ANNALES SCIENTIFIQUES DE L'É.N.S.

E. ESTEVES

Wronski algebra systems on families of singular curves

Annales scientifiques de l'É.N.S. 4e série, tome 29, nº 1 (1996), p. 107-134
<http://www.numdam.org/item?id=ASENS_1996_4_29_1_107_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 29, 1996, p. 107 à 134.

WRONSKI ALGEBRA SYSTEMS ON FAMILIES OF SINGULAR CURVES

By E. ESTEVES (1)

Abstract. – We replace the sheaves of principal parts on a family of reduced, local complete intersection curves by natural sheaves of algebras that are locally free. Our motivation is to be able to associate to any linear system on the family a Wronski system, as defined by Laksov and Thorup. By applying their general theory of Wronski systems, we obtain in particular a Weierstrass divisor on the family, in case there are no degenerate components on a general fibre.

1. Introduction

Linear systems on smooth curves in characteristic 0 have been extensively studied classically, with strong results on the projective geometry of smooth curves being discovered by the Italian school of Castelnuovo and others. A good part of their results involved the analysis of the ramification points of a linear system, sometimes called Weierstrass points, especially if the linear system is the canonical system.

In char. $p > 0$, the study of Weierstrass points began with F. K. Schmidt ([17] and [18]), who nevertheless considered only the canonical system. The study of Weierstrass points of a general linear system began only relatively recently, basically initiated by Matzat [15] and Laksov ([7] and [8]). Much work has been done since in trying to understand the peculiarities of the positive characteristic case.

Generalizing the theory in another direction, in 1984 Widland defined Weierstrass points for the canonical system on any Gorenstein, irreducible curve [20]. His definition was later extended to any linear system by Lax [12]. Around 1986 Eisenbud and Harris considered the question of Weierstrass points on a curve of compact type [2]. Very recently Garcia and Lax [4] and Laksov and Thorup [10] extended Widland's and Lax's definition to the positive characteristic case.

One of the main goals in extending the notion of Weierstrass points to the singular case is to improve the understanding of smooth curves. Analysing smooth curves by analysing

(1) Supported by CNPq, Proc. 202151/90-5, and a Starr fellowship from the MIT Japan Program.

their degenerations to singular curves has always been a very fruitful idea, as shown for instance by the recent work of Eisenbud and Harris (*see* a summary and references in [2]). Hence the need of not only a theory of Weierstrass points on a singular curve, but more generally a theory of Weierstrass points on families of curves.

The theory of Weierstrass points on families of smooth curves in characteristic 0 has been developed classically. Recently, Laksov and Thorup developed a framework for understanding Weierstrass points on families of smooth curves in arbitrary characteristic ([9] and [10]). More precisely, they show how to canonically associate a system of maps between locally free sheaves, called a *Wronski system,* to a linear system on the family. A Wronski system gives rise to a *Wronskian determinant,* whose zero locus gives a subscheme structure to the set of Weierstrass points of the linear system. To obtain locally free sheaves in the Wronski system, it is necessary that the sheaves of principal parts be locally free themselves. The smoothness of the family is thus essential, as it implies the latter property.

The purpose of this article is to overcome the "defficiency" in the case of singular curves of the sheaves of principal parts in being locally free. On a singular, Gorenstein curve the natural invertible sheaf replacing the sheaf of Kähler differentials Ω^1 is the dualizing sheaf ω. Therefore, a natural idea is to use the canonical map $\eta^1 : \Omega^1 \to \omega$ on a family X/S of Gorenstein curves to replace the sheaves of principal parts P^i by sheaves of P^i-algebras Q^i that are locally free. More precisely, we wish to obtain sheaves of algebras Q^i for $i = 0, 1, \ldots$ fitting into commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \Omega^i \arrow[r] \arrow[d, "\eta^i"'] & P^i \arrow[r] \arrow[d, "\psi^i"'] & P^{i-1} \arrow[r] \arrow[d, "\psi^{i-1}"'] & 0 \\
0 \arrow[r] & \omega^{\otimes i} \arrow[r, "\alpha^i"] & Q^i \arrow[r, "q^i"] & Q^{i-1} \arrow[r] & 0
\end{tikzcd}$$

for $i = 1, 2, \ldots$, where the first row is canonical, both rows are exact, the right hand side square is a diagram of algebra homomorphisms, and η^i is induced from η^1 in a canonical way (*see* Section 2). The data given by the sequence of sheaves Q^i for $i = 0, 1, \ldots$ and the maps in the above diagram is called a *Wronski algebra system* on X/S.

The method developed in the present article produces a canonical and natural Wronski algebra system for a general family X/S of reduced, local complete intersection curves, not necessarily irreducible or complete, in arbitrary characteristic. There are actually several Wronski algebra systems on a single curve (*see* Section 4), but if we require the formation of the sheaves Q^i to be natural, that is, to commute under base change in a fairly large class of families X/S, then we get a unique, natural Wronski algebra system.

By replacing the sheaves of principal parts P^i by the sheaves Q^i we can readily apply Laksov's and Thorup's method (in [9] or [10]) to associate to each linear system on X/S a canonical Wronski system of modules on X (Section 7).

There are a few novelties introduced in this article. First, our set-up includes reducible curves in any characteristic, as long as they are local complete intersections. In particular, we are able to consider families of Deligne-Mumford stable curves. However, we are faced with the same problem Eisenbud and Harris pointed out in [2], p. 339; namely, some components of the curve might be degenerate with respect to the linear system in consideration without the whole curve being degenerate. In our set-up, every point in a degenerate component is a Weierstrass point. By contrast, there are at most a finite number of Weierstrass points on an irreducible curve. Excluding the case of degenerate components, the theory developed here adequately defines Weierstrass points and weights on a family of reducible curves, as explained in Section 7.

Second, we are able to consider families of singular curves in any characteristic, instead of a single curve as in the previous literature (*see* however [13] in characteristic 0). In [10] Laksov and Thorup independently introduce substitutes for the sheaves of principal parts on an integral, Gorenstein curve. Nevertheless, it is not clear whether their method would extend to families, since they made use of the normalization map of the curve via Rosenlicht's local characterization of the dualizing sheaf on the curve [19]. It is not clear either that their substitutes coincide with ours in the case of an integral, local complete intersection curve.

Third, a Wronski system of modules gives a priori more information than the Wronskian determinant obtained from it. For instance, we are able to give a structure of determinantal subscheme to a subset of a family of curves defined by a condition of Weierstrass type. More precisely, given a Wronski system of modules $(\mathcal{W}, Q^i, q^i, v^i, i \geq 0)$ on a scheme Y, the k-th degeneracy locus of the map of vector bundles v^i gives a subscheme structure for the locus of points $y \in Y$ whose k-th order is greater than i. Hence, it is desirable to obtain a Wronski system of modules for each linear system, as done in the present article, instead of just a Weierstrass divisor.

There are important questions still open. First, is it possible to construct a Wronski algebra system on a general family of reduced, Gorenstein curves? If so, is it possible to construct it in a natural way? Is the Wronski algebra system unique in some sense?

Second, how can we explain limits of Weierstrass points when an irreducible curve approaches a reducible curve with degenerate components? Eisenbud and Harris have developed the technique of limit linear series when the reducible curve is of compact type [2]. Is there a way to handle the problem at least for stable curves? If so, valuable information could be obtained about the moduli of smooth curves, since it has a compactification by stable curves.

We now give a brief summary of the contents of this article. In Section 2 we define the notion of a Wronski algebra system, and state the main (and only) theorem of the article, Theorem 2.6. We also start an induction argument for the proof of the theorem, which will be completed in the next four sections. In Section 3 we give a local description of a Wronski algebra system, and prove a local criterion for its existence (Criterion 3.8.) The criterion applies to any family of reduced, Gorenstein curves. In Section 4 we restrict our attention to families of reduced, local complete intersection curves. We prove the existence and uniqueness of a Wronski algebra system on a "general" family, that is, a family of curves satisfying what we call the depth condition. In Section 5 we introduce the necessary tools to induce locally on any family a Wronski algebra system from a larger "general" family, and then to patch the induced local systems together. In Section 6 we use the existence and uniqueness of a Wronski algebra system on a "general" family, proved in Section 4, and the tools developed in Section 5 to wrap up the proof of Theorem 2.6. In Section 7 we show how the theory developed by Laksov and Thorup in [9] and [10] can be applied almost verbatim, once one has good substitutes for the sheaves of principal parts. Since we also allow for reducible curves, we make the necessary minor modifications to their set-up.

All schemes considered will be assumed noetherian, and all morphisms separated and of finite type.

This work is part of the author's Ph.D. thesis at M.I.T. [3]. The author would like to thank Professor Kleiman for his endless support during the research and preparation of this work; and Professors Eisenbud and Mattuck for several interesting comments on this work. The author would also like to thank Waseda University for its hospitality during the period this work was completed.

2. The Wronski algebra system

Let $f : X \to S$ be a flat morphism whose geometric fibres are reduced, Gorenstein curves. We will often refer to f as the family X/S. Let $\Omega^1_{X/S}$ denote the sheaf of relative Kähler differentials of f, and $P^n_{X/S}$ the sheaf of relative n-th order principal parts for each $n \geq 0$. If I denotes the ideal sheaf of the diagonal $X \to X \times_S X$, then

$$\Omega^1_{X/S} = \frac{I}{I^2} \qquad \text{and} \qquad P^n_{X/S} = \frac{\mathcal{O}_{X \times_S X}}{I^{n+1}}.$$

Denote by $\Omega^n_{X/S}$ the $\mathcal{O}_X$-module I^n/I^{n+1} for every $n \geq 0$. There is a canonical exact sequence

$$0 \to \Omega^n_{X/S} \to P^n_{X/S} \xrightarrow{p^n_{X/S}} P^{n-1}_{X/S} \to 0 \tag{2.1}$$

for every $n > 0$. In addition, the formation of $P^n_{X/S}$ and $p^n_{X/S}$ commute with base change and open embeddings. Note that $P^n_{X/S}$ is a sheaf of $\mathcal{O}_X$-algebras in two ways, induced by the two $\mathcal{O}_X$-algebra structures of $\mathcal{O}_{X \times_S X}$. We will distinguish between the two by calling one the left structure and the other the right structure. For general information on the sheaves of principal parts the reader may consult [5], Section 16.7, p. 36.

