ANNALES SCIENTIFIQUES DE L'É.N.S.

GENADI LEVIN

Disconnected Julia set and rotation sets

Annales scientifiques de l'É.N.S. 4e série, tome 29, nº 1 (1996), p. 1-22

<http://www.numdam.org/item?id=ASENS_1996_4_29_1_1_0>

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1996, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (http://www.elsevier.com/locate/ansens) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques

http://www.numdam.org/

Ann. scient. Éc. Norm. Sup.,
4e série, t. 29, 1996, p. 1 à 22.

DISCONNECTED JULIA SET AND ROTATION SETS ([1])

BY GENADI LEVIN

ABSTRACT. – Let Ψ be the conformal isomorphism from the complement of the unit disc to the complement of the Mandelbrot set M. We study the map Ψ at the roots of the hyperbolic components of M. It is shown that the function $(\log|w - w_0|^{-1})^{-1}$ describes Ψ at each periodic point w_0 of the map $w \mapsto w^2$.

1. Introduction

Let $f_c(z) = z^2 + c$ and let $M = \{c : \{f_c^n(0)\}_{n=0}^{\infty}$ *is bounded*$\}$ be the Mandelbrot set (f^n is n-iterated map f). It is a connected compact set in $\mathbb{C}$ with connected complement [9]. The boundary of M is the bifurcation diagram for the simplest nonlinear holomorphic dynamical system $f_c : \mathbb{C} \to \mathbb{C}$. The Mandelbrot set is an example of a fractal and it contains infinitely many copies of itself. It is treated as the universal bifurcational set for analytic one-parameter families [10]. The famous MLC conjecture says M is locally connected, or, equivalently, uniformizing conformal isomorphism $\Psi : \mathbb{D}^* = \{w : |w| > 1\} \to \mathbb{C} \setminus M$ extends to a continuous map on $S^1 = \partial\mathbb{D}^*$. J.-C.Yoccoz proved that ∂M is locally connected at many points (see [15]). In particular, it is locally connected at the points of the boundaries of the hyperbolic components of M (see definitions in [9]-[11] and in Sect. 7).

We will use some principal results of the Douady-Hubbard's theory of the Mandelbrot set [8]-[11] (they will be stated at the corresponding places). Let Δ be a hyperbolic component. According to this theory, the root c_Δ of Δ admits exactly two external arguments of M: t_Δ and t'_Δ, $0 < t_\Delta < t'_\Delta < 1$, which are periodic points of the map $\sigma : t \to 2t(\mathrm{mod}1)$ (except for the main cardioid, where we set $t_\Delta = 0, t'_\Delta = 1$). It means that the Uniformizing Map Ψ has the radial limit c_Δ at the points $w_\Delta = \exp(2\pi i t_\Delta)$, $w'_\Delta = \exp(2\pi i t'_\Delta)$ (we consider the map Ψ constructed in [9] so that $\Psi(w) \sim w$ at infinity). Conversely, for each periodic point t_0 of the map σ, the radial limit of Ψ at $w_0 = \exp(2\pi i t_0)$ exists and is the root of a hyperbolic component. A hyperbolic component is called *primitive* iff its root is not a point in the boundary of another hyperbolic component. Note that the radial limits of the map Ψ at the points $w = \exp(2\pi i t)$, t rational, play a special role in the theory of the Mandelbrot set (*see* e.g. [11]) .

([1]) This is a revised version of the Preprint No.15,1991/1992, Hebrew University of Jerusalem (1992).

Denote: $a(x) \asymp b(x)$ as $x \to x_0$, if $a(x)/b(x)$ and $b(x)/a(x)$ are bounded in a neighbourhood of x_0.

THEOREM A. – *Let* w_0, $|w_0| = 1$, *be a periodic point of the map* $w \mapsto w^2$, i.e. w_0 *is either* w_Δ *or* w'_Δ, *for some hyperbolic component* Δ *with the root* $c_0 = c_\Delta$.

If Δ *is not a primitive one, then*

$$|\Psi(w) - c_0| \asymp (-\log|w - w_0|)^{-1}$$

as $w \to w_0$.

If Δ *is a primitive hyperbolic component, then*

$$|\Psi(w) - c_0| \asymp (-\log|w - w_0|)^{-2}$$

as $w \to w_0$ *and* $\arg w/2\pi \in [t_\Delta, t'_\Delta]$.

We derive it studying the behavior out M of the multiplier $\lambda(c)$ of the repelling periodic point $a(c)$, which was attracting in Δ. It turns out that this behavior is universal in some sense (*see* Theorem 7.4). In the combinatorial part in Sect. 7 we describe rotation sets and rotation numbers of the repelling periodic point $a(c)$, when $c \in \mathbb{C} \setminus M$. This generalizes a well-known fact on the rotation sets and the rotation numbers of the fixed point [9], [14]. A corollary is (*see* Theorem 7.3): *the function* $\lambda(c)$ *extends from* Δ *to a holomorphic function in the wake* $W(\Delta)$ *of the hyperbolic component* Δ *so that* $|\lambda(c)| > 1$ *if* $c \in W(\Delta) \setminus \overline{\Delta}$ (the wake $W(\Delta)$ is the domain in $\mathbb{C}$ containing Δ and bounded by the external rays of M to the root c_Δ).[2]

Main tools for us are *the hedgehog* and *the Yoccoz-type inequality.*

The paper is organized as follows.

In Sects. 2-4 we recall the notion of the hedgehog of equal slope τ ("slanting" hedgehogs) for polynomials. In particular, we describe the change of the slanting hedgehogs in the language of wringing complex structures [6] and stretching rays [4]. The hedgehogs of the standard slope $\tau = \pi/2$ for quadratic polynomials f_c with $c < -2$ appeared in [5], [23]. This construction was studied and described in [20], [18] for the polynomials whose Julia set is not connected. Note that for polynomials the Green function of the basin of infinity agrees with dynamics, that is why the hedgehog of polynomial is, in fact, a realization of a general construction of so-called Green's star region [3], [24]. For polynomial-like mappings *see* [19].

In Sect. 5 an estimate for multiplier is considered. Such kind of estimates were obtained first in [21] and independendly in [26] and [17]. The Yoccoz's inequality in [26], [15], [22] relates the multiplier of a repelling fixed point of a polynomial and its rotation number provided the Julia set of the polynomial is connected. In the present paper we prove a similar (Yoccoz-type) inequality for polynomials whose Julia set is not necessarily connected (Sect. 5). For this purpose we introduce a geometric characteristic of the rotation set in the hedgehog: so called angle of access. To every slope τ there corresponds a

([2]) Another proof of this fact, in the spirit of Douady-Hubbard-Lavaurs' theory [11], [16], can be found now in: D. Schleicher, Internal Addresses in the Mandelbrot set and irreducibility of polynomials. Dissertation, Cornell University, 1994.

disc $D_\tau(a)$ of possible values of log for the multiplier of the fixed point a. The estimate itself does not depend on the slope: we take the intersection of the discs $D_\tau(a)$ over all τ. Later on we use this estimate to pass from combinatorial characteristics of periodic point (rotation sets) to a nearness of a given polynomial to the polynomials with neutral point. We would like to note that one can write down the corresponding inequality for polynomial-like mappings with disconnected Julia set.

Sect. 6 is devoted to a description of mainly known results on the rotation sets with two symbols [25], [7].

In Sect. 7 the hyperbolic components of the Mandelbrot set are considered and main results are proved. We apply here the slanting hedgehogs and the Yoccoz-type inequality. The combinatorial part is resumed in Theorem 7.1 that describes the rotation sets and numbers along boundaries of hyperbolic components. To every periodic point t_* of $\sigma : t \to 2t(\mathrm{mod} 1)$ of period m we associate other periodic point t'_* of σ and the periodic point $a(c)$ of f_c of the same period m such that, when the external argument t_c goes from t_* to t'_*, the rotation number of $a(c)$ changes between 0 and 1 monotonically and the rotation set of $a(c)$ contains exactly two digits in 2^m-expansion of its points. In fact, t_* and t'_* are the external arguments of a root of the hyperbolic component of $a(c)$, and the two-digit property is closely related to the description of small copies of M in M in terms of external arguments [8].

The short Sect. 8 extracts some ideas of proofs.

In conclusion let us note that the main results of Sect. 7 hold for the bifurcation set of the family $z \mapsto z^d + c$, for each integer $d \geq 2$.

2. The Böttcher function and the hedgehogs of polynomials

Fix a polynomial T of degree d. Let $A(\infty) = \{z : T^n(z) \to \infty, n \to \infty\}$ be the basin of infinity. We always assume that the Julia set $J = \partial A(\infty)$ is not connected. Denote by $u(z) = u_T(z)$ Green function of the domain $A(\infty)$ with the pole at infinity. Define $u = 0$ outside $A(\infty)$, then

$$u(T(z)) = d \cdot u(z), \qquad z \in \mathbb{C}.$$

Let $B(z)$ be the Böttcher function of T in a neighbourhood of infinity (*i.e.*, $B \circ T(z) = [B(z)]^2$ there), so that $u(z) = \log |B(z)|$ in the neighbourhood. We choose B such that $B(z) \sim z, z \to \infty$. Denote:

C is the set of all critical points of T in $A(\infty)$,

$u_{\max} = \max\{u(q) : q \in C\}$,

$K(r) = \{z : u(z) \leq r\}$,

$\Gamma(r) = \partial K(r) = \{z : u(z) = r\}$,

$G(r) = \mathbb{C} \setminus K(r)$.

