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UNITARY REPRESENTATIONS OF SOLVABLE LIE GROUPS (1)

BY L. PUKANSZKY.

Du kennst doch das Schillersche Gedicht " Spruch des Konfucius " und weisst, dass ich da besonders die Zeilen liebe : Nur die Fülle führt zur Klarheit und im Abgrund wohnt die Wahrheit.

N. BOHR, quoted in W. HEISENBERG,
Der Teil und das Ganze
(R. Piper Co., München, 1969, p. 284).

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INTRODUCTION (2).

The investigations of the present paper started with an examination by the present author, through special examples, of the possibility to extend the recent theory of type I solvable Lie groups by L. Auslander and B. Kostant (*cf.* [1]) to arbitrary Lie groups. These authors, carrying forward by an essential step the line of research started by A. A. Kirillov [22] and continued by P. Bernat [3], using results by C. C. Moore, gave a neccessary and sufficient condition in order that a connected and simply connected solvable Lie group be of type I. Furthermore they provided in this case a complete description, by aid of the orbit space of the coadjoint representation, of the dual. Thus, in particular, if G is such a group and $\mathfrak{g}$ is its Lie algebra, then G is of type I if and only if : (1) Any orbit of G on $\mathfrak{g}'$ (= dual of the underlying space of $\mathfrak{g}$) is locally

(2) The Introduction and the Summary, in front of each chapter, intends to give only an outline of the results of this paper. For a precise formulation of these as well as for complete references to the literature we refer to the corresponding point of the detailed discussion.

closed and (2) The de Rham class of the canonical 2-form is always integral (and hence zero; *cf.* Th. V. 3.2, *loc. cit.*). Two examples, due to J. Dixmier (*cf.* [10]) and F. I. Mautner resp., of solvable groups which are not of type I, are particularly well known in the literature (for the definition of these *cf.* the Summary of Chapters I and II). From among these Dixmier's group satisfies the first of the above conditions but not the second, Mautner's violates the first, but satisfies the second. A closer inspection of Dixmier's example led us to the conclusion, that by aid of a natural extension of the procedure of Auslander and Kostant one can associate to each orbit in the general position a well determined factor representation of type II_∞. More importantly it turned out, that this relationship admits a description modelled after Kirillov's theory of characters of a connected and simply connected nilpotent group. Let G be such a group and $\mathfrak{g}$ its Lie algebra. We can identify the underlying manifold of G to that of $\mathfrak{g}$ by means of the exponential map. The measure, corresponding on G to a translation invariant measure *dl* on $\mathfrak{g}$ is biinvariant. Let T be an irreducible unitary representation of G, φ an element of $C_c^\infty(\mathfrak{g})$ and let us form the operator $T(\varphi) = \int_{\mathfrak{g}} \varphi(l)\, T(l)\, dl$. It is of trace class, and Kirillov's formula, which is the natural analogue (*cf.* [30], p. 258-264) of the character formula of H. Weyl for compact semi-simple groups, provides the following expression for its trace. Let us write $\langle l, l' \rangle$ ($l \in \mathfrak{g}$, $l' \in \mathfrak{g}'$) for the canonical bilinear form of the underlying abelian group of $\mathfrak{g}$. We define the Fourier transform of φ by

$$\hat{\varphi}(l') = \int_{\mathfrak{g}} \varphi(l) \langle l, l' \rangle\, dl \qquad (l' \in \mathfrak{g}').$$

Then there is a uniquely determined orbit O of G on $\mathfrak{g}'$ such that

$$\mathrm{Tr}\,(T(\varphi)) = \int_{O} \hat{\varphi}(l')\, dv \tag{1}$$

where *dv* is an appropriately normalized invariant measure on O. Let us observe, incidentally, that in the case envisaged here, the proof of the absolute convergence of the right hand side is relatively simple. Conversely, to each orbit O there is a unitary equivalence class, corresponding to O by virtue of formula (1). In other words (1) can be used to define a bijection between elements of the orbit space and of the dual of G resp. (*cf.* for all these e. g. [29]). Returning to the example of Dixmier we found, that with the factor representations we constructed (1) substantially retains its validity, provided on the left hand side by $\mathrm{Tr}(\mathrm{T}(\varphi))$ we mean the value on $\mathrm{T}(\varphi)$ of an appropiately normalized trace in the sense of the II_{∞} factor generated by T. We obtained similar conclusion for the group of Mautner with the difference, that in place of O we had to substitute closures of orbits of the coadjoint representation of G.

The main consequence of the above observation for us was a concrete suggestion, that perhaps for all connected solvable Lie groups the left (or right) regular representation is a continuous direct sum of semifinite factor representations, or what amounts to the same, the left (or right) ring, that is the von Neumann algebra generated by the left (or right) regular representation is semifinite. Let us recall, that this was shown by I. E. Segal to be the case for any separable locally compact unimodular group (*cf.* [34]) but was disproved by R. Godement in the general case. This conclusion, in fact, imposes itself by assuming, that for any connected solvable group, too, sufficiently many semifinite factor representations can be constructed, such that the essential features of (1) be preserved, and by observing the mechanism of the Plancherel formula in the nilpotent case. In fact, let us write $\Lambda = \mathfrak{g}'/G$, and let us set T_λ, O_λ and $d\nu_\lambda$ resp. for the objects, as in (1), corresponding to $\lambda \in \Lambda$. Then to show, that the representations $\{\mathrm{T}_\lambda ; \lambda \in \Lambda\}$ provide a central continuous direct sum decomposition of the left regular representation, one has to prove, that the value $\varphi(0)$ of φ at zero can be reconstructed from the values $\mathrm{Tr}(\mathrm{T}_\lambda(\varphi))$ by aid of a formula of the type

$$\varphi(0) = \int_{\Lambda} \mathrm{Tr}(\mathrm{T}_\lambda(\varphi))\, d\mu(\lambda). \tag{2}$$

But if dl' is an appropriately normalized translation invariant measure on $\mathfrak{g}'$, we have

$$\varphi(0) = \int_{\mathfrak{g}'} \hat{\varphi}(l')\, dl'.$$

From this we conclude, that to obtain a formula as (2), it suffices to represent dl' as a continuous direct sum of the measures $d\nu_\lambda$ by aid of a measure $d\mu$ on Λ.

Although much progress has recently been made toward a clarification of the possibilities of a formula as (1) for type I groups (*cf.* [15]), unfortunately already in this case any attempt to obtain a theory as for the nilpotent groups is confronted with great difficulties. Their reasons, among others are, that a bijection along Kirillov's lines is limited to groups with simply connected orbits, and that it seems to be exceedingly difficult to establish the convergence of integrals as in (1) for a sufficiently ample family of functions. We wish to observe, incidentally, that these problems do not at all appear to increase by abandoning the assumption, that our group be of type I. In this fashion, to follow up the indications carried out above, we had to look for a different tool which we found in the theory of quasi unitary algebras due to J. Dixmier (*cf.* [7]). As a result, we succeeded in establishing the purely global result, that the left (or right) ring of any connected but not necessarily simply connected Lie group carries a faithful trace (3), such that the corresponding family of generalized Hilbert-Schmidt operators generates the whole ring (*cf.* Theorem 4, Chapter IV of this paper). Hence, in particular, the left (or right) ring of any group of the said sort is semifinite. This conclusion has been shown in the mean time by J. Dixmier to retain its force for an arbitrary connected topological group (*cf.* [14]).

This result of ours, however, leaves open the problem of the possibility of an « orbitwise » theory of factor representations. One can namely raise the question, if the procedure of Auslander and Kostant, through an appropriate modification, leads to a class of factor representations, which can claim some special interest. In this paper we show, that this is indeed the case as already suggested, incidentally, by the examples of Dixmier and Mautner discussed above. Our main result concerning this point (*cf.* Theorems 2 and 3, Chapter III) provides a family of factor representations parametrized by certain geometrical objects, generalizing the orbits of the coadjoint representation in such a fashion, that the regular representation admits a central continuous direct sum decomposition involving only these representations. The necessity to consider more than one representation for one orbit, and thus to go beyond these in a search for objects parametrizing the dual, arises already in the case of the universal covering group of the motion group of the Euclidean plane. For the general type I group, according to the algorithm of Auslander and Kostant, the irreducible representations, belonging to the same orbit, can be parametrized by a torus, the dimension of which is equal to the first Betti number this orbit. Our construction proceeds in two major steps. First (*cf.* Chapter I) we associate with any orbit a family of semifinite factor representations, the members of which are in a one to one correspondence with the underlying set of a torus. The dimension of the latter, however, is in general different from that of the

(3) For our definition of the trace *cf.* e. g. Section 7 in Chapter IV.

type I theory. Example for this situation is given by an orbit in the general position of Dixmier's group. Here our torus is zerodimensional, while the Betti number in question is 2. If all orbits are locally closed, as is the case, in particular, for the type I groups, the collection of all these representations already provides a central decomposition of the regular representation. For a type I group this step essentially reproduces the algorithm, defining the orbit-representation relation, of Auslander and Kostant. The only difference is, that our representations are not necessarily irreducible ones, but one or $+\infty$ fold multiples of such representations. If, however, there are orbits, which are not locally closed, as in the case of the group of Mautner, to obtain the « central components » of the regular representation a more involved construction is necessary. In Chapter II we introduce a generalization of the orbit concept leading to certain solvmanifolds, which are transformation spaces of our group, such that any orbit of the latter is dense. Also, these spaces carry invariant Borel measures. From here, by virtue of a classical principle (*cf.* Lemma 2.3.1, Chapter III) we obtain our « central factors » in Chapter III by forming continuous direct sums over the said manifolds of appropriate subcollections of the representations of Chapter I. Groups, violating simultaneously both conditions of Auslander and Kostant, may at this point display additional difficulties, not indicated by the examples of Dixmier and Mautner (*cf.* Section 8, Chapter II). Finally, using the previous theory we show, that if our group is simply connected, the left (or right) ring coincides with its type I or type II component (*cf.* Theorem 5, Chapter IV). In other words, the left (or right) regular representation of any such group admits a central continuous direct sum decomposition into factor representations, all of which are either of type I or of type II only. Let us also observe, that our results imply the necessity of the conditions of Auslander and Kostant quoted at the start. In fact, at once one of these is not satisfied, there appears in our list a factor representation, which is not of type I.

It is clear from the beginning, that our construction cannot aim at a complete classification of the factor representations of these groups. For example, in the case of the groups of Dixmier and Mautner our procedure provides semifinite factor representations only. On the other hand, since these groups are not of type I, by virtue of the results of J. Glimm they admit type III representations. We shall say, that the unitary representation T is of trace class, if on the von Neumann algebra $\mathbf{R}(T)$ it generates there exist a faithful, normal and semifinite trace [3], such that the set of generalized Hilbert-Schmidt operators in the range of the associated representation of the group C* algebra generates $\mathbf{R}(T)$

(*cf.* Section 7, Chapter IV). For instance, by what we saw above, the left (or right) regular representation of a connected solvable Lie group always has this property. Our results imply, that an « overwhelming » majority of the representations appearing in our list are trace class representations and hence, in particular, generate semifinite factors. But we leave in this paper the problem of an individual characterization of these representation open. While admitting, that certain points of the following programm, at the present stage of the research, might appear overly ambitious, we still believe, that ultimately it turn out, that our representations, up to quasi-equivalence, give precisely the collection of all trace class representations. In addition we conjecture, that the factors they generate are always approximately finite. The significance of the last point is, that in this fashion one could show, that by considering factor representations, which are not of type I, one does not get involved in the algebraic type problem of factors. Or, to put it more succintly, this widening of the view point should not place one in a situation worse, than in the type I theory. The author is indebted to C. C. Moore for the following suggestion of a collective characterization of our representations. One could try to show, that upon forming the kernels of the associated representations of the group C* algebra, one obtains precisely once each primitive ideal of the latter. Let us observe, that recently R. Howe obtained results along these lines for a class of discrete nilpotent groups (*cf.* [21]).

As far as the prerequisites for the reading of the present paper are concerned, our exposition of the necessary results of the geometry of orbits of linear solvable groups is self contained. On the other hand, we assume a relatively advanced knowledge of the theory of induced representations by G. W. Mackey. In fact, we shall use the basic results of [23] and [25] often without special reference. For a summary we refer to [2], Sections 9-10 (p. 50-63). Also, some preliminary familiarity with the notion of holomorphic induction is necessary (*cf.* the references of Section 4, Chapter I). The reader is advised to consult carefully the list of notational conventions at the end of the paper.

The results of Chapter IV, Sections 1-7 were announced in [33], those of the rest of this paper in the author's conference at the International Conference of Mathematicians, Nice, September 1970.

The author is much indebted to B. Kostant for introduction in his joint work with L. Auslander, and also for discussion in his seminair at the Massachusets Institute of Technology, Fall 1968, of several parts of Chapter IV.

CHAPTER I.

THE TRANSITIVE THEORY.

SUMMARY. — Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$. As already stated in the Introduction, in this chapter we assign to each orbit of the coadjoint representation of G on $\mathfrak{g}'$ a family of semifinite factor representations. Our discussion follows at many points the treatment of the type I case by Auslander and Kostant in [1]. One of the major differences appears, however, already at the start. The purpose of Sections 1-2 is to discuss certain factor representations of a group, which is central extension by a one dimensional torus $\mathbf{T}$ of a direct product of a free abelian group of finite rank with a vector group. The necessity to consider such groups arises in the following fashion. We denote by L the first derived group of G (or $L = [G, G]$); L is nilpotent and thus of type I. Let π be an irreducible unitary representation of L; then the corresponding Mackey group $\mathbf{M}_\pi$ (*cf.* the begin of Section 3 for the definition) has the indicated structure. Let Γ be a group of this class, U the centralizer of the connected center and $U^\natural$ the center of U. The main result of this part (*cf.* Proposition 2.1) states, that if χ is a character of $U^\natural$, such that its restriction to $\mathbf{T} \subset U^\natural$ is not trivial, then the unitary representation, induced by χ in Γ, is a semifinite factor representation, and gives a necessary and sufficient condition that it be of type I. Let π be as above, G_π its stabilizer in G, π^e an appropriately chosen projective extension of π to G_π, and G_π^e the corresponding central extension of G by a one dimensional torus. The collection of the factor representations of this chapter coincides with the family of all representations of the form $\operatorname{ind}_{G_\pi \uparrow G} (V \otimes \pi^e)$, where V is a representation, arising by lifting to G_π^e a representation of $\mathbf{M}_\pi (= \Gamma)$ obtained as above by aid of a χ, which on $\mathbf{T} \subset U^\natural$ coincides with the conjugate of the identity map of $\mathbf{T}$ onto itself, for all possible choice of π in the dual $\hat{L}$ of L and χ. In order, that G be of type I, in particular, $\mathbf{M}_\pi$ has to be of type I for all $\pi \in \hat{L}$. In this case our procedure yields one or infinite fold multiples of the collection of all irreductible representations of G. Section 3 gives a description, not directly involving the Mackey group, of our representations. It is shown (*cf.* Lemma 3.5) that each $\pi \in \hat{L}$ uniquely determines a closed subgroup $K_\pi \supseteq L$, such that π admits a proper extension ρ to K_π, and that our representations coincide with the collection of all representations of the form $\operatorname{ind}_{K_\pi \uparrow G} \rho$ $\left(\text{for all possible choice of } \pi \in \hat{L} \text{ and of } \rho \in \hat{K}_\pi,\ \rho \mid L = \pi\right)$. We give a necessary and sufficient condition that such a representation be of type I, and that two of them be quasi-equivalent (in which case they are also unitarily equivalent; *cf.* for all this Lemmas 3.7, 3.8 and Remark 3.3). These considerations do not at all depend on the assumption, that G be solvable, provided L is appropriately chosen. In an effort to bring this to expression, in this section (but only here) we allow G to be an arbitrary simply connected Lie group and take in place of [G, G] a closed, connected, invariant and type I subgroup L, such that G/L abelian. In Section 4, beside summarizing the necessary prerequisites of holomorphic induction and of the Kirillov theory (this we take for granted), we present the definition of the reduced stabilizer. Let g be an element of $\mathfrak{g}'$, G_g the stabilizer of g in G with respect to the coadjoint representations, $(G_g)_0$ the connected component of the identity and $\mathfrak{g}_g \subset \mathfrak{g}$ the Lie algebra of the latter. Since G is solvable and simply connected, $(G_g)_0$,