Assume that the fibres of f are local complete intersections. In addition, assume for the moment that f is quasi-projective. Denote by $\iota : X \hookrightarrow Y$ an S-embedding of X into an S-smooth scheme Y with pure relative dimension m over S (for instance, we could take Y to be a projective space over S.) Since the geometric fibres of X/S are local complete intersections and Y is S-smooth, the embedding ι is transversally regular relative to S (*see* [5], Section 19.3.7, p. 196). As a consequence, if J_Y denotes the ideal sheaf of X in a neighbourhood of X in Y, then J_Y/J_Y^2 is locally free of rank $m-1$. From the canonical exact sequence of sheaves on X,

$$\frac{J_Y}{J_Y^2} \xrightarrow{d_Y} \Omega^1_{Y/S} \otimes \mathcal{O}_X \xrightarrow{\pi_Y} \Omega^1_{X/S} \to 0,$$

we construct the map

$$\mu_Y : \Omega^1_{X/S} \otimes \bigwedge^{m-1} \frac{J_Y}{J_Y^2} \longrightarrow \bigwedge^{m} \Omega^1_{Y/S} \otimes \mathcal{O}_X,$$

locally defined on an affine open subset U of X by

$$\pi_Y(\lambda) \otimes g_1 \wedge \ldots \wedge g_{m-1} \longmapsto \lambda \wedge d_Y g_1 \wedge \ldots \wedge d_Y g_{m-1},$$

where λ is a section of $\Omega^1_{Y/S} \otimes \mathcal{O}_X$ on U and $g_1, \ldots, g_{m-1}$ are sections of J_Y/J_Y^2 on U. The map μ_Y is well-defined since $\frac{J_Y}{J_Y^2}$ is a locally free $\mathcal{O}_X$-module of rank $m-1$. Let

$$\omega_{X/S} := \bigwedge^m \Omega^1_{Y/S} \otimes \left(\bigwedge^{m-1} \frac{J_Y}{J_Y^2}\right)^{-1}. \tag{2.2}$$

By tensoring μ_Y with $(\bigwedge^{m-1}(J_Y/J_Y^2))^{-1}$ we obtain a map

$$\eta^1_{X/Y/S} : \Omega^1_{X/S} \to \omega_{X/S}.$$

It is clear from (2.2) that the formation of $\omega_{X/S}$ commutes with base change and open embeddings. Moreover, the above description shows that $\eta^1_{X/Y/S}$ is also natural, that is, for any S-scheme S_1 and any open subscheme Y_1 of $Y \times_S S_1$, the diagram

$$\begin{tikzcd}
\Omega^1_{X/S} \otimes \mathcal{O}_{X_1} \arrow[r, "\eta^1_{X/Y/S} \otimes \mathcal{O}_{X_1}"] \arrow[d, "\cong"'] & \omega_{X/S} \otimes \mathcal{O}_{X_1} \arrow[d, "\cong"'] \\
\Omega^1_{X_1/S_1} \arrow[r, "\eta^1_{X_1/Y_1/S_1}"] & \omega_{X_1/S_1}
\end{tikzcd}$$

commutes, where $X_1 = Y_1 \cap X \times_S S_1$.

The homomorphism $\eta^1_{X/Y/S}$ does not depend on a particular embedding of X into an S-smooth scheme Y. The proof of this statement is a simple modification of the proof found in [11] for the independence of $\omega_{X/S}$ with respect to embeddings of X into S-smooth schemes. Therefore, we can define

$$\eta^1_{X/S} := \eta^1_{X/Y/S}.$$

If $f : X \to S$ is a general flat morphism whose geometric fibres are reduced, local complete intersection curves, then we can cover X with open subschemes X_λ in such a way that X_λ is quasi-projective over S. Because $\eta^1_{X_\lambda/S}$ does not depend on a particular S-embedding of X_λ into an S-smooth scheme Y_λ of pure relative dimension over S, and the formation of $\eta^1_{X_\lambda/Y_\lambda/S}$ is natural, then we can glue the homomorphisms $\eta^1_{X_\lambda/S}$ together to obtain a global homomorphism $\eta^1_{X/S}$. It is clear that the global $\eta^1_{X/S}$ is natural, that is, its formation commutes with base change and open embeddings.

We remark that by [6], Corollary 23, p. 56 the sheaf $\omega_{X/S}$ is a dualizing sheaf for the family X/S. However, no dualizing property of $\omega_{X/S}$ will be used in the remaining of the article. For our purposes all we need is that $\omega_{X/S}$ is defined by (2.2).

It is worth mentioning that it would actually be possible to obtain a comparison homomorphism between the sheaf of Kähler differentials of f and a certain dualizing sheaf of f without the assumption that the geometric fibres be local complete intersections. As a matter of fact, we will only need this assumption in Section 4.

Let $\omega^n_{X/S} := \omega^{\otimes n}_{X/S}$.

PROPOSITION 2.3. – *$\eta^1_{X/S}$ induces canonical and natural homomorphisms*

$$\eta^i_{X/S} : \Omega^i_{X/S} \to \omega^i_{X/S}$$

for every $i \geq 1$, which are isomorphisms on the smooth locus of X/S.

Proof. – $\eta^1_{X/S}$ induces

$$(\eta^1_{X/S})^2 : \frac{I}{I^2} \otimes \frac{I}{I^2} \to \omega^2_{X/S}.$$

We will show that $(\eta^1_{X/S})^2$ factors through the multiplication homomorphism

$$m : \frac{I}{I^2} \otimes \frac{I}{I^2} \to \frac{I^2}{I^3}.$$

For this we just need to show that the support of the kernel of m does not include any associated points of X, since $\omega_{X/S}$ is invertible. But an associated point of X is an associated point of the fibre over S where it lies [5], 6.3.1, p. 138. Since the geometric fibres of f are reduced, then any associated point of X lies on the smooth locus of X/S, where m is an isomorphism. The construction of $\eta^2_{X/S}$ is thereby completed. Note that $\eta^1_{X/S}$ is an isomorphism on the smooth locus of X/S since both $\eta^1_{X/S}$ and m are. The construction of the remaining homomorphisms is analogous. The naturality is obvious from the construction and the naturality of $\eta^1_{X/S}$. It is also clear that the $\eta^i_{X/S}$ are isomorphisms on the smooth locus of X/S. $\square$

DEFINITION 2.4. – *A* Wronski algebra system *on X/S is a collection $\{Q^n_{X/S}; n \geq 0\}$ of sheaves of algebras on X together with algebra homomorphisms*

$$\psi^n_{X/S} : P^n_{X/S} \to Q^n_{X/S},$$

$$q^n_{X/S} : Q^n_{X/S} \to Q^{n-1}_{X/S},$$

and an $\mathcal{O}_X$-bimodule homomorphism

$$\alpha^n_{X/S} : \omega^n_{X/S} \to Q^n_{X/S}$$

for every $n \geq 0$, satisfying the following properties:

(1) $Q^0_{X/S} = P^0_{X/S}$;

(2) *the diagram of maps*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^n_{X/S} & \longrightarrow & P^n_{X/S} & \xrightarrow{p^n_{X/S}} & P^{n-1}_{X/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta^n_{X/S}} & & \downarrow{\scriptstyle \psi^n_{X/S}} & & \downarrow{\scriptstyle \psi^{n-1}_{X/S}} & & \\ 0 & \longrightarrow & \omega^n_{X/S} & \xrightarrow{\alpha^n_{X/S}} & Q^n_{X/S} & \xrightarrow{q^n_{X/S}} & Q^{n-1}_{X/S} & \longrightarrow & 0 \end{array} \tag{2.4.1}$$

is commutative with exact rows for every $n > 0$.

The homomorphism $\psi^n_{X/S}$ induces left and right $\mathcal{O}_X$-algebra structures on $Q^n_{X/S}$ for every $n \geq 0$. By definition, the homomorphism $\alpha^n_{X/S}$ is $\mathcal{O}_X$-linear with respect to both $\mathcal{O}_X$-algebra structures on $Q^n_{X/S}$. Because of the invertibility of $\omega_{X/S}$, the sheaf $Q^n_{X/S}$ is locally free of rank $n+1$ for each of its $\mathcal{O}_X$-algebra structures. Note also that Proposition 2.3 and the above property (1) imply that $\psi^n_{X/S}$ is an isomorphism on the smooth locus of X/S for every $n \geq 0$.

We will denote by

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

a Wronski algebra system on X/S. For simplicity we will sometimes denote a Wronski algebra system on X/S by $(Q^n_{X/S}, n \geq 0)$, leaving the homomorphisms implicit.

Two Wronski algebra systems,

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0),$$

$$(\tilde{Q}^n_{X/S}, \tilde{\psi}^n_{X/S}, \tilde{q}^n_{X/S}, \tilde{\alpha}^n_{X/S}, n \geq 0)$$

on X/S are *equivalent* if there are isomorphisms

$$\nu^n_{X/S} : Q^n_{X/S} \to \tilde{Q}^n_{X/S}$$

for all $n \geq 0$ such that $\tilde{\psi}^n_{X/S} = \nu^n_{X/S} \circ \psi^n_{X/S}$ and the diagram of maps

$$\begin{array}{ccccc}
\omega^n_{X/S} & \xrightarrow{\alpha^n_{X/S}} & Q^n_{X/S} & \xrightarrow{q^n_{X/S}} & Q^{n-1}_{X/S} \\
\| & & {\scriptstyle \nu^n_{X/S}}\downarrow & & {\scriptstyle \nu^{n-1}_{X/S}}\downarrow \\
\omega^n_{X/S} & \xrightarrow{\tilde{\alpha}^n_{X/S}} & \tilde{Q}^n_{X/S} & \xrightarrow{\tilde{q}^n_{X/S}} & \tilde{Q}^{n-1}_{X/S}
\end{array}$$

commutes for $n > 0$. The systems will be called *uniquely equivalent* if the $\nu^n_{X/S}$ are all unique.

Let $h_1 : S_1 \to S$ be any morphism of schemes, and let X_1 denote an open subscheme of $X \times_S S_1$. Let $h : X_1 \to X$ denote the induced morphism. If

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

is a Wronski algebra system on X/S, then it is easy to see from the naturality of $p^n_{X/S}$ and $\eta^n_{X/S}$ for $n \geq 1$ that

$$(h^*Q^n_{X/S}, h^*\psi^n_{X/S}, h^*q^n_{X/S}, h^*\alpha^n_{X/S}, n \geq 0)$$

is a Wronski algebra system on X_1/S_1. The system $(h^*Q^n_{X/S}, n \geq 0)$ on X_1/S_1 will be called the *restriction* of $(Q^n_{X/S}, n \geq 0)$ to X_1/S_1.

On the other hand, if $(Q^n_{X_1/S_1}, n \geq 0)$ is a Wronski algebra system on X_1/S_1, then $(Q^n_{X_1/S_1}, n \geq 0)$ is said to be *induced* from the system $(Q^n_{X/S}, n \geq 0)$ if $(Q^n_{X_1/S_1}, n \geq 0)$ is equivalent to the restriction of $(Q^n_{X/S}, n \geq 0)$ to X_1/S_1.

