ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 24, nº 1 (1991), p. 113-137

<http://www.numdam.org/item?id=ASENS_1991_4_24_1_113_0>

Ann. scient. Éc. Norm. Sup.,
4[e] série, t. 24, 1991, p. 113 à 137.

POLYNOMIAL BOUNDS FOR THE NUMBER OF AUTOMORPHISMS OF A SURFACE OF GENERAL TYPE

BY ALESSIO CORTI (*)

ABSTRACT. — We study Weierstraβ points and discriminants on algebraic surfaces of general type and we give applications to a polynomial type estimate in c_2 for the order of the automorphism group.

1. Introduction

In this paper we bound the order of the group of automorphisms of a complex surface of general type by an effective polynomial function of degree 10 in the second Chern class (Theorem 7.10). From this respect, our result is completely analogous to the classical bound 84 $(g-1)$ for curves of genus at least two. We do not know of any example where the growth is more than linear, but perhaps the belief that a linear bound should hold is too naïve. It would be nice to see an example of quadratic growth.

Andreotti [1] gave an estimation of exponential type in the geometric genus. The problem of giving polynomial bounds has been more recently attacked by Howard and Sommese [8], and by Horstmann [7] in his *Ph. D. thesis*, but as far as we know, no one was able to prove the result in its generality.

We use the same method as Howard and Sommese, who understood that the problem can be solved producing an invariant locus on the surface. That such a locus exists is proved in sections 5-6.

The search for an invariant locus led us to study the notion of Weierstraβ points. They were introduced, at least to our knowledge, by Iitaka [9] and Ogawa [15]. Unfortunately here the theory can not be as rich as one would like. In general Weierstraβ points do not exist. We have been able to bound their associated classes in the Chow group. This is done is sections 3-4. The only reason why we include here these estimations is that we think they could be of some technical interest. The problem is to bound the zero cycle of a section of a vector bundle of rank bigger than the base space. Our methods

(*) This paper was written while the author was supported by a government scholarship as a student of the graduate school in mathematics at the Scuola Normale Superiore, Pisa (Italy). It was revised while the author was supported by a teaching fellowship as a graduate student at the University of Utah.

could perhaps be used in different or more general situations. The reader that is not interested can skip section 3 and 4.

It is a pleasure for me to thank my friend and teacher Fabrizio Catanese. Without his stimulating conversations and his help, both human and professional, this paper would have never been written.

I also wish to thank the referee of the *Annales scientifiques de l'École Normale Supérieure*, for pointing out some mistakes in the first version of this paper, and C. Peters for valuable discussions during its revision.

Added to the last version (Nov. 15, 1989): we received a few weeks ago a preprint by Huckleberry and Sauer. They also obtain polynomial bounds, that actually are a little better than ours [of the order $\log(c_2)\,c_2^{15/2}$]. Their method is completely different and ultimately relies on finite group theory.

NOTATIONS. — In this paper S denotes a smooth algebraic surface over the field of complex numbers. In sections 3 to 6, S is a minimal surface of general type with ample canonical sheaf. In section 7, S is a surface of general type. For the definition and standard properties of algebraic surfaces of general type we refer to Barth, Peters and Van de Ven [2].

We use the following notations freely.

- $\mathbf{C}$: the field of complex numbers.
- $\mathbf{P}^n$: the n-dimensional projective space.
- $\mathcal{O}_X$: the structure sheaf of a non singular algebraic variety.
- $\mathrm{rg}(\mathcal{F})$: the rank of a coherent sheaf on an algebraic variety.
- $\mathcal{F}^*$: the dual of a coherent sheaf.
- $\det(\mathcal{F})$: the determinant line bundle of a coherent sheaf.
- $\mathcal{F} \subsetneq \mathcal{G}$: an injective homomorphism of coherent sheaves.
- $\mathcal{F} \cong \mathcal{G}$: a natural isomorphism of coherent sheaves.
- $\mathcal{O}(E)$: the sheaf of sections of a vector bundle E.
- $\mathrm{Gr}(E.)$: the graded bundle associated to a filtration $0=E_0\subset \ldots \subset E_k=E$. Namely, $\mathrm{Gr}(E.)=\bigoplus_{i=1}^{k} E_i/E_{i-1}$.
- E^*: the dual of a vector bundle E.
- $\bigwedge^r E$: the r-th exterior power of E.
- $S^r E$: the r-th symmetric power of E.
- $E\otimes E'$: the tensor product of two vector bundles E and E'.
- $\mathbf{P}(E)$: the projective bundle associated to a vector bundle. In our notation, that we take from Fulton [5], $\mathbf{P}(E)=\mathbf{P}roj(\bigoplus_{n\leq 0}\mathcal{O}(S^n E^*))$.
- $E|_Y$: the restriction of a vector bundle to a subscheme $Y \subsetneq X$.
- $c_i(E)$: the i-th Chern class of E.
- $J^r L$: the r-th jet bundle of a line bundle L.

$\mathcal{O}(D)$: the sheaf of sections of the line bundle associated to a divisor D.

$\mathcal{O}(D+D')$: D and D′ being two divisors, the sheaf $\mathcal{O}(D)\otimes\mathcal{O}(D')$.

$D\equiv D'$: the divisors D and D′ are linearly equivalent.

$|D|$: the complete linear system of effective divisors linearly equivalent to D.

For a smooth algebraic surface S we have the following notations:

Aut (S): the group of automorphisms of S.

I(C): the inertia group of a curve C on S. It is the subgroup of Aut(S) of elements fixing pointwise C. Namely, $I(C)=\{g\in \mathrm{Aut}(S)\ s.t.\ g(p)=p, \forall p\in C\}$.

$A_*(S)$: the Chow ring of cycles mod. rational equivalence. For $\delta\in A_*(S)$ we have a natural decomposition $\delta=\delta_0+\delta_1+\delta_2$ according to the dimension. For a subscheme $Y \subsetneq S$, we write [Y] for its class in $A_*(S)$ (*see* Fulton [5]).

D.D′: the intersection product of two divisors on S.

Ω^1: the cotangent bundle of S [and not the cotangent sheaf, which is denoted by the symbol $\mathcal{O}(\Omega^1)$].

K: the canonical bundle of S, or *a* canonical divisor.

$f_{|nK|}$: the n-th canonical map $f_{|nK|}: S - \to |nK|^*$.

p_g: the geometric genus $h^0(K)$.

p_n: the n-th plurigenus $h^0(nK)$.

q: the irregularity $h^0(\Omega^1)$.

$h^{1,1}$: the dimension over **C** of the vector space $H^1(\Omega^1)$.

$\chi(\mathcal{O}_S)$: the Euler characteristic of the structure sheaf: $\chi(\mathcal{O}_S)=1-q+p_g$.

c_1, c_2: the first and second Chern classes of the cotangent bundle.

$p_a(D)$: the arithmetic genus of a divisor D on S.

$\deg(L|_C)$: for a reduced irreducible curve C on S, and a line bundle L on S, the intersection number L.C.

For a smooth algebraic curve C:

$g(C)$: the geometric genus of C.

For a homology manifold M:

$e(M)$: the topological Euler characteristic of M.

For a finite set Z:

$|Z|$: the order of Z.

For a finite group G acting on S:

[G : G′]: for a subgroup G′ of G, the index of G′ in G.

G_p: for a point $p\in S$, the stabilizer of p in G. Namely, $G_p=\{g\in G\ s.t.\ g(p)=p\}$.

$G(p)$: the orbit of p: $G(p)=\{g(p), g\in G\}$

$x=O(y^k)$: there is a constant c such that $x\leqq cy^k$. We always use this notation meaning that we know how to compute the constant c, that is universal independent of the situation, but we are too lazy to do so.

2. Definition of Weierstraß points

We give in this section the basic definitions of canonical Weierstraß points, following Ogawa [15]. He defined Weierstraß points for algebraic varieties of any dimension relatively to any line bundle L. We simply specialize to surfaces and to the canonical bundle. *Throughout this section* S *denotes an algebraic surface.* We begin with the fundamental:

DEFINITION 2.1. — Let S be an algebraic surface, and $\omega_1, \ldots, \omega_{p_g}$ a basis for $H^0(K)$. It is then defined a homomorphism of vector bundles $v^k : \mathcal{O}_S^{p_g} \to \mathcal{O}(J^k K)$, s. t. $v^k(e_i) = j^k(\omega_i)$, where $j^k(\omega_i)$ is the k-th jet of ω_i. Let $m = \min\left\{p_g, \binom{k+2}{2}\right\}$. The k-th canonical Weierstraß locus $W_k(S)$ is the subscheme of S defined by the vanishing of the sheaf homomorphism $\overset{m}{\wedge} v^k : \overset{m}{\wedge} \mathcal{O}_S^{p_g} \to \mathcal{O}(\overset{m}{\wedge} J^k K)$.

There is a natural stratification of $W_k(S)$:

DEFINITION 2.2. — Let u be an integer, $0 \leqq u \leqq \binom{k+2}{2}$, and $m = \min\{u, p_g\}$. $W_k^u(S)$ is the subscheme of S defined by the vanishing of the sheaf homomorphism $\overset{m}{\wedge} v^k : \overset{m}{\wedge} \mathcal{O}_S^{p_g} \to \mathcal{O}(\overset{m}{\wedge} J^k K)$.

We say that a Weierstraß scheme is non-trivial if it is different from $\varnothing$, S, and in this case its associated class $[W_k^u(S)]$ in $A_*(S)$ is $[W_k^u(S)]_0 + [W_k^u(S)]_1$, where $[W_k^u(S)]_0$ is the class of a zero dimensional cycle and $[W_k^u(S)]_1$ is a divisor. Let us study W_0 and W_1 first. We state a lemma which we shall use a plenty of times in this paper:

LEMMA 2.3. — *We have, for all integers k, an exact sequence* (*called the principal parts exact sequence*):

$$0 \to S^k \Omega^1 \otimes K \to J^k K \to J^{k-1} K \to 0.$$

Proof. — There are many sources. *See* for instance Ogawa [15].

Q.E.D.

