}|,\dots,|I_{\mathrm{du}}|-|I_{\mathrm{sp}}|\}$ and

$$\dim_{\mathbf{F}_q} H_l(\bar{K}.(I,I'))^{(k)} = \begin{cases} \dbinom{|I_{\mathrm{du}}|-|I'_{\mathrm{du}}|}{l+|I_{\mathrm{sp}}|-|I'_{\mathrm{du}}|} & \text{if } l\in\{|I'_{\mathrm{du}}|-|I_{\mathrm{sp}}|,\dots,|I_{\mathrm{du}}|-|I_{\mathrm{sp}}|\} \text{ and } k=l+|I_{\mathrm{sp}}|,\\ 0 & \text{otherwise.}\end{cases}$$

We record a special case of this result, an analogue of Corollary 6.18.

COROLLARY 7.4. – *Suppose that g is nondegenerate and f_i is commode for $i = 1, \ldots, r$. Let $I' \subseteq I \subseteq S$ with $I'_{\mathrm{sp}} = I_{\mathrm{sp}}$ and $|I_{\mathrm{du}}| = |I'_{\mathrm{du}}| + 1 = |I_{\mathrm{sp}}| + 1$. Then $H_l(\bar{K}.(I, I')) = 0$ if $l > 1$, $\dim_{\mathbf{F}_q} H_1(\bar{K}.(I, I'))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(I, I'))^{(k-1)}$, and*

$$\dim_{\mathbf{F}_q} H_0(\bar{K}.(I, I'))^{(k)} = \begin{cases} 1 & \text{if } k = |I_{\mathrm{sp}}|, \\ 0 & \text{if } k \neq |I_{\mathrm{sp}}|. \end{cases}$$

Following the proof of Corollary 6.19, we arrive at the following.

COROLLARY 7.5. – *Suppose that $1 \leq r \leq N + 2$, g is nondegenerate, and f_i is commode for $i = 1, \ldots, r$. Then $H_l(\bar{K}.(S, S \setminus \{N + r\})) = 0$ if $l > 1$,*

$$\dim_{\mathbf{F}_q} H_1(\bar{K}.(S, S \setminus \{N + r\}))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(S, S \setminus \{N + r\}))^{(k-1)},$$

and $\dim_{\mathbf{F}_q} H_0(\bar{K}.(S, S \setminus \{N + r\}))^{(k)}$ is the coefficient of t^k in

$$(-1)^{N+r} t^{r-1} + \sum_{J \subseteq S_{\mathrm{sp}}} (-1)^{N+1-|J|} \sum_{\substack{e_1 + \cdots + e_r \leq |J| - 1 \\ e_j \geq 1 \text{ for all } j}} q^J_{e_1 \cdots e_r}(t).$$

We finally arrive at the analogue of Theorem 6.5.

THEOREM 7.6. – *Suppose that $1 \leq r \leq N$, g is nondegenerate, and f_i is commode for $i = 1, \ldots, r$. Then*

(i) *$\dim_{\mathbf{F}_q} H_l(\bar{K}.(S, S)) = 0$ for $l \neq 0, 1, N + 1 - r$.*

(ii) *$\dim_{\mathbf{F}_q} H_0(\bar{K}.(S, S))^{(k)}$ is the coefficient of t^k in*

$$(7.7) \quad (-1)^{N-r+1}(t^r + t^{r+1} + \cdots + t^N) + \sum_{J \subseteq S_{\mathrm{sp}}} (-1)^{N+1-|J|} \sum_{\substack{e_1 + \cdots + e_r \leq |J| - 1 \\ e_j \geq 1 \text{ for all } j}} q^J_{e_1 \cdots e_r}(t).$$

(iii) *If $N - r > 0$, then $\dim_{\mathbf{F}_q} H_1(\bar{K}.(S, S))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(S, S))^{(k-1)}$ and*

$$\dim_{\mathbf{F}_q} H_{N+1-r}(\bar{K}.(S, S))^{(k)} = \begin{cases} 1 & if\ k = N + 1, \\ 0 & otherwise. \end{cases}$$

(iv) *If $N - r = 0$, then*

$$\dim_{\mathbf{F}_q} H_1(\bar{K}.(S, S))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(S, S))^{(k-1)} + \begin{cases} 1 & if\ k = N + 1, \\ 0 & otherwise. \end{cases}$$

The generalization of the rest of sections 3 and 4 is straightforward. Let $d(k)$ be the coefficient of t^k in expression (7.7).