Let $\mathcal{C}$ be any class consisting of families X/S whose geometric fibres are reduced, Gorenstein curves, and such that $\mathcal{C}$ is closed under base change and open embeddings. In particular, we can consider the class $\mathcal{C}_{l.c.i.}$ consisting of families whose geometric fibres are reduced, local complete intersection curves.

DEFINITION 2.5. – *A* Wronski algebra system *on $\mathcal{C}$ consists of a Wronski algebra system*

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

for every family X/S in $\mathcal{C}$ such that if $h : S_1 \to S$ is any morphism and X_1 is an open subscheme of $X \times_S S_1$, then $(Q^n_{X_1/S_1}, n \geq 0)$ is induced from $(Q^n_{X/S}, n \geq 0)$.

We denote a Wronski algebra system on $\mathcal{C}$ by $(Q^n, \psi^n, q^n, \alpha^n, n \geq 0)$, or simply by $(Q^n, n \geq 0)$, leaving the homomorphisms implicit.

Two Wronski algebra systems, say $(Q^n, n \geq 0)$ and $(\tilde{Q}^n, n \geq 0)$, on $\mathcal{C}$ are *(uniquely) equivalent* if for every family X/S in $\mathcal{C}$ the Wronski algebra systems $(Q^n_{X/S}, n \geq 0)$ and $(\tilde{Q}^n_{X/S}, n \geq 0)$ are (uniquely) equivalent.

The goal of the present article is to show the following theorem.

THEOREM 2.6. – *There is a Wronski algebra system on $\mathcal{C}_{l.c.i.}$. Moreover, any two Wronski algebra systems are uniquely equivalent.*

We note that it makes perfect sense to talk about a *truncated in order N* Wronski algebra system on a family X/S, as being the data

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, 0 \leq n \leq N)$$

satisfying the conditions in Definition 2.4 up to order N. Likewise, all the concepts introduced so far make perfect sense for truncated Wronski algebra systems.

As the first step in proving Theorem 2.6, we define the truncated in order 0 Wronski algebra system as

$$(Q^0, \psi^0) := (P^0, \mathrm{id}_{P^0}).$$

We can also easily define for each family X/S the truncated in order 1 Wronski algebra system as the push-out of the infinitesimal $\mathcal{O}_X$-algebra extension (2.1) under $\eta^1_{X/S}$. Because of the categorical nature of the push-out construction, we can easily verify that the conditions in Definition 2.5 are met.

However, the push-out construction will not produce a truncated in higher order Wronski algebra system. The actual proof of Theorem 2.6 will be completed in the next four sections. We will often use in the proof the following trivial lemma and its corollary.

LEMMA 2.7. – *If two $\mathcal{O}_X$-linear maps $\beta_1, \beta_2 : E \to F$, where F is locally free, are equal on the smooth locus of X over S, then they are equal.*

Proof. – The lemma is a trivial consequence of the fact that the associated points of X lie on the smooth locus of X/S, as pointed out in the proof of Proposition 2.3. □

LEMMA 2.8. – *If two (truncated in order N) Wronski algebra systems are locally equivalent, then they are globally and uniquely equivalent.*

Proof. – The sheaves $Q^i_{X/S}$ of a Wronski algebra system are locally free and isomorphic to the $P^i_{X/S}$ on the smooth locus of X/S. Since the smooth locus of X/S is dense in X, then Lemma 2.7 guarantees the feasibility of patching together the local equivalences between two Wronski algebra systems. The uniqueness of the global equivalence follows likewise from Lemma 2.7. □

3. A local criterion

We first describe the simple local structure of a Wronski algebra system. Let $(Q^n_{X/S}, 0 \leq n < N)$ be a truncated in order $(N-1)$ Wronski algebra system on a family X/S of Gorenstein, reduced curves. Assume that X is an affine scheme and $\omega_{X/S}$ is free, say generated by τ. Of course $\omega^n_{X/S}$ is free, generated by τ^n for every $n \geq 0$. Pick a global section ζ_{N-1} of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$ under

$$q^2_{X/S} \circ \ldots \circ q^{N-1}_{X/S}.$$

Let ζ_n be the image of ζ_{N-1} in $Q^n_{X/S}$ under

$$q^{n+1}_{X/S} \circ \ldots \circ q^{N-1}_{X/S}$$

for each positive $n < N$.

Proposition 3.1. – *The homomorphisms of left $\mathcal{O}_X$-algebras,*

$$\mu^n : \frac{\mathcal{O}_X[T]}{(T^{n+1})} \to Q^n_{X/S},$$

sending T to ζ_n are isomorphisms, making

$$\begin{array}{ccccc}
\mathcal{O}_X & \xrightarrow{\cdot T^n} & \dfrac{\mathcal{O}_X[T]}{(T^{n+1})} & \xrightarrow{r^n} & \dfrac{\mathcal{O}_X[T]}{(T^n)} \\
{\scriptstyle \cdot\tau^n}\downarrow & & {\scriptstyle \mu^n}\downarrow & & {\scriptstyle \mu^{n-1}}\downarrow \\
\omega^n_{X/S} & \xrightarrow{\alpha^n_{X/S}} & Q^n_{X/S} & \xrightarrow{q^n_{X/S}} & Q^{n-1}_{X/S},
\end{array}$$

where r^n is the canonical quotient map, into a commutative diagram of maps for all $n < N$.

Proof. – The main observation here is that the above local description is known for the sheaves of principal parts $P^n_{X/S}$ in the case the family X/S is smooth [9], 2.4, p. 139. So it is natural to expect the same description to hold for good substitutes of the sheaves of principal parts. In fact, the above diagram is commutative on the smooth locus of X/S by the above observation. By Lemma 2.7, since $Q^n_{X/S}$ is locally free for every $n < N$, then the diagram is commutative everywhere. By the snake lemma and a simple induction argument on the above commutative diagrams, we can prove that μ^n is an isomorphism for every $n < N$. The proof is complete. □

Next, we will give a criterion for the local existence of a truncated in order N Wronski algebra system extending $(Q^n_{X/S}, 0 \leq n < N)$. To this purpose we can assume that S and X are affine, and X is a closed subscheme of an S-smooth affine scheme Y of pure relative dimension m over S. Let J be the ideal sheaf of X in Y. Let $f_1, \ldots, f_t$ be regular functions on Y generating J globally. As a convention, we will denote by $\bar{c}$ the restriction to X of a regular function c on Y. We will also assume that there are regular functions $u_1, \ldots, u_m$ on Y such that $du_1, \ldots, du_m$ form a basis for $\Omega^1_{Y/S}$. In particular, their respective restrictions $\bar{u}_1, \ldots, \bar{u}_m$ to X are such that $d\bar{u}_1, \ldots, d\bar{u}_m$ generate $\Omega^1_{X/S}$. For convenience, we let $v_i := \bar{u}_i$ for each $i = 1, \ldots, m$. In addition, assume that $\omega_{X/S}$ is free, generated by τ, and pick a global section ζ of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$. For convenience, we will use the same notation ζ for its images in $Q^n_{X/S}$ for $0 < n < N$.

If c is a section of $\mathcal{O}_Y$ (resp. of $\mathcal{O}_X$), then we will denote again by c its image in $P^n_{Y/S}$ (resp. $P^n_{X/S}$) under the left $\mathcal{O}_Y$-algebra (resp. $\mathcal{O}_X$-algebra) structure of $P^n_{Y/S}$ (resp. $P^n_{X/S}$), and by $\tilde{c}$ its image in $P^n_{Y/S}$ (resp. $P^n_{X/S}$) under the right algebra structure. Note the abuse of notation we make by not distinguishing between the several sheaves of principal parts.

Let $a_{i,j}$ be regular functions on X defined by

$$\psi^n_{X/S}(\tilde{v}_j) := v_j + a_{1,j}\zeta + \ldots + a_{n,j}\zeta^n$$

for every $n < N$ and each $j = 1, \ldots, m$. Note that

$$\eta^1_{X/S}(dv_j) = a_{1,j}\tau$$

for every $j = 1, \ldots, m$.

By Proposition 3.1, we need a criterion for the existence of a left $\mathcal{O}_X$-algebra homomorphism

$$\psi^N_{X/S} : P^N_{X/S} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

making

$$(3.2) \qquad \begin{array}{ccccc} \Omega^N_{X/S} & \longrightarrow & P^N_{X/S} & \xrightarrow{p^N_{X/S}} & P^{N-1}_{X/S} \\ {\scriptstyle \eta^N_{X/S}}\downarrow & & {\scriptstyle \psi^N_{X/S}}\downarrow & & {\scriptstyle \psi^{N-1}_{X/S}}\downarrow \\ \omega^N_{X/S} & \xrightarrow{\alpha^N_{X/S}} & \dfrac{\mathcal{O}_X[T]}{(T^{N+1})} & \xrightarrow{q^N_{X/S}} & Q^{N-1}_{X/S} \end{array}$$

into a commutative diagram, where $q^N_{X/S}$ and $\alpha^N_{X/S}$ are defined by $q^N_{X/S}(T) := \zeta$ and $\alpha^N_{X/S}(\tau^N) := T^N$, respectively.

For each $l \geq 0$, let

$$\Gamma_l := \{\gamma := (\gamma_1, \ldots, \gamma_m); \gamma_i \in \mathbf{Z}_{\geq 0} \text{ and } \gamma_1 + \ldots + \gamma_m = l\}.$$

Let

$$(3.3) \qquad \mathcal{O}_Y[Z_1, \ldots, Z_m] \to P^N_{Y/S}$$

be the left $\mathcal{O}_Y$-algebra homomorphism mapping Z_i to $\tilde{u}_i$ for every $i = 1, \dots, m$. Since Y is S-smooth and $du_1, \dots, du_m$ form a basis for $\Omega^1_{X/S}$, then (3.3) is surjective, with kernel generated by the products

$$(Z_1 - u_1)^{\gamma_1} \dots (Z_m - u_m)^{\gamma_m}$$

for all $\gamma \in \Gamma_{N+1}$. In addition, the kernel of the surjective map $P^N_{Y/S} \to P^N_{X/S}$ (induced by the quotient map $\mathcal{O}_Y \to \mathcal{O}_X$) is the ideal generated by f_k and $\tilde{f}_k$ for all $k = 1, \dots, t$. Note that $\tilde{f}_k$ can be expressed as

$$\begin{aligned}\tilde{f}_k =& f_k + \sum_{i=1}^{m} D_i f_k(\tilde{u}_i - u_i) + \dots \\ &+ \sum_{\gamma \in \Gamma_N} D_1^{\gamma_1} \dots D_m^{\gamma_m} f_k (\tilde{u}_1 - u_1)^{\gamma_1} \dots (\tilde{u}_m - u_m)^{\gamma_m}\end{aligned}$$

in $P^N_{Y/S}$ for every $k = 1, \dots, t$, where D_i^l is the Hasse derivation on Y/S of order l associated to u_i.