LEMMA 2.4. — *Let* S *be a surface of general type. If $p_g \geqq 1$, $W_0(S)$ is the base locus of the complete canonical system $|K|$, and the following estimates hold:*

$$\deg([W_0]_0) \leqq K^2 + c_2, \qquad [W_0]_1^2 \leqq [W_0]_1 . K \leqq K^2.$$

If $p_g \geqq 3$, $W_1(S)$ is the locus of points where the first canonical map $f_{|K|}$ is not a local immersion, and the following estimates hold:

$$\deg([W_1]_0) \leqq 16 K^2 + 4 c_2, \qquad [W_1]_1^2 \leqq 4 [W_1]_1 . K \leqq 16 K^2.$$

Proof. — The assertions about W_0 are clear. Let indeed $K = F + M$ where F is the fixed divisor, and M is the moving part. Clearly $[W_0]_1 = F$ and $K^2 = K.F + K.M \geqq K.F = M.F + F^2 \geqq F^2$. If K has isolated fixed points, they are the fixed points of the moving part M, and there are M^2 of them, counted with multiplicity. To bound M^2, observe first that C being any irreductible curve, $K.C \geqq 0$ unless C is an exceptional-1 curve (here $K.C=-1$), and there are at most c_2 such curves. So

$$K^2 = K.F + K.M \geqq K.M - c_2 = F.M + M^2 - c_2 \geqq M^2 - c_2.$$

We now prove that $f_{|K|}$ is a local immersion at $x \in S$ if and only if v^1 is surjective at x. Let U be a neighborhood of x and $\tilde{f}: U \to \mathbf{C}^{p_g}$ a local lifting of $f_{|K|}$. The differential $df_{|K|}$ is injective at x if and only if $d\tilde{f}$ is injective at x and $\tilde{f}(x) \notin V$, where $\tilde{f}(x)+V$ is the tangent space to $\tilde{f}(U)$ at x. This means exactly that v^1 is surjective at x. Now, remember that we have the principal parts exact sequence (lemma 2.3):

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Now W_1 is the zero scheme of a global section $\sigma = (\sigma_1, \ldots, \sigma_m)$ of $(4K)^m$, for some m. If σ vanishes on a divisor C, $C+E \cong 4K$ for some effective divisor E. Note that C contains all the exeptional curves of S. Since $4K$ is numerically connected, $C.E \geqq 0$ and

$$16K^2 = 4K(C+E) = 4K.C + 4K.E \geqq 4K.C = C^2 + C.E \geqq C^2.$$

To bound $\deg([W_1]_0)$, we proceed as follows. Let D_i be the divisor of zeros of σ_i, then assume $D_1 = F + D'_1$, $D_2 = F + D'_2$, with D'_1, D'_2 tranverse. Then clearly

$$\begin{aligned}\deg([W_1]_0) &\leqq D'_1.D'_2 + \deg Z(\sigma_3|_F, \ldots, \sigma_m|_F) \leqq D'_1.D'_2 + 4K.F + 4c_2 \\ &\leqq D'_1.D'_2 + F.D'_2 + 4K.F + 4c_2 = 4K.D'_2 + 4K.F + 4c_2 = 16K^2 + 4c_2.\end{aligned}$$

Q.E.D.

Let us say something about another extreme situation:

LEMMA 2.5. — *The map* $v^{p_g}: \mathcal{O}_S^{p_g} \to J^{p_g} K$ *is injective.*

Proof. — It follows easily from the theory of Wronskians. *See* Iitaka [9][1].

Q.E.D.

The aim of the next two sections is to give estimates for the fundamental classes of Weierstraß schemes. — In section 3 we give an upper bound for $[W_k^u(S)]_1.K$, and in section 4 for $\deg[W_k^u(S)]_0$. We wish now to observe that, despite Iitaka's conjecture (*cf. Iitaka* [9]), surfaces with trivial Weierstraß schemes do exist, and it seems likely that the generic surface has trivial Weierstraß schemes. In fact we have the following very elementary:

EXAMPLE 2.7. — *A generic quintic surface in* $\mathbf{P}^3$ *has trivial Weierstraß schemes.*

Proof. — To prove the assertion, it suffices to show that for S generic and smooth the natural $v^4: \mathcal{O}_S^4 \to J^2(K)$ is everywhere injective. It is easy to show that this means exactly

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Proof. — There are many sources. *See* for instance Ogawa [15].

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$$\deg([W_1]_0) \leqq 16 K^2 + 4 c_2, \qquad [W_1]_1^2 \leqq 4 [W_1]_1 . K \leqq 16 K^2.$$

Proof. — The assertions about W_0 are clear. Let indeed $K = F + M$ where F is the fixed divisor, and M is the moving part. Clearly $[W_0]_1 = F$ and $K^2 = K.F + K.M \geqq K.F = M.F + F^2 \geqq F^2$. If K has isolated fixed points, they are the fixed points of the moving part M, and there are M^2 of them, counted with

ERRATUM

Une erreur a été commise dans la pagination du N° 1 - 1991, T. 24 des *Annales Scientifiques de l'École Normale Supérieure.*

Les deux pages suivantes remplacent les pages 118 et 119 de ce numéro et sont à insérer en lieu et place.

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A mistake has been made in the pagination of N° 1 - 1991, Vol. 24 of the *Annales Scientifiques de l'École Normale Supérieure.*

The following two pages replace pages 118 and 119 of this number and should be inserted in place.

that $\forall P \in S$, $T_P(S) \cap S$ has not a triple point at P. An easy dimension count now shows that the locus of quintics having Weierstraβ points is of codimension one in the $\mathbf{P}^{55}$ of all quintics.

Q.E.D.

3. Estimates for the dimension one component

Throughout this section, S *indicates a minimal surface of general type over the field of complex numbers, and with a ample canonical sheaf.* – We give an upper estimate for the intersection number $[Z(\sigma)]_1 . K$, where σ is a global section of $\overset{s}{\Lambda} J^r K$. We use K-semistability in the sense of Mumford-Takemoto of the cotangent bundle (theorem 3.2). An upper estimate for $[W_k^u]_1 . K$ then easily follows (*cf.* theorem 4.6).

We first recall the following basic definition:

DEFINITION 3.1. — Let S be a projective surface and H an ample divisor on S. Let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_S$-modules, we define $\mu(\mathscr{F}) = c_1(\mathscr{F}) . H/\mathrm{rg}(\mathscr{F})$. A torsion-free coherent sheaf $\mathscr{E}$ of $\mathscr{O}_S$-modules is called H-semistable if for all coherent subsheaves of $\mathscr{O}_S$-modules $\mathscr{F}$ of $\mathscr{E}$, we have $\mu(\mathscr{F}) \leqq \mu(\mathscr{E})$. Otherwise we say that $\mathscr{E}$ is H-unstable.

THEOREM 3.2. — *The cotangent bundle* Ω^1 *of a surface of general type with ample canonical sheaf is* K-*semistable.*

Proof. – Yau's theorem (Yau [17]) gives a Hermite-Einstein metric on S. The tangent bundle endowed with this metric is by definition Kähler-Einstein with respect to the Käher class, hence K-semistable by a result of Kobayashi and Lübke (*see* Kobayashi [11]).

Q.E.D.

We now show how K-semistability can be used in our situation. We remark that jet bundles are not semistable, since the principal parts exact sequence (lemma 2.3) is destabilizing, as follows by straightforward computations. First of all we state a lemma which will be used several times in this paper.

LEMMA 3.3. — *An exact sequence of vector bundles:*

$$0 \to E' \to E \to E'' \to 0$$

determines in a natural way:

– *A filtration* $E. : 0 = E_0 \subset \ldots \subset E_{r+1} = S^r E$ *with* $\mathrm{Gr}(E.) = \bigoplus_{i=1}^{r+1} S^{i-1} E'' \otimes S^{r-i+1} E'$.

– *A filtration* $E. : 0 = E_0 \subset \ldots \subset E_{r+1} = \overset{r}{\Lambda} E$ *with* $\mathrm{Gr}(.) = \bigoplus_{i=1}^{r+1} \overset{i-1}{\Lambda} E'' \otimes \overset{r-i+1}{\Lambda} E'$.

Q.E.D.

Proof. – See Hirzebruch [6].

THEOREM 3.4. — *Let* σ *be a global section of* $\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K}$, *and* $\mathrm{Z}(\sigma)$ *its zero scheme. We then have :*

$$[\mathrm{Z}(\sigma)]_1 . \mathrm{K} \leqq \frac{s(r+2)}{2} \mathrm{K}^2.$$

Proof. — σ defines an injection $\mathcal{O}_{\mathrm{S}} \hookrightarrow \mathcal{O}(\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K})$. By saturating this inclusion we obtain a dévissage:

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_S & \to & \mathcal{O}(\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K}) & \to & \mathcal{G} & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & \mathcal{O}(\mathrm{D}) & \to & \mathcal{O}(\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K}) & \to & \mathcal{E} & \to 0 \end{array}$$

where D is an effective divisor, $[\mathrm{D}] = [\mathrm{Z}(\sigma)]_1$. It suffices then to show that for any subcoherent sheaf $\mathcal{F}$ of $\mathcal{O}(\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K})$ the following inequality holds:

$$\mu(\mathcal{F}) \leqq \frac{s(r+2)}{2} \mathrm{K}^2.$$

Suppose we have a vector bundle E and a filtration $0 = \mathrm{E}_0 \subset \ldots \subset \mathrm{E}_m \cong \mathrm{E}$ with $\mathcal{O}(\mathrm{E}_i/\mathrm{E}_{i-1})$ K-semistable. We claim that for any subcoherent sheaf $\mathcal{F}$ of $\mathcal{O}(\mathrm{E})$ the following inequality holds:

$$\mu(\mathcal{F}) \leqq \max_{1 \leqq i \leqq m} \mu(\mathrm{E}_i/\mathrm{E}_{i-1}).$$

In fact, taking $\mathcal{F}_i = \mathcal{F} \cap \mathcal{O}(\mathrm{E}_i)$, we obtain a filtration $0 = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_m \cong \mathcal{F}$ with $\mathcal{F}_i/\mathcal{F}_{i-1} \hookrightarrow \mathcal{O}(\mathrm{E}_i/\mathrm{E}_{i-1})$. The claim then follows observing that:

$$\mu(\mathcal{F}) = \sum_{i=1}^{m} \frac{\operatorname{rg}(\mathcal{F}_i/\mathcal{F}_{i-1})}{\operatorname{rg}(\mathcal{F})} \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \leqq \max \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$$

and using K-semistability of $\mathcal{O}(\mathrm{E}_i/\mathrm{E}_{i-1})$.

