MICHAEL E. GAGE

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Annales scientifiques de l'É.N.S. 4e série, tome 23, nº 2 (1990), p. 229-256
<http://www.numdam.org/item?id=ASENS_1990_4_23_2_229_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 23, 1990, p. 229 à 256.

CURVE SHORTENING ON SURFACES

By Michael E. GAGE

What happens when a simple closed curve on a surface M is allowed to move so that the instant instantaneous velocity at each point is proportional to the gepdesic curvature k of the curve at that point? This evolution is suggested by the equation of motion of an idealized elastic band along the surface (assuming that the friction term is relatively large and the mass relatively small). The only fixed curves are geodesics—where the geodesic curvature is identically zero. Using our intuition about elastic bands it seems reasonable to predict that at least some simple closed curves near geodesics will evolve towards the geodesics. Other curves will "slide off the surface", *i.e.* collapse to points.

To better understand the evolution of the curve, focus on these three questions:

(1) Does there exist a family of smooth curves $X: S^1 \times [0, T) \rightarrow M$ which satisfies the evolution equation

$$X_t = k N \tag{0.1}$$

and for how large a T can this solution be defined?

(2) Do the simple closed curves remain simple?

(3) What is the "final shape" the limiting shape that the curve approaches as time t approaches T? Initial research on this problem concentrated on the case where M is the Euclidean plane. The results obtained include:

(*a*) For convex curves the isoperimetric ratio L^2/A decreases under this evolution and convex curves become asymptotically circular as they shrink to a point. No corners or other singularities develop before the curve shrinks to a point ([G1], [G2], [G-H]).

(*b*) Simple nonconvex curves evolve smoothly to convex curves without any self intersections occurring; from there they evolve to "round" points according to the result above [Gr].

(*c*) Closed curves with self intersections are likely to develop cusps (*see* [A-L] and [E-G]).

Partially supported by NSF grant DMS-8604856.

This paper extends some of the results obtained in the plane to surfaces:

The main result of this paper is that a solution to equation (0.1) exists on a smooth complete surface M^2 with bounded Gauss curvature as long as the quantity k_π^* defined in the next section remains bounded. Informally, this means that the solution exists until a cusp forms; singularities such as corners cannot occur. A special case of this theorem was proved for (locally) convex plane curves in [G-H], Theorem 3.2.1 and is one of the cornerstones of the regularity theory described in that paper and in [Gr] and [E-G]. This paper extends the result to arbitrary closed immersed curves on complete surfaces of bounded curvature ([1]).

Secondly, we prove that if the initial curve is a closed simple curve, it remains simple during the evolution as long as k_π^* remains bounded.

Our third result is that if a solution exists for an infinite time then the curvature of the curves decreases uniformly to zero. If in addition the one parameter family of curves remains within a compact subset of M^2 then a subsequence of the family converges to a geodesic in the C^∞ metric. The hypothesis is necessary since otherwise there are examples where the curve slides off to infinity.

As an application we show that a simple closed curve on the unit two sphere which divides the sphere into two regions with equal areas and for which the total space curvature is less than 3π converges under the curve shortening evolution to a great circle. $\left[\text{The total space curvature is given by } \int (k^2+1)^{1/2}\, ds\right]$ In this case we are able to show directly that no cusps occur, and therefore the conclusions of the main theorem are valid. We can also show in this case that the limiting geodesic is unique – the entire one parameter family of curves converges to a single great circle. This is significant because *a priori* one must allow the possibility that the curve converges to a slowly rotating great circle with different subsequences converging to different fixed great circles. This does not occur.

Returning to the original questions, what remains to be done?

(4) One would like to generalize Matt Grayson's result [Gr] for plane curves to surfaces; *i.e.* Show that if the geodesic curvature is zero at some point then k_π^* must be bounded. Therefore the curve is smooth with finite geodesic curvature and the evolution can continue ([2]).

(5) If the curvature of the curve is everywhere positive and the curve is null homotopic show that the curve converges to a "round" point. (If one can show that such curves shrink smoothly to points then it seems likely that one can show that they are asymptotically circular by comparing with the Euclidean case).

(6) Does every family of curves satisfying equation (0.1) which has a subsequence converging to a geodesic converge uniquely to this geodesic? The application to spheres

([1]) See also [Gr2] for a similar theorem.

([2]) Added in proof: This has been proved by M. Grayson in [Gr2].

in this paper is a small step in this direction. Uniqueness is also easy to prove if the surface in the neighborhood of the geodesic has negative curvature. The question for geodesics on surfaces with positive or mixed Gauss curvature remains open.

Result (4) above together with the main theorem in this paper would complete the answer to question (1). Question (2) is essentially answered in this paper – only cusp singularities can introduce self intersections. The other results above would complete the answer to question (3) about the final shape of the curve.

I am particularly grateful to the referee whose patient and thorough persual of the first version of this manuscript caught many careless and typographical errors. I wish to thank Wolfgang Ziller, Herman Gluck and the rest of the geometry faculty at the University of Pennsylvania who first suggested studying this evolution of curves and gave it is name. Thanks also to Richard Hamilton and Matt Grayson whose comments, suggestions, and ideas on this subject have been invaluable. I am grateful to l'Institut des Hautes Études Scientifiques for their hospitality while much of this paper was being written.

1. Basic calculations

In this section we calculate the evolution equations for several quantities associated with a curve which evolves on a surface M according to the evolution equation

$$X_t = k N$$

where X is the position vector, k is the curvature and N is the normal to the curve. The subscript indicates differentiation and the family of curves is parameterized by u and by t. The arc length s is defined by $ds = |X_u|\, du$ and we set $v = |X_u|$.

Let the unit tangent vector be defined by $T = X_s = 1/v\, X_u$ and let $W = X_t = k N$. The Frenet formulas are $T_s = \nabla_T T = k N$ and $\nabla_T N = -k T$ where ∇ is covariant differentiation. This defines k, since N is chosen so that T, N is a properly oriented frame on the surface M. The commutator of W and N is given by $[W, T] = -(v_t/v) T$. This is easily checked: if $f : M \to \mathbb{R}$ then

$$WTf = \frac{\partial^2 f}{\partial t\, \partial s} = \frac{\partial}{\partial t}\frac{1}{v}\frac{\partial f}{\partial u} = -\frac{1}{v^2}\frac{\partial v}{\partial t}\frac{\partial f}{\partial u} + \frac{1}{v}\frac{\partial^2 f}{\partial u\, \partial t} = TWf - \frac{v_t}{v} Tf.$$

To calculate v_t we consider

$$\nabla_W T = W\left(\frac{1}{v}\right) X_u + \frac{1}{v}\nabla_W X_u = -\frac{v_t}{v^2} X_u + \nabla_T (k N) = -\frac{v_t}{v} T + k_s N - k^2 T.$$

Since $\langle \nabla_W T, T \rangle = 0$ we have $v_t = -k^2 v$. We conclude that $[W, T] = k^2 T$. We collect these and related results from these calculations in

LEMMA 1.1.

$$[W, T] = k^2 T$$
$$\nabla_W T = k_s N$$
$$\nabla_W N = -k_s T$$
$$v_t = -k^2 v.$$

Furthermore, for any function f we write $(f_s)_t$ *as* f_{st} *and observe that* $f_{st} = f_{ts} + k^2 f_s$.

To calculate the evolution for k we use the curvature tensor

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z = R(X, Y) Z.$$

The sectional curvature of a plane spanned by orthonormal vectors X and Y is $K(X, Y) = \langle R(X, Y) Y, X \rangle$. For surfaces, of course, this defines the Gauss curvature K.

From Lemma 1.1 we observe that $\nabla_W (k N) = k_t N + k(-k_s T)$, while from the definition of Gauss curvature we derive

$$\nabla_W (kN) = \nabla_W \nabla_T T = R(W, T) T + \nabla_T \nabla_W T + \nabla_{[W, T]} T = k R(N, T) T + k_{ss} N - k_s k T + k^3 N.$$

Comparing components in these two expressions yields.

LEMMA 1.2. — *The curvature evolves according to*

$$k_t = k_{ss} + k^3 + k K.$$

For later use we derive the evolution equations satisfied by arc length and the total curvature of the curve.

LEMMA 1.3

$$L_t = -\int k^2 \, ds, \qquad \left(\int k ds\right)_t = \int k K \, ds.$$

Proof. — These follow from the formulae is Lemma 1.1 and Lemma 1.2.

PROPOSITION 1.4. — *On any surface with bounded curvature, curves which are convex* ($k > 0$ *everywhere*) *remain convex under the curve shortening evolution. In fact the minimum value of the curvature decreases at most exponentially.*

Proof. — We derive a minimum principle for $k_t = k_{ss} + k^3 + k K$. Let $U = k e^{\alpha t}$ where α will be chosen below. U satisfies the equation:

$$U_t = U_{ss} + (k^2 + K + \alpha) U.$$

Choose α so that $K + \alpha > 0$. If U is initially greater than $\delta > 0$ there is no subsequent time at which $0 < U < \delta$, *i.e.*, the minimum value of U is non-decreasing in time. Suppose that U achieves the value ε, $0 < \varepsilon < \delta$. From continuity and compactness this occurs at some first time and place (u_0, t_0) where

$$U_t(u_0, t_0) \leqq 0, \qquad U_{ss}(u_0, t_0) \geqq 0 \qquad \text{and} \qquad (k^2+K+\alpha)U>0.$$

This contradicts the fact that U satisfies the equation above. Hence $U \geqq \delta$ and $k \geqq e^{-\alpha t} k_{\min}(0)>0$.

It is important to have some control over the magnitude of the curvature k. This is obtained by considering the quantity $w=(C^2+k^2)^{1/2}$ where C is chosen so that $C^2 \geqq |K(x)|$ for every point x on the surface M.