To construct a left $\mathcal{O}_X$-algebra homomorphism,

$$\psi^N_{X/S} : P^N_{X/S} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})},$$

it is enough to construct a left $\mathcal{O}_Y$-algebra homomorphism

$$\phi^N : \mathcal{O}_Y[Z_1, \dots, Z_m] \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

factoring through $P^N_{X/S}$. Since

$$\begin{array}{ccc} P^N_{X/S} & \xrightarrow{p^N_{X/S}} & P^{N-1}_{X/S} \\ \psi^N_{X/S} \downarrow & & \psi^{N-1}_{X/S} \downarrow \\ \dfrac{\mathcal{O}_X[T]}{(T^{N+1})} & \xrightarrow{q^N_{X/S}} & Q^{N-1}_{X/S} \end{array}$$

must be commutative, we must have

$$\phi^N(Z_j) = v_j + a_{1,j}T + \dots + a_{N-1,j}T^{N-1} - c_j T^N \tag{3.4}$$

for some regular function c_j on X for every $j = 1, \dots, m$. As a consequence of (3.4), the commutativity of

$$\begin{array}{ccc} \Omega^N_{X/S} & \longrightarrow & P^N_{X/S} \\ \eta^N_{X/S} \downarrow & & \psi^N_{X/S} \downarrow \\ \omega^N_{X/S} & \xrightarrow{\alpha^N_{X/S}} & \dfrac{\mathcal{O}_X[T]}{(T^{N+1})} \end{array}$$

is actually guaranteed for any choice of c_j, as it can be easily checked.

In order that ϕ^N, as defined by (3.4), factor through $P^N_{X/S}$ it is necessary and sufficient that

$$(3.5)\quad f_k^N(a_{1,1}T+\ldots+a_{N-1,1}T^{N-1}-c_1T^N,\ldots,a_{1,m}T+\ldots+a_{N-1,m}T^{N-1}-c_mT^N)$$

be divisible by T^{N+1} for all $k=1,\ldots,t$, where

$$(3.6)\quad f_k^N(Z_1,\ldots,Z_m):=\sum_{i=1}^m (D_if_k)^- Z_i+\ldots+\sum_{\gamma\in\Gamma_N}(D_1^{\gamma_1}\ldots D_m^{\gamma_m}f_k)^- Z_1^{\gamma_1}\ldots Z_m^{\gamma_m}$$

for each $k=1,\ldots,t$. Note that (3.5) is already divisible by T^N, since $\psi^{N-1}_{X/S}$ is assumed to be defined. Hence, one may define:

$$(3.7)\qquad d_k^N:=\left.\frac{f_k^N(\ldots,a_{1,j}T+\ldots+a_{N-1,j}T^{N-1},\ldots)}{T^N}\right|_{T=0}$$

as a regular function on X for every $k=1,\ldots,t$. Let also

$$M:=\begin{bmatrix} D_1f_1 & D_2f_1 & \ldots & D_mf_1\\ D_1f_2 & D_2f_2 & \ldots & D_mf_2\\ \vdots & \vdots & \ddots & \vdots\\ D_1f_t & D_2f_t & \ldots & D_mf_t\end{bmatrix}.$$

Then we have the following criterion.

CRITERION 3.8. – *There exists a homomorphism $\psi^N_{X/S}$ making diagram (3.2) commutative if and only if the linear system*

$$\bar{M}\begin{bmatrix} c_1\\ \vdots\\ c_m\end{bmatrix}=\begin{bmatrix} d_1^N\\ \vdots\\ d_t^N\end{bmatrix}$$

is solvable by regular functions $c_1,\ldots,c_m$ on X.

Proof. – By combining (3.5) and (3.6) we obtain that

$$\begin{aligned} &f_k^N(\ldots,a_{1,j}T+\ldots+a_{N-1,j}T^{N-1}-c_jT^N,\ldots)\\ &\quad=\left(d_k^N-\sum_{i=1}^m(D_if_k)^-c_i\right)T^N+(\text{higher order terms})\end{aligned}$$

for every $k=1,\ldots,t$. The criterion follows immediately. $\square$

4. The Wronski algebra system for "general" families

Recalling the set-up of last section, S is an affine scheme, Y is an affine S-smooth scheme whose sheaf of differentials $\Omega^1_{Y/S}$ is free of rank m with basis $du_1, \ldots, du_m$, where $u_1, \ldots, u_m$ are regular functions on Y, and $X \subset Y$ is the closed subscheme defined by regular functions $f_1, \ldots, f_t$ on Y. Assume now that $t = m-1$, and $f_1, \ldots, f_{m-1}$ form a regular sequence on Y relative to S. In other words, we assume that for every fibre of Y/S the restrictions of $f_1, \ldots, f_{m-1}$ to the fibre form a regular sequence. It follows from this assumption and (2.2) that the sheaf $\omega_{X/S}$ is free, generated by $du_1 \wedge \ldots \wedge du_m \otimes \sigma$, where σ is the dual to $f_1 \wedge \ldots \wedge f_{m-1}$. Hence, we can and will assume that $\tau = du_1 \wedge \ldots \wedge du_m \otimes \sigma$. The data

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

is called a *local data.*

Let M_j denote the maximal minor of M, obtained by deleting the j-th column of M. Let

$$\delta_j := (-1)^{j+1} \det M_j \quad \text{for} \quad j = 1, \ldots, m.$$

From the construction of $\eta^1_{X/S}$ shown in Section 2, the fact that $\eta^1_{X/S}(dv_j) = a_{1,j}\tau$, and the above choice of τ it is easy to check that $\bar{\delta}_j = a_{1,j}$ for $j = 1, \ldots, m$. Let

$$\Delta := (\bar{\delta}_1, \ldots, \bar{\delta}_m)\mathcal{O}_X.$$

As just seen, $\Delta\tau$ is the image of $\eta^1_{X/S}$ in $\omega_{X/S}$; hence $\Delta\tau$ is an intrinsic subsheaf of $\omega_{X/S}$.

LEMMA 4.1. – *If depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, then there is a homomorphism $\psi^N_{X/S}$ making diagram (3.2) commutative. Moreover, $\psi^N_{X/S}$ is unique in the following sense: if*

$$\theta^N : P^N_{X/S} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

is another homomorphism making diagram (3.2) commutative, then there is a unique isomorphism

$$\lambda^N : \frac{\mathcal{O}_X[T]}{(T^{N+1})} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})},$$

with $\lambda^N(T^N) = T^N$ and $q^N_{X/S} \circ \lambda^N = q^N_{X/S}$, such that $\lambda^N \circ \psi^N_{X/S} = \theta^N$.

Proof. – We first claim that $a_{i,j} \in \Delta(X)$ for all i, j. The proof will be by induction. We have already remarked that $a_{1,j} \in \Delta(X)$. Assume that $a_{i,j} \in \Delta(X)$ for $i < s$, where $1 < s < N$. For each $k = 1, \ldots, m-1$, we have

$$\text{(4.1.1)} \quad f^N_k(\ldots, a_{1,j}T + \ldots + a_{N-1,j}T^{N-1}, \ldots) = \left(\sum_{j=1}^{m}(D_j f_k)^- a_{1,j}\right)T + \ldots$$
$$+ \left(\sum_{l=1}^{r}\sum_{\gamma\in\Gamma_l}(D_1^{\gamma_1}\ldots D_m^{\gamma_m} f_k)^- \sum_{\substack{s(\alpha)=r\\ |\alpha_j|=\gamma_j}} \prod_{j=1}^{m}\binom{\gamma_j}{\alpha_j}\prod_{i=1}^{N-1} a_{i,j}^{\alpha_{i,j}}\right)T^r + \ldots,$$

where

$$\begin{aligned}\alpha &:= (\alpha_{i,j})_{\substack{1\leq i<N\\ 1\leq j\leq m}};\ \alpha_{i,j}\in \mathbf{Z}_{\geq 0},\\ \alpha_j &:= (\alpha_{1,j},\ldots,\alpha_{N-1,j}),\\ |\alpha_j| &:= \alpha_{1,j}+\ldots+\alpha_{N-1,j},\\ s(\alpha) &:= \sum_{i,j} i\alpha_{i,j},\\ \binom{\gamma_j}{\alpha_j} &:= \frac{\gamma_j!}{\prod_{i=1}^{N-1}\alpha_{i,j}!}.\end{aligned}$$

Since T^N divides (4.1.1), as remarked in Section 3, the coefficient of T^s in the expression (4.1.1) must be 0. But this same coefficient may be expressed from (4.1.1) as

$$\sum_{j=1}^{m}(D_j f_k)^{-} a_{s,j} + E_{k,s},$$

where $E_{k,s}$ does not involve any $a_{i,j}$ with $i \geq s$. Hence, from the induction hypothesis we obtain that $E_{k,s} \in \Delta^2(X)$, and therefore

$$\sum_{j=1}^{m}(D_j f_k)^{-} a_{s,j} \in \Delta^2(X)$$

for every $k = 1,\ldots,m-1$.

Since depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, by the Buchsbaum-Eisenbud criterion [1], Theorem, p. 260, the complex

$$(4.1.2)\qquad 0 \to \mathcal{O}_X^{\oplus m-1} \xrightarrow{\bar{M}^t} \mathcal{O}_X^{\oplus m} \xrightarrow{(\bar{\delta}_1,\ldots,\bar{\delta}_m)} \mathcal{O}_X \to \frac{\mathcal{O}_X}{\Delta} \to 0$$

is exact. By dualizing (4.1.2), we obtain the sequence

$$(4.1.3)\qquad 0 \to \mathcal{O}_X \xrightarrow{(\bar{\delta}_1,\ldots,\bar{\delta}_m)} \mathcal{O}_X^{\oplus m} \xrightarrow{\bar{M}} \mathcal{O}_X^{\oplus m-1} \to \underline{\mathrm{Ext}}^2_{\mathcal{O}_X}\left(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X\right) \to 0,$$

which is exact since

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X\right) = \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}\left(\frac{\mathcal{O}_X}{\Delta}, \mathcal{O}_X\right) = 0,$$

again by the condition on the depth [14], 16.7, p. 130. Since the $\mathcal{O}_X$-module $\underline{\mathrm{Ext}}^2_{\mathcal{O}_X}(\mathcal{O}_X/\Delta, \mathcal{O}_X)$ is annihilated by Δ, then the image of $\bar{M}$ contains $\Delta^{\oplus m-1}$. Since the kernel of $\bar{M}$ lies inside $\Delta^{\oplus m}$, then any section of $\mathcal{O}_X^{\oplus m}$ mapping into $(\Delta^2)^{\oplus m-1}$ under $\bar{M}$ must be in $\Delta^{\oplus m}$. Therefore,

$$a_{s,1},\ldots,a_{s,m} \in \Delta(X),$$

finishing the induction argument and the proof of the claim.