THEOREM 7.8. – *Suppose that $1 \leq r \leq N$, g is nondegenerate, and f_i is commode for $i = 1, \ldots, r$. Then the Newton polygon of the polynomial $P(t) = \det\left(I - t\bar{\alpha}_0 \mid B^S_S / \sum_{i=0}^{N+r} D_i B^{S \setminus \{i\}}_S\right)$ lies on or above the Newton polygon of the polynomial $\prod_{k=r}^{N} (1 - q^k t)^{d(k)}$ and their endpoints coincide.*

Remark. – The corresponding factor of $Z(V/\mathbf{F}_q;t)$ is $P(q^{-r}t)$, which has lower bound determined by the polynomial $\prod_{k=0}^{N-r}(1-q^kt)^{d(k+r)}$.

We now describe the Hodge polygon in the projective case. Let $F_1,\ldots,F_r\in\mathbf{C}[x_0,\ldots,x_N]$ be the generic homogeneous polynomials of degrees $d_1,\ldots,d_r$, respectively, and let $X\subseteq\mathbf{P}^N_{\mathbf{C}}$ be the smooth complete intersection $F_1=\cdots=F_r=0$. For $J\subseteq S_{\rm sp}$, let $U_J\subseteq\mathbf{P}^N_{\mathbf{C}}$ be the subset consisting of those points whose homogeneous coordinates $(x_0,\ldots,x_N)$ satisfy $x_j\neq 0$ if and only if $j\in J$. Thus $U_J\simeq\mathbf{T}^{|J|-1}_{\mathbf{C}}$ and there is a decomposition of $\mathbf{P}^N_{\mathbf{C}}$ as $\mathbf{P}^N_{\mathbf{C}}=\bigcup_{\emptyset\neq J\subseteq S_{\rm sp}}U_J$. Putting $X_J=X\cap U_J$, we get a decomposition

$$X=\bigcup_{\emptyset\neq J\subseteq S_{\rm sp}}X_J. \tag{7.9}$$

For each $J\subseteq S_{\rm sp}$, $J\neq\emptyset$, let $\tilde{F}_i^J$ be the dehomogenization of F_i with respect to any x_j, $j\in J$, and let $\tilde{\Delta}_i^J\subseteq\mathbf{R}^{|J|-1}$ be the polytope and $\tilde{q}^J_{e_1\cdots e_r}(t)$ be the polynomials associated to $\tilde{F}_i^J$ as in section 3. The genericity of F_i implies that these are independent of the choice of x_j. In fact, $\tilde{\Delta}_i^J$ is the $(|J|-1)$-simplex with vertices at the origin and $(d_i,0,\ldots,0),\ldots,(0,\ldots,0,d_i)$. By Proposition 5.9, $(-1)^{|J|-1-r}e^k(X_J)$ is the coefficient of t^k in

$$\sum_{\substack{e_1+\cdots+e_r\leq|J|-1\\ e_j\geq 1\text{ for all }j}}\tilde{q}^J_{e_1\cdots e_r}(t).$$