LEMMA 1.5. — *The function* $w=(C^2+k^2)^{1/2}$ *satisfies the equation*

$$w_t = w_{ss} - \frac{C^2 w_s^2}{w(w^2-C^2)} + \frac{(w^2-C^2)(w^2-C^2+K)}{w}.$$

The quantity $w_s^2/(w^2-C^2)=(k_s)^2/w^2$ *and is therefore well defined everywhere despite the apparent division by zero.*

Proof. — We calculate that

$$w_t = \frac{k k_t}{w} = \frac{k(k_{ss}+k^3+kK)}{w}, \qquad w_s = \frac{k k_s}{w}, \qquad w_{ss} = \frac{k_s^2}{w} - \frac{k w_s k_s}{w^2} + \frac{k k_{ss}}{w}.$$

Simplifying the last formula gives

$$w_{ss} = \frac{C^2 w_s^2}{w(w^2-C^2)} + \frac{k k_{ss}}{w},$$

which when substituted in the formula for w_t proves the lemma.

COROLLARY 1.6. — *The quantity* $\int w\,ds$ decreases with time provided $|K(x)| \leqq C^2$ *for all* x *in* M.

Proof:

$$\left(\int w\,ds\right)_t = \int w_t - w k^2\,ds = \int -\frac{w_s^2 C^2}{w(w^2-C^2)} + \frac{(w^2-C^2)(K-C^2)}{w}\,ds \leqq 0.$$

The derivative is zero only where w is identically C, *i. e.* for geodesics.

Remark 1.7. — Corollary 1.6 provides an analog to the fact total absolute curvature decreases with time for plane curves evolving according to (0.1).

2. The cusp theorem

In this section we obtain upper bounds on the maximum absolute curvature in terms of the initial curve, the time interval [0, T) over which the system evolves and the median curvature k_π^* which will be defined below. A consequence of these bounds is that the curve shortening evolution can always be continued until a cusp forms.

DEFINITION 2.1. — *Let* $k_\varphi^*(t)=\sup\{\beta\,|\,\{u\,|\,.\,|k(u,t)|>\beta\}$ *contains an interval* I *satisfying* $\left\{\int_I |k|\,ds \geqq \varphi\right\}$. *If the set is empty* $\left(\text{which occurs if } \int |k|\,ds<\varphi\right)$ *then let* $k_\varphi^*(t)=\min\{|k(u,t)|\,|\,u\in S^1\}$.

The integral $\int_I |k|\,ds$ is the *turning angle* of the interval I. A cusp occurs if the curvature approaches infinity uniformly on an interval whose turning angle remains greater than π.

The quantity w_φ^* is defined in a slightly different way so that lemma 2.3 will hold.

DEFINITION 2.2. — *Let* $w_\varphi^*(t)=\sup\{\beta\,|\,\{u\,|\,|w(u,t)|>\beta\}$ *contains an interval* I *satisfying* $\left\{\int_I k^2/w\,ds>\varphi\right\}$. *If the set is empty, let* $w_\varphi^*(t)=\min\{w(u,t)\,|\,u\in S^1\}$.

We have the following inequality between w_φ^* and k_φ^*.

LEMMA 2.3. — $k_\varphi^* \geqq (w_\varphi^{*2}-C^2)^{1/2}$.

Proof. – Let $\beta>w_\varphi^*(t)-\delta$ then on some interval I, $(k^2+C^2)^{1/2}\geqq\beta$ hence $k^2\geqq\beta^2-C^2\geqq w_\varphi^{*2}-2\,\delta\,w_\varphi^*-C^2$ on I and $\varphi\leqq\int_I k^2/w\,ds<\int_I |k|\,ds$. Taking the limit as δ goes to 0 proves the lemma.

Remark 2.4. – This modified definition is used rather than taking the supremum over intervals satisfying $\int_I |w|\,ds>\varphi$ because, with the modified definition, when w_φ^* approaches infinity one can conclude that the curvature goes to infinity on an interval for which the turning angle is *more* than φ. The turning angle decreases to φ as w_φ^* becomes large. If we used the definition $\int_I |w|\,ds>\varphi$ one could conclude only that the curvature goes to infinity on an interval with turning angle (possibly less than φ) whose turning angle approaches φ as w_φ^* becomes large. Experience with curves in the plane indicates that this small difference is very important (*see* [Gr]).

We will make use of the following properties of w_φ^*.

LEMMA 2.5. — *The inequalities* $\inf\{|w(u,t)|\,|\,u\in S^1\}\leqq w_\varphi^*(t)\leqq\sup\{|w(u,t)|\,|\,u\in S^1\}$ *hold. The set* $\{u\,|\,w(u,t)>w_\varphi^*(t)\}$ *is the countable union of disjoint open intervals* I_i *satisfying* $w=w_\varphi^*$ *at both endpoints and* $\int_{I_i} k^2/w\,ds\leqq\varphi$.

Proof. – The inequalities are obvious. In the second statement the set is a proper open subset of S^1 since the points where $w=\inf\{|w(u,t)|\}$ are in the complement. Every open subset of the line is the union of countably many disjoint intervals. Suppose that on one of these open intervals $\int_a^b k^2/w\,ds > \varphi$; then since w is continuous at each endpoint there is smaller compact interval on which the integral is still strictly greater than φ and on which $w > w^*_\varphi(t) + \varepsilon$ for some positive ε. This contradicts the definition of w^*_φ.

A similar lemma holds for w^*_φ.

PROPOSITION 2.6. — *If* $w^*_\pi < \mathrm{B}$ *on* $[0, \mathrm{T})$ *then* $\int w \,.\log(w/\mathrm{C})\,ds$ *is bounded uniformly for all* $t \in [0, \mathrm{T})$. *In fact, we have* $\int w\,.\log(w/\mathrm{C})\,ds \leqq \mathrm{C}_1\,\mathrm{B} + \mathrm{C}_2$ *where* C_1 *and* C_2 *are constants which depend on the initial curve. We have chosen* C *so that* $\mathrm{C}^2 \geqq \sup|\mathrm{K}|$ *on the surface.*

Proof. — We calculate using Lemmas 1.1 and 1.5

$$\left(\int w\,.\log(w/\mathrm{C})\,ds\right)_t = \int w_t\,.\log(w/\mathrm{C}) + w_t - w\,.\log(w/\mathrm{C})(w^2-\mathrm{C}^2)\,ds$$

$$= \int w_t(\log(w/\mathrm{C})+1) - w\,.\log(w/\mathrm{C})(w^2-\mathrm{C}^2)\,ds$$

$$= \int w_{ss}(\log(w/\mathrm{C})+1) - \frac{\mathrm{C}^2 w_s^2}{w(w^2-\mathrm{C}^2)}(\log(w/\mathrm{C})+1)$$

$$+ \frac{(w^2-\mathrm{C}^2)(w^2-\mathrm{C}^2+\mathrm{K})}{w}(\log(w/\mathrm{C})+1) - w\log(w/\mathrm{C})(w^2-\mathrm{C}^2)\,ds$$

$$= \int -\frac{w_s^2}{w} - \frac{\mathrm{C}^2 w_s^2}{w(w^2-\mathrm{C}^2)}(\log(w/\mathrm{C})+1) + \frac{(w^2-\mathrm{C}^2)(w^2-\mathrm{C}^2+\mathrm{K})}{w}(\log(w/\mathrm{C})+1)$$

$$- w\log(w/\mathrm{C})(w^2-\mathrm{C}^2)\,ds = \int -\frac{w_s^2\,w}{(w^2-\mathrm{C}^2)} - \frac{\mathrm{C}^2 w_s^2}{w(w^2-\mathrm{C}^2)}\log(w/\mathrm{C})$$

$$+ \frac{(w^2-\mathrm{C}^2)(\mathrm{K}-\mathrm{C}^2)}{w}(\log(w/\mathrm{C})+1) + w(w^2-\mathrm{C}^2)\,ds.$$

To obtain an upper bound for the left hand side we can drop the second and third terms on the right since $w \geqq \mathrm{C}$. Then for a fixed time t we divide the space parameter into disjoint open intervals I_i, whose union is the set $\{u \mid w(u, t) > w^*_\pi\}$; and the complementary set U. Notice that U is never empty since it contains the points where $w = w_{\min}$. On each I_i we change variables, letting $d\theta = (w^2-\mathrm{C}^2)/w\,ds = k^2/w\,ds$ to obtain

$$\int_\mathrm{I} -w_\theta^2 + w^2\,d\theta = \int_\mathrm{I} -w_\theta^2 + (w-w^*_\pi)^2 + 2\,w\,w^*_\pi - w^{*2}_\pi\,d\theta \leqq 2\,w^*_\pi \int_\mathrm{I} w\,d\theta = 2\,w^*_\pi \int_\mathrm{I} k^2\,ds.$$

The inequality follows from Wirtinger's inequality which states that if $f=0$ at the endpoints of an interval and the length of the interval is less than π then

$\int_I (f')^2 - f^2\,dx \geqq 0$. In the case above the function $w - w_\pi^* = 0$ at the endpoints of I and $\int_I d\theta \leqq \pi$ follows from the definition of w_π^*.

On U the inequality $w \leqq w_\pi^*$ holds and we have that

$$\int -\frac{w_s^2\, w}{w^2 - C^2} + w(w^2 - C^2)\,ds \leqq w_\pi^* \int_U k^2\,ds \leqq 2\,w_\pi^* \int_U k^2\,ds.$$

Adding these inequalities yields

$$\left(\int w\,.\log(w/C)\,ds\right)_t \leqq 2\,w_\pi^* \int k^2\,ds = -2\,w_\pi^*\,L_t \leqq -2\,B\,L_t.$$

using the formula for L_t given in Lemma 1.3. We integrate this to obtain

$$\int w \log(w/C)\,ds\Big|_T \leqq w \log(w/C)\,ds\Big|_0 + 2\,B\,(L(0) - L(T))$$

which completes the proof of the inequality.

We can now easily obtain a bound for k_φ^* with $\varphi < \pi$. This will be needed in Lemma 2.9.

COROLLARY 2.7. — *If* $w_\pi^* < B$ *on* $[0, T)$ *then for any* $0 < \varphi \leqq \pi$ *we have*

$$k_\varphi^* \leqq C \exp\left(\frac{C_1\,B + C_2}{\varphi}\right).$$

Proof. — Let β be a constant such that $\{u \mid k(u, t) > \beta\}$ contains an interval I on which $\int |k|\,ds \geqq \varphi$, then

$$\varphi\,.\log\left(\frac{\sqrt{\beta^2 + C^2}}{C}\right) \leqq \left(\int w ds\right) \log\left(\frac{\sqrt{\beta^2 + C^2}}{C}\right) \leqq \int_I w\,.\log\left(\frac{w}{C}\right) ds.$$

From Proposition 2.6, the right hand side is bounded by $C_1\,B + C_2$ and the estimate follows by taking the supremum of such β.