We study now jet bundles. The principal parts exact sequence (lemma 2.3) induces a filtration of $\mathcal{O}(\overset{s}{\Lambda}\, \mathrm{J}^r\mathrm{K})$ (lemma 3.3) with quotients isomorphic to:

$$\overset{i_r}{\Lambda}(\mathrm{S}^r\Omega^1 \otimes \mathrm{K}) \otimes \ldots \otimes \overset{i_0}{\Lambda}\, \mathrm{K}$$

where i_j are non negative integers s. t. $i_0 + \ldots + i_r = s$. We recall that by Maruyama [12] wedge and symmetric powers, and tensor products of semistable bundles are semistable. By our claim we thus only need to compute μ. We first compute $\mu(\overset{i_r}{\Lambda}(\mathrm{S}^r\Omega^1 \otimes \mathrm{K}))$. By straightforward computations we have

$$c_1(\mathrm{S}^r\Omega^1 \otimes \mathrm{K}) = ((r+1)(r+2)/2)\,\mathrm{K}$$

multiplicity. To bound M^2, observe first that C being any irreductible curve, $K.C\geqq 0$ unless C is an exceptional-1 curve (here $K.C=-1$), and there are at most c_2 such curves. So

$$K^2=K.F+K.M\geqq K.M-c_2=F.M+M^2-c_2\geqq M^2-c_2.$$

We now prove that $f_{|K|}$ is a local immersion at $x\in S$ if and only if v^1 is surjective at x. Let U be a neighborhood of x and $\tilde{f}:U\to \mathbf{C}^{p_g}$ a local lifting of $f_{|K|}$. The differential $df_{|K|}$ is injective at x if and only if $d\tilde{f}$ is injective at x and $\tilde{f}(x)\notin V$, where $\tilde{f}(x)+V$ is the tangent space to $\tilde{f}(U)$ at x. This means exactly that v^1 is surjective at x. Now, remember that we have the principal parts exact sequence (lemma 2.3):

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$$\begin{aligned}\deg([W_1]_0)&\leqq D'_1.D'_2+\deg Z(\sigma_3|_F, \ldots, \sigma_m|_F)\leqq D'_1.D'_2+4K.F+4c_2\\ &\leqq D'_1.D'_2+F.D'_2+4K.F+4c_2=4K.D'_2+4K.F+4c_2=16K^2+4c_2.\end{aligned}$$

Q.E.D.

Let us say something about another extreme situation:

LEMMA 2.5. — *The map $v^{p_g}:\mathcal{O}_S^{p_g}\to J^{p_g}K$ is injective.*

Proof. — It follows easily from the theory of Wronskians. *See* Iitaka [9](1).

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Proof. — To prove the assertion, it suffices to show that for S generic and smooth the natural $v^4:\mathcal{O}_S^4\to J^2(K)$ is everywhere injective. It is easy to show that this means exactly

(1) At a first glance, Iitaka's definitions look different from ours, but what he actually defines is our $W_{p_g}^{p_g}$.

and, since $c_1({}^{p}\,\mathrm{E})=\binom{\mathrm{rg}(\mathrm{E})-1}{p-1}c_1(\mathrm{E})$ we obtain:

$$\mu({}^{i}\,(\mathrm{S}^r\Omega^1\otimes\mathrm{K}))=\frac{i!\,(r+1-i)!}{(r+1)!}\;\frac{r!}{(i-1)!\,(r+1-i)!}\;\frac{(r+1)(r+2)}{2}\mathrm{K}^2=\frac{i(r+2)}{2}\mathrm{K}^2.$$

Finally, since $\mu(\mathrm{E}'\otimes\mathrm{E}'')=\mu(\mathrm{E}')+\mu(\mathrm{E}'')$:

$$\mu({}^{i_r}\,(\mathrm{S}^r\Omega^1\otimes\mathrm{K})\otimes\ldots\otimes{}^{i_0}\,\mathrm{K})\leqq\frac{s(r+2)}{2}\mathrm{K}^2.$$

Q.E.D.

4. Estimate for the dimension zero component

Notations are as in section 3. In particular, S *is a surface of general type over* **C** *and with ample canonical sheaf.* Let σ be a global section of ${}^{s}\,\mathrm{J}^r\mathrm{K}$, and let $\mathrm{Z}(\sigma)$ be the zero-scheme of σ. We assume that σ is not identically zero and we put $[\mathrm{Z}(\sigma)]=[\mathrm{Z}(\sigma)]_0+[\mathrm{Z}(\sigma)]_1$ in $\mathrm{A}_*(\mathrm{S})$, where $[\mathrm{Z}(\sigma)]_0$ is the A_0-component, and $[\mathrm{Z}(\sigma)]_1$ the A_1-component. We give in this section an upper bound for $\deg[\mathrm{Z}(\sigma)]_0$, in term of the invariants of S and the integers r, s (theorem 4.4). This will lead to an estimate for the degree of the A_0-component of $[\mathrm{W}^u_k(\mathrm{S})]$, which will be stated in theorem 4.6. The idea is to take a filtration $0=\mathrm{E}_0\subset\ldots\subset\mathrm{E}_m\cong{}^{s}\,\mathrm{J}^r\mathrm{K}$, with $\mathrm{E}_i/\mathrm{E}_{i-1}$ line bundles, and then put $\deg([\mathrm{Z}(\sigma)]_0)\leqq\max_{i,\,j}\{\mathrm{E}_i/\mathrm{E}_{i-1}\,.\,\mathrm{E}_j/\mathrm{E}_{j-1}\}$. To obtain such a filtration we begin with a filtration of Ω^1.

LEMMA 4.1. — *There exists a global section* $\tau\in\mathrm{H}^0(\Omega^1\otimes 10\,\mathrm{K})$ *with zero scheme* Z *smooth of pure dimension zero, giving thus a filtration:*

$$0\to\mathcal{O}(-10\,\mathrm{K})\to\mathcal{O}(\Omega^1)\to\mathcal{O}(11\,\mathrm{K})\otimes\mathscr{I}_{\mathrm{Z}}\to 0 \tag{4.1}$$

where $\mathscr{I}_{\mathrm{Z}}$ *is the ideal of* Z *and* $\deg(\mathrm{Z})=110\,\mathrm{K}^2+c_2$.

Proof. — Just note that since $\Omega^1_{\mathbf{P}^n}(2)$ is generated by global sections, and the complete linear system $|5\,\mathrm{K}|$ embeds S in projective space, we then have that $\Omega^1\otimes 10\,\mathrm{K}$ is generated by global sections and we may apply Kleiman [10], to obtain a global section τ whose zero scheme Z is smooth of pure dimension zero. τ gives an injection $\mathcal{O}_{\mathrm{S}}\subsetneq\mathcal{O}(\Omega^1)\otimes\mathcal{O}(10\,\mathrm{K})$, which tensored by $-10\,\mathrm{K}$ gives $\mathcal{O}(-10\,\mathrm{K})\subsetneq\mathcal{O}(\Omega^1)$. We have that the cokernel is isomorphic to $\mathscr{L}\otimes\mathscr{I}_{\mathrm{Z}}$, and $\mathscr{L}\cong\mathcal{O}(11\,\mathrm{K})$ by properties of the determinant. It is clear that $\deg(\mathrm{Z})=c_2(\Omega^1\otimes 10\,\mathrm{K})=110\,\mathrm{K}^2+c_2$.

Q.E.D.

We need the following slight refinement of lemma 3.3.

LEMMA 4.2. — *Let* E′, E *be vector bundles on* S. *Suppose we are given an exact sequence:*

$$0\to\mathcal{O}(\mathrm{E}')\to\mathcal{O}(\mathrm{E})\to\mathscr{F}\to 0,$$

where $\mathscr{F}$ is a torsion-free, rank one, coherent sheaf. We are then naturally given:

– *An exact sequence:*

$$0 \to \mathcal{O}(E' \otimes S^{r-1} E) \to \mathcal{O}(S^r E) \to Q \to 0$$

and an inclusion $Q \subsetneq Q^{**} \cong r\mathscr{F}^{**}$.

– *An exact sequence:*

$$0 \to \mathcal{O}(\overset{r}{\Lambda} E') \to \mathcal{O}(\overset{r}{\Lambda} E) \to Q \to 0$$

and an inclusion $Q \subsetneq Q^{**} \cong \overset{r-1}{\Lambda} E' \otimes \mathscr{F}^{**}$.

Proof. – For the first assertion we have (Hirzebruch [6]) an exact sequence on $S' = S - \operatorname{sing}(\mathscr{F})$:

$$0 \to \mathcal{O}(E'|_{S'} \otimes S^{r-1} E|_{S'}) \xrightarrow{i} \mathcal{O}(S^r E|_{S'}) \to r\mathscr{F}|_{S'} \to 0.$$

Now, E and E′ are locally free sheaves, and by Hartogs theorem i extends to an inclusion $\tilde{i}$:

$$\tilde{i}: \quad \mathcal{O}(E' \otimes S^{r-1} E) \subsetneq \mathcal{O}(S^r E).$$

The quotient Q is a torsion-free coherent sheaf, and therefore injects in its bidual Q^{**}. On S′, $Q^{**}|_{S'} \cong r\mathscr{F}^{**}|_{S'}$. Since we are on a non singular surface, Q^{**} and $\mathscr{F}^{**}$ are both locally free, and by Hartogs theorem the isomorphism extends to an isomorphism on all of S.

For the proof of the second assertion, proceed the same way.

Q.E.D.

In the proof of the main theorem 4.4 of this section, we will need to restrict filtration (4.1) to a curve, so we now prove:

LEMMA 4.3. — *Let* C *be a reduced irreducible curve on* S. *Filtration* (4.1) *induces a filtration:*

(4.2) $$0 \to -10K|_{\tilde{C}} + L \to \Omega^1|_{\tilde{C}} \to 11K|_{\tilde{C}} - L \to 0$$

where $\tilde{C}$ *is the normalization of* C *and* L *an effective divisor on* C *s. t.:*

$$\deg(L) \leqq \sum_{P \in Z \cap C} \mu_P(C)$$

where Z *is as in lemma* 4.1 *and* $\mu_P(C)$ *is the multiplicity of* C *at* P.