It is easily checked that $\tilde{q}^J_{e_1\cdots e_r}(t)=q^J_{e_1\cdots e_r}(t)$ for all J and all $e_1,\ldots,e_r$. Furthermore, the decomposition (7.9) implies that $e^k(X)=\sum_{\emptyset\neq J\subseteq S_{\rm sp}}e^k(X_J)$, hence $(-1)^{N-r}e^k(X)$ is the coefficient of t^k in

$$\sum_{\emptyset\neq J\subseteq S_{\rm sp}}(-1)^{N+1-|J|}\sum_{\substack{e_1+\cdots+e_r\leq|J|-1\\ e_j\geq 1\text{ for all }j}}q^J_{e_1\cdots e_r}(t).$$

Using well-known facts about the cohomology of smooth complete intersections in $\mathbf{P}^N_{\mathbf{C}}$, we conclude that

$$h^k(PH_c^{N-r}(X,\mathbf{C}))=d(k+r).$$

COROLLARY 7.10. – *The lower bound for the Newton polygon of the polynomial $P(q^{-r}t)$ given by Theorem 7.8 is the Hodge polygon of $PH_c^{N-r}(X,\mathbf{C})$.*

8. Appendix

We begin by proving an elementary result about Koszul complexes for which we do not know a reference. Let A be a commutative ring, M an A-module, and $\sigma_1, \dots, \sigma_n$ commuting endomorphisms of M as A-module. We compare the homology of the Koszul complexes $K.(M, \{\sigma_i\}_{i=1}^{n})$ and $K.(M, \{\sigma\}_{i=1}^{n-1})$ under the assumption that there is a relation

$$\sigma_n = \sum_{i=1}^{n-1} a_i \sigma_i, \qquad a_i \in A. \tag{8.1}$$

Clearly this implies that $H_0(K.(M, \{\sigma_i\}_{i=1}^{n})) = H_0(K.(M, \{\sigma_i\}_{i=1}^{n-1}))$. It also implies that the map

$$m\, e_{\emptyset} \mapsto m\, e_{\{n\}} + \sum_{i=1}^{n-1} (-a_i) m\, e_{\{i\}}$$

induces a homomorphism $\phi : H_0(K.(M, \{\sigma_i\}_{i=1}^{n})) \to H_1(K.(M, \{\sigma_i\}_{i=1}^{n}))$.

PROPOSITION 8.2. – *Suppose* $H_l(K.(M, \{\sigma_i\}_{i=1}^{n-1})) = 0$ *for* $l > 0$. *Then* $H_l(K.(M, \{\sigma_i\}_{i=1}^{n})) = 0$ *for* $l > 1$ *and* ϕ *is an isomorphism.*

Proof. – Let $K.^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1})$ be the complex $K.(M, \{\sigma_i\}_{i=1}^{n-1})$ shifted one position to the left, *i.e.*,

$$K_l^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1}) = K_{l-1}(M, \{\sigma_i\}_{i=1}^{n-1}).$$

There is a short exact sequence of complexes

$$0 \to K.(M, \{\sigma_i\}_{i=1}^{n-1}) \to K.(M, \{\sigma_i\}_{i=1}^{n}) \to K.^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1}) \to 0,$$

where the map $K_l(M, \{\sigma_i\}_{i=1}^{n-1}) \to K_l(M, \{\sigma_i\}_{i=1}^{n})$ is the natural inclusion defined by $m\, e_T \mapsto m\, e_T$ for $T \subseteq \{1, \dots, n-1\}$, $|T| = l$, and the map

$$K_l(M, \{\sigma_i\}_{i=1}^{n}) \to K_l^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1}) = K_{l-1}(M, \{\sigma\}_{i=1}^{n-1})$$

is defined by

$$m\, e_T \mapsto \begin{cases} 0 & \text{if } n \notin T, \\ m\, e_{T \setminus \{n\}} & \text{if } n \in T, \end{cases}$$

for $T \subseteq \{1, \dots, n\}$, $|T| = l$. Since $H_l(K.(M, \{\sigma_i\}_{i=1}^{n-1})) = 0$ for $l > 0$, the associated long exact homology sequence immediately implies that $H_l(K.(M, \{\sigma_i\}_{i=1}^{n})) = 0$ for $l > 1$. The connecting homomorphism