The next three lemmas will enable us to bound the sup norm $\|k\|_\infty$ of the curvature on $S^1 \times [0, T)$ in terms of the initial curve, the bound on the Gauss curvature of M, the bound on w_π^* and on T.

LEMMA 2.8. — *If* $\int |k|^6\,ds$ is bounded on $[0, T)$ *by* $\bar{B}$ *then* $\int k_s^2\,ds\,|_t \leqq C_5 + C_3\,C^4\,L(0) + C_4\,\bar{B}\,t$ where $t \in [0, T)$ *and* C_3 *and* C_4 *are universal constants,*

C^2 *bounds the Gauss curvature,* $L(0)$ *is the initial length of the curve, and* C_5 *depends on the initial curve.*

Proof. – Choose $C^2 \geqq |K(x)|$ for all $x \in M$. As in [G-H] we use Lemma 1.1, Lemma 1.2 and integration by parts to obtain:

$$\left(\int k_s^2\, ds\right)_t \leqq \int -2k_{ss}^2 - \frac{7}{3}k^3 k_{ss} - 2k k_{ss} K\, ds$$

$$\leqq \int -2k_{ss}^2 + \frac{7\varepsilon}{3}k_{ss}^2 + \frac{7}{12\varepsilon}k^6 + 2\delta C^2 k_{ss}^2 + \frac{1}{2\delta}C^2 k^2\, ds.$$

The inequality $ab \leqq \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ is used twice. Choosing δ and ε small we have

$$\left(\int k_s^2\, ds\right)_t \leqq C_4 \int |k|^6\, ds - C_3 C^2 L_t$$

from which the lemma follows by integration.

LEMMA 2.9. — *If* w_π^* *is bounded on* $[0, T)$ *then are constants* C_7 *and* C_6 *depending on the bound for* w_π^**, on the maximum curvature of the manifold and on the initial curve, such that* $\int |k|^6\, ds \leqq (C_7 + C_6 T)^4$.

Proof. – We calculate

$$\left(\int |k|^6\, ds\right)_t = \int -30 k^4 k_s^2 + 5k^8 + 6k^6 K\, ds.$$

We wish to absorb the k^6 term. Choose δ such that $\delta^2/\pi^2 = 20/49$ and therefore Wirtinger's inequality $\int_I f^2 \leqq (\delta/\pi)^2 \int_I f_\theta^2$ holds when $f = 0$ at the endpoints and $\int_I d\theta \leqq \delta$.

We choose, B_1 so that $B_1^2 \geqq \sup(6|K|)$ and $B_1 \geqq k_\delta^*(t)$ for $t \in [0, T)$. (From Corollary 2.7 we see that such a constant B_1 exists.)

Now divide the space parameter into disjoint open intervals whose union is $\{u \,|\, |k(u, t)| > B_1\}$ and the complement U of this set. Changing variables with $\partial/\partial\theta = (1/k)\,\partial/\partial s$ and $d\theta = k\,ds$ yields

$$\int_I -30k^4 k_s^2 + 5k^8 + 6k^6 K\, ds = \int_I -\frac{120}{49}((k^{7/2})_\theta)^2 + 5k^7 + 6k^5 K\, d\theta$$

$$\leqq \int_I -120/49\,((k^{7/2})_\theta)^2 + 6k^7\, d\theta \leqq 12 B_1^{7/2} \int_I k^{7/2}\, d\theta = 12 B_1^{7/2} \int_I k^{9/2}\, ds$$

where the Wirtinger inequality has been used as in Lemma 2.5. On the set U we have

$$\int_U -30k^4 k_s^2 + 5k^8 + 6k^6 K\, ds \leqq 12 B_1^{7/2} \int_U k^{9/2}\, ds.$$

Using Holder's inequality, we bound the last integral by a power $\int |k|^6\, ds$ times a constant which depends on the length of the initial curve. This yields

$$\left(\int |k|^6\, ds\right)_t \leqq \tilde{C}_6 \left(\int |k|^6\, ds\right)^{3/4}.$$

The proposition follows by integration.

COROLLARY 2.10. — *If* w_π^* *is bounded on* $[0, T)$ (*where* T *is finite*) *then* $\| k \|_\infty$ *is bounded on* $[0, T)$.

Proof. — From the definition 2.2 it is clear that $|k|$ assumes the value $\sqrt{w_\pi^{*2} - C^2}$ and using the Schwarz and triangle inequalities we find that

$$\| k \|_\infty \leqq \left| \| k \|_\infty - \sqrt{w_\pi^{*2} - C^2} \right| + \sqrt{w_\pi^{*2} - C^2}$$
$$\leqq \left(\int k_s^2\, ds\right)^{1/2} L^{1/2} + \sqrt{w_\pi^{*2} - C^2} \leqq (C_7 + C_8 T^5)^{1/2} L^{1/2} + w_\pi^*.$$

The last inequality follows from lemmas 2.8 and 2.9.

LEMMA 2.11. — *If* $w_\pi^*(t)$ is bounded on $[0, T)$ *then all of the derivatives of* k *with respect to* s *are bounded on this time interval.*

Proof. — The proof is by induction using the maximum principle. We will denote the n-th derivative by $f^{(n)}$.

We first remark that since $T < \infty$ and k is bounded, the family of curves lies in a bounded portion of the surface M. Since K is smooth this means that K and all of its covariant derivatives of any order are bounded.

Next observe that the derivatives of K along any curve gamma can always be written as $K^{(n)} = P(k, k^{(1)}, \ldots, k^{(n-2)})$ where P is a polynomial in the first $(n-2)$ derivatives of k with coefficients which are smooth functions on M. If $n < 2$ then P doesn't involve k at all. The coefficients of P are the covariant derivatives of K evaluated on unit vectors. For example,

$$K_s = \langle \nabla K, T \rangle, \qquad K_{ss} = \operatorname{Hess} K(T, T) + (\nabla K, \nabla_T T \rangle = \operatorname{Hess} K(T, T) + k \langle \nabla K, N \rangle,$$

and

$$K_{sss} = \nabla \operatorname{Hess} K(T, T, T) + k_s (\nabla K, N \rangle + k \operatorname{Hess} K(T, N) - k^2 \langle \nabla K, T \rangle.$$

This observation and the one in the paragraph above imply that $K^{(n)}$ is bounded on $[0, T)$ if $k, k^{(1)}, \ldots, k^{(n-2)}$ are all bounded on this time interval.

We now proceed to the induction: The induction statement is that $(k^{(n)})_t = (k^{(n)})_{ss} + ((3+n) k^2 + K)\ k^{(n)} + P(k, k^{(1)}, \ldots, k^{(n-1)})$ for some polynomial P and that $k^{(n)}$ is uniformly bounded on the time interval $[0, T)$.

For $n=1$ we have

$$k_{st}=k_{ts}+k^2 k_s=(k_{ss}+k^3+\mathrm{K}\,k)_s+k^2 k_s=k_{sss}+4k^2 k_s+k\,\mathrm{K}_s+\mathrm{K}\,k_s$$
$$(k^{(1)})_t=(k^{(1)})_{ss}+(4k^2+\mathrm{K})\,k^{(1)}+\mathrm{K}_s k.$$

Since the curves lie in a compact domain of M both K and K_s are bounded uniformly and k is bounded by corollary 2.10. Let $u=e^{-\alpha t}k^{(1)}-\mu t$ then $u_t=u_{ss}+(4k^2+\mathrm{K})\,u+\mathrm{K}_s k e^{-\alpha t}-\alpha u-\mu$ and choosing α and μ sufficiently large insures that the coefficient of u and the term $\mathrm{K}_s k e^{-\alpha t}-\mu$ are negative. From the maximum principle we see that u must be bounded by its initial value at $t=0$ and that $k^{(1)}$ is therefore bounded on the finite time interval [0, T).

If the induction statement holds for $k^{(1)}, \ldots, k^{(n-1)}$ we observe that

$$\begin{aligned}(k^{(n)})_t&=(k^{(n-1)})_{ts}+k^2 k^{(n)}\\&=[(k^{(n-1)})_{ss}+((n+2)k^2+\mathrm{K})\,k^{(n-1)}+\mathrm{P}(k, k^{(1)}, \ldots, k^{(n-2)})]_s+k^2 k^{(n)}\\&=(k^{(n)})_{ss}+((n+3)k^2+\mathrm{K})\,k^{(n)}+\tilde{\mathrm{P}}(k, k^{(1)}, \ldots, k^{(n-1)}).\end{aligned}$$

Applying the maximum principle to $u=e^{-\alpha t}k^{(n)}-\mu t$ as in the case where $n=1$ proves that $k^{(n)}$ is bounded.

We will need the *short time existence theorem* from [G-H] section 2.

THEOREM 2.12. — *For any smooth initial curve* $\mathrm{X}(u, 0)$ *there exists a positive* ε *and a one parameter family of curves* $\mathrm{X}(u, t)$ *with* $0 \leqq t \leqq \varepsilon$ *satisfying the curve shortening equation* (0.1)

THEOREM 2.13. Extension theorem. — *Assume that on a smooth complete surface* M *with bounded Gaussian curvature a solution to the curve shortening equation* $\mathrm{X}(u, t)$ *exists for* $t \in [0, \mathrm{T})$. *If* w_π^* *is bounded on* [0, T) *and* T *is finite then the solution of the evolution equation can be extended to* $[0, \mathrm{T}+\varepsilon)$ *for some positive* ε.

Proof. — The bound on w_π^* permits a uniform bound on k and all its derivatives, hence using the Arzela-Ascoli theorem k can be defined smoothy on [0, T]. Furthermore, the curves $\mathrm{X}(u, t)$ converge to a smooth limit curve $\mathrm{X}(u, \mathrm{T})$. Now use the short time existence theorem to extend the solution of the interval $[0, \mathrm{T}+\varepsilon)$.