Proof. – Let $\mathcal{O}(-10K)|_{\tilde{C}} \subsetneq \mathcal{O}(\Omega^1)|_{\tilde{C}}$ be the restriction to $\tilde{C}$ of the inclusion in (4.1). We saturate this inclusion obtaining an exact sequence of vector bundles on $\tilde{C}$:

$$0 \to -10K|_{\tilde{C}} + L \to \Omega^1|_{\tilde{C}} \to L' \to 0$$

with L an effective line bundle on $\tilde{C}$ and L′ a line bundle on $\tilde{C}$. $L' \cong K|_{\tilde{C}} - L$ by functoriality of the determinant bundle: since $K|_{\tilde{C}} = (\overset{2}{\Lambda}\Omega^1)|_{\tilde{C}} = \overset{2}{\Lambda}(\Omega^1|_{\tilde{C}})$, we have

$K|_{\tilde{C}} \cong -10K|_{\tilde{C}} + L + L'$ giving $L' \cong 11K|_{\tilde{C}} - L$. The estimate on $\deg(L)$ comes from the following local computation. Remember that S is locally factorial. This implies that locally in $P \in Z$ the inclusion $\mathcal{O}(-10K) \subsetneq \mathcal{O}(\Omega^1)$ is given by:

$$0 \to \mathcal{O}_S(U) \xrightarrow{i} \mathcal{O}_S(U)^2 \xrightarrow{\pi} (f, g)\, \mathcal{O}_S(U) \to 0$$

where $i(1) = (f, g)$; $\pi(e_1) = -g$, $\pi(e_2) = f$ and $f, g \in \mathcal{O}_S(U)$ are coprime. Let Q be a point of $\tilde{C}$ over P, t a uniformizing parameter at Q. Then $i|_{\tilde{C}} : \mathcal{O}(-10K|_{\tilde{C}}) \subsetneq \mathcal{O}(\Omega^1|_{\tilde{C}})$ writes locally at Q:

$$0 \to \mathcal{O}_{\tilde{C}}(V) \xrightarrow{i} \mathcal{O}_{\tilde{C}}(V)^2, \qquad i(1) = (t^{\nu_1(Q)}, t^{\nu_2(Q)})$$

where $\nu_1(Q) = \mathrm{ord}_Q(f|_{\tilde{C}}) = (F.R_Q)_P$ and $\nu_2(Q) = \mathrm{ord}_Q(g|_{\tilde{C}}) = (G.R_Q)_P$ with $F = \{f = 0\}$, $G = \{g = 0\}$ and R_Q the place of C corresponding to Q. Since $(F.G)_P = 1$ (recall that Z is smooth!), we have $\nu(Q) = \min\{\nu_1(Q), \nu_2(Q)\} \leq \mu_P(R_Q)$ (*cf.* Fulton [5]). To conclude observe that $\sum_{Q \to P} \mu_P(R_Q) = \mu_P(C)$ and that L corresponds to the divisor $\sum_{Q \to Z} \nu(Q)Q$ on $\tilde{C}$.

Q.E.D.

We can now prove:

THEOREM 4.4. — *Let* σ *be a not identically zero global section of* $\bigwedge^s J^r K$. *Then:*

$$\deg[Z(\sigma)]_0 = O(s^3 r^3 K^2 + s^2 r^2 K^2 c_2).$$

Proof. — We first need to isolate the cycle $[Z(\sigma)]_0$ from $[Z(\sigma)]_1$. If $D = [Z(\sigma)]_1$, σ factors through $\tau : \mathcal{O}_S \to \bigwedge^s J^r K(-D)$, and $[Z(\sigma)]_0 = [Z(\tau)]$. Now lemma 4.1 and 4.2, together with the principal parts exact sequence, give us a filtration $\mathcal{O}(\bigwedge^s J^r K)(-D) = \mathcal{O}(E_k) \supset \ldots \supset \mathcal{O}(E_0) = 0$, such that each quotient $\mathcal{O}(E_i)/\mathcal{O}(E_{i-1})$ injects in a sheaf of the form:

$$\mathcal{O}(sK - 10 r_1 K + 11 r_2 K - D) \tag{4.3}$$

with r_1, r_2 non negative integers, $r_1 + r_2 \leqq sr$. Note that "moving" the filtration (4.1), we may assume that $\mathcal{O}(E.)$ is non degenerate on $[Z(\sigma)]_0$. Now let k' be the smallest integer such that $\tau \in \mathcal{O}(E_{k'})$. Reducing modulo $\mathcal{O}(E_{k'-1})$ we get a non zero section in $H^0(sK - 10 r_1 K + 11 r_2 K - D)$, some r_1, r_2. Let $C = \sum n_i C_i$ be the decomposition in irreducible components of its divisor of zeros. Then clearly:

$$\deg([Z(\sigma)]_0) \leqq \sum n_i \deg[Z(\tau|_{C_i})]. \tag{4.4}$$

So now we bound $\deg[Z(\tau|_{C_i})]$ and sum up. This can be done with the aid of lemma 4.3. Coupled with the principal parts exact sequence it gives a filtration $0 = F_0 \subset \ldots \subset F_k = \bigwedge^s J^r K(-D)|_{\tilde{C}_i}$, with $\mathcal{O}(F_i/F_{i-1})$ a sheaf of the form:

$$\mathcal{O}(sK|_{\tilde{C}_i} + r_1(-10K|_{\tilde{C}_i} + L) + r_2(11K|_{\tilde{C}_i} - L) - D)$$

with r_1, r_2 non negative integers, $r_1+r_2 \leqq sr$. Note that lemma 4.3 gives also a bound on the degree of L. Obviously for any non zero section t of F_k, $\deg[Z(t)] \leqq \max\{\deg(F_i/F_{i-1})\}$. We apply this remark to $t=\tau|_{\tilde{C}_i}$. Summing up we get:

$$\deg[Z(\sigma)]_0 \leqq s\,K.C + sr\sum_i n_i(11\,K.C_i + \sum_{P \in C_i \cap Z} \mu_P(C_i)) - D.C$$
$$\leqq s(11\,r+1)\,K.C + sr\sum_i n_i \sum_{P \in C_i \cap Z} \mu_P(C_i)$$

(note that $D+C$ is a pluricanonical divisor, hence numerically connected, implying $D.C \geqq 0$). We easily conclude using the following lemma (use also the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leqq 3\,c_2$).

Q.E.D.

LEMMA 4.5. — *Notations being as in the proof of theorem* 4.4, *and* Z *being as in lemma* 4.1, *the following bound holds:*

$$\sum_i n_i \sum_{P \in C_i \cap Z} \mu_P(C_i) = O(sr\,K^2\,c_2 + s^2\,r^2\,K^2)$$

Proof. — *We write the sum as:*

$$\sum n_i \sum_{P \in C_i \cap Z} 1 + \sum n_i \sum_{P \in C_i \cap Z} (\mu_P - 1).$$

We bound the first summand as follows:

$$\sum n_i \sum_{P \in C_i \cap Z} 1 \leqq \deg(Z) \sum n_i \leqq \deg(Z)\,C.K.$$

Recall now that $C.K \leqq s(11\,r+1)\,K^2$, and lemma 4.1 where $\deg(Z)$ is computed. The second summand is certainly at most:

$$\sum n_i \sum_{P \in C_i} (\mu_P - 1) \leqq \sum n_i p_a(C_i) = \sum n_i \left(1 + \frac{1}{2}(C_i^2 + C_i.K)\right) \leqq C.K + \frac{1}{2}\sum n_i (C_i^2 + C_i.K).$$

Now $\sum n_i (C_i^2 + C_i.K) \leqq \max_{C' \leqq C} \{C'^2 + C'.K\}$, and since for some effective divisor D, $C+D \in |m\,K|$, with $m \leqq s(11\,r+1)$, everything follows.

Q.E.D.

The following is an easy consequence of theorems 3.4 and 4.4, and of definition 2.2:

THEOREM 4.6. — *Let* u *be an integer*, $0 \leqq u \leqq \binom{k+2}{2}$, *and* $m = \min\{u, p_g\}$. *The following inequalities then hold:*

$$[W_k^u]_1 . K \leqq \frac{m(k+2)}{2} K^2,$$
$$\deg([W_k^u]_0) = O(m^3\,k^3\,K^2 + m^2\,k^2\,K^2\,c_2).$$

Proof. — The second estimation needs perhaps some words of explanation. Now $[W_k^u]_0$ is the zero cycle of a section $(\sigma_1, \ldots, \sigma_N)$ of $(\overset{s}{\Lambda} J^r K(-D))^N$. For this section, the proof of theorem 4.4 works word by word.

Q.E.D.

5. Weierstraβ points along a foliation

We describe in this section a situation in which Weierstraβ points exist and are easily computed. In section 6 we show that on any surface of general type it is possible to find an invariant divisor *or* an invariant foliation (see the proof of theorem 6.5). The technique of Weierstraβ points along a foliation gives, once we have an invariant foliation, an invariant divisor. Therefore, the results in this section are of important use for us.

In this section S *indicates a minimal surface of general type over* **C** *with geometric genus* $p_g \geqq 1$ *and ample canonical sheaf.* We give the definition of a foliation.

DEFINITION 5.1. — Let S be a surface. A foliation on S is an inclusion of vector bundles:

$$L \subsetneq \Omega^1$$

where L is a line bundle. [In particular the corresponding inclusion $\mathcal{O}(L) \subsetneq \mathcal{O}(\Omega^1)$ is everywhere of constant rank 1].

Let L be a foliation and Q be the quotient bundle Ω^1/L. Note that taking the determinants we get $Q \cong K - L$. Applying lemma 3.3, together with the principal parts exact sequence, we get a filtration $\overset{p_g}{\Lambda} J^{p_g} K = E_k \supset \ldots \supset E_0 = 0$, such that each quotient E_i/E_{i-1} is isomorphic to a bundle of the form:

$$p_g K + r_1 L + r_2 (K - L)$$

with r_1, r_2 non negative integers, $r_1 + r_2 \leqq p_g^2$. We have a natural map

$$\overset{p_g}{\Lambda} \mathcal{O}_S^{p_g} \cong \mathcal{O}_S \to \mathcal{O}(\overset{p_g}{\Lambda} J^{p_g} K),$$

which is injective (lemma 2.5), and therefore corresponds to a non zero global section σ of $\overset{p_g}{\Lambda} J^{p_g} K$. Let k' be the smallest integer s.t. $\sigma \in \mathcal{O}(E_{k'})$. Since σ is non zero, $k' > 0$. Reducing modulo $E_{k'-1}$, σ gives a non zero section $[\sigma] \in H^0(E_{k'}/E_{k'-1})$. Therefore, it is well defined a couple r_1, r_2 such that σ gives a divisor:

$$W_L(K) \in \mathbf{P} H^0(p_g K + r_1 L + r_2 (K - L)).$$

DEFINITION 5.2. — We call the divisor $W_L(K)$ of the argument above, the canonical Weierstraβ divisor along L.