If $r \geq u_{\max}$, the Böttcher function $B(z)$ is well defined in $G(r)$ and gives a conformal map of $G(r)$ onto $\{w : |w| > r\}$. The function $B(z)$ satisfies the functional equation $B(T(z)) = [B(z)]^d$. This equation yields an analytic (infinitely valued) continuation of the function $B(z)$ on the whole domain $A(\infty)$. The continued function has branch points at the points of the set

$$C(\infty) = \bigcup_{m=0}^{\infty} T^{-m} C.$$

We obtain a *sinle-valued* analytic extention of $B(z)$ cutting the domain $A(\infty)$ along certain lines (τ-*cuts*). Fix an angle (slope) $\tau \in (0, \pi)$. Let $z \in A(\infty) \setminus C(\infty)$. There exists a unique maximal (*i.e.*, is not contained inside of other) C^1-curve, τ-*curve* $R(z)$, passing through z such that it meets any level line $\Gamma(r)$ crossing $R(z)$ at the same angle τ. The direction of the τ-curve is chosen so that the Green function $u(z)$ is decreasing along it. The origin of every τ-curve is either ∞ or some point from $C(\infty)$. In the former case the τ-curve is called the τ-*radius*, or *smooth* τ-*ray* (of T), in the latter case it is the τ-*cut*.

Bottcher function B extends along every smooth ray. Denote by A_τ a *set composed by all smooth* τ-*rays*. Then $A_\tau \cup \{\infty\}$ is simply-connected in the Riemann sphere. Its complement K_τ is the filled in Julia set of T completed by the τ-cuts. The extended univalent function B^τ maps the domain A_τ one-to-one onto a *hedgehog-like* domain U_τ. The boundary $S_\tau = \partial U_\tau$ is called the τ-**hedgehog** or **slanting hedgehog**.

It is convenient to straighten the hedgehog with the help of log-coordinates. Consider exterior $\mathbb{D}^*$ of the unit disk and its universal covering $H = \{\zeta : Im\zeta > 0\}$ with a covering anti-conformal projection $p : H \to \mathbb{D}^*$,

$$p : \omega \mapsto \exp(2\pi i \bar{\omega}).$$

For every $\omega \in H$ let L_ω be a straight line through ω which intersects the real line at the angle τ (τ-*straight line*). Let $X_\tau(\omega)$ be the point of the intersection. We define

$$arg_\tau(\omega) = X_\tau(\omega)(\mathrm{mod} 1).$$

The pre-image $H_\tau = p^{-1}(U_\tau)$ of the hedgehog-like domain U_τ is a universal covering of U_τ. Moreover,

$$H_\tau = \overline{H} \setminus Q_\tau,$$

where $Q_\tau = p^{-1}(S_\tau)$ is a one-periodic comb, τ-*comb*. The map

$$\Phi = (B^\tau)^{-1} \circ p : H_\tau \to A_\tau$$

is an analytic unbranched covering.

Let R be a τ-ray. Let us define its *external* τ-*argument*, or just τ-*argument* as follows. The full pre-image $\Phi^{-1}(R) = L + \mathbb{Z}$, where the straight ray L is a τ-straight line or part of such a line. *The* τ-*argument of* R is the arg_τ of the points of L. Conversely, to each $t \in \mathbb{T}$ there corresponds unique τ-radius R of the τ-argument t:

$$R = R_t.$$

Denote by $\Lambda_\tau(q)$ the set of the τ-arguments t of the radii R_t^τ with the end at $q \in C$.

Denote

$$\sigma_d(x) = d.x(mod 1), \varrho_d : \omega \mapsto d.\omega, \tilde{\varrho}_d(x+iy) = \sigma_d(x) + i.d.y, (x,y) \in [0,1) \times \mathbb{R}.$$

PROPOSITION 2.1 (*cf.* [20]).
1. *$T : A_\tau \to A_\tau$, $\varrho_d : H_\tau \to H_\tau$, and*

$$\Phi \circ \varrho_d = f \circ \Phi \,\text{in}\, H_\tau.$$

2. *The comb*

$$Q_\tau = \partial H_\tau = \left\{ [0,1) \bigcup \bigcup_{q \in C} \bigcup_{t \in \Lambda_\tau(q)} \bigcup_{n=0}^{\infty} \bigcup_{\tilde{\varrho}_d^n(x) = t + i.u(q)} N_x \right\} + \mathbb{Z}, \tag{2.1}$$

where $N_x = \{\omega : 0 \le \mathrm{Im}(\omega) \le \mathrm{Im}(x), \arg_\tau \omega = \arg_\tau x\}$.

The segments N_x, with x as in (2.1), will be called the *τ-needles* of the comb Q_τ. The ground of this comb is the real axis, and the ends x of the needles of Q_τ are the points with coordinates

$$\omega_{c,\tau}(n,k) = \frac{t+k}{d^n} + i\frac{1}{2\pi}\frac{u(c)}{d^n},$$

where $c \in C, t \in \Lambda_\tau(c), n \in 0 \bigcup \mathbb{N}$ and $k \in \mathbb{Z}$.

We will call the segments $N_{t+i.u(c)}$, with $c \in C$ and $t \in \Lambda_\tau(c)$ by the *generating segments* of the comb Q_τ. Note that there are finitely many generating segments and all needles are the pre-images of the generating segments under $\tilde{\varrho}_2$ shifted by $\mathbb{Z}$.

Define the *τ-rays* as follows. Note that usual (orthogonal) external rays will correspond to $\tau = \pi/2$. The τ-ray is just a τ-radius R, if R extends up to the Julia set (*i.e.* R does not end at a point of $C(\infty)$). Let the end point of R be a point of $C(\infty)$. Then the full pre-image $\Phi^{-1}(R) = L + \mathbb{Z}$, where the straight ray L lands at the top x of some needle N_x. The map Φ extends to a continuous injective map from either side of N_x to $\mathbb{C}$. It allows us to define the two τ-rays corresponding to the τ-radius R as the images of two sides of the τ-straight line $L \cup N_x = \{\omega \in H : 0 < \mathrm{Im}(\omega) < \infty, \arg_\tau \omega = t\}$, $t = \arg_\tau x$: we obtain the *right* R^+ and the *left* R^- *limit ray.*

Every τ-ray R has *external τ-argument*: this is the arg_τ of the points of $\Phi^{-1}(R)$. In particular, the right R^+ and the left R^- limit rays corresponding to the τ-radius R have the same external τ-argument which is the τ-argument of R.

DEFINITION 2.1. – *A τ-ray R lands at a point z of the Julia set $J(T)$ of T, if z is the unique limit point of R in $J(T)$. The τ-argument of R is called a τ-argument of z. Let z be a point of the Julia set $J(T)$. The set of all τ-arguments of the τ-rays landing at z will be denoted by $\Lambda_\tau(z)$.*

PROPOSITION 2.2. – *Let a be a point, such that the single-point set $\{a\}$ is a component of the Julia set $J(T)$. Then the set $\Lambda_\tau(a)$ is a non-empty compact of $\mathbb{T}$.*

Proof is easy and can be found in [14] or in [19].

Let now a be additionally a repelling fixed point of the map T. We have proved that the set $\Lambda_\tau(a)$ of the τ-external arguments of a is a non-empty closed, proper subset of $\mathbb{T}$. It is invariant under the action of $\sigma_d : \mathbb{T} \to \mathbb{T}$.

The set $\Lambda_\tau(a)$ is a *rotation set of the map* σ_d *with a rotation number* $\nu = \nu_\tau$.

Recall that *a compact* $\Lambda \subset \mathbb{T}$ *is said to be the rotation set of the map* σ_d *if* $\sigma_d(\Lambda) \subseteq \Lambda$ *and the restriction* $\sigma_d|_\Lambda$ *can be extended to a map of* $\mathbb{T}$ *to* $\mathbb{T}$ *which lifts to a non-decreasing continuous map* $F : \mathbb{R} \to \mathbb{R}$ *such that* $F - id$ *is 1-periodic.*

Under these conditions there exists

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}, \qquad x \in \mathbb{R},$$

and its fractional part ν depends only on (σ_d, Λ). The number $\nu \in [0, 1)$ is called the *rotation number of the set* Λ.

PROPOSITION 2.3. – *Given slope* τ, *the set* $\Lambda_\tau(a)$ *is a rotation set of* σ_d *with the rotation number* $\nu = \nu_\tau(a) \in \mathbb{T}$. *The number* ν *is rational (* $\nu = p/q$ *in reduced form) if and only if the set* $\Lambda_\tau(a)$ *is finite and consists of cycles of* σ_d *of the period* q.

Proof. – Choose a component K_0 of some $K(r)$, such that $a \in K_0$ and a branch $T^{-1} : K_0 \to K_0$, $T^{-1}(a) = a$, is well-defined. Denote by A_0, A_1 the sets of the external arguments of τ-rays, which cross the boundaries of K_0 and $T^{-1}(K_0)$ respectively. Then: (a) each A_i, $i = 0, 1$, is a union of a finite number of closed intervals $\{I^i\}$ in $\mathbb{T}$, (b) $A_1 \subset A_0$, $\sigma_d(A_1) = A_0$ and, moreover, for every different $I^1_1, I^1_2 \in \{I^1\}$ their images under σ can intersect each other only at endpoints, (c) the restriction $\sigma_0 = \sigma_d|_{A_1}$ preserves the cyclic order of the points in $\mathbb{T}$. The (a)-(c) yield that the restriction σ_0 extends to a degree one continuous map $\bar{\sigma}$ of the circle, which can be lifted to a non-decreasing map $F : \mathbb{R} \to \mathbb{R}$, such that $F - id$ is 1-periodic. Besides, $\bar{\sigma}$ is expanding on A_1 and $\Lambda_\tau(a) = \{t \in \mathbb{T} : \sigma_0^n(t) \in A_1, n \in \mathbb{N}\}$. From here it follows that the rotation number ν of the map $\bar{\sigma}$ (in the usual sense, *see* e.g. [1]) is well-defined. ν is rational if and only if $\bar{\sigma}^q$ has a fixed point (in this case $\bar{\sigma}^{qn}(t), t \in \Lambda_\tau(a)$, tends to one of the fixed points of σ_d^m as $n \to \infty$; but they are repelling, hence, such t is itself fixed by σ_d^m).

