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CONNECTIONS WITH PRESCRIBED CURVATURE AND YANG-MILLS CURRENTS: THE SEMI-SIMPLE CASE

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In this paper, we prove the local existence of real-analytic solutions of two closely related systems of real-analytic non-linear partial differential equations arising from geometry, in which the unknown is a connection in a principal bundle whose structure group is semi-simple. The main features of these two systems and of our ensuing discussion of them are extremely similar, and at times surprisingly so.

Let P be a principal bundle over a manifold M whose structure group G is semi-simple. The first problem we consider is to prescribe the curvature form of a connection on P, when M is three-dimensional. The second one is to solve the inhomogeneous Yang-Mills equation for a connection on P, when M is a Riemannian manifold of dimension $\geqq 3$.

We prove that, if F is an analytic 2-form on M with values in the Lie algebra of G, whose 1-jet at $x \in M$ satisfies a certain genericity condition, then there exists a connection on P whose curvature is determined by F on a neighborhood of x (Theorem 4.2). For the inhomogeneous Yang-Mills equation, our existence result may be expressed in a similar manner (Theorem 5.2).

Each of the problems is naturally cast as a system of partial differential equations with a connection as an unknown. The equations are difficult to solve because they are highly degenerate: every cotangent direction at every point of M is characteristic. This degeneracy stems from their equivariance under the infinite-dimensional pseudogroup of local gauge transformations of the bundle and changes of coordinates in the base manifold. Upon taking this invariance into account, we obtain identities which the (unknown) connection must satisfy in order that it be a solution of the original problem, and which involve the right-hand side and its covariant derivatives. From the point of view of power-series solutions, the new identities can be interpreted as obstructions to

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prolonging a formal solution of order k to one of order $k+1$. In general, for a system of non-linear partial differential equations, if one adds to the original system all of the equations which arise as obstructions to extending a formal solution of finite order to one of higher order, one might obtain a new system with no solutions. This is indeed the case for the equation for prescribed curvature when the dimension of M is >3 (*see* [10]). It is quite remarkable that for the dimension-three prescribed-curvature equation, and for the Yang-Mills equation, that one can actually compute these obstructions explicitly and determine their precise nature.

For both our systems, as in other situations in which one precribes a curvature tensor (*see* [2]), the Bianchi identity, an equation of order zero, is the obstruction to prolonging a formal solution of order k to one of order $k+1$. In contrast to other problems, where the solvability of the Bianchi identity is automatic, here there are obstructions to prolonging formal solutions of this equation. Indeed, the semi-simplicity of the group G leads to a set of further identities which the connection must satisfy. Each homogeneous invariant polynomial of positive degree on the Lie algebra $\mathfrak{g}$ of G gives us such an equation. We extract a complete set of these identities composed of r equations, where r is the rank of $\mathfrak{g}$; the nature of this set depends in a delicate way on the structure of the Lie algebra $\mathfrak{g}$. Using work of Kostant ([7], [8]) and Raïs on semi-simple Lie algebras, we are able to explain why these identities occur and why this complete set should provide us with all the obstructions to solvability. We then consider the system consisting of the original equations and the Bianchi identity, together with the r equations of such a complete set of identities. The task of extending a formal solution of order k of this new system to one of order $k+1$ reduces to a problem in linear algebra, which we solve explicitly using results from the structure theory of semi-simple Lie algebras. Thus we do not have to rely on the general theorems on the existence of formal solutions of [5], and are able to prove existence directly. The method of majorants then leads to the convergence of power-series solutions (*see* [9]).

We point out that, previous to our study, for $G=SL(2)$ or $SL(3)$, local existence of solutions for the prescribed-curvature equation in dimension 3, had already been proved in [10], using the Cartan-Kähler theory of [5] (*see* also [3]). R. Bryant has also obtained similar results for the group $SL(2)$, while S. Tsarev [11] has outlined another approach to this case.

This paper consists of two parts, which we now proceed to describe. The first one, consisting of Sections 1 and 2, is devoted to the algebraic results about semi-simple Lie algebras which we require in solving our non-linear equations. We rely greatly on the work of Kostant ([7], [8]) on complex semi-simple Lie algebras. Let $\mathfrak{g}$ be a real or complex semi-simple Lie algebra of rank r, and let $I(\mathfrak{g})$ denote the algebra of invariant polynomials on $\mathfrak{g}$. According to Chevalley's theorem, we may choose a set $\{p_1, \ldots, p_r\}$ of homogeneous generators of $I(\mathfrak{g})$, which are algebraically independent. The degrees of these polynomials depend only on $\mathfrak{g}$ and are $\geqq 2$. The differential $(dp_j)(X)$ of the polynomial p_j at $X \in \mathfrak{g}$ is a linear form on $\mathfrak{g}$ which we identify with an element of $\mathfrak{g}$, via the Killing form of $\mathfrak{g}$. An element of $\mathfrak{g}$ is said to be principal if the dimension of its centralizer is equal to the rank of $\mathfrak{g}$. The principal elements of $\mathfrak{g}$ form a non-empty

Zariski-open subset of $\mathfrak{g}$. The first result of Kostant, which is used constantly throughout Sections 1 and 2, is his criterion for recognizing when an element of $\mathfrak{g}$ is principal: an element X of $\mathfrak{g}$ is principal if and only if the r elements $(dp_1)(X), \ldots, (dp_r)(X)$ of $\mathfrak{g}$ are linearly independent (*see* [8]). An immediate consequence of this criterion is the following characterization of the image of ad $X : \mathfrak{g} \to \mathfrak{g}$, when X is principal: an element of $\mathfrak{g}$ belongs to this image if and only if it is orthogonal to the r vectors $(dp_1)(X), \ldots, (dp_r)(X)$ (*see* Theorem 1.1). The other result of Kostant which we need is his explicit construction, given in [7], of a set $\{H, X, Y\}$ of principal elements of a complex Lie algebra satisfying the commutation relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H,$$

with X, Y nilpotent and H regular and semi-simple; such a set of elements of a complex Lie algebra is called a principal S-triple. In Section 1, we use these two results together with work of Raïs to prove that certain Zariski-open subsets of various powers of $\mathfrak{g}$, defined in terms of the polynomials $\{p_1, \ldots, p_r\}$, are non-empty.

In Section 2, we develop a certain amount of linear algebra over the semi-simple Lie algebra $\mathfrak{g}$, which sould be of some independent interest. We solve various overdetermined systems of linear equations whose unknowns and coefficients are elements of the Lie algebra $\mathfrak{g}$. Just as Kostant's theorem, for a generic element X of $\mathfrak{g}$, characterizes the image of ad $X : \mathfrak{g} \to \mathfrak{g}$ in terms of the invariant polynomials $\{p_1, \ldots, p_r\}$, we are able to express the compatibility conditions for these systems by means of these polynomials under explicit genericity assumptions on the coefficients—certain of these should belong to one of the open subsets considered in Section 1. As consequences of the solvability of these systems, we obtain various results concerning Spencer cohomology, which are not needed in this paper, but which are interesting in their own right. They provide us with examples of subspaces whose Spencer cohomology can be explicitly computed and which is entirely concentrated in degree 2.

After reviewing in Section 3 the necessary material about connections in a principal bundle, we prove in Section 4 the existence of formal solutions for the equation of prescribed curvature in dimension 3. Each homogeneous polynomial of $I(\mathfrak{g})$ of degree $d+1$, with $d \geqq 1$, gives us a new scalar-valued equation of order $d-1$, obtained by repeatedly differentiating the Bianchi identity d-times and taking into account the original system. Let F be a $\mathfrak{g}$-valued 2-form on M, whose 1-jet at $x \in M$ satisfies a genericity condition expressed in terms of $\{p_1, \ldots, p_r\}$ and of one of the open subsets of Section 1. We seek a connection whose curvature form is F; then over an open neighborhood of x, the identities corresponding to the polynomials $\{p_1, \ldots, p_r\}$ form a complete set. Under these assumptions on F, we construct formal solutions of the system consisting of the original equation, the Bianchi identity and these r scalar-valued equations. The sub-system of order zero consisting of the Bianchi identity and the identities corresponding to the polynomials of degree 2 of the set $\{p_1, \ldots, p_r\}$ is highly non-trivial. In fact, the number of these polynomials of degree 2 is equal to the number of factors in a decomposition of the complexification of $\mathfrak{g}$ into minimal ideals, and is therefore always $\geqq 1$. We first show that this non-linear sub-system of order zero admits solutions at x

by solving a system of linear equations. Next, we extend any formal solution of order k of our completed system to one of order $k+1$ by solving one of the systems of linear equations of Section 2. Because we have included the identities corresponding to the set $\{p_1, \ldots, p_r\}$, the requisite compatibility condition for this linear system is satisfied.

Section 5 is devoted to the existence of formal solutions of the homogeneous and the inhomogeneous Yang-Mills equations. The proof for the inhomogeneous Yang-Mills equation follows the same lines at that of Section 4 for the equation of prescribed curvature. However, in this case a homogeneous polynomial of $I(\mathfrak{g})$ of degree d, with $d \geqq 2$, gives us a new identity of order d obtained by differentiating the Bianchi identity $(d+1)$-times. Thus the Bianchi identity is the only equation of order zero which needs to be added to the original system. It is quite remarkable that the linear system of Section 2, which we need to solve in order to extend a solution of order k to one of order $k+1$, is so closely related to the one which we consider in Section 4 for the analogous problem.

The genericity conditions imposed on the right-side of our equations are described in the remarks preceding Propositions 4.1 and 5.1. We wish to point out that the exactness of the sequences (2.8) or (2.18) corresponds to the completeness of the set of identities derived from $\{p_1, \ldots, p_r\}$; on the other hand, we use the exactness of the sequences of Corollary 2.3 or Corollary 2.5 to show that no further identities need be added to our systems. Relation (2.30) provides us with an unexpected link between our two problems and the systems of linear equations associated to them in Section 2.

The main substance of our existence proofs is to be found in Sections 4 and 5. We strongly recommend that the reader start directly with Sections 3, 4 and 5, referring back to the first two sections for the appropriate definitions and results whenever necessary.

We would like to thank H. Jacquet for several helpful discussions and M. Raïs for making us aware of the importance of principal S-triples in the theory of complex semi-simple Lie algebras and for kindly communicating to us the proof of Theorem 1.2.

1. Invariant polynomials on semi-simple Lie algebras

Let K be the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. If V is a real vector space, we write V_c for its complexification. Let $\mathfrak{g}$ be a semi-simple Lie algebra over K of rank r. The dimension of the centralizer $\mathfrak{g}_X$ of an element X of $\mathfrak{g}$ is $\geqq r$. We say that $X \in \mathfrak{g}$ is principal if $\dim \mathfrak{g}_X = r$; the set of all principal elements of $\mathfrak{g}$ is a non-empty Zariski-open subset of $\mathfrak{g}$, containing the regular semi-simple elements of $\mathfrak{g}$.

We now suppose that $\mathfrak{g}$ is a real semi-simple Lie algebra. We denote by $S^k \mathfrak{g}_c^*$ the k-th symmetric power of $\mathfrak{g}_c^*$. If $u \in \mathfrak{g}_c$, we denote by u^k the element of $\mathfrak{g}_c^k$ all of whose coordinates are equal to u. The rank of $\mathfrak{g}$ is equal to the rank of the complex Lie algebra $\mathfrak{g}_c$. Thus $X \in \mathfrak{g}$ is a principal element of $\mathfrak{g}$ if and only if it is a principal element of $\mathfrak{g}_c$. Let G be any connected Lie group with Lie algebra $\mathfrak{g}$. Let $I(\mathfrak{g}_c)$ [resp. $I(\mathfrak{g})$] be the algebra of all complex (resp. real) polynomials on $\mathfrak{g}_c$ (resp. $\mathfrak{g}$) invariant under G. Note that $I(\mathfrak{g}_c)$ and $I(\mathfrak{g})$ depend only on $\mathrm{Ad}(G)$ and that $I(\mathfrak{g}_c) \cong I(\mathfrak{g})_c$.

Let p be a homogeneous element of $\mathrm{I}(\mathfrak{g}_c)$ of degree k; to p corresponds a unique element of $\mathrm{S}^k \mathfrak{g}_c^*$ invariant under G, the polarization of p, which we shall also denote by p and consider as a function on $\mathfrak{g}_c^k$. Let B be the Killing form of $\mathfrak{g}_c$. If $k \geqq 1$, we associate to p the unique element $\tilde{p}$ of $\mathrm{S}^{k-1} \mathfrak{g}_c^* \otimes_{\mathbb{C}} \mathfrak{g}_c$ determined by the equality

$$p(u_1, \ldots, u_{k-1}, v) = \mathrm{B}(\tilde{p}(u_1, \ldots, u_{k-1}), v),$$

for $u_1, \ldots, u_{k-1}, v \in \mathfrak{g}_c$. If the restriction of the polynomial p to $\mathfrak{g}$ is real-valued, then $\tilde{p}(u_1, \ldots, u_{k-1})$ belongs to $\mathfrak{g}$, for all $u_1, \ldots, u_{k-1} \in \mathfrak{g}$.

According to Chevalley's theorem, we may choose r algebraically independent homogeneous polynomials $p_1, \ldots, p_r$ of $\mathrm{I}(\mathfrak{g}_c)$ which, together with 1, generate $\mathrm{I}(\mathfrak{g}_c)$. We may suppose that the p_a are real-valued on $\mathfrak{g}$; then $p_1, \ldots, p_r$, together with 1, generate $\mathrm{I}(\mathfrak{g})$. If $\deg p_a = d_a + 1$, the integers d_a depend only on $\mathfrak{g}$ and are $\geqq 1$ (*see* Varadarajan [13], Theorem 4.9.3 and p. 410).

For $\mathrm{X} \in \mathfrak{g}_c$, let

$$\pi_{\mathrm{X}} : \mathfrak{g}_c \to \mathbb{C}^r$$

be the mapping sending u into

$$(p_1(\mathrm{X}^{d_1}, u), \ldots, p_r(\mathrm{X}^{d_r}, u)).$$

The following result is due to Kostant [8], Theorem 9 (*see* also Varadarajan [12], Theorem 3).

THEOREM 1.1. — *Let* X *be an element of* $\mathfrak{g}_c$. *The complex*

$$\mathfrak{g}_c \xrightarrow{\mathrm{ad\,X}} \mathfrak{g}_c \xrightarrow{\pi_{\mathrm{X}}} \mathbb{C}^r \to 0$$

is exact if and only if X *is principal.*

Consequently, if $\mathrm{X} \in \mathfrak{g}$, the sequence

$$\mathfrak{g} \xrightarrow{\mathrm{ad\,X}} \mathfrak{g} \xrightarrow{\pi_{\mathrm{X}}} \mathbb{R}^r \to 0$$

is exact if and only if X is principal. The preceding theorem asserts that $\mathrm{X} \in \mathfrak{g}$ (resp. $\mathfrak{g}_c$) is principal if and only if the r elements $\{\tilde{p}_a(\mathrm{X}^{d_a})\}_{1 \leqq a \leqq r}$ of $\mathfrak{g}$ (resp. $\mathfrak{g}_c$) are linearly independent and constitute a basis for $\mathfrak{g}_{\mathrm{X}}$ (resp. $\mathfrak{g}_{c,\mathrm{X}}$). We set

$$q = r + \sum_{a=1}^{r} d_a, .$$

$$\mathrm{J} = \{(a, k) \mid 1 \leqq a \leqq r,\ 0 \leqq k \leqq d_a\}.$$

Let U_1 (resp. U_1') be the set of all $(\mathrm{X}, \mathrm{Y}) \in \mathfrak{g} \times \mathfrak{g}$ (resp. $\mathfrak{g}_c \times \mathfrak{g}_c$) for which the q elements

$$\{\tilde{p}_a(\mathrm{X}^k, \mathrm{Y}^{d_a - k})\}_{(a,\,k) \in \mathrm{J}}$$

of $\mathfrak{g}$ (resp. $\mathfrak{g}_c$) are linearly independent. Thus $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ belongs to U_1 if and only if the q elements

$$u \mapsto p_a(X^k, Y^{d_a-k}, u), \qquad u \in \mathfrak{g},$$

of $\mathfrak{g}^*$, with $(a, k) \in J$, are linearly independent. Clearly, by Theorem 1.1, if $(X, Y) \in U_1$, then X and Y are principal. The proof of the following theorem has been kindly been communicated to us by M. Raïs.

THEOREM 1.2. — *The set* U_1 *is a non-empty Zariski-open subset of* $\mathfrak{g} \times \mathfrak{g}$.

Let p be a homogeneous element of $I(\mathfrak{g}_c)$ of degree $d+1$, with $d \geqq 1$. From the invariance of p, we infer that

$$[X, \tilde{p}(X^k, Y^{d-k})] = (d-k)\, \tilde{p}(X^k, [X, Y], Y^{d-k-1}), \tag{1.1}$$

for $X, Y \in \mathfrak{g}_c$ and $0 \leqq k \leqq d$. Let X, Y be elements of $\mathfrak{g}_c$ satisfying

$$[Y, X] = \lambda X,$$

where $\lambda \in \mathbb{C}$, and consider the elements

$$v_{p,k} = \frac{1}{(d-k)!}\, \tilde{p}(X^k, Y^{d-k})$$

of $\mathfrak{g}_c$, for $0 \leqq k \leqq d$. We set $v_{p,k} = 0$, for $k > d$. From (1.1), we deduce that

$$[Y, v_{p,k}] = k\lambda\, v_{p,k}, \quad [X, v_{p,k}] = -\lambda\, v_{p,k+1}. \tag{1.2}$$

We set $v_{a,k} = v_{p_a,k}$, for $1 \leqq a \leqq r$. The following proposition, due to Raïs, is the crucial ingredient of the proof of Theorem 1.2.

PROPOSITION 1.1. — *If* X *is principal and* $\lambda \neq 0$, *the* q *elements*

$$\{v_{a,k}\}_{(a,k) \in J}$$

of $\mathfrak{g}_c$ *are linearly independent.*

Proof. — Suppose that we have a linear relation

$$\sum_{(a,l) \in J} b_{a,l}\, v_{a,l} = 0,$$

where the coefficients $b_{a,l} \in \mathbb{C}$ are not all zero. Let d be the largest integer for which there is a non-zero coefficient $b_{a,l}$, with $d_a = d$. Now let k be the smallest integer, with $0 \leqq k \leqq d$, for which there is a non-zero coefficient $b_{a,k}$, with $d_a = d$. Then $b_{i,l} = 0$ if $d_i > d$, or if $d_i = d$ and $l < k$. By (1.2), we have

$$0 = (\operatorname{ad} X)^{d-k} . \sum_{(a,l) \in J} b_{a,l}\, v_{a,l} = (-\lambda)^{d-k} \sum_{(a,l) \in J} b_{a,l}\, v_{a,l+d-k} = (-\lambda)^{d-k} (u_1 + u_2),$$

where

$$u_1 = \sum_{d_a = d} b_{a,k} v_{a,d}, \qquad u_2 = \sum_{\substack{d_a < d \\ 0 \leq l+d-k \leq d_a}} b_{a,l} v_{a,l+d-k}.$$

According to (1.2), we see that u_1 is an eigenvector of ad Y with eigenvalue $d\lambda$, while u_2 is a linear combination of eigenvectors of ad Y whose eigenvalues are different from $d\lambda$. It follows that

$$u_1 = \sum_{d_a = d} b_{a,k} \tilde{p}_a(X^d) = 0.$$

Since X is principal, according to Theorem 1.1, the elements $\{\tilde{p}_a(X^d)\}$ of $\mathfrak{g}_c$, with $d_a = d$, are linearly independent. Hence $b_{a,k} = 0$, for all $1 \leq a \leq r$, with $d_a = d$, which is a contradiction.

Since U'_1 is a Zariski-open subset of $\mathfrak{g}_c \times \mathfrak{g}_c$ and

$$U_1 = U'_1 \cap (\mathfrak{g} \times \mathfrak{g}),$$

to prove Theorem 1.2 it suffices to show that U'_1 is non-empty. According to Kostant [7], § 5.3, $\mathfrak{g}_c$ contains a principal nilpotent element X. Then the Jacobson-Morozov theorem gives us the existence of elements H, Y of $\mathfrak{g}_c$ such that

$$(1.3) \qquad [H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

By Proposition 1.1, the pair (H, X) belongs to U'_1.

Any set $\{H, X, Y\}$ of elements of $\mathfrak{g}_c$ satisfying the commutation relations (1.3), with X principal, is called a principal S-triple. The element H of such an S-triple is regular and semi-simple, and the element Y is also principal (*see* Kostant [7], § 5.2, 5.3). The following result is a consequence of Proposition 1.1.

PROPOSITION 1.2. — *If* $\{H, X, Y\}$ *is a principal* S-*triple of* $\mathfrak{g}_c$, *then*

$$\{\tilde{p}_a X^k, H^{d_a - k}), \tilde{p}_a (Y^l, H^{d_a - l})\}_{\substack{1 \leq a \leq r \\ 0 \leq k, l \leq d_a \\ l > 0}}$$

is a basis for $\mathfrak{g}_c$.

Proof. — Since Y is principal, by Proposition 1.1,

$$\{\tilde{p}_a(X^k, H^{d_a - k})\}_{(a,k) \in J}, \qquad \{\tilde{p}_a(Y^l, H^{d_a - l})\}_{\substack{(a,l) \in J \\ l > 0}}$$

are two sets of linearly independent elements of $\mathfrak{g}_c$. By (1.2), the first set consists of eigenvectors of ad H whose eigenvalues are ≥ 0, while the second set is entirely composed of eigenvectors of ad H with negative eigenvalues. As the dimension of $\mathfrak{g}_c$ is equal to $2q - r$ (*see* [13], § 4.15), the lemma follows from the preceding remark.

Let U_2 (resp. U'_2) be the set of all $(X_1, X_2, X_3) \in \mathfrak{g}^3$ (resp. $\mathfrak{g}_c^3$), with $(X_1, X_2) \in U_1$ (resp. U'_1), for which

$$\{\tilde{p}_a(X_1^{d_a}), \tilde{p}_a(X_3^{d_a})\}_{1 \leqq a \leqq r}, \qquad \{\tilde{p}_a(X_2^{d_a}), \tilde{p}_a(X_3^{d_a})\}_{1 \leqq a \leqq r}$$

are two sets of $2r$ linearly independent elements of $\mathfrak{g}$ (resp. $\mathfrak{g}_c$). If $\{H, X, Y\}$ is a principal S-triple of $\mathfrak{g}_c$, since Y is principal, by Proposition 1.2 we see that (H, X, Y) belongs to U'_2. As U'_2 is a Zariski-open set of $\mathfrak{g}_c^3$ and $U_2 = U'_2 \cap \mathfrak{g}^3$, we obtain:

PROPOSITION 1.3. — *The set* U_2 *is a non-empty Zariski-open subset of* $\mathfrak{g}^3$.

