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# A HODGE TYPE DECOMPOSITION FOR SPINOR VALUED FORMS

BY M. J. SLUPINSKI

ABSTRACT. – In this paper we define an action of the Lie algebra $\mathrm{sl}(2, \mathbb{R})$ on the space of spinor valued exterior forms $\Lambda \otimes S$ associated to a euclidean vector space $(V, g)$. This action commutes with the natural action of $\mathrm{Pin}(V, g)$ and we obtain a decomposition of $\Lambda \otimes S$ in terms of primitive elements analogous to the classical Hodge-Lefschetz pointwise decomposition of the exterior algebra of a Kähler manifold. This gives rise to Howe correspondences for the pair $(\mathrm{Pin}(V), \mathrm{sl}(2, \mathbb{R}))$ and Howe correspondences for the pair $(\mathrm{Spin}(V), \mathrm{sl}(2, \mathbb{R}))$ are also obtained. We prove some positivity results in this context, which are analogous to the classical, infinitesimal Hodge-Riemann bilinear relations.

## Introduction

For compact Kähler manifolds classical Hodge-Lefschetz theory gives a refined decomposition of the cohomology. This is implemented in two steps. First, in modern terminology, one proves that the representation in $\Lambda(\mathbb{R}^{2n}) \otimes \mathbb{C}$ of the unitary group $U(n)$ and the Hodge-Lefschetz $\mathrm{gl}(2, \mathbb{R})$ sets up a Howe correspondence (*cf.* [5]) for the pair $(U(n), \mathrm{gl}(2, \mathbb{R}))$ – *i.e.* $\mathrm{gl}(2, \mathbb{R})$ not only commutes with $U(n)$ but generates its full commutant. Then one globalises and proves that the above representation of $\mathrm{gl}(2, \mathbb{R})$ induces an action of $\mathrm{gl}(2, \mathbb{R})$ on the cohomology of the manifold, thereby providing decompositions of the De Rham and Dolbeault groups which refine their usual decompositions in terms of degree and bidegree.

A pin or spin structure on a riemannian manifold is a weaker geometric structure than a Kähler structure. In this paper we obtain an analogue of the first step above in these cases. We find an action of $\mathrm{sl}(2, \mathbb{R})$ on the spinor valued forms associated to a Euclidean vector space, which commutes with the action of $\mathrm{Pin}(n)$ and which gives rise to Howe correspondences. All of the results are representation theoretic in character and are closely related to the theory of dual pairs of R. Howe (*cf.* [5]). We do not consider any global aspects here, although, by $\mathrm{Pin}(n)$-invariance the $\mathrm{sl}(2, \mathbb{R})$ will act in the space of sections of the bundle of spinor valued forms over any pin manifold. Let us now give a more precise statement of the contents of this article.

If $V$ is a real, $n$-dimensional, Euclidian vector space, $\Lambda$ will be the exterior algebra on $V^*$ and $S$ a space of spinors (*see* § 0 for details). We define the operator $\Theta \in \operatorname{End}(\Lambda \otimes S)$ by

$$\Theta(\omega \otimes \psi) = \sum_{i=1}^{i=n} e_i \wedge \omega \otimes e_i.\psi$$

where $\omega \in \Lambda$ is an exterior form, $\psi \in S$ is a spinor, $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $V^*$ and $e_i.\psi$ is the action of $e_i$ on $\psi$ by Clifford multiplication. The key observation of this paper is that $\Theta$ and its adjoint $\Theta^*$ generate an $\mathrm{sl}(2, \mathbb{R})$, denoted $\mathrm{sl}_2(\Theta)$. Furthermore, $\mathrm{sl}_2(\Theta)$ commutes with the action of $\mathrm{Pin}(n)$ on $\Lambda \otimes S$. Applying the representation theory of $\mathrm{sl}(2, \mathbb{R})$ and some invariant theory, we prove the

THEOREM. – *For $k \leq n$,, the* $\mathrm{Pin}(n)$ *invariant decompositions*

$$\Lambda^k \otimes S = \bigoplus_{0 \leq r \leq \min(k, n-k)} \Theta^{k-r}(P_r) \qquad (n\ even)$$

$$(\Lambda^k \otimes S)^+ = \bigoplus_{0 \leq r \leq \min(k, n-k)} \Theta^{k-r}(P_r^+) \qquad (n\ odd)$$

$$(\Lambda^k \otimes S)^- = \bigoplus_{0 \leq r \leq \min(k, n-k)} \Theta^{k-r}(P_r^-) \qquad (n\ odd)$$

*are the decompositions into irreducible, non-isomorphic $Pin(n)$-nodules.*

Here the $P_r = \operatorname{Ker}\Theta^* \cap \Lambda^r \otimes S$ are the "primitive vectors" and, when $n$ is odd, $X^\pm$ denote the $\pm i$ eigenspaces of the central element of $\mathrm{Pin}(n)$ acting on $X \subseteq \Lambda \otimes S$. An alternative formulation (*see* 1.9) of this theorem says that the representation $\Lambda \otimes S$ when $n$ is even (resp. $(\Lambda \otimes S)^+$ or $(\Lambda \otimes S)^-$ when $n$ is odd) sets up a Howe correspondence for the pair $(\mathrm{Pin}(n), \mathrm{sl}_2(\Theta))$. In paragraph 2 and 3 we find Howe correspondences for the pair $(\mathrm{Spin}(n), \mathrm{sl}_2(\Theta))$ when $n$ is even and odd respectively.

As an application of the above theorems we prove some positivity results which are analogous to the infinitesimal Hodge-Riemann bilinear relations of the classical theory. Recall that there the basic result is (*see* [4] for example):

THEOREM. – *Let $V$ be a real $2m$-dimensional euclidean vector space with a given compatible complex structure and let $\Lambda^{p,q}$ be the space of exterior forms of type $(p, q)$. Let $L$ denote the operation of multiplication by the Kähler form. Then if $x \in P^{p,q} = \operatorname{Ker} L^* \cap \Lambda^{p,q}$ is non-zero, the real $(m, m)$-form $i^{p-q}(-1)^{\frac{1}{2}(p+q)(p+q-1)} x \wedge L^{m-(p+q)}(\bar{x})$ is a strictly positive multiple of the volume element.*

In our context, the corresponding result is the following (*see* § 5 for the notation):

THEOREM. – *Let $V$ be a real, oriented, $2m$-dimensional euclidean vector space. Then:*

(i) *if $x \in P_+^s$ is non-zero, the real $2m$-form $i^m(-1)^{\frac{1}{2}s(s-1)} x \underline{\wedge} \Theta^{2m-2s}(x)$ is a strictly positive multiple of the volume element;*

(ii) *if $x \in P_-^s$ is non-zero, the real $2m$-form $i^m(-1)^{\frac{1}{2}s(s-1)} x \underline{\wedge} \Theta^{2m-2s}(x)$ is a strictly positive multiple of the volume element.*

The departure point for the above results was a real vector space equipped only with a positive-definite inner product, or perhaps with an orientation for the results concerning

Spin $n$. In the case where the real (even dimensional) euclidean vector space also has an isometric complex structure, we show in paragraph 4 how the above action of $\mathrm{sl}_2(\mathbb{R})$ on $\Lambda \otimes S$ can be extended to an action of $\mathrm{sl}(3, \mathbb{C})$ which commutes with the action of $U'$, the double cover of the unitary group. This is in fact the Lie algebra generated by $\mathrm{sl}_2(\Theta)$ and the Hodge-Lefschetz $\mathrm{sl}(2, \mathbb{R})$ (acting on the exterior algebra factor of $\Lambda \otimes S$). There are also Howe correspondences in this situation, described in Theorem 4.10.

In this paper [5], R. Howe gives a general procedure which constructs many examples of dual pairs and Howe correspondences. Roughly speaking, he shows how one can construct a "double cover" of some known examples of dual pairs and Howe correspondences involving the complex classical groups, and obtain new Howe correspondences for essentially the same dual pairs. In particular, his method applies to the known dual pair of complex orthogonal groups $(O(n), O(2m)) \subset O(\mathbb{C}^n \otimes \mathbb{C}^{2m})$ acting in $\mathbb{C}^n \otimes \mathbb{C}^{2m}$, and produces a dual pair $(O(n), so(2m, \mathbb{C})) \subset so(\mathbb{C}^n \otimes \mathbb{C}^{2m})$ acting in $\Lambda(m\,\mathbb{C}^n)$, the exterior algebra on the direct sum of $m$-copies of $\mathbb{C}^n$. R. Howe has pointed out to the author that the pair $(\mathrm{Pin}(n), \mathrm{sl}_2(\Theta))$ obtained in the present paper is probably to be the thought of as (a real form of) a "double cover" of the known dual pair $(O(n), O(3)) \subset O(\mathbb{C}^n \otimes \mathbb{C}^3)$ acting in $\mathbb{C}^n \otimes \mathbb{C}^3$. This point is investigated and clarified in [8].

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## 0. Preliminaries

In this section we will give a summary of the basic properties of Clifford algebras and spinors which we will need in the rest of the paper. For more details and proofs, consult the book of E. Cartan [3] or the article of Atiyah, Bott and Shapiro [1].

0.1. CLIFFORD ALGEBRAS. – The Clifford algebra $C(V)$ associated to a finite-dimensional, real, positive-definite, inner product space $(V, g)$ is defined as the quotient of the tensor algebra $T(V) = \oplus V^{\otimes k}$ by the two-sided ideal $\mathfrak{g}$ of $T(V)$ generated by elements of the form $v \otimes v + 2g(v, v)\,\mathrm{Id}$. The natural map $\Lambda(V) \to T(V) \to C(V)$ (here $\Lambda(V)$ is the exterior algebra on $V$) is a vector space, but not algebra, isomorphism, equivariant for the natural action of the orthogonal group $O(V, g)$ and if $e_1, e_2, \ldots, e_n$ is an orthonormal basis of $V$, the algebra $C(V)$ is generated by their images subject to the relations

$$e_i\,e_j + e_j\,e_i = -2\,\delta_{ij}.$$

The natural $\mathbb{Z}_2$-grading of $T(V)$ into even and odd tensors induces a $\mathbb{Z}_2$-grading of the Clifford algebra $C(V) = C^+ \oplus C^-$. More generally, any automorphism or antiautomorphism of $T(V)$ preserving $\mathfrak{g}$ gives rise to an automorphism or antiautomorphism of $C(V)$. In particular, we will write $x \to x^*$ for the conjugate linear of the complexified Clifford algebra $C_c(V)$ induced by $v_1 \otimes v_2 \ldots \otimes v_k \to (-1)^k\,v_k \otimes v_{k-1} \ldots \otimes v_1$.

0.2. The even dimensional case. – In this section we will suppose that $V$ is of even dimension $2m$. The groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are defined as the following subsets of the real Clifford algebra.

0.2.1. Definition. – $\mathrm{Pin}(V) = \{ x \in C(V) : xx^* = 1$ *and* $xVx^{-1} = V \}$ *and* $\mathrm{Spin}(V)$ *is the connected component of the identity of* $\mathrm{Pin}(V)$.

The following facts are well known (*cf.* [1]:

(i) the image of the natural map $\pi : \mathrm{Pin}(V) \to \mathrm{End}(V)$ is exactly the orthogonal group of $(V, g)$;

(ii) if $x$ is a unit vector in $V$, then $x \in \mathrm{Pin}(V)$ and $\pi(x)$ is *minus* the reflection in the hyperplane orthogonal to $x$;

(iii) every element of $\mathrm{Pin}(V)$ can be written as the product of unit vectors in $V$ and every element of $\mathrm{Spin}(V)$ can be written as the product of an *even* number of unit vectors in $V$;

(iv) the map $\pi$ is a two to one covering map and if $m > 1$, $\pi : \mathrm{Spin}(V) \to SO(V)$ is a universal covering map;

(v) the group $O(V)$ preserves the ideal $\mathfrak{g}$ and therefore acts naturally by automorphisms on $C(V)$ and if $g \in O(V)$, this action is just the inner automorphism corresponding to $g' \in \mathrm{Pin}(V)$ where $\pi(g') = g$.

(vi) as representation spaces of $O(V)$, $C^+(V) \cong 1 \oplus \Lambda^2 \oplus \ldots \oplus \Lambda^{2m}$ and $C^-(V) = \Lambda^1 \oplus \Lambda^3 \oplus \ldots \oplus \Lambda^{2m-1}$ where $\Lambda^k$ denotes the $k$-th exterior power of $V$.

When the dimension of $V$ is even it can be shown that the *complex* Clifford algebra $C_c(V)$ is isomorphic to a full matrix algebra and if we choose a complex vector space $S$ of dimension $2^m$ and an algebra isomorphism $C_c(V) \cong \mathrm{End}(S)$, the space $S$ is called a space of spinors. It can be written $S = S_1 \oplus S_2$, making it into a graded module over the graded algebra $C = C^+ \oplus C^-$. (The spaces $S_1$ and $S_2$, which are of dimension $2^{m-1}$, are sometimes called semi-spinors). More generally, any natural automorphism or antiautomorphism of $C_c(V) \cong \mathrm{End}(S)$ can be realised by some geometrical structure on $S$ and in particular there is a unique (up to phase factor $e^{i\theta}$) positive-definite hermitian form $h$ on $S$ such that $h(x.\psi, \phi) = h(\psi, x^*.\phi)$ for all $x \in C_c(V)$ and for all $\psi, \phi \in S$. The hermitian form $h$ satisfies $h(S_1, S_2) = 0$.