Remark 2.1. – Let T be an arbitrary nonlinear polynomial of degree d and let R be a τ-ray of T. If its τ-argument is periodic under σ_d, then the ray R lands at a point of $J(T)$ and this point is either repelling or neutral rational periodic point of T (Sullivan, Douady and Hubbard). If $J(T)$ is connected, then, vice versa, each such a point z has a finite, but a non-empty, set $\Lambda_\tau(z)$ of its external arguments, and the rational rotation number is well-defined (it does not depend on the slope τ, if $J(T)$ connected): Douady and Yoccoz (*see*, for example, [22]). If $J(T)$ is not connected, again $\Lambda_\tau(z)$ is a non-empty compact in $\mathbb{T}$, for each repelling periodic point z (*see* [12] and [19]), but it can be infinite (*see* [14]) and the τ-rotation number can depend on τ (Sect. 7, Theorem 7.1).

3. A property of the hedgehog

Let us fix a slope τ and consider the comb Q_τ. Let $X = \{x_1, ..., x_n\}$ be a cycle of the map $\sigma_d : [0,1) \to [0,1)$, and $\tilde{X} = X + \mathbb{Z}$. Define two values (of angles) $\gamma^{(r)}(X) \in (0, \pi - \tau)$, $\gamma^{(\ell)}(X) \in (0, \tau)$ as follows. Look at a generating segment N_x of Q_τ with the end $x \in H$. Find the pont $x^{(r)}$ of the set $\tilde{X}$ closest to N_x from the right. Let us consider the triangle $\Delta^{(r)}$ with a vertex $x^{(r)}$ and the opposite side N_x. Then the angle $\gamma^{(r)}(X, N_x)$ is said to be an angle of $\Delta^{(r)}$ at vertex x. The angle $\gamma^{(\ell)}(X, N_x)$ is defined analogous but using the point $x^{(\ell)}$ of $\tilde{X}$ closest to N_x from the left, and the corresponding triangle $\Delta^{(\ell)}$.

Remark 3.1. – The value $\gamma(X, N_x) = \gamma^{(\ell)}(X, N_x) + \gamma^{(r)}(X, N_x)$ is the angle of *vision* of the interval $(x^{(\ell)}, x^{(r)})$ of $\mathbb{R}$ from the point $x \notin \mathbb{R}$.

We set

$$\gamma_\tau^{(r)} = \min_N \gamma^{(r)}(X, N), \qquad \gamma_\tau^{(\ell)} = \min_N \gamma^{(\ell)}(X, N),$$

where the minima are taken over the all generating segments N of the comb Q_τ.

PROPOSITION 3.1 (*cf.* [20], [18]). – *For every $\theta \in X$, the angles*

$$W_\tau^{(r)}(\theta) = \theta + \{\omega \in H : \pi - \tau - \gamma^{(r)}(X) < arg\omega < \pi - \tau\},$$
$$W_\tau^{(\ell)}(\theta) = \theta + \{\omega \in H : \tau - \gamma^{(\ell)}(X) < arg\omega < \tau\}$$

belong to H_τ, and they are the maximal open angles at the vertex $\theta \in X$ with this property (here $arg\omega$ is the standart, i.e. *$\pi/2$-argument of $\omega \in \mathbb{C} \setminus \{0\}$).*

If θ is not a base of any needle, then the angle

$$W_\tau(\theta) = W_\tau^{(r)}(\theta) \bigcup W_\tau^{(\ell)}(\theta) \bigcup \{\omega \in H : arg_\tau \omega = \theta\}$$

of the value $\gamma(X) = \gamma^{(\ell)}(X) + \gamma^{(r)}(X)$ also belongs to H_τ.

Proof. – Use that the map σ_d acts in H_τ.

4. The hedgehogs and the wringing complex structures

We describe here the slanting hedgehogs from a point of view of the wringing complex structures [6]. We will use this in Sect. 8.

Let T be a monic polynomial and its Julia set be not connected. Fix a slope $\tau \in (0, \pi)$, a coefficient (of stretching) $s > 0$ and an invariant under T measurable bounded complex structure μ on $\mathbb{C} \setminus A(\infty)$ (if the area of $\mathbb{C} \setminus A(\infty)$ is zero, then μ vanishs). Set $t = (1-s)/\mathrm{tg}\tau$ and $\xi = s + it$. Following [6], define the left multiplication by ξ in the right halfplane $\{x + iy | x > 0, \ y \in \mathbb{R}\}$:

$$\tilde{\Omega}_\xi(x + iy) = sx + i(tx + y).$$

This bijection projects to a diffeomorphism $\Omega_\xi : \mathbb{D}^* \to \mathbb{D}^*$.

Furthermore, Ω_ξ commutes with $P_0 : z \mapsto z^d$ and, moreover, preserves the set of curves in $\mathbb{D}^*$, which intersect each circle $|z| = r > 1$ at the angle τ. A complex structure $\mu_\xi = \Omega_\xi^*(\mu_0)$ in $\mathbb{D}^*$ is called the wringing one (μ_0 is the standard complex structure). The dilatation ratio for μ_ξ is equal to $(\xi - 1)/(\xi + 1)$ at every point. Define a new complex structure $\tilde{\mu} = \mu_{\xi,T}$ to be equal to μ on $\mathbb{C} \setminus A(\infty)$, $\tilde{\mu} = B^*(\mu_\xi)$ in a neighbourhood of infinity, and $(T^n)^*(\tilde{\mu})$, $n = 1, 2, \dots$, in the rest of $\mathbb{C}$ (almost everywhere). Then $\mu_{\xi,T}$ is bounded on the Riemann sphere $\bar{\mathbb{C}}$ and T-invariant. According to the Measurable Riemann Mapping Theorem [2], there exists a quasi-conformal homeomorphism $H = H_{\xi,\mu}$ of $\mathbb{C}$, $H(\infty) = \infty$, such that $H_*(\mu_{\xi,T}) = \mu_0$. Then

$$P = T_{\xi,\mu} = T_{\tau,s,\mu} = H \circ T \circ H^{-1}$$

is a polynomial and

$$B_P = \Omega_\xi \circ B \circ H^{-1}$$

is analytic and conjugates P and P_0 in a neighbourhood of infinity. One can choose H uniquely in such a way that P is monic and $B_P(z)/z \to 1$ as $z \to \infty$ [6]. The polynomial $P = T_{\tau,s,\mu}$ is conjugate to T by a quasi-conformal homeomorphism H in $\mathbb{C}$, which has a constant complex dilatation in $A(\infty)$ and such that H transfers the τ-rays for T to the τ-rays for P. In particular, $H(J_\tau^T) = J_\tau^P$.

Remark 4.1. – In [4] the *Stretching Ray* through T is defined as the set of polynomials $\{T_{\pi/2,s,\mu_0}\}_{s>0}$. We can define τ-stretching ray through T as $\{T_{\tau,s,\mu}\}_{s>0}$. Note that for $T(z) = z^2 + c$ the stretching ray are the usual external rays of the Mandelbrot set M and the τ-stretching rays are the curves in $\mathbb{C} \setminus M$ that crosses the equipotential curves of M at the angle τ.

5. The basic inequality

We consider a repelling fixed point a of the polynomial T. First, let the Julia set of T be connected. Then the Yoccoz inequality [26] describes a relation between the multiplier of a fixed point of T and the rotation number of this point:

$$\left|\log\lambda - \left(2\pi i\frac{p}{q} + \frac{\log d}{q}\right)\right| \le \frac{\log d}{q},$$

where λ is the multiplier of a and the rational number $\nu = \frac{p}{q}$ (in reduced form) is the rotation number of the fixed point a. It describes the order of permutations of nonequivalent paths to a from the basin of infinity $A(\infty)$[26], [22].

We obtain the inequality in the case when J is not connected and the set $\{a\}$ is a component of J. We know that the rotation number $\nu_\tau(a)$ and the rotation set $\Lambda_\tau(a)$ of the point a are well defined for every slope $\tau \in (0, \pi)$.

Let us fix the slope τ and consider the corresponding comb Q_τ.

Define a value called by the **angle of access to the point** a. Let $\theta \in \Lambda_\tau(a)$. It means the following: *there exists a τ-ray R_θ that lands at the point a; if R_θ contains a cut then it is either the right R_θ^+ or the left R_θ^- limit ray.*

Assume that the rotation number of a is rational: $\nu_\tau(a) = p/q$. Then θ is a point of a cycle $\bar\theta$ of σ_d. We will use notations from Proposition 3.1.

DEFINITION 5.1. – *The angle of access to the cycle $\bar\theta$ is: the value $\gamma(\bar\theta)$, if R_θ does not contain a τ-cut; otherwise this is either $\gamma^{(r)}(\bar\theta)$ or $\gamma^{(\ell)}(\bar\theta)$ depending on whether the right or the left ray lands at a. The angle of access to the fixed point a is the sum $\phi_\tau(a)$ of the angles of access to all different cycles of σ_d in $\Lambda_\tau(a)$.*

Remark 5.1. – If the rotation number irrational, we may define the value $\phi(\bar\theta)$ in the same way, but one can prove that it will be equal to zero.

The main result of this section is the following

THEOREM 5.1. – *Let $\nu_\tau(a) = \frac{p}{q}$ be rational (in reduced form). Then, for a branch of* $\log\lambda$ *of the multiplier $\lambda = T'(a)$,*

$$\log\lambda \in \left\{z : \left|z - \left(2\pi i\frac{p}{q} + \frac{\pi\log d}{q\phi_\tau(a)}\right)\right| < \frac{\pi\log d}{q\phi_\tau(a)}\right\} := D_\tau(a).$$

and, hence,

$$\log\lambda \in \bigcap_\tau D_\tau(a)$$

over all slopes τ, such that the rotation number $\nu_\tau(a)$ is rational.