LEMMA 1.1. — *If* $\{H, X, Y\}$ *is a principal* S-*triple of* $\mathfrak{g}_c$, *then the r elements*

$$w_a = 2(d_a - 1)\tilde{p}_a(X, Y, H^{d_a - 2}) - \tilde{p}_a(H^{d_a}), \qquad 1 \leqq a \leqq r,$$

of $\mathfrak{g}_c$ *are linearly independent.*

Proof. – Let p be a homogeneous element of $I(\mathfrak{g}_c)$ of degree $d+1$. By (1.3), we have

$$(\operatorname{ad} X)^k . \tilde{p}(X^l, H^{d-l}) = (-2)^k (d-l)(d-l-1)\ldots(d-l-k+1)\tilde{p}(X^{l+k}, H^{d-l-k}),$$

for $0 \leqq l \leqq d$ and $1 \leqq k \leqq d-l$; by the invariance of p, we also see that

$$\begin{aligned}\operatorname{ad} X . \tilde{p}(X^l, Y, H^{d-l-1}) &= \tilde{p}(X^l, [X, Y], H^{d-l-1}) + (d-l-1)\tilde{p}(X^l, Y, [X, H], H^{d-l-2}) \\ &= \tilde{p}(X^l, H^{d-l}) - 2(d-l-1)\tilde{p}(X^{l+1}, Y, H^{d-l-2}),\end{aligned}$$

for $0 \leqq l \leqq d-1$. Consider the positive integers $a_1, \ldots, a_d$ defined recursively by the equalities $a_1 = 1$ and

$$a_l = (d-l+1)a_{l-1} + (d-2)(d-3)\ldots(d-l),$$

for $2 \leqq l \leqq d$. It is easily verified that

$$\begin{aligned}&(\operatorname{ad} X)^l . \tilde{p}(X, Y, H^{d-2}) \\ &\qquad = (-2)^l (d-2)(d-3)\ldots(d-l-1)\tilde{p}(X^{l+1}, Y, H^{d-l-2}) + (-2)^{l-1} a_l \tilde{p}(X^l, H^{d-l}),\end{aligned}$$

for $1 \leqq l \leqq d-1$, and

$$(\operatorname{ad} X)^d . \tilde{p}(X, Y, H^{d-2}) = (-2)^{d-1} a_d \tilde{p}(X^d).$$

Hence, if $d \geqq 2$, we obtain

$$(\operatorname{ad} X)^d . (2(d-1)\tilde{p}(X, Y, H^{d-2}) - \tilde{p}(H^d)) = (-2)^d ((d-1)a_d + d!)\tilde{p}(X^d)$$

and

$$(\operatorname{ad} X)^{d+1} . (2(d-1)\tilde{p}(X, Y, H^{d-2}) - \tilde{p}(H^d)) = 0.$$

On the other hand, if $d=1$, we have

$$\operatorname{ad} X . \tilde{p}(H) = -2\tilde{p}(X), \qquad (\operatorname{ad} X)^2 . \tilde{p}(H) = 0.$$

Suppose that there is a linear relation

$$\sum_{a=1}^{r} b_a w_a = 0,$$

where the coefficients $b_a \in \mathbb{C}$ are not all zero. Let d be the largest integer for which there is a non-zero coefficient b_a, with $d_a = d$. Then $b_i = 0$ if $d_i > d$. From the above relations, we infer that

$$0 = (\operatorname{ad} X)^d . \sum_{a=1}^{r} b_a w_a = \sum_{d_a = d} b_a (\operatorname{ad} X)^d . w_a = -(-2)^d ((d-1) a_d + d!) \sum_{d_a = d} b_a \tilde{p}_a (X^d).$$

Since X is principal, by Theorem 1.1, the elements $\tilde{p}_a(X^{d_a})$ of $\mathfrak{g}_c$ are linearly independent. As

$$(d-1) a_d + d! \neq 0,$$

the b_a, with $d_a = d$, are necessarily all zero, which is a contradiction.

Let n be an integer $\geqq 2$. If $X_1, \ldots, X_n, A_1, \ldots, A_n$ are elements of $\mathfrak{g}_c$, we set

$$Z_a = \sum_{j=1}^{n} [A_j, \tilde{p}_a (X_1^{d_a - 1}, X_j)],$$

for $1 \leqq a \leqq r$, and consider the mapping

$$\lambda : \mathfrak{g}_c \to \mathfrak{g}_c \oplus \mathbb{C}^r,$$

sending $u \in \mathfrak{g}_c$ into

$$\lambda(u) = ([X_1, u], B(Z_1, u), \ldots, B(Z_r, u)).$$

Let V be the set of all $(X_1, \ldots, X_n, A_1, \ldots, A_n) \in \mathfrak{g}_c^{2n}$ for which the mapping λ is injective. We consider the following Zariski-open subsets of $\mathfrak{g}_c^{2n}$:

$$U'_3 = \{ (X_1, \ldots, X_n, A_1, \ldots, A_n) \in V \mid (X_1, X_2) \in U'_1 \},$$

and, when $n \geqq 3$,

$$U'_4 = \{ (X_1, \ldots, X_n, A_1, \ldots, A_n) \in V \mid (X_1, X_2, X_3) \in U'_2 \}.$$

The following proposition implies that U'_3 and U'_4 are non-empty.

PROPOSITION 1.4. — *Let* $\{H, X, Y\}$ *be a principal* S-*triple of* $\mathfrak{g}_c$, *and let* $X_3, \ldots, X_n$ *be arbitrary elements of* $\mathfrak{g}_c$. *Then:*

(i) *the element* $(H, X, X_3, \ldots, X_n, 0, Y, 0, \ldots, 0)$ *of* $\mathfrak{g}_c^{2n}$ *belongs to* U'_3;

(ii) *if* $n \geqq 3$, *the element* $(H, X, Y, X_4, \ldots, X_n, 0, Y, 0, \ldots, 0)$ *of* $\mathfrak{g}_c^{2n}$ *belongs to* U'_4.

Proof. – If $X_1 = H$, $X_2 = X$ and $A_j = 0$, for $j > 2$, then $Z_a = [Y, \tilde{p}_a(H^{d_a - 1}, X)]$, for $1 \leqq a \leqq r$. By the invariance of p_a, we see that Z_a is equal to the element w_a associated to $\{H, X, Y\}$ in Lemma 1.1. It is easily verified that w_a belongs to the centralizer $\mathfrak{h}$ of H in $\mathfrak{g}_c$. Since H is regular and semi-simple, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_c$ and the Killing form B is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$. From Lemma 1.1, it follows that λ is injective. The assertions of the proposition are now immediate consequences of Proposition 1.2.

We set $U_3 = U'_3 \cap \mathfrak{g}^{2n}$, and $U_4 = U'_4 \cap \mathfrak{g}^{2n}$ when $n \geqq 3$. From Proposition 1.4, we deduce the following result:

THEOREM 1.3. — *The set* U_3 (*resp.* U_4, *when* $n \geqq 3$) *is a non-empty Zariski-open subset of* $\mathfrak{g}^{2n}$.

Let $(X_1, \ldots, X_n, A_1, \ldots, A_n)$ be an element of U_3. If $B \in \mathfrak{g}$ and $\zeta \in \mathbb{R}^*$, since $\tilde{p}_a(X_1^{d_a})$ belongs to $\mathfrak{g}_{X_1}$, it is easily seen that

$$(X_1, X_2, \ldots, X_n, A_1 - [X_1, B], A_2, \ldots, A_n), \qquad (X_1, \ldots, X_n, \zeta A_1, \ldots, \zeta A_n)$$

are also elements of U_3. If p is a homogeneous element of $I(\mathfrak{g})$ of degree $d+1$, we have

$$\begin{aligned}\sum_{j=1}^{n} p(X_1^{d-1}, X_j, [A_j, u]) &= \sum_{j=1}^{n} B(\tilde{p}(X_1^{d-1}, X_j), [A_j, u]) \\ &= \sum_{j=1}^{n} B([\tilde{p}(X_1^{d-1}, X_j), A_j], u),\end{aligned}$$

for $u \in \mathfrak{g}$. If $\pi'_{X_1} : \mathfrak{g} \oplus \mathbb{R}^r \to \mathbb{R}^r$ is the mapping sending (u, z) into $\pi_{X_1}(u)$, for $u \in \mathfrak{g}$, $z \in \mathbb{R}^r$, according to Theorem 1.1, the sequence

$$0 \to \mathfrak{g} \xrightarrow{\lambda} \mathfrak{g} \oplus \mathbb{R}^r \xrightarrow{\pi'_{X_1}} \mathbb{R}^r \to 0$$

is exact; therefore, if $(c_1, \ldots, c_r)$ is an arbitrary element of $\mathbb{R}^r$, we can solve the system

$$\sum_{j=1}^{n} p_a(X_1^{d_a - 1}, X_j, [A_j, u]) = c_a, \qquad 1 \leqq a \leqq r,$$

for $u \in \mathfrak{g}_{X_1}$.

We now use Theorem 1.3 to prove the following result:

LEMMA 1.2. — *There exists a non-empty Zariski-open subset* U_5 *of* $\mathfrak{g}^{2n+1}$ *having the following properties:*

(i) *For all* $(X_1, \ldots, X_n, A_1, \ldots, A_n, B) \in U_5$, *there exists* $(B_1, \ldots, B_n) \in \mathfrak{g}^n$ *such that*

$$\sum_{j=1}^{n} [X_j, B_j] = B,$$

$$(X_1, \ldots, X_n, A_1 - [X_1, B_1], \ldots, A_n - [X_1, B_n]) \in U_3.$$

(ii) *If* $(X_1, \ldots, X_n, A_1, \ldots, A_n) \in U_3$, *then* $(X_1, \ldots, X_n, A_1, \ldots, A_n, 0)$ *belongs to* U_5.

Proof. — Set $s = q - r$; the dimension of $\mathfrak{g}$ is equal to $q + s = r + 2s$ (*see* [13], § 4.15). Choose an element $(X_1^0, \ldots, X_n^0, A_1^0, \ldots, A_n^0)$ of U_3. Fix s linearly independent elements $\{Z_1, \ldots, Z_s\}$ of $\mathfrak{g}$ which span a complement to the q-dimensional subspace of $\mathfrak{g}$ generated by the elements

$$\{\tilde{p}_a((X_1^0)^k, (X_2^0)^{d_a - k})\}_{(a,\, k) \in J}.$$

For $(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g}$, consider the mapping

$$\lambda(X_1, X_2): \mathfrak{g} \to \mathbb{R}^{q+s},$$

defined by

$$\lambda(X_1, X_2)(u) = (\pi_{X_1}(u), p_a(X_1^k, X_2^{d_a - k}, u), B(Z_1, u), \ldots, B(Z_s, u))_{\substack{(a,\, k) \in J, \\ k < d_a}}$$

for $u \in \mathfrak{g}$. When $(X_1, X_2) = (X_1^0, X_2^0)$, it is an isomorphism. Therefore by Cramer's rule, there exist polynomials λ', f' on $\mathfrak{g} \times \mathfrak{g}$ with values in $\mathrm{Hom}(\mathbb{R}^{q+s}, \mathfrak{g})$ and $\mathbb{R}$, respectively, such that

$$\lambda'(X_1, X_2) . \lambda(X_1, X_2) = f'(X_1, X_2) . \mathrm{id}_{\mathfrak{g}},$$

for $(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g}$, and $f'(X_1^0, X_2^0) \neq 0$. Let $\mathfrak{q}$ be a fixed complement to $\mathfrak{g}_{X_2^0}$ in $\mathfrak{g}$. The mapping

$$\mathrm{ad}\, X : \mathfrak{q} \to \mathfrak{g} \tag{1.4}$$

is injective when $X = X_2^0$. By Cramer's rule, there exist polynomials μ, f'' on $\mathfrak{g}$ with values in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{q})$ and $\mathbb{R}$, respectively, such that

$$\mu_X [X, u] = f''(X) u, \tag{1.5}$$

for $X \in \mathfrak{g}$, $u \in \mathfrak{q}$, and $f''(X_2^0) \neq 0$. If f'' does not vanish at $X \in \mathfrak{g}$, then the mapping (1.4) is injective. Let $\iota : \mathbb{R}^r \to \mathbb{R}^r \oplus \mathbb{R}^{2s}$ be the mapping defined by $\iota(x) = (x, 0)$, for $x \in \mathbb{R}^r$, and consider the $\mathrm{Hom}(\mathfrak{g}, \mathfrak{q})$-valued polynomial σ on $\mathfrak{g} \times \mathfrak{g}$ whose value at (X_1, X_2) is equal to the composition

$$\mathfrak{g} \xrightarrow{\pi_{X_1}} \mathbb{R}^r \xrightarrow{\iota} \mathbb{R}^{r+2s} \xrightarrow{\lambda'(X_1, X_2)} \mathfrak{g} \xrightarrow{\mu(X_2)} \mathfrak{q}.$$

We denote by f the polynomial on $\mathfrak{g}^{2n+1}$ whose value at the element

$$x=(X_1, \ldots, X_n, A_1, \ldots, A_n, B)$$

of $\mathfrak{g}^{2n+1}$ is $f(X_1, X_2)=f'(X_1, X_2).f''(X_2)$, and by $\Phi: \mathfrak{g}^{2n+1} \to \mathfrak{g}^{2n}$ the mapping sending x into

$$(X_1, \ldots, X_n, f(x)A_1, f(x)A_2-[X_2, \sigma(X_1, X_2)B], f(x)A_3, \ldots, f(x)A_n).$$

Then

$$U_5=\{x \in \mathfrak{g}^{2n+1} \mid f(x) \neq 0\} \cap \Phi^{-1}(U_3)$$

is a Zariski-open subset of $\mathfrak{g}^{2n+1}$; since $f(X_1^0, X_2^0) \neq 0$, the element

$$(X_1^0, \ldots, X_n^0, A_1^0, \ldots, A_n^0, 0)$$

of $\mathfrak{g}^{2n+1}$ belongs to U_5. It remains to show that U_5 satisfies property (i). Let $x=(X_1, \ldots, X_n, A_1, \ldots, A_n, B)$ be an element of U_5; then (X_1, X_2) belongs to U_1. Set $B_2=(1/f(x))\,\sigma(X_1, X_2)B$. Then

$$v=\frac{1}{f'(X_1, X_2)}\lambda'(X_1, X_2).\iota.\pi_{X_1}(B)$$

satisfies $\mu(X_2)v=f''(X_2)B_2$ and

$$p_a(X_1^{d_a}, v)=p_a(X_1^{d_a}, B), \qquad p_a(X_2^{d_a}, v)=0, \tag{1.6}$$

for $1 \leqq a \leqq r$. Since $f''(X_2) \neq 0$, by Theorem 1.1, the sequence

$$0 \to \mathfrak{q} \xrightarrow{\operatorname{ad} X_2} \mathfrak{g} \xrightarrow{\pi_{X_2}} \mathbb{R}^r \to 0$$

is exact. By (1.6), we know that $\pi_{X_2}(v)=0$. Therefore, from (1.5) we deduce that

$$[X_2, B_2]=v.$$

By Theorem 1.1 and (1.6), we are able to solve the equation

$$[X_1, B_1]=B-v=B-[X_2, B_2],$$

for $B_1 \in \mathfrak{g}$. Since

$$\Phi(x)=(X_1, \ldots, X_n, f(x)A_1, f(x)(A_2-[X_2, B_2]), f(x)A_3, \ldots, f(x)A_n),$$

by the remarks following Theorem 1.3, we see that

$$(X_1, \ldots, X_n, A_1 - [X_1, B_1], A_2 - [X_2, B_2], A_3, \ldots, A_n)$$

belongs to U_3. Thus the element $(B_1, B_2, 0, \ldots, 0)$ of $\mathfrak{g}^n$ has the desired properties.

If $n \geqq 3$, according to Theorem 1.3 and Lemma 1.2, the subset

$$U_6 = \{ (X_1, \ldots, X_n, A_1, \ldots, A_n, B) \in U_5 \mid (X_1, X_2, X_3) \in U_2 \}$$

of U_5 is also a non-empty Zariski-open subset of $\mathfrak{g}^{2n+1}$.

2. Linear algebra over semi-simple Lie algebras

Let n be an integer $\geqq 1$. Let $\mathfrak{g}$ be a real semi-simple Lie algebra, whose rank is equal to r. We consider the objects associated in Section 1 to the system of generators $\{p_1, \ldots, p_r\}$ of $I(\mathfrak{g})$, in particular the subsets U_1 of $\mathfrak{g} \times \mathfrak{g}$ and U_3 of $\mathfrak{g}^{2n}$, the subset U_5 of $\mathfrak{g}^{2n+1}$ given by Lemma 1.2, and, when $n \geqq 3$, the subsets

$$U_2 \subset \mathfrak{g}^3, \qquad U_4 \subset U_3, \qquad U_6 \subset U_5.$$

Let $X_1, \ldots, X_n$ be elements of $\mathfrak{g}$ and let T be a real vector space of dimension n. Choose a fixed basis $\{\partial_1, \ldots, \partial_n\}$ of T and consider the dual basis $\{dx^1, \ldots, dx^n\}$ of T^*. We denote by $\Lambda^k T^*$ and $S^k T^*$ the k-th exterior power and the k-th symmetric power of T^*; we set $S^k T^* = 0$, for $k < 0$. If $\xi \in T^*$, we write ξ^k for the k-th symmetric power of ξ, which is an element of $S^k T^*$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of length n, with $|\alpha| = \alpha_1 + \ldots + \alpha_n = k$, we set

$$X^\alpha = (X_1^{\alpha_1}, \ldots, X_n^{\alpha_n}) \in \mathfrak{g}^k,$$

$$dx^\alpha = (dx^1)^{\alpha_1} \cdot \ \ldots \ \cdot (dx^n)^{\alpha_n} \in S^k T^*.$$

For $1 \leqq i \leqq n$, we denote by ε_i the multi-index $(\delta_{i1}, \ldots, \delta_{in})$. If $u \in S^k T^*$, we may write

$$u = \sum_{|\alpha| = k} u_\alpha \frac{dx^\alpha}{\alpha!},$$

with $u_\alpha \in \mathbb{R}$; if $1 \leqq i_1, \ldots, i_k \leqq n$, then

$$u(\partial_{i_1}, \ldots, \partial_{i_n}) = u_\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and α_j is the number of the i_k's equal to j.

Let V, W be real vector spaces. If $k \geqq 1$ and $\Delta_{m,k} : S^{k+m} T^* \to S^m T^* \otimes S^k T^*$ is the natural inclusion, the m-th prolongation $(\lambda)_{+m}$ of a linear mapping

$$\lambda : S^k T^* \otimes V \to W$$

is equal to the composition

$$S^{k+m}T^*\otimes V \xrightarrow{\Delta_{m,k}\otimes \mathrm{id}} S^m T^*\otimes S^k T^*\otimes V \xrightarrow{\mathrm{id}\otimes\lambda} S^m T^*\otimes W.$$

For $m<0$, we let

$$(\lambda)_{+m} : S^{k+m}T^*\otimes V \to S^m T^*\otimes W$$

be the zero mapping. If g_k denotes the kernel of λ, for $m\geqq 0$, the kernel of $(\lambda)_{+m}$ is equal to

$$(S^{k+m}T^*\otimes V)\cap(S^m T^*\otimes g_k)$$

and is called the m-th prolongation of g_k. We set $g_m=S^m T^*\otimes E$, for $m<k$. The mapping

$$\delta : \Lambda^i T^*\otimes S^{m+1}T^*\otimes V \to \Lambda^{i+1}T^*\otimes S^m T^*\otimes V$$

equal to $\Delta_{1,m}\otimes\mathrm{id}$ when $i=0$, and determined by

$$\delta(\alpha\otimes u)=(-1)^i\alpha\wedge\delta u,$$

for $\alpha\in\Lambda^i T^*$, $u\in S^{m+1}T^*\otimes V$ when $i>0$, gives us by restriction a morphism

$$\delta : \Lambda^i T^*\otimes g_{k+m+1} \to \Lambda^{i+1}T^*\otimes g_{k+m}.$$

We thus obtain a complex

$$0\to g_{k+m}\xrightarrow{\delta} T^*\otimes g_{k+m-1}\xrightarrow{\delta}\Lambda^2T^*\otimes g_{k+m-2}\xrightarrow{\delta}\ldots\to\Lambda^n T^*\otimes g_{k+m-n}\to 0,$$

whose cohomology at $\Lambda^i T^*\otimes g_{k+m-i}$ is the Spencer cohomology group $H^{k+m-i,i}(g_k)$ of g_k; we have $H^{m,0}(g_k)=H^{m,1}(g_k)=0$, for all $m\geqq k$. If $H^{m,j}(g_k)=0$ for all $m\geqq k$ and $j\geqq 0$, we say that g_k is involutive (*see* [1], Chapter IX, [5]).

Let

$$\varphi : T^*\otimes\mathfrak{g}\to\mathfrak{g}$$

be the morphism defined by

$$\varphi\left(\sum_{i=1}^n dx^i\otimes u_i\right)=\sum_{i=1}^n [X_i, u_i],$$

for $u_1,\ldots,u_n\in\mathfrak{g}$, and consider the k-th prolongation

$$\varphi_k : S^{k+1}T^*\otimes\mathfrak{g}\to S^k T^*\otimes\mathfrak{g}$$

of $\varphi=\varphi_0$. If $u\in S^{k+1}T^*\otimes\mathfrak{g}$, $v\in S^kT^*\otimes\mathfrak{g}$ are expressed as

$$u=\sum_{|\alpha|=k+1}\frac{1}{\alpha!}dx^\alpha\otimes u_\alpha, \tag{2.1}$$

$$v=\sum_{|\beta|=k}\frac{1}{\beta!}dx^\beta\otimes v_\beta, \tag{2.2}$$

where u_α, $v_\beta\in\mathfrak{g}$, the equation $\varphi_k(u)=v$ is equivalent to the system

$$\sum_{i=1}^{n}[X_i, u_{\alpha+\varepsilon_i}]=v_\alpha, \tag{2.3}$$

for all α, with $|\alpha|=k$.