3. The embedding theorem

This section is devoted to proving the following.

THEOREM 3.1. — *Let* $\mathrm{X}: \mathrm{S}^1 \times [0, \mathrm{T}] \to \mathrm{M}^2$ *be a solution to the curve shortening evolution equation with* T *finite*. If $\mathrm{X}(\cdot, 0)$ *is an embedded curve and the curvature is uniformly bounded on* $\mathrm{S}^1 \times [0, \mathrm{T}]$ *then* $\mathrm{X}(\cdot, t)$ *is embedded for each* t.

This extends Theorem 3.2.1 in [G-H] to surfaces. The ideas are essentially the same, but the details of the proof are more complicated for surfaces.

The intuition behind the theorem comes from considering the "first" time t_0 when the curve has a self intersection point. The two portions of the curve are tangent at such a point and we assume that their curvatures are not equal. Consider the position of the curve just prior to the time of first contact, because the curvature of the "inner" curve segment is greater, the curve must have had a transversal intersection for times $t<t_0$. This contradicts the fact that t_0 was the first time of self intersection. We will use the minimum principle to give a rigorous version of this argument.

Ones first thought is to attempt to apply the minimum principle to the function $F(u_1, u_2, t)$ defined by the square of the distance from $X(u_1, t)$ to $X(u_2, t)$ on the entire secant set $\mathscr{D}=S^1\times S^1\times[0, T]$. This doesn't work however because F equals zero when $u_1=u_2$ even though there is no self-intersection. In addition F is not differentiable when $X(u_2, t)$ lies in the cut locus of $X(u_1, t)$. Instead we restrict the set by removing a set $\mathscr{E}$ which contains a neighborhood of the cut locus and of the diagonal and show that no self-intersections occur which correspond to points in $\mathscr{E}$. In the neighborhood of the diagonal we use the fact that the situation is nearly Euclidean to obtain a crude, but uniformly positive lower bound on F. On the remaining set $\mathscr{D}-\mathscr{E}$ we use the minimum principle to show that F is strictly positive and therefore there are no self-intersections.

Let D_1 be the infimum of the convexity radii over the smallest compact region of M^2 in which the family of curves $\{X(\cdot, t)\}$ lies. [Since $0\leqq t\leqq T<\infty$ and we have a uniform bound on the curvature the family of curves cannot travel too far from the original curve ($|X_t|=|kN|<B$); therefore the family lies in a compact set and the infimum of the convexity radius exists and is positive]. The convexity radius at a point is the radius of the largest convex geodesic disk about the point. It is smaller than the injectivity radius.

DEFINITION 3.2. — *Let*

$$\mathscr{E}=\{(u_1, u_2, t)\,|\,F(u_1, u_2, t)\geqq(D_1)^2\}\cup\left\{(u_1, u_2, t)\,\middle|\,\int_{u_1}^{u_2} ds<\delta\right\}.$$

δ will be chosen small enough so that the curve from u_1 to u_2 can be represented non-parametrically over the geodesic tangent to the curve at u_1 and the metric in the Fermi coordinate system about this geodesic is nearly Euclidean:

LEMMA 3.3. — *There exists positive* δ *and* ε *depending on the bounds on* $|k|<B$ *and on* $|K|<C^2$ *such that* $F\geqq\varepsilon>0$ *on the boundary of* $\mathscr{E}$. *In addition* $F(u_1, u_2, t)=0$ *for* $(u_1, u_2, t)\in\varepsilon$ *if and only if* $u_1=u_2$.

Proof. — We assume that $\varepsilon<D_1^2$ so we need only show that $F\geqq\varepsilon$ on the boundary of $\mathscr{E}$ near the diagonal.

Let l be the geodesic tangent to the curve $X(\cdot, t)$ at u_1 and consider the Fermi coordinate system based on l. Let y be the distance from a point $X(v, t)$ to the nearest point on l and let x be the distance along l of this point from u_1. Then the metric has the form $ds^2=dy^2+J^2\,dx^2$ where the function J, satisfies $J(x, 0)=1$, $J_y(x, 0)=0$ and $J_{yy}=-KJ$. The angle φ which the tangent to the curve $X(\cdot, t)$ makes with the x coordinate lines satisfies $\partial\varphi/\partial s=k-\cos\varphi\, J_y/J$ (*see* Appendix A for details.)

If $(u_1, v, t) \in \mathscr{E}$ then the distance of $X(v, t)$ from l is less than the arclength of the curve from u_1 *to* v which is in turn less than δ. It follows that if δ is small enough then the curve lies in a well defined coordinate neighborhood of l where

$$|J_y| < 1, |J-1| \leqq 1/2 \qquad \text{and} \qquad |\varphi(v)| \leqq \int_{u_1}^{v} |\varphi_s| \, ds \leqq (B+2)\,\delta < \pi/4.$$

This choice of δ can be made independent of the initial point on the curve u_1.

Restricting the metric to the curve X give us $J\,dx = |\cos\varphi|\,ds$ from which we calculate that $dx \geqq (\cos\varphi)/J \geqq (\sqrt{2/3})\,ds$. This is enough to prove the second statement of the lemma since $X(u_1, t)$ and $X(u_2, t)$ will have different x coordinates in this system unless $u_1 = u_2$.

If $\int_{u_1}^{u_2} ds = \delta$ the x coordinate of the point $X(u_2, t)$ is given by $x(u_2) = \int_{u_1}^{u_2} dx \geqq (\sqrt{2/3}) \int_{u_1}^{u_2} ds = (\sqrt{2/3})\,\delta$. The distance on the surface M between the point $X(u_1)$ and $X(u_2)$ is the infimum of the length of curves connecting the first point to the second. Let γ be such a curve. Then the length of γ is $\int_\gamma ds \geqq \int_\gamma J\,dx \geqq 1/2 \int_\gamma dx = x(u_2)/2 \geqq (\sqrt{2/6})\,\delta$, provided that it stays within the coordinate neighborhood. If the curve γ leaves the coordinate neighborhood then its length is greater than $\delta > (\sqrt{2}/6)\,\delta$. From this we conclude that $F(u_1, u_2, t) = (\mathrm{dist}(X(u_1), X(u_2)))^2 \geqq \delta^2/18$. Let $\varepsilon = \delta^2/18$.

This completes the proof of lemma 3.3.

To calculate the equation satisfied by F on the set $\mathscr{D} - \mathscr{E}$ we need to calculate the first and second variation for a 3 parameter family of geodesics. Let γ be the unique geodesic segment connecting $X(u_1, t)$ to $X(u_2, t)$. We parameterize γ proportionally to arclength with parameter α running from 0 to 1 that γ maps $(\mathscr{D} - \mathscr{E}) \times [0, 1]$ into M.

We define the following vector fields on $\mathscr{D} - \mathscr{E}$ with values in TM. Let $\dot\gamma = \gamma_*(\partial/\partial\alpha)$ and let $\tau = \dot\gamma/|\dot\gamma|$ while ν is the unit vector perpendicular to τ such that the orientation of τ, ν agrees with that of M. Let $l = |\dot\gamma|$ be the length of gamma.

Let $U_i = \gamma_*(\partial/\partial u_i)$. When restricted to the geodesic segment γ, U_1 is the unique Jacobi field which is 0 at $X(u_2, t)$ and agrees with $X_*(\partial/\partial u_1)$ at $X(u_1, t)$. U_2 is the Jacobi field which is 0 at $X(u_1, t)$ and agrees with $X_*(\partial/\partial u_2)$ at $X(u_2, t)$. Dividing each of these Jacobi fields by a constant yields the Jacobi fields $\tilde{T}_i = U_i/v(u_i, t)$ which agree with the unit tangent vectors of the curve X at the endpoints of gamma.

Finally let $W = \gamma_*(\partial/\partial t)$. When restricted to γ, W is the unique Jacobi field agreeing with $X_*(\partial/\partial t) = k(u_i, t)\,N(u_i, t)$ at each endpoint $X(u_i, t)$. We'll use the short hand notation $\dot\gamma_i$, k_i, N_i to represent vectors and quantifies at $X(u_i, t)$.

Proof of theorem (3.1). – To calculate the t derivative of F we use $\nabla_{\dot\gamma}\dot\gamma = 0$ and $[\dot\gamma, W] = 0$ where ∇ is the Levi-Cevita connection on M. From

$F(u_1, u_2, t) = \left(\int_0^1 |\dot\gamma|\, d\alpha\right)^2 = l^2$ we calculate

$$F_t = 2l\int_0^1 |\dot\gamma|_t\, d\alpha = 2l\int_0^1 \frac{\langle \dot\gamma, \nabla w \dot\gamma\rangle}{|\dot\gamma|}\, d\alpha = 2\int_0^1 (\dot\gamma, \nabla_{\dot\gamma} W)\rangle\, d\alpha$$

$$= 2\int_0^1 \frac{\partial}{\partial\alpha}\langle \dot\gamma, W\rangle\, d\alpha = 2(\dot\gamma, W\rangle|_0^1 = 2\langle \dot\gamma_2, k_2 N_2\rangle - 2(\dot\gamma_1, k_1 N_1\rangle.$$

The same calculation yields

$$F_{u_1} = 2\int_0^1 \frac{\partial}{\partial\alpha}\langle \dot\gamma, U_1\rangle\, d\alpha = -2\langle \dot\gamma_1, U_1(0)\rangle, \qquad F_{u_2} = 2\langle \dot\gamma_2, U_2(1)\rangle.$$

We have written $U_1(0)$ to emphasize that the vector field is evaluated at $X(u_1, t)$ where $\alpha = 0$.