The group $\mathrm{Pin}(V)$ acts on $S$ and this action clearly preserves the hermitian product $h$. In fact one can show that this representation is irreducible. The group $\mathrm{Spin}(V)$, however, preserves $S_1$ and $S_2$ and in fact these are irreducible, non-equivalent unitary representations of the same dimension.

0.3. The odd dimensional case. – When $\dim V = 2m - 1$ is odd, the situation is a little different. The natural action of the group $O(V)$ on $C(V)$ cannot be realised by inner automorphisms; in particular, $-\mathrm{Id} \in O(V)$ does not act by inner automorphism. Thus for notational convenience we will embed the Clifford algebra in a larger Clifford algebra in which the action of $O(V)$ is realised by inner automorphisms.

0.3.1. Definition. – *Let* $V_a$ *denote the Euclidean vector space obtained by taking the direct sum of* $(V, g)$ *and* $\mathbb{R}$, *equipped with its standard inner product. We will write* $e \in \mathbb{R}$ *for the canonical basis vector of unit length. Clearly the inclusion* $V \to V_a$ *extends to an* *inclusion* $C(V) \to C(V_a)$. *We will consider* $V$ *and* $V_a$ *as subsets of* $C(V_a)$. *The groups* $\mathrm{Pin}(V)$ *and* $\mathrm{Spin}(V)$ *are defined by*

$$\mathrm{Pin}(V) = \{x \in \mathrm{Pin}(V_a) : xVx^{-1} = V\}$$

*and*

$$\mathrm{Spin}(V) = \{x \in \mathrm{Spin}(V_a) : xVx^{-1} = V\}.$$

The following facts are the analogues of those stated in 0.1.1 for the even dimensional case but with significant changes in (iii) and (v):

(i) the image of the natural map $\pi : \mathrm{Pin}(V) \to \mathrm{End}(V)$ is exactly the orthogonal group of $(V, g)$;

(ii) if $x$ is a unit vector in $V$, then $x \in \mathrm{Pin}(V)$ and $\pi(x)$ is *minus* the reflection in the hyperplane orthogonal to $x$;

(iii) every element of $\mathrm{Spin}(V)$ can be written as the product of an *even* number of unit vectors in $V$; the element $e \in \mathrm{Pin}(V)$ is in the centre of $\mathrm{Pin}(V)$ and $\pi(e)$ is $-\mathrm{Id}_V$; every element of $\mathrm{Pin}(V)$ not in $\mathrm{Spin}(V)$ can be written as the product of $e$ with an *even* number of unit vectors in $V$;

(iv) the map $\pi$ is a two to one covering map and if $m > 2$, $\pi : \mathrm{Spin}(V) \to SO(V)$ is a universal covering map;

(v) the group $O(V)$ preserves the ideal $\mathfrak{g}$ and therefore acts naturally by automorphisms on $C(V)$ and if $g \in O(V)$, this action is just the restriction to $C(V)$ of the inner automorphism of $C(V_a)$ corresponding to $g' \in \mathrm{Pin}(V)$ where $\pi(g') = g$.

(vi) as representation spaces of $O(V)$, we have $C^+(V) \cong 1 \oplus \Lambda^2 \oplus \ldots \oplus \Lambda^{2m-2}$ and $C^-(V) = \Lambda^1 \oplus \Lambda^3 \oplus \ldots \oplus \Lambda^{2m-1}$ where $\Lambda^k$ denotes the $k$-th exterior power of $V$.

The algebra $C_c(V)$ is known not to be simple when $V$ is odd dimensional, but rather a product of two simple algebras. However, $C_c(V_a)$ is a simple algebra so let us choose a space of spinors for $V_a$, that is an algebra isomorphism $C_c(V_a) \cong \mathrm{End}(S)$. Then the $\mathrm{Pin}(V)$-module $S$ is not irreducible because the decomposition $S = \{\psi \in S : e(\psi) = i\psi\} \oplus \{\psi \in S : e(\psi) = -i\psi\} = S^+ \oplus S^-$ is $\mathrm{Pin}(V)$-invariant, the element $e \in \mathrm{Pin}(V)$ being central. It is easy to see that $S^+$ and $S^-$ are of the same dimension and it can also be shown that they are irreducible $\mathrm{Pin}(V)$-modules. They are inequivalent representations of $\mathrm{Pin}(V)$ since the central element $e$ takes different values in $S^+$ and $S^-$, but equivalent representations of $\mathrm{Spin}(V)$ since, for any orthonormal basis $\{e_1, e_2, \ldots, e_{2m-1}\}$ of $V$, the element $e_1 e_2, \ldots, e_{2m-1}$ of $C(V)$ is a $\mathrm{Spin}(V)$-intertwining operator.

The element $e$ is a unitary operator on $S$ and so $S^+$ and $S^-$ are orthogonal subspaces of $S$ and in fact $S^+$ and $S^-$ are non-isomorphic, dual representations of $\mathrm{Pin}(V)$. Further as $\mathrm{Pin}(n)$-modules, $(S^+)^* \otimes S^+ \cong (S^-)^* \otimes S^- \cong C_c^+(V) \cong 1 \oplus \Lambda^2 \oplus \Lambda^4 \oplus \ldots \oplus \Lambda^{2m-2}$.

## 1. Definition and basic properties of the operators $\Theta$ and $\Theta^*$

1.0. NOTATION. – We will now suppose that $\dim V = n$ and we will write $\Lambda^k$ for the space of real $k$-forms, equipped with the Euclidean metric for which the forms $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ are an orthonormal basis. Here, $e_1, e_2, \ldots, e_n$ is a real orthonormal basis of $V^*$, which space we will now write as $\Lambda^1$. The space $\Lambda$ will be the direct sum of exterior forms in all degrees. The real Clifford algebra of $\Lambda^1$ will be denoted $C$ and the complex Clifford algebra $C_c$. If $n$ is even, we choose a space of spinors $S$, that is an algebra isomorphism $C_c \stackrel{\sim}{=} \mathrm{End}\,(S)$, and we will choose a hermitian metric $h$ on $S$ as above. If $n$ is odd, we choose a space of spinors $S$ for $C\,(\Lambda^1_a)$ and we write $S = S^+ \oplus S^-$ as above. The groups $\mathrm{Pin}\,(\Lambda^1)$ and $\mathrm{Spin}\,(\Lambda^1)$ will be denoted $\mathrm{Pin}\,(n)$ and $\mathrm{Spin}\,(n)$ respectively. If $n$ is even they are subsets of $C\,(\Lambda^1)$ and if $n$ is odd they are subsets of $C\,(\Lambda^1_a)$, as explained above.

1.1. Definition. – *Define* $\Theta \in \mathrm{End}\,(\Lambda \otimes S)$ *by*

$$\Theta\,(\omega \otimes \psi) = \sum_{a=1}^{a=n} e_a \wedge \omega \otimes e_a.\psi$$

*where* $\omega \otimes \psi \in \Lambda^k \otimes S$ *and* $e_1, e_2, \ldots, e_n$ *is a real, orthonormal basis of 1-forms. It is easily verified that* $\Theta$ *does not depend on the choice of orthonormal basis. The operator* $\Theta^*$ *is its adjoint for the tensor product hermitian metric.*

1.2. Proposition. – *Let* $R : \mathrm{Pin}\,(n) \to \mathrm{End}\,(\Lambda \otimes S)$ *be the tensor product representation. Then we have*

$$\text{(i)} \qquad R\,(x)\,\Theta = \Theta\,R\,(x), \qquad \textit{where} \quad x \in \mathrm{Pin}\,(n).$$

*Thus the operator* $\Theta$ *commutes with the action of* $\mathrm{Pin}\,(n)$ *on* $\Lambda \otimes S$.

$$\text{(ii)} \qquad (\Theta\Theta^* - \Theta^*\,\Theta)\,(\omega \otimes \psi) = -(n - 2\,k)\,\omega \otimes \psi \qquad \textit{where} \quad \omega \otimes \psi \in \Lambda^k \otimes S.$$

*Proof.* – Part (i) is a straightforward calculation using the basis indpendence of $\Theta$ and the fact that for $x \in \mathrm{Pin}\,(n)$, the element $\pi\,(x) \in O\,(\Lambda^1)$ acts by $\pi\,(x)\,(v) = xvx^{-1}$ on $v \in \Lambda^1$.

To prove part (ii), recall that if $x \in \Lambda^1$, then the adjoint of exterior multiplication by $x$ is given by the map $i_x : \Lambda \to \Lambda$, where $i_x\,(\mathbb{C}) = 0$, $i_x\,(v) = (v|x)$ for $v \in \Lambda^1$ and $i_x$ is extended to give an antiderivation. Recall also that if $x \in \Lambda^1$ is of unit length, then $x$ is unitary and skew-adjoint when acting in S. Hence, taking tensor products of adjoints (and denoting $i_{e_a}$ by $i_a$) we have:

$$\Theta^*\,(\omega \otimes \psi) = -\sum_{a=1}^{a=n} i_a\,(\omega) \otimes e_a.\psi.$$

Hence,

$$\begin{aligned}
\Theta\,\Theta^*\,(\omega \otimes \psi) &= -\sum_{a \text{ and } b} e_a \wedge i_b\,(\omega) \otimes e_a\,e_b.\psi \\
&= -\sum_{a \neq b} e_a \wedge i_b\,(\omega) \otimes e_a\,e_b.\psi - \sum_a e_a \wedge i_a\,(\omega) \otimes e_a\,e_a.\psi \\
&= -\sum_{a \neq b} e_a \wedge i_b\,(\omega) \otimes e_a\,e_b.\psi + \sum_a e_a \wedge i_a\,(\omega) \otimes \psi \qquad (\text{as } e_a^2 = -1).
\end{aligned}$$

A simple calculation shows that $\sum_a e_a \wedge i_a\,(\omega) = k\,\omega$ if $\omega \in \Lambda^k$ and so finally:

$$\text{(1)} \qquad \Theta\,\Theta^*\,(\omega \otimes \psi) = -\sum_{a \neq b} e_a \wedge i_b\,(\omega) \otimes e_a\,e_b.\psi + k\,\omega \otimes \psi.$$

On the other hand,

$$(2)\ \Theta^* \Theta (\omega \otimes \psi) = -\sum_{a \text{ and } b} i_b (e_a \wedge \omega) \otimes e_b e_a.\psi$$
$$= -\sum_{a \text{ and } b} i_b (e_a) \wedge \omega \otimes e_b e_a.\psi + \sum_{a \text{ and } b} e_a \wedge i_b (\omega) \otimes e_b e_a.\psi$$
$$= n\,\omega \otimes \psi + \sum_{a=b} e_a \wedge i_b (\omega) \otimes e_b e_a.\psi + \sum_{a \neq b} e_a \wedge i_a (\omega) \otimes e_b e_a.\psi$$
$$= n\,\omega \otimes \psi - k\,\omega \otimes \psi + \sum_{a \neq b} e_a \wedge i_b (\omega) \otimes e_b e_a.\psi.$$

Subracting (2) from (1), we get:

$$[\Theta, \Theta^*] (\omega \otimes \psi) = (2k - n)\, \omega \otimes \psi \qquad (\text{since } e_a e_b + e_b e_a = 0 \text{ if } a \neq b).$$

1.3. COROLLARY. – *The following identities hold in* $\operatorname{End}(\Lambda \otimes S)$*:*

$$[[\Theta, \Theta^*], \Theta] = 2\Theta \qquad \textit{and} \qquad [[\Theta, \Theta^*], \Theta^*] = -2\Theta^*.$$

*Thus the real Lie subalgebra of* $\operatorname{End}(\Lambda \otimes S)$ *generated by the operators* $\Theta$ *and* $\Theta^*$ *is isomorphic to* $\mathrm{sl}_2(\mathbb{R})$*. As from now, it will be denoted by* $\mathrm{sl}_2(\Theta)$.

*Proof.* – This is an immediate consequence of Proposition 1.2 (ii).

1.4. DEFINITION. – *For* $0 \leq k \leq n$*, write* $P_k = (\Lambda^k \otimes S) \cap \operatorname{Ker} \Theta^*$*. Note that* $P_0 = S$ *and that the* $P_k$ *are* $\operatorname{Pin}(n)$*-invariant.*

The proof of the following proposition will be omitted and is a standard application of the representation theory of $\mathrm{sl}_2(\mathbb{C})$ (*cf.* [7], Ch. 4 for example), the point being that $[\Theta, \Theta^*]$ acts on $\Lambda^k \otimes S$ as $(2k - n)\operatorname{Id}$.

1.5. PROPOSITION. – *The following hold:*

(i) $P_k = \{0\}$ *if* $k > \frac{n}{2}$*;*

(ii) *the map* $\Theta^r : \Lambda^k \otimes S \to \Lambda^{k+r} \otimes S$ *restricted to* $P_k$ *is injective if* $r \leq n - 2k$ *and* $\Theta^{n-2k+1}(P_k) = 0$*;*

(iii)
$$\Lambda^k \otimes S = \bigoplus_{0 \leq r \leq \min(k,\, n-k)} \Theta^{k-r}(P_r);$$

(iv) $\operatorname{Ker} \Theta^* = \bigoplus_{0 \leq k \leq \frac{n}{2}} P_k$ *and the operator* $(\Theta^*)^{k-r}$ *maps* $\Theta^{k-r}(P_r)$ *isomorphically onto* $P_r$ *if* $0 \leq r \leq \min(k, n-k)$.