Let p be a homogeneous polynomial of $I(\mathfrak{g})$ of degree $d+1$. For $u\in\mathfrak{g}$, using the invariance of p, we see that

$$p(X^\alpha, [X_j, u])=\sum_{i=1}^{n}\alpha_i p([X_i, X_j], X^{\alpha-\varepsilon_i}, u),$$

where $1\leqq j\leqq n$ and $|\alpha|=d$, and hence that

$$p(X^\alpha, [X_j, u])=-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{\alpha_i}{\alpha_j+1}p(X^{\alpha+\varepsilon_j-\varepsilon_i}, [X_i, u]). \tag{2.4}$$

We consider the linear mapping

$$\psi_p : S^dT^*\otimes\mathfrak{g}\to\mathbb{R}$$

defined by

$$\psi_p(v)=\sum_{i_1,\ldots,i_d=1}^{n}p(X_{i_1},\ldots,X_{i_d}, v(\partial_{i_1},\ldots,\partial_{i_d})),$$

for $v\in S^dT^*\otimes\mathfrak{g}$. Then it is easily seen that

$$\psi_p(dx^\alpha\otimes v)=p(X^\alpha, v),$$

for $v\in\mathfrak{g}$, $|\alpha|=d$.

Proposition 2.1. — *If p is a homogeneous element of $I(\mathfrak{g})$ of degree $d+1$, with $d\geqq 1$, the sequence*

$$S^{d+1}T^*\otimes\mathfrak{g}\xrightarrow{\varphi_d}S^dT^*\otimes\mathfrak{g}\xrightarrow{\psi_p}\mathbb{R}$$

is a complex.

Proof. — If $u \in S^{d+1} T^* \otimes \mathfrak{g}$, then

$$\psi_p(\varphi_d(u)) = \sum_{j, i_1, \ldots, i_d = 1}^{n} p(X_{i_1}, \ldots, X_{i_d}, [X_j, u(\partial_j, \partial_{i_1}, \ldots, \partial_{i_d})]);$$

using the invariance of p, we see that

$$\psi_p(\varphi_d(u)) = d \sum_{j, k, i_1, \ldots, i_{d-1} = 1}^{n} p([X_k, X_j], X_{i_1}, \ldots, X_{i_{d-1}}, u(\partial_j, \partial_k, \partial_{i_1}, \ldots, \partial_{i_{d-1}})) = 0.$$

The m-th prolongation

$$(\psi_p)_{+m} : S^{d+m} T^* \otimes \mathfrak{g} \to S^m T^*$$

of ψ_p is given by

$$(\psi_p)_{+m}\left(\sum_{|\alpha| = d+m} \frac{dx^\alpha}{\alpha!} \otimes v_\alpha\right) = \sum_{\substack{|\beta| = m \\ |\gamma| = d}} \frac{1}{\gamma!} p(X^\gamma, v_{\gamma+\beta}) \frac{dx^\beta}{\beta!},$$

where $v_\alpha \in \mathfrak{g}$; it clearly satisfies $(\psi_p)_{+m} \circ \varphi_{d+m} = 0$.

We define a mapping

$$\psi_k : S^k T^* \otimes \mathfrak{g} \to \bigoplus_{1 \leqq a \leqq r} S^{k-d_a} T^*$$

by

$$\psi_k(v) = ((\psi_{p_a})_{+(k-d_a)}(v))_{1 \leqq a \leqq r},$$

for $v \in S^k T^* \otimes \mathfrak{g}$. If v is the element (2.2) of $S^k T^* \otimes \mathfrak{g}$ and

(2.5) $$w_a = \sum_{|\beta| = k-d_a} w_{a,\beta} \frac{dx^\beta}{\beta!} \in S^{k-d_a} T^*, \qquad 1 \leqq a \leqq r,$$

with $w_{a,\beta} \in \mathbb{R}$, the equation $\psi_k(v) = (w_1, \ldots, w_r)$ is equivalent to the system

(2.6) $$\sum_{|\alpha| = d_a} \frac{1}{\alpha!} p_a(X^\alpha, v_{\alpha+\beta}) = w_{a,\beta}, \qquad |\beta| = k - d_a \geqq 0, \quad 1 \leqq a \leqq r.$$

PROPOSITION 2.2. — *For $k \geqq 1$, if X_1 is principal, the mapping ψ_k is surjective.*

Proof. — Consider the elements $w_a \in S^{k-d_a} T^*$ given by (2.5), with $w_{a,\beta} \in \mathbb{R}$. If v is the element (2.2) of $S^k T^* \otimes \mathfrak{g}$ and m is an integer satisfying $1 \leqq m \leqq k$, the system S_m of equations

(2.7) $$\frac{1}{d_a!} p_a(X_1^{d_a}, v_{m\varepsilon_1+\beta}) = w_{a,(m-d_a)\varepsilon_1+\beta} - \sum_{\substack{|\alpha| = d_a \\ \alpha_1 < d_a}} \frac{1}{\alpha!} p_a(X^\alpha, v_{(m-d_a)\varepsilon_1+\alpha+\beta}),$$

with $d_a \leqq m$, $\beta = (0, \beta_2, \ldots, \beta_n)$, $|\beta| = k - m$ and $1 \leqq a \leqq r$, is a sub-system of (2.6). Note that the right-hand side of (2.7) only involves the $v_{l\varepsilon_1+\gamma}$, with $l < m$ and $\gamma = (0, \gamma_2, \ldots, \gamma_n)$, $|\gamma| = k - l$. We denote by S_0 the empty system of equations. If X_1 is regular, we now construct a solution v of the form (2.2) of the equation $\psi_k(v) = (w_1, \ldots, w_r)$, or equivalently of all the systems S_m, with $1 \leqq m \leqq k$. For $\beta = (0, \beta_2, \ldots, \beta_n)$, with $|\beta| = k$, let v_β be an arbitrary element of $\mathfrak{g}$. Let m be an integer satisfying $1 \leqq m \leqq k$ and assume that we have chosen elements $v_{l\varepsilon_1+\beta} \in \mathfrak{g}$, for all $0 \leqq l < m$, $\beta = (0, \beta_2, \ldots, \beta_n)$, $|\beta| = k - l$, satisfying the systems $S_0, S_1, \ldots, S_{m-1}$. Then, according to Theorem 1.1, for each $\beta = (0, \beta_2, \ldots, \beta_n)$, with $|\beta| = k - m$, we may choose $v_{m\varepsilon_1+\beta} \in \mathfrak{g}$ satisfying all the equations (2.7), with $d_a \leqq m$ and $1 \leqq a \leqq r$.

According to Proposition 2.1, we have the complexes

$$S^{k+1}T^* \otimes \mathfrak{g} \xrightarrow{\varphi_k} S^k T^* \otimes \mathfrak{g} \xrightarrow{\psi_k} \bigoplus_{1 \leqq a \leqq r} S^{k-d_a} T^* \to 0 \tag{2.8}$$

for $k \geqq 0$. If $n = 1$ and X_1 is principal and if $d = \sup(d_a)$, it is easily seen that, for $k \geqq d$, the sequence (2.8) is isomorphic to the complex

$$\mathfrak{g} \xrightarrow{\operatorname{ad} X_1} \mathfrak{g} \xrightarrow{\pi_{X_1}} \mathbb{R}^r \to 0,$$

and so is exact by Theorem 1.1; this theorem also shows that the sequence (2.8) is not exact at $S^k T^* \otimes \mathfrak{g}$, for $k < d$. The following theorem gives a condition for the exactness of these complexes.

THEOREM 2.1. — *If $n \geqq 2$ and $(X_1, X_2) \in U_1$, then the sequences* (2.8) *are exact for all* $k \geqq 0$.

Proof. — We know X_1, X_2 are principal. Thus by Proposition 2.2, ψ_k is surjective for all $k \geqq 0$. We now demonstrate the exactness of the sequences (2.8) at the middle position $S^k T^* \otimes \mathfrak{g}$ by induction on n. We first consider the case $n = 2$.

If $n = 2$ and V is a complex vector space, we identify $S^k T^* \otimes V$ with V^{k+1} in the following manner. If

$$u = \sum_{|\alpha| = k} \frac{dx^\alpha}{\alpha!} \otimes u_\alpha, \quad \text{with } u_\alpha \in V,$$

for $\alpha = (j, k-j)$, we write $u_j = u_{(j, k-j)}$ and then identify u with the $(k+1)$-tuple $(u_0, u_1, \ldots, u_k)$ of V^{k+1}. If $u = (u_0, \ldots, u_{k+1})$ belongs to $S^{k+1} T^* \otimes \mathfrak{g}$ and $v = (v_0, \ldots, v_k)$ belongs to $S^k T^* \otimes \mathfrak{g}$, the equation $\varphi_k(u) = v$ is equivalent to

$$[X_1, u_{l+1}] + [X_2, u_l] = v_l,$$

for $0 \leqq l \leqq k$, and $(\psi_{p_a})_{+(k-d_a)}(v)$ is the $(k - d_a + 1)$-tuple

$$\left(\sum_{l=0}^{d_a} \frac{1}{l!\,(d_a - l)!} p_a(X_1^l, X_2^{d_a - l}, v_{l+i}) \right)_{i=0, \ldots, k-d_a},$$

for $1 \leqq a \leqq r$ and $d_a \leqq k$.

The following proposition implies the exactness of the sequences (2.8) at the middle position when $n=2$.

PROPOSITION 2.3. — *Suppose that $n=2$ and that X_1, X_2 are principal. Given an element $v=(v_0, \ldots, v_k)$ of $S^k T^* \otimes \mathfrak{g}$ and $Y \in \mathfrak{g}$ satisfying*

$$\psi_k(v)=0, \qquad p_a(X_2^{d_a}, Y)=0,$$

$$(2.9) \qquad \frac{1}{m!(d_a-m)!} p_a(X_1^{d_a-m}, X_2^m, Y) = \sum_{l=0}^{m} \frac{1}{l!(d_a-l)!} p_a(X_1^{d_a-l}, X_2^l, v_{m-l}),$$

for all $1 \leqq a \leqq r$ and $0 \leqq m \leqq \min(k, d_a-1)$, there exists an element $u=(u_0, \ldots, u_{k+1})$ of $S^{k+1} T^* \otimes \mathfrak{g}$ such that $\varphi_k(u)=v$ and

$$[X_2, u_0]=Y.$$

Proof. — We proceed by induction on k. If $k=0$, the conditions imposed on Y are

$$p_a(X_2^{d_a}, Y)=0, \qquad p_a(X_1^{d_a}, Y)=p_a(X_1^{d_a}, v_0),$$

for $1 \leqq a \leqq r$. The first equalities and Theorem 1.1 give us the existence of an element u_0 of $\mathfrak{g}$ satisfying

$$[X_2, u_0]=Y.$$

The latter conditions and Theorem 1.1 tell us that we can solve the equation

$$[X_1, u_1]=v_0-Y=v_0-[X_2, u_0],$$

for $u_1 \in \mathfrak{g}$. Thus $\varphi(u_0, u_1)=v_0$. Suppose that the proposition holds for $k-1$, with $k \geqq 1$. Consider the element $v'=(v_0, \ldots, v_{k-1})$ of $S^{k-1} T^* \otimes \mathfrak{g}$. We easily see that the assumption $\psi_k(v)=0$ implies that $\psi_{k-1}(v')=0$. By our induction hypothesis and the condition imposed on Y, there exists an element $u'=(u_0, \ldots, u_k)$ of $S^k T^* \otimes \mathfrak{g}$ such that $[X_2, u_0]=Y$ and $\varphi_k(u')=v'$. We now wish to solve the equation

$$(2.10) \qquad [X_1, u_{k+1}]=v_k-[X_2, u_k],$$

for $u_{k+1} \in \mathfrak{g}$; a solution u_{k+1} of (2.10) determines a solution $u=(u_0, \ldots, u_k, u_{k+1})$ of the equation $\varphi_k(u)=v$. According to Theorem 1.1, in order to solve (2.10) it suffices to verify that

$$(2.11) \qquad p_a(X_1^{d_a}, v_k-[X_2, u_k])=0,$$

for $1 \leqq a \leqq r$. Since $\varphi_k(u')=v'$, according to (2.4), we have

$$\frac{1}{l!(d_a-l)!} p_a(X_1^{d_a-l}, X_2^l, [X_2, u_{k-l}])$$

$$= -\frac{1}{(l+1)!(d_a-l-1)!} p_a(X_1^{d_a-l-1}, X_2^{l+1}, v_{k-l-1}-[X_2, u_{k-l-1}]),$$

for $0 \leqq l < \min(k, d_a)$; hence, if $d_a > k$, we obtain

$$\frac{1}{d_a!} p_a(X_1^{d_a}, [X_2, u_k]) = \frac{1}{k!(d_a-k)!} p_a(X_1^{d_a-k}, X_2^k, Y) - \sum_{l=1}^{k} \frac{1}{l!(d_a-l)!} p_a(X_1^{d_a-l}, X_2^l, v_{k-l}),$$

and if $d_a \leqq k$

$$\frac{1}{d_a!} p_a(X_1^{d_a}, [X_2, u_k]) = -\sum_{l=1}^{d_a} \frac{1}{l!(d_a-l)!} p_a(X_1^{d_a-l}, X_2^l, v_{k-l}).$$

If $d_a > k$, equation (2.11) is therefore equivalent to our hypothesis (2.9), with $m=k$; on the other hand, when $d_a \leqq k$, the left-hand side of (2.11) is equal to the last component of the element $(\psi_{p_a})_{+(k-d_a)}(v)$ of $S^{k-d_a} T^*$. Thus (2.11) holds for all $1 \leqq a \leqq r$, and we have completed the proof of the proposition.

We now continue the proof of Theorem 2.1. Let $n \geqq 3$ and assume that the sequences (2.8), corresponding to $X_1, \ldots, X_{n-1}$ and the basis $\{dx^1, \ldots, dx^{n-1}\}$ generating a certain hypersurface T_0^* of T^*, are exact at the middle position. Let v be an element of $S^k T^* \otimes \mathfrak{g}$ given by (2.2) and satisfying $\psi_k(v)=0$. If u is the element (2.1) of $S^{k+1} T^* \otimes \mathfrak{g}$ and m is an integer satisfying $0 \leqq m \leqq k$, the system S_m of equations

$$\sum_{i=1}^{n-1} [X_i, u_{\alpha+\varepsilon_i+m\varepsilon_n}] = v_{\alpha+m\varepsilon_n} - [X_n, u_{\alpha+(m+1)\varepsilon_n}],$$

for all $\alpha=(\alpha_1, \ldots, \alpha_{n-1}, 0)$, with $|\alpha| = k-m$, is a sub-system of (2.3). This system S_m is of the form $\varphi_{k-m}(u') = v'$, where $u' \in S^{k-m+1} T_0^* \otimes \mathfrak{g}$, $v' \in S^{k-m} T_0^* \otimes \mathfrak{g}$ and φ is defined in terms of $X_1, \ldots, X_{n-1}$ and $\{dx^1, \ldots, dx^{n-1}\}$. We now construct a solution u of the form (2.1) of the equation $\varphi_k(u)=v$, or equivalently of all the systems S_m, with $0 \leqq m \leqq k$. Let $u_{(k+1)\varepsilon_n}$ be an arbitrary element of $\mathfrak{g}$. We denote by S_{k+1} the empty system of equations. Let m be an integer, with $0 \leqq m \leqq k$, and assume that we have chosen elements $u_{\beta+l\varepsilon_n} \in \mathfrak{g}$, for all $m+1 \leqq l \leqq k+1$, $\beta=(\beta_1, \ldots, \beta_{n-1}, 0)$, with $|\beta| = k-l+1$, satisfying the systems $S_{m+1}, \ldots, S_{k+1}$. Then we wish to solve S_m for $u_{\alpha+m\varepsilon_n}$, with $\alpha=(\alpha_1, \ldots, \alpha_{n-1}, 0)$, $|\alpha| = k-m+1$. According to our induction hypothesis, it suffices to verify that

$$\sum_{\substack{|\alpha|=d_a \\ \alpha_n=0}} \frac{1}{\alpha!} p_a(X^\alpha, v_{\alpha+m\varepsilon_n+\beta} - [X_n, u_{\alpha+(m+1)\varepsilon_n+\beta}]) = 0, \tag{2.12}$$

for $\beta=(\beta_1, \ldots, \beta_{n-1}, 0)$, $|\beta|=k-m-d_a$, and $1\leqq a\leqq r$, with $k-m\geqq d_a$. According to (2.4), if $k-m\geqq d_a$, for $0\leqq l<d_a$, we have

$$\sum_{\substack{|\alpha|=d_a-l\\ \alpha_n=0}} \frac{1}{\alpha!\,l!} p_a(X^\alpha, X_n^l, [X_n, u_{\alpha+(m+l+1)\varepsilon_n+\beta}])$$

$$= - \sum_{\substack{|\gamma|=d_a-l-1\\ \gamma_n=0}} \frac{1}{\gamma!\,(l+1)!} p_a(X^\gamma, X_n^{l+1}, v_{\gamma+(m+l+1)\varepsilon_n+\beta}-[X_n, u_{\gamma+(m+l+2)\varepsilon_n+\beta}]),$$

for $\beta=(\beta_1, \ldots, \beta_{n-1}, 0)$, with $|\beta|=k-m-d_a$. Hence, we see that the left-hand side of (2.12) is equal to the expression

$$\sum_{l=0}^{d_a} \sum_{\substack{|\alpha|=d_a-l\\ \alpha_n=0}} \frac{1}{\alpha!\,l!} p_a(X^\alpha, X_n^l, v_{\alpha+(m+l)\varepsilon_n+\beta}),$$

which vanishes because $(\psi_{p_a})_{+(k-d_a)}(v)=0$. We therefore can solve the system S_m, and the equation $\varphi_k(u)=v$.

We have completed the proof of Theorem 2.1. We denote by g_{k+1} be the kernel of φ_k. For $k\geqq 1$, let m_k be the number of the d_a's equal to k. Under the hypotheses of Theorem 2.1, from the exactness of the sequences (2.8), we deduce by a standard argument (*see* [4]) the following result:

COROLLARY 2.1. — *If* $n\geqq 2$ *and* $(X_1, X_2)\in U_1$, *we have*

$$H^{k,2}(g_1)\cong \mathbb{R}^{m_{k+1}}, \qquad H^{k,i}(g_1)=0,$$

for $k\geqq 0$ *and* $i>2$.

Thus if $n\geqq 2$ and $(X_1, X_2)\in U_1$, if $d=\sup(d_a)$, then

$$H^{m,j}(g_1)=0,$$

for all $m\geqq d$ and $j\geqq 0$.

LEMMA 2.1. — *Suppose that* $k\geqq 0$ *and that* X_1 *is principal. Assume that the elements* u *of* $S^{k+1}T^*\otimes \mathfrak{g}$ *and* v *of* $S^kT^*\otimes\mathfrak{g}$ *given by* (2.1) *and* (2.2) *satisfy the following conditions:*

(i) *for all* $2\leqq j\leqq n$ *and* $\alpha=(\alpha_1, \ldots, \alpha_n)$, *with* $|\alpha|=k+1$, $\alpha_1=0$, *we have*

$$[X_j, u_\alpha]=0;$$

(ii) *for all* $\alpha=(\alpha_1, \ldots, \alpha_n)$, *with* $|\alpha|=k$, $\alpha_1<k$, *we have*

$$\sum_{j=1}^{n} [X_j, u_{\alpha+\varepsilon_j}]=v_\alpha; \tag{2.13}$$

(iii) *for all* $1 \leqq a \leqq r$, *if* $d'_a=\min(d_a, k)$, *we have*

$$\sum_{l=0}^{d'_a} \sum_{\substack{|\beta|=l\\ \beta_1=0}} \frac{1}{(d_a-l)!\,\beta!} p_a(X_1^{d_a-l}, X^\beta, v_{(k-l)\varepsilon_1+\beta})=0. \tag{2.14}$$

Then there exists $u' \in \mathfrak{g}$ *such that*

$$[X_1, u']+\sum_{j=2}^{n} [X_j, u_{k\varepsilon_1+\varepsilon_j}]=v_{k\varepsilon_1}.$$

Proof. — We proceed by induction on n. For $n=1$, the result follows directly from Theorem 1.1. Assume that the lemma holds for $n-1$, with $n \geqq 2$. Let u and v be elements of $S^{k+1}T^* \otimes \mathfrak{g}$ and $S^k T^* \otimes \mathfrak{g}$, given by (2.1) and (2.2) respectively, satisfying the three conditions of the lemma. Then we have

$$\sum_{j=1}^{n-1} [X_j, u_{\beta+\varepsilon_j}]=v_\beta-[X_n, u_{\beta+\varepsilon_n}],$$

for all $\beta=(\beta_1, \ldots, \beta_{n-1}, 0)$, with $|\beta|=k$ and $\beta_1<k$. For $1 \leqq a \leqq r$ and $\alpha=(\alpha_1, \ldots, \alpha_n)$, with $|\alpha|=d_a$ and $\alpha_1+k \geqq d_a$, we set

$$\Phi_a(\alpha_1, \beta, \alpha_n; u)=\frac{1}{\alpha!} p_a(X^\alpha, [X_n, u_{\alpha+(k-d_a)\varepsilon_1+\varepsilon_n}]),$$

$$\Psi_a(\alpha_1, \beta, \alpha_n; v)=\frac{1}{\alpha!} p_a(X^\alpha, v_{\alpha+(k-d_a)\varepsilon_1}),$$

where $\beta=(0, \alpha_2, \ldots, \alpha_{n-1}, 0)$. By our induction hypothesis, in order to solve the equation

$$[X_1, u']+\sum_{j=2}^{n-1} [X_j, u_{k\varepsilon_1+\varepsilon_j}]=v_{k\varepsilon_1}-[X_n, u_{k\varepsilon_1+\varepsilon_n}],$$

it suffices to verify that

$$\sum_{l=0}^{d'_a} \sum_{\substack{|\beta|=l\\ \beta_1=\beta_n=0}} (\Psi_a(d_a-l, \beta, 0; v)-\Phi_a(d_a-l, \beta, 0; u))=0, \tag{2.15}$$

where $d'_a = \min(d_a, k)$, for all $1 \leqq a \leqq r$. According to (2.4), for $0 \leqq s < d'_a$ and $1 \leqq a \leqq r$, we have

$$\sum_{l=0}^{d'_a - s} \sum_{\substack{|\beta| = l \\ \beta_1 = \beta_n = 0}} \Phi_a(d_a - l - s, \beta, s; u)$$
$$= \sum_{l=0}^{d'_a - s - 1} \sum_{\substack{|\beta| = l \\ \beta_1 = \beta_n = 0}} (\Phi_a(d_a - l - s - 1, \beta, s+1; u) - \Psi_a(d_a - l - s - 1, \beta, s+1; v)).$$

Here, we have used condition (i) when $k \leqq d_a$, and the relation (2.13), with

$$\alpha = (k - l - 1)\varepsilon_1 + \beta + (s+1)\varepsilon_n,$$

for $0 \leqq l \leqq d'_a - s - 1$ and $\beta = (\beta_1, \ldots, \beta_n)$, with $|\beta| = l$ and $\beta_1 = \beta_n = 0$. Thus we see that the left-hand side of (2.15) is equal to

$$\sum_{s=0}^{d'_a} \sum_{l=0}^{d'_a - s} \sum_{\substack{|\beta| = l \\ \beta_1 = \beta_n = 0}} \Psi_a(d_a - l - s, \beta, s; v) - \Phi_a(d_a - d'_a, 0, d'_a; u).$$

The second term of the above expression vanishes, by the invariance of p_a when $k \geqq d_a$, and by condition (i) when $k < d_a$. Therefore the equality (2.15) is equivalent to the relation (2.14).