too, is simply connected, whence we conclude, that there is a well determined character χ_g on $(G_g)_0$ such that $d\chi_g(l) = i(l, g)$ $(l \in \mathfrak{g}_g)$. Let us put $\mathring{G}_g = \ker(\chi_g \mid (G_g)_0)$; this is an invariant subgroup of G_g, and the reduced stabilizer $\overline{G}_g$ of g is the complete inverse image, in G_g, of the center of $G_g/\mathring{G}_g$. Section 5 reproduces the proof of an important result of Auslander and Kostant establishing a relation between the obstruction cocycle belonging to $\pi \in \hat{L}$ and the Kirillov orbit of π in the dual $\mathfrak{d}'$ of the underlying space of the Lie algebra $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ of L. Using this in Section 6 we show, that if $\pi \in \hat{L}$ corresponds to the Kirillov orbit $Lf \subset \mathfrak{d}'$ $(f \in \mathfrak{d}')$, and g is any element of $\mathfrak{g}'$ such that $g \mid \mathfrak{d} = f$, then we have $K_\pi = L\overline{G}_g$, and $\operatorname{ind}_{K_\pi \uparrow G} \rho$ $(\rho \in K_\pi,\ \rho \mid L = \pi)$ is of type I if and only if the group $G_g/\overline{G}_g$ is finite. This condition can be shown (but we do not carry out this point) to be equivalent to the rationality of the de Rham class of the canonical 2-from on Gg (for a definition of the latter *cf.* e. g. [30], p. 256). The integrality of this form means, that $\overline{G}_g = G_g$ and conversely; in order, that G be of type I in particular, this must be valid for all $g \in \mathfrak{g}'$. The results of this section are used in an essential fashion, among others, in Chapter IV to estimate the « size » of the totality of type I representations in the central decomposition of the regular representation (*cf.* in particular Proposition 8.1, Chapter IV). Finally Section 7 brings the construction, along the lines laid down by Auslander and Kostant, of our representations as (in general) holomorphically induced representations. For a $g \in \mathfrak{g}'$ let us denote by $\hat{\overline{G}}_g$ the collection of all characters of $\overline{G}_g$ restricting on $(G_g)_0$ to χ_g (*cf.* above). Let us put $\mathcal{R} = \cup_{g \in \mathfrak{g}} \hat{\overline{G}}_g$; $\mathcal{R}$ is a transformation space of G. One of our main conclusions is, that there is a bijection between the set of all unitary equivalence classes of our representations and points of $\mathcal{R}/G$. Orbits, lying over Gg (g fix in $\mathfrak{g}'$), in $\mathcal{R}$ are parametrized by points of $\hat{\overline{G}}_g$. The underlying set of the latter admits a natural identification with the dual of $\overline{G}_g/(G_g)_0$, which is a free abelian group of finite rank. In the case of a type I group, since here $\overline{G}_g = G_g$, $\overline{G}_g/(G_g)_0$ is just the fundamental group of Gg. But, for instance, in the case of the group of Dixmier quoted in the Introduction, the situation is already completely different. This group belongs to the Lie algebra, spanned over the reals by the elements $\{ e_j ; 1 \leq j \leq 7 \}$ with the nonvanishing brackets

$$[e_1, e_2] = e_7, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = -e_3, \qquad [e_2, e_5] = e_6, \qquad [e_2, e_6] = -e_5.$$

For a general $g \in \mathfrak{g}'$ we have $\overline{G}_g = (G_g)_0$, while the rank of $G_g/(G_g)_0$ is two. Thus the dimension of the torus, parametrizing the representations belonging to the same G orbit in $\mathfrak{g}'$, is in general different from the first Betti number of the latter.

1. Proposition 1.1. — *Let Z be a free abelian group of finite rank, and let us consider a central extension $\overline{Z}$ of Z by a one dimensional torus* **T**. *Let χ be a character of the center $\overline{Z}^\natural$ of $\overline{Z}$, which, when restricted to* **T**, *reduces to the identity map of the circle group onto itself. The unitary representation*

$$\operatorname*{ind}_{\overline{Z}^\natural \uparrow \overline{Z}} \chi = U$$

of $\overline{Z}$ is a factor representation of finite class which, on **T**, *equals to a multiple of χ. U is of type I if and only if the index of $\overline{Z}^\natural$ in $\overline{Z}$ is finite.*

Proof. — *a.* We recall first (*cf.* [2], p. 188), that there exist a skew-symmetric bilinear form α from $Z \times Z$ into $\mathbf{T}$ (identified with the group of all complex numbers of absolute value one), such that $\overline{Z}$ is isomorphic to the group of all pairs (z, u) $(z \in Z)$, $u \in \mathbf{T})$ with the law of multiplication

$$(z, u)(z', u') = (z + z', u.u'.\alpha(z, z')).$$

Given a subgroup Γ of Z, we shall write $\overline{\Gamma}$ for the subgroup $\{(\gamma, u); \gamma \in \Gamma, u \in \mathbf{T}\}$ of $\overline{Z}$. Let us form now the subgroup $Z_0 = \{x; x \in Z, (\alpha(x, y))^2 = \text{for all } y \in Z\}$ of Z; one verifies easily, that $\overline{Z}_0$ coincides with the center $\overline{Z}^\natural$ of $\overline{Z}$. We denote by Z_1 the subgroup of all elements $\{x; \alpha(x, y) = 1 \text{ for all } y \text{ in } Z\}$, and by χ_1 the restriction of χ to $\overline{Z}_1$. Finally, we write $\mathcal{F}$ for the set of all characters of $\overline{Z}_0$, which on $\overline{Z}_1$ restrict to χ_1.

LEMMA 1.1. — *Putting*

$$V = \operatorname*{ind}_{\overline{Z}_1 \uparrow \overline{Z}} \chi_1 \qquad \textit{and} \qquad U_\varphi = \operatorname*{ind}_{\overline{Z}_0 \uparrow \overline{Z}} \varphi \qquad (\varphi \in \mathcal{F})$$

we have

$$V = \sum_{\varphi \in \mathcal{F}} \oplus U_\varphi$$

and U_φ, *when restricted to* $\overline{Z}_0$, *equals to a multiple of* φ.

Proof. — Let us observe first, that $\overline{Z}_0/\overline{Z}_1$ is finite; in fact, it is isomorphic to Z_0/Z_1, which is of a finite rank and any element in it has the order 2. We have thus

$$\operatorname*{ind}_{\overline{Z}_1 \uparrow \overline{Z}_0} \chi_1 = \sum_{\varphi \in \mathcal{F}} \oplus \varphi$$

whence, through induction by stages we conclude, that

$$V = \operatorname*{ind}_{\overline{Z}_1 \uparrow \overline{Z}} \chi_1 = \operatorname*{ind}_{\overline{Z}_0 \uparrow \overline{Z}} \left(\operatorname*{ind}_{\overline{Z}_1 \uparrow \overline{Z}_0} \chi_1 \right) = \sum_{\varphi \in \mathcal{F}} \oplus \operatorname*{ind}_{\overline{Z}_0 \uparrow \overline{Z}} \varphi = \sum_{\varphi \in \mathcal{F}} \oplus U_\varphi.$$

Finally, since $\overline{Z}_0$ is the center of $\overline{Z}$, U_φ on $\overline{Z}_0$ restricts to a multiple of φ.

Q. E. D.

b. Let $\mathcal{A}$ be a countable abelian group and β a skew symmetric bilinear form, with values in $\mathbf{T}$ (= circle group), on $\mathcal{A} \times \mathcal{A}$. Similarly as above, we write $\overline{\mathcal{A}}$ for the group defined on the set of all pairs $\{(a, u); a \in \mathcal{A}, u \in \mathbf{T}\}$ by the law of multiplication

$$(a, u)(b, v) = (a + b, u.v.\beta(a, b)).$$

We denote by χ_0 the character of $\mathbf{T} \subset \overline{\mathcal{A}}$ defined by $\chi_0((0, u)) \equiv u$.

LEMMA 1.2. — *With the above notations, the unitary representation* $\mathrm{W} = \operatorname{ind}_{\mathbf{T}\uparrow\overline{\mathcal{A}}} \chi_0$ *can be described as follows. There is a unitary map from the representation space* H (W) *of* W *onto* $\mathrm{L}^2(\mathcal{A})$ ($\mathcal{A}$ being taken with the discrete topology), *such that the von Neumann algebra* **R** (W) *generated by* W *goes over into the set of all bounded operators on* $\mathrm{L}^2(\mathcal{A})$ *which, with respect to the natural basis can be expressed in matrix form as*

$$\{a_{y-x}\beta(x, y); x, y \in \mathcal{A}\} \qquad (a_x \in \mathbf{C}, x \in \mathcal{A}).$$

The commutant of R (W) *goes over into the set of all bounded operators, which can be written as*

$$\{b_{y-x}\beta(y, x); \; x, y \in \mathcal{A}\}.$$

Proof. — In the following we shall write a and u in place of $(a, 1)$ and $(0, u)$ resp. $(a \in \mathcal{A}, u \in \mathbf{T})$ whenever convenient.

1° Choosing an invariant measure on $\overline{\mathcal{A}}$, by virtue of our definition of $\overline{\mathcal{A}}$, there is a natural isomorphism between the Hilbert spaces $\mathrm{L}^2(\overline{\mathcal{A}})$ and $\mathrm{L}^2(\mathbf{T}) \otimes \mathrm{L}^2(\mathcal{A})$. Let L and R be the left and right regular representation resp. of $\overline{\mathcal{A}}$ on $\mathrm{L}^2(\overline{\mathcal{A}})$. Since $(0, u)^{-1}(a, v) = (a, \overline{u}.v)$, writing **R** for the ring, generated by the regular representation of **T** on $\mathrm{L}^2(\mathbf{T})$, we see at once, that $\mathbf{R}(\mathrm{L} \mid \mathbf{T}) = \mathbf{R} \otimes \mathrm{I} \subset (\mathbf{R}(\mathrm{L}))^{\natural}$, from which, taking into account that $\mathbf{R}' = \mathbf{R}$, we conclude, that any operator in the left ring **R** (L) of $\mathcal{A}$ can be expressed as a matrix

$$\{\mathrm{A}_{x,y}; \; x, y \in \mathrm{A}\} \tag{1}$$

the entries taking their values in **R**. Similar observation applies to the right ring.

2° For $w \in \mathbf{T}$ and $f \in \mathrm{L}^2(\mathbf{T})$ let us write $\mathrm{L}_w f(u) \equiv f(wu)$. Since $(z, 1)(x, u) = (z + x, u\beta(z, x))$, we can conclude, that for $f \in \mathrm{L}^2(\overline{\mathcal{A}})$,

$$(\mathrm{L}(z) f)(x, u) \equiv (\mathrm{L}_{\beta(z,x)})_u f(x + z, u) \qquad [\mathrm{L}(z) = \mathrm{L}((z, 1)^{-1})].$$

In this fashion the right ring of $\overline{\mathcal{A}}$ coincides with the set of all operators in (1), which commute with any member of the family of operators $\{ L(z); z \in \mathcal{A} \}$. Similarly, putting for $z \in \mathcal{A}$ and $f \in L^2(\overline{\mathcal{A}}) = L^2(\mathbf{T}) \otimes L^2(\mathcal{A})$

$$(R(z) f)(x, u) \equiv (L_{\beta(x,z)})_u f(x + z, u)$$

the left ring is the collection of all elements in (1) commuting with every operator in $\{ R(z); z \in \mathcal{A} \}$.

3° Let now A be an arbitrary element of (1), and let us write out the condition, that it belong to the right ring. According to what we have just seen, in order that this happen, we must have for all $z \in \mathcal{A}$ and $f \in L^2(\mathcal{A})$

$$(L(z) A f)(x, u) \equiv (A L(z) f)(x, u).$$

But

$$(L(z) A f)(x, u) \equiv \sum_{\delta \in A} (L_{\beta(z,x)} A_{x+z,\delta})_u f(\delta, u)$$

and

$$(A L(z) f)(x, u) \equiv \sum_{\delta \in A} (A_{x,\delta} L_{\beta(z,\delta)})_u f(\delta + z, u) \equiv \sum_{\delta \in A} (A_{x,\delta-z} L_{\beta(z,\delta)})_u f(\delta, u).$$

Thus A belongs to the right ring if and only of we have for all $z, x, \delta \in \mathcal{A}$,

$$A_{x,\delta-z} L_{\beta(z,\delta)} = L_{\beta(z,x)} A_{x+z,\delta}$$

whence, putting $x = 0$, and writting $A_y = A_{0,y}$ $(y \in \mathcal{A})$ we conclude, that a necessary condition is the existence of a sequence $\{ A_y; y \in \mathcal{A} \} \subset \mathbf{R}$, such that

$$A_{x,y} = A_{y-x} L_{\beta(x,y)}. \tag{2}$$

One sees, however, at once, that this condition is also sufficient provided, of course, $\{ A_x; x \in \mathcal{A} \}$ is such, that (2) defines a bounded operator on $L^2(\overline{\mathcal{A}})$.

Similarly one finds, that the operators in the left ring are representable as $\{ A_{y-x} L_{\beta(y,x)}; x, y \in \mathcal{A}) \}$, and conversely.

4° We recall, that the representation space $\mathbf{H}(W)$ of $W = \operatorname{ind}_{\mathbf{T} \uparrow \overline{\mathcal{A}}} \chi_0$ consists of the collection of all complex-valued measurable functions, satisfying $f(u.\mathfrak{a}) \equiv u f(\mathfrak{a})$ for all $u \in \mathbf{T}$ and $\mathfrak{a} \in \overline{\mathcal{A}}$, for which

$$\sum |f(\mathfrak{a})|^2 < +\infty$$

provided $\mathfrak{a}$ runs through a residue system of $\overline{\mathcal{A}}$ mod $\mathbf{T}$.

On any such function the action of $W(\alpha)$ $(\alpha \in \overline{\mathfrak{A}})$ is obtained by translation on the right by α. Assume, as we can, that the total measure of $\mathbf{T} \in \overline{\mathfrak{A}}$, with respect to the invariant measure $d\tau$, equals 1, and let us form the central projection

$$P = \int_{\mathbf{T}} \bar{u}\, R(u)\, d\tau(u).$$

From what we have just said it is clear, that we have a natural identification of $\mathbf{H}(W)$ with $PL^2(\overline{\mathfrak{A}})$, such that W corresponds to the component of the right regular representation in $PL^2(\overline{\mathfrak{A}})$. On the other hand, $PL^2(\overline{\mathfrak{A}})$ is canonically identifiable with $L^2(\mathfrak{A})$. Putting $P_1 = \int_{\mathbf{T}} \bar{u} L_u\, d\tau(u)$, we have, that $P = \{P_1 \delta_{x,y};\, x, y \in \mathfrak{A}\}$. Since for any $A \in \mathbf{R}$, $P_1 A$ is a scalar multiple of P_1, bearing in mind what we have just seen in 3° we conclude, that the von Neumann algebra $\mathbf{R}(W)$ generated by W coincides in $L^2(\mathfrak{A})$ with the collection of all operators having a matrix expression of the form

$$\{a_{y-x}\, \beta(x, y);\, x, y \in \mathfrak{A}\} \qquad (a_y \in \mathbf{C} \text{ for all } y \in \mathfrak{A}).$$

The commutant of $\mathbf{R}(W)$, corresponding to the component of the left ring of $\overline{\mathfrak{A}}$ in $PL^2(\overline{\mathfrak{A}}) \sim L^2(\mathfrak{A})$ is given by the family of all bounded operators, which can be written as

$$\{a_{y-x}\, \beta(y, x);\; x, y \in \mathfrak{A}\}.$$

Q. E. D.

REMARK 1.1. — Observe, that the reasoning employed above implies, that $\operatorname{ind}_{\mathbf{T}\uparrow\overline{\mathfrak{A}}} \chi_0$ is the largest subrepresentation of the right regular representation of $\overline{\mathfrak{A}}$ with the property, that on $\mathbf{T} \subset \overline{\mathfrak{A}}$ it restricts to a multiple of the identity map of $\mathbf{T}$ into itself. Analogous statement holds true upon replacing χ_0 by $\bar{\chi}_0$.

c. LEMMA 1.3. — *With the previous notations,* $\mathbf{R}(W)$ *is a von Neumann algebra of a finite class.*

Proof. — For $A = \{a_{y-x}\, \beta(x, y);\, x, y \in \mathfrak{A}\}$ let us put $f(A) = a_0$. Evidently, f defines a linear form on $\mathbf{R}(W)$. To prove our lemma, it is enough to show, that $\operatorname{Tr}(AA^*) = 0$ implies, that $A = 0$, and that $\operatorname{Tr}(AA^*) = \operatorname{Tr}(A^*A)$ for all A in $\mathbf{R}(W)$. One sees at once, that $A^* = \{b_{y-x}\, \beta(x, y)\}$, where $b_x = \bar{a}_{-x}$ $(x \in \mathfrak{A})$ and thus

$$AA^* = \left\{\sum_z a_{z-x}\, \bar{a}_{z-y}\, \beta(x, z)\, \beta(z, y)\right\} \quad \text{and} \quad A^*A = \left\{\sum_z \bar{a}_{x-z}\, a_{y-z}\, \beta(z, x)\, \beta(z, y)\right\},$$

from which we infer, that

$$f(AA^*) = \sum_z |a_z|^2 = f(A^*A)$$

proving our lemma.

d. We recall (*cf*. the list of notational conventions at the end of this paper), that given a family of operators J on a Hilbert space, we shall write **R** (J) for the von Neumann algebra generated by the elements of J.