By Proposition 1.2, each one of the subspaces in 1.5 (iii) is invariant under the action of $\operatorname{Pin}(n)$ and hence we have decomposed $\Lambda^k \otimes S$ (and $\Lambda^{n-k} \otimes S$) into $k+1$ $\operatorname{Pin}(n)$-invariant subspaces when $k \leq \frac{n}{2}$. The natural question now is whether this is the decomposition of $\Lambda^k \otimes S$ into irreducible $\operatorname{Pin}(n)$-components. However we see that we have to distinguish the cases $n$ even and $n$ odd because the space $S (= P_0)$ is $\operatorname{Pin}(n)$-irreducible if $n$ is even but not if $n$ is odd (*cf.* 0.2 and 0.3 above). In fact when $n$ is odd, the central element $e \in \mathrm{Pin}(V)$ (*cf.* 0.3) acts by $R(e) = (-1\, d)^k \otimes e$ on $\Lambda^k \otimes S$ and so commutes with $\Theta$ and $\Theta^*$. Its eigenspaces in $\Lambda \otimes S$ are therefore invariant both by $\mathrm{Pin}(n)$ and the operators $\Theta$ and $\Theta^*$.

1.6. DEFINITION. – *Suppose $n$ is odd. For $0 \leq k \leq \frac{n}{2}$, define*

$$(\Lambda \otimes S)^{\pm} = \{ x \in \Lambda \otimes S \,:\, R(e)(x) = \pm i x \},$$
$$(\Lambda^k \otimes S)^{\pm} = \{ x \in \Lambda^k \otimes S \,:\, R(e)(x) = \pm i x \}$$

*and*

$$P_k^{\pm} = \{ x \in P_k \,:\, R(e)(x) = \pm i x \}.$$

*Note that* $\dim P_k^+ = \dim P_k^-$ *since the operator* $\mathrm{Id} \otimes e_1 e_2 \ldots e_n$ (cf. 0.3) *exchanges these spaces.*

We now have the following theorem.

1.7. THEOREM. – (i) *For $0 \leq k \leq n$, the $\mathrm{Pin}(n)$-invariant decompositions*

$$\Lambda^k \otimes S = \bigoplus_{0 \leq r \leq \min(k,\, n-k)} \Theta^{k-r}(P_r) \qquad (n\ even)$$
$$(\Lambda^k \otimes S)^+ = \bigoplus_{0 \leq r \leq \min(k,\, n-k)} \Theta^{k-r}(P_r^+) \qquad (n\ odd)$$
$$(\Lambda^k \otimes S)^- = \bigoplus_{0 \leq r \leq \min(k,\, n-k)} \Theta^{k-r}(P_r^-) \qquad (n\ odd)$$

*are the decompositions into irreducible, non-isomorphic $\mathrm{Pin}(n)$-modules.*

(ii) *Let $\mathrm{sl}_2(\Theta)$ be the Lie subalgebra of* $\mathrm{End}(\Lambda \otimes S)$ *(n even) or End $(\Lambda \otimes S)^{\pm}$ (n odd), generated by $\Theta$ and $\Theta^*$. Then, as a representation of the product* $\mathrm{Pin}(n) \times \mathrm{sl}_2(\Theta)$, *we have isomorphisms:*

$$\Lambda \otimes S \cong \bigoplus_{0 \leq k \leq \frac{n}{2}} (P_k \otimes \sigma_{n+1-2k}) \qquad (n\ even)$$
$$(\Lambda \otimes S)^+ \cong \bigoplus_{0 \leq k \leq \frac{n}{2}} (P_k^+ \otimes \sigma_{n+1-2k}) \qquad (n\ odd)$$
$$(\Lambda \otimes S)^- \cong \bigoplus_{0 \leq k \leq \frac{n}{2}} (P_k^- \otimes \sigma_{n+1-2k}) \qquad (n\ odd)$$

*where $\sigma_r$ denotes the unique irreducible* $\mathrm{sl}_2(\mathbb{C})$ *module of dimension $r$.*

*Proof.* – These results in the case $n$ even can also be deduced from Theorem 7 in [5].

(i) It is sufficient to prove (i) when $0 \leq k \leq \frac{n}{2}$ since by 1.3 (ii), $\Theta^{n-2k} : \Lambda^k \otimes S \to \Lambda^{n-k} \otimes S$ is a $\mathrm{Pin}(n)$-equivariant isomorphism which preserves the corresponding decompositions. We will give the argument for the case $(\Lambda^k \otimes S)^+$ and $n$ odd, and then indicate how to modify it in the other cases.

If $V \subset (\Lambda^k \otimes S)^+$ is a $\mathrm{Pin}(n)$ invariant subspace, then $\pi_V$, the orthogonal projection onto $V$, is a $\mathrm{Pin}(n)$ invariant element of $\mathrm{End}((\Lambda^k \otimes S)^+)$. If we have a $\mathrm{Pin}(n)$ invariant decomposition $(\Lambda^k \otimes S)^+ = V_1 \oplus V_2 \oplus \ldots \oplus V_s$, then the corresponding projections are linearly independent in $\mathrm{End}((\Lambda^k \otimes S)^+)$. This gives an upper bound for $s$ : $s \leq \dim \{\mathrm{End}((\Lambda^k \otimes S)^+)\}^{\mathrm{Pin}(n)}$. If we have equality $s = \dim \{\mathrm{End}((\Lambda^k \otimes S)^+)\}^{\mathrm{Pin}(n)}$, then the $V_i$ must be irreducible and pairwise non-isomorphic as $\mathrm{Pin}(n)$-modules.

Consider the canonical isomorphism

$$\mathrm{End}((\Lambda^k \otimes S)^+) \cong ((\Lambda^k \otimes S)^+)^* \otimes (\Lambda^k \otimes S)^+.$$

If $k$ is even, $(\Lambda^k \otimes S)^+ = \Lambda^k \otimes S^+$ and if $k$ is odd $(\Lambda^k \otimes S)^+ = \Lambda^k \otimes S^-$, since the actiof of the central element $e \in \mathrm{Pin}(n)$ on $\Lambda^k$ is $(-\mathrm{Id})^k$ and its action on $S^\pm$ is $\pm i\,\mathrm{Id}$. Hence this isomorphism becomes.

$$(1) \quad \begin{cases} \mathrm{End}((\Lambda^k \otimes S)^+) \cong (\Lambda^k \otimes S^+)^* \otimes \Lambda^k \otimes S^+ \cong (\Lambda^k)^* \otimes \Lambda^k \otimes (S^+)^* \otimes S^+ \\ \text{or} \\ \mathrm{End}((\Lambda^k \otimes S)^+) \cong (\Lambda^k \otimes S^-)^* \otimes \Lambda^k \otimes S^- \cong (\Lambda^k)^* \otimes \Lambda^k \otimes (S^-)^* \otimes S^- \end{cases}$$

depending on whether $k$ is even or odd respectively. Using the $\mathrm{Pin}(n)$-isomorphisms (*cf.* 0.3)

$$(2) \quad (S^+)^* \otimes S^+ \cong (S^-)^* \otimes S^- \cong C_c^+(V) \cong 1 \oplus \Lambda^2 \oplus \Lambda^4 \oplus \ldots \oplus \Lambda^{n-1} = \Lambda^{ev}$$

and the fact that $\Lambda^k$ is a self dual $\mathrm{Pin}(n)$-representation, we deduce a $\mathrm{Pin}(n)$-isomorphism:

$$(3) \qquad \mathrm{End}((\Lambda^k \otimes S)^+) \cong (\Lambda^k \otimes \Lambda^k \otimes \Lambda^{ev})^*,$$

which is valid for $k$ odd or even. This representation factors through $\pi : \mathrm{Pin}(n) \to O(n)$ and so the complex dimension of the space of invariants $\{\mathrm{End}((\Lambda^k \otimes S)^+)\}^{\mathrm{Pin}(n)}$ is the complex dimension of the space of $O(n)$-invariant linear maps from $\Lambda^k \otimes \Lambda^k \otimes \Lambda$ to $\mathbb{C}$, which is in turn the real dimension of the space of $O(n)$-invariant linear maps from $\Lambda^k_{\mathbb{R}} \otimes \Lambda^k_{\mathbb{R}} \otimes \Lambda_{\mathbb{R}}$ to $\mathbb{R}$. By the main theorem of H. Weyl for $O(n)$ (*cf.* [10]), the only possibilities are linear combinations of contractions of indices. In this case these are:

$$\Lambda^k_{\mathbb{R}} \otimes \Lambda^k_{\mathbb{R}} \otimes \Lambda^0_{\mathbb{R}} \to \mathbb{R} : \omega \otimes \sigma \otimes \lambda \to \omega_{a_1 a_2 \ldots a_k} \sigma_{a_1 a_2 \ldots a_k} \lambda$$
$$\Lambda^k_{\mathbb{R}} \otimes \Lambda^k_{\mathbb{R}} \otimes \Lambda^2_{\mathbb{R}} \to \mathbb{R} : \omega \otimes \sigma \otimes \lambda \to \omega_{a_1 a_2 \ldots a_{k-1} a} \sigma_{a_1 a_2 \ldots a_{k-1} b} \lambda_{ab}$$
$$\Lambda^k_{\mathbb{R}} \otimes \Lambda^k_{\mathbb{R}} \otimes \Lambda^4_{\mathbb{R}} \to \mathbb{R} : \omega \otimes \sigma \otimes \lambda \to \omega_{a_1 \ldots a_{k-2} ab} \sigma_{a_1 \ldots a_{k-2} cd} \lambda_{abcd}$$

upto

$$\Lambda^k_{\mathbb{R}} \otimes \Lambda^k_{\mathbb{R}} \otimes \Lambda^{2k}_{\mathbb{R}} \to \mathbb{R} : \omega \otimes \sigma \otimes \lambda \to \omega_{a_1 a_2 \ldots a_k} \sigma_{b_1 b_2 \ldots b_k} \lambda_{a_1 a_2 \ldots a_k b_1 b_2 \ldots b_k}.$$

Hence there are exactly $k+1$ possible contractions and so $\dim \{\mathrm{End}((\Lambda^k \oplus S)^+)\}^{\mathrm{Pin}(n)} \leq k+1$. Since the given decomposition already has $k+1$ components, we are in the limit case described at the beginning of the proof and therefore the components of this decomposition are irreducible, pairwise distinct $\mathrm{Pin}(n)$-modules. This completes the proof.

The argument for the case $(\Lambda^k \otimes S)^-$, and $n$ odd is completely analogous. For the case $n$ even we have to modify the equations (1), (2) and (3) slightly but then the rest of the argument stays the same.

(ii) If we sum 1.5 (iii) over all $k \leq n$, we get:

$$\Lambda^* \otimes S = \bigoplus_{0 \leq k \leq \frac{n}{2}} (P_k \oplus \Theta(P_k) \oplus \ldots \oplus \Theta^{n-2k}(P_k))$$

and more or less by definition, the $\mathrm{Pin}(n) \times \mathrm{sl}_2(\Theta)$ representation

$$P_k \oplus \Theta(P_k) \oplus \ldots \oplus \Theta^{n-2k}(P_k)$$

is isomorphic to the product representation $P_k \otimes \sigma_{n+1-2k}$ since every element of $P_k$ is primitive for $\mathrm{sl}_2(\mathbb{C})$ and $\Theta$ commutes with the action of $\mathrm{Pin}(n)$. The result follows.

1.8. Corollary. – *The ring of complex invariants of* $\mathrm{Pin}(n)$ *in* $\mathrm{End}(\Lambda \otimes S)$ *when* $n$ *is even, and in End* $((\Lambda \otimes S)^+)$ *or in End* $((\Lambda \otimes S)^-)$ *when* $n$ *is odd, is generated (over* $\mathbb{C}$*) by* $\mathrm{sl}_2(\Theta)$.

*Proof.* – This follows from a generalisation of Schur's Lemma:

1.9. Theorem (Folklore but *see* the appendix of [8] for example). – *Let* $X$ *be a finite dimensional, complex vector space. Let* $G \subseteq End(X)$ *be a real, semisimple or compact Lie group and let* $\mathfrak{h} \subseteq End(X)$ *be a real, reductive Lie algebra which commutes with* $G$. *Then the following properties are equivalent:*

(A) *as a representation of* $G \times \mathfrak{h}$, *there is is an isomorphism*

$$X \cong \bigoplus_{i \in I} R_i \otimes S_i,$$

*where the* $R_i\,(resp.\ S_i)$ *are distinct, complex irreducible representations of* $G\,(resp.\ \mathfrak{h})$*;*

(B) *the commutant in* $End(X)$ *of* $G$ *is generated (over* $\mathbb{C}$*) by* $\mathfrak{h}$.

1.10. Remark. – In the language of representation theory, one says that the representation $X$ sets up a Howe correspondence (*cf.* [5]) between irreducible representations of the group $G$ and the Lie algebra $\mathfrak{h}$ when condition (A) holds. Thus Theorem 1.7 provides us with some examples of Howe correspondences which in the case $n$ odd are new. Many other examples are to be found in [5] including the case $n$ even of 1.7. As pointed out to the author by R. Howe, the examples of this paper are probably members of a family of similar Howe correspondences involving real orthogonal groups of various signatures and their double covers. This point is examined in [8].