Changing to the arclength differentiation operators simplifies these equations to

$$F_{s_1} = -2\langle \dot\gamma_1, T_1\rangle|_{X(u_1, t)} \tag{3.1}$$

$$F_{s_2} = 2\langle \dot\gamma_2, T_2\rangle|_{X(u_2, t)}. \tag{3.2}$$

Now calculate the second derivatives:

$$(F_{s_1})_{u_1} = -2\frac{\partial}{\partial u_1}(\langle \dot\gamma_1, T_1\rangle|_{X(u_1, t)}) = -2\langle \nabla_{U_1}\dot\gamma_1, T_1\rangle - 2\langle \dot\gamma_1, \nabla_{U_1} T_1\rangle$$

$$= -2v(u_1, t)\langle \nabla_{\tilde\gamma_{.1}} \tilde T_1, T_1\rangle|_{X(u_1, t)} - 2v(u_1, t)\langle \dot\gamma_1, \nabla_{\tilde T_1} T_1\rangle|_{X(u_1, t)}$$

which becomes

$$F_{s_1 s_1} = -2\langle \nabla_{\tilde\gamma_{.1}} \tilde T_1, T_1\rangle|_{X(u_1, t)} - 2\langle \dot\gamma_1, k_1 N_1\rangle$$

when the arclength operators are used.

The Jacobi field $\tilde T_1$ agrees with the unit tangent of the curve at $X(u_1, t)$ and is 0 at $X(u_2, t)$ so it is given by

$$\tilde T_1(\alpha) = \langle T_1(u_1, t), \tau_1\rangle(1-\alpha)\tau + \langle T_1(u_1, t), \nu_1\rangle \frac{J_1(\alpha)}{J_1(0)}\nu \tag{3.3}$$

where J_1 is defined on [0, 1] by $J_1(1) = 0$, $J_1'(1) = -l$ and $J_1'' + l^2 KJ = 0$.

$$\nabla_{\dot\gamma} \tilde T_1|_{\alpha=0} = -\langle T_1, \tau_1\rangle\tau + \langle T_1, \nu_1\rangle \frac{J_1'(0)}{J_1(0)}\nu.$$

and

$$F_{s_1 s_1} = 2\langle T_1, \tau_1\rangle^2 - 2\langle T_1, \nu_1\rangle^2 \frac{J_1'(0)}{J_1(0)} - 2\langle \dot\gamma_1, k_1 N_1\rangle$$

In the same way one derives

$$F_{s_2 s_2}=2\langle T_2, \tau_2\rangle^2+2\langle T_2, \nu_2\rangle^2\frac{J_2'(1)}{J_2(1)}+2\langle \dot{\gamma}_2, k_2 N_2\rangle$$

and therefore

$$(3.4)\quad F_t=F_{s_1 s_1}+F_{s_2 s_2}-2\langle T_1, \tau_1\rangle^2+2\langle T_1, \nu_1\rangle^2\frac{J_1'(0)}{J_1(0)}$$
$$-2\langle T_2, \tau_2\rangle^2-2\langle T_2, \nu_2\rangle^2\frac{J_2'(1)}{J_2(1)}=F_{s_1 s_1}+F_{s_2 s_2}+A(u_1, u_2, t).$$

We will not be able to show that the minimum of F increases, but we will show that the minimum decreases exponentially, at worst, which is enough to prove the lemma. To do this let $\bar{F}=e^{\beta t}F$ so that

$$(3.5)\qquad \bar{F}_t=\bar{F}_{s_1 s_1}+\bar{F}_{s_2 s_2}+\beta\bar{F}+e^{\beta t}A(u_1, u_2, t).$$

We know that $\bar{F}\geqq F\geqq\varepsilon$ on the boundary of $\mathscr{D}-\mathscr{E}$. Suppose that $\bar{F}$ attains a positive value less than ε on the interior of $\mathscr{D}-\mathscr{E}$ then let $(\hat{u}_1, \hat{u}_2, \hat{t})$ be the first time that this occurs. At this point

$$(3.6)\qquad \bar{F}_t\leqq 0$$

$$(3.7)\qquad \bar{F}_{s_1}=\bar{F}_{s_2}=0$$

and

$$(3.8)\qquad \bar{F}_{s_1 s_1}+\bar{F}_{s_2 s_2}\geqq 2\sqrt{\bar{F}_{s_1 s_1}\bar{F}_{s_2 s_2}}\geqq 2|\bar{F}_{s_1 s_2}|$$

since it is a minimum point in the space variable. From (3.7) we see that $\langle T_i, \tau_i\rangle=0$ and that $\langle T_i, \nu_i\rangle=\pm 1$.

To estimate $\bar{F}_{s_1 s_2}=e^{\beta t}F_{s_1 s_2}$ we return to (3.1). Observe that in (3.1) $\dot{\gamma}_1$ changes as u_2 is varied with u_1 held fixed, but it is not well defined as a vector field on the target surface M at $X(u_1, t)$. For points on M close to $X(u_1, t)$ however this variation does give a well defined vector field so by taking limits we can still calculate the derivative using the Levi-Cevita connection on M in the following manner:

$$(F_{s_2})_{u_1}=\frac{\partial}{\partial u_1}\left(\lim_{\alpha\to 1} 2\langle\dot{\gamma}(\alpha), \tilde{T}_2\rangle\Big|_{\gamma(u_1, u_2, t, \alpha)}\right)=\lim_{\alpha\to 1} 2\frac{\partial}{\partial u_1}\langle\dot{\gamma}, \tilde{T}_2\rangle$$
$$=\lim_{\alpha\to 1}(\langle 2\nabla_{\dot{\gamma}}U_1, \tilde{T}_2\rangle+2\langle\dot{\gamma}, \nabla_{U_1}\tilde{T}_2\rangle).$$

Interchanging the limit and differentiation is permissible since all functions are C^∞. Now $U_1=0$ at $\alpha=1$ so the limit of the second term is zero. Changing to arclength differentiation operators and using (3.3) yields

$$F_{s_2 s_1} = -2\langle T_1, \tau_1\rangle\langle T_2, \tau_2\rangle + 2\langle T_1, \nu_1\rangle\langle T_2, \nu_2\rangle \frac{J_1'(1)}{J_1(0)}. \tag{3.9}$$

One can also show that

$$F_{s_1 s_2} = -2\langle T_1, \tau_1\rangle\langle T_2, \tau_2\rangle - 2\langle T_1, \nu_1\rangle\langle T_2, \nu_2\rangle \frac{J_2'(0)}{J_2(1)}. \tag{3.10}$$

and that $J_2'(0)/J_2(1) = -J_1'(1)/J_1(0)$.

At the point $(\hat{u}_1, \hat{u}_2, \hat{t})$ the left hand side of equation (3.5) is non-positive by (3.6). Using (3.7) through (3.10) shows that the right hand side of (3.5) is greater than or equal to

$$\begin{aligned} e^{\beta f}(\beta F + 2|F_{s_1 s_2}| + A) &\geqq e^{\beta f}\left(\beta F + 2\frac{J_1'(0)}{J_1(0)} - 2\frac{J_1'(1)}{J_1(0)} - 2\frac{J_2'(1)}{J_2(1)} + 2\frac{J_2'(0)}{J_2(1)}\right) \\ &= e^{\beta f}\left(\beta F - 2\int_0^1 -l^2 K \frac{J_1(\alpha)}{J_1(0)} d\alpha - 2\int_0^1 -l^2 K \frac{J_2(\alpha)}{J_2(1)} d\alpha\right) \\ &\geqq e^{\beta f}(\beta F - 4 l^2 C^2) = e^{\beta f}(\beta - 4C^2) F > 0. \end{aligned}$$

provided that β is chosen so that $\beta - 4C^2$ is positive. (Recall that $C^2 > \sup|K|$.) We have also used the fact that because the geodesic segment is always within the convexity radius of its endpoints the function J_1 is strictly decreasing on the interval [0, 1] while J_2 is strictly increasing.

This contradiction proves that $\bar{F}$ remains greater than ε on $\mathscr{D} - \mathscr{E}$ and therefore F remains positive.

This completes the proof of theorem (3.1).

4. Curves which converge to geodesics

In this section we determine the behavior of curves for which $w_\pi^* < B$ on an infinite time interval. This is the class of curves which we expect to converge to geodesics and except for some curves which move off to infinity, this is so. (Note that if $k_\pi^* \leqq \bar{B}$ then $w_\pi^* \leqq \sqrt{B^2 + C^2}$ by lemma 2.3.)

The estimates in section 2 are inadequate because the curvature bounds increase with T. In this section we obtain bounds which show that the curvature converges to zero; and under additional hypotheses show that subsequences of the curves themselves also converge. The first step is to show that the L_2 norm of the curvature converges to zero.

PROPOSITION 4.1. — *If* $w_\pi^* < B$ *for* $t \in [0, \infty)$ *then* $\int k^2\, ds$ converges to zero.

Remark. – The key to this proof is that $L_t = -\int k^2\, ds$, hence $\int_0^\infty \int k^2\, ds\, dt$ is finite.

Proof. – Given a positive ε with $\varepsilon < \min\{(4 L_0)^{-1}, (4 B^2 + 4 C)^{-1}\}$ we choose T_1 large enough so that

(1) $\int k^2\, ds|_{T_1} < \varepsilon/2$

(2) $\int_{T_1}^\infty \int k^2\, ds\, dt = L(T_1) - L_\infty < \varepsilon^2.$

L_∞ is the limit of the length as t goes to infinity. (2) holds for all large T_1 and from (2) we conclude that $\int k^2\, ds$ goes to zero for a subsequence $t_1, t_2, \ldots,$ of times, so both conditions can be simultaneously satisfied for some large T_1. Calculate that

$$\left(\int k^2\, ds\right)_t = \int -2 k_s^2 + k^4 + 2 k^2 K\, ds.$$

We now claim that when $\int k^2\, ds$ is small the growth is at most exponential:

$$\text{(4.1)} \qquad \text{if} \int k^2\, ds < \varepsilon \quad \text{then} \quad \left(\int k^2\, ds\right)_t < (4 B^2 + 2 C^2) \int k^2\, ds.$$

Using the Schwarz inequality we obtain

$$L(t) \int k_s^2\, ds \geqq (\|k\|_\infty - w_\pi^*)^2 = \|k\|_\infty^2 - 2\|k\|_\infty w_\pi^* + (w_\pi^*)^2 \geqq \|k\|_\infty^2 - \frac{1}{2}\|k\|_\infty^2 - (w_\pi^*)^2$$

and therefore $\|k\|_\infty^2 \leqq 2 L(t) \int k_s^2\, ds + 2 (w_\pi^*)^2$. From this it follows that

$$\left(\int k^2\, ds\right)_t \leqq \int -2 k_s^2\, ds + 2\left(\int k_s^2\, ds\right) L(t) \left(\int k^2\, ds\right) + (2 B^2 + 2 C^2) \int k^2\, ds \leqq (2 B^2 + 2 C^2) \int k^2\, ds$$

using $\int k^2\, ds < \varepsilon < (4 L(0))^{-1} < (4 L(t))^{-1}$ so that the second term can be absorbed by the first one. This proves the claim.