Let us form the subgroup $\mathfrak{A}_0 = \{ x; (\beta(x, y))^2 = 1 \text{ for all } y \text{ in } \mathfrak{A} \}$ of $\overline{\mathfrak{A}}$. Similarly as in (*a*) we write $\overline{\mathfrak{A}}_0$ for the corresponding subgroup of $\overline{\mathfrak{A}}$. Observe, that $\overline{\mathfrak{A}}_0$ coincides with the center $\overline{\mathfrak{A}}^\natural$ of $\overline{\mathfrak{A}}$.

Lemma 1.4. — *We have* $(\mathbf{R}(W))^\natural = \mathbf{R}(W \,|\, \overline{\mathfrak{A}}_0)$.

Proof. — 1° By virtue of Lemma 1.2, if A belongs to

$$(\mathbf{R}(W))^\natural = \mathbf{R}(W) \cap (\mathbf{R}(W))'$$

we have

$$A = \{ a_{y-x}\, \beta(x, y) \} = \{ b_{y-x}\, \beta(y, x) \}$$

from which we conclude at once, that $a_x \equiv b_x$ $(x \in \mathfrak{A})$ and that $a_z \equiv \beta(x, z)^2 a_z$ for all x and z in $\mathfrak{A}$. This implies, that if a_z is nonzero, z is an element of $\mathfrak{A}_0$.

2° To obtain the identity claimed in our lemma, it is now sufficient to observe, that by virtue of what we saw in the proof of lemma 1.2, we have for any $z \in \mathfrak{A}$,

$$W((z, 1)) = R(z) \,|\, \mathbf{H}(W) = \{ \delta_{y-x-z}\, \beta(x, y); x, y \in \mathfrak{A} \}.$$

These two observations imply, that $(R(W))^\natural = R(W \,|\, \overline{\mathfrak{A}}_0)$. The opposite inclusion being trivial, our lemma is proved.

Q. E. D.

From now on we shall assume, that $\mathfrak{A}_0$ is finite, in which case $\overline{\mathfrak{A}}_0$ is compact. We write **E** for the collection of all characters of $\overline{\mathfrak{A}}_0$, which, when restricted to $\mathbf{T} \subset \overline{\mathfrak{A}}_0$ coincide with χ_0 [*cf*. (*b*)]. With this notation we have

Corollary of lemma 1.4. — *Writing* $W \,|\, \overline{\mathfrak{A}}_0 = \sum_{\chi \in E} \cdot \chi P_\chi$, *and* W_χ *for the part of* W *in* $P_\chi H(W)$, W_χ *is a factor representation of finite class, and*

$$W = \sum_{\chi \in E} \oplus W_\chi.$$

Proof. — To obtain the desired conclusion, it is enough to remark, that by virtue of lemma 1.4 the center of $\mathbf{R}(W)$ is identical with the collection of all linear combinations of the family $\{P_\chi; \chi \in \mathbf{E}\}$. $\mathbf{R}(W_\chi)$ is a factor of finite class since, by Lemma 1.3, $\mathbf{R}(W)$ is of finite class.

Q. E. D.

e. LEMMA 1.5. — *With the previous notations we have* $\dim P_\chi = m$ $(\chi \in \mathbf{E})$, *where m is the* (finite or infinite) *index of* $\mathfrak{A}_0$ *in* $\mathfrak{A}$.

Proof. — We recall [*cf.* 4°) in (*b*) above], that $W | \overline{\mathfrak{A}}_0$ is just the part of $R | \overline{\mathfrak{A}}_0$ in $PL^2(\overline{\mathfrak{A}})$. We write R_0 for the regular representation of $\overline{\mathfrak{A}}_0$ and recall, that $R | \overline{\mathfrak{A}}_0$ is unitarily equivalent to mR_0, where m equals the (finite or infinite) index of $\overline{\mathfrak{A}}_0$ in $\overline{\mathfrak{A}}$, which is the same as the index of $\mathfrak{A}_0$ in $\mathfrak{A}$. From here to obtain, that $\dim P_\chi = m$ $(\chi \in \mathbf{E})$ it suffices to observe, that $\overline{\mathfrak{A}}_0$ is isomorphic to the direct product of the circle group and of a finite abelian group.

Q. E. D.

f. LEMMA 1.6. — $\mathbf{R}(W_\chi)$ *is a factor of type* I *or* II *according to whether* $\mathfrak{A}$ *is finite or infinite resp.* $(\chi \in \mathbf{E})$.

Proof. — Let us consider the involution S of $L^2(\mathfrak{A}) = \mathbf{H}(W)$ defined by $(Sf)(x) \equiv \overline{f(-x)}$ $[f \in \mathbf{H}(W)]$. One sees at once, that if

$$\mathbf{R}(W) \ni A = \{ a_{y-x}\, \beta(x, y);\, x, y \in \mathfrak{A} \},$$

we have

$$SAS = \{ b_{y-x}\, \beta(y, x);\, x, y \in \mathfrak{A} \}$$

where $b_x \equiv \overline{a}_{-x}$ $(x \in \mathfrak{A})$, and thus $S\mathbf{R}(W)S = (\mathbf{R}(W))'$. If A lies in $(\mathbf{R}(W))^{\natural}$, $a_u \neq 0$ implies, that u belongs to $\mathfrak{A}_0$ (*cf.* 1° in Lemma 1.4). But since $x \equiv y\ (\mathfrak{A}_0)$ entails $\beta(x, y) = \beta(y, x)$, we can conclude, that now $SAS = A^*$. Therefore, in particular, S leaves invariant the subspace $P_\chi \mathbf{H}(W)$ $(\chi \in \mathbf{E})$, and denoting its part in the latter by S_χ, we have

$$S_\chi \mathbf{R}(W_\chi) S_\chi = (\mathbf{R}(W_\chi))'.$$

In this fashion we obtain, that $\mathbf{R}(W_\chi)$ is of type (I_n, I_n) or (II_1, II_1), according to whether $m = \dim P_\chi$ is finite or infinite. But since m is the index of $\mathfrak{A}_0$ in $\mathfrak{A}$, and $\mathfrak{A}_0$ is assumed to be finite, we get the conclusion of our lemma.

Q. E. D.

Lemma 1.7. — *For each element of* $\mathbf{E}$ *there is a factor representation* W_χ $(\chi \in \mathbf{E})$, *such that* $W_\chi | \overline{\mathfrak{A}}_0 \equiv \chi I$, *and*

$$\operatorname*{ind}_{\mathbf{T}\uparrow\overline{\mathfrak{A}}} \chi_0 = \sum_{\chi \in \mathbf{E}} \oplus W_\chi.$$

The factors $\mathbf{R}(W_\chi)$ *are of type* I_n *if* $\mathfrak{A}$ *is finite; otherwise they are all of type* II_1.

Proof. — We have, similarly as in Lemma 1.1,

$$\operatorname*{ind}_{\mathbf{T}\uparrow\overline{\mathfrak{A}}_0} \chi_0 = \sum_{\chi \in \mathbf{E}} \oplus \chi,$$

and thus

$$\operatorname*{ind}_{\mathbf{T}\uparrow\overline{\mathfrak{A}}} \chi_0 = \sum_{\chi \in \mathbf{E}} \oplus \operatorname*{ind}_{\overline{\mathfrak{A}}_0\uparrow\overline{\mathfrak{A}}} \chi = \sum_{\chi \in \mathbf{E}} \oplus W_\chi,$$

whence the desired conclusion follows by virtue of lemma 1.6.

Q. E. D.

Remark 1.2. — Similar result holds true if we replace χ_0 by $\overline{\chi}_0$.

g. Using the previous considerations, we can complete the proof of Proposition 1.1 in the following fashion.

1° Let us consider again the character χ_1 of $\overline{Z}_1$ [*cf.* (*a*)]. The function, assigning to $x \in Z_1$ the complex number $\chi_1((x, 1))$, on Z_1 is obviously a character of the latter; we denote it by χ_1'. Let ψ be an arbitrary character of Z extending χ_1', and let us define a function ψ on $\overline{Z}$ by $\psi(a) = \psi(x).u$ if $a = (x, u) \in \overline{Z}$. One verifies easily, that $\psi(a)\psi(b) = \omega(a, b)\psi(a, b)$, where $\omega(a, b) = \overline{\alpha(x, y)}$ if $a = (x, u)$, $b = (y, u)$. We have evidently $\omega(aa_0, bb_0) \equiv \omega(a, b)$ if a_0 and b_0 are arbitrary elements in $\overline{Z}_1$.

2° We denote by $\overline{Z}^e$ the group defined on the set $\{(a, u); a \in \overline{Z}, u \in \mathbf{T}\}$ by the law of composition

$$(a, u)(b, v) = (a.b, uv\,\omega(a, b)).$$

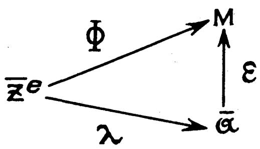
The subset $\{(a, 1); a \in \overline{Z}_1\}$ is a central subgroup of $\overline{Z}^e$, to be denoted again by $\overline{Z}_1$. We put $M = \overline{Z}^e/\overline{Z}_1$ and write Φ for the canonical homomorphism from $\overline{Z}^e$ onto M.

Let us define the function Ψ_e on $\overline{Z}^e$ by $\Psi_e((a, u)) = \Psi(a).u$; Ψ_e is a character of $\overline{Z}^e$. Denoting by R the right regular representation of M, we write R_1 for the part or R in the subspace, having the projection

$$\int_{\mathbf{T}} u.R(u)\,d\tau(u)$$

[*cf.* 4°) in (*b*)], of $L^2(M)$. The representation $(R_1 \circ \Phi) \otimes \Psi_e$ of $\bar{Z}^e$ is identically one on the subgroup $\{(e, u);\, u \in \mathbf{T}\} = \mathbf{T}_e$ of $\bar{Z}^e$ and, by virtue of Lemma 1, [26] (p. 325), *the corresponding representation on* $\bar{Z} = \bar{Z}^e/T_e$ *is unitarily equivalent to* $\operatorname{ind}_{\bar{Z}_1 \uparrow \bar{Z}} \chi_1$.

3° Let us put $\mathcal{A} = Z/Z_1$; we denote by μ the canonical homomorphism of Z onto $\mathcal{A}$. There is a function β on $\mathcal{A} \times \mathcal{A}$, such that for all x, y, in Z we have $\beta(\mu(x), \mu(y)) \equiv \alpha(x, y)$. We form $\bar{\mathcal{A}}$ as at the start of (*b*) with the β just defined. Define the map $\zeta : \bar{Z} \to \mathcal{A}$ by $\zeta((x, u)) = \mu(x)$; ζ is a homomorphism, and (*cf.* 1°) above) $\omega(a, b) \equiv \beta(\zeta(a), \zeta(b))$. Let us put for $(a, u) \in \bar{Z}^e : \lambda((a\ u)) = (\zeta(a), u) \in \bar{\mathcal{A}}$. By virtue of what we have just said, λ is a homomorphism of $\bar{Z}^e$ onto $\bar{\mathcal{A}}$, and its kernel coincides with $\bar{Z}_1^e \subset \bar{Z}^e$ (*cf.* 2°). Let ε be the isomorphism from $\bar{\mathcal{A}}$ onto $M = \bar{Z}^e/\bar{Z}_1$ such that the diagramm



is commutative. Then $R_1 \circ \varepsilon$ is the largest subrepresentation of the right regular representation of $\bar{\mathcal{A}}$ with the property. that on $\mathbf{T} \subset \bar{\mathcal{A}}^\natural$ it coincides with a multiple of $\bar{\chi}_0$ [for the latter *cf.* (*b*)], and hence, by virtue of Remark 1.1, we have $R_1 \circ \varepsilon = \operatorname{ind}_{T \uparrow \bar{\mathcal{A}}} \bar{\chi}_0$. Upon forming, as in (*d*) above, the subgroup $\mathcal{A}_0$ of $\mathcal{A}$, we find, that $\mathcal{A}_0 = \mu(Z_0) = Z_0/Z_1$, and thus $\mathcal{A}_0$ is of finite order. In this fashion, using Lemma 1.7 and Remark 1.2, with notations as *loc. cit.* we get, that $R_1 \circ \varepsilon = \sum_{\chi \in E} \oplus W_\chi$ and thus also

$$(R_1 \circ \Phi) \otimes \Psi_e = \sum_{\chi \in E} \oplus (W_\chi \circ \lambda) \otimes \Psi_e.$$

The representation, corresponding to $(W_\chi \circ \lambda) \otimes \Psi_e$, of $\bar{Z} = \bar{Z}^e/\mathbf{T}_e$ is a factor representation of the type of W_χ; when restricted to $\bar{Z}_0 = \bar{Z}^\natural$, it coincides with a multiple of a character φ in $\mathcal{F}$ [*cf.* (*a*)]. Denoting it by U'_φ, and by $\mathcal{F}_1$ the subset of $\mathcal{F}$ formed by the φ's so obtained, we get, that in the sense of unitary equivalence $V = \operatorname{ind}_{\overline{Z}_1 \uparrow \overline{Z}} \chi_1 = \sum_{\varphi \in \mathcal{F}_1} \oplus U'_\varphi$. On the other hand, we have (*cf.* Lemma 1.1), $V = \sum_{\varphi \in \mathcal{F}_1} \oplus U_\varphi$ and U_φ is identical on $\mathbf{T} \subset \overline{Z}^\natural$ to a multiple of φ. Hence we conclude finally, that $\mathcal{F}_1 = \mathcal{F}$ and $U_\varphi = U'_\varphi$. By Lemma 1.7 and by what we have just said U'_φ is a factor representation of finite class. It is of type I if and only if $\mathcal{Cl} = Z/Z_1$ is of finite order. Since Z_0/Z_1 is finite, this is the case if and only if Z/Z_0 is finite, or, since $\overline{Z}/\overline{Z}_0$ is isomorphic to Z/Z_0, U'_φ is of type I if and only if the index of $\overline{Z}^\natural$ in $\overline{Z}$ is finite.

To complete the proof of Proposition 1.1, it is enough to observe, that χ as *loc. cit.* is contained in $\mathcal{F}$.

Q. E. D.

Remark 1.3. — Analogous statement holds true for a character χ of $\overline{Z}^\natural$, which on $\mathbf{T}$ coincides with the conjugate of the identity map of the circle group onto itself.

2. Proposition 2.1. — *Let* Z *be direct product of a vector group and of a free abelian group of finite rank, and let us consider a central extension* Γ *of* Z *by a one dimensional torus* $\mathbf{T}$. *We denote by* U *the centralizer of the center of the connected component of* Γ. *Let* χ *be a character of the center* $U^\natural$ *of* U *which, when restricted to* $\mathbf{T} \subset U^\natural$, *reduces to the identity map of the circle group onto itself. Let us put* $\operatorname{ind}_{U^\natural \uparrow \Gamma} \chi = V(\chi)$. *With these notations, the unitary representation* $V(\chi)$ *of* Γ *is a factor representation of type* I *or* II. *It is of type* I *if and only of the subgroup* $U^\natural \Gamma_0$ (Γ_0 = connected component of the identity in Γ) *is of a finite index in* U. *Finally, we have* $V(\chi) = V(\chi')$ *if and only if* χ *and* χ' *lie on the same orbit of* Γ *in the dual of* $U^\natural$.

Proof. — *a.* We recall first (*cf.* [2], p. 188), that there exist a skew symmetric bilinear form B on $Z \times Z$ with values in $\mathbf{R}$, such that, putting $\beta(x, y) = \exp[(i/2) B(x, y)]$ $(x, y \in Z)$, Γ is isomorphic to the group, defined on the set of all pairs (z, u) $(z \in Z,\ u \in \mathbf{T})$ by the law of multiplication

$$(x, u)(y, v) = (x + y,\ u.v.\beta(x, y)).$$

b. Given a in Γ, let us put $(a\chi)(g) \equiv \chi(a^{-1} ga)$ $(g \in U^\natural)$. We claim, that we have $a\chi = \chi$ if and only of a belongs to U. To this end let us note, that

$$(x, u)(y, v)(x, u)^{-1} = (y, (\beta(x, y))^2 . v) = (y, v)(0, (\beta(x, y))^2)$$

for any pair of elements (x, u) and (y, v) in Γ. In particular, they commute, if and only if, $(\beta(x, y))^2 = 1$. Let us assume now, that $a = (x, u)$ and that $a\chi = \chi$. Then, since $\Gamma_0^\natural \subset U^\natural$ and $\chi \mid \mathbf{T} =$ identity map of $\mathbf{T}$ onto itself ($\mathbf{T} = \{(0, u)\}$), we must have, in particular, $(\beta(x, y))^2 = 1$ whenever (y, v) belongs to $\Gamma_0^\natural$; in other words, a must lie in U, proving our assertion.