1.11. Remark. – If the representation $X$ sets up a Howe correspondence for the pair $(G, \mathfrak{h})$, any $G \times \mathfrak{h}$-invariant subspace of $X$ also sets up a Howe correspondence since it must be a sum of the $G \times \mathfrak{h}$-irreducible modules $R_i \otimes S_i$.

1.12. REMARK. – Instead of considering the operators $\Theta$, $\Theta^* \in \operatorname{End}(\Lambda \otimes S)$ as defined in 1.1, one could have considered their "graded" analogues:

$$\hat{\Theta}(\omega \otimes \psi) = \sum_{a=1}^{a=n} (-1)^{\omega}\, e_a \wedge \omega \otimes e_a.\psi$$

$$\hat{\Theta}^*(\omega \otimes \psi) = \sum_{a=1}^{a=n} (-1)^{\omega}\, i_a \wedge \omega \otimes e_a.\psi.$$

It is then easily verified that these operators commute with the action of $\operatorname{Pin}(n)$ on $\Lambda \otimes S$ and generate a Lie isomorphic to $\operatorname{sl}(2, \mathbb{R})$. Since $\operatorname{Ker}\Theta = \operatorname{Ker}\hat{\Theta}$ and $\operatorname{Ker}\Theta^* = \operatorname{Ker}\hat{\Theta}^*$, all of the theorems of this chapter remain true mutatis mutandis for the operators $\hat{\Theta}$ and $\hat{\Theta}^*$.

## 2. Decomposition of $\Lambda \otimes S$ for the action of $\operatorname{Spin}(2m) \times \operatorname{sl}_2(\Theta)$

In this section it will be shown that when the vector space $V$ is oriented and even-dimensional, we can write the space $\Lambda \otimes S$ as the sum of two subspaces which are invariant under $\operatorname{Spin}(2m)$ and the $\operatorname{sl}(2, \mathbb{R})$ generated by $\Theta$ and $\Theta^*$. The decomposition of either of these subspaces into irreducible components for the product action provides analogues of Theorem 1.7.

For the rest of this section, $V$ will be a real, oriented, Euclidean vector space of even dimension $2m$. The real Clifford algebra of $V^*$ will be denoted by $C$, the complex Clifford algebra by $C_c$ and we fix a space of spinors $S$ and an algebra isomorphism $C_c \overset{\sim}{=} \operatorname{End} S$. We will consider $V^*$ as a subset of its Clifford algebra.

2.1. DEFINITION AND NOTATION. – (i) *If $e_1, e_2, \ldots, e_{2m}$ is positively-oriented, orthonormal basis of $V^*$, define $\varepsilon \in \mathit{Spin}(2m) \subset C$ by $\varepsilon = e_1 e_2 \ldots e_{2m}$. Then it is easily verified that $\varepsilon^2 = (-1)^m$.*

(ii) *If $X$ is an $\mathit{Pin}(2m)$-module in which $-1 \in \mathit{Pin}(2m)$ acts as $\operatorname{Id}_X$, we set: $X^+ = \{\psi \in X : \varepsilon(\psi) = i^m \psi\}$ and $X^- = \{\psi \in X : \varepsilon(\psi) = -i^m \psi\}$.*

One verifies that $\varepsilon$ does not depend on the choice of positively-oriented, orthonormal basis and that $\varepsilon = (-1)^m \varepsilon^*$. The operator $\varepsilon$ is in the centre of the group $\operatorname{Spin}(2m) \subset C$ but not in the centre of group $\operatorname{Pin}(2m)$ and hence $X^+$ and $X^-$ are $\operatorname{Spin}(2m)$-invariant but not necessarily $\operatorname{Pin}(2m)$-invariant. Note also that $\varepsilon$ acts on $\Lambda^k \otimes S$ by $R(\varepsilon) = (-1)^k \otimes \varepsilon$.

The condition in (ii), that $-1 \in \operatorname{Pin}(2m)$ acts as $-\operatorname{Id}_X$, is equivalent to the condition that the representation of $\operatorname{Pin}(2m)$ does not factor through the covering map $\pi : \operatorname{Pin}(2m) \to O(2m)$. Thus, for example, the representation in $S$ satisfies this condition, whilst the representation in $V$ does not.

2.2. LEMMA. – (i) *Let $X$ be an irreducible, complex $\mathit{Pin}(2m)$-module in which $-1$ acts as $-\operatorname{Id}_X$. Then the decomposition*

$$X = \{x \in X : \varepsilon(x) = i^m x\} \oplus \{x \in X : \varepsilon(x) = -i^m x\} = X^+ \oplus X^-$$

*is the decomposition of $X$ into irreducible components as a $\mathit{Spin}(2m)$-module. The two factors are irreducible, non-isomorphic $\mathit{Spin}(2m)$-modules of the same dimension.*

(ii) *Let $X$ and $Y$ be two such Pin $(2m)$-modules. Then $X^+ \cong Y^+$ as Spin $(2m)$-modules if and only if $X \cong Y$ as Pin $(2m)$-modules.*

*Proof.* – If $\rho : \mathrm{Pin}(2m) \to \mathrm{End}\, X$ is the representation, then $\rho(\varepsilon)^2 = \rho(\varepsilon^2) = \rho((-1)^m) = (\rho(-1)^m) = (-1)^m \mathrm{Id}_X$ and so the possible eigenvalues of $\rho(\varepsilon)$ are $i^m$ and $-i^m$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\rho(\varepsilon)$, let $W_\lambda$ be the associated eigenspace and let $x \in W_\lambda$ be an eigenvector. If $v \in V^*$ is such that $v^2 = -1$, we have $\rho(\varepsilon)(\rho(v)(x)) = -\rho(v)(\rho(\varepsilon)(x)) = -\lambda\rho(v)(x)$, and hence $\rho(v)$ defines an isomorphism of $W_\lambda$ with $W_{-\lambda}$. Notice that this also implies that $W_\lambda$ is a proper subspace of $X$. The subspace $W_\lambda \oplus W_{-\lambda}$ is Pin $(2m)$-invariant since it is invariant by both the element $v$ and the group Spin $(2m)$, which together generate Pin $(2m)$. Hence $X = W_\lambda \oplus W_{-\lambda}$ by Pin $(2m)$-irreducibility. The subspaces $W_\lambda$ and $W_{-\lambda}$ are Spin $(2m)$-invariant and easily seen to be Spin $(2m)$-irreducible. As representations of Spin $(2m)$ they are not isomorphic because the central element $\varepsilon$ takes the values $i^m$ in one and the value $-i^m$ in the other. This proves (i).

In one direction (ii) is obvious so suppose that $f : X^+ \to Y^+$ is a Spin $(2m)$-isomorphism. Then it is straightforward to check that $g : X^- \to Y^-$ defined by $g(x) = z.f(z^{-1}.x)$ is a Spin $(2m)$-isomorphism and that $f + g : X \to Y$ is a Pin $(2m)$-isomorphism. Here $z$ is any element of $V^* \cap \mathrm{Pin}(2m)$.

The representation of Pin $(2m)$ in $\Lambda^k \otimes S$ satisfies the condition of 2.1 (ii), *i.e.* the element $-1 \in \mathrm{Pin}(2m)$ acts as $-\mathrm{Id}$. Parts (i) and (ii) of the following theorem are immediate consequences of Theorem 1.7 and Lemma 2.2 and part (iii) follows from Theorem 1.9.

2.3. THEOREM. – (i) *For $0 \leq k \leq 2m$, the decomposition*

$$\Lambda^k \otimes S = \bigoplus_{0 \leq k \leq \min(k,\, n-k)} (\Theta^{k-r}(P_r^+) \oplus \Theta^{k-r}(P_k^-))$$

*is the decomposition into Spin $(2m)$-irreducible components. No multiplicities occur.*

(ii) *We have the following isomorphisms of Spin $(2m) \times \mathrm{sl}_2(\Theta)$ modules*

$$(\Lambda \otimes S)^+ = \bigoplus_{k=0}^{k=m} P_k^+ \otimes \sigma_{2m+1-2k} \qquad \text{and} \qquad (\Lambda \otimes S)^- \cong \bigoplus_{k=0}^{k=m} P_k^- \otimes \sigma_{2m+1-2k},$$

*where $\sigma_r$ denotes the unique irreducible $\mathrm{sl}_2(\Theta)$-module of dimension of dimension $r$. The $P_k^+$ (resp. $P_k^-$) are distinct irreducible representations of Spin $(2m)$.*

(iii) *The ring of Spin $(2m)$-invariants in End $(\Lambda \otimes S)$ is generated by $\mathrm{sl}_2(\Theta)$ and $R(\varepsilon)$. The ring of Spin $(2m)$-invariants in End $((\Lambda \otimes S)^+)$ or End $((\Lambda \otimes S)^-)$ is generated by $\mathrm{sl}_2(\Theta)$.*

When the vector space $V$ is oriented, one defines the Hodge star operator $* : \Lambda^k \to \Lambda^{2m-k}$, which is an isomorphism of $SO(2m)$-modules, but not of $O(2m)$-modules since for $g \in O(2m)$ and $x \in \Lambda$, we have $g(*x) = (\det g)(*g(x))$. By taking the tensor product with the identity on the spinors $S$, we get a Spin $(2m)$-equivariant isomorphism, which we will also denote by $* : \Lambda^k \otimes S \to \Lambda^{2m-k} \otimes S$. If we restrict this to a Spin $(2m)$-irreducible subspace of $\Lambda^k \otimes S$, the image must be an isomorphic subspace of

$\Lambda^{2m-k} \otimes S$ and hence by Theorem 2.3 (i), we have Spin $(2m)$-equivariant isomorphisms (for $0 \le k \le m$ and for $0 \le s \le k$):

$$* : \Theta^s (P^+_{k-s}) \to \Theta^{2m-2k+s} (P^+_{k-s}) \qquad \text{and} \qquad * : \Theta^s (P^-_{k-s}) \to \Theta^{2m-2k+s} (P^-_{k-s}).$$

Since the operator $\Theta^{2m-2k}$ is another Spin $(2m)$-equivariant isomorphism between the same pairs of irreducible Spin $(2m)$-modules, it must be proportional to the Hodge star operator when acting on them. We will postpone the calculation of the constants of proportionality to paragraph 5 but we can make the following observation:

2.4. COROLLARY. – *If for all $x^+ \in \Theta^s (P^+_{k-s})$, $\Theta^{2m-2k} (x^+) = \lambda (* x^+)$, where $\lambda \in \mathbb{C}$, then for all $x^- \in \Theta^s (P^-_{k-s})$, we have $\Theta^{2m-2k} (x^-) = -\lambda (* x^-)$.*

*Proof.* – Let $v \in V^*$ be a real 1-form such that $v^2 = -1$, considered as an element of Pin $(2m)$ and let $R_v \in$ End $(\Lambda \otimes S)$ denote its representant in End $(\Lambda \otimes S)$. If $x^+ \in P^+_{k-s}$, then $R_v (x^+) \in P^-_{k-s}$ as in the proof of Lemma 2.2. (i). Hence,

$$\Theta^{2m-2k} (R_v (x^+)) = R_v \Theta^{2m-2k} (x^+) = \lambda R_v (* x^+) = -\lambda * R_v (x^+).$$

The first equality follows because $\Theta$ commutes with Pin $(2m)$ and the final equality because $R_v$ acts as a reflection on $\Lambda$.

2.5. REMARK. – The referee has pointed out that this corollary means that the Hodge * together with $\mathrm{sl}_2 (\Theta)$ generate the full commutant of Spin $(2m)$ in End $(\Lambda \otimes S)$.

## 3. Decomposition of $\Lambda \otimes S$ for the action of Spin $(n) \times \mathrm{sl}_2 (\mathbb{C})$ ($n$ odd)

When the dimension of $V$ is odd, the decomposition of $(\Lambda^k \otimes S)^\pm$ obtained in Theorem 1.7 (i) is in fact irreducible for the group Spin $(n)$. This is a consequence of the following lemma:

3.1. LEMMA. – *Suppose $n$ is odd and let $X$ be a irreducible, complex Pin $(n)$-module. Then $X$ is an irreducible Spin $(n)$-module.*

*Proof.* – The central element $e \in$ Pin $(n)$ (*cf.* 0.3 (iii)) acts as a scalar in $X$ by Schur's lemma. Any Spin $(n)$-invariant subspace of $X$ is therefore Pin $(n)$-invariant since every element of Pin $(n)$ can be written as a product of $e$ and an element of Spin $(n)$.

This lemma, Theorem 1.7 and Theorem 1.9 imply:

3.2. THEOREM. – *Let $n$ be an odd integer. Then:*

(i)
$$(\Lambda^k \otimes S)^+ = \bigoplus_{0 \le r \le \min(k, n-k)} \Theta^{k-r} (P^+_r)$$
$$(\Lambda^k \otimes S)^- = \bigoplus_{0 \le r \le \min(k, n-k)} \Theta^{k-r} (P^-_r)$$

*are the decompositions into Spin $(n)$-irreducible components and no multiplicities occur;*

(ii) *we have the following isomorphisms of Spin* $(n) \times \mathrm{sl}_2(\Theta)$*-modules*

$$(\Lambda \otimes S)^+ \cong \bigoplus_{k=0}^{k=\frac{n-1}{2}} P_k^+ \otimes \sigma_{n+1-2k} \qquad \textit{and} \qquad (\Lambda \otimes S)^- \cong \bigoplus_{k=0}^{k=\frac{n-1}{2}} P_k^- \otimes \sigma_{n+1-2k},$$

*where* $\sigma_r$ *denotes the unique irreducible* $\mathrm{sl}_2(\Theta)$*-module of dimension of dimension* $r$. *The* $P_k^+$ *(resp.* $P_k^-$*) are distinct irreducible representations of Spin* $(n)$.