The following lemma is an immediate consequence of Lemma 2.1.

LEMMA 2.2. — *Suppose that $k \geqq 0$ and that X_1 is principal. Assume that the element u of $S^{k+1} T^* \otimes \mathfrak{g}$ given by* (2.1) *satisfies the following conditions:*

(i) *for all $2 \leqq j \leqq n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = k+1$, $\alpha_1 = 0$, we have*

$$[X_j, u_\alpha] = 0;$$

(ii) *for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = k$, $\alpha_1 < k$, we have*

$$\sum_{j=1}^{n} [X_j, u_{\alpha + \varepsilon_j}] = 0.$$

Then there exists $u' \in \mathfrak{g}$ such that

$$[X_1, u'] + \sum_{j=2}^{n} [X_j, u_{k\varepsilon_1 + \varepsilon_j}] = 0.$$

LEMMA 2.3. — *Let p be a homogeneous element of $I(\mathfrak{g})$ of degree $d+1$, with $d \geqq 1$, and u be an element of $\mathfrak{g}_d$ given by*

$$u = \sum_{|\alpha| = d} \frac{1}{\alpha!} dx^\alpha \otimes u_\alpha,$$

with $u_\alpha \in \mathfrak{g}$. *Then, for all* $Y \in \mathfrak{g}$, *we have*

$$\sum_{|\alpha|=d} \frac{1}{\alpha!} p(X^\alpha, [Y, u_\alpha]) = 0.$$

Proof. — By the relation (2.4) corresponding to the $n+1$ elements $\{X_1, \ldots, X_n, Y\}$ of $\mathfrak{g}$, we obtain

$$\sum_{|\alpha|=d} \frac{1}{\alpha!} p(X^\alpha, [Y, u_\alpha]) = -\sum_{\substack{|\alpha|=d \\ 1 \leqq j \leqq n}} \frac{\alpha_j}{\alpha!} p(X^{\alpha-\varepsilon_j}, Y, [X_j, u_\alpha])$$
$$= -\sum_{\substack{|\beta|=d-1 \\ 1 \leqq j \leqq n}} \frac{1}{\beta!} p(X^\beta, Y, [X_j, u_{\beta+\varepsilon_j}]) = 0.$$

We shall require the following result to prove Lemma 2.5 and Theorem 2.2.

LEMMA 2.4. — *Assume that* $n \geqq 3$ *and that, for all* $1 \leqq i < j \leqq 3$, *the* $2r$ *elements*

$$\{\tilde{p}_a(X_i^{d_a}), \tilde{p}_a(X_j^{d_a})\}_{1 \leqq a \leqq r}$$

of $\mathfrak{g}$ *are linearly independent. Then, for all* $w \in \mathfrak{g}$, *there exist* $u, v \in \mathfrak{g}_{X_3}$ *satisfying*

$$[X_1, u] + [X_2, v] = w.$$

Proof. — Since $\{\tilde{p}_a(X_j^{d_a})\}_{1 \leqq a \leqq r}$ is a basis of $\mathfrak{g}_{X_j}$, for $j=1, 2, 3$, we see that

$$\mathfrak{g}_{X_1} \cap \mathfrak{g}_{X_3} = \mathfrak{g}_{X_2} \cap \mathfrak{g}_{X_3} = 0.$$

Hence, by Theorem 1.1, the sequence

$$0 \to \mathfrak{g}_{X_3} \xrightarrow{\operatorname{ad} X_j} \mathfrak{g} \xrightarrow{\pi_{X_j}} \mathbb{R}^r \to 0$$

is exact for $j=1, 2$. Let w be an element of $\mathfrak{g}$; according to our hypothesis, there exists $Y \in \mathfrak{g}$ satisfying

$$p_a(X_1^{d_a}, Y) = p_a(X_1^{d_a}, w), \qquad p_a(X_2^{d_a}, Y) = 0.$$

Then $\pi_{X_2}(Y) = 0$ and we can solve the equation

$$[X_2, v] = Y$$

for $v \in \mathfrak{g}_{X_3}$. Finally, since $\pi_{X_1}(w - Y) = 0$, we can find an element $u \in \mathfrak{g}_{X_3}$ such that

$$[X_1, u] = w - Y.$$

If $n \geqq 3$ and (X_1, X_2, X_3) belongs to U_2, the hypotheses of the preceding lemma are satisfied.

We denote by g the unique scalar product on T for which $\{\partial_1, \ldots, \partial_n\}$ is an orthonormal basis of T. Let

$$\text{(2.16)} \qquad \begin{aligned} &\mathrm{Tr} : S^k T^* \to S^{k-2} T^*, \\ &\mathrm{Tr}_k : S^k T^* \otimes T^* \to S^{k-1} T^* \end{aligned}$$

be the trace mappings depending only on g and defined by

$$(\mathrm{Tr}\, u)(\eta_1, \ldots, \eta_{k-2}) = \sum_{j=1}^{n} u(\partial_j, \partial_j, \eta_1, \ldots, \eta_{k-2}),$$

$$(\mathrm{Tr}_k\, v)(\eta_1, \ldots, \eta_{k-1}) = \sum_{j=1}^{n} u(\eta_1, \ldots, \eta_{k-1}, \partial_j, \partial_j),$$

for $u \in S^k T^*$, $v \in S^k T^* \otimes T^*$ and $\eta_1, \ldots, \eta_{k-1} \in T$. Both these mappings are surjective. We denote by $S_0^k T^*$ the kernel of (2.16). If

$$u = \sum_{|\alpha| = k} u_\alpha \frac{dx^\alpha}{\alpha!},$$

with $u_\alpha \in \mathbb{R}$, is an element of $S^k T^*$, then

$$\mathrm{Tr}\, u = \sum_{|\beta| = k-2} \left(\sum_{i=1}^{n} u_{\beta + 2\varepsilon_i} \right) \frac{dx^\beta}{\beta!}.$$

Let

$$\sigma : S^2 T^* \otimes T^* \to T^*$$

be the mapping $\mathrm{Tr} \otimes \mathrm{id} - \mathrm{Tr}_2$; its k-th prolongation

$$\sigma_k : \quad S^{k+2} T^* \otimes T^* \to S^k T^* \otimes T^*$$

is equal to $\mathrm{Tr} \otimes \mathrm{id} - \Delta_{k,1} . \mathrm{Tr}_{k+2}$. For $k \geqq 2$, we denote by h_k the kernel of σ_{k-2}; clearly, $S^{k+1} T^*$ is a subspace of h_k.

PROPOSITION 2.4. — *If* $n \geqq 3$, *the subspace* h_2 *of* $S^2 T^* \otimes T^*$ *is involutive and the sequences*

$$\text{(2.17)} \qquad 0 \to h_{k+2} \to S^{k+2} T^* \otimes T^* \xrightarrow{\sigma_k} S^k T^* \otimes T^* \xrightarrow{\mathrm{Tr}_k} S^{k-1} T^* \to 0$$

are exact, for $k \geqq 1$.

Proof. — It is easily verified that (2.17) is a complex and that Tr_k is the $(k-1)$-th prolongation of Tr_1, for $k \geqq 1$. Let $O(T)$ be the orthogonal group of the Euclidean vector space (T, g). From the relation

$$\sigma_1 (dx^1 \cdot (dx^3)^2 \otimes dx^2) = 2\, dx^1 \otimes dx^2,$$

we infer that the irreducible $O(\mathrm{T})$-modules $\Lambda^2 \mathrm{T}^*$ and $\mathrm{S}_0^2 \mathrm{T}^*$ of $\mathrm{T}^* \otimes \mathrm{T}^*$ are both contained in the image of σ_1. On the other hand, Tr_1 is non-zero and we have the decomposition

$$\mathrm{T}^* \otimes \mathrm{T}^* = \Lambda^2 \mathrm{T}^* \oplus \mathrm{S}_0^2 \mathrm{T}^* \oplus \mathbb{R} g$$

of $O(\mathrm{T})$-modules. It follows that (2.17) is exact when $k=1$. For $1 \leqq j \leqq n$, let V_j denote the subspace of $\mathrm{S}^2 \mathrm{T}^*$ composed of those elements u of $\mathrm{S}^2 \mathrm{T}^*$ satisfying

$$u(\partial_i, \eta) = 0,$$

for all $1 \leqq i \leqq j$ and all $\eta \in \mathrm{T}$; set $\mathrm{V}_0 = \mathrm{S}^2 \mathrm{T}^*$. For $1 \leqq i, j \leqq n$, with $i \neq j$, we have

$$\sigma((dx^i)^2 \otimes dx^j) = 2\, dx^j, \qquad \sigma((dx^j)^2 \otimes dx^j) = 0;$$

hence we obtain

$$\sigma(\mathrm{V}_j \otimes \mathrm{T}^*) = \mathrm{T}^*,$$

for $1 \leqq j < n-1$, and we see that $\sigma(\mathrm{V}_{n-1} \otimes \mathrm{T}^*)$ is equal to the subspace of T^* spanned by $\{dx^1, \ldots, dx^{n-1}\}$. Since $\dim \mathrm{V}_{n-1} = 1$, we have

$$\sum_{j=0}^{n-1} \dim(h_2 \cap \mathrm{V}_j) = n \sum_{j=0}^{n-2} (\dim \mathrm{V}_j - 1) + 1 = n \sum_{j=0}^{n-1} \dim \mathrm{V}_j - (n^2 - 1)$$
$$= \dim \mathrm{S}^3 \mathrm{T}^* \otimes \mathrm{T}^* - \dim \mathrm{T}^* \otimes \mathrm{T}^* - \dim \mathbb{R} = \dim h_3,$$

by the exactness of (2.17) with $k=1$. This equality implies that h_2 is involutive (*see* [1], Theorem 2.14, Chapter IX). By a standard argument (*see* [4]), from the involutivity of h_2 and the exactness of (2.17) with $k=1$, if follows that the sequences (2.17) are exact at $\mathrm{S}^k \mathrm{T}^* \otimes \mathrm{T}^*$ for all $k \geqq 1$.

For $k \geqq 2$, consider the mapping

$$\mathrm{id} \otimes \varphi : \mathrm{S}^k \mathrm{T}^* \otimes \mathrm{T}^* \otimes \mathfrak{g} \to \mathrm{S}^k \mathrm{T}^* \otimes \mathfrak{g}.$$

Let p be a homogeneous element of $\mathrm{I}(\mathfrak{g})$ of degree $d+1$. If $u \in h_{k+2} \otimes \mathfrak{g}$, with $k \geqq d$, then, for $i_{d+1}, \ldots, i_k \leqq n$, we have

$$(\mathrm{Tr}\,.\,(\psi_p)_{+(k-d+2)} \cdot (\mathrm{id} \otimes \varphi)\, u)(\partial_{i_{d+1}}, \ldots, \partial_{i_k})$$
$$= \sum_{i_1, \ldots, i_d, j, l = 1}^{n} p(\mathrm{X}_{i_1}, \ldots, \mathrm{X}_{i_d}, [\mathrm{X}_l, u(\partial_{i_1}, \ldots, \partial_{i_k}, \partial_j, \partial_j, \partial_l)])$$
$$= \sum_{i_1, \ldots, i_d, j, l = 1}^{n} p(\mathrm{X}_{i_1}, \ldots, \mathrm{X}_{i_d}, [\mathrm{X}_l, u(\partial_{i_1}, \ldots, \partial_{i_k}, \partial_l, \partial_j, \partial_j)])$$
$$= ((\psi_p)_{+(k-d)} \cdot \varphi_k \cdot \mathrm{Tr}_{k+2}\, u)(\partial_{i_{d+1}}, \ldots, \partial_{i_k}) = 0,$$

by Proposition 2.1. Therefore, if

$$\mathrm{Tr} : \bigoplus_{1 \leqq a \leqq r} \mathrm{S}^{k-d_a} \mathrm{T}^* \to \bigoplus_{1 \leqq a \leqq r} \mathrm{S}^{k-d_a-2} \mathrm{T}^*$$

is the surjective mapping sending $(w_1, \ldots, w_r)$ into $(\operatorname{Tr} w_1, \ldots, \operatorname{Tr} w_r)$, with $w_a \in S^{k-d_a} T^*$, we obtain the complexes

$$h_k \otimes \mathfrak{g} \xrightarrow{\mathrm{id} \otimes \varphi} S^k T^* \otimes \mathfrak{g} \xrightarrow{\mathrm{Tr}.\psi_k} \bigoplus_{1 \leq a \leq r} S^{k-d_a-2} T^* \to 0, \tag{2.18}$$

for $k \geqq 2$. The following theorem gives a condition for their exactness.

THEOREM 2.2. — *If* $n \geqq 3$ *and* $(X_1, X_2, X_3) \in U_2$, *then the sequences* (2.18) *are exact for all* $k \geqq 2$.

We require the following lemma for the proof of Theorem 2.2.

LEMMA 2.5. — *Assume that* $n \geqq 3$ *and that, for all* $1 \leqq i < j \leqq 3$, *the* $2r$ *elements*

$$\{ \tilde{p}_a(X_i^{d_a}), \tilde{p}_a(X_j^{d_a}) \}_{1 \leqq a \leqq r}$$

of $\mathfrak{g}$ *are linearly independent. Then we have*

$$S_0^k T^* \otimes \mathfrak{g} \subset (\mathrm{id} \otimes \varphi)(h_k \otimes \mathfrak{g}),$$

for $k \geqq 2$.

Proof. — Let v be an element of $S_0^k T^* \otimes \mathfrak{g}$, with $k \geqq 2$, given by (2.2). We seek an element

$$u = \sum_{\substack{|\alpha| = k \\ 1 \leqq j \leqq n}} \frac{1}{\alpha!} dx^\alpha \otimes dx^j \otimes u_{\alpha, j} \in S^k T^* \otimes T^* \otimes \mathfrak{g}_{X_3},$$

with $u_{\alpha, j} \in \mathfrak{g}_{X_3}$, satisfying

$$\sum_{j=1}^{n} [X_j, u_{\alpha, j}] = v_\alpha \tag{2.19}$$

for $|\alpha| = k$, and

$$\sum_{i=1}^{n} (u_{\alpha+\varepsilon_i+\varepsilon_j, i} - u_{\alpha+2\varepsilon_i, j}) = 0 \tag{2.20}$$

for all $1 \leqq j \leqq n$, $|\alpha| = k-2$. First, using Lemma 2.4, for $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = k$ and $\alpha_3 = 0$ or 1, we choose elements $u_{\alpha, 1}$ and $u_{\alpha, 2}$ of $\mathfrak{g}_{X_3}$ satisfying the equation

$$[X_1, u_{\alpha, 1}] + [X_2, u_{\alpha, 2}] = v_\alpha. \tag{2.21}$$

For $j = 1, 2$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = k$ and $\alpha_3 \geqq 2$, we define elements $u_{\alpha, j} \in \mathfrak{g}_{X_3}$ recursively on α_3 by

$$u_{\beta+2\varepsilon_3, j} = - \sum_{\substack{1 \leqq i \leqq n \\ i \neq 3}} u_{\beta+2\varepsilon_i, j},$$

where $\beta=(\beta_1, \ldots, \beta_n)$, $|\beta|=k-2$. Then, since $(\mathrm{Tr}\otimes\mathrm{id})\, v=0$, we see that (2.21) holds for all $\alpha=(\alpha_1, \ldots, \alpha_n)$, with $|\alpha|=k$; we also have

$$\sum_{i=1}^{n} u_{\beta+2\varepsilon_i,\, j}=0, \tag{2.22}$$

for $j=1$, 2 and $\beta=(\beta_1, \ldots, \beta_n)$, with $|\beta|=k-2$. For all $j>3$ and $|\alpha|=k$, let $u_{\alpha, j}=0$. For any choice of the $u_{\alpha, 3}\in \mathfrak{g}_{X_3}$, the equations (2.19) will be satisfied. For $|\beta|=k-1$, with $\beta\neq(k-1)\,\varepsilon_3$, we set

$$u_{\beta+\varepsilon_3,\, 3}=-u_{\beta+\varepsilon_1,\, 1}-u_{\beta+\varepsilon_2,\, 2}$$

and $u_{k\varepsilon_3,\, 3}=0$. Then equation (2.20) holds for $j\neq 3$ and all $|\alpha|=k-2$. On the other hand, equation (2.20), with $j=3$ and $\alpha_3\geqq 1$, is a consequence of (2.22). Finally, the surjectivity of the mapping (2.16), with T replaced by its subspace of dimension $n-1$ generated by $\{\partial_1, \partial_2, \partial_4, \ldots, \partial_n\}$, gives us the existence of elements $u_{\alpha, 3}\in \mathfrak{g}_{X_3}$, with $|\alpha|=k$ and $\alpha_3=0$, satisfying

$$\sum_{\substack{i=1\\ i\neq 3}}^{n} u_{\beta+2\varepsilon_i,\, 3}=u_{\beta+\varepsilon_1+\varepsilon_3,\, 1}+u_{\beta+\varepsilon_2+\varepsilon_3,\, 2},$$

for all $|\beta|=k-2$; thus (2.20) also holds when $j=3$ and $\alpha_3=0$.

We now give the proof of Theorem 2.2. By Proposition 2.2, we need only demonstrate that (2.18) is exact at $S^k T^*\otimes\mathfrak{g}$, for $k\geqq 2$. In fact, the diagram

$$\begin{array}{ccccc}
S^{k+1}T^*\otimes\mathfrak{g} & \xrightarrow{\varphi_k} & S^k T^*\otimes\mathfrak{g} & \xrightarrow{\psi_k} & \bigoplus_{1\leqq a\leqq r} S^{k-d_a}T^* \\
\downarrow{\scriptstyle \mathrm{Tr}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{Tr}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\
S^{k-1}T^*\otimes\mathfrak{g} & \xrightarrow{\varphi_{k-2}} & S^{k-2}T^*\otimes\mathfrak{g} & \xrightarrow{\psi_{k-2}} & \bigoplus_{1\leqq a\leqq r} S^{k-d_a-2}T^*
\end{array}$$

commutes. Thus, if $v\in S^kT^*\otimes\mathfrak{g}$, with $k\geqq 2$, satisfies $\mathrm{Tr}\,.\,\psi_k(v)=0$, then $\psi_{k-2}(\mathrm{Tr}\, v)=0$ and, by Theorem 2.1, there exists $u\in S^{k-1}T^*\otimes\mathfrak{g}$ such that $\varphi_{k-2}(u)=\mathrm{Tr}\, v$. Since the mappings (2.16) are surjective, there is an element $u'\in S^{k+1}T^*\otimes\mathfrak{g}$ such that $\mathrm{Tr}\, u'=u$. Then u' is an element of $h_k\otimes\mathfrak{g}$ such that

$$v-(\mathrm{id}\otimes\varphi)\, u'\in S^k_0\, T^*\otimes\mathfrak{g}.$$

The hypotheses of Lemma 2.5 are satisfied, and hence v belongs to $(\mathrm{id}\otimes\varphi)(h_k\otimes\mathfrak{g})$.

For $k\geqq 2$, the kernel of

$$\mathrm{id}\otimes\varphi\colon h_k\otimes\mathfrak{g}\to S^k T^*\otimes\mathfrak{g}$$

is equal to the kernel of

$$(\sigma_{k-2}\otimes\mathrm{id})\oplus(\mathrm{id}\otimes\varphi)\colon S^k T^*\otimes T^*\otimes\mathfrak{g}\to(S^{k-2}T^*\otimes T^*\otimes\mathfrak{g})\oplus(S^k T^*\otimes\mathfrak{g});$$

hence h'_{k+2} is the k-th prolongation of h'_2, for $k \geqq 2$. Under the hypotheses of Theorem 2.2, from the exactness of the sequences (2.18) and from the involutivity of h_2, we deduce by a standard argument (*see* [4]) the following result:

COROLLARY 2.2. — *If* $n \geqq 3$ *and* $(X_1, X_2, X_3) \in U_2$, *we have*

$$H^{k,2}(h'_2) \cong \mathbb{R}^{m_k}, \qquad H^{k,i}(h'_2) = 0,$$

for $k \geqq 2$ *and* $i > 2$.

Thus, under the hypotheses of Corollary 2.2, if $d = \sup(d_a)$, then

$$H^{m,j}(h'_2) = 0,$$

for all $m \geqq d+1$ and $j \geqq 0$.