As with $E_{k,s}$ above, it is easy to see from (4.1.1) that $d_k^N \in \Delta^2(X)$ for every $k = 1, \ldots, m-1$. As the image of $\bar{M}$ contains $\Delta^{\oplus m-1}$, there are $c_1, \ldots, c_m \in \Delta(X)$ such that

$$\sum_{j=1}^{m} (D_j f_k)^{-} c_j = d_k^N \quad \text{for} \quad k = 1, \ldots, m-1.$$

By Criterion 3.8, the existence of $\psi_{X/S}^N$ follows.

As for uniqueness, from (4.1.3) any two solutions $(c_1, \ldots, c_m)$ and $(c'_1, \ldots, c'_m)$ to the linear system of Criterion 3.8, corresponding to homomorphisms $\psi_{X/S}^N$ and θ^N, respectively, differ by a multiple of $(\bar{\delta}_1, \ldots, \bar{\delta}_m)$, say

$$(c'_1, \ldots, c'_m) = (c_1, \ldots, c_m) + e(\bar{\delta}_1, \ldots, \bar{\delta}_m),$$

where e is a regular function on X. If we let

$$\lambda^N : \frac{\mathcal{O}_X[T]}{(T^{N+1})} \to \frac{\mathcal{O}_X[T]}{(T^{N+1})}$$

be the $\mathcal{O}_X$-algebra morphism defined by $\lambda^N(T) := T - eT^N$, then $\lambda^N(T^N) = T^N$,

$$q_{X/S}^N \circ \lambda^N(T) = q_{X/S}^N(T - eT^N) = \zeta = q_{X/S}^N(T),$$

and

$$\begin{aligned} \lambda^N \circ \psi_{X/S}^N(\tilde{v}_j) &= \lambda^N(v_j + a_{1,j}T + \ldots + a_{N-1,j}T^{N-1} - c_jT^N) \\ &= v_j + a_{1,j}T + \ldots + a_{N-1,j}T^{N-1} - (c_j + ea_{1,j})T^N \\ &= v_j + a_{1,j}T + \ldots + a_{N-1,j}T^{N-1} - c'_jT^N \\ &= \theta^N(\tilde{v}_j) \end{aligned}$$

for $j = 1, \ldots, m$. In addition, $\lambda_{X/S}^N$ is unique by Lemma 2.8 (alternatively, because the homomorphism $(\bar{\delta}_1, \ldots, \bar{\delta}_m)$ is injective.) The proof of the lemma is complete. $\square$

LEMMA 4.2. – *If depth $(\Delta(X), \mathcal{O}_X(X)) = 2$, then there is a Wronski algebra system on X/S. Moreover, any two Wronski algebra systems are uniquely equivalent.*

Proof. – From Section 2, we know already that there is a unique truncated in order 1 Wronski algebra system on X/S. Assume by induction that there is a unique truncated in order $N-1$ Wronski algebra system $(Q_{X/S}^n, 0 \leq n < N)$ on X/S. By Lemma 4.1 there is a truncated in order N Wronski algebra system extending $(Q_{X/S}^n, 0 \leq n < N)$. The first statement of the corollary follows now by induction. As for the second statement, we can assume by induction that any two Wronski algebra systems on X/S agree up to order $N-1$, say with $(Q_{X/S}^n, 0 \leq n < N)$. Moreover, since X is affine and $\omega_{X/S}$ is free, by Proposition 3.1 any extension $(Q_{X/S}^N, \psi_{X/S}^N, q_{X/S}^N, \alpha_{X/S}^N)$ of $(Q_{X/S}^n, 0 \leq n < N)$ is equivalent to an extension of the form

$$\left(\frac{\mathcal{O}_X[T]}{(T^{N+1})}, \psi_{X/S}^N, q_{X/S}^N, \alpha_{X/S}^N \right),$$

where $q^N_{X/S}(T) = \zeta$ and $\alpha^N_{X/S}(\tau^N) = T^N$, with $\tau := du_1 \wedge \ldots \wedge du_m \otimes \sigma$, and ζ a chosen global section of $Q^{N-1}_{X/S}$ mapping to $\alpha^1_{X/S}(\tau)$ in $Q^1_{X/S}$. Therefore, by Lemma 4.1 any two extensions are equivalent. The induction argument is complete. The uniqueness of the equivalence follows from Lemma 2.8. □

Let $f : X \to S$ denote now a flat morphism whose geometric fibres are reduced, local complete intersection curves. The family X/S is said to satisfy the *depth condition* if

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathrm{Coker}\ \eta^1_{X/S}, \mathcal{O}_X) = 0$$

for $i = 0, 1$.

PROPOSITION 4.3. – *If X/S satisfies the depth condition, then there is a Wronski algebra system on X/S. Moreover, any two Wronski algebra systems on X/S are uniquely equivalent.*

Proof. – By the first statement of Lemma 4.2, we can cover X/S by local Wronski algebra systems. The local Wronski algebra systems can be patched together to yield a global Wronski algebra system by the second statement of Lemma 4.2. The uniqueness of the global Wronski algebra system follows likewise. □

By proving Theorem 2.6 we will have shown that the depth condition is not necessary for the existence of a Wronski algebra system. Hence, the depth condition is not necessary for the first statement in Lemma 4.1. But the condition is necessary for the second statement therein. In fact, consider the affine plane curve $X \subset \mathbf{A}^2_k$ defined by $f := y^3 - x^4$, where k is an algebraically closed field of characteristic different from 2. We have

$$a_{1,1} = (D_y f)^- = 3y^2 \qquad \text{and} \qquad a_{1,2} = -(D_x f)^- = 4x^3$$

on X. Hence,

$$d^2 = \left.\frac{(y + 4x^3T)^3 - (x + 3y^2T)^4}{T^2}\right|_{T=0} = -6x^2y^4 = -6x^6y$$

on X. Since

$$[-4x^3 \quad 3y^2]\begin{bmatrix} 0 \\ -2x^2y^2 \end{bmatrix} = -6x^2y^4,$$

then by Criterion 3.8 there is a homomorphism ψ^2 making diagram (3.2) commute, namely,

$$\psi^2(\tilde{x}) := x + 3y^2T \qquad \text{and} \qquad \psi^2(\tilde{y}) := y + 4x^3T + 2x^2y^2T^2.$$

However, the homomorphism

$$\theta^2 : P^2_X \to \frac{\mathcal{O}_X[T]}{(T^3)},$$

given by

$$\theta^2(\tilde{x}) := x + 3y^2T - \frac{3}{2}x^3yT^2 \qquad \text{and} \qquad \theta^2(\tilde{y}) := y + 4x^3T,$$

is well-defined and also makes diagram (3.7) commute, since

$$[-4x^3 \quad 3y^2]\begin{bmatrix}\frac{3}{2}x^3y\\ 0\end{bmatrix} = -6x^6y$$

on X. We claim there is no automorphism λ^2 of $\mathcal{O}_X[T]/T^3$ such that $\lambda^2 \circ \psi^2 = \theta^2$ and $\lambda^2(T^2) = T^2$. In fact, such λ^2 would be defined by $\lambda^2(T) = \pm T + eT^2$ for some regular function e on X. If $\lambda^2 \circ \psi^2 = \theta^2$, then

$$x \pm 3y^2T + 3y^2eT^2 = \lambda^2 \circ \psi^2(\tilde{x}) = \theta^2(\tilde{x}) = x + 3y^2T - \frac{3}{2}x^3yT^2.$$

But there is no polynomial $p(x,y)$ such that $3y^2p(x,y) + \frac{3}{2}x^3y$ is divisible by $y^3 - x^4$. Hence, the truncated in order 2 Wronski algebra systems defined by ψ^2 and θ^2 are not equivalent.

5. Some lemmas

Of course the hypothesis on the depth cannot be satisfied if, for instance, the family X/S consists simply of a curve over a field. The general idea to overcome such a problem is to "enlarge" locally the family X/S in such a way that the larger family satisfies the depth condition, apply Proposition 4.3, and then restrict the Wronski algebra system in the larger family to the family X/S. One must also take care that two different "enlargements" do not yield two different Wronski algebra systems, if we are to glue the local systems together. The purpose of this section is to build enough tools to handle the above process.

Assume there is a commutative diagram of morphisms of schemes,

$$\begin{array}{ccc} S_1 & \xrightarrow{h_2} & S_3 \\ {\scriptstyle h_0}\uparrow & & {\scriptstyle h_3}\uparrow \\ S_0 & \xrightarrow{h_1} & S_2. \end{array} \tag{5.1}$$

Let X_3 be a flat scheme over S_3 whose geometric fibres are reduced, local complete intersection curves. For $i = 1, 2$, let $X_i \subset X \times_{S_3} S_i$ be an open subscheme. Let X_0 be an open subscheme of

$$X_1 \times_{S_1} S_0 \cap X_2 \times_{S_2} S_0.$$

LEMMA 5.2. – *For $i = 2, 3$, let $(Q^n_{X_i/S_i}, n \geq 0)$ be a Wronski algebra system on X_i/S_i. For $i = 0, 1$, let $(Q^n_{X_i/S_i}, n \geq 0)$ be a Wronski algebra system on X_i/S_i induced from $(Q^n_{X_{i+2}/S_{i+2}}, n \geq 0)$. If X_2/S_2 satisfies the depth condition, then $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_1/S_1}, n \geq 0)$.*

Proof. – Since X_2/S_2 satisfies the depth condition, then by Proposition 4.3 the system $(Q^n_{X_2/S_2}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$. Hence, $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$ via $h_3 \circ h_1$. Since $h_2 \circ h_0 = h_3 \circ h_1$ and $(Q^n_{X_1/S_1}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$, then the system $(Q^n_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_1/S_1}, n \geq 0)$. The proof is complete. $\square$

We now return to the local case, that is, assume that

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

is a local data. We will give an explicit "enlargement" for X/S that satisfies the depth condition. Let $S' := S \times \operatorname{Spec} \mathbf{Z}[T_{i,j}]$, where $\mathbf{Z}[T_{i,j}]$ is the polynomial ring over the integers $\mathbf{Z}$ in the variables $T_{i,j}$, with $1 \leq i \leq m-1$ and $1 \leq j \leq m+1$. Let $Y' := Y \times_S S'$. We will make an abuse of notation by not distinguishing between a regular function on Y and its pull-back to Y'. The scheme Y' is smooth over S', and $\Omega^1_{Y'/S'}$ is free of rank m, with basis $du_1, \ldots, du_m$. Let $Z' \subset Y'$ be the closed subscheme whose sheaf of ideals J' is generated by the regular functions

$$f'_k := f_k + \sum_{j=1}^{m} u_j T_{k,j} + T_{k,m+1} \quad \text{for} \quad k = 1, \ldots, m-1$$

on Y'. If we let $h_S : S \to S'$ be the closed embedding obtained by setting $T_{i,j} = 0$ for all i, j, then it is clear that

$$Y = Y' \times_{S'} S \qquad \text{and} \qquad X = Z' \times_{S'} S$$

under h_S. Let $U' \subset Y'$ be the open subscheme of Y' where $f'_1, \ldots, f'_{m-1}$ is a regular sequence relative to S' (*see* [5], 11.1.4, p. 118). Since $f_1, \ldots . f_{m-1}$ is a regular sequence on Y relative to S, then $Y = U' \times_{S'} S$. Let $X' \subset Z' \cap U'$ be the open subscheme of points which are reduced in their geometric fibres [5], 12.1.1, p. 174. Since the geometric fibres of X/S are reduced, then $X = X' \times_{S'} S$. It is clear that the embedding $X' \subset U'$ is transversally regular relative to S'. Since U' is smooth over S', then X'/S' is a flat family whose geometric fibres are reduced, local complete intersection curves.