(iii) *The ring of Spin* $(2m)$*-invariants in End* $((\Lambda\otimes S)^+)$ *or End* $((\Lambda\otimes S)^-)$ *is generated by* $\mathrm{sl}_2(\Theta)$.

## 4. The case of a Euclidean vector space with compatible complex structure

When the Euclidean vector space $V$ has additional geometric structures the group of symmetries of the situation becomes smaller but the space of invariant objects becomes larger. In this section we suppose that $V$ has an isometric complex structure and consider a Lie subalgebra of End $(\Lambda \otimes S)$ which contains $\mathrm{sl}_2(\Theta)$ and which is $U'$-invariant, where $U'$ is the (connected) subgroup of Pin $(V)$ which covers the unitary subgroup of $O(V)$ defined by $J$.

4.1. Decomposition into types and the Clifford algebra in the presence of a complex structure. – Suppose that $V$ is a real, $2m$-dimensional, vector space equipped with a positive-definite inner product, $g$, and a compatible almost complex structure, $J : V \to V$ ($J^2 = -\mathrm{Id}$ and $J$ is isometric for $g$). We define $\Lambda^{p,q}$, the space of forms of type $(p, q)$, and its natural hermitian metric in the standard way (*cf.* [4] or [9]). Thus, for example, $(\alpha|\beta) = g(\alpha, \bar\beta)$ where $\alpha$, $\beta$ are complex 1-forms, defines the hermitian form on $V^* \otimes \mathbb{C}$. Hence if $\{z_1, z_2, \ldots, z_m\}$ is an orthonormal basis of $\Lambda^{1,0}$, the forms $\{\bar z_1, \bar z_2, \ldots, \bar z_m\}$ give an orthonormal basis for $\Lambda^{0,1}$ and the real forms $\left\{\frac{z_a+\bar z_a}{\sqrt2}, \frac{z_a-\bar z_a}{i\sqrt2}\right\}_{1\le a\le m}$ give a real orthonormal basis of $V^*$.

Now as in 0.1, let $C_c$ denote the complex Clifford algebra associated to $(V^*, g)$ and let $S$ be a space of spinors for $C_c$. If $\{z_1, z_2, \ldots, z_m\}$ is an orthonormal basis of $\Lambda^{1,0}$, the following relations hold in $C_c$:

$$\begin{aligned} z_i\, z_j + z_j\, z_i &= 0 && \text{for } 1 \le i, j \le m,\\ z_i\, z_i^* + z_i^*\, z_i &= 2\,\mathrm{Id} && \text{for } 1 \le i \le m,\\ z_i\, z_j^* + z_j^*\, z_i &= 0 && \text{for } 1 \le i \ne j \le m. \end{aligned}$$

The "number" operator $N$ is defined by

$$N = \frac12 \sum_{a=1}^{a=m} z_a^*\, z_a$$

and if $S_s = \{\psi \in S : N(\psi) = s\,\psi\}$ then $S = \bigoplus_{s=0}^{s=m} S_s$. This operator does not depend on the choice of orthonormal basis of $\Lambda^{1,0}$. It is well known that dim $S_0 = 1$ and that if $\psi_0 \in S_0$, then $\{ z^*_{i_1} z^*_{i_2} \dots z^*_{i_k}.\psi_0 \, : \, 1 \leq i_1 < i_2 < \dots < i_k \leq m \}$ is a basis for $S_k$. In the language of E. Cartan, $\psi_0$ is a pure spinor and the subspace $S_0$ of $S$ is uniquely characterised by the property: $\psi \in S_0$ if and only if $z_i.\psi = 0$ for $1 \leq i \leq m$. More generally, the operators $z_i$ map $S_s$ to $S_{s-1}$ and the operators $z^*_i$ map $S_s$ to $S_{s+1}$.

4.2. Definition and basic properties of the operators $Z$ and $\bar{Z}$.

4.3. Definition. – *Let* $\{ z_1, z_2, \dots, z_m \}$ *be an orthonormal basis of* $\Lambda^{1,0}$. *Define* $Z$, $\bar{Z} \in \mathrm{End}(\Lambda \otimes S)$ *by:*

$$Z(\omega \otimes \psi) = \sum_{a=1}^{a=m} z_a \wedge \omega \otimes \bar{z}_a.\psi \qquad \textit{and} \qquad \bar{Z}(\omega \otimes \psi) = \sum_{a=1}^{a=m} \bar{z}_a \wedge \omega \otimes z_a.\psi$$

*where* $\omega \in \Lambda^*$ *and* $\psi \in S$. *It is easily checked that these operators do not depend on the choice of orthonormal basis of* $\Lambda^{1,0}$. *Their adjoints are given by:*

$$Z^*(\omega \otimes \psi) = -\sum_{a=1}^{a=m} i_a \, \omega \otimes z_a.\psi \qquad \textit{and} \qquad \bar{Z}^*(\omega \otimes \psi) = -\sum_{a=1}^{a=m} i_{\bar{a}} \, \omega \otimes \bar{z}_a.\psi$$

*where* $i_a, i_{\bar{a}} : \Lambda \to \Lambda$ *are the interior products along* $z_a$ and $\bar{z}_a$ *respectively.*

4.3.1. *Remark.* – In terms of the decomposition into types and spin states, we see that the operator $Z$ maps $\Lambda^{p,q} \otimes S_s$ to $\Lambda^{p+1,q} \otimes S_{s+1}$ and that the operator $\bar{Z}$ maps $\Lambda^{p,q} \otimes S_s$ to $\Lambda^{p,q+1} \otimes S_{s-1}$.

The operators $Z$ and $\bar{Z}$ do not commute with the action of $\mathrm{Pin}(2m)$ on $\Lambda \otimes S$ since the complex structure $J$ is invariant only under the unitary group $U(V, g, J)$ and not under the full orthogonal group $O(V, g)$. However if $U'$ denotes the subgroup of $\mathrm{Pin}(2m)$ covering $U(V, g, J)$, we have the following

4.4. Proposition. – (i) $\Theta = Z + \bar{Z}$ *where* $\Theta$ *is defined in* 1.1.

(ii) *Let* $u \in U'$ *be an element of the (non-trivial) double cover of the unitary group (acting on* $\Lambda \otimes S$ *by the tensor product representation). Then* $Z u = u Z$ *and* $\bar{Z} u = u \bar{Z}$ *in* $End(\Lambda \otimes S)$.

*Proof.* – (i) By definition,

$$\Theta = \sum_{a=1}^{a=m} \frac{z_a + \bar{z}_a}{\sqrt{2}} \wedge \omega \otimes \frac{z_a + \bar{z}_a}{\sqrt{2}}.\psi + \sum_{a=1}^{a=m} \frac{z_a - \bar{z}_a}{i\sqrt{2}} \wedge \omega \otimes \frac{z_a - \bar{z}_a}{i\sqrt{2}}.\psi$$

since $\left\{ \frac{z_a + \bar{z}_A}{\sqrt{2}}, \frac{z_a - \bar{z}_A}{i\sqrt{2}} \right\}_{1 \leq a \leq m}$ is a real orthonormal basis. This simplifies immediately to give the result.

(ii) Since $U'$ is contained in $\mathrm{Pin}(V)$ and since the group $\mathrm{Pin}(V)$ commutes with $\Theta$ by Proposition 1.2, we have $u\Theta = \Theta u$ and so $u(Z + \bar{Z}) = (Z + \bar{Z})u$. Decomposing the forms into types and comparing components, we see that $Z u = u Z$ and $\bar{Z} u = u \bar{Z}$.

Now we would like to identify the Lie subalgebra of $\mathrm{End}(\Lambda \otimes S)$ generated by the operators $Z$, $\bar{Z}$, $Z^*$ and $\bar{Z}^*$. The first step is the following proposition.

4.5. PROPOSITION. – *On $\Lambda^{p,q}(S_s)$ the following identities hold:*

$$[Z,\, Z^*] = Z\, Z^* - Z^*\, Z = 2\,(p + s - m)\,\mathrm{Id}$$
$$[\bar Z,\, \bar Z^*] = \bar Z\, \bar Z^* - \bar Z^*\, \bar Z = 2\,(q - s)\,\mathrm{Id}.$$

*Proof.* – If $\omega \otimes \psi \in \Lambda \otimes S$, we have

$$(1)\qquad \bar Z\, \bar Z^*\,(\omega \otimes \psi) = -\sum_{a \text{ and } b} \bar z_a \wedge i_{\bar b}\,(\omega) \otimes z_a.\bar z_b\,\psi$$
$$= -\sum_a \bar z_a \wedge i_{\bar a}\,(\omega) \otimes z_a.\bar z_a.\psi - \sum_{a \neq b} \bar z_a \wedge i_{\bar b}\,(\omega) \otimes z_a.\bar z_b.\psi.$$

On the other hand,

$$(2)\qquad \bar Z^*\, \bar Z\,(\omega \otimes \psi) = -\sum_{a \text{ and } b} i_{\bar b}\,(\bar z_a \wedge \omega) \otimes \bar z_b.z_a.\psi$$
$$= -\sum_{a \text{ and } b} i_{\bar b}\,(\bar z_a)\,\omega \otimes \bar z_b.z_a.\psi + \sum_{a \text{ and } b} \bar z_a \wedge i_{\bar b}\,(\omega) \otimes \bar z_b.z_a.\psi$$
$$= -\omega \otimes \sum_{a=1}^{a=n} \bar z_a.z_a.\,\psi + \sum_a \bar z_a \wedge i_{\bar a}\,(\omega) \otimes \bar z_a.z_a.\psi$$
$$+ \sum_{a \neq b} \bar z_a \wedge i_{\bar b}\,(\omega) \otimes \bar z_b.z_a.\psi.$$

Subtracting the equation (2) from (1), we get:

$$(3)\qquad [\bar Z,\, \bar Z^*]\,(\omega \otimes \psi) = \omega \otimes \sum_{a=1}^{a=m} \bar z_a.z_a.\,\psi - \sum_a \bar z_a \wedge i_{\bar a}\,(\omega) \otimes (\bar z_a.z_a + z_a.\bar z_a)\psi.$$

(i) The number operator $N$ is given by $N = \dfrac{1}{2} \sum_{a=1}^{a=m} z_a^*\, z_a$ and $N\,(\psi) = s\,\psi$ if and only if $\psi \in S_s$ where $0 \leq s \leq m$. Using the fact that $\bar z_a = -z_a^*$, this implies that $\sum_{a=1}^{a=m} \bar z_a.z_a.\psi = -2\,s\,\psi$.

(ii) By definition, $\sum_a \bar z_a \wedge i_{\bar a}\,(z_b) = 0$ and $\sum_a \bar z_a \wedge i_{\bar a}\,(\bar z_b) = \bar z_b$. Now since the interior product is an antiderivation, we deduce that $\sum_a \bar z_a \wedge i_{\bar a}\,(\omega) = q\,\omega$ if $\omega \in \Lambda^{p,q}$.

(iii) For each $a\,(1 \leq a \leq m)$, we have $\bar z_a\, z_a + z_a\, \bar z_a = -z_a\, z_a^* - z_a^*\, z_a = -2\,\mathrm{Id}$.

Substituting (i), (ii) and (iii) in the equation (3), we find

$$(\bar Z\, \bar Z^* - \bar Z^*\, \bar Z)\,(\omega \otimes \psi) = 2\,(q - s)\,(\omega \otimes \psi).$$

This proves half of the proposition, the other half following from a similar calculation.

4.6. COROLLARY. – *The following equations hold in* $End(\Lambda \otimes S)$:

$$[\bar{Z}\bar{Z}^* - \bar{Z}^*\bar{Z},\, \bar{Z}] = 4\,Z \qquad \text{and} \qquad [\bar{Z}\bar{Z}^* - \bar{Z}^*\bar{Z},\, \bar{Z}^*] = -4\,\bar{Z}^*$$
$$[\bar{Z}\bar{Z}^* - \bar{Z}^*\bar{Z},\, \bar{Z}] = 4\,\bar{Z} \qquad \text{and} \qquad [\bar{Z}\bar{Z}^* - \bar{Z}^*\bar{Z},\, \bar{Z}^*] = -4\,\bar{Z}^*.$$

*Thus the complex Lie subalgebra of* $End(\Lambda \otimes S)$ *generated by the operators* $Z$, $Z^*$ *(or, taking conjugates, by the operators* $\bar{Z}$, $\bar{Z}^*$*) is isomorphic to* $\mathrm{sl}_2(\mathbb{C})$.

*Proof.* – This is immediate from 4.5.

This means that there are two $\mathrm{sl}_2(\mathbb{C})$'s acting in $\Lambda \otimes S$ but these actions do not commute, as the next proposition shows.