Let

$$(A_{ij}, B_j)_{1 \leqq i, j \leqq n}$$

be an element of $\mathfrak{g}^{n(n+1)}$. Let p be a homogeneous polynomial of $I(\mathfrak{g})$ of degree $d+1$. We define a linear mapping

$$\chi_p : S^d T^* \otimes \mathfrak{g} \to \mathbb{R} \tag{2.23}$$

by

$$\chi_p(u) = \sum_{\substack{|\alpha| = d-1 \\ 1 \leqq j, l \leqq n}} \frac{1}{\alpha!} \{ p(X^{\alpha+\varepsilon_j}, [u_{\alpha+\varepsilon_l}, A_{jl}]) - p(X^\alpha, [X_j, X_l], [u_{\alpha+\varepsilon_l}, B_j]) \},$$

where

$$u = \sum_{|\alpha| = d} \frac{dx^\alpha}{\alpha!} \otimes u_\alpha \in S^d T^* \otimes \mathfrak{g},$$

with $u_\alpha \in \mathfrak{g}$. We now derive an expression for $\chi_p(u)$, which we shall need in Section 4. First, we have

$$\sum_{\substack{|\alpha| = d-1 \\ 1 \leqq j, l \leqq n}} \frac{1}{\alpha!} p(X^{\alpha+\varepsilon_j}, [[u_{\alpha+\varepsilon_j}, B_l] + [B_j, u_{\alpha+\varepsilon_l}], X_l])$$

$$\begin{aligned} &= \sum_{\substack{|\alpha| = d-1 \\ 1 \leqq j, l \leqq n}} \frac{1}{\alpha!} \{ p(X^\alpha, [X_j, X_l], [u_{\alpha+\varepsilon_l}, B_j] - [u_{\alpha+\varepsilon_j}, B_l]) \\ &\qquad + \sum_{i=1}^{n} \alpha_i p(X^{\alpha+\varepsilon_j-\varepsilon_i}, [X_i, X_l], [u_{\alpha+\varepsilon_l}, B_j] - [u_{\alpha+\varepsilon_j}, B_l]) \} \\ &= 2 \sum_{\substack{|\alpha| = d-1 \\ 1 \leqq j, l \leqq n}} \frac{1}{\alpha!} p(X^\alpha, [X_j, X_l], [u_{\alpha+\varepsilon_l}, B_j]) \\ &+ \sum_{\substack{|\beta| = d-2 \\ 1 \leqq i, j, l \leqq n}} \frac{1}{\beta!} p(X^{\beta+\varepsilon_j}, [X_i, X_l], [u_{\beta+\varepsilon_i+\varepsilon_l}, B_j] - [u_{\beta+\varepsilon_i+\varepsilon_j}, B_l]). \end{aligned}$$

Since $u_{\beta+\varepsilon_i+\varepsilon_l}$ is a symmetric function of i and l, we see that

$$\sum_{1 \leqq i,\, l \leqq n} p(X^{\beta+\varepsilon_j}, [X_i, X_l], [u_{\beta+\varepsilon_i+\varepsilon_l}, B_j]) = 0,$$

for $1 \leqq j \leqq n$ and $|\beta| = d-2$. On the other hand, we obtain

$$\sum_{\substack{|\beta|=d-2 \\ 1 \leqq i,\, j,\, l \leqq n}} \frac{1}{\beta!} p(X^{\beta+\varepsilon_j}, [X_i, X_l], [u_{\beta+\varepsilon_i+\varepsilon_j}, B_l])$$

$$= \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq i,\, j,\, l \leqq n}} \frac{\alpha_j}{\alpha!} p(X^\alpha, [X_i, X_l], [u_{\alpha+\varepsilon_i}, B_l])$$

$$= (d-1) \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq i,\, l \leqq n}} \frac{1}{\alpha!} p(X^\alpha, [X_i, X_l], [u_{\alpha+\varepsilon_i}, B_l]).$$

We therefore have verified the equality

$$\sum_{\substack{|\alpha|=d-1 \\ 1 \leqq j,\, l \leqq n}} \frac{1}{\alpha!} p(X^{\alpha+\varepsilon_j}, [[u_{\alpha+\varepsilon_j}, B_l] + [B_j, u_{\alpha+\varepsilon_l}], X_l])$$

$$= (d+1) \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq j,\, l \leqq n}} \frac{1}{\alpha!} p(X^\alpha, [X_j, X_l], [u_{\alpha+\varepsilon_l}, B_j]),$$

from which we deduce the relations

$$(2.24) \quad \chi_p(u) = \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq j,\, l \leqq n}} \frac{1}{\alpha!} p\left(X^{\alpha+\varepsilon_j}, [u_{\alpha+\varepsilon_l}, A_{jl}] - \frac{1}{d+1} [[u_{\alpha+\varepsilon_j}, B_l] + [B_j, u_{\alpha+\varepsilon_l}], X_l]\right)$$

$$= \sum_{\substack{|\alpha|=d \\ 1 \leqq j,\, l \leqq n}} \frac{\alpha_j}{\alpha!} p\left(X^\alpha, [u_{\alpha-\varepsilon_j+\varepsilon_l}, A_{jl}] - \frac{1}{d+1} [[u_\alpha, B_l] + [B_j, u_{\alpha-\varepsilon_j+\varepsilon_l}], X_l]\right).$$

We define a mapping

$$\chi_k : S^k T^* \otimes \mathfrak{g} \to \bigoplus_{1 \leqq a \leqq r} S^{k-d_a} T^*$$

by

$$\chi_k(v) = ((\chi_{p_a})_{+(k-d_a)}(v))_{1 \leqq a \leqq r},$$

for $v \in S^k T^* \otimes \mathfrak{g}$. If v is the element (2.2) of $S^k T^* \otimes \mathfrak{g}$ and w_a is the element (2.5) of $S^{k-d_a} T^*$, for $1 \leqq a \leqq r$, the equation $\chi_k(v) = (w_1, \ldots, w_r)$ is equivalent to the system

$$(2.25) \quad \sum_{\substack{|\alpha|=d_a-1 \\ 1 \leqq j,\, l \leqq n}} \frac{1}{\alpha!} \{ p_a(X^{\alpha+\varepsilon_j}, [v_{\alpha+\beta+\varepsilon_l}, A_{jl}]) - p_a(X^\alpha, [X_j, X_l], [v_{\alpha+\beta+\varepsilon_l}, B_j]) \} = w_{a,\beta},$$

with $|\beta|=k-d_a\geqq 0$, $1\leqq a\leqq r$. For $1\leqq a\leqq r$, $1\leqq l\leqq n$ and $u\in\mathfrak{g}$, we set

$$\tau_{a,l}(u)=\frac{1}{(d_a-1)!}\sum_{j=1}^{n}\{p_a(X_1^{d_a-1}, X_j, [u, A_{jl}])-p_a(X_1^{d_a-1}, [X_j, X_l], [u, B_j])\};$$

if $u\in\mathfrak{g}_{X_1}$, we see that

$$\tau_{a,1}(u)=\frac{1}{(d_a-1)!}\sum_{j=1}^{n}p_a(X_1^{d_a-1}, X_j, [u, A_{j1}-[X_1, B_j]]). \tag{2.26}$$

PROPOSITION 2.5. — *If* $n\geqq 2$ *and*

$$(X_1, X_2, \ldots, X_n, A_{11}-[X_1, B_1], A_{21}-[X_1, B_2], \ldots, A_{n1}-[X_1, B_n])$$

belongs to U_3, *then the mapping*

$$\chi_k\colon g_k\to\bigoplus_{1\leqq a\leqq r}S^{k-d_a}T^*$$

is surjective for $k\geqq 1$.

Proof. — For $1\leqq a\leqq r$, consider the elements $w_a\in S^{k-d_a}T^*$ given by (2.5), with $w_{a,\beta}\in\mathbb{R}$. We wish to solve the system S of equations consisting of

$$\sum_{j=1}^{n}[X_j, v_{\alpha+\varepsilon_j}]=0,\qquad |\alpha|=k-1,$$

and (2.25), with $|\beta|=k-d_a\geqq 0$, $1\leqq a\leqq r$, for an element v of $S^kT^*\otimes\mathfrak{g}$ of the form (2.2). Let m be an integer satisfying $1\leqq m\leqq k$. The system S_m of equations

$$[X_1, v_{m\varepsilon_1+\beta}]+\sum_{j=2}^{n}[X_j, v_{(m-1)\varepsilon_1+\beta+\varepsilon_j}]=0, \tag{2.27}$$

$$\begin{aligned}\tau_{a,1}(v_{m\varepsilon_1+\beta})={}&w_{a,(m-d_a)\varepsilon_1+\beta}-\sum_{l=2}^{n}\tau_{a,l}(v_{(m-1)\varepsilon_1+\beta+\varepsilon_l})\\&-\sum_{\substack{|\alpha|=d_a-1\\ \alpha_1<d_a-1\\ 1\leqq j,l\leqq n}}\frac{1}{\alpha!}\{p_a(X^{\alpha+\varepsilon_j}, [v_{\alpha+(m-d_a)\varepsilon_1+\beta+\varepsilon_l}, A_{jl}])\\&\qquad-p_a(X^{\alpha}, [X_j, X_l], [v_{\alpha+(m-d_a)\varepsilon_1+\beta+\varepsilon_l}, B_j])\},\end{aligned} \tag{2.28}$$

with $d_a\leqq m$, $\beta=(0, \beta_2, \ldots, \beta_n)$, $|\beta|=k-m$ and $1\leqq a\leqq r$, is a sub-system of S. Note that the right-hand sides of (2.27) and (2.28) only involve the $v_{l\varepsilon_1+\gamma}$, with $l<m$ and $\gamma=(0, \gamma_2, \ldots, \gamma_n)$, $|\gamma|=k-l$. We denote by S_0 the empty system of equations. Under our hypotheses, we now construct a solution v of the form (2.2) of S, or equivalently of all the systems S_m, with $1\leqq m\leqq k$. For $\beta=(0, \beta_2, \ldots, \beta_n)$, with $|\beta|=k$, let $v_\beta=0$. Let m be an integer satisfying $1\leqq m\leqq k$ and assume that we have chosen elements $v_{l\varepsilon_1+\beta}\in\mathfrak{g}$, for all $0\leqq l<m$, $\beta=(0, \beta_2, \ldots, \beta_n)$, $|\beta|=k-l$, satisfying the systems $S_0, S_1, \ldots, S_{m-1}$. Let

$\beta=(0, \beta_2, \ldots, \beta_n)$, with $|\beta|=k-m$. Since X_1 is principal, we may apply Lemma 2.2 to the element u of $S^m T^* \otimes \mathfrak{g}$ defined by

$$u=\sum_{|\alpha|=m} \frac{dx^\alpha}{\alpha!} \otimes v_{\alpha+\beta},$$

and we may choose $u'_\beta \in \mathfrak{g}$ satisfying

$$[X_1, u'_\beta]+\sum_{j=2}^{n} [X_j, v_{(m-1)\varepsilon_1+\beta+\varepsilon_j}]=0.$$

If $w'_{a,(m-d_a)\varepsilon_1+\beta}$ denotes the right-hand side of (2.28), by (2.26) and the remark preceding Lemma 1.2, our hypothesis enables us to solve the equations

$$\tau_{a,1}(u_\beta)=w'_{a,(m-d_a)\varepsilon_1+\beta}-\tau_{a,1}(u'_\beta), \qquad 1 \leqq a \leqq r,$$

for $u_\beta \in \mathfrak{g}_{X_1}$. Then $v_{m\varepsilon_1+\beta}=u_\beta+u'_\beta$ is a solution of the equations (2.27) and (2.28), and we have thus constructed a solution of S_m.

If W is a real vector space, we define the mapping

$$\psi'_k: (S^k T^* \otimes \mathfrak{g}) \oplus W \to \bigoplus_{1 \leqq a \leqq r} S^{k-d_a} T^*$$

by $\psi'_k(v, w)=\psi_k(v)$, for $v \in S^k T^* \otimes \mathfrak{g}$ and $w \in W$. From Theorem 2.1 and Proposition 2.5, we deduce the following:

Corollary 2.3. — *If $n \geqq 2$ and*

$$(X_1, X_2, \ldots, X_n, A_{11}-[X_1, B_1], A_{21}-[X_1, B_2], \ldots, A_{n1}-[X_1, B_n])$$

belongs to U_3, then the sequences

$$S^{k+1} T^* \otimes \mathfrak{g} \xrightarrow{\varphi_k \oplus \chi_{k+1}} (S^k T^* \otimes \mathfrak{g}) \oplus \bigoplus_{1 \leqq a \leqq r} S^{k-d_a+1} T^* \xrightarrow{\psi'_k} \bigoplus_{1 \leqq a \leqq r} S^{k-d_a} T^* \to 0$$

are exact for $k \geqq 0$.

Let $d=\sup(d_a)$ and, for $m \geqq 0$, denote by g'_{d+m} the subspace of $S^{d+m} T^* \otimes \mathfrak{g}$ equal to the kernel of $\varphi_{d+m-1} \oplus \chi_{d+m}$. Then g'_{d+m} is the m-th prolongation of g'_d. From Corollary 2.3, we deduce by a standard argument (*see* [4]) the following result:

Corollary 2.4. — *If $n \geqq 2$ and*

$$(X_1, X_2, \ldots, X_n, A_{11}-[X_1, B_1], A_{21}-[X_1, B_2], \ldots, A_{n1}-[X_1, B_n])$$

belongs to U_3, then

$$H^{d+m, j}(g'_d)=0,$$

for all $m, j \geqq 0$.

Let

$$\chi_p' : S^{d+1}T^* \otimes T^* \otimes \mathfrak{g} \to \mathbb{R} \tag{2.29}$$

be the linear mapping defined by

$$\chi_p'(u) = \sum_{\substack{|\alpha|=d \\ 1 \leqq i,\, j \leqq n}} \frac{1}{\alpha!} p\Big(X^\alpha, [X_j, [u_{\alpha+\varepsilon_i, i}, B_j] + 2[B_i, u_{\alpha+\varepsilon_i, j}]] - 2[A_{ij}, u_{\alpha+\varepsilon_i, j}] - \sum_{l=1}^{n} \alpha_l [A_{lj}, u_{\alpha+2\varepsilon_i-\varepsilon_l, j}]\Big),$$

where

$$u = \sum_{|\alpha|=d+1} \frac{dx^\alpha}{\alpha!} \otimes dx^j \otimes u_{\alpha, j} \in S^{d+1}T^* \otimes T^* \otimes \mathfrak{g},$$

with $u_{\alpha, j} \in \mathfrak{g}$. Let

$$u = \sum_{|\alpha|=d+2} \frac{dx^\alpha}{\alpha!} \otimes u_\alpha,$$

with $u_\alpha \in \mathfrak{g}$, be an element of $S^{d+2}T^* \otimes \mathfrak{g}$. By Proposition 2.1, we have

$$\sum_{\substack{|\alpha|=d \\ 1 \leqq j \leqq n}} \frac{1}{\alpha!} p(X^\alpha, [X_j, [B_i, u_{\alpha+\varepsilon_i+\varepsilon_j}]]) = 0,$$

for all $1 \leqq i \leqq n$. On the other hand, if u belongs to g_{d+2}, by Lemma 2.3, we see that

$$\sum_{|\alpha|=d} \frac{1}{\alpha!} p(X^\alpha, [A_{ij}, u_{\alpha+\varepsilon_i+\varepsilon_j}]) = 0,$$

for all $1 \leqq i, j \leqq n$. Therefore, if u is an element of g_{d+2}, we obtain

$$\begin{aligned}
\chi_p'(u) &= \sum_{\substack{|\alpha|=d \\ 1 \leqq i,\, j \leqq n}} \frac{1}{\alpha!} p\Big(X^\alpha, [X_j, [u_{\alpha+2\varepsilon_i}, B_j] + 2[B_i, u_{\alpha+\varepsilon_i+\varepsilon_j}]] - 2[A_{ij}, u_{\alpha+\varepsilon_i+\varepsilon_j}] - \sum_{l=1}^{n} \alpha_l [A_{lj}, u_{\alpha+2\varepsilon_i-\varepsilon_l+\varepsilon_j}]\Big) \\
&= \sum_{\substack{|\alpha|=d \\ 1 \leqq i,\, j \leqq n}} \frac{1}{\alpha!} p(X^\alpha, [X_j, [u_{\alpha+2\varepsilon_i}, B_j]]) - \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq i,\, j,\, l \leqq n}} \frac{1}{\alpha!} p(X^{\alpha+\varepsilon_l}, [A_{lj}, u_{\alpha+2\varepsilon_i+\varepsilon_j}]) \\
&= \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq i,\, j,\, l \leqq n}} \frac{1}{\alpha!} \{ p(X^{\alpha+\varepsilon_j}, [u_{\alpha+2\varepsilon_i+\varepsilon_l}, A_{jl}]) - p(X^\alpha, [X_j, X_l], [u_{\alpha+2\varepsilon_i+\varepsilon_l}, B_j]) \} \\
&= \chi_p(v),
\end{aligned}$$

where $v=(\mathrm{Tr}\otimes\mathrm{id})\,u$ belongs to g_d; thus we have shown that

$$\chi'_p(u)=\chi_p.(\mathrm{Tr}\otimes\mathrm{id})(u), \tag{2.30}$$

for all $u\in g_{d+2}$.

We define a mapping

$$\chi'_k: \mathrm{S}^k\mathrm{T}^*\otimes\mathrm{T}^*\otimes\mathfrak{g}\to \bigoplus_{1\leqq a\leqq r} \mathrm{S}^{k-d_a-1}\mathrm{T}^*$$

by

$$\chi'_k(u)=((\chi'_{p_a})_{+(k-d_a-1)}(u))_{1\leqq a\leqq r},$$

for $u\in \mathrm{S}^k\mathrm{T}^*\otimes\mathrm{T}^*\otimes\mathfrak{g}$. Then by (2.30), we obtain

$$\chi'_{k+1}(u)=\chi_k.(\mathrm{Tr}\otimes\mathrm{id})(u), \tag{2.31}$$

for all $u\in g_{k+2}$.

PROPOSITION 2.6. — *If* $n\geqq 2$ *and*

$$(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n, \mathrm{A}_{11}-[\mathrm{X}_1, \mathrm{B}_1], \mathrm{A}_{21}-[\mathrm{X}_1, \mathrm{B}_2], \ldots, \mathrm{A}_{n1}-[\mathrm{X}_1, \mathrm{B}_n])$$

belongs to U_3, *then the mapping*

$$\chi'_{k+1}: g_{k+2}\to \bigoplus_{1\leqq a\leqq r} \mathrm{S}^{k-d_a}\mathrm{T}^*$$

is surjective for $k\geqq 0$.

Proof. — Let $k\geqq 1$ and $w_a\in \mathrm{S}^{k-d_a}\mathrm{T}^*$, for $1\leqq a\leqq r$. According to Proposition 2.5 and its proof, there exists an element v of g_k given by (2.2), with $v_\alpha=0$ whenever $\alpha_1=0$, satisfying $\chi_k(v)=(w_1, \ldots, w_r)$. We define an element

$$u=\sum_{|\alpha|=k+2}\frac{dx^\alpha}{\alpha!}\otimes u_\alpha,$$

with $u_\alpha\in\mathfrak{g}$, of $\mathrm{S}^{k+2}\mathrm{T}^*\otimes\mathfrak{g}$ as follows. For $\beta=(0, \beta_2, \ldots, \beta_n)$, with $|\beta|=k-m+2$, we let $u_{m\varepsilon_1+\beta}=0$ if $m=0$, 1 or 2, and we define the $u_{m\varepsilon_1+\beta}$, with $3\leqq m\leqq k+2$, recursively on m by

$$u_{m\varepsilon_1+\beta}=v_{(m-2)\varepsilon_1+\beta}-\sum_{j=2}^{n} u_{(m-2)\varepsilon_1+\beta+2\varepsilon_j}.$$

Clearly, we have $(\mathrm{Tr}\otimes\mathrm{id})\,u=v$. To prove that u belongs to g_{k+2}, we now verify by induction on $1\leqq m\leqq k+2$ that

$$\sum_{j=1}^{n}[\mathrm{X}_j, u_{(m-1)\varepsilon_1+\beta+\varepsilon_j}]=0, \tag{2.32}$$

for all $\beta=(0, \beta_2, \ldots, \beta_n)$, with $|\beta|=k-m+2$. This is obviously true for $m=1$ or 2. Assume that $m \geqq 3$ and that (2.32) holds when m is replaced by $m-2$. The left-hand side of (2.32) is equal to

$$\sum_{j=1}^{n} [X_j, v_{(m-3)\varepsilon_1+\beta+\varepsilon_j}] - \sum_{\substack{i, j=1 \\ i>1}}^{n} [X_j, u_{(m-3)\varepsilon_1+\beta+2\varepsilon_i+\varepsilon_j}];$$

the first term of this expression vanishes because v belongs to g_k, and the second one vanishes by our induction hypothesis. Hence, u is an element of g_{k+2} and so, by (2.31), we see that $\chi'_{k+1}(u)=(w_1, \ldots, w_r)$.

Since g_{k+2} is a subspace of $h_{k+1} \otimes \mathfrak{g}$, from Theorem 2.2 and Proposition 2.6, we deduce the following:

Corollary 2.5. — *If $n \geqq 3$ and*

$$(X_1, X_2, \ldots, X_n, A_{11}-[X_1, B_1], A_{21}-[X_1, B_2], \ldots, A_{n1}-[X_1, B_n])$$

belongs to U_4, then the sequences

$$h_{k+1} \otimes \mathfrak{g} \xrightarrow{(\mathrm{id}\otimes\varphi)\oplus\chi'_{k+1}} (S^{k+1}T^* \otimes \mathfrak{g}) \oplus \bigoplus_{1 \leqq a \leqq r} S^{k-d_a}T^* \xrightarrow{\mathrm{Tr}\,.\,\psi'_{k+1}} \bigoplus_{1 \leqq a \leqq r} S^{k-d_a-1}T^* \to 0$$

are exact for $k \geqq 1$.

For $k \geqq 2$, we denote by h''_k the kernel of the mapping

$$(\sigma_{k-2} \otimes \mathrm{id}) \oplus (\mathrm{id} \otimes \varphi) \oplus \chi'_k \colon S^k T^* \otimes T^* \otimes \mathfrak{g} \to (S^{k-2}T^* \otimes T^* \otimes \mathfrak{g}) \oplus (S^k T^* \otimes \mathfrak{g}) \oplus \bigoplus_{1 \leqq a \leqq r} S^{k-d_a-1}T^*.$$

Let $d=\sup(d_a)$; then for $m \geqq 0$ we see that h''_{d+m+1} is the m-th prolongation of h''_{d+1}. From Corollary 2.5, we deduce the following result:

Corollary 2.6. — *If $n \geqq 3$ and*

$$(X_1, X_2, \ldots, X_n, A_{11}-[X_1, B_1], A_{21}-[X_1, B_2], \ldots, A_{n1}-[X_1, B_n])$$

belongs to U_4, then

$$H^{d+m+1, j}(h''_{d+1})=0,$$

for all $m, j \geqq 0$.