Note that

$$Z' \cong Y \times \operatorname{Spec} \mathbf{Z}[T_{k,j}]_{\substack{1 \leq k \leq m-1 \\ 1 \leq j \leq m}} \tag{5.3}$$

and

$$D_j f'_k = D_j f_k + T_{k,j} \tag{5.4}$$

for every $k = 1, \ldots, m-1$ and $j = 1, \ldots, m$. Let

$$M' := \begin{bmatrix} D_1 f'_1 & D_2 f'_1 & \ldots & D_m f'_1 \\ D_1 f'_2 & D_2 f'_2 & \ldots & D_m f'_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f'_{m-1} & D_2 f'_{m-1} & \ldots & D_m f'_{m-1} \end{bmatrix},$$

and $\delta'_j := \det M'_j$, where M'_j is the maximal minor of M' obtained by deleting the j-th column for every $j = 1, \ldots, m$. Let

$$\Delta' := (\bar{\delta}'_1, \ldots, \bar{\delta}'_m)\mathcal{O}_{Z'}.$$

Because of (5.3) and (5.4), the matrix M' is "generic." As a consequence,

$$\operatorname{depth}(\Delta'(Z'), \mathcal{O}_{Z'}(Z')) = 2,$$

or equivalently,

$$\underline{\operatorname{Ext}}^i_{\mathcal{O}_{Z'}}\left(\frac{\mathcal{O}_{Z'}}{\Delta'}, \mathcal{O}_{Z'}\right) = 0$$

for $i = 0, 1$ (*see* [16], Prop. 1, p. 195). Hence, X'/S' satisfies the depth condition. By Proposition 4.3, there is a Wronski algebra system $(Q^n_{X'/S'}, n \geq 0)$ on X'/S'. Its restriction to X/S is a Wronski algebra system on X/S, as we remarked in Section 2.

Since we will often use the above construction, we make the following definition. The data

$$(S', Y', Z', X', f'_1, \ldots, f'_{m-1})$$

will be called the *enlargement* of the local data

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1}).$$

There are of course several ways of extending the family X/S to a larger family where the depth condition holds. Nevertheless, for the sake of proving Theorem 2.6, we will stick to the above construction.

The construction of the enlargement is "functorial" in the following sense. Let

$$(S_1, Y_1, X_1, u_1, \ldots, u_m, f_1, \ldots, f_{m-1})$$

be a local data. Let $h : S_2 \to S_1$ be any morphism of affine schemes, and let $Y_2 \subset Y_1 \times_{S_1} S_2$ be an affine open subscheme. Let $X_2 := Y_2 \cap X_1 \times_{S_1} S_2$. We will make an abuse of notation by not distinguishing between a regular function on Y_1 and its pull-back to Y_2. Of course, the sheaf $\Omega^1_{Y_2/S_2}$ is free, with basis $du_1, \ldots, du_m$, and the ideal sheaf of X_2 in Y_2 is generated by $f_1, \ldots, f_{m-1}$. For $i = 1, 2$, let

$$(S'_i, Y'_i, Z'_i, X'_i, f'_1, \ldots, f'_{m-1})$$

denote the enlargement of the local data

$$(S_i, Y_i, X_i, u_1, \ldots, u_m, f_1, \ldots, f_{m-1}).$$

It is easy to see from the construction of the enlargement that h lifts to a morphism h' making the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{h_{S_1}} & S'_1 \\ {\scriptstyle h}\uparrow & & {\scriptstyle h'}\uparrow \\ S_2 & \xrightarrow{h_{S_2}} & S'_2 \end{array}$$

Cartesian, that Y_2' is an affine open subscheme of $Y_1' \times_{S_1'} S_2'$, and that

$$Z_2' = Y_2' \cap Z_1' \times_{S_1'} S_2'.$$

In addition, we have

$$X_2' = Y_2' \cap X_1' \times_{S_1'} S_2'.$$

For $i = 1, 2$, let $(Q^n_{X_i'/S_i'}, n \geq 0)$ be a Wronski algebra system on X_i'/S_i', and let $(Q^n_{X_i/S_i}, n \geq 0)$ be a system on X_i/S_i induced from $(Q^n_{X_i'/S_i'}, n \geq 0)$. Since X_2'/S_2' satisfies the depth condition, by Lemma 5.2 we have that the Wronski algebra system $(Q^n_{X_2/S_2}, n \geq 0)$ is induced from $(Q^n_{X_1/S_1}, n \geq 0)$.

There might be two different enlargements for a family X/S, depending on the choice of local data. Nevertheless, two enlargements of X/S can be obtained as subfamilies of a bigger family, as we observe below. Let $Y_0 \to S_0$ be any morphism of finite type of affine schemes. Let S_1 and S_2 be affine spaces over S_0 together with sections

$$S_0 \to S_1 \quad \text{and} \quad S_0 \to S_2.$$

For $i = 1, 2$, let $Y_i := Y_0 \times_{S_0} S_i$, and let $f_1^i, \ldots, f_t^i$ be a sequence of regular functions on Y_i. Assume that these sequences restrict to the same sequence $f_1^0, \ldots, f_t^0$ on Y_0 under the above sections.

LEMMA 5.5. – *There are an affine space S_3 over $S_1 \times_{S_0} S_2$ together with sections*

$$S_1 \to S_3 \quad \textit{and} \quad S_2 \to S_3$$

making the diagram of maps

$$\begin{array}{ccc} S_1 & \longrightarrow & S_3 \\ \uparrow & & \uparrow \\ S_0 & \longrightarrow & S_2 \end{array} \tag{5.5.1}$$

commutative, and regular functions $f_1^3, \ldots, f_t^3$ on $Y_3 := Y_0 \times_{S_0} S_3$ restricting to $f_1^i, \ldots, f_t^i$ on Y_i for $i = 1, 2$.

Proof. – The proof, which is standard, is left to the reader. □

For $0 \leq i \leq 3$, let Z_i be the closed subscheme of Y_i defined by $f_1^i, \ldots, f_t^i$. Of course,

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z_3 \\ \uparrow & & \uparrow \\ Z_0 & \longrightarrow & Z_2 \end{array}$$

is a commutative diagram which is Cartesian over (5.5.1), that is,

$$\begin{array}{ccc} Z_i & \longrightarrow & Z_j \\ \downarrow & & \downarrow \\ S_i & \longrightarrow & S_j \end{array}$$

is Cartesian whenever defined. Moreover, for $0 \leq i \leq 3$ let U_i denote the open subscheme of Y_i where $f_1^i, \ldots, f_t^i$ form a regular sequence relative to S_i. Let also $X_i \subset Z_i \cap U_i$ be the open subscheme of points which are reduced in their geometric fibres over S_i. Then we obtain a commutative diagram, again Cartesian over (5.5.1),

$$\begin{array}{ccc} X_1 & \longrightarrow & X_3 \\ \uparrow & & \uparrow \\ X_0 & \longrightarrow & X_2. \end{array}$$

6. Existence and uniqueness of the Wronski algebra system

Consider a flat, quasi-projective morphism $f : X \to S$, whose geometric fibres are reduced, local complete intersection curves. Fix an S-embedding $\iota : X \to Y$ into an S-smooth scheme Y of pure relative dimension m over S.

PROPOSITION 6.1. – *There is a Wronski algebra system on X/S.*

Proof. – Since ι is transversally regular over S, we can cover S with affine open subscheme S_λ, and then $X \times_S S_\lambda$ with affine open subschemes $Y_\mu \subset Y \times_S S_\lambda$ in such a way that for every λ, μ:

(1) there are regular functions $u_{\mu,1}, \ldots, u_{\mu,m}$ on Y_μ such that their differentials $du_{\mu,1}, \ldots, du_{\mu,m}$ form a basis for $\Omega^1_{Y_\mu/S_\lambda}$;

(2) if we let $X_\mu := X \cap Y_\mu$, then X_μ is the closed subscheme of Y_μ given by a regular sequence $f_{\mu,1}, \ldots, f_{\mu,m-1}$ on Y_μ relative to S_λ.

Let

$$(S'_\lambda, Y'_\mu, Z'_\mu, X'_\mu, f'_{\mu,1}, \ldots, f'_{\mu,m-1})$$

be the enlargement of the local data

$$(S_\lambda, Y_\mu, X_\mu, u_{\mu,1}, \ldots, u_{\mu,m}, f_{\mu,1}, \ldots, f_{\mu,m-1}) \tag{6.1.1}$$

for every λ, μ. Let $(Q^n_{X'_\mu/S'_\lambda}, n \geq 0)$ be a Wronski algebra system on X'_μ/S'_λ, and let $(Q^n_{X_\mu/S_\lambda}, n \geq 0)$ be its restriction to X_μ/S_λ for every λ, μ.

We will prove that the above local systems can be patched together to form a global Wronski algebra system on X/S. For $i = 1, 2$, let

$$(S_i, Y_i, X_i, u_{i,1}, \ldots, u_{i,m}, f_{i,1}, \ldots, f_{i,m-1})$$

be one of the local data (6.1.1). Let $S_0 \subset S_1 \cap S_2$ and

$$Y_0 \subset Y_1 \times_{S_1} S_0 \cap Y_2 \times_{S_2} S_0$$

be affine open subschemes such that there is an $(m-1) \times (m-1)$ invertible matrix C_0 whose entries are regular functions on Y_0 and

$$\begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m-1} \end{bmatrix} = C_0 \begin{bmatrix} f_{2,1} \\ \vdots \\ f_{2,m-1} \end{bmatrix}. \tag{6.1.2}$$

It is clear that we can cover $S_1 \cap S_2$ and $X_1 \times_{S_1} S_0 \cap X_2 \times_{S_2} S_0$ with such subschemes. Let $X_0 := X \cap Y_0$. We will show that the restrictions of $(Q^n_{X_1/S_1}, n \geq 0)$ and $(Q^n_{X_2/S_2}, n \geq 0)$ to X_0/S_0 are (uniquely) equivalent. The uniqueness of the equivalence allows us to perform the above-mentioned patching, thereby finishing the proof.