Let us put $U(\chi) = \operatorname{ind}_{U^\natural \uparrow U} \chi$; we have $V(\chi) = \operatorname{ind}_{U \uparrow \Gamma} U(\chi)$, and $U(\chi) \mid U^\natural$ is a multiple of χ. We observe next, that to prove Proposition 2.1 *it is enough to establish, that* $U(\chi)$ *is a factor representation of type* I *or* II, *and that we have the first case if and only if* $U/U^\natural\Gamma_0$ *is finite.* In fact, since, as we saw above, U is the stable group of χ in Γ, $\operatorname{ind}_{U \uparrow \Gamma} U(\chi) = V(\chi)$ is a factor representation of the type of $U(\chi)$. Thus to complete the proof of our proposition it suffices to show, that $V(\chi) = V(\chi')$ (in the sense of unitary equivalence) if and only if χ and χ' differ by an action of Γ. Since $V(\chi) \mid U^\natural$ is a multiple of the direct sum of members of the (countable) subset $\Gamma\chi$ of the dual of $U^\natural$, the condition is evidently necessary. If, on the other hand, $\chi' = a\chi$ $(a \in \Gamma)$, then $V(\chi') = aV(\chi) = V(\chi)$ $[aV(\chi)(g) \equiv V(\chi)(a^{-1}ga);\ g \in \Gamma]$ completing the proof of our statement.

c. If W is some subset of Z, we shall write sometimes also W for the subset $\{(w, 1); w \in W\}$ of Γ. On the other hand, $W_B^\perp$ will stand for $\{z; z \in Z,\ B(z, w) = 0$ for all w in $W\} \subset Z$.

Let us put Γ_1 for the centralizer of Γ_0 in Γ. Writting Z_0 for the connected component of zero in Z, and using the above notations we get easily $\Gamma_1 = (Z_0)_B^\perp . \mathbf{T}$. If Z_1 is the radical of the restriction of B to $Z_0 \times Z_0$, that is $Z_1 = (Z_0)_B^\perp \cap Z_0$, we obtain in the same fashion $U = (Z_1)_B^\perp . \mathbf{T}$. Since $(Z_1)_B^\perp = Z_0 + (Z_0)_B^\perp$, we have

$$\Gamma_1 . \Gamma_0 = ((Z_0)_B^\perp + Z_0) . \mathbf{T} = U.$$

Clearly, Z_1 is the connected component of zero in $(Z_0)_B^\perp$. Let Σ be a closed subgroup, such thet $(Z_0)_B^\perp$ is the direct sum of Z_1 and of Σ. Then we have also $(Z_1)_B^\perp = Z_0 + \Sigma$ (direct sum). Let $\psi : (Z_1)_B^\perp \to \Sigma$ be the projection onto Σ. Then the map $\Psi : U \to \Sigma$ defined by $\Psi((x, u)) = \psi(x)$ is a homomorphism. Let us define the map $\omega : U \times U \to \mathbf{T}$ by

$$\omega(a, b) = \overline{\beta(\Psi(a), \Psi(b))} \qquad (a, b \in U).$$

The law of composition $(a, u)(b, v) = (ab, \omega(a, b).uv)$ defines a group U^e on the set of all pairs $\{(a, u); u \in \mathbf{T}, a \in U\}$ (topologized in the obvious fashion).

Given a subgroup V of U, we shall write V^e for its complete inverse image in U^e. Given a subset $S \subset U$, we shall often use the same notation for $\{(s, 1); s \in S\} \subset U^e$. We denote by $\mathbf{T}_e$ the central subgroup $\{(e, u); u \in \mathbf{T}\}$ of U^e.

By virtue of the definition of ω, we have

$$\omega(a.a_0, b.b_0) \equiv \omega(a, b) \qquad (a, b \in U;\ a_0, b_0 \in \Gamma_0),$$

implying that Γ_0 $[= \{(a, 1); a \in \Gamma_0\} \subset U^e]$ is an invariant subgroup of U^e. In fact, one verifies at once, that for $a \in \Gamma_0$ and $(b, u) \in U^e$:

$$(b, u)(a, 1)(b, u)^{-1} = (bab^{-1}, 1).$$

d. Let us consider the subgroup $\Gamma_1^e \subset U^e$. Since $\Gamma_0^{\natural} = Z_1 . \mathbf{T}$, we have

$$\Gamma_1^e = \Gamma_1 \mathbf{T}_e = \Sigma \Gamma_0^{\natural} \mathbf{T}_e,$$

and the map, assigning to the triple (σ, a, u) $(\sigma \in \Sigma,\ a \in \Gamma_0^{\natural},\ u \in \mathbf{T}_e)$ the element σau of Γ_1^e, is a bijection between the set $\Sigma \times \Gamma_0^{\natural} \times \mathbf{T}_e$ and Γ_1^e. [Observe, that here σ stands for $((\sigma, 1), 1) \in U^e$ if $\sigma \in \Sigma \subset Z$, etc.] We write now χ_0 for the restriction of χ (the latter as in the proposition) to $\Gamma_0^{\natural} \subset U^{\natural}$, and define a map $\chi^e : \Gamma_1^e \to \mathbf{T}$ by $\chi^e(\sigma au) = \chi_0(a).u$. We claim, that χ^e is a character of Γ_1^e. In fact, we have

$$\sigma au . \tau bv = \sigma a \tau buv = \sigma\tau(\tau^{-1} a \tau) buv \qquad (\sigma, \tau \in \Sigma;\ a, b \in \Gamma_0^{\natural};\ u, v \in \mathbf{T}_e),$$

and by what we saw above, $\tau^{-1} a \tau = a$. On the other hand

$$\sigma\tau = ((\sigma, 1), 1)((\tau, 1), 1) = \big((\sigma + \tau, \beta(\sigma, \tau)), \overline{\beta(\sigma, \tau)}\big)$$

and thus, writing $\alpha(\sigma, \tau) = ((0, \beta(\sigma, \tau)), 1)$ we can conclude, that

$$\sigma au . \tau bv = \sigma' a' v',$$

where

$$\sigma' = \sigma + \tau \in \Sigma, \qquad a' = \alpha(\sigma, \tau).a.b \in \Gamma_0^{\natural} \quad \text{and} \quad v' = \overline{\beta(\sigma, \tau)}.uv.$$

In this fashion, since $\chi_0 | \mathbf{T}$ is the identity map, we get, that

$$\chi^e(\sigma au . \tau bv) = \chi_0(\alpha(\sigma, \tau)\, a . b\ ).u.v.\overline{\beta(\sigma, \tau)} = \chi_0(a)\, u . \chi_0(b)\, v = \chi^e(\sigma au).\chi^e(\tau bv)$$

proving our statement.

e. Let us denote by γ the Lie algebra of $\Gamma_0 = Z_0 . \mathbf{T} \subset U$. Denoting by v the element of γ, such that $\exp(tv) \equiv (0, \exp(it))$ $(t \in \mathbf{R})$, we can identify γ with $Z_0 + \mathbf{R} v$, such that $[z + cv, z' + c'v] = B(z, z')v$, and that $\exp(z + cv) = (z, \exp(ic)) \in \Gamma_0$.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of γ, and let us put H for the corresponding connected subgroup of Γ_0. If d is any element of the dual γ' of γ, there is a well determined character χ_d of H satisfying $\chi_d(\exp l) = \exp[i(l, d)]$ $(l \in \mathfrak{h})$. We recall finally. that $\operatorname*{ind}_{H \uparrow \Gamma_0} \chi_d$ is an irreducible representation of Γ_0, which we denote by V_d.

Let us consider now the unitary representation $W = \operatorname*{ind}_{\Gamma_1^e \uparrow U^e} \chi^e$ of U^e. Our next objective is to show, that if $d \in \gamma'$ is such, that $\chi_d \mid \Gamma_0^\natural \equiv \chi_0 (\equiv \chi \mid \Gamma_0^\natural)$, then W *is multiple of a representation* ρ of U^e, *such that* $\rho \mid \Gamma_0$ *is unitarily equivalent to* V_d. To prove this, let us put $K = \Gamma_1^e H = \Sigma H \mathbf{T}_e$ and let us observe, that there is a character χ'_d of K, such that $\chi'_d \mid \Gamma_1^e \equiv \chi^e$ and $\chi'_d \mid H \equiv \chi_d$. In fact, to this end it is enough to take into consideration, that 1° if $a \in \Gamma_1^e$ and $b \in H$, then $ab = ba$, 2° $\Gamma_0^\natural = \Gamma_1^e \cap H$, 3° by virtue of our choice of d, $\chi_d \mid \Gamma_0^\natural = \chi^e \mid \Gamma_0^\natural$. We put $L = K/\Gamma_1^e$ and denote by λ the canonical homomorphism from K onto L. Observe, that we have

$$L = K/\Gamma_1^e = \Gamma_1^e H/\Gamma_1^e = H/\Gamma_1^e \cap H = H/\Gamma_0^\natural,$$

and thus L is isomorphic to a vector group. The unitary representation $\operatorname*{ind}_{\Gamma_1^e \uparrow K} \chi^e$ of K is of the form $(S \circ \lambda) \otimes \chi'_d$, where S is a continous direct sum of all characters of L with respect to the absolutely continous measure on $\hat{L}$ (*cf.* [26], Lemma 1, p. 325). Let ω be an element in L, and let us put $\varphi = \omega \circ \lambda$. We claim, that there is an element $m \in \Gamma_0$, such that $\varphi \chi'_d \equiv m \chi'_d$. In fact, since $K = \Sigma H \mathbf{T}_e$, and since elements of Σ and Γ_0 commute with each other (both in Γ and U^e) it is enough to find an element m in Γ_0, such that $m \chi_d \equiv \chi_d (\varphi \mid H)$. We can write the right hand side, by an appropriate choice of $d' \in \gamma'$, as $\chi_{d'}$, and thus it suffices to show, that d and d' are on the same orbit of Γ_0 in γ'. But since $\varphi \mid \Gamma_0^\natural \equiv 1$, we have $\chi_d \mid \Gamma_0^\natural \equiv \chi_{d'} \mid \Gamma_0^\natural$ and thus $d \mid \gamma^\natural = d' \mid \gamma^\natural$. Since $[\gamma, \gamma] = \mathbf{R}\, v$, we have the desired conclusion. Let us put $\rho = \operatorname*{ind}_{K \uparrow U^e} \chi'_d$. By what we have just seen, $\operatorname*{ind}_{K \uparrow U^e} \varphi . \chi'_d$, is unitarily equivalent to ρ, and thus, by virtue of what we said above about S, $W = \operatorname*{ind}_{\Gamma_1^e \uparrow U^e} \chi^e = \operatorname*{ind}_{K \uparrow U^e} (S \circ \lambda) \otimes \chi'_d$ is unitarily equivalent to a multiple of ρ. Therefore, to complete our proof of the above assertion, it suffices to show, that $\rho \mid \Gamma_0$ is unitarily equivalent to $V_d = \operatorname*{ind}_{H \uparrow \Gamma_0} \chi_d$. To this end we shall use the following proposition, which is a trivial consequence of Theorem 12.1 in [25] (p. 127). *Assume, that* G *is a separable locally compact group,* G_1 *and* G_2 *closed* *subgroups of* G, *such that* $G = G_1 . G_2$. *Let* χ *be a character of* G_1. *Then, putting* $\chi' = \chi \mid G_1 \cap G_2$, *we have*

$$\Big(\operatorname*{ind}_{G_1 \uparrow G} \chi\Big) \mid G_2 = \operatorname*{ind}_{G_1 \cap G_2 \uparrow G} \chi'.$$

Taking U^e for G, K for G_1, χ'_d for χ and Γ_0 for G_2 we get

$$G_1 . G_2 = K . \Gamma_0 = \Sigma \Gamma_0^{\natural} \Gamma_0 . \mathbf{T}_e = \Sigma \Gamma_0 . \mathbf{T}_e = U^e = G,$$

$$G_1 \cap G_2 = H \qquad \text{and} \qquad \chi'_d \mid G_1 \cap G_2 \equiv \chi_d.$$

Therefore

$$\rho \mid \Gamma_0 = \Big(\operatorname*{ind}_{K \uparrow U^e} \chi'_d\Big) \mid \Gamma_0 = \operatorname*{ind}_{H \uparrow \Gamma_0} \chi_d = V_d$$

which is the desired conclusion.

f. Let us put $M = U^e/\Gamma_0$, and let us denote by Φ the canonical homomorphism from U^e onto M. Recalling [*cf.* (*c*)], that $U^e = U\,\mathbf{T}_e = \Sigma \Gamma_0 \mathbf{T}_e$, it is easy to see, that M is identifiable to the set $\{(\sigma, u);\ \sigma \in \Sigma,\ u \in \mathbf{T}\}$ with the multiplication $(\sigma, u)(\tau, v) = (\sigma + \tau, uv\,\overline{\beta(\sigma, \tau)})$, such that $\Phi((x, u), v)) = (\Psi(x), v)$. We show now, that $\overset{-1}{\Phi}(M^{\natural}) = (U^{\natural})^e . \Gamma_0$. To this end let us observe first, that putting $\Sigma_1 = \{\sigma; \sigma \in \Sigma$ and $(\beta(\sigma, \tau))^2 = 1$ for all τ in $\Sigma\}$, we have

$$M^{\natural} = \{(\sigma, u); \sigma \in \Sigma_1, u \in \mathbf{T}\}.$$

In this fashion, to arrive at the desired conclusion it is enough to establish, that (z, u) in U belongs to $U^{\natural}$ if and only if $z \in \Sigma_1 + Z_1$. Let us write $z = \sigma + z_0$ $(\sigma \in \Sigma,\ z_0 \in Z_0)$. Then for $\tau \in \Sigma$ and $z' \in Z_0$ we have

$$(\beta(z, \tau + z'))^2 \equiv (\beta(\sigma, \tau))^2 (\beta(z_0, z'))^2,$$

and, evidently, the right hand side is identically one for all τ and z' if and only if $\sigma \in \Sigma_1$, and $z_0 \in Z_1$.

We observe, that M is a central extension of a free abelian group of finite rank by a one dimensional torus. Let ω be a character of $M^{\natural}$, such that $\omega((0, u)) = \bar{u}$. By virtue of Proposition 1.1 (*cf.* also Remark 1.3) $\operatorname*{ind}_{M^{\natural} \uparrow M} \omega$ is a factor representation of a finite class, and it is of type I if and only if $M/M^{\natural}$ is finite. By what we saw above, we have $M/M^{\natural} = U^e/(U^{\natural})^e \Gamma_0$. Therefore we can conclude, that

$$\operatorname*{ind}_{(U^{\natural})^e . \Gamma_0 \uparrow U^e} (\omega \circ \Phi) = \Big(\operatorname*{ind}_{M^{\natural} \uparrow M} \omega\Big) \circ \Phi \qquad (= B, \text{say})$$

is a factor representation of finite class, and it is, since $U^e/(U^\natural)^e.\Gamma_0$ is isomorphic to $U/U^\natural.\Gamma_0$, of type I if and only if $U^\natural\Gamma_0$ has a finite index in U.

g. Let us consider now the representation $B \otimes W$ of U^e $\left[W = \operatorname{ind}_{\Gamma_1^e \uparrow U^e} \chi^e;\right.$ *cf.* (e) above$\left.\right]$. The von Neumann algebra $\mathbf{R}(B \otimes W)$ it generates is a multiple of $\mathbf{R}(B \otimes \rho) = \mathbf{R}(B) \otimes \mathfrak{B}$, where $\mathfrak{B}$ is the full ring of the representation space $\mathbf{H}(\rho)$ of ρ. Therefore, $\mathbf{R}(B \otimes W)$ is a factor of type I or II, and we have the first case if and only if $\mathbf{R}(B)$ is of type I, or if and only if $U/U^\natural\Gamma_0$ is finite.