Proof of the theorem (*cf.* [26], [22], [17], [15]). – Let $\theta \in \Lambda_\tau(a)$ and W be the sector of the angle $\gamma(\bar\theta)$ (*i.e.* W is $W_\tau^{(r)}(\theta)$, $W_\tau^{(\ell)}(\theta)$, or $W_\tau(\theta)$: *see* Definition 5.1). The W is invariant under σ_d^q. Define two families of curves $\tilde E$ and E. $\tilde E = \{\tilde e_\alpha\}$ is said to be the family of all intervals in the sector W, such that the interval $\tilde e_\alpha$ joins a point V, $\mathrm{Im}V = h$, with the point $\theta + (V-\theta)d^{-q}$. Here $h > 0$ is small and fixed and α is the angle between $\tilde e_\alpha$ and $\mathbb{R}$. The $\tilde E$ is projected by $B^{-1}\circ p$ to a family of curves near the point a, and after that to the family E of curves on a torus S, which is given by dynamics of T near a. Every two curves $e_1, e_2 \in \Gamma$ are disjoint, because the level h is small and the point θ is periodic of period q. Moreover, the curves $e \in \Gamma$ are closed. The torus S is conformally equivalent to $\mathbb{C}/\Pi$, $\Pi = \log\lambda\cdot\mathbb{Z}\times 2\pi i\cdot\mathbb{Z}$. Every e is lifted to a curve γ in $\mathbb{C}$, which joins a point z with $z + q\log\lambda - p2\pi i$, with some choice of $\log\lambda$. This is because the cycle $\bar\theta$ has the rotation number p/q, *i.e.* exactly p curves among $\{\gamma + k\log\lambda\}_{k=0}^{q-1}$ in $\mathbb{C}/2\pi i\mathbb{Z}$ are disposed between γ and $\gamma + \log\lambda$ (including γ). The listed geometric properties of $\tilde E$ and E lead to the following estimates. First, introduce the metric ρ on the torus S, which is induced by the Euclidean one using representation $S \cong \mathbb{C}/\Pi$. In its turn, the metric ρ (or the corresponding metric in a punctured neighbourhood of the point a) induces a metric $\tilde\rho$ in the sector W with the help of the map $p^{-1}\circ B$. Let now $M = AL^{-2}$, where A is the area of the set of the points $z \in e$, $e \in E$, and L is infimum of lengths of the

$e \in E$ with respect to the metric ρ. The number $\tilde{M} = \tilde{A}\tilde{L}^{-2}$ is defined similarly, but for the family $\tilde{E}$ and the metric $\tilde{\rho}$. Then

$$\frac{\phi(\bar{\theta})}{q \log d} \leq \tilde{M} = M \leq \frac{A}{|q \log \lambda - 2\pi i p|^2}.$$

Summing up these inequalities over the all orbits $\bar{\theta}$ in $\Lambda_\tau(a)$, we obtain:

$$\frac{\phi_\tau(a)}{q \log d} \leq \frac{2\pi \log |\lambda|}{|q \log \lambda - 2\pi i p|^2},$$

where the equality is attained if and only if the metric $\tilde{\rho}$ is logarithmic one (i.e. $\tilde{\rho}(w) = |dw/|w - \theta||$), what is impossible, if the Julia set is not an analytic arc.

6. Rotation sets with two symbols

Fix an integer $d \geq 2$. We say that a rotation set Λ of $\sigma_d : t \to d.t (\text{mod} 1)$ is with *two symbols* $\alpha, \beta \in \{0, 1, ..., d-1\}$, $\alpha < \beta$, if one can write every point $\theta \in \Lambda$ as

$$\theta = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{d^i} := 0.\varepsilon_1 \varepsilon_2 ... \varepsilon_i ...,$$

with $\varepsilon_i \in \{\alpha, \beta\}$. The following information can be found in [25], [7]. Given $\nu \in [0, 1]$, define a set $\Lambda_d^{\alpha,\beta}(\nu)$ invariant under the map $\sigma_d : \mathbb{T} \to \mathbb{T}$, and points $\theta^{\pm}_{\alpha,\beta,d}(\nu)$,$\theta_{\alpha,\beta,d}(\nu)$ as follows. For every $\nu \in \mathbb{T}$ there exists the unique minimal closed rotation set $\Lambda_d^{\alpha,\beta}(\nu)$ of the map $\sigma_d : \mathbb{T} \to \mathbb{T}$ with two symbols α and β and with the rotation number ν (sometimes we will omit d).

If $\nu = 0$ or $\nu = 1$, then $\Lambda^{\alpha,\beta}$ is $\{\frac{\alpha}{d-1}\}$ or $\{\frac{\beta}{d-1}\}$ respectively.

Let $\nu \in (0, 1)$ and let $\theta^{(\ell)}_{\alpha,\beta}(\nu)$ and $\theta^{(r)}_{\alpha,\beta}(\nu)$ be the extreme left and the extreme right points of $\Lambda^{\alpha,\beta}(\nu)$,

$$\frac{\alpha}{d-1} < \theta^{(\ell)}_{\alpha,\beta}(\nu) < \theta^{(r)}_{\alpha,\beta}(\nu) < \frac{\beta}{d-1}.$$

If ν irrational then

$$\sigma_d(\theta^{(\ell)}_{\alpha,\beta}(\nu)) = \sigma_d(\theta^{(r)}_{\alpha,\beta}(\nu)) := \theta_{\alpha,\beta,d}(\nu).$$

If $\nu = \frac{p}{q}$ rational then

$$\theta^{-}_{\alpha,\beta,d}(\nu) := \sigma_d(\theta^{(r)}_{\alpha,\beta}(\nu)) < \sigma_d(\theta^{(\ell)}_{\alpha,\beta}(\nu)) := \theta^{+}_{\alpha,\beta,d}(\nu),$$

and $\theta^{\pm}_{\alpha,\beta,d}(\nu)$ is the pair of adjacent and the nearest points of the cycle $\Lambda_d^{\alpha,\beta}(\nu)$ of σ_d of the period q.

Remark 6.1. – It is convenient to consider, that for ν irrational

$$\theta^-_{\alpha,\beta,d}(\nu) = \theta^+_{\alpha,\beta,d}(\nu) = \theta_{\alpha,\beta,d}(\nu).$$

On the other hand, for every $\theta \in (\frac{\alpha}{d-1}, \frac{\beta}{d-1})$, there exists the unique $\nu \in \mathbb{T}$, such that either $\theta = \theta_{\alpha,\beta,d}(\nu)$, with ν irrational, or $\theta \in [\theta^-_{\alpha,\beta,d}(\nu), \theta^+_{\alpha,\beta,d}(\nu)]$, with ν rational. Moreover, $\nu \in [0,1]$ is a nondecreasing function of θ (it is a devil's staircase).

For every two adjoint rationals p/q and P/Q, $Qp - Pq = 1$, we have from [7]:

$$(6.1)\qquad \begin{aligned} \theta^-_{\alpha,\beta,d}\left(\frac{p}{q}\right) - \theta^+_{\alpha,\beta,d}\left(\frac{P}{Q}\right) &= \frac{(\beta-\alpha)(d-1)}{(d^q-1)(d^Q-1)},\\ \theta^+_{\alpha,\beta,d}\left(\frac{p}{q}\right) - \theta^-_{\alpha,\beta,d}\left(\frac{p}{q}\right) &= \frac{(\beta-\alpha)(d-1)}{d^q-1}. \end{aligned}$$

Remark 6.2. – The numbers $\theta^{\pm}_{\alpha,\beta,d}(\nu)$,$\theta_{\alpha,\beta,d}(\nu)$ are constructed by an explicit algorithm [7]. For instance, for each $N = 2,3,...,$

$$\theta^-_{\alpha,\beta,d}\left(\frac{1}{N}\right) = 0.(\alpha\alpha...\alpha\beta), \qquad \theta^+_{\alpha,\beta,d}\left(\frac{1}{N}\right) = 0.(\alpha...\alpha\beta\alpha)$$

(α repeated $N-1$ times in the first equality and $N-2$ times in the second one).

Here and below

$$0.(\epsilon) = 0.\epsilon\epsilon....$$

Let

$$\begin{aligned} I^\alpha &= (\frac{\alpha}{d-1}, \frac{\alpha+1}{d}) = (0.(\alpha), 0.\alpha(\beta)),\\ I^\beta &= (\frac{\beta}{d}, \frac{\beta}{d-1}) = (0.\beta(\alpha), 0.(\beta)),\\ I^{\alpha\beta} &= (\frac{\alpha}{d-1}, \frac{\beta}{d-1}) = (0.(\alpha), 0.(\beta)). \end{aligned}$$

Remark 6.3. – Observe that $\sigma_d : I^\alpha \to I^{\alpha\beta}$ and $\sigma_d : I^\beta \to I^{\alpha\beta}$ is one-to-one.

The rotation set $\Lambda^{\alpha,\beta}_d(\nu)$ for $\nu \in (0,1)$ is decomposed to two non-empty parts $\Lambda^\alpha(\nu)$ and $\Lambda^\beta(\nu)$:

$$\Lambda^\varepsilon(\nu) = \Lambda^{\alpha,\beta}_d(\nu) \bigcap I^\varepsilon, \quad \varepsilon \in \{\alpha,\beta\}.$$

Denote by $I^\nu_\alpha = [\theta^{(\ell)}_{\alpha,\beta}(\nu), \theta^\alpha(\nu)]$ and $I^\nu_\beta = [\theta^\beta(\nu), \theta^{(r)}_{\alpha,\beta}(\nu)]$ two minimal closed intervals containing Λ^ν_α and Λ^ν_β. It is easy to understand that $\sigma_d(\theta^\alpha(\nu)) = \theta^{(r)}_{\alpha,\beta}(\nu)$ and $\sigma_d(\theta^\beta(\nu)) = \theta^{(\ell)}_{\alpha,\beta}(\nu)$.