$$H_1(K.^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1})) = H_0(K.(M, \{\sigma_i\}_{i=1}^{n-1})) \to H_0(K.(M, \{\sigma_i\}_{i=1}^{n-1}))$$

sends the homology class of $m \in M$ to the homology class of $\sigma_n(m) \in M$. But $\sigma_n(m) = \sum_{i=1}^{n-1} a_i\sigma_i(m)$, so this homology class is trivial, *i.e.*, the connecting homomorphism is zero. Thus the long exact homology sequence gives isomorphisms

$$H_1(K.(M,\{\sigma_i\}_{i=1}^n)) \simeq H_1(K.^{(1)}(M,\{\sigma_i\}_{i=1}^{n-1})) = H_0(K.(M,\{\sigma_i\}_{i=1}^{n-1})),$$

$$H_0(K.(M,\{\sigma_i\}_{i=1}^n)) \simeq H_0(K.(M,\{\sigma_i\}_{i=1}^{n-1})),$$

hence an isomorphism

$$H_1(K.(M,\{\sigma_i\}_{i=1}^n)) \simeq H_0(K.(M,\{\sigma_i\}_{i=1}^n)).$$

From the definitions, one checks that this isomorphism sends a 1-cycle $\sum_{i=1}^n m_i\, e_{\{i\}}$ to the 0-cycle $m_n\, e_\emptyset$. The map ϕ defined above is inverse to this isomorphism.

We apply this result in the setting of section 6. Assume that g is nondegenerate and that f_i is homogeneous of degree d_i and commode, $i = 1,\ldots,r$. Consider the Koszul complex $K.(S,\emptyset) = K.(B,\{D_i\}_{i=0}^{N+r})$. The homogeneity condition implies that $\dim\Delta = N + r$, hence by [1, Theorem 2.9] there is a subset $S' \subset S$ of cardinality $N+r$, say, $S' = S\setminus\{i_0\}$, such that $K.(B,\{D_i\}_{i\in S'})$ is acyclic in positive dimension and D_{i_0} is a linear combination of the D_i's, $i \in S'$, say $D_{i_0} = \sum_{i\in S'} a_iD_i$, $a_i \in \Omega_0$. Thus by Proposition 8.2, the map (where $\xi \in B$)

$$\xi\, e_\emptyset \mapsto \xi_{i_0}\, e_{\{i_0\}} + \sum_{i\in S'}(-a_i)\xi_i\, e_{\{i\}}$$

is an isomorphism of $H_0(K.(S,\emptyset))$ and $H_1(K.(S,\emptyset))$. From (6.7) we see that the map $\phi_S^\emptyset$ defined by (6.9) is just a nonzero scalar multiple of this isomorphism (in fact, the scalar is ϵ_{i_0}, where ϵ_i is defined following (6.9)), hence $\phi_S^\emptyset$ is also an isomorphism. More generally, one has the following.

PROPOSITION 8.3. – *Suppose that g is nondegenerate and f_i is commode for $i = 1,\ldots,r$. Then for $I \subseteq S$ with $I_{\rm sp} \neq \emptyset$ and $I_{\rm du} \neq \emptyset$ we have $H_l(K.(I,\emptyset)) = 0$ for $l > 1$ and $\phi_I^\emptyset$ is an isomorphism.*

Proof. – The hypothesis on I implies that $\dim\Delta_I = |I| - 1$, hence one can repeat the argument given above for the case $I = S$.

There is a modification for the Koszul complex $\bar{K}. = K.(R,\{g_i\}_{i=0}^{N+r})$ of section 7. The results of [1], together with Proposition 8.2, imply that there is an isomorphism (homogeneous of degree 1) from $H_0(\bar{K}.)$ onto $H_1(\bar{K}.)$. The dimension of $H_0(\bar{K}.)^{(k)}$ can then be calculated from (7.1). However, note that the map $\phi_S^\emptyset$ defined by (6.9) need not be an isomorphism unless $d_1,\ldots,d_r$ are all prime to p. So in general, we have the following weaker version of Proposition 8.3.