By virtue of what we saw in (b), we shall have completed the proof of Proposition 2.1 at once we can show, that $\mathbf{R}(B \otimes W)$ is unitarily equivalent to $\mathbf{R}(U(\chi))$, for an appropriate choice of ω in the character group of $M^\natural$ [such that, as before, $\omega((0, u)) = \bar{u}$; *cf.* (f) above]. To this end we shall use the following assertion, which is a trivial consequence of theorem 12.2 in [25] (p. 128). *Assume, that* G *is a separable locally compact group*, G_1 *and* G_2 *closed subgroups of* G, *such that* $G = G_1 . G_2$. *Let* χ_1 *and* χ_2 *be characters of* G_1 *and* G_2 *resp., and let us write*

$$\chi' = (\chi_1 \mid G_1 \cap G_2)(\chi_2 \mid G_1 \cap G_2).$$

Then we have

$$\operatorname{ind}_{G_1 \uparrow G} \chi_1 \otimes \operatorname{ind}_{G_2 \uparrow G} \chi_2 = \operatorname{ind}_{G_1 \cap G_2 \uparrow G} \chi'.$$

Let us choose for G the group U^e, and for G_1 and G_2 $(U^\natural)^e\,\Gamma_0$ and Γ_1^e resp. We have

$$G_1 = \Sigma_1\,\Gamma_0\,\mathbf{T}_e \qquad \text{and} \qquad G_2 = \Sigma\Gamma_0^\natural\,\mathbf{T}_e$$

and thus

$$G_1 . G_2 = \Sigma\Gamma_0\,\mathbf{T}_e = U^e = G.$$

Also,

$$G_1 \cap G_2 = \Sigma_1\,\Gamma_0^\natural\,\mathbf{T}_e = U^\natural\,\mathbf{T}_e = (U^\natural)^e.$$

Let σ be an element of Σ_1. Then (σ, u) $(u \in \mathbf{T})$ can be viewed as an element of $U^\natural = \Sigma_1\,\Gamma_0^\natural$ but also as an element of $U^\natural = \Sigma_1\,\Gamma_0^\natural$. Let us define the map $\omega : M^\natural \rightarrow \mathbf{T}$ by $\omega((\sigma, u)) \equiv \chi((\sigma, 1)) . \bar{u}$; ω is a character of $M^\natural$, such that $\omega((0, u)) \equiv \bar{u}$ $(u \in \mathbf{T})$. In fact, if $\tau \in \Sigma_1$, and $\sigma \in \mathbf{T}$, we have

$$(\sigma, u)(\tau, v) = (\sigma + \tau, \beta(\sigma, \tau) . uv),$$

and thus

$$\omega((\sigma, u)(\tau, v)) = \chi((\sigma + \tau, 1))\,\overline{\beta(\sigma, \tau)} . \bar{u}\,\bar{v}.$$

On the other hand, in $U^\natural$:

$$(\sigma, 1)(\tau, 1) = (\sigma + \tau, \beta(\sigma, \tau)),$$

and hence, since $\chi \mid \mathbf{T} \equiv$ identity map,

$$\chi((\sigma, 1)) . \chi((\tau, 1)) = \chi((\sigma + \tau, 1)) \beta(\sigma, \tau)$$

proving our statement.

Let us suppose, that $\chi_1 = \omega \circ \Phi$ (ω being chosen as above). Then $\chi' \mid \mathbf{T}_e$ is identically one, $\chi' \mid \Gamma_0^\natural \equiv \chi^e \mid \Gamma_0^\natural \equiv \chi_0 \equiv \chi \mid \Gamma_0^\natural$, $\chi' \mid \Sigma \equiv \omega \circ \Phi \mid \Sigma_1$. and thus, by virtue of the definition of ω, is the same as χ lifted from $U^\natural$ to $(U^\natural)^e$. But then, by virtue of what we said above we can conclude, that $B \otimes W$ is unitarily equivalent to $U(\chi)$ lifted to U^e, and thus the rings $\mathbf{R}(B \otimes W)$ and $\mathbf{R}(U(\chi))$ are spatially isomorphic.

Q. E. D.

Remark 2.1. — Analogous conclusion holds true for a character χ of $U^\natural$ which, on $\mathbf{T} \subset U^\natural$, reduces to the conjugate of the identity map of the circle group onto itself.

Remark 2.2. — Let us observe, that the previous reasonings imply, that $U^\natural = \Gamma_1^\natural$, an that $U/\Gamma_0 U^\natural$ is isomorphic to $\Gamma_1/\Gamma_1^\natural$ (for this *cf.* also Lemma 6.5 below). Hence $V(\chi)$ *is of type* I *if and only if the index of the center of the centralizer of the connected component of the identity of* $\Gamma (= \Gamma_1)$ *in* Γ_1 *is finite.*

3. In this section G will denote a connected and simply connected Lie group, and L a closed, connected, type I invariant subgroup of G, such that G/L is abelian. Let us recall, that by virtue of a recent result of J. Dixmier, a choice, of the indicated sort, of L is always possible (*cf.* [14]).

In the following, given two unitary representations ρ_1 and ρ_2, we shall often write $\rho_1 \sim \rho_2$ to express, that they are unitarily equivalent, but not necessarily identical as concrete representations. Given a set S of equivalence classes of unitary representations, we shall denote by S_c the set of the corresponding concrete representations. For a summary of the results concerning projective extensions etc. used in the sequel, the reader is referred to Section 4 (p. 18) in [2].

Let π be a fixed element in $\hat{L}$; we shall denote by the same letter a fixed concrete representation of the class π. Let G_π be the stable group of $\pi \in \hat{L}$ in G, and let us denote by π^e a projective extension of π to G_π such that

$$\pi^e(a)\,\pi^e(b) = \alpha(a, b)\,\pi^e(ab)\ (a, b \in G_\pi) \qquad \text{and} \qquad \alpha(al_1, bl_2) \equiv \alpha(a, b) \quad (l_1, l_2 \in L).$$

By virtue of our assumptions bearing on G and L, G_π/L, being a closed subgroup of the vector group G/L, is isomorphic to $\mathbf{R}^a \times \mathbf{Z}^b$; thus the extension cocycle α is cohomologous to a skew symmetric bilinear form, with values in $\mathbf{T}$ (= circle group, lifted from G_π/L to G_π (*cf.* [2], p. 188)).

We denote by G_π^e the group defined on the set $G_\pi \times \mathbf{T}$ by the law of multiplication

$$(a, u)(b, v) = (ab, \alpha(a, b).uv) \qquad (a, b \in G_\pi;\ u, v \in \mathbf{T}).$$

We assume, as we can, that α is continuous and take G_π^e with the product topology on $G_\pi \times \mathbf{T}$. One verifies at once, thet the subset $\{(l, 1);\ l \in L\}$ is a closed invariant subgroup of G_π^e; we denote it again by L. Let us put $\mathbf{M}_\pi = G_\pi^e/L$; $\mathbf{M}_\pi$ is called the Mackey group belonging to $\pi \in \hat{L}$. By virtue of what preceeds, $\mathbf{M}_\pi$ satisfies the exact sequence

$$1 \to \mathbf{T} \xrightarrow{\text{central}} \mathbf{M}_\pi \to \mathbf{R}^a \times \mathbf{Z}^b \to 1.$$

LEMMA 3.1. — *Let* A *be a closed subgroup of* G, *such that* $A \supseteq L$. *If* ρ *is a unitary representation of* A, *such that* $\rho \mid L \sim \pi$, *we have* $G_\pi \supseteq A$.

Proof. — Given $a \in G$, let us put $a\rho(x) \equiv \rho(a^{-1}xa)$ $(a \in A)$. Then we have for any a in A : $a\pi \sim a\rho \mid L \sim \rho \mid L \sim \pi$, proving our statement.

Q. E. D.

We write Φ for the canonical homomorphism from G_π^e onto $\mathbf{M}_\pi = G_\pi^e/L$. Given a subset S of G_π, we shall put $S^e = \{(s, u);\ s \in S,\ u \in \mathbf{T}\}$. A being assumed as above, we have

LEMMA 3.2. — *There is* $\rho \in (\hat{A})_c$ *such that* $\rho \mid L \sim \pi$, *if and only if* $\Phi(A^e)$ *is abelian in* $\mathbf{M}_\pi$.

Proof. — *a.* We show first the necessity. We can assume, that $\rho \mid L = \pi$, Then there is a continuous map $f : A \to \mathbf{T}$ such that $\pi^e(a) \equiv f(a).\rho(a)$ from where $\alpha(a, b) \equiv f(a).f(b)/f(ab)$ $(a, b \in A)$. We have furthermore $f(al) \equiv f(a)$ $(a \in A,\ l \in L)$ implying, since G/L is abelian, $f(ab) = f(ba)$ and $\alpha(a, b) = \alpha(b, a)$ $(a, b \in A)$. But then

$$(a, u)(b, v) = (ab, \alpha(a, b).uv) = (ba, \alpha(b, a).uv)(l, 1) = (b, v)(a, u)(l, 1)$$
$$(a, b \in A,\ l = a^{-1}b^{-1}ab \in L),$$

proving, that $\Phi(A^e)$ is abelian.

b. Let us put $\mathcal{F} = G_\pi/L$, and let us denote by Ψ the canonical homomorphism $G_\pi \to \mathcal{F}$. We denote by β the cocycle on $\mathcal{F} \times \mathcal{F}$ which, when lifted to G_π, coincides with α. Then $\mathbf{M}_\pi$ can be realized as the group defined on the set of pairs $\{(c, u);\ c \in \mathcal{F},\ u \in \mathbf{T}\}$ by the law of multiplication

$$(c, u)(d, v) = (cd, \beta(c, d).uv) \qquad (c, d \in \mathcal{F}),$$

and we have

$$\Phi((a, u)) = ((\Psi(a), u) \qquad (a \in G_\pi).$$

c. To show the sufficiency of our condition, let us now assume, that $\Phi(A^e)$ is abelian. Then, writing $\mathbf{T}_\pi$ for $\{(e, u);\ u \in \mathbf{T}\} \subset \Phi(A^e)$, there is a closed subgroup B of $\mathbf{M}_\pi$, such that $\Phi(A^e) = B \times \mathbf{T}_\pi$. Let τ be the projection of $\Phi(A^e)$ onto B and let us put

$$\tau((b, 1)) = (b, g(b)) \qquad (b \in \Psi(A)).$$

Then, writing h for $1/g$ we obtain

$$\beta(c, d) = h(c).h(d)/h(cd) \qquad (c, d \in \Psi(A))$$

implying, that $\alpha(a, b) = f(a).f(b)/f(ab)$, where $f(a) \equiv h(\Psi(a))\ (a, b \in A)$. Putting, finally, $\rho(a) \equiv \pi^e(a)/f(a)\ (a \in A)$, ρ is a representation, restricting on L to π, of A.

Q. E. D.

Given a subgroup U of G^e_π, such that U contains $\mathbf{T} = \{(e, u)\} \subset G^e_\pi$ we shall write $U/\mathbf{T}$ for the canonical image of U in G_π. If Λ is some subgroup of $\mathbf{M}_\pi$, we denote by Λ^+ its centralizer in $\mathbf{M}_\pi$.

Lemma 3.3. — *Let* A *and* ρ *be as in the previous lemma, and* G_ρ *the stabilizer of the image of* ρ *in* $\hat{A}$. *We have* $G_\rho = \overset{-1}{\Phi}((\Phi(A^e))^+)/\mathbf{T}$.

Proof. — *a.* Let a be some element of G_π, and let us assume, that $\Phi((a, 1))$ commutes with $\Phi(A^e)$. We show first, that this assumption implies, that a belongs to G_π. We denote by B the smallest closed subgroup, containing a and A, of G_π. Evidently, $\Phi(B^e)$ is abelian and hence, by virtue of Lemma 3.2, there is $\sigma \in (\hat{B})_c$ with $\sigma \mid L = \pi$. Also, we can find a character φ of A, such that $\rho \sim \varphi.(\sigma \mid A)$ $(\varphi \mid L \equiv 1)$, and therefore

$$a\rho \sim \varphi.a(\sigma \mid A) = \varphi.(a\sigma \mid A) = \varphi.(\sigma \mid A) \sim \rho,$$

implying, that a belongs to G_ρ.

b. We assume next, that $a\rho \sim \rho$; by virtue of Lemma 3.1 this implies $a \in G_\pi$. We shall show, that $\Phi((a, 1))$ commutes with $\Phi(A^e)$. We suppose again, as in (*a*) of Lemma 3.2, that $\rho \mid L = \pi$ and $\pi^e \mid A \equiv f\rho$ (cf. *loc. cit.*). Then we can conclude, that $a\pi^e \mid A \sim \pi^e \mid A$. Given any fixed a in G_π, an easy computation, the details of which we leave to the reader, shows, that

$$(a\,\pi^e)(b) \equiv \eta(b)\,(\pi^e(a))^{-1}\,\pi^e(b)\,\pi^e(a)$$

where

$$\eta(b) \equiv \alpha(a, a^{-1}).\overline{\alpha(a^{-1}, b)}.\overline{\alpha(a^{-1}b, a)} \qquad (b \in G_\pi).$$

By virtue of what we saw above we infer from this, that with a satisfying $a\rho \sim \rho$ we get $\eta(b) \equiv 1$ for all b in A, or $\alpha(a, a^{-1}) = \alpha(a^{-1}, b)\,\alpha(a^{-1}b, a)$ $(b \in A)$ But this implies at once that $(a, 1)^{-1}(b, 1)(a, 1) = (a^{-1}ba, 1)$, and thus the left hand side is of the form $(l, 1)(b, 1)$ $(l = a^{-1}b^{-1}ab,\ b \in A)$, from where the conclusion is clear.

Q. E. D.

LEMMA 3.4. — *Let us denote by* $\mathfrak{A}$ *the family of all those closed, connected subgroups, containing* L, *of* G, *to which* π *admits a trivial extension, invariant under* $(G_\pi)_0$ (= connected component of the identity in G). *Then* $\mathfrak{A}$ *contains a well defined maximal element.*

Proof. — *a*. Let us start by observing, that the elements of $\mathfrak{A}$ are contained in G_π (*cf.* Lemma 3.1). We put $\Gamma = M_\pi$, and show, that if A belongs to $\mathfrak{A}$ we have $\Phi(A^e) \subset \Gamma_0^\natural$. To this end we take into account, that obviously $\Phi((G_\pi^e)_0) = \Gamma_0$ and therefore, by virtue of our definition of $\mathfrak{A}$ and Lemma 3.3, $\Phi(A^e)$ is contained in the centralizer Γ_1 of the connected component of the identity in Γ. But since A, and hence also A^e, is connected we obtain, that $\Phi(A^e) \subset (\Gamma_1)_0 = \Gamma_0^\natural$ [*cf.* (*c*) in the proof of Proposition 2.1].

b. To complete our proof of Lemma 3.4, it will now be sufficient to establish, that the subgroup $\Pi = \overset{-1}{\Phi}(\Gamma_0^\natural)/\mathbf{T}$ of G_π belongs to $\mathfrak{A}$. But : 1° Evidently Π is closed and connected; 2° $\Phi(\Pi^e)$ being abelian, π extends to Π trivially (*cf.* Lemma 3.2); 3°) If ρ is any such extension, by Lemma 3.3, since $\Phi(\Pi^e)$ and $\Phi((G_\pi^e)_0)$ commute, we have $(G_\pi)_0 \subset G_\rho$.

Q. E. D.

We denote, as in the previous section, the centralizer of the connected center $\Gamma_0^\natural$ of Γ by U. If Π and $\rho \in (\hat{\Pi})_c$ are as above, putting $\mathfrak{U} = G_\rho$, we conclude by aid of Lemma 3.3. that $\mathfrak{U} = \Phi(U)/\mathbf{T}$.

Lemma 3.5. — *Let us denote by* $\mathfrak{B}$ *the family of all those closed, not necessarily connected subgroups, containing* L, *of* G, *to which* π *admits a trivial extension* ρ, *the stabilizer of which contains* $\mathfrak{U}$. *Then* $\mathfrak{B}$ *contains a well defined maximal element uniquely determined by* $\pi \in \hat{L}$.

Proof. — *a.* Let A be an element of $\mathfrak{B}$; we claim, that $\Phi(A^e)$ is contained in $U^{\natural}$. Let $\rho \in (\hat{A})_c$ be such, that $\rho \mid L = \pi$. Since $G_\rho \supset \mathfrak{U}$, any two element of $\Phi(A^e)$ and $\Phi(\mathfrak{U}^e) = U$ commute. Hence, in particular, $\Phi(A^e)$ is contained in the centralizer of $\Gamma_0^{\natural}$, that is in U, and therefore $\Phi(A^e) \subset U^{\natural}$.

b. By virtue of what preceeds, to complete the proof of our lemma, it suffices to show, that the subgroup $B = \overset{-1}{\Phi}(U^{\natural})/\mathbf{T}$ of G_π is an element of $\mathfrak{B}$. But again, since $\Phi(B^e) = U^{\natural}$ is abelian, π extends trivially to B. If ρ is any such extension, we have $G_\rho \supset \mathfrak{U}$, since the elements of $\Phi(B^e) = U^{\natural}$ and of $\Phi(\mathfrak{U}^e) = U$ pairwise commute.