Suppose now that $\int k^2\, ds > \varepsilon$ for some $t \in [T_1, \infty)$. Since $\int k^2\, ds$ is a continuous function of time there will be a first time $T_2 > T_1$ at which $\int k^2\, ds = \varepsilon$ with $\int k^2\, ds \leqq \varepsilon$ on

$[T_1, T_2]$. Integrating equation (4.1) from T_1 to T_2 and using (1) and (2) we have

$$\frac{\varepsilon}{2} < \int k^2\, ds\Big|_{T_2} - \int k^2\, ds\Big|_{T_1} \leqq (2B^2 + 2C^2)\int_{T_1}^{T_2}\int k^2\, ds\, dt \leqq (2B^2 + 2C^2)\,\varepsilon^2$$

hence $(4B^2 + 4C^2)^{-1} \leqq \varepsilon$ which contradicts our choice of ε. This proves the proposition.

COROLLARY 4.2. — *If the curve continues to evolve for an infinite time then* $\int |k|\, ds$ *converges to zero.*

Proof. — Using the Schwarz inequality the square of the L_1 norm is less than the L_2 norm times the length of the curve.

Once the L_2 norm of k goes to zero we can prove that the L_2 norm of k_s remains bounded:

PROPOSITION 4.3. — *If* $\int k^2\, ds \to 0$ *on* $[0, \infty)$ *then* $\int k_s^2\, ds$ *remains bounded.*

Proof.

$$\left(\int k_s^2\, ds\right)_t = \int -2k_{ss}^2 + 7k^2 k_s^2 - 2kk_{ss}K\, ds \leqq \int -2k_{ss}^2 - 2kk_{ss}K + 7(\sup|k_s|)^2 k^2\, ds.$$

Since $k_s = 0$ at some point on the curve we have $(\sup|k_s|)^2 \leqq \left(\int k_{ss}^2\, ds\right) L$ using the Schwartz inequality. Now applying $2ab < a^2 + b^2$ to the second term and taking t_0 large enough so that $\int k^2\, ds < 1/(7L(0))$ we have

$$\left(\int k_s^2\, ds\right)_t \leqq \int -2k_{ss}^2 + k_{ss}^2 + k^2K^2\, ds + \frac{L(t)}{L(0)}\int k_{ss}^2\, \mathrm{ds} \leqq C^4\int k^2\, ds = -C^4 L_t(t)$$

Now integrate to prove that $\int k_s^2\, ds$ remains bounded for $t \in [0, \infty)$.

If the length decreases to a positive lower limit, L_∞ then we can probe that the maximum curvature converges to zero. This assumption on the length is necessary.

LEMMA 4.4. — *The following inequality holds:*

$$\|k\|_\infty \leqq \max\left\{\left(8\int k_s^2\, ds\int |k|\, ds\right)^{1/3}, \frac{2}{L}\int |k|\, ds\right\}.$$

Proof. — Let $|k|_{\min} = \min\{|k(s)|\}$. If $|k|_{\min} \leqq 1/2\|k\|_\infty$ then there is an interval $I = [a, b]$, $|k(b)| = \|k\|_\infty$ and $|k(a)| = \|k\|_\infty/2$ and $|k| \geqq \|k\|_\infty/2$ on I. Using the Schwarz inequality, it is clear that:

$$\left(\|k\|_\infty - \frac{\|k\|_\infty}{2}\right)^2 \leqq \int k_s^2\,ds \int_I ds \qquad \text{where} \quad I=\{u \,\|\, |k|>\|k\|_\infty/2\}.$$

On the interval I

$$\frac{\|k\|_\infty}{2}\int_I ds \leqq \int |k|\,ds$$

holds hence

$$\frac{\|k\|_\infty^2}{4} \leqq \frac{2\int k_s^2\,ds \int |k|\,ds}{\|k\|_\infty}$$

and

$$\|k\|_\infty^3 \leqq 8 \int k_s^2\,ds \int |k|\,ds.$$

If $\|k\|_\infty \leqq 2|k|_{\min}$ then

$$\|k\|_\infty \leqq 2|k|_{\min} \leqq \frac{2}{L}\int |k|\,ds \leqq \frac{2}{L_\infty}\int |k|\,ds.$$

This proves the lemma.

Corollary 4.5. — *If* $L_\infty > 0$ *and* $w_\pi^* < B$ *on* $[0, \infty)$ *then* $\|k\|_\infty$ *goes to* 0.

Proof. – From Lemma 4.3 we see that $\int k_s^2\,ds$ is bounded and from Lemma 4.2 that $\int |k|\,ds$ goes to 0. The result now follows from Lemma 4.4.

Proposition 4.6. — *If* $L_\infty > 0$, *the higher covariant derivatives of k are uniformly bounded and* $w_\pi^* < B$ *for all t then all of the derivatives of k with respect to s converge uniformly to* 0 *as t approaches infinity.*

Proof. – The proof is by induction. We will let $f^{(i)}$ stand for the ith derivative of f with respect to s. Let $S(n)$ be the statement: $\lim_{t\to\infty} \|k^{(i)}\|_\infty = 0$ $i=0, 1, \ldots, n-1$ and $\|k^{(n)}\|_2 \leqq M < \infty$ for $t \in [0, \infty)$. $S(1)$ is proved by Proposition 4.3 and Corollary 4.5. Assuming that $S(n-1)$ has been proven, we prove $S(n)$. The first step is to show that for all large t $\left(\int (k^{(n)})^2\,ds\right)_t$ is negative and therefore $\|k^{(n)}\|_2$ remains bounded. This involves the same tricks of balancing terms which are potentially positive by terms which are negative definite $\left(\int(-2k^{(n+1)})^2\,ds \text{ in this case}\right)$ and integrable terms $\left(\int k^2\,ds=-L_t\right)$. The following observations will be useful:

Remark 4.7. – By applying the Wirtinger inequality repeatedly we obtain

$$\int (k^{(m)})^2\,ds \leqq \left(\frac{L}{2\pi}\right)^{2n-2m} \int (k^{(n)})^2\,ds \qquad \text{with} \quad n>m\geqq 1.$$

The length of the curve L is always less than the initial length.

Remark 4.8. – Using integration by parts and $ab\leqq \varepsilon a^2+b^2/4\varepsilon$ repeatedly as well as remark 4.7 we obtain an interpolation formula for $m\leqq n$:

$$\int (k^{(m)})^2\,ds \leqq \varepsilon \int (k^{(n+1)})^2\,ds + D\int k^2\,ds$$

where $\varepsilon>0$ is arbitrarily small and D is some large constant, which depends on ε.

Remark 4.9. – Assuming $S(n-1)$ with $n\geqq 2$, then terms of the form $k^{(n+1)}k^{(a)}k^{(b)}k^{(c)}$ with $0\leqq a\leqq b\leqq c$ and $a+b+c=n-1$ can be bounded by $\mu\int (k^{(n+1)})^2\,ds$ for any $\mu>0$ if t is sufficiently large. Observe that since $n\geqq 2$ we have $b<c\leqq n-1$ and therefore

$$\int k^{(n+1)}k^{(a)}k^{(b)}k^{(c)}\,ds \leqq \|k^{(a)}\|_\infty \|k^{(b)}\|_\infty (\|k^{(c)}\|_2^2+\|k^{(n+1)}\|_2^2)/2 \leqq \mu \|k^{(n+1)}\|_2^2 \quad \text{for large } t$$

follows from the inductive hypothesis and remark 4.7.

Assume $S(n-1)$. Using lemmas 1.1 and 1.2 we calculate

$$(4.2)\quad \left(\int (k^{(n)})^2\,ds\right)_t = \int 2k^{(n)}(k^{(n-1)})_{ts}+k^2(k^{(n)})^2$$

$$= \int -2k^{(n+1)}(k^{(n-1)})_t + k^2(k^{(n)})^2$$

$$= \int -2k^{(n+1)}((k_t)^{(n-1)}+P(k,\,\ldots,\,k^{(n-1)}))+k^2(k^{(n)})^2$$

The polynomial P arises from the interchange of the operators t and s; each term is of the form $k^{(a)}k^{(b)}k^{(c)}$ with $a+b+c=n-1$. After multiplying through by $k^{(n+1)}$ in (4.2) the terms arising from the polynomial can be bound by $\mu\int (k^{(n+1)})^2\,ds$ according to remark 4.9. The last term can also be bounded by $\mu\int (k^{(n+1)})^2\,ds$ using remark 4.7

and the inductive hypothesis. We are left with

$$\left(\int (k^{(n)})^2\, ds\right)_t = \int -2\, k^{(n+1)} (k_{ss} + k^3 + \mathrm{K}\, k)^{n-1} + \mu\, (k^{(n+1)})^2$$

$$= \int -2\, (k^{(n+1)})^2 + \mathrm{D}_1\, ((\mathrm{K}\, k)^{n-1})^2 + 3\, \mu\, (k^{(n+1)})^2.$$

where the $k^{(n+1)}\, (k^3)^{(n-1)}$ terms are bounded as in remark 3 and the third term is split using $ab \leqq \varepsilon\, a^2 + b^2/4\, \varepsilon$.