4.7. PROPOSITION. – *The following identities hold:*

(i) $$[Z,\, \bar{Z}^*] = 0 \qquad \text{and} \qquad [Z^*,\, \bar{Z}] = 0;$$

(ii) $$[Z,\, \bar{Z}] = 2\,i\,L,$$

*where* $L$ *is multiplication by the Kähler form* $k = i\sum_{a=1}^{a=m} z_a \wedge \bar{z}_a$: $L(\omega \otimes \psi) = k \wedge \omega \otimes \psi$.

*Proof.* – (i) This is straightforward.

(ii) We have:

$$\begin{aligned} Z\,\bar{Z}\,(\omega \otimes \psi) &= \sum_{a \text{ and } b} z_b \wedge \bar{z}_a \wedge \omega \otimes \bar{z}_b.z_a.\psi \\ &= \sum_a z_a \wedge \bar{z}_a \wedge \omega \otimes \bar{z}_a.z_a.\psi + \sum_{a \neq b} z_b \wedge \bar{z}_a \wedge \omega \otimes \bar{z}_b.z_a.\psi \end{aligned}$$

In the same way,

$$\bar{Z}\,Z\,(\omega \otimes \psi) = \sum_a \bar{z}_a \wedge z_a \wedge \omega \otimes z_a.\bar{z}_a.\psi + \sum_{a \neq b} \bar{z}_b \wedge z_a \wedge \omega \otimes z_b.\bar{z}_a.\psi.$$

Hence,

$$\begin{aligned} [Z,\, \bar{Z}]\,(\omega \otimes \psi) &= \sum_a z_a \wedge \bar{z}_a \wedge \omega \otimes (\bar{z}_a.z_a + z_a.\bar{z}_a).\psi \qquad (\text{since } \bar{z}_a \wedge z_a + z_a \wedge \bar{z}_a = 0) \\ &= -2\Big(\sum_a z_a \wedge \bar{z}_a\Big) \wedge \omega \otimes \psi \qquad (\text{since } \bar{z}_a\,z_a + z_a\,\bar{z}_a = -2\,\mathrm{Id}) \\ &= 2\,ik \wedge \omega \otimes \psi. \end{aligned}$$

We now have the following 8 operators in the Lie subalgebra of $\mathrm{End}(\Lambda \otimes S)$ generated by $Z$, $\bar{Z}^*$ $Z^*$ and $\bar{Z}$:

$$Z,\, \bar{Z}^*,\, Z^*,\, \bar{Z},\, H_1 = [Z,\, Z^*], \quad H_2 = [\bar{Z},\, \bar{Z}^*], \quad 2\,i\,L = [Z,\, \bar{Z}], \quad 2\,i\,L^* = [Z^*,\, \bar{Z}^*]$$

and we have calculated some but not all of the possible commutators. The following proposition gives the remaining commutators but the proof is omitted since all of the calculations are straightforward.

4.8. PROPOSITION. – (i) *The operators* $H_1$ *and* $H_2$ *are self adjoint and* $[H_1, H_2] = 0$. *The following identities and their adjoints hold in End* $(\Lambda \otimes S)$*:*

$$\text{(ii)} \qquad [H_1, Z] = 4\,Z, \qquad [H_1, \bar{Z}] = -2\,\bar{Z}, \qquad [H_1, L] = 2\,L;$$
$$\text{(iii)} \qquad [H_2, \bar{Z}] = 4\,\bar{Z}, \qquad [H_2, Z] = -2\,Z, \qquad [H_2, L] = 2\,L;$$
$$\text{(iv)} \qquad [L, L^*] = \frac{1}{2}(H_1 + H_2), \qquad [L, Z] = [L, \bar{Z}] = 0, \qquad [L, Z^*] = i\,\bar{Z}.$$

We can now conclude that the linear subspace of End $(\Lambda \otimes S)$ generated by the operators $Z$, $\bar{Z}^*$, $Z^*$, $\bar{Z}$, $H_1$, $H_2$, $L$ and $L^*$ is closed under Lie bracket.

4.9. THEOREM. – (i) *The complex Lie subalgebra of End* $(\Lambda \otimes S)$ *generated by the operators* $Z$, $\bar{Z}^*$, $Z^*$, $\bar{Z}$, *– which will be denoted by* $\mathrm{sl}_3(Z)$ *– is isomorphic to* $\mathrm{sl}_3(\mathbb{C})$.

(ii) *The real form* $X \to \bar{X}$ *corresponds to the real form* $su(2, 1)$ *of* $\mathrm{sl}_3(\mathbb{C})$ *and the real form* $X \to -X^*$ *corresponds to the compact real form* $su(3)$ *of* $\mathrm{sl}_3(\mathbb{C})$.

(iii) *The subalgebra* $\mathrm{sl}_2(\Theta) = \langle \Theta, \Theta^*, [\Theta, \Theta^*] \rangle$ *is a principal* $\mathrm{sl}_2(\mathbb{C})$. *The associated grading of* $\mathrm{sl}_3(Z)$*:*

$$\mathrm{sl}_3(Z) = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \qquad (\textit{with } [\mathcal{G}_n, \mathcal{G}_m] \subset \mathcal{G}_{n+m}),$$

*where* $H = [\Theta, \Theta^*]$ *and* $\mathcal{G}_n = \{X \in \mathrm{sl}_3(Z) : [H, X] = 2\,n\,X\}$, *is in order, the decomposition:*

$$\mathrm{sl}_3(Z) = \langle L^* \rangle \oplus \langle Z^*, \bar{Z}^* \rangle \oplus \langle H_1, H_2 \rangle \oplus \langle Z, \bar{Z} \rangle \oplus \langle L \rangle.$$

*The subalgebra* $\mathcal{G}_0$ *is a Cartan subalgebra, the subalgebra* $\mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$ *is a Borel subalgebra and* $Z$, $\bar{Z}$ *are simple root vectors.*

*Proof.* – (i) Consider $\mathbb{C}^3$ with hermitian forms (in the canonical basis) $h = z_1\,\bar{z}_1 + z_2\,\bar{z}_2 + z_3\,\bar{z}_3$ and $g = z_1\,\bar{z}_1 + z_2\,\bar{z}_2 - z_3\,\bar{z}_3$. If $A$ is $3 \times 3$ complex matrix, then adjunction with respect to the first is given by $A \to \bar{A}^t$ and with respect to the second by $A = \begin{pmatrix} X & v \\ w^t & a \end{pmatrix} \to A' = \begin{pmatrix} \bar{X}^t & -\bar{w} \\ -\bar{v}^t & \bar{a} \end{pmatrix}$, where $X$ is a $2 \times 2$ complex matrix, $v$ and $w$ are $2 \times 1$ complex matrices and $a$ is a complex number. Define $\phi : \mathrm{sl}_3(\mathbb{C}) \to \mathcal{G}$ by:

$$\phi\left(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = Z; \qquad \phi\left(\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = Z^*;$$

$$\phi\left(\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \bar{Z}; \qquad \phi\left(\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \bar{Z}^*;$$

$$\phi\left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right) = H_1; \qquad \phi\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\right) = H_2;$$

$$\phi\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\right) = 2\,i\,L; \qquad \phi\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}\right) = 2\,i\,L^*$$

Calculation now shows that the matrices above form a basis of $\mathrm{sl}_3(\mathbb{C})$ and that the map $\phi : \mathrm{sl}_3(\mathbb{C}) \to \mathrm{sl}_3(Z)$ is a homomorphism of Lie algebras; since $\mathrm{sl}_3(\mathbb{C})$ is simple, it is necessarily an isomorphism. This proves (i) of the proposition.

(ii) To prove part (ii), we remark that $\phi(\bar{A}^t) = \phi(A)^*$ and that $\phi(-A') = \overline{\phi(A)}$.

(iii) The subalgebra $\mathrm{sl}_2(\Theta) = \{\Theta, \Theta^*, [\Theta, \Theta^*]\}$ is isomorphic to $\mathrm{sl}_2(\mathbb{C})$ by Proposition 1.3. To prove that it is principal in the sense of Kostant (*cf.* [2]), we have to show that $\Theta$ is a nilpotent element of the Lie algebra $\mathrm{sl}_3(Z)$ and that its centraliser in $\mathrm{sl}_3(Z)$ is of the same dimension as that of a Cartan subalgebra, namely 2 in our case. Now $\Theta = Z + \bar{Z}$ so that $\phi^{-1}(\Theta) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The result follows by direct calculation. Note that the operators $\Theta$ and $L$ give a basis of the centraliser of $\Theta$ in $\mathrm{sl}_3(Z)$. The rest of the proposition also follows by calculation.

Analogues of Theorems 1.7 and 1.8 hold in this situation.

4.10. THEOREM. – *The Lie algebra* $\mathrm{sl}_3(Z) \oplus \langle C \rangle$ $(\cong \mathrm{gl}_3(\mathbb{C}))$ *generates the commutant of* $U'$ *in End* $(\Lambda \otimes S)$ *(and therefore* (cf. 1.11) *also in End* $((\Lambda \otimes S)^+)$ *and in End* $((\Lambda \otimes S)^-)$. (Here $C \in \mathrm{End}(\Lambda \otimes S)$, defined by: $C(\omega \otimes \psi) = i(p-q)\,\omega \otimes \psi + i\,\omega \otimes \left(\frac{m}{2} - s\right)\psi$ on $\Lambda^{p,q} \otimes S_s$, is the action of the complex structure of $V$ viewed in the centre of $u'$. By the theorem, the representations $(\Lambda^k \otimes S)^+$ $(\Lambda^k \otimes S)^-$ set up Howe correspondences (*cf.* 1.10) not only for the dual pair $(\mathrm{Spin}(2m), \mathrm{sl}_2(\Theta))$ (by 2.3) but also for "see-saw" pair (*cf.* [6]) $((U', \mathrm{sl}_3(Z) \oplus \langle C \rangle)$.)

*Proof.* – Following a suggestion of the referee, we rewrite the $U'$-representation $\Lambda \otimes S$ as follows:

$$\begin{aligned}\Lambda \otimes S &\cong \Lambda(\Lambda^{1,0} \oplus \Lambda^{1,0}) \otimes \Lambda(\Lambda^{0,1}) \otimes K^{\frac{1}{2}} \cong \Lambda(\Lambda^{1,0}) \otimes \Lambda(\Lambda^{0,1}) \otimes \Lambda(\Lambda^{0,1}) \otimes K^{\frac{1}{2}} \\ &\cong K \otimes \Lambda(\Lambda^{0,1}) \otimes \Lambda(\Lambda^{0,1}) \otimes \Lambda(\Lambda^{0,1}) \otimes K^{\frac{1}{2}} \\ &\cong K \otimes \Lambda(\Lambda^{0,1} \otimes \mathbb{C}^3) \otimes K^{\frac{1}{2}}.\end{aligned}$$

Here $K^{\frac{1}{2}}$ is the square root of the $U'$-representation $K = \Lambda^{m,0}$ and we have used the $U'$-isomorphisms $\Lambda^{p,0} \cong K \otimes \Lambda^{0,m-p}$ (realised by the complex linear Hodge star operator) and $S \cong \Lambda(\Lambda^{0,1}) \otimes K^{\frac{1}{2}}$ (*cf.* 4.1 above). The Lie algebra $\mathrm{gl}(\mathbb{C}^3)$ acts naturally on $K \otimes \Lambda(\Lambda^{0,1} \otimes \mathbb{C}^3) \otimes K^{\frac{1}{2}}$ and commutes with the action of $U'$. One can recognise this as precisely the action of $\mathrm{sl}_3(Z) \oplus \langle \mathbb{C} \rangle$. Since $\mathrm{End}(K \otimes \Lambda(\Lambda^{0,1} \otimes \mathbb{C}^3) \otimes K^{\frac{1}{2}}) \cong \mathrm{End}(\Lambda(\Lambda^{0,1} \otimes \mathbb{C}^3))$ and since the action of $U'$ on $\Lambda^{0,1}$ factors through the action of its quotient the unitary group, proving the theorem is equivalent to proving that the $U'(\Lambda^{0,1})$ invariants in $\mathrm{End}(\Lambda(\Lambda^{0,1} \otimes \mathbb{C}^3))$ are generated by $\mathrm{gl}(\mathbb{C}^3)$.

Applying Theorem 7 in Howe [5] (with $U = (0)$, $W = \Lambda^{0,1} \otimes \mathbb{C}^3$, $G = GL(\Lambda^{0,1})$ $\Gamma = \mathrm{gl}(\Lambda^{0,1})$, $\Gamma' = \mathrm{gl}(\mathbb{C}^3)$, $O = O(W \oplus W^*)$ (= the complex orthogonal group preserving the canonical symmetric bilinear form) and $\mathrm{End}^0 = \mathrm{End}(\Lambda^{0,1} \otimes \mathbb{C}^3)$), we deduce that $\mathrm{gl}(\mathbb{C}^3)$ generates the algebra of $GL(\Lambda^{0,1})$ invariants in End $(\Lambda^{0,1} \otimes \mathbb{C}^3)$. Since this representation of $GL(\Lambda^{0,1})$ is holomorphic, the algebra of $GL(\Lambda^{0,1})$ invariants is the same as the algebra of $U(\Lambda^{0,1})$ invariants.