Q. E. D.

Lemma 3.6. — *Denoting by* K *the maximal element of the previous lemma, and by* ρ *a trivial extension of* π *to* K, *we have* $\mathfrak{U} = G_\rho$. *The maximal element of Lemma* 3.4 *is the connected component of the identity in* K.

Proof. — The first statement is clear since, by (*b*) in the proof of Lemma 3.5, we have $K = \overset{-1}{\Phi}(U^{\natural})/\mathbf{T}$. Let Π be the maximal element of Lemma 3.4 [*cf.* (*b*), *loc. cit.*]. Since $\Pi = \overset{-1}{\Phi}(\Gamma_0^{\natural})/\mathbf{T}$, the desired conclusion follows by observing, that $(U^{\natural})_0 = \Gamma_0^{\natural}$.

Q. E. D.

Remark 3.1. — Observe, that upon replacing π by $a\,\pi$ $(a \in G)$, K and $\mathfrak{U}$ do not change.

Lemma 3.7. — *Let* K *and* ρ *be as in the previous lemma. The unitary representation* $\operatorname{ind}_{K \uparrow G} \rho$ *of* G *is a semifinite factor representation. It is of type I if and only if the group* $\mathfrak{U}/(G_\pi)_0 K$ *is finite.*

Proof. — We write again $\Gamma = \mathbf{M}_\pi$; Γ, as just defined, satisfies the conditions of Proposition 2.1. Let us observe immediately, that by virtue of what we saw above, the group $U/\Gamma_0 U^{\natural}$ is isomorphic to $\mathfrak{U}/(G_\pi)_0 K$. If χ is a character of $U^{\natural}$, such that $\chi \mid \mathbf{T}$ coincides with the conjugate of the identity map of the circle group onto itself, by virtue of Proposition 2.1 (*cf.* also Remark 2.1) the unitary representation $V(\chi) = \operatorname{ind}_{U^{\natural} \uparrow \Gamma} \chi$

is a semifinite factor representation. Hence the same holds true for $\tau(\chi) = \operatorname{ind}_{K^e \uparrow G^e_\pi} (\chi \circ \Phi)$, since we have $\tau(\chi) = V(\chi) \circ \Phi$. Let us write μ for the character $\chi \circ \Phi$ of K^e. We have

$$\pi^e \otimes \tau(\chi) = \pi^e \otimes \operatorname{ind}_{K^e \uparrow G^e_\pi} \mu = \operatorname{ind}_{K^e \uparrow G^e_\pi} (\mu . \pi^e).$$

Let us denote by σ and σ_1 the representations of $K = K^e/\mathbf{T}$ and of $G_\pi = G^e_\pi/\mathbf{T}$ arising from $\mu\pi^e$ and $\pi^e \otimes \tau(\chi)$ resp. We have $\sigma_1 = \operatorname{ind}_{K \uparrow G_\pi} \sigma$, and σ_1 is a semifinite factor representation of the type of $V(\chi)$, restricting on L to a multiple of π. Let us form the representation $T(\sigma) = \operatorname{ind}_{G_\pi \uparrow G} \sigma_1 = \operatorname{ind}_{K \uparrow G} \sigma$. It, too, is a semifinite factor representation of the type of $V(\chi)$. Hence by a remark made above $T(\sigma)$ is of type I if and only if the group $\mathfrak{U}/(G_\pi)_0 K$ is finite.

We have evidently $\sigma \mid L = \pi$. Therefore, there is a character φ of K, $\varphi \mid L \equiv 1$, such that $\rho = \varphi\sigma$. Then, if ψ is any character of G, such that $\psi \mid K = \varphi$, we have $\operatorname{ind}_{K \uparrow G} \rho = \psi T(\sigma)$, completing the proof of our lemma.

Q. E. D.

Remark 3.2. — Observe, that the above proof implies. that the representation $\operatorname{ind}_{K \uparrow G} \rho$ restricts on L to the transitive quasi-orbit carried by $G\pi$ (*cf.* [2], Theorem 6.2, p. 58). More precisely, if $\{\pi(\zeta); \zeta \in \hat{L}\}$ is a Borel measurable field of irreducible representations on $\hat{L}$, such that $\pi(\zeta)$ is of the equivalence class of $\zeta \in \hat{L}$, then $\operatorname{ind}_{K \uparrow G} \rho \mid L$ is a multiple of $\int_{\hat{L}} \oplus \pi(\zeta)\, d\mu(\zeta)$, where $d\mu(\zeta)$, is quasi-invariant under G and is carried by $G\pi$ (cf. *loc. cit.*, p. 57).

We recall, that the unitary representations T_1 and T_2 of G are said to be quasi-equivalent, if there is a $\star$-isomorphism Φ from $\mathbf{R}(T_1)$ onto $\mathbf{R}(T_2)$ such that $\Phi(T_1(a)) \equiv T_2(a)$ $(a \in G)$ (*cf.* [2], 5.3.2, Definition, p. 106 and 13.14, p. 250). In this case we shall write $T_1 \approx T_2$. Given $\pi \in \hat{L}$ and K as above, we put

$$\mathfrak{S}(\pi) = \{\rho; \rho \in \hat{K},\ \rho \mid L = \pi\}.$$

Lemma 3.8. — *Assume, that for* $\sigma_j \in (\mathfrak{S}(\pi_j))_c$ $(\pi_j \in \hat{L},\ j = 1, 2)$ *we have* $\operatorname{ind}_{K_1 \uparrow G} \sigma_1 \approx \operatorname{ind}_{K_2 \uparrow G} \sigma_2$, *where* K_j *corresponds to* π_j, $(j = 1, 2)$ *as* K *does to* π *in Lemma* 3.6. *Then* $K_1 = K_2$, *and there is an element a of* G *such a* $\sigma_1 \sim \sigma_2$.

Proof. — *a*. Let us put $T_j = \text{ind}\, \sigma_j$ $(j = 1, 2)$. We start by observing, that our assumptions imply, that $G\pi_1 = G\pi_2$. In fact, we conclude from $T_1 \approx T_2$ that $T_1 | L \approx T_2 | L$. With the notations of Remark 3.2, $T_j | L$ is quasi-equivalent to a representation

$$\int_{\hat{L}} \oplus \pi(\zeta)\, d\mu_j(\zeta)$$

where $d\mu_j(\zeta)$ is carried by $G\pi_j$ $(j = 1, 2)$. They can be quasi-equivalent only if μ_1 and μ_2 are equivalent ([12], Proposition 8.4.4, p. 151 and 18.7.6, p. 325). But then we must have $G\pi_1 = G\pi_2$.

b. By virtue of Remark 3.1 we can now conclude, that $K_1 = K_2 = K$, say.

c. For some fixed $\pi \in \hat{L}$ let ρ, K and $\mathfrak{U}$ be as in Lemma 3.6. We put $T = \underset{K \uparrow G}{\text{ind}}\, \rho$. Let $\overset{\cap}{K}$ be the set of all quasi-equivalence classes of factor factor representations of K with its usual Borel structure (*cf*. [12], 18.6.2, p. 323). There is a standard measure μ on $\overset{\cap}{K}$, uniquely determined up to equivalence, a μ measurable field $\{ T(\zeta);\ \zeta \in \overset{\cap}{K} \}$ of factor representations, such that $T(\zeta)$ is of the quasi-equivalence class of $\zeta \in \overset{\cap}{K}$, and such that $T | K = \int_{\hat{K}} \oplus T(\zeta)\, d\mu(\zeta)$ in the sense of unitary equivalence, the decomposition being central, that is $\mathbf{R}(T | K)$ contains the ring of diagonalisable operators (*cf*. [12], 8.4.2, Théorème, p. 149 and 18.7.6, p. 325). *We are going to show, that* μ *is carried by* $G\tau$, *where* τ *is the image of* ρ *in* $\hat{K} \subset \overset{\cap}{K}$. Let us put $A = G/K$; let f be a Borel cross section from A into G and da an element of the invariant measure on A. Then we have

$$T | K = \int_{A} \oplus f(a)\, \rho\, da.$$

Let us put $B = \mathfrak{U}/K \subset A$ and $\Lambda = A/B$ $(\sim G/\mathfrak{U})$; we denote by db and $d\lambda$ elements of the invariant measures on B and Λ resp. If φ is a Borel cross section from Λ into A, the map $\omega : \Lambda \times B \to A$ defined by $\omega(\lambda, b) = \varphi(\lambda) + b$ is one-to-one and Borel, and hence establishes a Borel isomorphism between $\Lambda \times B$ and A (*cf*. [2], Proposition 2.5, p. 7). The image of $d\lambda\, db$ under ω is an invariant measure on A; we can assume, that it coincides with da. We write $g = f \circ \varphi$; it is a Borel cross section from $\Lambda = G/\mathfrak{U}$ into G. For an arbitrary (λ, b) in $\Lambda \times B$, the elements

$g(\lambda) f(b)$ and $f(\omega(\lambda, b))$ belong to the same residue class according to K, and thus we have $f(\omega(\lambda, b)) \sim g(\lambda) f(b) \rho$ and hence also

$$T \mid K = \int_{\Lambda \times B} \oplus g(\lambda) f(b) \rho \, d\lambda \, db$$

in the sense of unitary equivalence. Let us put $U = \int_B \oplus f(b) \rho \, db$. Since $f(b) \in \mathfrak{U} = G_\rho$, we have $U \sim m\rho$, where $m = +\infty$ if $\mathfrak{U} \underset{\neq}{\supset} K$ and $m = 1$ if $\mathfrak{U} = K$. Writing $T(\lambda) = g(\lambda) U$, we obtain, that

$$T \mid K = \int_\Lambda \oplus T(\lambda) \, d\lambda.$$

Let us denote by $\mathfrak{Z}$ the ring of all diagonalisable operators of the last decomposition; we are going to show, that $\mathbf{R}(T \mid K) \supset \mathfrak{Z}$. To this end we denote by C the subgroup $G_\pi / U \subset \Lambda$ and put $H = \Lambda / C$. Let $d\eta$ and dc be elements of the invariant measures on H and C resp. We denote by ψ a Borel cross section from H into Λ and set $h = g \circ \psi$; h is a Borel cross section from $H = G/G_\pi$ into G. Reasoning as above with H, C and ψ in place of Λ, B and φ resp., we conclude, that

$$\int_\Lambda \oplus T(\lambda) \, d\lambda = \int_{H \times C} \oplus h(\eta) \, g(c) \, U \, d\eta \, dc.$$

Let us observe now, that if a belongs to $\mathfrak{U} = G_\rho$, we have $a\varphi \sim \varphi_a$, where φ_a is a continuous character of K, the kernel of which contains L. Moreover we have $\varphi_a = \varphi_b$ if and only if a and b belong to the same residue class according to $\mathfrak{U}$. Let us put $\omega_c = \varphi_{g(c)}$ $(c \in C)$; by virtue of what we have just said, if $\omega_{c_1} = \omega_{c_2}$, then $c_1 = c_2$. Observing, that $C = G_\pi / \mathfrak{U}$ is countable, we set $T_1 = \sum_{c \in C} \oplus \omega_c$. The ring of diagonalisable operators of this decomposition, $\mathfrak{Z}_1$ say, is equal to $\mathbf{R}(T_1)$. Let us put

$$T_2 = \int_H \oplus h(\eta) \rho \, d\eta,$$

and $\mathfrak{Z}_2$ for the ring of this decomposition. Since $U \sim m\rho$, we conclude, that there is a unitary equivalence between $T \mid K$ and the m-fold multiple of $T_1 \otimes T_2$, which makes correspond $\mathfrak{Z}$ and $m(\mathfrak{Z}_1 \otimes \mathfrak{Z}_2)$ to each other. In this fashion, to establish, that $\mathbf{R}(T \mid K) \supset \mathfrak{Z}$, it will be sufficient to show, that $\mathbf{R}(T_1 \otimes T_2) \supset \mathfrak{Z}_1 \otimes \mathfrak{Z}_2$. To this end we prove first, that $\mathbf{R}(T_2 \mid L) = \mathfrak{Z}_2'$. In fact, the map $a \mapsto a\pi$ $(a \in G)$ is Borel from G into $\hat{L}$ (*cf.* [2], p. 57, top) and hence the correspondence $\eta \mapsto h(\eta)\pi$ establishes a Borel isomorphism betwen H and the Borel subset $G\pi$ of $\hat{L}$ (*cf.* [2], Proposition 2.5, p. 7). Let us write $d\nu(\zeta)$ for the measure on $\hat{L}$, which is carried by $G\pi$ and there coincides with the image of $d\eta$. With $\{\pi(\zeta);\ \zeta \in \hat{L}\}$ as in Remark 3.2 let us put $\overline{T}_2 = \int_{\hat{L}} \oplus \pi(\zeta)\, d\nu(\zeta)$; we write $\overline{\mathfrak{Z}}_2$ for the ring of diagonalisable operators. There is a unitary correspondence between $T_2 \mid L$ and $\overline{T}_2$, which maps $\mathfrak{Z}_2$ onto $\overline{\mathfrak{Z}}_2$. Thus, to arrive at the desired conclusion, it is enough to recall, that $\mathbf{R}(\overline{T}_2) = \overline{\mathfrak{Z}}'_2$ (*cf.* [12], 8.6.4, Proposition, p. 155 and 18.7.6, p. 325). Since $T_2(k) \in \mathfrak{Z}'_2$, we have $T_1(k) \otimes I \in \mathbf{R}(T_1 \otimes T_2)$ $(k \in K)$ and hence $\mathfrak{Z}_1 \otimes I \subset \mathbf{R}(T_1 \otimes T_2)$. On the other hand

$$I \otimes \mathfrak{Z}_2 \subset I \otimes \mathbf{R}(T_2 \mid L) = \mathbf{R}((T_1 \otimes T_2) \mid L) \subset \mathbf{R}(T_1 \otimes T_2),$$

and in this fashion finally $\mathfrak{Z}_1 \otimes \mathfrak{Z}_2 \subset \mathbf{R}(T_1 \otimes T_2)$. Summing up the previous discussion we have shown that, putting $T(\lambda) = g(\lambda)\, U$ $(\lambda \in \Lambda)$, the direct integral decomposition into factor representations $T \mid K = \int_\Lambda \oplus T(\lambda)\, d\lambda$ is central. Let us recall, that g is a Borel cross section from $\Lambda = G/\mathfrak{U}$ into G. From this, using a reasoning employed above we infer, that the map $\lambda \mapsto g(\lambda)\tau$, where τ is the unitary equivalence class of ρ, establishes a Borel isomorphism between Λ and the Borel subset $G\tau$ of $\hat{K}$. Let us add, that $\hat{K}$ being Borel in $\check{K}$ (*cf.* [12], 18.6.3, p. 324), $G\tau$ is a Borel subset of $\check{K}$. We denote by $d\mu(\zeta)$ the measure on $\check{K}$, which is carried by $G\tau$, and there coincides with the image of $d\lambda$. Putting

$$T(\zeta) = T(\lambda) = g(\lambda)\, U \qquad \text{if} \quad \check{K} \ni \zeta = g(\lambda)\tau,$$

we obtain finally, that

$$T \mid K = \int_{\check{K}} \oplus T(\zeta)\, d\mu(\zeta)$$

in the sense of unitary equivalence, the direct sum decomposition being central, and $T(\zeta)$ is of the quasi-equivalence class of $g(\lambda)\tau = \zeta$, completing the proof of the assertion formulated at the start of (*c*).

d. Using the previous considerations, we finish proving Lemma 3.8 as follows. With notations as in (*a*) we have by assumption $T_1 \approx T_2$, and hence also $T_1 \mid K \approx T_2 \mid K$ [*cf.* (*b*)]. Let τ_j be the quasi-equivalence class of σ_j $(j = 1, 2)$. We form the central decompositions

$$T_j \mid K = \int_{\hat{K}} \oplus\, T_j(\zeta)\, d\mu_j(\zeta) \qquad (j = 1, 2).$$

By virtue of what we saw above in (*c*), the measure μ_j is carried by $G\tau_j$ $(j = 1, 2)$. On the other hand, by virtue of $T_1 \mid K \approx T_2 \mid K$, μ_1 must be equivalent to μ_2 (*cf.* [12], 8.4.4, Proposition, p. 151) and hence $G\tau_1 = G\tau_2$. In this fashion there is an $a \in G$, such that $a\sigma_1 \sim \sigma_2$, completing the proof of Lemma 3.8.

Q. E. D.

REMARK 3.3. — Let us observe, that by what we have just seen, $\operatorname{ind}_{K\uparrow G} \sigma_1 \approx \operatorname{ind}_{K\uparrow G} \sigma_2$ implies, that these representations are unitarily equivalent.