In the next section we will use the following. For every $t \in I^{\alpha\beta}$, there exist unique points $t^{(\ell)} \in I^\alpha$ and $t^{(r)} \in I^\beta$, such that

$$\sigma_d(t^{(\ell)}) = \sigma_d(t^{(r)}) = t.$$

Then we can find unique points $t^\alpha \in I^\alpha$ and $t^\beta \in I^\beta$, such that

$$\sigma_d(t^\alpha) = t^{(r)} \quad \text{and} \quad \sigma_d(t^\beta) = t^{(\ell)}.$$

$t^{(\ell)}$, $t^{(r)}$, t^α, t^β are increasing functions of t, and $t^{(\ell)} < t^\alpha < t^\beta < t^{(r)}$. Set

$$L_{\alpha,\beta}(t) = (t^{(\ell)}, t^\alpha) \cup (t^\beta, t^{(r)}).$$

We have:

$$\theta^{(\ell)}_{\alpha,\beta}(\nu) = (\theta^+_{\alpha,\beta,d}(\nu))^{(\ell)}, \quad \theta^{(r)}_{\alpha,\beta}(\nu) = (\theta^-_{\alpha,\beta,d}(\nu))^{(r)}, \theta^\alpha(\nu)$$
$$= (\theta^-_{\alpha,\beta,d}(\nu))^\alpha, \quad \theta^\beta(\nu) = (\theta^+_{\alpha,\beta}(\nu))^\beta.$$

Now, if $t \in [\theta^-_{\alpha,\beta,d}(\nu), \theta^+_{\alpha,\beta,d}(\nu)]$, then $\Lambda^{\alpha,\beta}_d(\nu) \subset \overline{L_{\alpha,\beta}(t)}$. Moreover, $I^\nu_\alpha \subseteq [t^{(\ell)}, t^\alpha]$, $I^\nu_\beta \subseteq [t^\beta, t^{(r)}]$, and we have the equalities iff $t = \theta^-_{\alpha,\beta,d}(\nu) = \theta^+_{\alpha,\beta,d}(\nu)$, i.e. ν is irrational.

7. The Mandelbrot set

We consider the polynomial family

$$f_c(z) = z^2 + c$$

and the Mandelbrot set M in the space of the parameter $c \in \mathbb{C}$. Fix for a moment $c \in \mathbb{C} \backslash M$.

We will keep the notation

$$\sigma = \sigma_2 : t \to 2t(\text{mod} 1).$$

Let u_c and B_c be Green function and Böttcher function in the basin of infinity of f_c. Parameters h_c and t_c are defined by

$$B_c(c) = e^{2\pi(h_c + it_c)}, \quad h_c = u_c(c) > 0, \ t_c \in [0, 1).$$

According to Douady-Hubbard theorem [9], the correspondence

$$c \mapsto B_c(c)$$

is one-to-one conformal map (Riemann map) of the complement $\mathbb{C} \backslash M$ onto the complement $\mathbb{D}^*$ of the unit disc. We denote by $\Psi : \mathbb{D}^* \to \mathbb{C} \setminus M$ the inverse map to $c \to B_c(c)$. The *external ray of* M at the argument $t \in \mathbb{T}$ is the curve $R(M, t) = \Psi(\{\rho \exp(2\pi i t) : \rho > 1\})$. If $R(M, t)$ lands at a unique point $c = \lim_{\rho \to 1} \Psi(\rho \exp(2\pi i t))$, the angle t is called an *external argument* of $c \in \partial M$.

The *hyperbolic component* Δ is a component of *int*M such that f_c has an attracting cycle of a period m for $c \in \Delta$. Let Δ be a hyperbolic component and $\lambda(c)$ be the multiplier of the cycle attracting for $c \in \Delta$. Then it is the Riemann map $\Delta \to \{|\lambda| < 1\}$ and extends to a homeomorphism $\partial\Delta \to \{|\lambda| = 1\}$ (theorem of Douady-Hubbard-Sullivan). The unique point $c_\Delta \in \partial\Delta$ such that $\lambda(c_\Delta) = 1$ is called *the root* of Δ. The point $c_\Delta \in \partial M$ has exactly two external arguments t_Δ and t'_Δ, which are periodic points of the map $\sigma : t \mapsto 2.t(mod 1)$ of $\mathbb{T}$ [11] (if $t_\Delta = 0$, we set $t'_\Delta = 1$). Conversely, Ψ has a radial limit along every periodic (under $w \mapsto w^2$) radius, and the limit point is a root of some hyperbolic component [11]. Hyperbolic component Δ is called *primitive* iff its root is not a point in the boundary of another hyperbolic component. The conjugated rationals t_Δ and t'_Δ are described in [16].

In this section we give a different description of the conjugated pairs, with some additional properties of them (*see* Theorem 7.1).

As we know, the external rays are defined also in the dynamical plane of the map $f_c : z \mapsto z^2 + c,\ c \in \mathbb{C}$.

From now on, up to Theorem 7.1, we shall consider only usual, i.e. $\frac{\pi}{2}$*-radii and rays and their external angles. In particular,* t_c *is the* $(\frac{\pi}{2}-)$ *external angle of the critical value* c *of* f_c.

Let us fix a periodic point t_* of the map σ:

$$t_* = \frac{\alpha}{2^m - 1},$$

where m is the minimal period of t_* and $\alpha \in \{0, 1, \dots, 2^m - 2\}$. We define now other digit

$$\beta \in \{0, 1, \dots, 2^m - 1\}, \quad \beta \neq \alpha,$$

as follows.

First, $\alpha = 0$ and $\beta = 1$, if $m = 1$.

Let $m > 1$. Fix such $c_* \in \mathbb{C} \setminus M$ that $t_{c_*} = t_*$. Consider the external radius R_{t_*} in the dynamic plane $z \mapsto f_{c_*}(z)$. It ends at a point z_0 such that $f^m_{c_*}(z_0) = c_*$. The smooth curve R_{t_*} extends by other external radius R_{t_1} up to a smooth curve L_0 going from infinity to infinity so that the Green function u_{c_*} decreases from $+\infty$ to $u_{c_*}(z_0)$ as z goes along R_{t_*} to z_0, and then again increases to $+\infty$ along R_{t_1}. We have:

$$\sigma^m(t_1) = t_*,$$

because $f^m_{c_*}(R_{t_*}) = f^m_{c_*}(R_{t_1})$ and is the part of R_{t_*} joining c_* with infinity.

The digit β is defined from the equality

$$t_1 - t_* = \frac{\beta - \alpha}{2^m}.$$

We will use the notations:

$$0.\varepsilon_1\varepsilon_2\ldots = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{(2^m)^k}; \qquad (\varepsilon) = \varepsilon\varepsilon\ldots.$$

Then equivalent definition for β is: $t_1 = 0.\beta(\alpha)$.

In the sequel we assume that $\beta > \alpha$. If $\beta < \alpha$, the proofs hold with obvious changes of notations.

Definition 7.1. – *Set* $t'_* = 0.(\beta) = \beta/(2^m - 1)$ *and let* $I^{\alpha\beta} = (t_*, t'_*)$ *be the open interval in* $(0,1)$ *with the ends* t_* *and* t'_*. *Denote* $I^\alpha = (0.(\alpha), 0.\alpha(\beta))$ *and* $I^\beta = (0.\beta(\alpha), 0.(\beta))$. *According to the previous notations* $t_* = 0.(\alpha)$, $t'_* = 0.(\beta)$, $t_1 = 0.\beta(\alpha)$. *Denote also* $t'_1 = 0.\alpha(\beta)$.

Remark 7.1. – The map $\sigma^m : I^\alpha \to I^{\alpha\beta}$ and $\sigma^m : I^\beta \to I^{\alpha\beta}$ is one-to-one.

Lemma 7.1. – *The points*

$$t_*, \sigma(t_*), \sigma^2(t_*), \ldots, \sigma^{m-1}(t_*) \tag{7.1}$$

lie outside the interval $I^{\alpha\beta}$.

The points

$$t_1, \sigma(t_1), \sigma^2(t_1), ..., \sigma^{m-1}(t_1) \tag{7.2}$$

lie outside the interval (t_*, t_1).

Proof. – The period $m > 1$, because otherwise there is nothing to prove. It is enough to prove (7.1) for smaller interval $I = (t_*, t_1)$, because all points (7.1) are fixed by σ^m and t'_* is the closest to t_1 fixed point of σ^m out I. Each curve

$$L_0 = R_{t_*} \bigcup \{z_0\} \bigcup R_{t_1}, \quad L_1 = f_{c_*}(L_0), ..., \quad L_{m-1} = f_{c_*}(L_0)$$

is smooth, does not intersect others and splits the complex plane into two unbounded parts. Note that $0 = f_{c_*}^{m-1}(z_0) \in L_{m-1}$ and $f_{c_*}(L_{m-1}) \subset L_0$. In particular, all iterations of zero belong to the curves L_k. For $i = 0, 1, ..., m-2$, we denote by Ω_i the domain bounded by L_i and does not containing 0. We want to show that

$$L_k \subset \mathbb{C} \setminus \Omega_0, \qquad k = 1, ..., m-1. \tag{7.3}$$

There exists $i \in \{0, 1, ..., m-2\}$ such that all curves L_k lie out Ω_i. If $i = 0$, we are done. If $i \neq 0$, one can choose a univalent branch

$$F = f_{c_*}^{-i} : \Omega_i \to \mathbb{C}$$

such that $F(f_{c_*}^i(z_0)) = z_0$. Then $F(\Omega_i) = \Omega_0$ since otherwise Ω_i contains an iteration of 0. By the same reason, $L_k \bigcap \Omega_0 = \emptyset$, $k = 1, ..., m-1$, that is (7.3) is proved. In particular, $f_{c_*}^k(z_0)$, $k = 0, ..., m-1$, are in the complement to Ω_0. In the language of external arguments it means that $\sigma^k(t_*)$ $\sigma^k(t_1)$ don't lie in (t_*, t_1), for $k = 0, 1, ..., m-1$. The proof is finished.