Finally $((\mathrm{K}\, k)^{(n-1)})^2 \leqq \mathrm{C}_2\, \Sigma\, (\mathrm{K}^{(a)})^2\, (k^{(b)})^2$ with $a+b=n-1$. The derivatives with respect to s of K are polynomials in $\mathrm{P}(k, \ldots, k^{(n-2)})$ with bounded coefficients (*see* proof of proposition 2.11) hence the $\mathrm{K}^{(a)}$'s are bounded in the supremum norm. Further $\|k^{(b)}\|_2^2$ can be bounded by $\varepsilon\, \|k^{(n+1)}\|_2^2 + \mathrm{D}\, \|k\|_2^2$ according to remark 4.8. This proves that for sufficiently large t and small μ and ε

$$\left(\int (k^{(n)})^2\, ds\right)_t \leqq \int (-2 + 3\, \mu + \varepsilon)\, (k^{(n+1)})^2 + \mathrm{D}_2\, k^2\, ds \leqq -\mathrm{CL}_t$$

hence $\int (k^{(n)})^2\, ds$ is bounded. This proves the first part of $\mathrm{S}(n)$.

Since $\|k^{(n)}\|_2$ is bounded $k^{(n-1)}$ is equicontinuous and converges uniformly to some function f. The antiderivative of f is 0 since by the induction hypothesis $k^{(n-2)} \to 0$; therefore f is also 0. The proposition now follows from the principle of mathematical induction.

THEOREM 4.10. — *If w_π^* remains bounded for infinite time and if all the curves meet a fixed compact set on the complete surface* M *then a subsequence of the one parameter family of curves which solves the evolution equation converges to a geodesic.*

Proof. – If the curves intersect a compact region on the surface, then they lie inside a slightly larger compact region since the length (which is bounded) is greater than twice the diameter of the curves. Since $v = |\mathrm{X}_u|$ decreases we can apply the Arzela-Ascoli theorem to conclude that a subsequence converges to a limit curve γ_∞.

The length cannot decrease to zero, for if it did a subsequence would converge to a point, each curve in the subsequence would bound a disk, and from the Gauss-Bonnet theorem we could estimate the total curvature by

$$2\,\pi = \int \mathrm{K}\, d\mathrm{A} + \int k ds \leqq \mathrm{C}^2\, \mathrm{A} + \int |k|\, ds.$$

Since both the area and $\int |k|\, ds$ converge to zero we have a contradiction.

From Corollary 4.5 we have that k and its derivatives converge uniformly to zero hence γ_∞ is a geodesic.

Remark 4.11. — Surfaces of revolution with negative curvature can be constructed which show that the hypotheses in Corollary 4.5 and Theorem 4.10 are necessary.

THEOREM 4.12. — *If the surface* M *is oriented then the limiting geodesic is embedded, and the subsequence of curves converges to a single covering of the limiting geodesic.*

Proof. — The image set of the limiting curve is embedded for if it had transversal self intersections then the sequence converging to it would have transversal self-intersections. Neither can it have tangential intersections because as a geodesic, it satisfies a second order differential equation and two closed geodesics with a common point and tangent direction must be identical or one must be a multiple covering of the other.

From lemma A.1 we see that once the curvature of the curves is uniformly small and the converging subsequence of curves is uniformly close to the limiting closed geodesic then the curves can be written non-parametrically over the geodesic and converge to a single covering of the geodesic.

5. An application to curves on round spheres

THEOREM 5.1. — *A simple closed curve* γ *on the sphere of radius* $1/C$ *which divides the sphere into two pieces of equal area and whose total space curvature* $\int (k^2+C^2)^{1/2}\,ds$ *is less than* 3π *converges to a great circle under the evolution* $X_t = kN$.

Remark. — It is interesting that in this case we can show that the entire one parameter family, not just a subsequence, converges to a single geodesic. In particular, this means the flow of the curve can not approach a slowly rotating geodesic.

Proof. — From the Gauss-Bonnet theorem we have $\int k\,ds = 0$ for the initial curve and this is preserved by the flow since $\left(\int k\,ds\right)_t = \int k\,K\,ds$ according to Lemma 1.3. From the isoperimetric inequality, we have $L^2 \geqq A(4\pi - KA) = 4\pi^2 C^{-2}$ for any curve dividing the area of the sphere in half. Let I be an interval on which $k > k^*_\pi - \varepsilon$ and $\int |k|\,ds = \pi$. Then the length of I is less than $\pi(k^*_\pi - \varepsilon)^{-1}$ and the length of the rest of the curve is greater than $2\pi C^{-1} - \pi(k^*_\pi - \varepsilon)^{-1}$. We, therefore, have the inequality

$$\int_\gamma w\,ds \geqq \int_I w\,ds + \int_{\gamma - I} w\,ds \geqq \pi + C\left(\frac{2\pi}{C} - \frac{\pi}{k^*_\pi - \varepsilon}\right) \geqq 3\pi - \frac{C\pi}{k^*_\pi - \varepsilon}$$

which after rearrangement yields:

$$\infty > \frac{\pi C}{(3\pi - \int_\gamma w\, ds)} + \varepsilon \geqq k_\pi^*.$$

The quantity $\int w\, ds$ decreases by Corollary 1.6 so from Theorem 4.10 we conclude that the maximum of the curvature of the curves converges to zero and that a subsequence of the curves in the evolution converges to a fixed geodesic.

To show that the evolution converges to a single great circle, consider the set $G(t)$ of great circles which intersect the curve at time t in 4 or more points.

LEMMA 5.2. — *Let γ be a closed geodesic on an oriented surface* M *and let* $X(u, t)$ *be a one parameter family of curves following the curve shortening flow then the number of intersections $z(t)$ of $X(\cdot, t)$ with γ is a non-increasing function of time.*

COROLLARY 5.3. — *The set* $G(t)$ *decreases with time*, i.e. *if* $t_2 > t_1$, *then* $G(t_2) \subseteqq G(t_1)$.

Proof of corollary. — Follows immediately from Lemma 5.2.

The lemma follows from a theorem due to Sigur Angenent on the zero sets of parabolic partial differential equations. (See [An 1], Theorem C for the proof and [An 2] for further applications of this result to the curve shortening flow.) We state a slightly restricted version of Angement's theorem:

THEOREM 5.4 (Angenent). — *Let* $u: [0, 1] \times [0, T] \to \mathbb{R}$ *be a bounded solution to*

$$u_t = a(x, t)\, u_{xx} + b(x, t)\, u_x + c(x, t)\, u$$

with periodic boundary conditions. The assumptions on the coefficients are that a is positive and that a, a^{-1}, a_t, a_x, a_{xx}, b, b_x, and c are all bounded periodic functions. Let $z(t)$ denote the number of zeros of $u(\cdot, t)$ in $[0, 1]$. *Then*

(1) *for $t > 0$ $z(t)$ is well defined and is finite;*

(2) *if (x_0, t_0) is a multiple zero of u (i.e. u and u_x are both zero) then for all $t_1 < t_0 < t_2$ we have $z(t_1) > z(t_2)$.*

Proof of lemma 5.2. — We observe that if $X(\cdot, t)$ and γ intersect transversally at time t_0, then they intersect transversally on some small time interval $(t_0 - \delta, t_0 + \delta)$. Using the implicit function theorem the intersection points x_i can each be described by a continuous function $x_i(t)$ and in particular the number of intersection points does not change in this interval. $z(t)$ is constant on $(t_0 - \delta, t_0 + \delta)$.

If $X(\cdot, t_0)$ intersects γ tangentially at some point and the geodesic curvature of X is uniformly small then by lemma A.3 X can be written non-parametrically over γ for all t in some interval $(t_0 - \delta, t_0 + \delta)$. In local coordinates X satisfies (A.5) to which Angenent's theorem applies. Hence $z(t_1) > z(t_2)$ for $t_0 - \delta < t_1 < t_0 < t_2 < t_0 + \delta$.

This completes the proof that $z(t)$ is a decreasing function.

Remark 5.5. — Observe that this part of the proof of theorem 5.1 works for any complete C^∞ surface with bounded curvature.

LEMMA 5.6. — *The 'diameter' of the set* G(*t*) *decreases to zero.*

Proof. — We associate each great circle in G(*t*) with its polar point and let G(*t*) stand also for this set of polar points. Let H(*t*) be the set of polar points associated with the tangents to the curve X(*u*, *t*). Assume that the evolution has continued for a sufficiently long time so that $\int |k| ds$ is small and the curve lies close to an equatorial great circle and the polar curve of X lies in a small region of the north pole. The length of the polar curve is given by the total curvature of X, namely $\int |k| ds$. We claim that G(*t*) is contained in the convex hull of H(*t*) together with the convex hull of $-$H(*t*).

If a great circle intersects X in 4 or more points, choose antipodal points on the great circle so that at least two points of intersection lie in each arc between the antipodal points. Now rotate the great circle through the antipodal points until two intersections points come together in a point of tangency. Do the same in the other direction. Keep in mind that if $\int |k| ds$ is small, the curve X lies in a small tubular method of some geodesic and is even non-parametrically represented over that geodesic. This shows that the polar point to the original great circle lies on a (short) geodesic between two points of H(*t*) [or $-$H(*t*)] and therefore lies in the convex hull of H(*t*) or $-$H(*t*). Since the diameter of H(*t*) decreases to zero the diameter of G(*t*) must also decrease to zero.

From these lemmas it is clear that a curve satisfying the hypotheses converges to a single great circle. This completes the proof of Theorem 5.1.

A. APPENDIX

Here we summarize the local differential geometric formulae we need to describe the curve shortening flow in the neighborhood of a geodesic.

We construct a coordinate system for the collar neighborhood of a geodesic γ in M. This is a local construction and holds for either a closed geodesic or a geodesic segment. Let $F: [0, L_0] \times (-\varepsilon, \varepsilon) \to M$ give the normal coordinate system. The image of $F(x, 0)$ is the geodesic γ and for fixed x, $F(x, y)$ is geodesic segment which meets γ perpendicularly at $F(x, 0)$. L_0 is the length of the geodesic segment. The metric on M is expressed in these coordinates by $ds^2 = J^2\, dx^2 + dy^2$ where $J(x, y)$ satisfies $J_{yy} = -KJ$ (K is the Gauss curvature of the surface) with initial conditions $J(x, 0) = 1$ and $J_x(x, 0) = \kappa = 0$ (κ is the geodesic curvature of γ).

ε can be chosen small enough so that the map F is a diffeomorphism. In the case of a closed geodesic on an oriented manifold M the map F becomes a diffeomorphism from the cylinder obtained by identifying the endpoints of $[0, L_0]$.