3. Connections on a principal bundle

Let M be a manifold of dimension n. We shall denote by T^* the cotangent bundle of M. By $\Lambda^k T^*$ and $S^k T^*$, we shall mean the k-th exterior product and the k-th symmetric product of T^*, respectively. Let E be a vector bundle over M. We denote by $E^{\otimes k} = \bigotimes^k E$ the k-th tensor power of E, by $\mathscr{E}$ [resp. $J_k(E)$] the sheaf (resp. the vector bundle of k-jets) of sections of E over M, and by $\pi_k : J_{k+l}(E) \to J_k(E)$ the natural projection. We identify $J_0(E)$ with E and set $J_k(E)=0$, for $k<0$. If s is a section of E over a neighborhood of $x \in M$, then $j_k(s)(x)$ is the k-jet of s at x. For $k \geqq 0$, we have the exact sequence

$$0 \to S^k T^* \otimes E \xrightarrow{\varepsilon} J_k(E) \xrightarrow{\pi_{k-1}} J_{k-1}(E) \to 0,$$

given by Lemma 2.1 of [4].

If V is a finite-dimensional vector space, we also denote by V the trivial vector bundle $M \times V$ over M, and we write $\Lambda^k \mathscr{T}^* \otimes V$ for the sheaf of sections of $\Lambda^k T^* \otimes V$ over M. If

$$d: \Lambda^j \mathscr{T}^* \otimes V \to \Lambda^{j+1} \mathscr{T}^* \otimes V$$

is the exterior derivative, there exists a unique morphism of vector bundles

$$\sigma_k(d): S^{k+1} T^* \otimes \Lambda^j T^* \otimes V \to S^k T^* \otimes \Lambda^{j+1} T^* \otimes V$$

such that

$$j_k(du')(x) = j_k(du)(x) + \sigma_k(d)\, \varepsilon^{-1} j_{k+1}(u'-u)(x),$$

where u, u' are sections of $\Lambda^j T^* \otimes V$ over a neighborhood of $x \in M$ satisfying $j_k(u)(x) = j_k(u')(x)$.

Let G be a Lie group and let P be a principal bundle over M with structure group G. We denote by R_g the right-action of an element $g \in G$ on P and by V(P) the bundle of all vertical tangent vectors of P. Consider the vector bundle

$$E = P \times_G \mathfrak{g}$$

over M associated to P corresponding to the adjoint representation Ad of G on $\mathfrak{g}$. It is easily seen that the bracket of $\mathfrak{g}$ induces a bracket

$$E \otimes E \to E;$$

thus we also have brackets

$$(\Lambda^j T^* \otimes E) \otimes (\Lambda^k T^* \otimes E) \to (\Lambda^{j+k} T^* \otimes E),$$
$$(\Lambda^j T^* \otimes \mathfrak{g}) \otimes (\Lambda^k T^* \otimes \mathfrak{g}) \to (\Lambda^{j+k} T^* \otimes \mathfrak{g}),$$

determined by

$$[\alpha \otimes u, \beta \otimes v] = (\alpha \wedge \beta) \otimes [u, v],$$

for $\alpha \in \Lambda^j T^*$, $\beta \in \Lambda^k T^*$, $u, v \in E$ or $u, v \in \mathfrak{g}$.

The bundle of all connections on P is an affine bundle over M modeled on the vector bundle $T^* \otimes E$. In fact, if Γ, Γ' are two connections on P, which we identify with their connection forms, then $\omega = \Gamma - \Gamma'$ is a $\mathfrak{g}$-valued 1-form on P satisfying

$$R_g^* \omega = \mathrm{Ad}\, g^{-1} . \omega, \qquad \langle \xi, \omega \rangle = 0,$$

for all $g \in G$ and $\xi \in V(P)$. Such a $\mathfrak{g}$-valued 1-form ω on P can be identified with a section of $T^* \otimes E$. The covariant differential corresponding to the connection Γ gives rise to a first-order differential operator

$$d^{\Gamma}: \Lambda^j \mathscr{T}^* \otimes \mathscr{E} \to \Lambda^{j+1} \mathscr{T}^* \otimes \mathscr{E}.$$

The curvature of Γ is a $\mathfrak{g}$-valued 2-form F_{Γ} on P satisfying

$$R_g^* F_{\Gamma} = \operatorname{Ad} g^{-1} . F_{\Gamma}, \qquad i(\xi) F_{\Gamma} = 0,$$

for all $g \in G$, $\xi \in V(P)$, and can be identified with a section of $\Lambda^2 T^* \otimes E$. Then we have

$$d^{\Gamma} . d^{\Gamma} u = [F_{\Gamma}, u],$$

for $u \in \Lambda^j \mathscr{T}^* \otimes \mathscr{E}$, and the equation

$$d^{\Gamma} F_{\Gamma} = 0$$

is equivalent to the Bianchi identity for Γ (*see* [6]).

A section s of P over an open subset U of M gives us a trivialization

$$U \times G \xrightarrow{\sim} P_{|U}$$

sending $(x, g) \in U \times G$ into $s(x) g$, and a corresponding trivialization

$$U \times \mathfrak{g} \xrightarrow{\sim} E_{|U}. \tag{3.1}$$

If Γ' is the connection on $P_{|U}$ induced by s, whose horizontal spaces are equal to

$$\{ R_{g*} s_* (T_x) \mid x \in U, g \in G \},$$

in terms of this trivialization of E, we see that $d^{\Gamma'}$ corresponds to the exterior derivative

$$d: \Lambda^j \mathscr{T}^* \otimes \mathfrak{g} \to \Lambda^{j+1} \mathscr{T}^* \otimes \mathfrak{g},$$

and that ω is identified with a section of $T^* \otimes \mathfrak{g}$ and d^{Γ} with the differential operator

$$d^{\omega}: \Lambda^j \mathscr{T}^* \otimes \mathfrak{g} \to \Lambda^{j+1} \mathscr{T}^* \otimes \mathfrak{g},$$

defined by

$$d^{\omega} u = du + [\omega, u],$$

for $u \in \Lambda^j \mathscr{T}^* \otimes \mathfrak{g}$, and F_{Γ} with the section

$$F_{\omega} = d\omega + \frac{1}{2} [\omega, \omega]$$

of $\Lambda^2 T^* \otimes \mathfrak{g}$. The Bianchi identity for Γ is now written as

$$d^{\omega} F_{\omega} = dF_{\omega} + [\omega, F_{\omega}] = 0.$$

Given a section F of $\Lambda^2 T^* \otimes E$ over U, finding a connection Γ on $P_{|U}$, whose curvature F_Γ is equal to F, is equivalent to solving the equation

$$(3.2) \qquad d\omega + \frac{1}{2}[\omega, \omega] = F$$

for a section ω of $T^* \otimes \mathfrak{g}$, where F is identified with a section of $\Lambda^2 T^* \otimes \mathfrak{g}$ by means of the trivialization (3.1). A solution ω of this equation must also satisfy the Bianchi identity

$$(3.3) \qquad dF + [\omega, F] = 0.$$

Let p be a linear function on $\bigotimes^{d+1} \mathfrak{g}$. We consider the morphism of vector bundles

$$\tau_p : (\Lambda^2 T^* \otimes \mathfrak{g})^{\otimes d} \otimes \bigotimes^d T^* \otimes \Lambda^3 T^* \otimes \mathfrak{g} \to (\Lambda^3 T^*)^{\otimes (d+1)}$$

determined by

$$\tau_p((\alpha_1 \otimes u_1) \otimes \ldots \otimes (\alpha_d \otimes u_d) \otimes \beta_1 \otimes \ldots \otimes \beta_d \otimes \gamma \otimes v)$$
$$= p(u_1, \ldots, u_d, v)(\alpha_1 \wedge \beta_1) \otimes \ldots \otimes (\alpha_d \wedge \beta_d) \otimes \gamma,$$

for $\alpha_1, \ldots, \alpha_d \in \Lambda^2 T^*$, $\beta_1, \ldots, \beta_d \in T^*$, $\gamma \in \Lambda^3 T^*$, $u_1, \ldots, u_d, v \in \mathfrak{g}$. If ∇ is a connection in $T^*_{|U}$, a solution ω of (3.2) on U also satisfies the equation

$$(3.4) \qquad \tau_p(F^d \otimes \nabla^d(dF + [\omega, F])) = 0$$

of order d, where $F^d \in (\Lambda^2 T^* \otimes \mathfrak{g})^{\otimes d}$ is the d-th tensor power of F.

4. Connections with prescribed curvature

Let $(x^1, \ldots, x^n)$ be a coordinate system on an open subset U of M. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of length n, with $|\alpha| = k$, we consider the section

$$dx^\alpha = (dx^1)^{\alpha_1} \cdot \ldots \cdot (dx^n)^{\alpha_n}$$

of $S^k T^*$ over U and we set

$$\partial^\alpha = \frac{\partial^k}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

Let ω be a section of $T^* \otimes \mathfrak{g}$ over U given by

$$\omega = \sum_{i=1}^n dx^i \otimes \omega_i,$$

where the ω_i are $\mathfrak{g}$-valued functions on U. For $\beta=(\beta_1, \ldots, \beta_n)$, with $|\beta|=k+1$, we set

$$u_\beta=\frac{1}{k+1}\sum_{j=1}^{n}\beta_j\,\partial^{\beta-\varepsilon_j}\omega_j;$$

then

$$u=\sum_{|\beta|=k+1}\frac{dx^\beta}{\beta!}\otimes u_\beta$$

is a section of $S^{k+1}T^*\otimes\mathfrak{g}$. For $\alpha=(\alpha_1, \ldots, \alpha_n)$, with $|\alpha|=k$, and $1\leqq l\leqq n$, we have

$$u_{\alpha+\varepsilon_l}=\frac{1}{k+1}\left(\partial^\alpha\omega_l+\sum_{j=1}^{n}\alpha_j\,\partial^{\alpha-\varepsilon_j+\varepsilon_l}\omega_j\right)$$

and

$$\partial^\alpha\omega_l=u_{\alpha+\varepsilon_l}+\frac{1}{k+1}\sum_{j=1}^{n}\alpha_j\,\partial^{\alpha-\varepsilon_j}\left(\frac{\partial\omega_l}{\partial x^j}-\frac{\partial\omega_j}{\partial x^l}\right). \tag{4.1}$$

A section F of $\Lambda^2T^*\otimes\mathfrak{g}$ over U can be written as

$$F=\frac{1}{2}\sum_{i,j=1}^{n}dx^i\wedge dx^j\otimes F_{ij},$$

where the F_{ij} are $\mathfrak{g}$-valued functions on U satisfying $F_{ij}=-F_{ji}$. The equation (3.2) is equivalent to

$$\frac{\partial\omega_j}{\partial x^i}-\frac{\partial\omega_i}{\partial x^j}+[\omega_i, \omega_j]=F_{ij}, \qquad 1\leqq i, j\leqq n.$$

Henceforth, in this section we suppose that $n=3$. If F is a section of $\Lambda^2T^*\otimes\mathfrak{g}$ over U, we write

$$F=F_1\,dx^2\wedge dx^3+F_2\,dx^3\wedge dx^1+F_3\,dx^1\wedge dx^2,$$

$$\operatorname{div}F=\sum_{j=1}^{3}\frac{\partial F_j}{\partial x^j};$$

if $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, with $|\alpha|=k$, we consider the $\mathfrak{g}^k$-valued function $F^\alpha=(F_1^{\alpha_1}, F_2^{\alpha_2}, F_3^{\alpha_3})$ on U. For $x\in U$, let

$$\varphi_x\colon J_1(\Lambda^2T^*\otimes\mathfrak{g})_x\to\mathfrak{g}^7$$

be the surjective mapping sending $j_1(F)(x)$ into

$$\left(F_1(x), F_2(x), F_3(x), \frac{\partial F_1}{\partial x^1}(x), \frac{\partial F_1}{\partial x^2}(x), \frac{\partial F_1}{\partial x^3}(x), (\operatorname{div}F)(x)\right).$$

Let F be a fixed section of $\Lambda^2 T^* \otimes \mathfrak{g}$ over U. We have

$$[\omega, F] = (dx^1 \wedge dx^2 \wedge dx^3) \otimes \sum_{j=1}^{3} [\omega_j, F_j],$$

$$dF = (dx^1 \wedge dx^2 \wedge dx^3) \otimes \operatorname{div} F;$$

the Bianchi identity (3.3) is equivalent to

$$\sum_{j=1}^{3} [F_j, \omega_j] = \operatorname{div} F.$$

Let ∇ be the flat connection in $T^*_{|U}$ determined by

$$\nabla\, dx^j = 0,$$

for $j=1, 2, 3$. If p is linear function on $\bigotimes^{d+1} \mathfrak{g}$, the equation (3.4) is equivalent to

$$(4.2) \qquad \sum_{|\alpha|=d} \frac{1}{\alpha!} p\left(F^\alpha, \partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega_j, F_j]\right)\right) = 0.$$

We assume for the remainder of this section that $\mathfrak{g}$ is a semi-simple Lie algebra, whose rank is equal to r. We consider the objects associated in Section 1 to the system of generators $\{p_1, \ldots, p_r\}$ of $I(\mathfrak{g})$, in particular the subsets U_1 of $\mathfrak{g} \times \mathfrak{g}$ and U_3 of $\mathfrak{g}^6$, and the subset U_5 of $\mathfrak{g}^7$ given by Lemma 1.2. We suppose that $d_1 \leqq d_2 \leqq \ldots \leqq d_r$. For $x \in U$, since φ_x is surjective, we see that $\mathcal{O}_x = \varphi_x^{-1}(U_5)$ is a non-empty Zariski-open subset of $J_1(\Lambda^2 T^* \otimes \mathfrak{g})_x$.

Let p be a homogeneous element of $I(\mathfrak{g})$ of degree $d+1$, with $d \geqq 1$. By Proposition 2.1, with $X_j = F_j(x)$, for $x \in U$ and for $j=1, 2, 3$, and by (4.1), we obtain

$$\sum_{\substack{|\alpha|=d \\ 1 \leqq l \leqq 3}} \frac{1}{\alpha!} p\left(F^\alpha, [\partial^\alpha \omega_l, F_l]\right) = \frac{1}{d+1} \sum_{\substack{|\alpha|=d \\ 1 \leqq j, l \leqq 3}} \frac{\alpha_j}{\alpha!} p\left(F^\alpha, \left[\partial^{\alpha-\varepsilon_j}\left(\frac{\partial \omega_l}{\partial x^j} - \frac{\partial \omega_j}{\partial x^l}\right), F_l\right]\right).$$

Thus, we may write

$$(4.3) \qquad \sum_{|\alpha|=d} \frac{1}{\alpha!} p\left(F^\alpha, \partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega_j, F_j]\right)\right)$$
$$= \Phi_p(\omega) + \frac{1}{d+1} \sum_{\substack{|\alpha|=d \\ 1 \leqq j, l \leqq 3}} \frac{\alpha_j}{\alpha!} p\left(F^\alpha, \left[\partial^{\alpha-\varepsilon_j}\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right)_{jl}, F_l\right]\right),$$

where $\Phi_p(\omega)$ depends only on the derivatives of ω up to order $d-1$. In fact, for $d \geqq 2$, we have

$$(4.4)\quad \Phi_p(\omega) = \sum_{\substack{|\alpha|=d \\ 1 \leqq j, l \leqq 3}} \frac{\alpha_j}{\alpha!} p\left(F^\alpha, \left[\partial^{\alpha-\varepsilon_j}\omega_l, \frac{\partial F_l}{\partial x^j}\right] - \frac{1}{d+1}[[\partial^{\alpha-\varepsilon_j}\omega_j, \omega_l] + [\omega_j, \partial^{\alpha-\varepsilon_j}\omega_l], F_l]\right) + \rho_p(F, \omega),$$

where $\rho_p(F, \omega)$ is an expression which only involves derivatives of ω up to order $d-2$; if $d=1$, we see that

$$(4.5)\quad \Phi_p(\omega) = \sum_{1 \leqq j, l \leqq 3} p\left(F_j, \left[\omega_l, \frac{\partial F_l}{\partial x^j}\right] - \frac{1}{2}[[\omega_j, \omega_l], F_l]\right) - 3p(F_1, [F_2, F_3]) + \sum_{j=1}^{3} p\left(F_j, \frac{\partial \operatorname{div} F}{\partial x^j}\right).$$

If ω is a solution of (3.2), clearly we also have the equality $\Phi_p(\omega)=0$.

Fix $x \in U$ and set

$$X_j = F_j(x), \qquad A_{jl} = \frac{\partial F_l}{\partial x^j}(x), \qquad B = (\operatorname{div} F)(x),$$

for $j, l = 1, 2, 3$. For $d \geqq 1$ and $\omega_0 \in T_x^* \otimes \mathfrak{g}$, we define a linear mapping

$$\sigma(\Phi_p)_{\omega_0} : S^{d-1} T_x^* \otimes T_x^* \otimes \mathfrak{g} \to \mathbb{R}$$

as follows. If $d \geqq 2$, then it is the unique linear mapping satisfying the relation

$$\Phi_p(\omega')(x) = \Phi_p(\omega)(x) + \sigma(\Phi_p)_{\omega_0}(u),$$

whenever ω, ω' are sections of $T^* \otimes \mathfrak{g}$ over U and u is an element of $S^{d-1} T_x^* \otimes T_x^* \otimes \mathfrak{g}$ such that

$$j_{d-1}(\omega')(x) = j_{d-1}(\omega)(x) + \varepsilon u$$

and $\omega(x) = \omega_0$. If $d=1$, then $\sigma(\Phi_p)_{\omega_0}$ is the differential at ω_0 of Φ_p along the fibers of the vector bundle $T^* \otimes \mathfrak{g}$ over M. For $d \geqq 1$, if

$$(4.6)\qquad \omega_0 = \sum_{j=1}^{3} dx^j \otimes B_j,$$

with $B_j \in \mathfrak{g}$, and

$$u = \sum_{\substack{|\alpha|=d-1 \\ 1 \leqq j \leqq 3}} \frac{dx^\alpha}{\alpha!} \otimes dx^j \otimes u_{\alpha, j},$$

with $u_{\alpha,j} \in \mathfrak{g}$, then

$$(4.7) \quad \sigma(\Phi_p)_{\omega_0}(u) = \sum_{\substack{|\alpha|=d \\ 1 \leq j, l \leq 3}} \frac{\alpha_j}{\alpha!} p\left(X^\alpha, [u_{\alpha-\varepsilon_j, l}, A_{jl}] - \frac{1}{d+1}[[u_{\alpha-\varepsilon_j, j}, B_l] + [B_j, u_{\alpha-\varepsilon_j, l}], X_l]\right).$$

The m-th prolongation

$$\sigma_m(\Phi_p)_{\omega_0} : S^{d+m-1} T_x^* \otimes T_x^* \otimes \mathfrak{g} \to S^m T_x^*$$

of $\sigma(\Phi_p)_{\omega_0}$ has the following property: whenever $m+d \geqq 2$, if ω, ω' are sections of $T^* \otimes \mathfrak{g}$ over U and u is an element of $S^{d+m-1} T_x^* \otimes T_x^* \otimes \mathfrak{g}$ satisfying $\omega(x) = \omega_0$ and

$$j_{d+m-1}(\omega')(x) = j_{d+m-1}(\omega)(x) + \varepsilon u,$$

then

$$(4.8) \qquad j_m(\Phi_p(\omega'))(x) = j_m(\Phi_p(\omega))(x) + \varepsilon \sigma_m(\Phi_p)_{\omega_0}(u)$$

(*see* [5]). Let ω_0 be the element (4.6) of $T_x^* \otimes \mathfrak{g}$ and let

$$\chi_p : S^d T_x^* \otimes \mathfrak{g} \to \mathbb{R}$$

be the mapping (2.23), with $T^* = T_x^*$, defined in terms of the dx^j, X_j, A_{jl} and B_j. Then by (4.7) and (2.24), we see that

$$\sigma(\Phi_p)_{\omega_0 \mid S^d T_x^* \otimes \mathfrak{g}} = \chi_p;$$

therefore, we also have the equality

$$(4.9) \qquad \sigma_m(\Phi_p)_{\omega_0 \mid S^{d+m} T_x^* \otimes \mathfrak{g}} = (\chi_p)_{+m}.$$

For $k \geqq 0$, let $R_{k,x}$ be the subset of $J_k(T^* \otimes \mathfrak{g})_x$ consisting of all k-jets $j_k(\omega)(x)$, where ω is a section of $T^* \otimes \mathfrak{g}$ over U satisfying the equations

$$(4.10) \qquad \begin{gathered} j_{k-1}\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right)(x) = 0, \\ j_{k-d_a+1}(\Phi_{p_a}(\omega))(x) = 0, \qquad j_k(dF + [\omega, F])(x) = 0, \end{gathered}$$

for $d_a \leqq k+1$, $1 \leqq a \leqq r$.

We recall that $j_1(F)(x)$ belongs to $\mathcal{O}_x$ if and only if

$$(X_1, X_2, X_3, A_{11}, A_{21}, A_{31}, B)$$

belongs to the subset U_5 of $\mathfrak{g}^7$ given by Lemma 1.2 (with $n=3$) and defined in terms of the subset U_3 of $\mathfrak{g}^6$. An element

$$(Y_1, Y_2, Y_3, v_1, v_2, v_3)$$

of $\mathfrak{g}^6$ belongs to U_3 if and only if the q elements

$$\{ \tilde{p}_a(Y_1^k, Y_2^{d_a-k}) \}_{(a,k)\in J}$$

of $\mathfrak{g}$ are linearly independent and if the mapping

$$\lambda : \mathfrak{g} \to \mathfrak{g} \oplus \mathbb{R}^r,$$

sending $u \in \mathfrak{g}$ into

$$\lambda(u) = ([Y_1, u], B(Z_1, u), \ldots, B(Z_r, u)),$$

where

$$Z_a = \sum_{j=1}^{3} [v_j, \tilde{p}_a(Y_1^{d_a-1}, Y_j)],$$

is injective.

PROPOSITION 4.1. — *Assume that* $j_1(F)(x)$ *belongs to the open subset* $\mathcal{O}_x$ *of* $J_1(\Lambda^2 T^* \otimes \mathfrak{g})_x$. *Then:*

(i) *There exists an element* $\omega_0 \in R_{0,x}$ *given by* (4.6) *satisfying*

$$(4.11) \qquad (X_1, X_2, X_3, A_{11} - [X_1, B_1], A_{21} - [X_1, B_2], A_{31} - [X_1, B_3]) \in U_3.$$

(ii) *If* $v \in R_{k,x}$ *and the element* $\omega_0 = \pi_0 v$, *given by* (4.6), *satisfies* (4.11), *then there exists* $v' \in R_{k+1,x}$ *such that* $\pi_k v' = v$.