We now show that it never may happen that $W_L(K)$ is the zero divisor:

THEOREM 5.3. — $W_L(K)$ *is not the zero divisor.*

Proof. — Suppose by contradiction we have r_1, r_2 such that:

$$p_g K + r_1 L + r_2 (K - L) = (p_g + r_2) K + (r_1 - r_2) L \equiv 0.$$

First of all we have $r_1 - r_2 \neq 0$ (remember that in our hypothesis $p_g \geqq 1$). In particular $L^2 > 0$. But then $c_2 = L.(K - L) = L.K - L^2 < L.K$, which is absurd since by Miyaoka [14] we always have $L.K \leqq c_2$ for all L s. t. $\mathcal{O}(L) \subsetneq \Omega^1$.

Q.E.D.

To conclude, we make two rather trivial remarks:

REMARK 5.4. — Let L be an invariant foliation, in the sense that the natural lifted action of the automorphism group Aut(S) of S on the total space Ω^1 restricts to an action on L. Then the divisor $W_L(K)$ is an invariant divisor.

REMARK 5.5. — We have $W_L(K).K \leqq p_g K^2 + p_g^2 \max\{L.K, (K-L).K\}$. But since Ω^1 is K-semistable (theorem 3.2), $L.K \leqq 1/2\, K^2$. We therefore have:

$$W_L(K).K \leqq p_g (p_g + 1) K^2 - p_g^2 L.K.$$

6. Discriminants and discriminantal divisors

As we already pointed out in section 2, surfaces without Weierstraβ points do exist. However, to reach our main goal, which is to prove a polynomial bound in c_2 on the order of the automorphism group of a surface of general type, we need non trivial invariant loci. In this section we introduce discriminantal divisors. Despite the lack of a clear geometric interpretation, discriminantal divisors exist and are computable. The idea is to generalize the notion of the "parabolic curve", which for a quintic surface S in $\mathbf{P}^3$ is the intersection of S with its hessian surface H(S), and is a divisor linearly equivalent to 12 K.

We begin with a preliminar argument, leading to Lemma-Definition 6.1. First of all, let us fix the notation. In this section S indicates a minimal surface of general type over the field of complex numbers. *We make the additional assumption that the canonical bundle is ample and base point free.* We have, as in Definition 2.1, natural maps $v^k : \mathcal{O}_S^{p_g} \to J^k K$. The kernel of v^k is a 2-syzygy coherent sheaf, and it is therefore the sheaf of sections of a vector bundle E_{k+1} (*see* Kobayashi [1], Chap. V, Cor. 5.11), so that for all integers k, $0 \leqq k \leqq p_g$, we have an exact sequence of sheaves:

$$0 \to \mathcal{O}(E_{k+1}) \to \mathcal{O}_S^{p_g} \to \mathcal{O}(J^k K).$$

The principal parts exact sequence (lemma 2.3) induces then natural inclusions:

$$0 \to \mathcal{F}_k \to \mathcal{O}(S^k \Omega^1 \otimes K)$$

where $\mathscr{F}_k \cong \mathcal{O}(\mathrm{E}_k)/\mathcal{O}(\mathrm{E}_{k+1})$. Note that $\mathscr{F}_k$ is a subbundle in $\mathrm{S}^k(\Omega^1)\otimes\mathrm{K}$, for all k, if and only if v^k is of constant rank for all k, that is to say that S has trivial Weierstraß schemes. Note also that $\mathscr{F}_0 \cong \mathrm{K}$, since K is base point free. We also remember that v^{p_g} is injective (lemma 2.5). We may therefore state the following:

LEMMA-DEFINITION 6.1. — We have a natural filtration

$$\mathcal{O}_{\mathrm{S}}^{p_g} = \mathcal{O}(\mathrm{E}_0) \supset \ldots \supset \mathcal{O}(\mathrm{E}_{p_g+1}) = 0$$

and natural inclusions:

$$\mathscr{F}_k = \mathcal{O}(\mathrm{E}_k)/\mathcal{O}(\mathrm{E}_{k+1}) \subsetneq \mathcal{O}(\mathrm{S}^k\Omega^1\otimes\mathrm{K})$$

We call $\mathcal{O}(\mathrm{E}.)$ the fundamental filtration and $\mathscr{F}_k$ the k-th fundamental subsheaf.

The following is well known [remember our conventions about $\mathbf{P}(\mathrm{E})$]:

LEMMA 6.2. — *Let* E *be a rank two vector bundle over* S, $\mathbf{P}=\mathbf{P}(\mathrm{E})$ *the associated projective bundle,* $p:\mathbf{P}\to\mathrm{S}$ *the natural projection,* $\mathcal{O}_{\mathbf{P}}(1)$ *the dual of the tautological subbundle. An effective divisor* Δ *in* $\mathbf{P}$ *corresponds to a non zero global section of a line bundle of the form* $\mathcal{O}_{\mathbf{P}}(k)\otimes p^*(\mathrm{L})$ (L *is a line bundle on* S) *or, equivalently, to an inclusion of sheaves:*

$$\mathcal{O}(k\det(\mathrm{E})-\mathrm{L}) \subsetneq \mathcal{O}(\mathrm{S}^k\mathrm{E}).$$

The next lemma is central in what follows.

LEMMA 6.3. — *Let k be an integer,* $0\leqq k\leqq p_g$, *let* $\mathscr{F}_k$ *be the k-th fundamental subsheaf,* r_k *its rank, suppose moreover that* $r_k\neq 0$. *There is a natural inclusion:*

$$\mathcal{O}\left(\det\mathscr{F}_k\otimes-\frac{r_k(r_k+1)}{2}\mathrm{K}\right)\subsetneq\mathcal{O}(\mathrm{S}^{r_k(k-r_k+1)}\Omega^1) \tag{6.1}$$

which corresponds to an invariant divisor Δ_k *of relative degree* $r_k(k-r_k+1)$ *in* $\mathbf{P}(\Omega^1)$. Δ_k *is the zero divisor if and only if* $\mathscr{F}_k\cong\mathrm{S}^k\Omega^1\otimes\mathrm{K}$.

Proof. — Let us begin with a few remarks. The universal exact sequence:

$$0\to\mathcal{O}_{\mathbf{P}}(-1)\to p^*\Omega^1\to\mathrm{Q}\to 0$$

induces a filtration $p^*\mathrm{S}^k\Omega^1=\mathrm{E}_{k+1}\supset\ldots\supset\mathrm{E}_0=0$ with

$$\mathrm{E}_{i+1}/\mathrm{E}_i\cong i\mathrm{Q}\otimes\mathcal{O}_{\mathbf{P}}(i-k)\cong\mathcal{O}_{\mathbf{P}}(2i-k)\otimes p^*(i\mathrm{K})$$

(*cf.* Lemma 3.3). E. induces a filtration

$$p^*\bigwedge^{r_k}\mathrm{S}^k\Omega^1=\mathrm{F}_{m+1}\supset\ldots\supset\mathrm{F}_0=0$$

with

$$\mathrm{F}_{m+1}/\mathrm{F}_m\cong\mathcal{O}_{\mathbf{P}}(r_k(k-r_k+1))\otimes p^*(r_k(2k-r_k+1)/2)\mathrm{K}$$

(*see* again Lemma 3.3). *We now claim that* $p^* \det(\mathscr{F}_k \otimes (-\mathrm{K}))$, *which injects in* $p^* \bigwedge^{r_k} \mathrm{S}^k \Omega^1$, *injects in* $\mathrm{F}_{m+1}/\mathrm{F}_m$, *giving an inclusion:*

$$\mathcal{O}(\det \mathscr{F}_k \otimes -r_k \mathrm{K}) \subsetneq \mathcal{O}\left(\mathrm{S}^{r_k(k-r_k+1)} \Omega^1 \otimes \frac{r_k(r_k-1)}{2} \mathrm{K}\right).$$

This comes trivially from the relativization of the following argument in $\mathbf{P}^n$.

Let V be a two dimensional vector space over the field of complex numbers, $\mathbf{P} = \mathbf{P}(\mathrm{V})$. Let us call V the trivial bundle $\mathrm{V} \times \mathbf{P}$. We have the universal exact sequence:

$$0 \to \mathcal{O}_{\mathbf{P}}(-1) \to \mathrm{V} \to \mathcal{O}_{\mathbf{P}}(1) \to 0$$

which gives (lemma 3.3) a filtration:

$$\mathrm{S}^k \mathrm{V} \supset \mathrm{S}^{k-1} \mathrm{V} \otimes \mathcal{O}_{\mathbf{P}}(-1) \supset \ldots \supset \mathcal{O}_{\mathbf{P}}(-k) \supset 0. \tag{6.2}$$

The projective space $\mathbf{P}(\mathrm{S}^k \mathrm{V})$ may be canonically identified with the complete linear system of divisors of degree k on the projective line. For any point $x \in \mathbf{P}$, (6.2) gives a flag $\mathbf{P}(\mathrm{S}^k \mathrm{V}) = \mathrm{F}_k(x) \supset \ldots \supset \mathrm{F}_0(x)$ in projective k-space. This flag can be geometrically interpreted as follows. $\mathrm{F}_{k-1}(x)$ is the linear system of divisors passing through x. More generally, F_{k-i} is the linear system of divisors passing i times through x. In particular, $\mathrm{F}_0(x)$ is just the divisor kx. Note that the locus $\{kx, x \in \mathbf{P}\}$ is the rational normal curve C in $\mathbf{P}^k$, From this point of view, $\mathrm{F}_{k-i}(x)$ is the $(k-i)$-dimensional tangent space to C at x. Now, let W be a rank r subspace of $\mathrm{S}^k \mathrm{V}$. $\mathbf{P}\mathrm{W}$ is a $(r-1)$-dimensional linear space in $\mathbf{P}\mathrm{S}^k \mathrm{V}$ and, since C is not contained in any hyperplane, we have that for $x \in \mathbf{P}$ generic the $(k-r)$-tangent space to C at x does not intersect $\mathbf{P}\mathrm{W}$ (otherwise, projecting from $\mathbf{P}\mathrm{W}$ we get a curve all of whose points are inflectional). This can be rephrased saying that $\mathrm{F}_{k-i}(x) \cap \mathbf{P}\mathrm{W}$ is of codimension i in $\mathbf{P}\mathrm{W}$, for all i, $i \leqq r-1$, or, equivalently, that being W the trivial bundle $\mathrm{W} \times \mathbf{P}$, $\mathcal{O}(\det \mathrm{W})$ injects in

$$\mathcal{O}((\mathrm{S}^k \mathrm{V}/\mathrm{S}^{k-1} \mathrm{V} \otimes \mathcal{O}_{\mathbf{P}}(-1)) \otimes \ldots$$
$$\otimes (\mathrm{S}^{k-r+1} \mathrm{V} \otimes \mathcal{O}_{\mathbf{P}}(-r+1)/\mathrm{S}^{k-r} \mathrm{V} \otimes \mathcal{O}_{\mathbf{P}}(-r))) \cong \mathcal{O}_{\mathbf{P}}(r(k-r+1)).$$