Definition 7.2. – *Define two periodic points* $a(c)$ *and* $b(c)$ *of* f_c *as follows. Let* R_*^+ *and* R_*^- *be external rays of* f_{c_*} *which are the limit of rays* R_t *as t tends to* t_* *within the interval* $I^{\alpha\beta}$ *and outside this interval respectively. The points* a_* *and* b_* *are said to be the landing points of the periodic rays* R_*^+ *and* R_*^- *respectively. Set*

$$E = \{c \in \mathbb{C} \setminus M : 0 < h_c < \infty, 0 < t_c < 1\}.$$

Define $a(c)$ *and* $b(c)$ *as holomorphic functions in* E *such that* $a(c_*) = a_*$ *and* $b(c_*) = b_*$, *and* $a(c)$,$b(c)$ *are periodic points of* f_c.

Remark 7.2. – The minimal period of a_* (and, hence, $a(c)$) is m, because $a_* \in \Omega_0$ and the radii $f_{c_*}^k(R_{t_*})$, $1 \leq k \leq m-1$, don't intersect Ω_0 (see the proof of Lemma 7.1).

The following statement will allow us to estimate the multiplier of $a(c)$.

THEOREM 7.1. – *I. If* $m > 1$, *then the intervals* $\sigma^i(I^\alpha)$ *and* $\sigma^i(I^\beta)$, $i = 1, \cdots, m-1$, *dont intersect* $I^{\alpha\beta}$.

II. Let $t_c \in (0,1)$, $h_c > 0$ *correspond to* $c \in \mathbb{C} \setminus M$. *Let* $\tau \in (0, \pi)$ *be chosen so that*

$$t_c^\tau := \arg_\tau(t_c + ih_c) \in \overline{I^{\alpha\beta}}$$

and $\nu \in [0,1]$ *is uniquely defined by the condition:*

$$t_c^\tau = \theta_{\alpha,\beta,2^m}(\nu) \text{ or } t_c^\tau \in [\theta^-_{\alpha,\beta,2^m}(\nu), \theta^+_{\alpha,\beta,2^m}(\nu)].$$

Then the τ*-rotation number of* $a(c)$,

$$\nu_\tau(a(c)) = \nu,$$

and the τ*-rotation set of* $a(c)$,

$$\Lambda_\tau(a(c)) \supseteq \Lambda_{2^m}^{\alpha,\beta}(\nu).$$

Proof. – I. There are two steps in the proof of p.I.

Step 1. – Let us consider a point x and an integer $1 \leq i \leq m$ such that $\sigma^i(x) = t_*$. Assume that $x \in I^\beta$. It follows from Remark 7.1 together with $\sigma^m(t_1) = t_*$ that $i < m$. Then, again from Remark 7.1,

$$\sigma^i(t_1) < \sigma^i(x) = t_* < \sigma^i(t'_*),$$

therefore

$$t_* = \sigma^m(t_1) < \sigma^{m-i}(t_*) < \sigma^m(t'_*) = t'_*,$$

what contradicts to Lemma 7.1. *We have proved that* x *lies outside the interval* I^β. The similar considerations show that x *lies outside the interval* I^α.

Step 2. – Now assume that the statement is false. By Remark 7.1 and Step 1, it is possible, if, for some $1 \leq i \leq m-1$, either

$$\sigma^i(I^\alpha) \ni t'_*,$$

or

$$\sigma^i(I^\beta) \ni t'_*.$$

The first inclusion implies that $\sigma^i(t_*) \in I^{\alpha\beta}$: contradiction with Lemma 7.1. The second one leads to the inclusion $y = \sigma^i(t_1) \in I^{\alpha\beta}$. Taking into account that y is not in (t_*, t_1) by Lemma 7.1, the only case remains $y \in I^\beta$. But $\sigma^{m-i}(y) = t_*$. It contradicts to Step 1.

II. Denote $c_0 = c$ and $\tau_0 = \tau$. Join c_* and c_0 by an open arc $\ell \subset E$ and find a continuous function $\tau(c) : \overline{\ell} = \ell \bigcup \{c_*\} \bigcup \{c_0\} \to (0, \pi)$, such that $\tau(c_*) = \pi/2, \tau(c_0) = \tau_0$ and $t_c^{\tau(c)} \in I^{\alpha\beta}$, if $c \in \ell$. It follows from the proven p.I that, for each $t \in I^{\alpha\beta}$ the set $L_{\alpha,\beta}(t)$ (*see* Sect. 6) does not contain any point θ such that $\sigma^k(\theta) = t_c$, $k = 1, ..., m-1$. By virtue of this, for every $c \in \ell$ and for corresponding $\tau = \tau(c)$, the open intervals

$$I_c^\alpha = \left\{ \omega = t + ih : h = \frac{h_c}{2^m}, (t_c^\tau)^{(\ell)} < t < (t_c^\tau)^\alpha \right\},$$

$$I_c^\beta = \left\{ \omega = t + ih : h = \frac{h_c}{2^m}, (t_c^\tau)^\beta < t < (t_c^\tau)^{(r)} \right\}$$

belong to comb-domain H_τ of f_c. Then we can apply the map $\Phi_c = (B_c^\tau)^{-1} \circ p : H_\tau \to (A_c)_\tau$, and obtain that the following subsets of $\Gamma(\frac{h_c}{2^m}) = \{z : u_c(z) = h_c/2^m\}$:

$$\Gamma_1 = \Phi_c(I_c^\alpha), \qquad \Gamma_2 = \Phi_c(I_c^\beta)$$

are curves and change continuously as $c \in \ell$. Let $K_1(c)$ be a component of $intK(h_c/2^m) = \{z : u_c(z) < h_c/2^m\}$ which contains $a(c)$. For $c = c_*$, Γ_1 and Γ_2 are arcs of the boundary of the component $K_1(c_*)$. Therefore, for every $c \in \ell$, Γ_1 and Γ_2 are the arcs of the boundary of the component $K_1(c)$. We have proved that every τ_0-ray R_θ of f_{c_0} with $\theta \in L_{\alpha,\beta}(t_{c_0})$ crosses $K_1(c_0)$. Take ν from the condition of the theorem. Then all τ_0-rays R_θ, $\theta \in \Lambda_{2^m}^{\alpha,\beta}(\nu)$ land inside the $K_1(c_0)$. Hence, a landing point z of such a ray never leaves $K(c_0)$ under the iterates $f_{c_0}^{mi}$, $i \in \mathbb{N}$. Thus $z = a(c_0)$. The proof of p.II is completed.

COROLLARY 7.1. – *The $\pi/2$-ray $R_{t'_*}$ of the map f_{c_*} lands at the point $b_* = b(c_*)$.*

Proof. – We consider only $\pi/2$-rays. Let $R_{t_1}^+$ be the right limit ray to the radius R_{t_1}. The rays $R_{t_1}^+$ and $R_{t_*}^-$ land in the same component $K_2(c_*)$ of the set $intK(h_c/2^m)$ since the radii R_{t_*} and R_{t_1} end at the same point z_0. But the rays $R_{t_1}^+$ and $R_{t'_*}$ also land in same component of $intK(h_c/2^m)$. This is because of p.I, Theorem 7.1: the interval I^β does not cover any point θ such that $\sigma^k(\theta) = t_*$. Thus, the rays $R_{t_*}^-$ and $R_{t'_*}$ land inside $K_2(c_*)$, and, hence, at the same point b_*, because they both are fixed by $f_{c_*}^m$ and this map is injective on $K_2(c_*)$.

THEOREM 7.2. – *Let $t_c \in (0,1)$. For a branch $\log\lambda(c)$ of the multiplier $\lambda(c) = (f_c^m)'(a(c))$,*

$$\log\lambda(c) \in \bigcap_{\substack{0<\frac{p}{q}<1 \\ \text{in reduced} \\ \text{form}}} \left\{ z : \left| z - \left(2\pi i \frac{p}{q} + \frac{\pi \log 2^m}{q\phi_{p/q}(V_c)} \right) \right| < \frac{\pi \log 2^m}{q\phi_{p/q}(V_c)} \right\},$$

where $\phi_{p/q}(V_c)$ is the angle of vision of the interval $[\theta^-_{\alpha,\beta,2^m}(\frac{p}{q}), \theta^+_{\alpha,\beta,2^m}(\frac{p}{q})]$ from the point $V_c = t_c + ih_c$.

Proof. – We need to use the slanting hedgehogs. Recall $arg_\tau V$ denotes the τ-argument of V in the upper half-plane H. Fix $p/q \in (0,1)$ and choose τ_c such that $arg_{\tau_c} V_c \in [\theta^-_{\alpha,\beta,2^m}(p/q), \theta^+_{\alpha,\beta,2^m}(p/q)]$, where $V_c = t_c + ih_c$, $t_c \in [t_*, t'_*]$. By Theorem 7.1 and

Proposition 3.1, the angle of access $\phi_{\tau_c}(a(c))$ to the point $a(c)$ is not less than the angle $\phi_{p/q}(V_c)$. The application of Theorem 5.1 completes the proof.

The Theorem 7.2 is constructive. Some applications will be given now.