## 5. The Hodge star operator and positivity

Recall that in section 2 we remarked that if $\dim V = 2m$ and if $0 \leq k \leq m$, the two $\mathrm{Spin}(V)$-equivariant maps $\Theta^{2m-2k}$ and the complex Hodge star operator are proportional when restricted to $\Theta^s(P^+_{k-s})$ (or $\Theta^s(P^-_{k-s})$). In this section we will calculate the constants of proportionality by evaluation on particular elements of these spaces and deduce some positivity results, analogous to the infinitesimal Hodge-Riemann bilinear relations on a Kähler manifold. In order to do this we fix an isometric complex structure $J$ on $V$ and use the results of paragraph 4.

5.1. PROPOSITION. – (i) *If $\omega \in \Lambda \otimes S$ and $\bar{Z}(\omega) = 0$, then $\Theta^s(\omega) = p_s(Z, L)(\omega)$, where $s \in \mathbb{N}$ and $p_s(z, k)$ is a complex polynomial in two variables. (The operators $Z$ and $L$ commute by Proposition 4.9 so that this makes sense.)*

(ii) *The polynomials $p_s$ satisfy $p_{s+1} = zp_s - 2ik \dfrac{\partial p_s}{\partial z}$ and $p_0 = 1$.*

*Proof.* – The proof depends on the following lemma:

LEMMA. – *For all $s \in \mathbb{N}$, we have $[\bar{Z}, Z^s] = -2is\, LZ^{s-1}$.*

*Proof of lemma.* – If $s = 1$, from Proposition 4.7 we have $[\bar{Z}, Z] = -2i\,L$ and the formula is true. Now suppose that it is true for $0 \leq s \leq n$ and proceed by induction. We have

$$\begin{aligned}
\bar{Z}\,Z^{n+1} - Z^{n+1}\,\bar{Z} &= \bar{Z}\,Z^{n+1} - Z^n\,\bar{Z}Z + Z^n\,\bar{Z}Z - Z^{n+1}\,\bar{Z} \\
&= [\bar{Z},\, Z^n]\,Z + Z^n\,[\bar{Z},\, Z] \\
&= -2in\,L\,Z^n - 2i\,L\,Z^n \qquad \text{(by the induction hypothesis)} \\
&= -2i\,(n+1)\,L\,Z^n.
\end{aligned}$$

Now we can prove the proposition by induction. Since $\Theta = Z + \bar{Z}$, applying $\Theta$ to $\omega \in \mathrm{Ker}\,\bar{Z}$ gives $\Theta(\omega) = Z(\omega)$ and the righthand side is a polynomial in $Z$ acting on $\omega$.

Suppose now that for $0 \leq s \leq n$, $\Theta^s(\omega) = p_s(Z, L)(\omega)$, where $p_s$ is a complex polynomial in two variables. Then $\Theta^{n+1}(\omega) = (Z + \bar{Z})\,\Theta^n(\omega) = Z\,p_n(Z, L)(\omega) + \bar{Z}\,p_n(Z, L)(\omega)$. The first term is clearly polynomial in $Z$ and $L$; as for the second term, by the lemma $\bar{Z}\,Z^s(\omega) = -2is\,L\,Z^{s-1}(\omega)$ and so $\bar{Z}\,p_n(Z, L)(\omega) = -2i\,L\,\dfrac{\partial p_n}{\partial z}(Z, L)(\omega)$. Thus the second term is also polynomial and by induction the proposition is proved.

5.1.1. COROLLARY. – *For $s \in \mathbb{N}$ the polynomial $p_s(z, k)$ of 5.1 above is given by:*

$$p_s(z, k) = (-2ik)^s\, e^{\frac{z^2}{4ik}}\, \frac{\partial^s}{\partial z^s}\,(e^{\frac{-z^2}{4ik}}).$$

*In particular,* $p_{2s}(0, k) = (-i)^s\, \dfrac{(2s)!}{s!}\, k^s$.

*Proof.* – The solutions of the differential equation (ii) are more or less classical Hermite polynomials.

5.1.2. Corollary. – *If* $\omega \in \Lambda \otimes S$ *and* $\bar{Z}(\omega) = 0$, *then*

$$\Theta^{2s}(\omega) = \left( Z^{2s} + \sum_{r=1}^{r=s} a_r\, L^r\, Z^{2s-2r} \right)(\omega) \quad (\textit{where } a_r \in \mathbb{C}).$$

*Proof.* – This is immediate from the formula for $p_s$.

5.1.3. Corollary. – *Suppose* $0 \leq s \leq m$. *If* $\omega \in \Lambda^{s,0} \otimes S$ *and* $\bar{Z}(\omega) = 0$, *then* $\Theta^{2m-2s}(\omega) = (-i)^{m-s} \dfrac{(2m-2s)!}{(m-s)!} L^{m-s}(\omega)$.

*Proof.* – From 5.1.2, we have $\Theta^{2m-2s}(\omega) = \left( Z^{2m-2s} + \sum_{r=1}^{r=m-s} a_r\, L^r\, Z^{2m-2s-2r} \right)(\omega)$. The form part of the term $L^r\, Z^{2m-2s-2r}(\omega)$ is of type $(s + r + 2m - 2s - 2r,\, r) = (2m - s - r,\, r)$ and so vanishes when $r + s < m$, that is when $0 \leq r < m - s$. Hence only the last term in the series is left, namely $a_{m-s}\, L^{m-s}(\omega)$, and from Proposition 5.1.1 we know the value of $a_{m-s}\,(= p_{2m-2s}(0, k))$.

5.2. Definition. – *If* $z_1, z_2, \ldots, z_m$ is an orthonormal basis of $\Lambda^{1,0}$, the associated orientation/volume form is defined as $\varepsilon = i^m\, z_1 \wedge \bar{z}_1 \wedge z_2 \wedge \bar{z}_2 \wedge \ldots z_m \wedge \bar{z}_m$. This is a real $2m$-form type $(m, m)$ which does not depend on the choice of orthonormal basis. If we set $e_a = \dfrac{z_a + \bar{z}_a}{\sqrt{2}}$ and $J\, e_a = \dfrac{z_a - \bar{z}_a}{i\sqrt{2}}$, then $\varepsilon = e_1 \wedge J\, e_1 \wedge e_2 \wedge J\, e_2 \ldots e_m \wedge J\, e_m$ in terms of the real orthonormal basis $\{ e_1,\, J\, e_1 \ldots e_m,\, J\, e_m \}$. We will also write $\varepsilon$ for this form viewed as an element of the Clifford algebra or the group $\mathrm{Spin}(V)$ (*cf.* 2.1.1).

5.3. Proposition. – *Let* $\psi_0 \in S_0$ *be a pure spinor* (cf. 4.1) *and let* $\omega \in \Lambda^{s,0} \otimes S$ *be the element defined by* $\omega = z_1 \wedge z_2 \wedge \ldots \wedge z_s \otimes \psi_0$. *Then the following hold:*

(i) $\varepsilon.\psi_0 = (-i)^m\, \psi_0$ *and* $R(\varepsilon)(\omega) = (-1)^{m+s}\, i^m\, \omega$.

(ii) $\omega \in \mathrm{Ker}\,\Theta^* \cap \mathrm{Ker}\,\bar{Z}$.

(iii) $k^{m-s} \wedge z_1 \wedge z_2 \wedge \ldots \wedge z_s = (-1)^{\frac{1}{2}s(s-1)}\, i^s\, (m-s)!\, *\, (z_1 \wedge z_2 \wedge \ldots \wedge z_s)$, *where* $k = i \sum_{a=1}^{a=m} z_a \wedge \bar{z}_a$ *is the Kähler form and* $*$ *denotes the complex linear Hodge star operator.*

5.4. Corollary. – *For* $0 \leq s \leq m$, *the following identities hold:*

$$\Theta^{2m-2s}(x^+) = (-1)^{\frac{1}{2}s(s-1)}\, i^m\, (2m-2s)!\,(*\, x^+) \qquad \textit{for all } x^+ \in P_s^+;$$
$$\Theta^{2m-2s}(x^-) = -(-1)^{\frac{1}{2}s(s-1)}\, i^m\, (2m-2s)!\,(*\, x^-) \qquad \textit{for all } x^- \in P_s^-.$$

*Proof of* 5.3. – (i) The identity $\varepsilon.\psi_0 = (-i)^m\, \psi_0$ is straightforward. The element $\varepsilon$ of $\mathrm{Spin}(V)$ acts on $V^*$ as $-\mathrm{Id}$ and hence on $\Lambda^s$ as $(-\mathrm{Id})^s$. Taking the tensor product we get $R(\varepsilon)(\omega) = R(\varepsilon)(z_1 \wedge z_2 \wedge \ldots \wedge z_s \otimes \psi_0) = (-1)^s\, (-i)^m = (-1)^{m+s}\, i^m$.

(ii) From the defining formulas of 4.2, we see that $\bar{Z}(\omega) = \bar{Z}^*(\omega) = Z^*(\omega) = 0$ because $z_a.\psi_0 = 0$ and $i_{\bar{a}}(z_1 \wedge z_2 \wedge \ldots \wedge z_s) = 0$ for all $1 \leq a \leq m$.

(iii) This is a standard calculation for the complex linear Hodge star operator (*cf.* Weil).

*Proof of* 5.4. – Suppose that $m+s$ is even. Then $\omega \in P_s^+$ by Proposition 5.3 (ii) and Definition 2.1. By Corollary 5.1.3 and Proposition 5.3 (ii), we have:

$$\begin{aligned}\Theta^{2m-2s}(\omega) &= (-i)^{m-s}\,\frac{(2m-2s)!}{(m-s)!}\,L^{m-s}(\omega)\\ &= (-i)^{m-s}\,\frac{(2m-2s)!}{(m-s)!}\,(-1)^{\frac{1}{2}s(s-1)}\,i^s\,(m-s)!\,*\,(\omega)\\ &= i^m\,(2m-2s)!\,(-1)^{\frac{1}{2}s(s-1)}\,*\,(\omega) \qquad (\text{since } m-s \text{ is even}).\end{aligned}$$

Since we already know that $\Theta^{2m-2s}$ and $*$ are proportional when restricted to $P_s^+$, this gives what we want on $P_s^+$ and thus by Corollary 2.4, also on $P_s^-$.

These formulas give the action of the Hodge star operator on primitive elements of fixed degree $s$ in terms of the operator $\Theta$. In order to give its action on the other Pin $(n)$-irreducible components of $\Lambda^s \otimes S$ we need the following lemma, whose proof is left as an exercise to the reader:

5.5. Lemma. – *Let $\Lambda^k$ denote the space of real k-forms on $V$, an $m$-dimensional vector space. Let $x \in \Lambda^1$ be a non-zero 1-form and let $i_x : \Lambda^* \to \Lambda^{*-1}$ denote the interior product along $x$. Then, if $y \in \Lambda^k$,* we have

$$*(x \wedge y) = (-1)^k\, i_x\,(*y).$$

5.6. Corollary. – *If $\omega \otimes \psi \in \Lambda^k \otimes S$, then the following hold:*

$$(*(\Theta\,(\omega \otimes \psi)) = (-1)^{k+1}\,\Theta^*\,(*(\omega \otimes \psi)$$

*and*

$$*(\Theta^{2s}\,(\omega \otimes \psi)) = (-1)^s\,(\Theta^*)^{2s}\,(*(\omega \otimes \psi)).$$

*Proof.* – By definition of the operators $\Theta$, $\Theta^*$ and the above lemma, we have:

$$\begin{aligned}*(\Theta\,(\omega \otimes \psi)) &= \sum_{a=1}^{a=2m} *(e_a \wedge \omega) \otimes e_a.\psi = (-1)^k \sum_{a=1}^{a=2m} i_a\,(*\,\omega) \otimes e_a.\psi\\ &= (-1)^{k+1}\,\Theta^*\,(*(\omega \otimes \psi)).\end{aligned}$$

The other identify follows by iteration.

One can now generalise Corollary 5.4 to obtain the following (compare Weil [9], Théorème 1.4.2):

5.7. Proposition. – *Let $x^+ \in P_s^+$ and $x^- \in P_s^-$ be primitive elements and let $r$ be an integer such that $0 \le r \le 2m-2s$. Then the following identities hold:*

(i) $$*\Theta^r\,(x^+) = (-i)^m\,\frac{r!}{(2m-2s-r)!}\,(-1)^{\frac{1}{2}s(s-1)+\frac{1}{2}r(r+1)+rs}\,\Theta^{2m-2s-r}\,(x^+);$$

(ii) $$*\Theta^r\,(x^-) = -(-i)^m\,\frac{r!}{(2m-2s-r)!}\,(-1)^{\frac{1}{2}s(s-1)+\frac{1}{2}r(r+1)+rs}\,\Theta^{2m-2s-r}\,(x^-)$$

*Proof.* – This depends on the following calculations:

5.7.1. LEMMA (*See* Weil [9], 1.4 formula (11)). – *Let* $\rho : \mathrm{sl}_2(\Theta) \to End V$ *be an irreducible representation of* $\mathrm{sl}_2(\Theta)$ *of dimension* $n+1$ *and let* $p \in V$ *be a primitive element (that is* $\rho(\Theta)(p) = 0$ *and* $p$ *is an eigenvector of* $\rho([\Theta^*, \Theta])$*. Then for* $0 \leq r \leq n$,

$$\rho(\Theta^*)^r \rho(\Theta)^n(p) = \frac{n!\,r!}{(n-r)!}\rho(\Theta)^{n-r}(p).$$

*Proof of Proposition* 5.7. – We have:

$$\begin{aligned} * \Theta^r(x^+) &= *\Theta(\Theta^{r-1}(x^+)) \\ &= (-1)^{s+r}\Theta^*(*\Theta^{r-1}(x^+)) \qquad \text{(by 5.6.1)} \\ &= (-1)^{(s+r)+(s+r-1)+\ldots(s+1)}(\Theta^*)^r(*x^+) \\ &= (-1)^{\frac{1}{2}r(r+1)+rs}(\Theta^*)^r\left((-i)^m \frac{1}{(2m-2s)!}(-1)^{\frac{1}{2}s(s-1)}\Theta^{2m-2s}(x^+)\right) \\ &\qquad\qquad \text{(by Corollary 5.4)} \\ &= (-i)^m \frac{r!}{(2m-2s-r)!}(-1)^{\frac{1}{2}s(s-1)+\frac{1}{2}r(r+1)+rs}\Theta^{2m-2s-r}(x^+). \end{aligned}$$

This last step comes from Lemma 5.7.1 and the fact complex vector space spanned by $x^+$, $\Theta(x^+)$, $\Theta^2(x^+)$, $\ldots$, $\Theta^{2m-2s}(x^+)$ is an irreducible representation of $\mathrm{sl}_2(\Theta)$ of dimension $2m-2s+1$. This completes the proof of 5.7 (i) and the proof of 5.7 (ii) is the same except for the fact that an extra minus sign is acquired when we apply Corollary 5.4.