Note that this implies that W induces canonically a divisor Δ of degree $r(k-r+1)$ on the projective line $\mathbf{P}\mathrm{V}$. It is easy to see that Δ has the following amusing geometric interpretation. Let us identify the projective line $\mathbf{P}$ with C by means of the morphism which sends x to kx. Let us project C from $\mathbf{P}\mathrm{W}$. Then Δ is just the (classical) Weierstraβ divisor associated to the linear system on C cut by $\mathbf{P}(\mathrm{W})$. Note that $\Delta \equiv 0$ if and only if $\mathrm{W} = \mathrm{S}^k \mathrm{V}$.

Relativizing the above argument we get immediately our claim and our divisor Δ_k in $\mathbf{P}(\Omega^1)$. It is trivial that Δ_k is an invariant divisor, in the sense that the natural lifted action of Aut(S) on $\mathbf{P}(\Omega^1)$ restricts to an action on Δ_k.

Q.E.D.

As a corollary we get:

COROLLARY 6.4. — *There is an* Aut(S)-*invariant invertible subsheaf* $\mathcal{O}(\mathrm{L}) \subsetneq \mathrm{S}^k\Omega^1$, *with* $1 \leqq k \leqq p_g^2$, $-p_g^2\mathrm{K}^2 < \mathrm{L}.\mathrm{K} < 0$, *or an invariant divisor* D *on* S, *such that* $\mathrm{D}.\mathrm{K} = O(p_g^2\mathrm{K}^2)$.

Proof. — Of course, L is going to be one of the $\det \mathscr{F}_k \otimes ((-r_k(r_k+1)/2)\mathrm{K})$ of lemma 6.3. Recall that $\mathscr{F}_k = \mathcal{O}(\mathrm{E}_{k+1})/\mathcal{O}(\mathrm{E}_k)$, where $\mathcal{O}(\mathrm{E}.)$ is the fundamental filtration of $\mathcal{O}_{\mathrm{S}}^{p_g}$ (lemma-definition 6.1), which implies, being $c_1(\mathscr{F}_0) = \mathrm{K}$ (K is base point free), that $\sum_{k=1}^{p_g} c_1(\mathscr{F}_k) = -\mathrm{K}$. So at least one of the $\mathscr{F}_k$'s, say $\mathscr{F}_{k'}$ is negative. On the other hand, by the K-semistability of the cotangent bundle, none of them can be too positive, and since their sum is given, none of them can be too negative either. It is easy to show that $-c_1(\mathscr{F}_k).\mathrm{K} = O(p_g^2\mathrm{K}^2)$ and in fact for all k:

$$c_1(\mathscr{F}_k).\mathrm{K} \geqq -\frac{p_g(p_g-1)}{2}\mathrm{K}^2. \tag{6.3}$$

Now if $\mathscr{F}_{k'}$ is negative and not generically of maximal rank,

$$\mathrm{L} = \det(\mathscr{F}_{k'}) - \frac{r_{k'}(r_{k'}+1)}{2}\mathrm{K}$$

will do. If $\mathscr{F}_{k'}$ has generically rank $k'+1$, the inclusion:

$$\mathcal{O}(\det \mathscr{F}_{k'}) \subsetneq \mathcal{O}(\det(\mathrm{S}^{k'}\Omega^1 \otimes \mathrm{K})) = \mathcal{O}\left(\frac{(k'+1)(k'+2)}{2}\mathrm{K}\right)$$

is given by a non zero global section σ of

$$\mathcal{O}\left(\frac{(k'+1)(k'+2)}{2}\mathrm{K} - \det \mathscr{F}_{k'}\right),$$

whose zero divisor D is a non zero invariant divisor. Moreover, equation (6.3) implies:

$$\mathrm{D}.\mathrm{K} = O(p_g^2\mathrm{K}^2).$$

Q.E.D.

We are now ready to prove the main theorem is this section:

THEOREM 6.5. — *Let* S *be a surface of general type over the field of complex numbers, with base point free canonical sheaf. There is on* S *an invariant divisor* D *with:*

$$\mathrm{D}.\mathrm{K} = O(p_g^4\mathrm{K}^2),$$

or an invariant finite set Z *of points with:*

$$|Z| = O(c_2 + p_g^2 K^2).$$

Proof. — Let L and k be as in Corollary 6.4. For reasons that will be clear in the sequel, we first of all need to "improve" this L. The inclusion $\mathcal{O}(L) \subsetneq S^k \Omega^1$ corresponds to a divisor Δ of relative degree k in $\mathbf{P} = \mathbf{P}(\Omega^1)$ (lemma 6.2). Let $\Delta = \sum h_i \Delta_i$ be the decomposition in irreducible reduced components. It is easy to see that $L = \sum h_i L_i$. Each individual Δ_i is not necessarily invariant, but $\Delta_{\text{red}} = \sum \Delta_i$ is. This is a divisor of relative degree $k' = \sum k_i$ in $\mathbf{P}$ and corresponds to an inclusion $\mathcal{O}(L') \subsetneq S^{k'} \Omega^1$, with $L' = \sum L_i$. I claim that:

$$-2 p_g^2 K^2 < L'.K < \frac{k'}{2} K^2. \tag{6.4}$$

We note that on the right it is very important to have $<$ and not $\leqq$ (*see* Case 1 below).

— By semistability $L'.K \leqq (k'/2) K^2$, but by the same reason also $L_i.K \leqq (k_i/2) K^2$ so if $L'.K = (k'/2) K^2$, that means $L_i.K = (k_i/2) K^2$ for all i's. This contradicts $L.K < 0$ (Corollary 6.4).

— The divisor $\Delta_0 = \sum (h_i - 1) \Delta_i$ has relative degree $\sum (h_i - 1) k_i < k$ and therefore by K-semistability $\sum (h_i - 1) L_i.K < (k/2) K^2$. Therefore, using again Corollary 6.4:

$$L'.K = L.K - \sum (h_i - 1) L_i.K > -p_g^2 K^2 - \frac{k}{2} K^2 > -2 p_g^2 K^2.$$

To produce our invariant locus we now distinguish two cases:

Case 1. — Δ_{red} has relative degree $k' \geqq 2$ in $\mathbf{P}$. The idea is now define an invariant locus studying the ramification of Δ_{red} over S. This can be done using the discriminant. Let V be a two dimensional vector space over the field of complex numbers. Viewing the points of $S^n(V)$ as homogeneous polynomials of degree n in two variables, the discriminant D is by definition the homogeneous set of polynomials with no more than $n-1$ distinct roots. Let us explain how elimination theory provides a natural equation for D. Let $p = \sum_{i=0}^{n} a_i x^{n-i} y^i \in S^n V$, and take the $2(n-1) \times 2(n-1)$ square resultant matrix of the partial derivatives p_x and p_y of p:

$$R(p) = \begin{pmatrix} 0 & 0 & \cdots & na_0 & (n-1)a_1 & \cdots & & a_{n-1} \\ \vdots & \vdots & \iddots & & & & \iddots & 0 \\ 0 & na_0 & \cdots & & & a_{n-1} & & \vdots \\ na_0 & (n-1)a_1 & \cdots & & a_{n-1} & 0 & \cdots & 0 \\ a_1 & 2a_2 & \cdots & & na_n & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & & & na_n & & \vdots \\ \vdots & \vdots & \ddots & & & & \ddots & 0 \\ 0 & 0 & \cdots & a_1 & 2a_2 & \cdots & & na_n \end{pmatrix}$$

Clearly $p \in \mathrm{D}$ if and only if p_x and p_y have a common zero. This set can be described at least on the open $a_n \neq 0$ by the equation $\det(\mathrm{R}(p)) = d(p) = 0$ (elimination theory). But since D is of degree $2(n-1)$ in $\mathbf{P}(\mathrm{S}^n \mathrm{V})$, D is then globally defined by $d(p)=0$. The natural action of GL(V) on $\mathrm{S}^n \mathrm{V}$ restricts to an action on D. This implies that d is a degree $n-1$ semi-invariant under the action of GL(V), or $d(g(p)) = (\det g)^{n-1} d(p)$ for all $g \in \mathrm{GL}(\mathrm{V})$. We may therefore extend the notion of discriminant to the n-th symmetric power $\mathrm{S}^n(\mathrm{E})$ of a rank two vector bundle. Returning to our $\mathcal{O}(\mathrm{L}') \subsetneq \mathcal{O}(\mathrm{S}^{k'} \Omega^1)$, we have then a discriminant:

$$d(\mathrm{L}') \in \mathrm{H}^0(k'(k'-1)\mathrm{K} - 2(k'-1)\mathrm{L}').$$

The zero set of this section describes exactly the set of points in S where Δ_{red} restricted to the fibre of p in $\mathbf{P}$ is not k' *distinct* points. But since Δ_{red} is a *reduced* divisor, this section can not be identically zero, while it cannot be a constant by equation (6.4), second inequality. So it has a non trivial divisor of zeros D, and $\mathrm{D}.\mathrm{K} = O(p_g^4 \mathrm{K}^2)$, by equation (6.4).