DEFINITION 7.3. – *For a rational number $\nu = p/q \in (0,1)$, let*

$$[\nu] = [a_1, a_2, \ldots, a_k], \qquad a_i \in \mathbb{N}$$

be a continued fraction expansion of ν (there are two of them: with $a_k = 1$ and with $a_k > 1$). Let

$$[\nu](N) = [a_1, a_2, \ldots, a_k, N] = \frac{P(N)}{Q(N)}, \qquad N = 1, 2, \ldots.$$

(Here, for example, $Q(N) = Nq + q_{k-1}$, where q_{k-1} is denominator of $[a_1, \ldots, a_{k-1}]$). Define

$$r([\nu], N) = \frac{(\beta - \alpha)(2^m - 1)2^{mq}}{(2^{mQ(N)} - 1)(2^{mq} - 1)}, \qquad 0 < \nu < 1, \quad N = 1, 2, \ldots,$$

and

$$r([0], N) = r([1], N) = \frac{(\beta - \alpha)2^m}{(2^{mN} - 1)}, \qquad N = 2, 3, \ldots$$

First of all, we want to deduce using Theorem 7.2, that t_* and t'_* are the external arguments of the same hyperbolic component Δ, and $|\lambda(c)| = 1$ on $\partial\Delta$.

We know from Douady-Hubbard's theory that the ray $R(M, t_c)$, for $t_c = \theta^-_{\alpha,\beta,2^m}(p/q)$ or $t_c = \theta^+_{\alpha,\beta,2^m}(p/q)$, lands at a point on M (like every ray with rational argument).

LEMMA 7.2. – *If $t_c = \theta^{\pm}_{\alpha,\beta,2^m}(p/q)$ and $h_c \to 0$, then $\lambda(c) \to \exp(2\pi i p/q)$.*

Proof. – Let, for example, $t_c = \theta^-_{\alpha,\beta,2^m}(\nu)$, with $\nu = p/q \neq 0$. Let us consider the continued fraction expansion of ν,

$$[\nu] = [a_1, a_2, \ldots, a_k],$$

such that $k \geq 1$ is odd. We have: $[\nu](N) < \nu$, $[\nu](N) \to \nu$ as $N \to \infty$. It follows from (6.1) that

$$t_c - \theta^-_{\alpha,\beta,2^m}([\nu](N)) = \frac{(\beta - \alpha)(2^m - 1)2^{mq}}{(2^{mQ(N)} - 1)(2^{mq} - 1)} = r([\nu], N)$$

and

$$\theta^+_{\alpha,\beta,2^m}([\nu](N)) - \theta^-_{\alpha,\beta,2^m}([\nu](N)) = \frac{(\beta - \alpha)(2^m - 1)}{(2^{mQ(N)} - 1)} := r'([\nu], N).$$

In particular,

$$\frac{r'([\nu](N))}{r([\nu](N))} = \frac{2^{mq} - 1}{2^{mq}} \in (0.5, 1).$$

Consider the angle of vision $\phi_{[\nu](N)}(V_c)$ (*see* Theorem 7.2). Then, for each $V_c = \theta^-_{\alpha,\beta,2^m}(p/q) + ih_c$, if

$$r([\nu], N+1) \leq h_c \leq r([\nu], N), \tag{7.4}$$

this angle is not less than an absolute constant $\Phi_* \in (0, \pi/2)$. Denote

$$R(N) = \frac{\log(2^m)}{\Phi_*.Q(N)}.$$

By Theorem 7.2, $\log \lambda(c)$ lies in the disc of radius $R(N)$ that tangents the imaginary axis at the point $2\pi i[\nu](N)$. For each $h_c \leq r([\nu], 2)$ let us choose N such that (7.4) holds. If $h_c \to 0$, then $Q(N) \to \infty$, and the disc tends to the point $2\pi i\nu$. The end of the proof.

Now let Δ' be any hyperbolic component of the period m, and $\lambda'(c)$ be its multiplier so that $|\lambda'(c)| = 1$ on $\partial\Delta'$. The root $c_{\Delta'}$ is a landing point of two external rays of M. Denote their arguments

$$t_{\Delta'} = \frac{\alpha'}{2^m - 1} \quad \text{and} \quad t'_{\Delta'} = \frac{\beta'}{2^m - 1},$$

for some "digits" $\alpha', \beta' \in \{0, 1, 2, \ldots, 2^m - 1\}$, $\alpha' < \beta'$.

Let $\lambda'(c) = \exp(2\pi i\nu)$, for some $c \in \partial\Delta'$. According to Douady-Hubbard description of small copies of M [8], *the point $c \in \partial M$ has the following external arguments: if ν is irrational, the external argument is $\theta_{\alpha',\beta',2^m}(\nu)$, and if ν is rational, the external arguments are $\theta^{\pm}_{\alpha',\beta',2^m}(\nu)$.*

DEFINITION 7.4. – *The wake $W(\Delta')$ is the domain containing Δ' and bounded by the rays $R(M, t_{\Delta'}), R(M, t'_{\Delta'})$. Given rational $\nu \in (0,1)$, the ν-wake $W_\nu(\Delta')$ is a closed domain, which does not contain Δ' and is bounded by the rays*
$R(M, \theta^-_{\alpha',\beta',2^m}(\nu)), R(M, \theta^+_{\alpha',\beta',2^m}(\nu))$.

THEOREM 7.3. – *I. The numbers t_*, t'_* are the external arguments of a hyperbolic component Δ of M,* i.e. *the rays $R(M, t_*), R(M, t'_*)$ land at the root c_Δ of Δ.*

II. The functions $a(c)$ and $\lambda(c)$ extend to analytic functions in the wake $W(\Delta)$. Moreover, $|\lambda(c)| > 1$ for $c \in W_\Delta \setminus \overline{\Delta}$

III. Given $c \in W(\Delta)\setminus\overline{\Delta}$, the periodic point $a(c)$ has the rational rotation number ν if and only if $c \in W_\nu(\Delta)$, and irrational rotation number ν if and only if $c \in R(M, \theta_{\alpha',\beta',2^m}(\nu))$.

Proof. – I. It follows immediately from the previous lemma, that every ray $R(M, \theta^{\pm}_{\alpha,\beta,2^m}(1/N))$, $N = 2, 3, \ldots$, lands at a point $c(N)$ of a hyperbolic component $\Delta(N)$ of the *period* m. Besides, the 2^m-expansion of the external argument of $c(N)$ contains exactly the digits α, β. From the Douady-Hubbard description of the external arguments we conclude, that the external arguments of the root of Δ' are t_* and t'_*, that is all $\Delta(N)$ coincide with the hyperbolic component Δ such that $|\lambda(c)| = 1$ on $\partial\Delta$ (another proof of this fact uses the theorem of Douady-Hubbard-Sullivan that each hyperbolic component has only one root).

II. If the function $a(c)$ is not extended to $W(\Delta)$, it has a singularity (algebraic ramification point) in the root c' of a hyperbolic component Δ' different from Δ, but with the same period

m. Then an external argument t' of c' is rational. Hence, $t' \in [\theta^-_{\alpha,\beta,2^m}(\nu), \theta^+_{\alpha,\beta,2^m}(\nu)]$, ν rational (in fact ν should be $1/2$). Then, by Theorem 7.2, for $c \in R(M,t')$, the value of $\log\lambda(c)$ lies in a disc of a radius $\rho(c)$, which is tangent to the imaginary axis at the point $2\pi i\nu$. If $c \to c'$ along the ray $R(M,t')$, the radius of the disc $\rho(c)$ tends to a finite value. Therefore, $\log\lambda(c)$ cannot tend to 1. This is a contradiction.

The rest of p.II and p.III are easily follow now.

The theorem below describes the behavior of the multiplier $\lambda(c)$ as a function of the uniformizing variable close to the rational points $(\lambda(c))^n = 1$, $n = 1, 2, \cdots$. Set

$$K^\pm_\nu(w) = |\lambda(c) - \exp(2\pi i\nu)|.(-\log|w - w_\nu|),$$

where $c = \Psi(w)$, ν rational ($0 = \frac{0}{1}$, $1 = \frac{1}{1}$) and $w_\nu = \exp(2\pi i\theta^\pm_{\alpha,\beta,2^m}(\nu))$.

THEOREM 7.4. – *There exist absolute constants K_1, K_2, $0 < K_1 < K_2 < \infty$, such that, for every $\nu = p/q \in [0,1]$,*

$$\frac{K_1}{q^2} < \liminf K^\pm_\nu(w) \le \limsup K^\pm_\nu(w) < K_2(\log d^m)^2, \tag{7.5}$$

as $w \to w_\nu$ and $t_c \in I^{\alpha\beta}$.

Remark 7.3. – Recall that $w = \exp(2\pi(h_c + it_c))$ so the condition $t_c \in I^{\alpha\beta}$ is satisfied automatically if only $\nu \neq 0$ and $\nu \neq 1$.

Proof. – Let, for example, $w_\nu = \exp(2\pi i t_0)$, $t_0 = \theta^-_{\alpha,\beta,2^m}(\nu))$. First, we estimate $|\log\lambda(c) - 2\pi i\nu|$ in a left "quarter"-neighborhood $E^{(l)} = \{(t,h) : t \le t_0, h > 0, |h + i(t_0 - t)| \le \delta\}$ of t_0 (if only $\nu \neq 0$) using Theorem 7.2 like in Lemma 7.2. To do this, consider the rotation sets $\Lambda^{\alpha,\beta}_{2^m}([\nu](N))$ of σ^m, where

$$[\nu] = \frac{p}{q} = [a_1, a_2, \ldots, a_k]$$

is the unique continued fraction expantion of $\nu = p/q$, such that $k \ge 1$ is odd. We have: $[\nu](N) < \nu$, $[\nu](N) \to \nu$ as $N \to \infty$. Let us divide the set $E^{(l)}$ into the "quarter" annuli (7.6).

$$E_N = \{(t,h) : h > 0, t \le t_0, r([\nu], N+1) \le |h + i(t_0 - t)| \le r([\nu], N)\}, N = 1, 2, ...,$$

For every point $V = t + ih \in \partial E_N$, except for the part $I = [r([\nu], N+1), r([\nu], N)] \times \{h = 0\} \subset \partial E_N$ on the axis $h = 0$, the angle $\phi_{\nu_N}(V)$ or the angle $\phi_{\nu_{N+1}}(V)$ (*see* Theorem 7.2) is not less than some absolute constant $\Phi_* \in (0, \pi/2)$. Denote

$$R_N = \frac{\log(2^m)}{\Phi_*.Q(N)}.$$

We want to prove that:
for $V_c = t_c + ih_c \in E_N$

$$|R_{N+1} + 2\pi i(\nu - \nu_{N+1})| - R_{N+1} \le |\log\lambda(c) - 2\pi i\nu| \le |R_N + 2\pi i(\nu - \nu_N)| + R_N. \tag{7.7}$$

Taking into account that the function $\log\lambda(c)$, $c = \Psi(w)$, is holomorphic and non-vanished for $w \in E_N$, it is enough to show the following: that for $V_c \to \partial E_N$ the $\log\lambda(c)$ lies in the convex hull of two discs D_N, D_{N+1}, where $D_i = \{z : |z-(R_i+2\pi i\nu_i)| \le R_i\}$. If $V_c \in \partial E_N$, except the interval $I \subset \partial E_N$, this follows from Theorem 7.2. Now, for every V_c close enough to the part I of the boundary, one can find a rational $\nu' \in [[\nu](N), [\nu](N+1)]$ such that the angle $\phi_{\nu'}(V_c)$ is larger than Φ_*. Then, by Theorem 7.2, $\log\lambda(c)$ belongs to the above convex hull.