Proof. — (i) Our hypothesis implies that

$$(X_1, X_2, X_3, A_{11}, A_{21}, A_{31}, B)$$

is an element of U_5. According to Lemma 1.2, there exist v_1, v_2, $v_3 \in \mathfrak{g}$ such that

$$\sum_{j=1}^{3} [X_j, v_j] = B$$

and

$$(X_1, X_2, X_3, A_{11} - [X_1, v_1], A_{21} - [X_1, v_2], A_{31} - [X_1, v_3]) \in U_3.$$

If $p_1, \ldots, p_s$ are the elements of $\{ p_1, \ldots, p_r \}$ of degree 2, we set

$$c_a = 3\, p_a(X_1, [X_2, X_3]) - \sum_{j=1}^{3} p_a\left(X_j, \frac{\partial \operatorname{div} F}{\partial x^j}(x)\right) - \sum_{j,l=1}^{3} p_a\left(X_j, [v_l, A_{jl}] - \frac{1}{2}[[v_j, v_l], X_l]\right),$$

for $1 \leq a \leq s$. According to the remarks preceding Lemma 1.2, we are able to solve the equations

$$\sum_{j=1}^{3} p_a(X_j, [w, A_{j1} - [X_1, v_j]]) = c_a,$$

for $w \in \mathfrak{g}_{X_1}$ and all $1 \leqq a \leqq s$. We set $w_1 = w$, $w_2 = w_3 = 0$ and $B_j = v_j + w_j$, for $j = 1, 2, 3$; then $[w_j, w_l] = 0$, for $j, l = 1, 2, 3$. Since $[X_1, w] = 0$, we have

$$\sum_{j=1}^{3} [X_j, B_j] = B;$$

on the other hand, we obtain

$$\begin{aligned}
&\sum_{j,l=1}^{3} p_a\left(X_j, [B_l, A_{jl}] - \frac{1}{2}[[B_j, B_l], X_l]\right) \\
&\quad = \sum_{j,l=1}^{3} \left\{ p_a\left(X_j, [v_l, A_{jl}] - \frac{1}{2}[[v_j, v_l], X_l]\right) \right. \\
&\qquad\qquad \left. + p_a\left(X_j, [w_l, A_{jl}] - \frac{1}{2}[[v_j, w_l] + [w_j, v_l], X_l]\right) \right\} \\
&\quad = \sum_{j,l=1}^{3} p_a\left(X_j, [v_l, A_{jl}] - \frac{1}{2}[[v_j, v_l], X_l]\right) + \sum_{j=1}^{3} p_a(X_j, [w, A_{j1} - [X_1, v_j]]) \\
&\quad = 3 p_a(X_1, [X_2, X_3]) - \sum_{j=1}^{3} p_a\left(X_j, \frac{\partial \operatorname{div} F}{\partial x^j}(x)\right).
\end{aligned}$$

Hence $\omega_0 \in R_{0,x}$; since $w \in \mathfrak{g}_{X_1}$, the relation (4.11) also holds.

(ii) Let $v_1 = j_{k+1}(\omega)(x)$ be an element of $J_{k+1}(T^* \otimes \mathfrak{g})_x$ satisfying $\pi_k v_1 = v$, where ω is a section of $T^* \otimes \mathfrak{g}$ over U. Then (4.10) holds and we have the following equalities among elements of $S^{k-1} T_x^* \otimes \Lambda^3 T_x^* \otimes \mathfrak{g}$:

$$\begin{aligned}
\sigma_{k-1}(d)\, \varepsilon^{-1} j_k\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right)(x) &= \varepsilon^{-1} j_{k-1}\left(d\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right)\right)(x) \\
&= -\varepsilon^{-1} j_{k-1}(dF + [\omega, F])(x) \\
&= 0.
\end{aligned}$$

By the exactness of the sequence

$$0 \to S^{k+2} T^* \otimes \mathfrak{g} \to S^{k+1} T^* \otimes T^* \otimes \mathfrak{g} \xrightarrow{\sigma_k(d)} S^k T^* \otimes \Lambda^2 T^* \otimes \mathfrak{g} \xrightarrow{\sigma_{k-1}(d)} S^{k-1} T^* \otimes \Lambda^3 T^* \otimes \mathfrak{g},$$

there exists $w \in S^{k+1} T_x^* \otimes T_x^* \otimes \mathfrak{g}$ such that

$$\sigma_k(d)\, w = -\varepsilon^{-1} j_k\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right)(x).$$

Let

$$u = \sum_{|\alpha| = k+2} \frac{dx^\alpha}{\alpha!} \otimes u_\alpha$$

be an element of $S^{k+2}T^*_x \otimes \mathfrak{g}$, with $u_\alpha \in \mathfrak{g}$, and

$$\omega' = \sum_{j=1}^{3} dx^j \otimes \omega'_j, \qquad \omega'' = \sum_{j=1}^{3} dx^j \otimes \omega''_j$$

be sections of $T^* \otimes \mathfrak{g}$ over U satisfying

$$j_{k+1}(\omega')(x) = v_1 + \varepsilon w, \qquad j_{k+1}(\omega'')(x) = v_1 + \varepsilon u + \varepsilon w;$$

then we have

$$j_k\left(d\omega' + \frac{1}{2}[\omega', \omega'] - F\right)(x) = j_k\left(d\omega'' + \frac{1}{2}[\omega'', \omega''] - F\right)(x) = 0. \tag{4.12}$$

We now will choose the element u in such a way that the equations

$$j_{k+1}(dF + [\omega'', F])(x) = 0, \qquad j_{k-d_a+2}(\Phi_{p_a}(\omega''))(x) = 0, \tag{4.13}$$

with $d_a \leqq k+2$, $1 \leqq a \leqq r$, also hold. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $|\alpha| = k+1$, we have

$$\partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega''_j, F_j]\right)(x) = \partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega'_j, F_j]\right)(x) + \sum_{j=1}^{3} [u_{\alpha+\varepsilon_j}, X_j]. \tag{4.14}$$

By our hypothesis on $j_1(F)(x)$, we know that (X_1, X_2) belongs to U_1. Therefore, according to Theorem 2.1, if

$$\sum_{|\alpha| = d_a} \frac{1}{\alpha!} p_a\left(F^\alpha, \partial^{\alpha+\beta}\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega'_j, F_j]\right)\right)(x) = 0, \tag{4.15}$$

for all $1 \leqq a \leqq r$, with $d_a \leqq k+1$, and all β, with $|\beta| = k+1-d_a$, we may choose the element u of $S^{k+2}T^*_x \otimes \mathfrak{g}$ in such a way that the right-hand side of (4.14) vanishes. Since

$$j_k(dF + [\omega', F])(x) = j_k(dF + [\omega, F])(x) = 0,$$

by (4.3), (4.12) and (4.10), the left-hand side of (4.15) is equal to

$$\partial^\beta\left(\sum_{|\alpha| = d_a} \frac{1}{\alpha!} p_a\left(F^\alpha, \partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega'_j, F_j]\right)\right)\right)(x) = (\partial^\beta \Phi_{p_a}(\omega'))(x) = (\partial^\beta \Phi_{p_a}(\omega))(x) = 0.$$

If $d_a \leqq k+2$ and $1 \leqq a \leqq r$, we set $m_a = k+2-d_a$; according to (4.10), (4.8) and (4.9), we have the following equalities among elements of $S^{m_a}T^*_x$:

$$\begin{aligned} \varepsilon^{-1} j_{m_a}(\Phi_{p_a}(\omega''))(x) &= \varepsilon^{-1} j_{m_a}(\Phi_{p_a}(\omega))(x) + \sigma_{m_a}(\Phi_{p_a})_{\omega 0}(u+w) \\ &= \varepsilon^{-1} j_{m_a}(\Phi_{p_a}(\omega))(x) + \sigma_{m_a}(\Phi_{p_a})_{\omega 0}(w) + (\chi_{p_a})_{+m_a}(u). \end{aligned}$$

As (4.15) holds for all $1 \leqq a \leqq r$, with $d_a \leqq k+1$, and all β, with $|\beta| = k+1-d_a$, by (4.11) and Corollary 2.3, we are able to find a solution $u \in S^{k+2}T^*_x \otimes \mathfrak{g}$ of the system of equations

$$\sum_{j=1}^{3} [X_j, u_{\alpha+\varepsilon_j}] = \partial^\alpha\left(\operatorname{div} F + \sum_{j=1}^{3} [\omega'_j, F_j]\right)(x),$$

$$(\chi_{p_a})_{+m_a}(u) = -\varepsilon^{-1} j_{m_a}(\Phi_{p_a}(\omega))(x) - \sigma_{m_a}(\Phi_{p_a})_{\omega_0}(w),$$

for all α, with $|\alpha| = k+1$, and for all $1 \leqq a \leqq r$, with $d_a \leqq k+2$ and $m_a = k+2-d_a$. With this choice of u, the section ω'' of $T^* \otimes \mathfrak{g}$ satisfies the equations (4.13), and so $j_{k+1}(\omega'')(x)$ belongs to $R_{k+1,x}$.

The preceding proposition together with Theorem 2.2, Chapter IX of [1] (due to Malgrange [9], Appendix) applied to the system of equations

$$(4.16)\qquad \begin{gathered} j_{k-1}\left(d\omega + \frac{1}{2}[\omega, \omega] - F\right) = 0, \\ j_{k-d_a+1}(\Phi_{p_a}(\omega)) = 0, \qquad j_k(dF + [\omega, F]) = 0 \end{gathered}$$

of order k for a section ω of $T^* \otimes \mathfrak{g}$ over U, where $k = \sup(d_r - 1, 1)$ and $1 \leqq a \leqq r$, yields the following result:

THEOREM 4.1. — *Suppose that* F_1, F_2, F_3 *are real-analytic functions of* (x^1, x^2, x^3) *on* U. *Let* $x \in U$ *and assume that* $j_1(F)(x)$ *belongs to the open subset* $\mathcal{O}_x$ *of* $J_1(\Lambda^2 T^* \otimes \mathfrak{g})_x$. *Then there exists a solution*

$$\omega = \sum_{j=1}^{3} dx^j \otimes \omega_j$$

of the system (4.16) *and of the equation* (3.2) *on a neighborhood* $V \subset U$ *of* x, *where the* ω_j *are real-analytic* $\mathfrak{g}$*-valued functions of* (x^1, x^2, x^3) *on* V.

Remark. — Under the assumption that $j_1(F)(y)$ belongs to $\mathcal{O}_y$, for all $y \in U$, using Proposition 4.1, Corollary 2.2 and results from [5], it is easily seen that the subset R_k of $J_k(T^* \otimes \mathfrak{g})_{|U}$, corresponding to the system (4.16), with $k = \sup(d_r - 1, 1)$, is a formally integrable differential equation, whose symbol is involutive, in the sense of [5].

Assume that M is a real-analytic manifold and that P is a real-analytic principal G-bundle over M. Suppose that the section s of P over U is real-analytic and that $(x^1, \ldots, x^n)$ is a real-analytic coordinate system on U. Let $\tilde{\mathcal{O}}_x$ be the non-empty Zariski-open subset of $J_1(\Lambda^2 T^* \otimes E)_x$ equal to the image of $\mathcal{O}_x$ under the isomorphism

$$J_1(\Lambda^2 T^* \otimes \mathfrak{g})_{|U} \to J_1(\Lambda^2 T^* \otimes E)_{|U}$$

determined by the trivialization (3.1). From Theorem 4.1, we deduce:

THEOREM 4.2. — *Assume that* M *is a real-analytic manifold of dimension* 3 *and that* P *is a real-analytic principal* G*-bundle over* M. *Then, for any real-analytic section* F *of* $\Lambda^2 T^* \otimes E$ *over a neighborhood of* x, *with* $j_1(F)(x) \in \tilde{\mathcal{O}}_x$, *there exist a neighborhood* V *of* x *and a real-analytic connection* Γ *on* $P_{|V}$ *whose curvature* F_Γ *is equal to* F *on* V.

5. The Yang-Mills equation

We consider the principal bundle P over M with structure group G of Section 3, and the objects associated to it. We suppose that $\mathfrak{g}$ is a semi-simple Lie algebra. It is easily seen that the Killing form B of $\mathfrak{g}$ induces a scalar product on the vector bundle E. Let g be a Riemannian metric on M. We denote by $\langle\ ,\ \rangle$ the scalar product on the vector bundles T^* and $\Lambda^j T^* \otimes E$ induced by g and this scalar product on E. If Γ is a connection on P, we denote by

$$\delta^\Gamma : \Lambda^{j+1} \mathscr{T}^* \otimes \mathscr{E} \to \Lambda^j \mathscr{T}^* \otimes \mathscr{E}$$

the formal adjoint of d^Γ with respect to the metric g and the scalar product of E. The section $\delta^\Gamma F_\Gamma$ of $T^* \otimes E$ over M is called the current of the connection Γ; later, we shall verify that it satisfies the Bianchi identity

$$\delta^\Gamma . \delta^\Gamma F_\Gamma = 0. \tag{5.1}$$

Given a section C of $T^* \otimes E$ over M, we consider the inhomogeneous Yang-Mills equation

$$\delta^\Gamma F_\Gamma = C \tag{5.2}$$

for a connection Γ on P. In view of (5.1), a solution Γ of this equation must also satisfy

$$\delta^\Gamma C = 0. \tag{5.3}$$

We endow the trivial vector bundle $\mathfrak{g}$ over M with the scalar product induced by the Killing form B. Let ω be a section of $T^* \otimes \mathfrak{g}$ over M. We denote by

$$\delta^\omega : \Lambda^{j+1} \mathscr{T}^* \otimes \mathfrak{g} \to \Lambda^j \mathscr{T}^* \otimes \mathfrak{g}$$

the formal adjoint of the differential operator

$$d^\omega : \Lambda^j \mathscr{T}^* \otimes \mathfrak{g} \to \Lambda^{j+1} \mathscr{T}^* \otimes \mathfrak{g}$$

and by

$$\omega^* : \Lambda^{j+1} T^* \otimes \mathfrak{g} \to \Lambda^j T^* \otimes \mathfrak{g}$$

the morphism of vector bundles equal to the adjoint of the morphism

$$\Lambda^j T^* \otimes \mathfrak{g} \to \Lambda^{j+1} T^* \otimes \mathfrak{g},$$

sending $u \in \Lambda^j T^* \otimes \mathfrak{g}$ into $[\omega, u]$. If d^* is the formal adjoint of d, then we have

$$\delta^\omega u = d^* u + \omega^*(u),$$

for $u \in \Lambda^{j+1} \mathscr{T}^* \otimes \mathfrak{g}$. If $x \in M$ and $k \geqq 0$, the mappings

$$\sigma_k(\delta^\omega) = \sigma_k(d^*) : S^{k+1} T_x^* \otimes T_x^* \otimes \mathfrak{g} \to S^k T_x^* \otimes \mathfrak{g},$$

$$\sigma_k(\delta^\omega d) = \sigma_k(d^* d) : S^{k+2} T_x^* \otimes T_x^* \otimes \mathfrak{g} \to S^k T_x^* \otimes \mathfrak{g},$$

are equal to the mappings $-\mathrm{Tr}_k \otimes \mathrm{id}$ and $-\sigma_k \otimes \mathrm{id}$ respectively, where Tr_k and σ_k are the mappings defined in Section 2, corresponding to the vector space T_x endowed with the metric g. Let $h_{k+2,x}$ be the subspace of $S^{k+2} T_x^* \otimes T_x^*$ equal to the kernel of this mapping σ_k. If $n \geqq 3$, Proposition 2.4 implies that the sequences

$$(5.4)\quad 0 \to h_{k+1,x} \otimes \mathfrak{g} \to S^{k+1} T_x^* \otimes T_x^* \otimes \mathfrak{g} \xrightarrow{\sigma_{k-1}(d^* d)} S^{k-1} T_x^* \otimes T_x^* \otimes \mathfrak{g} \xrightarrow{\sigma_{k-2}(\delta^\omega)} S^{k-2} T_x^* \otimes \mathfrak{g} \to 0$$

are exact, for $k \geqq 1$.

Let s be a section of P over an open subset U of M; consider the connection Γ' on $P_{|U}$ induced by s and the trivialization (3.1). Let Γ be a connection on P. In terms of this trivialization of E, we identify $\omega = \Gamma - \Gamma'$ with a section of $T^* \otimes \mathfrak{g}$; then we see that $\delta^{\Gamma'}$ corresponds to δ and that δ^Γ is identified with δ^ω. Let $x \in U$ and $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of T_x^* and $u \in \mathfrak{g}$. If

$$F = \sum_{1 \leqq i < j \leqq n} \xi_i \wedge \xi_j \otimes F_{ij},$$

with $F_{ij} \in \mathfrak{g}$, is an element of $\Lambda^2 T^* \otimes \mathfrak{g}$, then we see that

$$\langle [F, u], F \rangle = \sum_{1 \leqq i < j \leqq n} B([F_{ij}, u], F_{ij}) = 0.$$

Therefore, if u is a section of $\mathfrak{g}$ over U, we have

$$\langle d^\omega . d^\omega u, F_\omega \rangle = \langle [F_\omega, u], F_\omega \rangle = 0;$$

from this equality, we deduce that

$$(5.5)\qquad \delta^\omega . \delta^\omega F_\omega = 0$$

and that (5.1) holds.

Let $(x^1, \ldots, x^n)$ be a coordinate system on U and let C be a section of $T^* \otimes E$ over U. Finding a connection Γ on $P_{|U}$ satisfying the equation (5.2) is equivalent to solving the equation

$$(5.6)\qquad \delta^\omega \left(d\omega + \frac{1}{2} [\omega, \omega] \right) = C$$

for a section ω of $T^* \otimes \mathfrak{g}$ over U, where C is identified with a section of $T^* \otimes \mathfrak{g}$ over U by means of the trivialization (3.1).

Let

$$C = \sum_{i=1}^{n} dx^i \otimes C_i$$

be a section of $T^* \otimes \mathfrak{g}$ over U, where the C_i are $\mathfrak{g}$-valued functions on U. For $1 \leqq i \leqq n$, we set

$$C'_i = \sum_{j=1}^{n} g^{ij} C_j,$$

and for $\alpha=(\alpha_1, \ldots, \alpha_n)$, with $|\alpha|=k$, we consider the $\mathfrak{g}^k$-valued functions

$$C^\alpha=(C_1^{\alpha_1}, \ldots, C_n^{\alpha_n}), \qquad C'^\alpha=(C_1'^{\alpha_1}, \ldots, C_n'^{\alpha_n})$$

on U. Let ω be a section of $T^*\otimes\mathfrak{g}$ over U given by

$$\omega=\sum_{i=1}^n dx^i\otimes\omega_i,$$

where the ω_i are $\mathfrak{g}$-valued functions on U; for $1\leqq i, j\leqq n$, we set

$$\psi_{ij}=\frac{\partial\omega_j}{\partial x^i}-\frac{\partial\omega_i}{\partial x^j}+[\omega_i, \omega_j].$$

Then it is easily verified that

$$\delta^\omega\left(d\omega+\frac{1}{2}[\omega, \omega]\right)=-\sum_{i=1}^n dx^j\otimes\theta_j,$$

where

$$\theta_j=\sum_{i,k=1}^n\left\{g^{ik}\left(\frac{\partial\psi_{ij}}{\partial x^k}+[\omega_k, \psi_{ij}]\right)+\frac{\partial g^{ik}}{\partial x^k}\psi_{ij}+\sum_{l,m=1}^n g_{ij}g^{lm}\frac{\partial g^{ik}}{\partial x^l}\psi_{mk}\right\}.$$

Hence, for $|\alpha|=d$ and $1\leqq j\leqq n$, we may write

$$(5.7)\qquad \sum_{i,k=1}^n g^{ik}\partial^{\alpha+\varepsilon_i+\varepsilon_k}\omega_j=\sum_{i,k=1}^n\left\{g^{ik}(\partial^{\alpha+\varepsilon_j+\varepsilon_k}\omega_i-[\partial^{\alpha+\varepsilon_k}\omega_i, \omega_j]-2[\omega_i, \partial^{\alpha+\varepsilon_k}\omega_j]\right.$$
$$\left.+[\omega_i, \partial^{\alpha+\varepsilon_j}\omega_k])+\sum_{l=1}^n\frac{\partial g^{ik}}{\partial x^l}\varphi(\alpha, i, k, l;\omega)\right\}+\partial^\alpha\theta_j+\varphi(\alpha, j;\omega),$$

where the $\varphi(\alpha, i, k, l;\omega)$ only involve derivatives of ω of order $d+1$ and the $\varphi(\alpha, j;\omega)$ depend only on the derivatives of ω up to order d.

If ω is a solution of (5.6), by (5.5) it must also satisfy the identity

$$(5.8)\qquad \delta^\omega C=d^*C+\sum_{j=1}^n[C_j', \omega_j]=0,$$

where

$$d^*C=-\sum_{j=1}^n\frac{\partial C_j'}{\partial x^j}.$$

Clearly, the equation (5.8) is equivalent to

$$\sum_{j=1}^n[C_j', \omega_j]=\sum_{j=1}^n\frac{\partial C_j'}{\partial x^j}.$$

We suppose that the semi-simple Lie algebra $\mathfrak{g}$ has rank r and consider the objects associated in Section 1 to the system of generators $\{ p_1, \ldots, p_r \}$ of $I(\mathfrak{g})$, in particular the subsets

$$U_2 \subset \mathfrak{g}^3, \qquad U_4 \subset \mathfrak{g}^{2n}, \qquad U_6 \subset \mathfrak{g}^{2n+1},$$

when $n \geqq 3$. For $x \in U$, let

$$\varphi'_x : J_1(T^*_x \otimes \mathfrak{g}) \to \mathfrak{g}^{2n+1}$$

be the surjective mapping sending $j_1(\theta)(x)$, where

$$\theta = \sum_{j=1}^{n} dx^j \otimes \theta_j$$

is a section of $T^* \otimes \mathfrak{g}$ over U, into

$$\left(\theta'_1(x), \ldots, \theta'_n(x), \frac{\partial \theta'_1}{\partial x^1}(x), \ldots, \frac{\partial \theta'_1}{\partial x^n}(x), \sum_{j=1}^{n} \frac{\partial \theta'_j}{\partial x^j}(x) \right),$$

with $\theta'_i = \sum_{j=1}^{n} g^{ij} \theta_j$. If $n \geqq 3$, for $x \in U$, since φ'_x is surjective, we see that $\mathcal{O}'_x = \varphi'^{-1}_x(U_6)$ is a non-empty Zariski-open subset of $J_1(T^*_x \otimes \mathfrak{g})_x$.