We will make an abuse of notation by not distinguishing between regular functions on Y_i and their restrictions to Y_0, for $i = 1, 2$. For $i = 1, 2$, let

$$(S^i_0, Y^i_0, Z^i_0, X^i_0, f'_{i,1}, \dots, f'_{i,m-1})$$

be the enlargement of the local data

$$(S_0, Y_0, X_0, u_{i,1}, \dots, u_{i,m}, f_{i,1}, \dots, f_{i,m-1}).$$

For $i = 1, 2$, let $(Q^n_{X^i_0/S^i_0}, n \geq 0)$ be a Wronski algebra system on X^i_0/S^i_0, and let $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ be its restriction to X_0/S_0. As shown in Section 5, for $i = 1, 2$ the Wronski algebra system $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_i/S_i}, n \geq 0)$.

Since C_0 is invertible, Z^2_0 is the subscheme of Y^2_0 cut out by the components of

$$C_0 \begin{bmatrix} f'_{2,1} \\ \vdots \\ f'_{2,m-1} \end{bmatrix}.$$

By (6.1.2), the restriction of the above vector to Y_0 is

$$\begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m-1} \end{bmatrix}.$$

Therefore, from Lemma 5.5 we can easily find a family X_3/S_3 satisfying the depth condition, morphisms

$$S^1_0 \to S_3 \quad \text{and} \quad S^2_0 \to S_3$$

making

$$\begin{array}{ccc} S^1_0 & \longrightarrow & S_3 \\ \uparrow & & \uparrow \\ S_0 & \longrightarrow & S^2_0 \end{array}$$

commutative, and such that

$$X^1_0 = X_3 \times_{S_3} S^1_0 \quad \text{and} \quad X^2_0 = X_3 \times_{S_3} S^2_0.$$

Let $(Q^n_{X_3/S_3}, n \geq 0)$ be a Wronski algebra system on X_3/S_3. Since X^i_0/S^i_0 satisfies the depth condition, then $(Q^n_{X^i_0/S^i_0}, n \geq 0)$ is induced from $(Q^n_{X_3/S_3}, n \geq 0)$ for $i = 1, 2$. By

Lemma 5.2, the Wronski algebra system $(Q^{2,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X^1_0/S^1_0}, n \geq 0)$. Since $(Q^{1,n}_{X_0/S_0}, n \geq 0)$ is also induced from $(Q^n_{X^1_0/S^1_0}, n \geq 0)$ by construction, then

$$(Q^{1,n}_{X_0/S_0}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X_0/S_0}, n \geq 0)$$

are equivalent. Since $(Q^{i,n}_{X_0/S_0}, n \geq 0)$ is induced from $(Q^n_{X_i/S_i}, n \geq 0)$ for $i = 1, 2$, then the restrictions of

$$(Q^n_{X_1/S_1}, n \geq 0) \quad \text{and} \quad (Q^n_{X_2/S_2}, n \geq 0)$$

to X_0/S_0 are (uniquely) equivalent. The proof is complete. $\square$

It is worth remarking that the above construction does not depend on the covering of S, X, Y by local data. In fact, we could have taken a covering consisting of all possible local data in the above proof. We would still need to prove that the Wronski algebra system constructed above does not depend on the choice of Y and the embedding $\iota : X \to Y$. However, since such a proof follows a standard argument, similar to the one used to prove the naturality of $\omega_{X/S}$ as constructed in Section 2 (*see* [11]), the proof is left to the reader.

Proof of Theorem 2.6. – We first claim that there is a Wronski algebra system on $\mathcal{C}_{l.c.i.}$. In fact, let X/S be a family in $\mathcal{C}_{l.c.i.}$. One can cover X with open subschemes X_λ in such a way that for every λ there is an S-embedding $X_\lambda \hookrightarrow Y_\lambda$ into an S-smooth scheme Y_λ of pure relative dimension over S. Let

$$(Q^n_{X_\lambda/S}, n \geq 0) := (Q^n_{X_\lambda/Y_\lambda/S}, n \geq 0)$$

be the system constructed in Proposition 6.1. Because $(Q^n_{X/S}, n \geq 0)$ is independent of Y_λ and the S-embedding $X_\lambda \to Y_\lambda$, we can glue the above local systems together to obtain a Wronski algebra system $(Q^n_{X/S}, n \geq 0)$ on X/S. It is easy to check that the conditions in Definition 2.5 are met.

We now show that two Wronski algebra systems on $\mathcal{C}_{l.c.i.}$ are uniquely equivalent. In fact, for $i = 1, 2$, let $(Q^{i,n}, n \geq 0)$ be a Wronski algebra system on $\mathcal{C}_{l.c.i.}$. Let X/S be any family in $\mathcal{C}_{l.c.i.}$. By Lemma 2.8, we can restrict ourselves to the local case, that is, we can assume that there is a local data

$$(S, Y, X, u_1, \ldots, u_m, f_1, \ldots, f_{m-1}).$$

Let

$$(S', Y', Z', X', f'_1, \ldots, f'_{m-1})$$

be the enlargement of the above local data. By Proposition 4.3, the Wronski algebra systems

$$(Q^{1,n}_{X'/S'}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X'/S'}, n \geq 0)$$

are equivalent. Since by Definition 2.5 the system $(Q^{i,n}_{X/S}, n \geq 0)$ is induced from $(Q^{i,n}_{X'/S'}, n \geq 0)$ for $i = 1, 2$, then the Wronski algebra systems

$$(Q^{1,n}_{X/S}, n \geq 0) \quad \text{and} \quad (Q^{2,n}_{X/S}, n \geq 0)$$

are (uniquely) equivalent. The proof is complete. $\square$

7. Wronski systems and Wronskians

Let $f : X \to S$ be a flat morphism whose geometric fibres are reduced, Gorenstein curves. Assume there is a Wronski algebra system

$$(Q^n_{X/S}, \psi^n_{X/S}, q^n_{X/S}, \alpha^n_{X/S}, n \geq 0)$$

on X/S. The existence of such system has been shown here only for local complete intersection curves though. If no confusion is likely, then we will ommit the subscript X/S.

Let L be an invertible sheaf on X. Denote by $P^n(L)$ (resp. $Q^n(L)$) the tensor product of L by P^n (resp. Q^n) with respect to the right $\mathcal{O}_X$-algebra structure of P^n (resp. Q^n.) The sheaf $P^n(L)$ (resp. $Q^n(L)$) will be regarded as an $\mathcal{O}_X$-module via the left $\mathcal{O}_X$-algebra structure of P^n (resp. Q^n.)

Tensoring diagram (2.4.1) on the right by L we obtain a commutative diagram of left $\mathcal{O}_X$-modules, namely,

$$\begin{array}{ccc} P^n(L) & \xrightarrow{p^n(L)} & P^{n-1}(L) \\ {\scriptstyle \psi^n(L)}\downarrow & & {\scriptstyle \psi^{n-1}(L)}\downarrow \\ Q^n(L) & \xrightarrow{q^n(L)} & Q^{n-1}(L) \end{array} \tag{7.1}$$

for each $n > 0$. On the other hand, there are canonical homomorphisms

$$\nu^n : f^* f_* L \to P^n(L)$$

for each $n \geq 0$ such that

$$p^n(L) \circ \nu^n = \nu^{n-1} \tag{7.2}$$

for every $n > 0$.

Since L is invertible, then $Q^n(L)$ is locally free of rank $n+1$ for every $n \geq 0$. Let W be a locally free $\mathcal{O}_S$-module of constant rank $r+1$. Let

$$\gamma : W \to f_* L$$

be an $\mathcal{O}_S$-linear map. By composing $f^*\gamma$ with ν^n and $\psi^n(L)$ we obtain a map

$$v^n : f^* W \to Q^n(L)$$

for each $n \geq 0$, such that

$$q^n(L) \circ v^n = v^{n-1} \tag{7.3}$$

for every $n > 0$, because of (7.2) and the commutativity of (7.1). Expression (7.3) shows that $(f^*W, Q^n(L), q^n(L), v^n, n \geq 0)$ is a Wronski system, as defined in [9] or [10]. For convenience, we will use the following shorter notation,

$$\mathcal{W}_{X/S}(L, \gamma) := (f^*W, Q^n(L), q^n(L), v^n, n \geq 0),$$

or simply $\mathcal{W}(L, \gamma)$, when no confusion is likely.

Associated to a Wronski system we have the concepts of a gap sequence at a point of X and of a Weierstrass point, for which the reader may consult [10], Section 2. The *order sequence* at a point of X is the gap sequence shifted by -1.

Laksov and Thorup have shown how to associate to a Wronski system certain maps, called Wronskians, whose zero schemes consist of Weierstrass points of the Wronski system. These maps are presented below. Let $n_0, n_1, \ldots$ be the sequence of integers defined inductively by

$$\mathrm{rk} v^i := n_0 + \ldots + n_i$$

for $i = 0, 1, \ldots$, where $\mathrm{rk} v^i$ denotes the rank of the map of vector bundles v^i. Note that either n_i is 0 or 1 for all $i \geq 0$ (*see* [9] , 1.4, p. 134.) Denote by $\epsilon_0, \epsilon_1, \ldots$ the increasing sequence of indices ϵ for which $n_\epsilon = 1$, or in other words: the *generic order sequence* of $\mathcal{W}(L, \gamma)$. Then, for every non-negative integer h there is a canonical homomorphism

$$w_h : f^* \bigwedge^{r_h+1} W \to L^{r_h+1} \otimes \omega^{\epsilon_0 + \ldots + \epsilon_{r_h}}, \tag{7.4}$$

where $\epsilon_0, \ldots, \epsilon_{r_h}$ is the increasing sequence of generic orders ϵ less than or equal to h (*see* [9], 1.5.2, p. 135.) The map w_h is called the *Wronskian* of rank $r_h + 1$ of the Wronski system $\mathcal{W}(L, \gamma)$. The homomorphism w_h is in fact defined by r_h rather than by h.

The importance of w_h is that its zero scheme Z_h parametrizes Weierstrass points of $\mathcal{W}(L, \gamma)$ (*see* [9], 1.7, p. 136.) In fact, the Z_h form an increasing sequence of closed subsets of X,

$$Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset X,$$

which becomes eventually stationary, and the set of Weierstrass points of the Wronski system is the union of the Z_h (*see* [10], 3.4.)