Elementary, but rather long calculations show that (7.6)-(7.7) lead to (7.5) in the left "quarter"-neighborhood $E^{(l)}$. On the other hand, $\theta^-_{\alpha,\beta}(\nu)$ is a periodic point of σ of period mq, and we can define another periodic point $a^\nu(c)$ of f_c of the period mq using Definition 7.2. Then we estimate its multiplier $|\log\lambda^\nu(c)|$, as above, in a right "quarter"-neighborhood of t_0 and obtain (7.5) for this multiplier with $\nu = 0$, $q = 1$ and mq instead of m. To pass from $\lambda^\nu(c)$ to $\lambda(c)$, we observe that, by Lemma 7.2, $\lambda(c) \to \exp(2\pi i\nu)$ and $\lambda^\nu(c) \to 1$ as $h_c \to 0$ and the condition $t_c = \theta^-_{\alpha,\beta,2^m}(\nu)$ holds. That is the external ray $t_c = \theta^-_{\alpha,\beta,2^m}(\nu)$ of M lands at a limit point c_0 of ∂M and the periodic point $a(c)$ of the period m collides with the periodic point $a^\nu(c)$ of the period mq at $c = c_0$. Now we use a relation, due to J. Guckenheimer [13]: $\frac{d}{dc}\lambda^\nu(c) = q^2\frac{d}{dc}\lambda(c)$ at $c = c_0$, and obtain (7.5). If $\nu = 0$ or $\nu = 1$, only a semi-neighbourhood is under the consideration.

Theorem A announced in Introduction is a simple corollary of the previous Theorem 7.4.

Proof of Theorem A. – Let $\lambda(c)$ be the multiplier corresponding to Δ. It is known [11, part II] that, for $c \to c_0$, $c - c_0 \sim const(\lambda(c) - \lambda(c_0))$, if Δ is not a primitive one, and $\sim const(\lambda(c) - \lambda(c_0))^2$ otherwise. The statement follows from Theorem 7.4.

8. Conclusion

Let T be a polynomial of degree d, and its Julia set J be not connected. Let a be its fixed point such that $\{a\}$ is a component of J. A reason of the phenomenon when the multiplier of a goes to a point of the unit circle as the polynomial changes is as follows.

Consider the set of polynomials $\{T_{x+iy}\}_{x>0,y\in\mathbb{R}}$ (*see* Sect. 4). Let a_{x+iy} be the corresponting to a fixed point of T_{x+iy} so that $T_1 = T$ and $a_1 = a$. Let now $x \to 0$ and $y \to 0$. In general, there is no reason for the fixed point a_{x+iy} turns into a neutral one. In this paper we considered a case when this occures. Namely, suppose that one of the following combinatorial conditions hold:

(CC') for every slope τ from a semi-neighborhood of $\pi/2$ the rotation set $\Lambda_\tau(a)$ for the fixed point a of T contains a rotation set $\Lambda^{\alpha,\beta}(\nu)$ of the map $\sigma : t \to d.t(\mathrm{mod} 1)$ such that ν does varies as $\tau \to \pi/2$,

or

(CC") for every y from a semi-neighborhood of zero the rotation set $\Lambda_{\pi/2}(a_{1+iy})$ for the fixed point a_{1+iy} of T_{1+iy} contains a rotation set $\Lambda^{\alpha,\beta}(\nu)$ of σ such that ν varies as $y \to 0$.

Then the multiplier $\lambda = T'(a_{x+iy})$ tends to $\exp(2\pi i\nu_{\pi/2}(a))$ as $x \to 0$ and $y \to 0$ (in the semi-neighborhood of zero).

Sketch of the proof. – There exists a critical value c of T as follows. Let x and y be close to zero. Then one can choose $\nu = p/q$ with q large such that the angle of vision of the interval $[\theta^-_{\alpha,\beta,d}(\nu), \theta^+_{\alpha,\beta,d}(\nu)]$ from the point $B_{x+iy}(c_{x+iy})$ is not small. This angle is the angle of access to the fixed point a_{x+iy} in an appropriate slanting hedgehog. Theorem 5.1 completes the proof.

In the present paper we proved the conditions (CC') or (and) (CC") for the fixed point of f_c^m corresponding to a hyperbolic component of the Mandelbrot set M.

REFERENCES

[1] V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, Springer-Verlag, 1983.

[2] L. Ahlfors and L. Bers, *The Riemann mapping theorem for variable metrics* (*Annals of Mathematics*, Vol. 72, 1960, pp. 385-404).

[3] M. G. Arsove and G. Jr. Johnson, *A conformal mapping technique for infinitely connected regions* (*Memoirs of the AMS*, Vol. 91, 1970).

[4] B. Branner, *Cubic polynomials: turning around the connectedness locus*, The Technical University of Denmark, Mat-report 1992-05.

[5] M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, *Geometrical and electrical properties of some Julia sets* (*Lecture Notes in Pure and Applied Mathematics*, Vol. 92, 1988).

[6] B. Branner and J. H. Hubbard, *The iteration of cubic polynomials, Part I: The global topology of parameter space* (*Acta Mathematica*, Vol. 160, 1988, pp. 143-206).

[7] S. Bullett and P. Sentenac, *Ordered orbits of the shift, square roots, and the devil's staircase* (*Preprint, Université Paris-Sud Mathématiques*, 91-39, 1991).

[8] A. Douady, *Algorithms for computing angles in the Mandelbrot set* (*Caotic Dynamics and Fractals*, ed. Barnsley and Demko, Acad. Press, 1986, pp. 155-168).

[9] A. Douady and J. H. Hubbard, *Iteration of complex quadratic polynomials* (*C.R.A.S.*, Vol. 294, 1982, pp. 123-126).

[10] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like mappings* (*Ann. École Norm. Sup.*, (4), Vol. 18, 1985, pp. 287-343).

[11] A. Douady and J. H. Hubbard, *Étude dynamique des polynomes complexes I, II* (*Publication Mathématiques d'Orsay*, no. 84-02, 1984, no. 85-04, 1985).

[12] A. Eremenko and G. Levin, *On periodic points of polynomials* (*Ukrainskii Math. Journal*, v.41, n.11, 1989).

[13] J. Guckenheimer, preprint of Cornell University.

[14] L. R. Goldberg and J. Milnor, *Fixed Points of Polynomial Maps II. Fixed Point Portraits* (*Ann. Scient. École Norm. Sup.*, (4), Vol. 26, 1993, pp. 51-98).

[15] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, preprint IHES, October 1992.

[16] P. Lavaurs, *Une description combinatoire de l'involution définie par M sur les rationnels à dénominateur impair* (*C.R.A.S.*, Vol. 303, 1986, pp. 143-146).

[17] G. Levin, *On Pommerenke's inequality for the eigenvalues of fixed points* (*Colloquium Mathematicum*, Vol. LXII, fasc. 1, 1991, pp. 168-177).

[18] G. Levin, *On the complement of the Mandelbrot set* (*Israel Journal of Mathematics*, Vol. 88, 1994, pp. 189-212).

[19] G. Levin and F. Przytycki, *External rays to periodic points* (*Israel Journal of Mathematics*, to appear).

[20] G. Levin and M. Sodin, *Polynomials with disconnected Julia sets and Green maps* (*The Hebrew University of Jerusalem*, preprint no. 23, 1990/91).

[21] Ch. Pommerenke, *On conformal mapping and iteration of rational functions* (*Complex Variables*, Vol. 5(2-4), 1986, pp. 117-126).

[22] C. L. Petersen, *On the Pommerenke-Levin-Yoccoz inequality* (*Ergodic Theory and Dynamical Systems*, Vol. 13, 1993, pp. 785-806).

[23] M. Sodin and P. Yuditski, *The limit-periodic finite difference operator on $\ell^2(\mathbb{Z})$ associated with iterations of quadratic polynomials* (*J. Stat. Phys.*, Vol. 60, 1990, pp. 863-873).
[24] L. Sario and M. Nakai, *Classification theory of Riemann Surfaces*, Springer-Verlag, 1970.
[25] P. Veerman, *Symbolic dynamics and rotation numbers* (*Physica 13A*, 1986, pp. 543-576).
[26] J.-C. Yoccoz, *Sur la taille des membres de l'ensemble de Mandelbrot*, manuscript (1987).

(Manuscript received June 4, 1993;
revised June 9, 1995;
accepted June 30, 1995.)

G. Levin
Institute of Mathematics,
The Hebrew University of Jerusalem,
Givat Ram 91904 Jerusalem, Israel.