Let p be a homogeneous element of $I(\mathfrak{g})$ of degree $d+1$, with $d \geqq 1$. We consider the morphism of vector bundles

$$\tau'_p : (T^* \otimes \mathfrak{g})^{\otimes d} \otimes \bigotimes^{d+2} T^* \otimes \mathfrak{g} \to \mathbb{R}$$

determined by

$$\tau'_p((\alpha_1 \otimes u_1) \otimes \ldots \otimes (\alpha_d \otimes u_d) \otimes \beta_1 \otimes \ldots \otimes \beta_{d+2} \otimes v)$$
$$= p(u_1, \ldots, u_d, v) \langle \alpha_1, \beta_1 \rangle \ldots \langle \alpha_d, \beta_d \rangle \langle \beta_{d+1}, \beta_{d+2} \rangle,$$

for $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_{d+2} \in T^*$, $u_1, \ldots, u_d, v \in \mathfrak{g}$. Let ∇ be a connection in $T^*_{|U}$; if ω is a solution of (5.6), it also satisfies the equation

$$\tau'_p(C^d \otimes \nabla^{d+2}(\delta^\omega C)) = 0$$

of order $d+2$, where $C^d \in (T^* \otimes \mathfrak{g})^{\otimes d}$ is the d-th tensor power of C.

Assume that ∇ is the flat connection in $T^*_{|U}$ determined by

$$\nabla dx^j = 0,$$

for $1 \leqq j \leqq n$. Then we have

$$(5.9)\quad \tau'_p(C^d \otimes \nabla^{d+2}(\delta^\omega C))$$

$$= \sum_{\substack{|\alpha|=d \\ 1\leqq i,k\leqq n}} \frac{1}{\alpha!} p\left(C'^\alpha, g^{ik}\partial^{\alpha+\varepsilon_i+\varepsilon_k}\left(d^* C + \sum_{j=1}^n [C'_j, \omega_j]\right)\right)$$

$$= \sum_{\substack{|\alpha|=d \\ 1\leqq i,j,k\leqq n}} \frac{1}{\alpha!} p\left(C'^\alpha, g^{ik}\left([C'_j, \partial^{\alpha+\varepsilon_i+\varepsilon_k}\omega_j] + 2\left[\frac{\partial C'_j}{\partial x^i}, \partial^{\alpha+\varepsilon_k}\omega_j\right]\right.\right.$$

$$\left.\left. + \sum_{l=1}^n \alpha_l \left[\frac{\partial C'_j}{\partial x^l}, \partial^{\alpha+\varepsilon_i+\varepsilon_k-\varepsilon_l}\omega_j\right]\right)\right) + \rho_p(C, \omega),$$

where $\rho_p(C, \omega)$ is an expression which only involves derivatives of ω up to order d. By Proposition 2.1, for $1 \leqq i,\ k \leqq n$, we see that

$$\sum_{|\alpha|=d} \frac{1}{\alpha!} p\left(C'^\alpha, \sum_{j=1}^n [C'_j, g^{ik}\partial^{\alpha+\varepsilon_j+\varepsilon_k}\omega_i]\right) = 0,$$

$$\sum_{|\alpha|=d} \frac{1}{\alpha!} p\left(C'^\alpha, \sum_{j=1}^n [C'_j, g^{ik}[\omega_i, \partial^{\alpha+\varepsilon_j}\omega_k]]\right) = 0.$$

Hence, by (5.7), we may write

$$(5.10)\qquad \tau'_p(C^d \otimes \nabla^{d+2}(\delta^\omega C)) = -\Phi'_p(\omega) + \sum_{\substack{|\alpha|=d \\ 1\leqq j\leqq n}} \frac{1}{\alpha!} p(C'^\alpha, [C'_j, \partial^\alpha(\theta_j + C_j)]),$$

where $\Phi'_p(\omega)$ depends only on the derivatives of ω up to order $d+1$. In fact, we have

$$\Phi'_p(\omega) = \sum_{\substack{|\alpha|=d \\ 1\leqq i,j,k\leqq n}} \frac{1}{\alpha!} p\left(C'^\alpha, g^{ik}\left([C'_j, [\partial^{\alpha+\varepsilon_k}\omega_i, \omega_j] + 2[\omega_i, \partial^{\alpha+\varepsilon_k}\omega_j]] - 2\left[\frac{\partial C'_j}{\partial x^i}, \partial^{\alpha+\varepsilon_k}\omega_j\right]\right.\right.$$

$$\left.\left. - \sum_{l=1}^n \alpha_l \left[\frac{\partial C'_j}{\partial x^l}, \partial^{\alpha+\varepsilon_i+\varepsilon_k-\varepsilon_l}\omega_j\right]\right) - \sum_{l=1}^n \left[C'_j, \frac{\partial g^{ik}}{\partial x^l}\varphi(\alpha, i, k, l; \omega)\right]\right)$$

$$+ \rho'_p(C, \omega),$$

where $\rho'_p(C, \omega)$ is an expression which only involves derivatives of ω up to order d. If ω is a solution of (5.6), clearly we also have the equality $\Phi'_p(\omega) = 0$.

Fix $x \in U$ and assume henceforth that $(x^1, \ldots, x^n)$ are normal coordinates at x. For $1 \leqq j,\ l \leqq n$, we set

$$X_j = C_j(x) = C'_j(x), \qquad A_{jl} = \frac{\partial C_l}{\partial x^j}(x) = \frac{\partial C'_l}{\partial x^j}(x),$$

$$B = \sum_{j=1}^n \frac{\partial C_j}{\partial x^j}(x);$$

then the equality

$$(5.11)\quad \Phi'_p(\omega)=\sum_{\substack{|\alpha|=d\\ 1\leq i,j\leq n}}\frac{1}{\alpha!}p\Big(C^\alpha, [C_j, [\partial^{\alpha+\varepsilon_i}\omega_i, \omega_j]+2[\omega_i, \partial^{\alpha+\varepsilon_i}\omega_j]] -2\left[\frac{\partial C_j}{\partial x^i}, \partial^{\alpha+\varepsilon_i}\omega_j\right]-\sum_{l=1}^{n}\alpha_l\left[\frac{\partial C_j}{\partial x^l}, \partial^{\alpha+2\varepsilon_i-\varepsilon_l}\omega_j\right]\Big)+\rho'_p(C,\omega)$$

holds at x.

Let

$$(5.12)\qquad \omega_0=\sum_{j=1}^{n} dx^j\otimes B_j$$

be an element of $T^*_x\otimes g$, with $B_j\in g$, and let

$$\chi'_p: S^{d+1}T^*_x\otimes T^*_x\otimes g\to\mathbb{R}$$

be the mapping (2.29), with $T^*=T^*_x$, defined in terms of the dx^j, X_j, A_{jl} and B_j. Then, whenever ω, ω' are sections of $T^*\otimes g$ over U and u is an element of $S^{d+m+1}T^*_x\otimes T^*_x\otimes g$ satisfying $\omega(x)=\omega_0$ and

$$j_{d+m+1}(\omega')(x)=j_{d+m+1}(\omega)(x)+\varepsilon u,$$

from (5.11) it follows that

$$(5.13)\qquad j_m(\Phi'_p(\omega'))(x)=j_m(\Phi'_p(\omega))(x)+\varepsilon(\chi'_p)_{+m}(u)$$

(*see* [5]).

For $k\geq 0$, let $R'_{k,x}$ be the subset of $J_k(T^*\otimes g)_x$ consisting of all k-jets $j_k(\omega)(x)$, where ω is a section of $T^*\otimes g$ over U satisfying the equations

$$(5.14)\qquad \begin{gathered} j_{k-2}\left(\delta^\omega\left(d\omega+\frac{1}{2}[\omega,\omega]\right)-C\right)=0,\\ j_{k-d_a-1}(\Phi'_{p_a}(\omega))=0,\qquad j_k(\delta^\omega C)=0,\end{gathered}$$

for $d_a\leq k-1$, $1\leq a\leq r$.

We recall that $j_1(C)(x)$ belongs to $\mathcal{O}'_x$ if and only if

$$(X_1,\ldots,X_n,A_{11},\ldots,A_{n1},B)$$

belongs to the subset U_6 of g^{2n+1} given by Lemma 1.2 and defined in terms of the subset U_4 of g^{2n}. An element

$$(Y_1,\ldots,Y_n,v_1,\ldots,v_n)$$

of g^{2n} belongs to U_4 if and only if the q elements

$$\{\tilde{p}_a(Y_1^k, Y_2^{d_a-k})\}_{(a,k)\in J}$$

of $\mathfrak{g}$ are linearly independent, if

$$\{\tilde{p}_a(Y_1^{d_a}), \tilde{p}_a(Y_3^{d_a})\}_{1 \leqq a \leqq r}, \qquad \{\tilde{p}_a(Y_2^{d_a}), \tilde{p}_a(Y_3^{d_a})\}_{1 \leqq a \leqq r}$$

are two sets of $2r$ linearly independent elements of $\mathfrak{g}$ and if the mapping

$$\lambda : \mathfrak{g} \to \mathfrak{g} \oplus \mathbb{R}^r,$$

sending $u \in \mathfrak{g}$ into

$$\lambda(u) = ([Y_1, u], B(Z_1, u), \ldots, B(Z_r, u)),$$

where

$$Z_a = \sum_{j=1}^{n} [v_j, \tilde{p}_a(Y_1^{d_a - 1}, Y_j)],$$

is injective.

PROPOSITION 5.1. — *Assume that* $n \geqq 3$ *and that* $j_1(C)(x)$ *belongs to the open subset* $\mathcal{O}'_x$ *of* $J_1(T^* \otimes \mathfrak{g})_x$. *Then:*

(i) *There exists an element* $\omega_0 \in R'_{0,x}$ *given by* (5.12) *satisfying*

$$(5.15) \qquad (X_1, X_2, \ldots, X_n, A_{11} - [X_1, B_1], A_{21} - [X_1, B_2], \ldots, A_{n1} - [X_1, B_n]) \in U_4.$$

(ii) *If* $v \in R'_{k,x}$ *and the element* $\omega_0 = \pi_0 v$, given by (5.12), *satisfies* (5.15), *then there exists* $v' \in R'_{k+1,x}$ *such that* $\pi_k v' = v$.

Proof. — (i) Our hypothesis implies that

$$(X_1, \ldots, X_n, A_{11}, \ldots, A_{n1}, B)$$

is an element of U_6. According to Lemma 1.2, there exist elements $B_1, \ldots, B_n \in \mathfrak{g}$ such that

$$\sum_{j=1}^{n} [X_j, B_j] = B$$

and such that (5.15) holds. Then the element ω_0 given by (5.12) satisfies the required conditions.

(ii) Let $k \geqq 0$ and $v_1 = j_{k+1}(\omega)(x)$ be an element of $J_{k+1}(T^* \otimes \mathfrak{g})_x$ satisfying $\pi_k v_1 = v$, where

$$\omega = \sum_{j=1}^{n} dx^j \otimes \omega_j$$

is a section of $T^* \otimes \mathfrak{g}$ over U. Then the equations (5.14) hold. If $k = 0$, we have $(\delta^\omega C)(x) = 0$ and there exists an element

$$u = \sum_{i,j=1}^{n} dx^i \otimes dx^j \otimes u_{ij}$$

of $T^*_x \otimes T^*_x \otimes \mathfrak{g}$, with $u_{ij} \in \mathfrak{g}$, satisfying

$$\sum_{j=1}^{n} [X_j, u_{ij}] = -\frac{\partial}{\partial x^i}\left(d^* C + \sum_{j=1}^{n} [C'_j, \omega_j]\right)(x),$$

for $1 \leqq i \leqq n$; then, if ω' is a section of $T^* \otimes \mathfrak{g}$ over U such that

$$j_1(\omega')(x) = v_1 + \varepsilon u,$$

we have $j_1(\delta^{\omega'} C)(x) = 0$, and so $j_1(\omega')(x)$ belongs to $R'_{1,x}$. We now suppose that $k \geqq 1$. By (5.5), we have the following equalities among elements of $S^{k-2} T^*_x \otimes \mathfrak{g}$:

$$\begin{aligned} \sigma_{k-2}(\delta^\omega)\, \varepsilon^{-1} j_{k-1}\left(\delta^\omega\left(d\omega + \frac{1}{2}[\omega, \omega]\right) - C\right)(x) &= \varepsilon^{-1} j_{k-2}\left(\delta^\omega\left(\delta^\omega\left(d\omega + \frac{1}{2}[\omega, \omega]\right) - C\right)\right)(x) \\ &= -\varepsilon^{-1} j_{k-2}(\delta^\omega C)(x) \\ &= 0. \end{aligned}$$

The exactness of the sequence (5.4) gives us an element w of $S^{k+1} T^*_x \otimes T^*_x \otimes \mathfrak{g}$ such that

$$\sigma_{k-1}(d^* d)\, w = -\varepsilon^{-1} j_{k-1}\left(\delta^\omega\left(d\omega + \frac{1}{2}[\omega, \omega]\right) - C\right)(x).$$

Let

$$u = \sum_{|\alpha| = k+1} \frac{dx^\alpha}{\alpha!} \otimes dx^j \otimes u_{\alpha, j}$$

be an element of $h_{k+1, x} \otimes \mathfrak{g}$, with $u_{\alpha, j} \in \mathfrak{g}$, and

$$\omega' = \sum_{j=1}^{n} dx^j \otimes \omega'_j, \qquad \omega'' = \sum_{j=1}^{n} dx^j \otimes \omega''_j$$

be sections of $T^* \otimes \mathfrak{g}$ over U satisfying

$$j_{k+1}(\omega')(x) = v_1 + \varepsilon w, \qquad j_{k+1}(\omega'')(x) = v_1 + \varepsilon u + \varepsilon w.$$

Since $h_{k+1, x} \otimes \mathfrak{g}$ is the kernel of $\sigma_{k-1}(d^* d)$, we see that

$$\begin{aligned} (5.16)\quad j_{k-1}\left(\delta^{\omega''}\left(d\omega'' + \frac{1}{2}[\omega'', \omega'']\right) - C\right)(x) \\ &= j_{k-1}\left(\delta^{\omega'}\left(d\omega' + \frac{1}{2}[\omega', \omega']\right) - C\right)(x) \\ &= j_{k-1}\left(\delta^{\omega}\left(d\omega + \frac{1}{2}[\omega, \omega]\right) - C\right)(x) + \varepsilon \sigma_{k-1}(d^* d)\, w \\ &= 0. \end{aligned}$$

We now will choose the element u of $h_{k+1, x} \otimes \mathfrak{g}$ in such a way that the equations

$$(5.17) \qquad j_{k+1}(\delta^{\omega''} C)(x) = 0, \qquad j_{k-d_a}(\Phi'_{p_a}(\omega''))(x) = 0,$$

with $d_a \leqq k$, $1 \leqq a \leqq r$, also hold. If $|\alpha| = k+1$, we have

$$(5.18)\qquad \partial^\alpha\Big(d^*C + \sum_{j=1}^n [C_j, \omega''_j]\Big)(x) = \partial^\alpha\Big(d^*C + \sum_{j=1}^n [C'_j, \omega'_j]\Big)(x) + \sum_{j=1}^n [X_j, u_{\alpha, j}].$$

By our hypothesis on $j_1(C)(x)$, we know that (X_1, X_2, X_3) belongs to U_2. Therefore, according to Theorem 2.2, if

$$(5.19)\qquad \sum_{\substack{|\alpha| = d_a \\ 1 \leqq i, k \leqq n}} \frac{1}{\alpha!} p_a\Big(C'^\alpha, g^{ik}\,\partial^{\alpha+\beta+\varepsilon_i+\varepsilon_k}\Big(d^*C + \sum_{j=1}^n [C'_j, \omega'_j]\Big)\Big)(x) = 0,$$

for all $1 \leqq a \leqq r$, with $d_a \leqq k-1$, and all β, with $|\beta| = k - d_a - 1$, we may choose the element u of $h_{k+1, x} \otimes \mathfrak{g}$ in such a way that the right-hand side of (5.18) vanishes. Since

$$j_k(\delta^{\omega'} C)(x) = j_k(\delta^\omega C)(x) = 0,$$

by (5.9), (5.10), (5.16) and (5.14), the left-hand side of (5.19) is equal to

$$\partial^\beta\Big(\sum_{\substack{|\alpha| = d_a \\ 1 \leqq i, k \leqq n}} \frac{1}{\alpha!} p_a\Big(C'^\alpha, g^{ik}\,\partial^{\alpha+\varepsilon_i+\varepsilon_k}\Big(d^*C + \sum_{j=1}^n [C'_j, \omega'_j]\Big)\Big)\Big)(x)$$
$$= -(\partial^\beta \Phi'_{p_a}(\omega'))(x) = -(\partial^\beta \Phi'_{p_a}(\omega))(x) = 0.$$

As (5.19) holds for all $1 \leqq a \leqq r$, with $d_a \leqq k-1$, and all β, with $|\beta| = k - d_a - 1$, by (5.15) and Corollary 2.5, we are able to find a solution $u \in h_{k+1, x} \otimes \mathfrak{g}$ of the system of equations

$$\sum_{j=1}^n [X_j, u_{\alpha, j}] = -\partial^\alpha\Big(d^*C + \sum_{j=1}^n [C'_j, \omega'_j]\Big)(x),$$
$$(\chi'_{p_a})_{+(k-d_a)}(u) = -\varepsilon^{-1} j_{k-d_a}(\Phi'_{p_a}(\omega'))(x),$$

for all α, with $|\alpha| = k+1$, and for all $1 \leqq a \leqq r$, with $d_a \leqq k$. By (5.13), with this choice of u, the section ω'' of $T^* \otimes \mathfrak{g}$ satisfies the equations (5.17), and so $j_{k+1}(\omega'')(x)$ belongs to $R'_{k+1, x}$.

Let $d = \sup(d_a)$. The preceding proposition together with Theorem 2.2, Chapter IX of [1] (due to Malgrange [9], Appendix) applied to the system of equations

$$(5.20)\qquad \begin{gathered} j_{d-1}\Big(\delta^\omega\Big(d\omega + \frac{1}{2}[\omega, \omega]\Big) - C\Big) = 0, \\ j_{d-d_a}(\Phi'_{p_a}(\omega)) = 0, \qquad j_{d+1}(\delta^\omega C) = 0, \end{gathered}$$

with $1 \leqq a \leqq r$, of order $d+1$ for a section ω of $T^* \otimes \mathfrak{g}$ over U yields the following result:

THEOREM 5.1. — *Suppose that* $n \geqq 3$ *and that* g *and* $C_1, \ldots, C_n$ *are real-analytic functions of* $(x^1, \ldots, x^n)$ *on* U. *Let* $x \in U$ *and assume that* $j_1(C)(x)$ *belongs to the open* *subset* $\mathcal{O}'_x$ *of* $J_1(T^* \otimes \mathfrak{g})_x$. *Then there exists a solution*

$$\omega = \sum_{j=1}^{n} dx^j \otimes \omega_j$$

of the system (5.20) *and of the equation* (5.6) *on a neighborhood* $V \subset U$ *of* x, *where the* ω_j *are real-analytic* $\mathfrak{g}$-*valued functions of* $(x^1, \ldots, x^n)$ *on* V.

Assume that (M, g) is a real-analytic Riemannian manifold and that P is a real-analytic principal G-bundle over M. Suppose that the section s of P over U is real-analytic and that $(x^1, \ldots, x^n)$ is a real-analytic coordinate system on U. Let $\widetilde{\mathcal{O}}'_x$ be the non-empty Zariski-open subset of $J_1(T^* \otimes E)_x$ equal to the image of $\mathcal{O}'_x$ under the isomorphism

$$J_1(T^* \otimes \mathfrak{g})_{|U} \to J_1(T^* \otimes E)_{|U}$$

determined by the trivialization (3.1). From Theorem 5.1, we deduce:

THEOREM 5.2. — *Assume that* (M, g) *is a real-analytic Riemannian manifold of dimension* $n \geqq 3$ *and that* P *is a real-analytic principal G-bundle over* M. *Then, for any real-analytic section* C *of* $T^* \otimes E$ *over a neighborhood of* x, *with* $j_1(C)(x) \in \widetilde{\mathcal{O}}'_x$, *there exist a neighborhood* V *of* x *and a real-analytic connection* Γ *on* $P_{|V}$ *satisfying the equation* (5.2).

Let x be an arbitrary point of M. For $k \geqq 2$, let $N_{k,x}$ be the subset of $J_k(T^* \otimes \mathfrak{g})_x$ consisting of all k-jets $j_k(\omega)(x)$, where ω is a section of $T^* \otimes \mathfrak{g}$ over M satisfying the equation

$$j_{k-2}(\delta^\omega F_\omega)(x) = 0.$$

The first part of the proof of Proposition 5.1, (ii), with $C = 0$, also shows that:

PROPOSITION 5.2. — *For* $x \in M$, *the mappings*

$$\pi_1 : N_{2,x} \to J_1(T^* \otimes \mathfrak{g})_x, \qquad \pi_k : N_{k+1,x} \to N_{k,x}$$

are surjective, for $k \geqq 2$.

The preceding proposition together with Theorem 2.2, Chapter IX of [1], applied to the equation

$$\delta^\omega F_\omega = 0 \tag{5.21}$$

of order 2 for a section ω of $T^* \otimes \mathfrak{g}$ over M, yields solutions of the homogeneous Yang-Mills equation and the following results:

PROPOSITION 5.3. — *Assume that* (M, g) *is a real-analytic Riemannian manifold of dimension* $n \geqq 3$, *and let* $x \in M$.

(i) *If* q *is an element of* $J_1(T^* \otimes \mathfrak{g})_x$, *there exists a real-analytic solution* ω *of* (5.21) *over a neighborhood of* x *such that* $j_1(\omega)(x) = q$.

(ii) *Assume that* P *is a real-analytic principal* G-*bundle over* M. *Then there exist a neighborhood* V *of* x *and a real-analytic connection* Γ *on* $P_{|V}$ *satisfying the equation*

$$\delta^\Gamma F_\Gamma = 0.$$

Remark. — Using Proposition 5.2 and results from [5], it is easily seen that the subset N_2 of $J_2(T^* \otimes g)$, whose fiber at $x \in M$ is equal to $N_{2,x}$, is a formally integrable differential equation, whose symbol is involutive, in the sense of [5].

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