PROPOSITION 8.4. – *Suppose that g is nondegenerate and f_i is commode for $i = 1,\ldots,r$. Then for $I \subseteq S$ with $I_{\rm sp} \neq \emptyset$ and $I_{\rm du} \neq \emptyset$ we have $H_l(\bar{K}.(I,\emptyset)) = 0$ for $l > 1$,*

$\dim_{\mathbf{F}_q} H_1(\bar{K}.(I,\emptyset))^{(k)} = \dim_{\mathbf{F}_q} H_0(\bar{K}.(I,\emptyset))^{(k-1)}$, *and* $\dim_{\mathbf{F}_q} H_0(\bar{K}.(I,\emptyset))^{(k)}$ *is the coefficient of* t^k *in*

$$\sum_{\substack{e_1+\cdots+e_r=|I_{\rm sp}|-1 \\ e_i=0 \text{ if } N+i\notin I_{\rm du}}} q^{I_{\rm sp}}_{e_1\cdots e_r}(t).$$

We prove another simple lemma on complexes. Let

$$0 \to L. \to M. \to N. \to 0$$

be a short exact sequence of complexes of A-modules. Suppose that $\rho : L_0 \to L_1$, $\phi : M_0 \to M_1$, and $\psi : N_0 \to N_1$ are A-module homomorphisms with image contained in the space of 1-cycles and sending 0-boundaries to 1-boundaries (hence they induce homomorphisms $\bar{\rho} : H_0(L.) \to H_1(L.)$, $\bar{\phi} : H_0(M.) \to H_1(M.)$, and $\bar{\psi} : H_0(N.) \to H_1(N.)$) such that the diagram

$$\begin{array}{ccccc} L_1 & \to & M_1 & \to & N_1 \\ {\scriptstyle\rho}\uparrow & & {\scriptstyle\phi}\uparrow & & {\scriptstyle\psi}\uparrow \\ L_0 & \to & M_0 & \to & N_0 \end{array} \tag{8.5}$$

commutes.

PROPOSITION 8.6. – *If* $\bar{\psi}$ *is surjective, then the connecting homomorphism* $H_1(N.) \to H_0(L.)$ *is zero. If in addition* $H_2(N.) = 0$ *and* $\bar{\phi}$ *and* $\bar{\psi}$ *are isomorphisms, then* $\bar{\rho}$ *is also an isomorphism.*

Proof. – We first show that the conecting homomorphism is zero. Let $\xi \in N_1$ be a 1-cycle. Since $\bar{\psi}$ is surjective, we may choose $\eta \in N_0$ such that $\psi(\eta)$ is homologous to ξ. Choose $\zeta \in M_0$ such that ζ maps to η under the surjection $M_0 \to N_0$. By the commutativity of (8.5), $\phi(\zeta) \in M_1$ maps to $\psi(\eta) \in N_1$ under the surjection $M_1 \to N_1$. But $\phi(\zeta)$ is a 1-cycle in M_1, so by definition of the connecting homomorphism, ξ maps to 0 in $H_0(L.)$. If $H_2(N.) = 0$, one then has a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & H_1(L.) & \to & H_1(M.) & \to & H_1(N.) & \to & 0 \\ & & {\scriptstyle\bar\rho}\uparrow & & {\scriptstyle\bar\phi}\uparrow & & {\scriptstyle\bar\psi}\uparrow & & \\ 0 & \to & H_0(L.) & \to & H_0(M.) & \to & H_0(N.) & \to & 0. \end{array}$$

Since $\bar{\phi}$ and $\bar{\psi}$ are both isomorphisms, $\bar{\rho}$ must also be an isomorphism.

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