If $U \subset X$ is an open subscheme, then the restriction of the system $\mathcal{W}_{X/S}(L, \gamma)$ to U is a Wronski system; more precisely, the restriction is equal to $\mathcal{W}_{U/S}(L_U, \gamma_U)$, where L_U is the restriction of L to U and γ_U is the composition of γ with the push-forward by f of the canonical map $L \to L_U$ on X. If U contains all the associated points of X, then the rank of the restriction v_U^n of v^n to U is equal to the rank of v^n for each $n \geq 0$. Therefore, the Wronskians of $\mathcal{W}_{U/S}(L_U, \gamma_U)$ are the restrictions to U of the Wronskians of $\mathcal{W}_{X/S}(L, \gamma)$ (*see* [9], 1.5, p. 135.) In particular, if we let $U := X^{sm}$, where X^{sm} is the S-smooth locus of X, then the restrictions of the Wronskians of $\mathcal{W}_{X/S}(L, \gamma)$ to X^{sm} are equal to the Wronskians of $\mathcal{W}_{X^{sm}/S}(L_{X^{sm}}, \gamma_{X^{sm}})$. In addition, by Lemma 2.7 the maps w_h are determined by their restrictions to X^{sm}, what allows us to compare the maps w_h and the Wronskians obtained in the previous literature.

Assume from now on that γ is injective. If X/S is smooth, then Laksov and Thorup have shown that the map v^i is injective for sufficiently large i (*see* [9], 4.6, p. 146.) Their result carries over immediately to the case where the fibres of X/S are geometrically integral. Moreover, since v^i is injective if and only if $v^i(\xi)$ is injective for every associated point ξ of X, then we can even easily claim the following proposition.

PROPOSITION 7.5. – *If the fibres of X/S over associated points of S are geometrically integral, then v^i is injective for sufficiently large i.* □

It is not true in general that v^i is injective for i sufficiently large. The reason is that although γ may be injective for a reducible curve X over a field, there might be relations of linear dependence among the sections of L in W when restricted to an irreducible component of X. In more geometrical terms, the rational map $X \to \mathbf{P}(W)$ defined by (L, γ) will map X to a non-degenerate curve in $\mathbf{P}(W)$ if γ is injective, but may map some of the components of X into proper subspaces of $\mathbf{P}(W)$. Easy examples of the non-injectivity of v^i for all i can thus be found by considering non-degenerate reducible curves in projective space with some degenerate components, together with the linear system given by the hyperplane sections. Actually, this is in fact the only way that v^i may fail to be injective for sufficiently large i (Proposition 7.6.).

If v^i is injective for sufficiently large i, then there must be $r+1$ generic gaps, $\epsilon_0, \ldots, \epsilon_r$. In this case, we can consider the Wronskian of rank $r+1$,

$$w := w_h : f^* \bigwedge^{r+1} W \to L^{r+1} \otimes \omega^{\epsilon_0 + \ldots + \epsilon_r},$$

for h sufficiently large. The map w will be called simply the *Wronskian* of (L, γ). As remarked above, the Wronskian is in fact determined by its restriction to the smooth locus of X/S. Hence, in the case $S = \operatorname{Spec} k$, where k is an algebraically closed field, and X is an irreducible curve, the Wronskian obtained above coincides with the one defined by Lax and Widland in characteristic zero [12], or by Garcia and Lax in arbitrary characteristic [4]. The zero scheme Z of the Wronskian w is called the *Weierstrass subscheme* of X associated to (L, γ).

The most natural question we are left with is whether Z is a Cartier divisor on X. Since Z is the zero scheme of w, then Z is a Cartier divisor if and only if w is not zero at the associated points of X. Equivalently, Z is a Cartier divisor if and only if Z does not contain any irreducible component of any fibre of X/S over an associated point of S. Moreover, in order for Z to be a relative Cartier divisor over S, then we need to impose the above condition on every geometric fibre of X/S. More precisely, Z is a relative Cartier divisor if and only if Z does not contain any irreducible component of any fibre of X/S.

For each $s \in S$, let $\gamma'(s)$ be the composition

$$\gamma'(s) : W(s) \xrightarrow{\gamma(s)} (f_* L)(s) \to H^0(X(s), L(s)),$$

where the second homomorphism is given by base change. Laksov and Thorup ([9], 4.7, p. 147) called (L, γ) a linear system on X/S if $\gamma'(s)$ is injective for each $s \in S$. In our more general situation we have to modify their definition to take into account all the irreducible components of the fibres, in order to prevent a situation like the one described after Proposition 7.5.

Let $s \in S$, and let $Y \subset X(s)$ be an irreducible component. Let

$$\gamma'(s)_Y : W(s) \xrightarrow{\gamma'(s)} H^0(X(s), L(s)) \to H^0(Y, L(s)_Y),$$

where the second homomorphism is given by restriction to Y. The pair (L,γ) is called a *linear system* on X/S if $\gamma'(s)_Y$ is injective at each point $s \in S$, for each irreducible component Y of $X(s)$.

PROPOSITION 7.6. – *If $\gamma'(s)_Y$ is injective at a point $s \in S$, for each irreducible component $Y \subset X(s)$, then $v^i(s)$ is injective for sufficiently large i.*

Proof. – The map $v^i(s)$ is injective if $v^i(\xi)$ is injective for all generic points $\xi \in X(s)$. Therefore, the proof of [9], 4.5, p. 145 may be easily adapted to yield the proof of the proposition, since the stronger hypothesis that $\gamma'(s)_Y$ be injective for every irreducible component Y of $X(s)$ takes care of all generic points of $X(s)$. The proof is complete. □

COROLLARY 7.7. – *If $\gamma'(s)_Y$ is injective at every associated point s of S, for each irreducible component $Y \subset X(s)$, then v^i is injective for sufficiently large i. In particular, the Weierstrass subscheme Z of X associated to (L,γ) is defined. If in addition the characteristic of the residue field $k(s)$ is 0 for all associated points $s \in S$, then Z is a Cartier divisor.*

Proof. – The injectivity of v^i may be checked at the associated points ξ of X, which are the generic points of the fibres $X(s)$ over associated points $s \in S$. Hence, the proof of the first statement follows directly from Proposition 7.6. As for the last statement, since v^i is injective for i sufficiently large, then the number of gaps at every associated point ξ of X is $r+1$. Since $\mathrm{char}.\ k(s) = 0$ for the point $s \in S$ lying under ξ, then the sequence of orders at ξ is classical, that is, the sequence is $0, 1, \ldots, r$ (*see* [9], 4.5, p. 145). This sequence must also be the generic order sequence. Hence, ξ is not contained in Z. In other words, the subscheme Z is a Cartier divisor. □

PROPOSITION 7.8. – *If (L,γ) is a linear system on X/S and the characteristic of $k(s)$ is 0 for all $s \in S$, then the Weierstrass subscheme Z is a relative Cartier divisor.*

Proof. – Since (L,γ) is a linear system on X/S, then we can apply Proposition 7.6 to all fibres of X/S. The proof is then analogous to the one given for Corollary 7.7. □

The hypothesis on the characteristics of the residue fields of points in S is necessary, even when X/S is a smooth family. The reason is that the sequence of generic orders of a non-singular curve in positive characteristic need not be classical, as several examples in the literature show. Hence, it might be the case that the general fibre of a smooth family is classical but a special fibre is not. In this case the whole special fibre is contained in Z. In order to obtain a Cartier divisor (resp. a relative Cartier divisor), one must thus impose conditions on the sequence of orders at every generic point of every fibre of X/S over an associated point of S (resp. of every fibre of X/S.)

PROPOSITION 7.9. – *If the sequence of orders of (L,γ) at all generic points of all fibres of X/S (resp. of all the fibres of X/S over the associated points of S) are equal, then the associated Weierstrass subscheme of X is a relative Cartier divisor (resp. a Cartier divisor.)*

Proof. – As in Corollary 7.7. □

REFERENCES

[1] D. Buchsbaum and D. Eisenbud, *What makes a complex exact?* (*J. Algebra*, Vol. 25, 1973, pp. 259-268).

[2] D. Eisenbud and J. Harris, *Limit linear series: basic theory* (*Inventiones Math.*, Vol. 85, 1986, pp. 337-371).

[3] E. Esteves, *The presentation functor and Weierstrass theory for families of local complete intersection curves* (*M.I.T. Ph. D. Thesis*, 1994).

[4] A. Garcia and R. Lax, *Weierstrass points on Gorenstein curves in arbitrary characteristic* (*Preprint*).

[5] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique* (*Publ. Math. I.H.E.S.*, Vol. 24, 28, 32, 1965-67).

[6] S. Kleiman, *Relative duality for quasi-coherent sheaves* (*Compositio Mathematica*, Vol. 41, 1980, pp. 39-60).

[7] D. Laksov, *Weierstrass points on curves* (*Astérisque*, Vol. 87, 1981, pp. 221-247).

[8] D. Laksov, *Wronskians and Plücker formulas for linear systems on curves* (*Ann. Sci. École Norm. Sup.*, Vol. 17, 1984, pp. 45-66).

[9] D. Laksov and A. Thorup, *The Brill-Segre formula for families of curves* (*Contemporary Mathematics*, Vol. 123, 1991, pp. 131-148).

[10] D. Laksov and A. Thorup, *Weierstrass points and gap sequences for families of curves* (To appear in *Ark. Math.*).

[11] S. Lang, *Introduction to Arakelov theory* (Springer-Verlag, 1988).

[12] R. Lax, *On the distribution of Weierstrass points on singular curves* (*Israel J. Math.*, Vol. 57, 1987, pp. 107-115).

[13] R. Lax, *Weierstrass weights and degenerations* (*Proc. Amer. Math. Soc.*, Vol. 101, 1987, pp. 8-10).

[14] H. Matsumura, *Commutative ring theory* (*Cambridge studies in advanced mathematics*, Vol. 8, 1986).

[15] B. Matzat, *Ein Vortrag über Weierstrasspunkte* (*Karlsruhe*, 1975).

[16] D. G. Northcott, *Some remarks on the theory of ideals defined by matrices* (*Q. Jl. Math. Oxford*, (2), Vol. 14, 1963, pp. 193-204).

[17] F. K. Schmidt, *Die Wronskische Determinante in beliebigen differenzierbaren Funktionenkörpern* (*Math. Z.*, Vol. 45, 1939, pp. 62-74).

[18] F. K. Schmidt, *Zur arithmetischen Theorie der algebraischen Funktionen II* (*Math. Z.*, Vol. 45, 1939, pp. 75-96).

[19] J.-P. Serre, *Groupes algébriques et corps de classes* (Hermann, Paris, 1959).

[20] C. Widland, *On Weierstrass points of Gorenstein curves* (*Louisiana State University Ph. D. Thesis*, 1984).

(Manuscript received September 5, 1994.)

E. Esteves
Dept. of Mathematics,
Waseda University,
3-4-1 Ohkubo,
Shinjuku-ku,
Tokyo 169, Japan.
New address:
I.M.P.A.
Estrada Dona Castorina, 110
22460-Rio de Janeiro-RJ
Brazil.