We define an orthonormal frame field in this neighborhood by $e_1 = (1/J) F_*(\partial/\partial x)$ and $e_2 = F_*(\partial/\partial y)$. If a curve $X(u)$ lies within the collar neighborhood then it can be expressed parametrically in polar coordinates by $F^{-1} \circ X = (x(u), y(u))$.

The Levi-Cevita covariant derivative satisfies $\nabla_{e_1} e_1 = -(J_y/J) e_2$, $\nabla_{e_1} e_2 = (J_y/J) e_1$, $\nabla_{e_2} e_1 = 0$, and $\nabla_{e_2} e_2 = 0$. Let φ be the angle between the unit tangent T to the curve and the vector e_1. Then expressing $k = \langle \nabla_T T, N \rangle$ in terms of e_1 and e_2 and φ and simplifying yields the expression

$$k = \varphi_s - \cos\varphi \frac{J_y}{J} \tag{A.1}$$

where s is the arclength parameter along the curve.

In addition we observe that $\tan\varphi = dy/(J\,dx)$ and differentiating this yields.

$$\varphi_x = \frac{J^2}{J^2 + y_x^2} \left(\frac{y_x}{J} \right)_x \tag{A.2}$$

Finally we have $ds = \sqrt{J^2\,dx^2 + dy^2} = \sqrt{J^2\,dx^2 + J^2 \tan^2\varphi\,dx^2} = |\sec\varphi|\,J\,dx$ and from this, using the chain rule, we derive $\varphi_x = \varphi_s\,ds/dx = J \sec\varphi\varphi_s$.

Changing the independent variable from s to x in equation (A.1) and using (A.2) gives us

$$k = \frac{\cos\varphi}{J} \varphi_x - \frac{\cos\varphi}{J} J_y = \frac{J\cos\varphi}{J^2 + y_x^2} \left(\frac{y_x}{J} \right)_x - \frac{\cos\varphi}{J} J_y. \tag{A.3}$$

The equation of motion of the curve under the curve shortening flow can now be derived for those portions of the curve lying non-parametrically in the coordinate neighborhood by writing $X_t = kN$ in the local coordinates x and y and then changing variables so that the differentiation with respect to t is taken while keeping x fixed rather than u (*see* [G-H], p. 79 or [E-H], p. 22 for more details). The equation is

$$y_t = \frac{k}{\cos\varphi} = \frac{1}{J} \frac{\partial\varphi}{\partial x} - \frac{J_y}{J} = \frac{y_{xx}}{J^2 + y_x^2} - \frac{1}{J^2 + y_x^2} \frac{J_x}{J} y_x - \frac{J_y}{J}. \tag{A.4}$$

Which we rewrite as

$$y_t = \frac{y_{xx}}{J^2 + y_x^2} - \frac{1}{J^2 + y_x^2} \frac{J_x}{J} y_x - cy \tag{A.5}$$

with $c = J_y/(y\,J)$. Notice that $\lim_{y \to 0} J_y/y = \lim_{y \to 0} J_{yy}/1 = -K$ so the function c is well defined and bounded.

We now proceed to derive the estimates needed in the paper:

LEMMA A.1. — *Let* $B > \sup|K|$ *then*

$$-\sqrt{B} \tan\left(\sqrt{B}\,y\right) \leqq \frac{J_y}{J} \leqq \sqrt{B} \tanh\left(\sqrt{B}\,y\right).$$

hold for $y \geqq 0$ and the reverse inequalities hold when $y \leqq 0$. Hence for any δ and B it is possible to choose an ε collar neighborhood of a geodesic so that $|J_y/J| < \delta$ in the entire neighborhood.

Proof. — Let $\psi = J_y/J$ and observe that ψ satisfies $\psi_y = -K - \psi^2$ with $\psi(0) = 0$. We choose comparison functions which correspond to surfaces with constant curvature B. Let $\underline{J} = \cos\left(\sqrt{B}\, y\right)$, and let $\underline{\varphi} = \underline{J}_y/\underline{J}$ so that $\underline{\varphi}_y = -B - \underline{\varphi}^2$ with $\underline{\varphi}(0) = 0$. Similarly let $\bar{J} = \cosh\left(\sqrt{B}\, y\right)$, and let $\bar{\varphi} = \bar{J}_y/\bar{J}$ so that $\bar{\varphi}_y = B - \bar{\varphi}^2$ with $\bar{\varphi}(0) = 0$. It follows from a standard comparison theorem for ordinary differential equations [A], p. 17 that $\underline{\varphi} \leqq \psi \leqq \bar{\varphi}$ for $y \geqq 0$. For example, suppose that y_0 is the first y for which $\underline{\varphi}(y) \geqq \psi(y)$. Then $\underline{\varphi}_y(y_0) \geqq \psi_y(y_0)$, but from the equation it follows that $\underline{\varphi}_y = -B - \underline{\varphi}^2 < -K - \psi^2 = \psi_y$. This contradiction shows that $\underline{\varphi} \leqq \psi$ for $y \geqq 0$. A similar argument proves the other half of the inequality. For $y \leqq 0$ the comparison theorem yields $\underline{\varphi} \geqq \psi \geqq \bar{\varphi}$. This completes the proof.

LEMMA A.2. — *If the surface M is oriented and the curve X lies completely in an ε neighborhood of a closed geodesic γ and if the geodesic curvature satisfies $|k| \leqq \delta$ where δ and ε depend only on B and γ then the curve can be described non-parametrically over γ.*

Proof. — Choose ε small enough so that the normal coordinate system of γ is one-to-one and so that $|J_y/J| \leqq \pi/(8\,L_0)$. Let $\delta = \pi/(8\,L_0)$. From (A.1). We see that $|\varphi_s| \leqq |k| + \sup|J_y/J| \leqq \pi/(4\,L_0)$. At the point where X is furthest from γ, φ equals 0 or π. (If $\varphi = \pi$ reverse the orientation of X so that $\varphi = 0$.) Integrating this inequality proves that $|\varphi(s)| \leqq \pi/4$ hence locally the curve is represented non-parametrically in the normal coordinate system.

Because X has no self-intersections restricting the projection map $(x, y) \to (x, 0)$ to X yields a differentiable covering map π from X to γ. The covering is finite because X is compact. Since M is oriented the local coordinate system near γ is homeomorphic to a cylinder and it makes sense to define $f(\hat{x}, 0) = (\hat{x}, \hat{y})$ where $\hat{y}$ is the supremum of the y coordinates of points of X whose x coordinate is $\hat{x}$. Locally f equals the local homeomorphism $(\pi|_U)^{-1}$ so f is a continuous and open function. X has only one component hence the image of f is all of X. Since $\pi \circ f = \mathrm{id}$, f is one to one with inverse π. f gives the non-parametric representation of X.

LEMMA A.3. — *If the curve X intersects γ in an angle less than δ and if $|k| \leqq \eta$ where δ and η depend only on B, ε and L_0 then X lies entirely within the ε neighborhood of γ and by the previous lemma can be written non-parametrically over γ.*

Remark A.4. — The intuition behind this is that geodesics close to γ diverge from γ more slowly than similar geodesics on a surface with large negative curvature B. Furthermore a curve with small geodesic curvature should behave approximately like a geodesic. In this lemma we prove an easier and weaker result which is sufficient for our present purposes.

Proof. — We use the comparison function

$$\bar{w}(s)=\eta\left(\cosh\left(\sqrt{C}\,s\right)-1\right)+\delta/\sqrt{C}\sinh\left(\sqrt{C}\,s\right)$$

which satisfies $\bar{w}(0)=0$, $\bar{w}_s(0)=\delta>0$ and $\bar{w}_{ss}=C\bar{w}+\eta$. Given C, L_0 and ε it is clearly possible to choose δ and η sufficiently small so that $0\leqq\bar{w}(s)\leqq\varepsilon$ for $s\in[0, L_0]$. Let $\underline{w}=-\bar{w}$.

Choose C so that $-Cy\leqq\sqrt{B}\tan\left(\sqrt{B}\,y\right)\leqq\sqrt{B}\tanh\left(\sqrt{B}\,y\right)\leqq Cy$ for $\varepsilon>y\geqq 0$. Then from equation (A.6) it is clear that $|J_y/J|\leqq C|y|$.

We now prove a comparison theorem similar to the one in [A], p. 17. Let $\bar{\psi}=\bar{w}_s$ and $\underline{\psi}=-\bar{\psi}=\underline{w}_s$. Let $s_1=\sup\{s\,|\,s\in[0, L_0]$ and for all $\sigma\in[0, s]\,\underline{\psi}(\sigma)<\varphi(\sigma)<\bar{\psi}(\sigma)\}$.

Since $y_s=T[y]=\cos\varphi\, e_1[y]+\sin\varphi\, e_2[y]=\sin\varphi$ we have $\underline{\psi}\leqq-\varphi\leqq y_s\leqq\varphi\leqq\bar{\psi}$ for $s\in[0, s_1]$ and therefore $\underline{w}\leqq y\leqq\bar{w}$ on the same interval. Suppose now that $s_1<L_0$ then by continuity we have either $\bar{\psi}(s_1)=\varphi(s_1)$ or $\underline{\psi}(s_1)=\varphi(s_1)$. Assuming the former we see that $\varphi_s(s_1)\geqq\bar{\psi}_s(s_1)$ on the other hand

$$\varphi_s(s_1)=k+\cos\varphi\,(J_y/J)<\eta+C\,|y(s_1)|<\eta+C\bar{w}(s_1)=\bar{\psi}_s(s_1)$$

which is a contradiction. A similar contradiction with inequalities in the opposite direction is obtained if $\underline{\psi}(s_1)=\varphi(s_1)$. It follows that $s_1=L_0$ and that $\underline{w}(s)<y(s)<\bar{w}(s)$ for $s\in[0, L_0]$. This completes the proof.

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(Manuscript received November 22, 1988,
in revised form: june 27, 1989).

Michael E. GAGE,
University of Rochester,
Department of Mathematics,
Ray P. Hylan Building,
Rochester, New-York 14627
and
Institut des Hautes Études Scientifiques,
35, route de Chartres,
91440 Bures-sur-Yvette.