Case 2. — Δ_{red} has relative degree $k'=1$. It then corresponds to an inclusion $i: \mathcal{O}(\mathrm{L}') \subsetneq \mathcal{O}(\Omega^1)$. We have three possibilities:

1. If i is not saturated, we get saturating (and using the K-semistability of the cotangent sheaf) an invariant divisor D such that:

$$(6.5) \qquad \mathrm{D}.\mathrm{K} = O(p_g^2 \mathrm{K}^2).$$

2. If i is saturated, but not of constant rank, let Q be the quotient sheaf $\mathcal{O}(\Omega^1)/\mathcal{O}(\mathrm{L}')$. We have $\mathrm{Q} \cong (\mathrm{K}-\mathrm{L}') \otimes \mathcal{I}_\mathrm{Z}$, where $\mathcal{I}_\mathrm{Z}$ is the ideal sheal of a subscheme of pure dimension zero. We have $c_2 = \mathrm{L}'.(\mathrm{K}-\mathrm{L}') + \deg \mathrm{Z}$, therefore $\deg \mathrm{Z} = c_2 + \mathrm{L}'^2 - \mathrm{L}'.\mathrm{K}$. By the index theorem, and the K-semistability of the cotangent bundle (theorem 3.2) and equation (6.4), we finally have:

$$(6.6) \qquad \deg \mathrm{Z} = O(c_2 + p_g^2 \mathrm{K}^2).$$

3. Let i be of constant rank. We may now apply theorem 5.3 on Weierstraβ points along a foliation, and remarks 5.4, 5.5. We get that $\mathrm{W}_{\mathrm{L}'}(\mathrm{K})$ is an invariant divisor such that:

$$(6.7) \qquad \mathrm{W}_{\mathrm{L}'}(\mathrm{K}).\mathrm{K} = p_g^2 \mathrm{K}^2.$$

Summing up, and using the proof of case 1, equations (6.5), (6.7), (6.6), and corollary 6.4, we have the theorem.

Q.E.D.

7. Automorphisms of surfaces of general type

In this section, S *indicates a surface of general type over the field of complex numbers*. We reach our main goal, which is a polynomial estimate in c_2 for the order of the automorphism group Aut(S). This group is well known to be finite for varieties of general type of any dimension (*see* Matsumura [13]), and it is therefore natural to ask for a bound on the order. While proving our main theorem (theorem 7.10) we also give some sharper bounds is special cases (theorems 7.6, 7.8, 7.9). Some results of this kind are also given in Horstmann [7]. Of particular interest, in our opinion, are theorem 7.8 about surfaces whose canonical map is composed with a pencil, and theorem 7.9 about surfaces whose canonical map is not everywhere a local holomorphic immersion. Moreover, we have a sharper bound for surfaces having non trivial Weierstraβ schemes (theorems 7.11 and 7.12). We essentially follow the method of Howard and Sommese [8]. The idea is simple. We use the invariant subschemes whose existence was proved in section 6 to find a subgroup of bounded index in Aut(S), fixing a point $p \in S$. Such a group has a faithful representation in $GL(T_P(S)) \cong GL(2, \mathbf{C})$. A finite subgroup of $GL(2, \mathbf{C})$ has an abelian subgroup of index at most 12. Finally, it is easy to deal with abelian groups. *We remark that our result is completely effective*, though we do not compute explicitly the constants. Let us collect some simple facts which we shall use several times, even without explicit mention:

1. The Hurwitz-Schwarz-Klein theorem, which asserts that for a complex curve C of genus $g > 2$, $|\mathrm{Aut}(C)| \leqq 84(g-1)$. We also have the period of any $\gamma \in \mathrm{Aut}(C)$ is at most $4g+2$.

2. The counting principle, also called "Lagrange theorem": let the finite group G act on the set Z, and let $p \in Z$. We then have $|G| = |G_p||G(p)|$.

3. The finite subgroups of $\mathbf{P}GL(2, \mathbf{C})$ are cyclic, dihedral or they are finite of order 12, 24, or 60 (this last three orders corresponding respectively to the automorphism groups of the tethraedron, the cube and the dodecahedron). This fact follows easily from the Hurwitz formula: *see* Du Val [4]. In particular any such group has a cyclic subgroup of index $\leqq 12$. By simple group theoretical considerations we can also argue (*see* again Du Val [4]) that a finite subgroup of $GL(2, \mathbf{C})$ has an abelian subgroup of index $\leqq 12$.

4. Let G be a finite group acting faithfully on S with a fixed point p, then the tangent representation of G in $GL(T_p(S))$ is faithful (it follows immediately from the existence of an invariant metric).

We begin with some technical lemmas:

LEMMA 7.1. — *Let* S *be a surface of general type and* $f: S \to \mathbf{P}^1$ *a holomorphic connected fibration. Let* $F_1, \ldots, F_r$ *be the singular fibres of* f *and* F *a general fibre. Then:*

$$3 \leqq r \leqq c_2 + 4(g(F) - 1).$$

Moreover, let $f: S \to E$ *be a connected holomorphic fibration,* E *an elliptic curve. Then,* r *being as above:*

$$1 \leqq r \leqq c_2.$$

Proof. — Let us prove the first assertion. Suppose by contradiction f have less than two singular fibres. Removing the singular fibres we get a smooth morphism $f': S' \to \mathbf{C}^*$. Passing to a finite covering $\pi: \mathbf{C}^* \to \mathbf{C}^*$ we may eliminate the monodromy, so that the pull back family $f'': S'' = S' \times_\pi \mathbf{C}^* \to \mathbf{C}^*$ is rigidified. Now the universal covering space of $\mathbf{C}^*$ is $\mathbf{C}$ itself, and using the Torelli theorem and the fact that the Siegel upper half space $\mathbf{H}^g$ is biholomorphic to a bounded domain, we get that f'' is of constant moduli, and being rigidified it is isomorphic to $\mathbf{C}^* \times \mathrm{F}$ where F is any fibre. But then S is uniruled, hence ruled, which is absurd. We thus proved $3 \leqq r$. Now, remember that for all holomorphic fibrations $f: \mathrm{S} \to \mathrm{B}$ we have the formula:

$$c_2 = e(\mathrm{F})\, e(\mathrm{B}) + \sum_{i=1}^{r} (e(\mathrm{F}_i) - e(\mathrm{F})), \tag{7.1}$$

being F a general fibre. Remember also that $e(\mathrm{F}_i) > e(\mathrm{F})$, $\forall\, i$ (*see* Barth, Peters Van de Ven [2]). But in our case $e(\mathrm{B}) = 2$, giving $r \leqq c_2 + 4(g(\mathrm{F}) - 1)$.

Let us prove the second assertion. Just note that in this case $e(\mathrm{B}) = 0$ in equation (7.1) and that $c_2 > 0$ on a surface of general type.

Q.E.D.

LEMMA 7.2. — *There is a universal effective constant c with the following property. Let* S *be any surface of general type with ample and base point free canonical sheaf. Then one of the following is true:*

1. *There exists a reduced irreducible curve* C *such that* C *is not rational and:*

$$[\mathrm{Aut}(\mathrm{S}) : \mathrm{I}(\mathrm{C})] \leqq c c_2^9,$$

or C *is rational and there is a subgroup* G *of* Aut(S) *acting on* C *and such that:*

$$[\mathrm{Aut}(\mathrm{S}) : \mathrm{G}] \leqq c c_2^5,$$

2. *There exists a point $p \in \mathrm{S}$ such that:*

$$[\mathrm{Aut}(\mathrm{S}) : \mathrm{G}_p] \leqq c c_2^3.$$

Proof. — According to theorem 6.5, we have two possibilities:

1. There exists an invariant divisor D on S with:

$$\mathrm{D}\,.\,\mathrm{K} = O(p_g^4\, \mathrm{K}^2) = O(c_2^4\, \mathrm{K}^2)$$

[remember the Nöther inequality $p_g \leqq (1/2)\, c_1^2 + 2$ and the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leqq 3\, c_2$]. Let $\mathrm{C}_1, \ldots, \mathrm{C}_n$ be the irreducible components of D. It is clear that there is an integer i_0 such that $\mathrm{C}_{i_0}\,.\,\mathrm{K} \leqq (1/n)\, \mathrm{D}\,.\,\mathrm{K}$. Now let C be C_{i_0}, with the reduced induced structure. Since K is numerically effective, $n \leqq \mathrm{D}\,.\,\mathrm{K} = O(c_2^4\, \mathrm{K}^2)$ and Aut(S) has a subgroup G acting on C with $[\mathrm{Aut}(\mathrm{S}) : \mathrm{G}] \leqq n$. If C is rational, we are done. If C is elliptic, let $\tilde{\mathrm{C}}$ be the normalization. G has a subgroup G′ of index at most 6, acting on $\tilde{\mathrm{C}}$ by translations. $\mathrm{G}'/\mathrm{I}(\mathrm{C})$ acts faithfully on $\tilde{\mathrm{C}}$, and also acts on the projective space $\mathbf{P} = \mathbf{P}\,\mathrm{H}^0(\mathrm{K}\,|_{\tilde{\mathrm{C}}})$. Since $\mathrm{G}'/\mathrm{I}(\mathrm{C})$ is abelian, the action on $\mathbf{P}$ can be simultaneously diagonalized and there exists a fixed divisor on $\tilde{\mathrm{C}}$ of degree $\mathrm{K}\,.\,\tilde{\mathrm{C}}$. We get $|\mathrm{G}'/\mathrm{I}(\mathrm{C})| \leqq \mathrm{K}\,.\,\tilde{\mathrm{C}}$, since a translation with a fixed point acts as the identity. Now,

$$[\mathrm{Aut}(\mathrm{S}) : \mathrm{I}(\mathrm{C})] = [\mathrm{Aut}(\mathrm{S}) : \mathrm{G}]\,[\mathrm{G} : \mathrm{G}']\,[\mathrm{G}' : \mathrm{I}(\mathrm{C})] = O(c_2^4\, \mathrm{K}^2)$$

and we are done using the Bogomolov-Miyaoka-Yau inequality. If C has genus $\geqq 2$, $g(\mathrm{C})=1+1/2(\mathrm{C}^2+\mathrm{C}.\mathrm{K})$, and by the Hurwitz-Schwarz-Klein theorem $[\mathrm{G}:\mathrm{I}(\mathrm{C})]\leqq 84(g-1)$. We get

$$[\mathrm{Aut}(\mathrm{S}):\mathrm{I}(\mathrm{C})]\leqq 84n(g(\mathrm{C})-1)\leqq 84n\max\{\mathrm{C}.\mathrm{K},\ \mathrm{C}^2\}=O(c_2^9),$$

by the index theorem, and the Bogomolov-Miyaoka-Yau inequality.