5.8. DEFINITION. – *If* $x = \omega \otimes \psi \in \Lambda^s \otimes S$ *and* $y = \sigma \otimes \phi \in \Lambda^r \otimes S$, *then the hermitian exterior product* $x \underline{\wedge} y \in \Lambda^{s+r}$ *is defined by:*

$$x \underline{\wedge} y = \omega \wedge \bar{\sigma}(\psi|\phi).$$

*This can clearly be extended to give a complex linear map*

$$\underline{\wedge} : \Lambda^s \otimes S \otimes \overline{\Lambda^r \otimes S} \to \Lambda^{s+r}$$

*with the properties:*

$$\overline{x \underline{\wedge} y} = (-1)^{sr} x \underline{\wedge} y \qquad \text{and} \qquad x \underline{\wedge} * x' = (x|x')\,\mathbf{v},$$

*where* $\mathbf{v} \in \Lambda^{2m}$ *is the volume form and* $x$, $x' \in \Lambda^s \otimes S$, *and* $y \in \Lambda^r \otimes S$.

5.8.1. LEMMA. – *For* $\alpha \in \Lambda^a \otimes S$ *and* $\beta \in \Lambda^b \otimes S$, *we have*

$$\Theta^k(\alpha) \underline{\wedge} \beta = (-1)^{ka+\frac{1}{2}k(k+1))}\alpha \underline{\wedge} \Theta^k(\beta).$$

*Proof.* – Let $\alpha = \omega \otimes \psi$ and $\beta = \sigma \otimes \phi$. Then we have:

$$\Theta(\alpha) \underline{\wedge} \beta = \sum_{a=1}^{a=2m} e_a \wedge \omega \wedge \bar{\sigma}(e_a.\psi|\phi)$$

and

$$\alpha \underline{\wedge} \Theta(\beta) = \sum_{a=1}^{a=2m} \omega \wedge e_a \wedge \bar{\sigma}(\psi|e_a.\phi).$$

Now Clifford multiplication by the unit vector $e_a$ is skew adjoint so that

$$\Theta(\alpha) \underline{\wedge} \beta = (-1)^{a+1} \alpha \underline{\wedge} \Theta(\beta).$$

By induction, the lemma follows. □

We can now formulate a positivity result which is analogous to the classical Hodge-Riemann bilinear relations as stated in [Weil], p. 77, corollaire IV.7.6.

5.9. THEOREM. – *Let $p$ be an integer such that $0 \leq p \leq 2m$ and let $r$ be an integer such that $(p-m)^+ \leq r \leq p$. Let $\mu(p, r)$ be strictly positive constants. Take $\alpha, \beta \in \Lambda^p \otimes S$ and let $\alpha = \sum_{(p-m)^+ \leq r \leq p} \Theta^r(\alpha_r)$ and $\beta = \sum_{(p-m)^+ \leq r \leq p} \Theta^r(\beta_r)$ be the canonical decompositions, where $\alpha_r$ and $\beta_r$ are primitive elements in $(\Lambda^{p-r} \otimes S)_+$. Define*

$$A^+ : (\Lambda^p \otimes S)^+ \otimes \overline{(\Lambda^p \otimes S)^+} \to \Lambda^{2m}$$

*by:*

$$A^+(\alpha, \beta) = \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, i^m \frac{r!}{(2m-2p-r)!} (-1)^{\frac{1}{2}(p-r)(p-r-1)} \alpha_r \underline{\wedge} \Theta^{2m-2p+2r}(\beta_r).$$

*Then we have:*

(i) *$A^+$ is $Spin(2m)$-invariant;*

(ii) $\overline{A^+(\alpha, \beta)} = A^+(\beta, \alpha)$;

(iii) *$A^+$ is a positive-definite hermitian form in the sense that $A^+(\alpha, \alpha)$ is a strictly positive multiple of the volume element if $\alpha$ is non-zero.*

5.9.1. *Remark.* – Similarly, the complex linear map

$$A^- : (\Lambda^p \otimes S)^- \otimes \overline{(\Lambda^p \otimes S)}^- \to \Lambda^{2m}$$

*defined by minus the formula above has the properties* (i), (ii) *and* (iii) *above.*

*Proof.* – Consider the expression:

$$A(\alpha, \beta) = \sum_{(p-m)^+ \leq r \leq p} \mu(p, r) (\Theta^r(\alpha_r) | \Theta^r(\beta_r))\, \mathbf{v},$$

where $\mathbf{v}$ is the volume form. This obviously has the properties (i), (ii) and (iii) of the theorem. We will show that this is precisely $A^+(\alpha, \beta)$.

$$\begin{aligned} A(\alpha, \beta) &= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r) (\Theta^r(\alpha_r) | \Theta^r(\beta_r))\, \mathbf{v}, \\ &= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, \Theta^r(\alpha) \underline{\wedge} * \Theta^r(\beta_r) \\ &= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, \Theta^r(\alpha) \\ &\quad \underline{\wedge} (-i)^m (-1)^{r(p-r)+\frac{1}{2}r(r+1)+\frac{1}{2}(p-r)(p-r-1)} \\ &\quad \times \frac{r!}{(2m-2p+r)!} \Theta^{2m-2p+r}(\beta_r) \end{aligned}$$

(by Proposition 5.7, since $\beta_r$ is primitive of degree $p-r$)

$$= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, i^m \frac{r!}{(2m-2p+r)!} (-1)^{\frac{1}{2}p(p-1)+r}\, \Theta^r(\alpha_r) \underline{\wedge}\, \Theta^{2m-2p+r}(\beta_r)$$

$$\left(\text{since } r(p-r) + \frac{1}{2} r(r+1) + \frac{1}{2}(p-r)(p-r-1) = \frac{1}{2} p(p-1) + r\right)$$

$$= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, i^m \frac{r!}{(2m-2p+r)!} \times (-1)^{\frac{1}{2}p(p-1)+r+(p-r)r+\frac{1}{2}r(r+1)}\, \alpha_r \underline{\wedge}\, \Theta^{2m-2p+2r}(\beta_r)$$

(by Lemma 5.8.1)

$$= \sum_{(p-m)^+ \leq r \leq p} \mu(p, r)\, i^m \frac{r!}{(2m-2p+r)!} \times (-1)^{\frac{1}{2}(p-r)(p-r-1)}\, \alpha_r \underline{\wedge}\, \Theta^{2m-2p+2r}(\beta_r)$$

since $\frac{1}{2}p(p-1) + r + (p-r)r + \frac{1}{2}r(r+1) = \frac{1}{2}(p-r)(p-r-1) + 2pr + r - r^2$. This proves the theorem. □

There is another way to express these positivity results which is the infinitesimal analogue of the way the Hodge-Riemann bilinear relations are formulated in [4] and which can be summarised in the

5.10. THEOREM. – *Let $p$ be an integer such that $0 \leq p \leq m$. Define*

$$A_p : (\Lambda^p \otimes S) \otimes \overline{(\Lambda^p \otimes S)} \to \Lambda^{2m} \text{ by : } A_p(\alpha, \beta) = i^m (-1)^{\frac{1}{2}p(p-1)}\, \alpha_r \underline{\wedge}\, \Theta^{2m-2p}(\beta)$$

*where $\alpha, \beta \in \Lambda^p \otimes S$. Then the following hold:*

(i) *$A_p$ is $Pin(2m)$-invariant. If we choose an orientation $\varepsilon \in Pin(2n)$* (cf. *Def.* 2.1*),* then *$A(x_+, x_-) = 0$ if $x_+ \in (\Lambda^p \otimes S)^+$ and $x_- \in (\Lambda^p \otimes S)^-$;*

(ii) *$A_{p+1}(\Theta(\alpha), \Theta(\beta)) = -A_p(\alpha, \beta)$ if $0 \leq p+1 \leq m$;*

(iii) *$A_p(\beta, \alpha) = \overline{A_p(\alpha, \beta)}$;*

(iv) *If $\alpha = \Theta^r(\alpha_{p-r})$ and $\beta = \Theta^s(\beta_{p-s})$, where $\alpha_{p-r} \in \Lambda^{p-r} \otimes S$ and $\beta_{p-s} \in \Lambda^{p-s} \otimes S$ are primitive, then $A_p(\alpha, \beta) = 0$ when $r \neq s$.*

(v) *If $x_+ \in P_p^+$ is primitive, then $A_p(x_+, x_+)$ is a positive multiple of the volume element.*
*If $x_- \in P_p^-$ is primitive, then $-A_p(x_-, x_-)$ is a positive multiple of the volume element.*

*Proof.* – It is clear that $A_p$ is Pin$(2m)$-invariant. By definition, if $x_+ \in (\Lambda^p \otimes S)^+$ and $x_- \in (\Lambda^p \otimes S)^-$ then $\varepsilon(x_+) = i^m x_+$ and $\varepsilon(x_-) = -i^m x_-$. Hence

$$\begin{aligned} A_p(x_+, x_-) &= A_p(\varepsilon(x_+), \varepsilon(x_-)) \\ &= A_p(i^m x_+, -i^m x_-) \\ &= -i^m (-i)^m = A_p(x_+, x_-) \\ &= -A_p(x_+, x_-). \end{aligned}$$

To prove (ii), we have:

$$A_{p+1}(\Theta(\alpha),\, \Theta(\beta)) = i^m (-1)^{\frac{1}{2}(p+1)p}\, \Theta(\alpha) \underline{\wedge}\, \Theta^{2m-2p-2}(\Theta(\beta))$$
$$= i^m (-1)^{\frac{1}{2}(p+1)p+p+1}\, \alpha \underline{\wedge}\, \Theta^{2m-2p}\, \beta \qquad \text{(by Lemma 5.8.1)}$$
$$= -A_p(\alpha,\, \beta).$$

To prove (iii), we have:

$$\beta \underline{\wedge}\, \Theta^{2m-2p}(\alpha) = (-1)^{p(2m-p)}\, \overline{\Theta^{2m-2p}(\alpha) \underline{\wedge}\, \beta} \qquad \text{(by Definition 5.8)}$$
$$= (-1)^{p(2m-2p)+p(2m-2p)+\frac{1}{2}(2m-2p)(2m-2p+1)}\, \overline{\alpha \underline{\wedge}\, \Theta^{2m-2p}(\beta)}$$
$$\text{(by Lemma 5.8.1)}$$
$$= (-1)^{p(2m-p)+(m-p)(2m-2p+1)}\, \overline{\alpha \underline{\wedge}\, \Theta^{2m-2p}(\beta)}$$
$$= (-1)^m\, \overline{\alpha \underline{\wedge}\, \Theta^{2m-2p}(\beta)}$$

since $m = p(2m-p) + (m-p)(2m-2p+1)$ modulo 2. The result follows directly from the definition of $A_p$.

To prove (iv), note that it is sufficient to prove that $A_p(\alpha, \beta) = 0$ when $\alpha, \beta \in (\Lambda^p \otimes S)_+$ (or $\alpha,\, \beta \in (\Lambda^p \otimes S)_-$) by part (i) above.

By definition,

$$A_p(\alpha,\, \beta) = i^m (-1)^{\frac{1}{2}p(p-1)}\, \Theta^r(\alpha_{p-r}) \underline{\wedge}\, \Theta^{2m-2p+s}(\beta_{p-s}).$$

By Proposition 5.7, $*\Theta^s(\beta_{p-s})$ is proportional to $\Theta^{2m-2p+s}(\beta_{p-s})$ and hence $A_p(\alpha, \beta)$ is proportional to $\alpha \underline{\wedge} * \beta$, that is to $(\alpha|\beta)\,\mathbf{v}$ (*cf.* Definition 5.8). But $(\alpha|\beta)$ is zero because the canonical decomposition is orthogonal for the hermitian metric $(.|\,.)$. This proves part (iv).

Part (v) is a direct consequence of Proposition 5.7 and the fact that $x_+ \underline{\wedge} * x_+$ is equal to $(x_+|x_+)\,\mathbf{v}$. □

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