Let us put $\mathfrak{G} = \bigcup_{\pi \in \hat{L}} \mathfrak{G}(\pi)$. Given $\tau \in \mathfrak{G}$ and a concrete representation σ of the unitary equivalence class of τ, we write $\eta(\tau)$ for the quasi-equivalence class of $\operatorname{ind}_{K\uparrow G} \sigma$ (for K *cf.* Lemma 3.6, with σ in place of ρ, *loc. cit.*). Writing $a\tau$ for the unitary equivalence class of $a\sigma$ $(a \in G)$, the correspondence $(a, \tau) \mapsto a\tau$ defines $\mathfrak{G}$ as a transformation space of G. With these notations we have

PROPOSITION 3.1. — *The map η just defined takes its values in $\hat{G}$. We have $\eta(\tau_1) = \eta(\tau_2)$ if and only if τ_1 and τ_2 lie on the same orbit of* G.

Proof. — This is an immediate consequence of Lemmas 3.7 and 3.8.

Q. E. D.

4. The purpose of this section is to collect several facts concerning real and holomorphic induction, and to present them in the manner we shall use them in the sequel (*cf.* [32], Section 3, p. 442-446). We should like to emphasize already at the start, that although we shall employ later in an essential fashion two deep results of [1] (*cf.* Lemma II.3.1 and Theorem III.3.1, *loc. cit.*), we apply the procedure of holomorphic induction, when compared with the treatment of [1], but to relatively special situations.

a. Let G be a separable locally compact group, A a closed subgroup of G and χ a continuous homomorphism of A into $\mathbf{T}$ (= group of complex numbers of absolute value 1). Let dg and da be an element of the right invariant Haar measure on G and A resp., and let us define the modular functions Δ_G and Δ_A by

$$d(g_0 g) = \Delta_G(g_0)\, dg \quad \text{and} \quad d(a_0 a) = \Delta_A(a_0)\, da \text{ resp.} \qquad (g_0 \in G,\ a_0 \in A).$$

Before proceeding we recall the following facts (*cf.* [5], chapter VII, § 2). Let $f(x)$ be an element of C (G) (= continuous functions, of a compact support, on G) and let us put $F(x) = \int_A f(ax)\, da$. Then the corresponding function on G/A, to be denoted by $F(p)$ ($p \in G/A$) lies in C (G/A) and any function of this sort can be so obtained. Next, if $d\nu(x)$ is a positive Borel measure on G satisfying $d\nu(ax) = \Delta_A(a)\, d\nu(x)$ for all a in A, there is a uniquely determined Borel measure $d\mu(p)$ on G/A such that

$$(1) \qquad \int_{G/A} F(p)\, d\mu(p) = \int_G f(x)\, d\nu(x)$$

for all f in C (G). We shall sometimes denote $d\mu(p)$ by $d\nu/A$ (or μ by ν/A resp.). For a in A let us set $\eta(a) = \Delta_A(a)/\Delta_G(a)$. If $l(x)$ is a non negative locally integrable function on G satisfying $l(ax) \equiv \eta(a)\, l(x)$ ($a \in A$, $x \in G$), then for the Borel measure $d\nu(x) = l(x)\, dx$ we shall have $d\nu(ax) = \Delta_A(a)\, d\nu(x)$. One defines the unitary representation U induced by χ, of G, in following fashion. Let $\mathcal{E}$ be the collection of all complex valued Borel measurable functions on G satisfying $f(ax) = (\eta(a))^{1/2} \chi(a) f(x)$ for all a and x in A and G resp., and for which $|f(x)|^2$ is locally integrable with respect to dx. By what we saw above, the measure $d\nu_f(x) = |f(x)|^2\, dx$ satisfies $d\nu_f(ax) = \Delta_A(a)\, d\nu_f(x)$ and hence we can form the measure $\mu_f = \nu_f/A$ on G/A. Let $\mathcal{F}$ be the collection of all those elements in $\mathcal{E}$ for which the total mass of G/A with respect to μ_f is finite. One can define (*cf.* [4], p. 80-83) on the quotient space, according to the linear variety of elements with $\mu_f(G/A) = 0$, of $\mathcal{F}$ the structure of a Hilbert space $\mathbf{H}(U)$ in such a fashion, that the square of the norm of the equivalence class containing $f \in \mathcal{F}$ is equal to $\mu_f(G/A)$. Finally, for g_0 in G the operator $U(g_0)$ on $\mathbf{H}(U)$ is obtained from the map $f(x) \mapsto f(xg_0)$ of $\mathcal{F}$ onto itself by taking the images in the quotient space $\mathbf{H}(U)$.

For later use we add the following observation. Let us assume, that there is a continuous homomorphism $k(x)$ of G into the multiplicative group of positive numbers extending η. Then : 1º There is a positive Borel measure $dv(p)$ on G/A satisfying $dv(pg) = k(g)\, dv(p)$ for all g in G. In fact, to see this it suffices to take $dv(p) = k(x)\, dx/A$. 2º For $f \in \mathcal{F}$ the function $|f(x)|^2/k(x)$ ($x \in G$) is invariant under translation on the left by elements of A, and we have

$$d\mu_f(p) = (|f(x)|^2/k(x))\, dv(p).$$

In fact, let h be some element of C (G); then we get with the previous notations

$$\int_{G/A} H(p)\, d\mu_f\,(p) = \int_G h(x)\,|\,f(x)\,|^2\, dx$$
$$= \int_G h(x)\,(\,|\,f(x)\,|^2/k(x))\,k(x)\,dx$$
$$= \int_{G/A} H(p)\,(\,|\,f(x)\,|^2/k(x))\,d\nu\,(p).$$

Hence, in particular, we obtain

$$\mu_f\,(G/A) = \int_{G/A} (\,|\,f(x)\,|^2/k(x))\,d\nu\,(p).$$

b. Let $\mathfrak{g}$ be a nilpotent Lie algebra over the real field, and f a nonzero element of the dual $\mathfrak{g}'$ of the underlying space of $\mathfrak{g}$ (to be considered also as an element of the dual of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$). We put $B_f\,(x, y) = ([x, y], f)$ $(x, y \in \mathfrak{g})$ and write again B_f for the corresponding skew symmetric bilinear form on $\mathfrak{g}_{\mathbf{C}} \times \mathfrak{g}_{\mathbf{C}}$. A complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbf{C}}$ will be called a polarization with respect to f, in symbols $\mathfrak{h} = \mathrm{pol}\,(f)$, provided the following conditions hold : 1° $\mathfrak{h}$ is maximal self orthogonal with respect to B_f; 2° $(\alpha)\,\mathfrak{h} + \bar{\mathfrak{h}}$ is also a subalgebra of $\mathfrak{g}_{\mathbf{C}}$, (β) for $x + iy \in \mathfrak{h}$ $(x, y \in \mathfrak{g})$ we have $B_f\,(x, y) \geq 0$ and $B_f\,(x, y) = 0$ if and only if $x, y \in \mathfrak{h} \cap \mathfrak{g}$. Let G be the connected and simply connected Lie group belonging to $\mathfrak{g}$. Assume, that K is a subgroup of Aut $(\mathfrak{g})$ such that $[K, K] \subset \mathrm{Ad}\,(G)$, and that $f \in \mathfrak{g}'$ is invariant under the contragredient action of K on $\mathfrak{g}'$. Then *there exists* $\mathfrak{h} = \mathrm{pol}\,(f)$, *which is invariant under the action of* K *on* $\mathfrak{g}_{\mathbf{C}}$ (*cf.* [1], Lemma II.3.1).

c. From now on, unless stated otherwise, $\mathfrak{g}$ will denote a real solvable Lie algebra and G the corresponding connected and simply connected Lie group. Given a subalgebra $\mathfrak{g}_1 \subseteq \mathfrak{g}$, we shall write exp $\mathfrak{g}_1$ for the connected subgroup, belonging to $\mathfrak{g}_1$, of G. We recall, that *in our case* exp $\mathfrak{g}_1$ *is closed and simply connected* (*cf.* [20], Theorem 2.2, p. 137).

In the following, given $a \in G$ and $l \in \mathfrak{g}$ we shall put $al = \mathrm{Ad}\,(a)\,l$ and similarly, if g belongs to $\mathfrak{g}'$, ag will stand for $(\mathrm{Ad}\,(a^{-1}))'g$. Let now g be a fixed nonzero element of $\mathfrak{g}'$, and G_g its stabilizer in G. We denote by $(G_g)_0$ the connected component of the identity in G_g, and by $\mathfrak{g}_g$ its Lie algebra. Thus, by virtue of the notational convention introduced above we have $(G_g)_0 = \exp \mathfrak{g}_g$. Let us observe now, that since $(G_g)_0$ is simply connected, there is a character χ_g of $(G_g)_0$ uniquely determined by

$$\chi_g\,(\exp\,(l)) = \exp\,[i\,(l, g)] \quad (l \in \mathfrak{g}_g) \qquad \text{or} \quad d\chi_g = i\,(g\,|\,\mathfrak{g}_g).$$

We write $\mathring{G}_g = \ker(\chi_g \mid (G_g)_0)$. If a is some element of G_g we have for all $l \in \mathfrak{g}_g$:

$$\chi_g(a.\exp(l).a^{-1}) = \chi_g(\exp(al)) = \exp[i(al, g)] = \exp[i(l, g)] = \chi_g(\exp(l))$$

and hence $\chi_g(aba^{-1}) \equiv \chi_g(b)$ $[b \in (G_g)_0]$ implying, that $\mathring{G}_g$ is invariant in G_g. With the previous notations we introduce the following

DEFINTION 4.1. — *The reduced stabilizer of g is the closed subgroup of* G_g *defined as* $\{a;\ a \in G_g,\ aba^{-1}b^{-1} \in \mathring{G}_g$ for all b in $G_g\}$. *It will be denoted by* $\overline{G}_g$.

Let us observe, that $\overline{G}_g$ could also be defined as the complete inverse image of $(G_g/\mathring{G}_g)^{\natural}$ (= center of $G_g/\mathring{G}_g$) in G_g.

We denote by $\hat{G}_g$ the collection of all characters of $\overline{G}_g$ which, when restricted to $(G_g)_0$, coincide with χ_g. Let us put $\overset{\ominus}{G}_g$ for the group of all characters of $\overline{G}_g$ which are identically one on $(G_g)_0$. Given a fixed element χ_0 of $\hat{G}_g$, any $\chi \in \hat{G}_g$ can uniquely be written as $\varphi\chi_0$ $(\varphi \in \overset{\ominus}{G}_g)$. We observe finally, that since $G_g/(G_g)_0$ is a free abelian group [*cf.* below (*d*)], so is $\overline{G}_g/(G_g)_0$, and hence $\overset{\ominus}{G}_g$ is isomorphic to a multitorus of a dimension equal to the rank of $\overline{G}_g/(G_g)_0$.

d. The following lemmas are well known, but because of their role in our subsequent considerations we include proofs for them here. Also, the reasonings employed below will be often referred to later.

Let $\mathfrak{g}$ and G be as above. We put $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ and $L = \exp \mathfrak{d} \subset G$; observe, that we have $L = [G, G]$ (*cf.* [20], Theorem 3.1, p. 138).

LEMMA 4.1. — *Let* A *be a closed subgroup of* G, *such that* AL *is closed, and* $A \cap L$ *is connected. Then* A/A_0 *is free abelian.*

Proof. — By virtue of our assumption $A \cap L = A_0 \cap L$, and hence

$$AL/A_0L = A/A \cap L/A_0/A_0 \cap L = A/A_0.$$

Let Φ be the canonical homomorphism from G onto $G/L \sim \mathbf{R}^n$. Then

$$A/A_0 = AL/A_0L = \Phi(A)/\Phi(A_0).$$

But $\Phi(A) = \Phi(AL)$ being closed, the right hand side is free abelian.

Q. E. D.

COROLLARY 4.1. — $G_g/(G_g)_0$ *is free abelian.*

Proof. — We show, that G_g verifies the condition imposed on A in Lemma 4.1. Since G is solvable, $L = [G, G]$ is nilpotent and hence the exponential map from $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ into L is onto. This implies at once, that $G_g \cap L$ is connected by virtue of the fact, that if E is a nilpotent endomorphism of a finite dimensional real vector space, and if exp (E) annihilates an element, then so does E. Next, $(\mathrm{Ad}(L))'$ is a unipotent group on $\mathfrak{g}'$ and thus the orbit $Lg \subset \mathfrak{g}'$ is closed (*cf.* Proposition 1.1, chapter II). Since, however, we have

$$G_g L = \{ a;\, a(Lg) \subseteq Lg \},$$

$G_g L$ is closed in G. Q. E. D.

LEMMA 4.2. — A *being as in Lemma* 3.1 *let us assume, that* M *is a connected subgroup of* L *containing* $L \cap A$. *Then* AM *is closed in* G.

Proof. — Let us put $A_0 = \exp \mathfrak{a}$. We denote by $\mathfrak{b}$ a supplementary subspace to $\mathfrak{a} \cap \mathfrak{d}$ in $\mathfrak{a}$, and by $\{ a_j \}$ a complete residue system in A according to A_0. We write S_0 for the image, under the exponential map, of $\mathfrak{b}$ and set $S = \bigcup_j a_j . S_0$. Since $d\Phi(\mathfrak{a}) = d\Phi(\mathfrak{b})$, we have $\Phi(S_0) = \Phi(A_0)$ and hence $\Phi(S) = \Phi(A)$. Since $A_1 = A \cap L$ is connected, $\Phi(a_i) = \Phi(a_j)$ implies $a_i = a_j$ from which we conclude, that $\Phi \mid S$ is a bijection with $\Phi(A)$. We are going to show now, that it is even a homeomorphism. To this end let us assume, that $\{ s_n \}$ is a sequence of elements in S such that $\Phi(s_n) \to \Phi(s_0)$ $(s_0 \in S)$; we claim, that in this case $s_n \to s_0$ in G. In fact, since $\Phi(A) = \Phi(AL)$ is closed in $\Phi(G)$, $\Phi(A_0)$ is open in $\Phi(A)$ and therefore we can assume that, for a suitable j, $\{ s_n \} \subset a_j . S_0$ and $s_0 \in a_j . S_0$. From here to obtain the desired conclusion it is enough to take into account, that $\Phi \mid S_0$ is certainly a homeomorphism with its image. By virtue of what we have just seen, the map $\varphi : S \times L \to SL$ $(= AL)$ defined by $\varphi(s, l) = sl$ $(s \in S,\, l \in L)$ is a homeomorphism. But then, if M is as in our lemma, we have, since $A = SA_1$, that $\varphi(S, M) = SM = SA_1 M = AM$ is closed in AL and hence also in G.