2. There is an invariant finite set of points Z with $|\mathrm{Z}|=O(c_2^3)$ (use again the Nöther inequality and the Bogomolov-Miyaoka-Yau inequality). Take any $p\in\mathrm{Z}$.

Q.E.D.

LEMMA 7.3. — *There exists a universal constant* c *with the following property. Let* S *be a surface such that one of the Weierstraß schemes* $\mathrm{W}_0(\mathrm{S})$ *or* $\mathrm{W}_1(\mathrm{S})$ *is non trivial. Then one of the following is true:*

1. *There exists a reduced irreducible curve* C *on* S *such that* C *is non rational and:*

$$[\mathrm{Aut}(\mathrm{S}):\mathrm{I}(\mathrm{C})]\leqq cc_2^2,$$

or C *is rational and there exists a subgroup* G *of* Aut (S) *acting on* C *and such that:*

$$[\mathrm{Aut}(\mathrm{S}):\mathrm{G}]\leqq cc_2.$$

2. *There exists a point* $p\in\mathrm{S}$ *such that:*

$$[\mathrm{Aut}(\mathrm{S}):\mathrm{G}_p]\leqq cc_2.$$

Proof. — If $[\mathrm{W}_0(\mathrm{S})]_0$ or $[\mathrm{W}_1(\mathrm{S})]_0$ is non trivial, the assertion is obvious by lemma 2.4.

Suppose now that $[\mathrm{W}_0(\mathrm{S})]_1$ be non trivial. By lemma 2.4, $[\mathrm{W}_0(\mathrm{S})]_1.\mathrm{K}\leqq\mathrm{K}^2$. Now it is clear that there are at most c_2-1 curves on S. On the other hand, on a minimal surface of general type $h^{1,\,1}\leqq 6c_2$ [reading c_2 on the Hodge diamond gives us

$$h^{1,\,1}=c_2-2+4q-2p_g=c_2-4\chi+2+2p_g\leqq c_2-2+2p_g,$$

since $\chi>0$. Now by the Nöther inequality and the Bogomolov-Miyaoka-Yau inequality, $p_g\leqq(1/2)c_1^2+2\leqq(3/2)c_2+2]$. So there are at most $6c_2-1$ and -2 curves. Therefore, the divisor $[\mathrm{W}_0(\mathrm{S})]_1$ has at most K^2+6c_2 irreducible components. Pick one of them, say C. Now argue exactly as in lemma 7.2, with the difference that now, to bound C^2 [and $g(\mathrm{C})$], we may use lemma 2.4.

If $[\mathrm{W}_1(\mathrm{S})]_1$ is non trivial, argue the same way.

Q.E.D.

LEMMA 7.4. — *Let* C *be a curve on* S. *Suppose* C *is not a* -2 *curve. Then:*

$$|\mathrm{I}(\mathrm{C})|\leqq 6(10\mathrm{K}^2+1).$$

Proof. — Consider the action of I (C) on $|5\mathrm{K}|$. Since I (C) is cyclic [for all $\mathrm{P}\in\mathrm{C}$ the tangent representation is fully faithful in GL (2, **C**) and the determinant gives an injection of I (C) in **C***], the action can be simultaneously diagonalized, with respect to a projective frame $p_1, \ldots p_{p_5}$ in $|5K|$. Let

$$L = \{ D \in |5K| \, s.\, t.\, f_{|5K|}(P) \in D, \; \forall P \in C \}.$$

L is a linear variety of codimension at least 2. Therefore, at least one line $l = p_i p_j$ avoids L. l gives an I(C)-invariant rational fibration $\varphi_l : S - \to \mathbf{P}^1$. Since $\varphi_l(C) = \mathbf{P}^1$, I(C) acts trivially on the base. Look at the generic fibre and remember that the period of an automorphism of a curve of genus $g \geqq 2$ is at most $4g+2$.

Q.E.D.

We recall the following theorem of Andreotti:

THEOREM 7.5. — *Let* S *be a surface of general type. Then:*

$$|\mathrm{Aut}(S)| \leqq (c_2 + 85\, K^2)^{p_5^2 - 1},$$

where p_5 *is the* 5*-th plurigenus of* S.

Proof. – *See* Andreotti [1].

Q.E.D.

We now give some polynomial bounds in special cases:

THEOREM 7.6. — *There is a universal effective constant* c *with the following property. For all* S *and abelian groups* G *acting on* S, *we have:*

$$|G| \leqq c c_2^2.$$

Proof. – *see* Howard and Sommese [8].

Q.E.D.

REMARK 7.7. – Xiao announced to have proved a linear bound in c_2 for abelian groups.

THEOREM 7.8. — *There is a universal effective constant* c *with the following property. For all surfaces of general type* S, *whose canonical map* $f_{|K|}$ *is composed with a pencil, the following estimate holds:*

$$|\mathrm{Aut}(S)| \leqq c c_2.$$

Proof. – If $\chi(\mathcal{O}(S)) < 21$, only a finite number of families occurs, and we get a universal bound using theorem 7.5.

If $\chi(\mathcal{O}(S)) \geqq 21$, by Beauville [3], the pencil is base point free and $2 \leqq g \leqq 5$, g being the genus of a generic fibre. Remember that is any case the fibres are connected, since the canonical curves are numerically connected, hence connected. Moreover, by Xiao [16], we have in any case $0 \leqq b \leqq 1$, b being the genus of the base curve $f_{|K|}(S)$. In any case it is clear that the fibration $f_{|K|}(S)$ is Aut(S)-equivariant, meaning that Aut(S) acts on the base compatibility with the action on S. We discuss two cases:

A. $b=0$. G being the subgroup of transformations fixing the base pointwise, Aut(S)/G has an abelian subgroup H of index $\leqq 12$. H has two fixed points on the base $\mathbf{P}^1$, let us call them 0 and ∞. We get that Aut(S) has a subgroup G′ fixing two fibres (not necessarily pointwise) and of index $\leqq 12$. But there are at least three singular fibres (lemma 7.1), so at least one of them, say F, has image different from 0, ∞. G′ interchanges the singular fibres, and since there are at most c_2+16 of them (lemma 7.1 and Beauville [3]), G′ has a subgroup G″ of index at most c_2+16 fixing F and therefore the base pointwise (a group acting on $\mathbf{P}^1$ with three fixed points, acts trivially). We get the result applying the Hurwitz-Schwarz-Klein theorem and Beauville [3] to a generic fibre.

B. $b=1$. By lemma 7.1 there is at least one singular fibre, and no more than c_2 singular fibres. Aut (S) has then a subgroup G of index $\leqq 6c_2$ fixing the base pointwise. Once again, apply the Hurwitz-Schwarz-Klein theorem and Beauville [3] to a generic fibre.

Q.E.D.

The following result improves theorem 4.6 of Howard and Sommese [8]:

THEOREM 7.9. — *There is a universal effective constant c with the following property. Let* S *be a surface of general type such that the first canonical map* $f_{|K|}$ *is not everywhere a local immersion. Then:*

$$|\mathrm{Aut}(S)|\leqq cc_2^3,$$

Proof. — If $p_g=0$, only a finite number of families is allowed, so we may apply theorem 7.5.

If the canonical map is composed with a pencil, we may use theorem 7.8.

Otherwise we are in the hypothesis of lemma 7.3. Notations being as in lemma 7.3, if C is non rational, just bound I (C) with lemma 7.4. If C is rational, G/I (C) has an abelian subgroup of index at most 12, which has therefore a fixed point $p\in C$. Aut (S) has therefore a subgroup G′ of index $O(c_2)$, which fixes p. The tangent representation of G′ at p is faithful in GL (2, $\mathbf{C}$). Therefore, G′ has an abelian subgroup of index at most 12. Use now theorem 7.6. Finally if S has a point p with $[\mathrm{Aut}(S):G_p]=O(c_2)$, G_p has a faithful representation in GL (2, $\mathbf{C}$) and hence it has an abelian subgroup of index $\leqq 12$. Now bound $|G_p|$ using theorem 7.6.

Q.E.D.

THEOREM 7.10. — *There is a unisersal effective constant c such that for all surfaces of general type:*

$$|\mathrm{Aut}(S)|\leqq cc_2^{10}.$$

Proof. — By theorem 7.9 we may assume K ample and base point free. According to lemma 7.2, we distinguish two cases:

1. Notations being as in lemma 7.2, if C is rational, G/I (C) has a subgroup of index at most 12 which is abelian, and has therefore a fixed point $p\in C$. Aut (S) has therefore a subgroup G′ of index $O(c_2^5)$, which fixes a point $p\in S$. The tangent representation of G′ at p is faithful. Therefore, G′ has an abelian subgroup of index at most 12. Apply theorem 7.6, and get $|\mathrm{Aut}(S)|=O(c_2^7)$. If C is not rational, $|\mathrm{Aut}(S)|=[\mathrm{Aut}(S):I(C)]\,|I(C)|=O(c_2^{10})$ (use lemma 7.2 and 7.4).

2. If S has a point p with $[\mathrm{Aut}(S):G_p]=O(c_2^3)$, G_p has a faithful representation in $GL(2, \mathbf{C})$ and hence it has an abelian subgroup of index $\leqq 12$. Now bound $|G_p|$ using theorem 7.6.

Q.E.D.

THEOREM 7.11. — *These exists a universal effective constant c with the following property. For all surfaces of general type* S *having a Weierstraß scheme with non trivial divisorial component, the following estimate holds:*

$$|\mathrm{Aut}(S)| \leqq c c_2^7,$$

Proof. — By assumption there is a $W_k^u(S)$ with non trivial $[W_k^u(S)]_1$. Using lemma 2.5, we find that there is a non trivial $[W_k^u(S)]_1$ for u, $k \leqq p_g$. It is easy now to prove the theorem following the proof of theorem 7.10, part 1, using theorem 4.6 instead of theorem 6.5.

Q.E.D.

THEOREM 7.12. — *There exists a universal effective constant c with the following property. For all surfaces of general type having non trivial Weierstraß schemes, the following estimate holds:*

$$|\mathrm{Aut}(S)| \leqq c c_2^9.$$

Proof. — By assumption there is a non trivial $W_k^u(S)$. Using lemma 2.5, we find that there is a non trivial $W_k^u(S)$ for some u, $k \leqq p_g$. Argue now as in the proof of theorem 7.10, using theorem 4.6 instead of theorem 6.5.

Q.E.D.

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(Manuscript received October 3, 1989,
in revised form March 23, 1990.)

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