Q. E. D.

e. Let us fix a g in $\mathfrak{g}'$; we put $f = g \mid \mathfrak{d}$ $(\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}])$ and assume, that $f \neq 0$. Since evidently $G_g \subset G_f$ and $[\overline{G}_g, \overline{G}_g] \subset L$, we can employ the result quoted at the end of (*b*) above with $K = \mathrm{Ad}(\overline{G}_g) \mid \mathfrak{d}$ and L in place of K and G as *loc. cit.* with the conclusion, that there is a complex subalgebra $\mathfrak{h} \subset \mathfrak{d}_{\mathbf{C}}$ such that $\mathfrak{h} = \mathrm{pol}(f)$ and $\overline{G}_g . \mathfrak{h} \subset \mathfrak{h}$. Assume now, that G_1 is a closed, but not necessarily connected subgroup of G, containing $\overline{G}_g L$. Given an element χ of $\hat{G}_g$ [*cf.* (*c*) above] we can construct a unitary representation ind $(\mathfrak{h}, \chi, g; G_1)$ of G_1 as follows. (If $G_1 = G$, we shall omit indicating it.) Let us put $d = \mathfrak{h} \cap \mathfrak{d}$ and $D = \exp(d) \subset L$; d being invariant under $\overline{G}_g$, D is normalized by the latter, and hence $\overline{G}_g D$ is a subgroup of G. We observe next, that it is a closed subgroup of G. To prove this we show, that $\overline{G}_g$ and D satisfy the conditions imposed on A and M resp. in Lemma 3.2. Since $G_g L$ is closed (*cf.* the proof of Corollary 4.1), so is $\overline{G}_g L$. We have also (cf. *loc. cit.*) $\overline{G}_g \cap L = G_g \cap L = \exp(\mathfrak{g}_g \cap \mathfrak{d})$. $\mathfrak{h}$ being maximal self orthogonal with respect to B_f [*cf.* (*b*)] we have evidently $\mathfrak{g}_g \cap \mathfrak{d} \subset \mathfrak{h}$ and hence $\mathfrak{g}_g \cap \mathfrak{d} \subset d$ and finally $\overline{G}_g \cap L \subset D$, which proves our statement. Since $d = \mathfrak{h} \cap \mathfrak{d}$ is self orthogonal with respect to B_f, there is a character χ_f of D uniquely determined by the condition, that

$$\chi_f(\exp(l)) = \exp[i, (l, f)] \qquad (l \in d).$$

Let us put $A = \overline{G}_g D$ and observe, that there is a character φ on A, such that $\varphi \mid \overline{G}_g \equiv \chi$, $\varphi \mid D \equiv \chi_f$. We have, in fact, $\overline{G}_g \cap D = \exp(\mathfrak{g}_g \cap d)$, and thus evidently $\chi \mid (\overline{G}_g \cap D) = \chi_f \mid (\overline{G}_g \cap D)$. In this fashion, to arrive at the desired conclusion it is enough to remark, that putting, for a fix in $\overline{G}_g$, $(a\chi_f)(b) \equiv \chi_f(a^{-1} ba)$ $(b \in D)$, we have $a\chi_f = \chi_f$. Taking in (*a*) G_1 for G and φ for χ resp., let us form the representation $\operatorname{ind}_{A \uparrow G} \varphi$. We denote again by χ_f the homomorphism of $H = \exp(\mathfrak{h}) \subset L_{\mathbf{C}}$ into the multiplicative group of nonzero complex numbers, determined by

$$\chi_f(\exp(l)) = \exp[i(l, f)] \qquad (l \in \mathfrak{h}).$$

Let us put $e = \mathfrak{h} + \overline{\mathfrak{h}} \cap \mathfrak{d}$ and $E = \exp(e)$. Let a_0 be a fixed element in G_1 and f some function in $\mathcal{F}$ [*cf.* (*a*)]. If $h, h_1 \in H$ and $k, k_1 \in E$ are such, that $hk = h_1 k_1$, we have

$$\chi_f(h) f(ka_0) = \chi_f(h_1) f(k_1 a_0).$$

In fact, since $H \cap E = \exp(\mathfrak{h} \cap e) = \exp(d) = D$, there is an element δ of D such that $h_1 = h\delta$ and $k_1 = \delta^{-1} k$, from which the conclusion follows by virtue of $f(k_1 a_0) = \overline{\chi_f(\delta)} f(ka_0)$. One shows easily, that $\mathfrak{h} + e = \mathfrak{h} + \overline{\mathfrak{h}} = e_{\mathbf{C}}$ implying, that EH is an open subset in $E_{\mathbf{C}}$. We denote by $\mathcal{H}$ the family of all those elements of $\mathcal{F}$, for which the map $hk \mapsto \chi_f(h) f(ka_0)$, for each fixed a_0 in G, is analytic on HE. One can show, that the image $\tilde{\mathbf{H}}$ of $\mathcal{H}$ in $\mathbf{H}(U)$ is a closed subspace (*cf.* [11], 1.9); it is evidently stable under U. *We define* ind $(\mathfrak{h}, \chi\, g; G_1)$ *as the part of* U *in* $\tilde{\mathbf{H}}$.

f. Let us assume now, that $\mathfrak{g}$ is nilpotent. We choose a nonzero element f in $\mathfrak{g}'$ and a subalgebra $\mathfrak{h} = \operatorname{pol}(f) \subset \mathfrak{g}_{\mathbf{C}}$. Let G be the connected and simply connected group determined by $\mathfrak{g}$. We shall denote by Ind $(\mathfrak{h}, f)$ the unitary representation of G, which we obtain by forming first the representation $\operatorname{ind}_{D \uparrow G} \chi_f$ and taking, imitating the procedure followed above at the end of (*e*), its « holomorphic part ». It *can be shown, that* Ind $(\mathfrak{h}, f)$ *is irreducible and, up to unitary equivalence is independent of the particular choice of* $\mathfrak{h}$ *and even of* f, *provided the latter is restricted to a fixed orbit of* G *in* $\mathfrak{g}'$. *Conversely, if* Ind $(\mathfrak{h}_1, f_1)$ *and* Ind $(\mathfrak{h}_2, f_2)$ *are unitarily equivalent, we have* $f_2 = af_1$ *with somme* a *in* G (*cf.* [22] Theorem 5.2 and [1], Lemma III.1). *Finally, any nontrivial irreducible representation of* G *is unitarily equivalent to some of the form* Ind $(\mathfrak{h}, f)$ (*cf.* [22], Theorem 5.1). Summing up, the map, assigning to the orbit Gf $(f \neq 0)$ the equivalence class of the irreducible representation Ind $(\mathfrak{h}, f)$, and to the orbit of the neutral element in $\mathfrak{g}'$ the trivial representation of G, establishes a bijection between the orbit space $\mathfrak{g}'/G$ and the dual $\hat{G}$ of G.

g. We shall also use the fact, that

$$\operatorname{ind}_{G_1 \uparrow G} (\operatorname{ind} (\mathfrak{h}, \chi, g; G_1)) = \operatorname{ind} (\mathfrak{h}, \chi, g) \qquad (cf. [11], 2.1).$$

h. We assume again, that G is a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. We let Aut (G) operate on $\mathfrak{g}$ by setting, for α in Aut (G) and l in $\mathfrak{g}$: $al = (d\alpha)\, l$ ($d\alpha$ = differential of α at the unity of G). If g is some element in $\mathfrak{g}'$, we shall put $\alpha g = [(d\,\alpha)']^{-1}\, g$. We have $\alpha\,(ag) = \alpha\,(a)\,\alpha\, g$, from which we conclude, that $\alpha\,(G_g) = G_{\alpha g}$ and hence also $\alpha\,((G_g)_0) = (G_{\alpha g})_0$ and $\alpha\,(\mathfrak{g}_g) = \mathfrak{g}_{\alpha g}$. Using the notations of (*c*) above, we have $\chi_{\alpha g} \circ \alpha \equiv \chi_g$ on $(G_g)_0$ implying first, that $\alpha\left(\mathring{G}_g\right) = \mathring{G}_{\alpha g}$ and hence also $\alpha\left(\overline{G}_g\right) = \overline{G}_{\alpha g}$. From all this we deduce, that if χ is some element of $\hat{\hat{G}}_g$ then, defining $\alpha\chi$ by $\chi\left(\overset{-1}{\alpha}(x)\right)\left(x \in \overline{G}_{\alpha g}\right)$, $\alpha\chi$ belongs to $\hat{\hat{G}}_{\alpha g}$, and the map, assigning $\alpha\chi$ to $\chi \in \hat{\hat{G}}_g$ [α fix in Aut (G)] is a bijection between $\hat{\hat{G}}_g$ and $\hat{\hat{G}}_{\alpha g}$. If ρ is some representation of G, we shall put

$$(\alpha\rho)\,(x) = \rho\left(\overset{-1}{\alpha}(x)\right) \qquad (x \in G).$$

The following lemma will be often used in the sequel.

LEMMA 4.3. — *With the previous notations we have*

$$\alpha \operatorname{ind} (\mathfrak{h}, \chi, g) = \operatorname{ind} (\alpha\, \mathfrak{h}, \alpha\chi, \alpha\, g).$$

Proof. — Let us put $\mathfrak{h}_1 = \alpha\,\mathfrak{h}$, $\chi_1 = \alpha\chi$, $g_1 = \alpha\, g$. We shall distinguish notions, associated by the construction of (e) with the triple $(\mathfrak{h}_1, \chi_1, g_1)$, by the index 1. Putting, for h in $\mathcal{F}$, $(\mathrm{V}h)(x) \equiv h\left(\overset{-1}{\alpha}(x)\right)$, we are going to show, that $\mathrm{V}\,\mathcal{F} = \mathcal{F}_1$, and $\mu_h(\mathrm{G}/\mathrm{A}) = \mu_{\mathrm{V}h}(\mathrm{G}/\mathrm{A}_1)$. To this end we observe first, that $\alpha(\mathrm{A}) = \mathrm{A}_1$. In fact, by definition $\mathrm{A} = \overline{\mathrm{G}}_g\,\mathrm{D}$; as already noted above $\alpha\left(\overline{\mathrm{G}}_g\right) = \overline{\mathrm{G}}_{\alpha g}$, but we have also $\alpha(\mathrm{D}) = \mathrm{D}_1$, since $\mathrm{D} = \exp(d)$ and $\alpha(d) = \alpha(\mathfrak{h} \cap \mathfrak{d}) = \alpha\mathfrak{h} \cap \mathfrak{d} = \mathfrak{h}_1 \cap \mathfrak{d} = d_1$, proving our assertion. From here to show, that $\mathrm{V}\,\mathcal{F} = \mathcal{F}_1$, it is enough to estabish, that if a is some element in A and $a_1 = \alpha(a) \in \mathrm{A}_1$, we have $\varphi(a) = \varphi_1(a_1)$ and $\eta(a) = \eta_1(a_1)$ [*cf.* (a)]. The first relation being certainly valid on $\overline{\mathrm{G}}_g$, we can assume, that a is in D and of the form $a = \exp(l)$ $(d \ni l)$. But then

$$\varphi_1(a) = \chi_{f_1}(a_1) = \exp[i(\alpha\, l, \alpha\, g)] = \exp[i(l, g)] = \chi_f(a) = \varphi(a).$$

As far as the second relation is concerned, we have by definition $\eta(a) = \Delta_{\mathrm{A}}(a)/\Delta_{\mathrm{G}}(a)$ and hence it suffices to show, that $\Delta_{\mathrm{A}}(a) = \Delta_{\mathrm{A}_1}(a_1)$ and $\Delta_{\mathrm{G}}(a) = \Delta_{\mathrm{G}}(a_1)$ [$a_1 = \alpha(a)$, a arbitrary in A]. Writing $\mathrm{A}_0 = \exp\mathfrak{a}$ we have

$$\Delta_{\mathrm{A}}(a) = \det(\mathrm{Ad}(a) \mid \mathfrak{a}) \qquad \text{and} \qquad \Delta_{\mathrm{A}_1}(a_1) = \det(\mathrm{Ad}(a_1) \mid \alpha(\mathfrak{a})),$$

whence the desired conclusion is obvious since, putting $\beta = d\,\alpha$, we have $\mathrm{Ad}(a_1) = \beta(\mathrm{Ad}\, a)\,\beta^{-1}$ and $\alpha(a) = \beta(a)$ by definition. One proves similarly, that $\Delta_{\mathrm{G}}(a) = \Delta_{\mathrm{G}}(a_1)$.

Let us write $\mathrm{K} = \mathrm{G}/\mathrm{A}$, and $\mathrm{K}_1 = \mathrm{G}/\mathrm{A}_1$; we denote by γ the homeomorphism from K onto K_1 assigning to $\mathrm{A}\,x$ the coset $\mathrm{A}_1\,\alpha(a)$. Given a Borel measure τ on K we shall write $\gamma\tau$ for its image on K_1. We show next, that if the right invariant measures da and da_1 on A and A_1 resp. are appropriately normalized, we have $\gamma\mu_h = \mu_{\mathrm{V}h}$ for all h in $\mathcal{F}$. This clearly implies, that $\mu_h(\mathrm{G}/\mathrm{A}) = \mu_{\mathrm{V}h}(\mathrm{G}/\mathrm{A}_1)$ as claimed above. Let f be a function in C (G) and let us form as in (a) :

$$\mathrm{F}(p_1) = \int_{\mathrm{A}_1} f(ax)\, da \qquad (p \in \mathrm{K}_1).$$

Since $\mu_{\mathrm{V}h} = \nu_{\mathrm{V}h}/\mathrm{A}$ [$d\nu_h(x) = |\, h(x)\,|^2\, dx$, cf. (a)] and $(\mathrm{V}h)(x) \equiv h\left(\overset{-1}{\alpha}(x)\right)$, we have

$$\int_{\mathrm{K}_1} \mathrm{F}(p_1)\, d\mu_{\mathrm{V}h}(p_1) = \int_{\mathrm{G}} f(x)\,|\, h(\overset{-1}{\alpha}(x))\,|^2\, dx = \int_{\mathrm{G}} f(\alpha(x))\,|\, h(x)\,|^2\, d\alpha(x).$$

Let c be a constant such that $cdx = d\,\alpha\,(x)$. Assuming, as we can, that c is also the ratio of da and of the image, under α, of da on $A_1 = \alpha\,(A)$ we can conclude, since at the same time $\mu_h = \nu_h/A$, that

$$\int_{K_1} F(p_1)\,d\mu_{\nu h}(p_1) = c\int_K\left(\int_A f(\alpha(a)\,\alpha(x))\,da\right)d\mu_h(p) = \int_K\left(\int_{A_1} f(a_1\,\alpha(x))\,da_1\right)d\mu_h(p)$$

$$= \int_K F(\gamma p)\,d\mu_h(p) = \int_{K_1} F(p_1)\,d(\gamma\mu_h)(p)$$

which is what we had to prove.

From here to complete the proof of Lemma 4.3 it is enough to show, that $V\mathcal{H} = \mathcal{H}_1$ [*cf.* (*e*)]. Let p be some element of $\mathcal{H}$ and let us put $P_1(h_1 k_1) \equiv \chi_{f_1}(h)\,V\,p\,(k_1 a_0)$ ($h_1 \in H_1$, $k_1 \in E_1$; a_0 fixed in G) and

$$P(hk) \equiv \chi_f(h)\,p\,(ka'_0)$$

$\left[h \in H,\ k \in E;\ a'_0 = \overset{-1}{\alpha}(a_0)\right]$. Since p belongs to $\mathcal{H}$, P is analytic on HE, and we have to show, that P_1 is analytic on $H_1 E_1$. But this follows at once from the easily verifiable facts, that $H_1 E_1 = \alpha\,(HE)$ and $P_1 \circ \alpha \equiv P$.

Q. E. D.

REMARK 4.1. — One proves similarly, that

$$\alpha\,\mathrm{ind}\,(\mathfrak{h}, \chi, g; G_1) = \mathrm{ind}\,(\alpha\,\mathfrak{h}, \alpha\chi, \alpha\,g; \alpha\,G_1) \qquad [cf.\ (e)]$$

and

$$\alpha\,\mathrm{Ind}\,(\mathfrak{h}, f) = \mathrm{Ind}\,(\alpha\,\mathfrak{h}, \alpha\,f) \qquad [cf.\ (f)].$$

5. The results of this section are due to L. Auslander and B. Kostant (*cf.* [1], Theorem IV.4.1). Here we follow closely the exposition given by B. Kostant in his course at the M. I. T., Spring, 1969.

Let $\pi \in \hat{L}$ be different from the trivial representation of L. By virtue of 4 (*f*) there is an $f \in \mathfrak{d}'$, $f \neq 0$, and $\mathfrak{h} = \mathrm{pol}\,(f) \subset \mathfrak{d}_{\mathbf{C}}$ such that $\mathrm{Ind}\,(\mathfrak{h}, f)$ belongs to the unitary equivalence class π. Below we shall also write π for the concrete representation $\mathrm{Ind}\,(\mathfrak{h}, f)$.

LEMMA 5.1. — *With the previous notations, we have* $G_\pi = G_f L$.

Proof. — Given $a \in G$ we set $af = (\mathrm{Ad}\,(a^{-1}) \mid \mathfrak{d})'f$, and if ρ is some representation of L, we write $(a\,\rho)\,(x) \equiv \rho\,(a^{-1}\,xa)$ $(x \in L)$. By virtue of Remark 4.1, $a\,\mathrm{Ind}\,(\mathfrak{h}, f)$ is unitarily equivalent to $\mathrm{Ind}\,(a\,\mathfrak{h}, af)$ which, by what we saw in 4 (*f*), implies, that a belongs to G_π if and only if af belongs to the orbit $L\,f$. But this is clearly equivalent to $a \in G_f\,L$.

Q. E. D.

Substituting G_f in place of A in the proof of Lemma 4.2, we obtain as *loc. cit.* a closed subset $S \subset G_f$ such that any $a \in G_f$ can uniquely be written as $a = sl$ ($s \in S$, $l \in L_f = G_f \cap L$) and the factors on the right hand side depend continuously on a. We have also $G_\pi = G_f L = SL$; if $a \in G_\pi$ we shall write $s(a)$ and $l(a)$ for the elements of S and L resp., with which $a = s(a)\, l(a)$. We observe next, that concerning $\mathfrak{h} = \text{pol}(f)$ as above we can assume, that it is invariant with respect to G_f. To see this, it suffices to apply the last remark of 4 (*b*) with L, $\mathfrak{d}$, $\text{Ad}(G_f) \mid \mathfrak{d}$ in place of G, $\mathfrak{g}$ and K *loc. cit.*. We put, as in 4 (*e*), $d = \mathfrak{h} \cap \mathfrak{d}$; evidently $G_f\, d \subset d$. The map $a \mapsto \det(\text{Ad}(a) \mid \mathfrak{d}/d)$ $(a \in G_f)$ is a homomorphism of G_f into the multiplicative group of positive numbers, containing in its kernel $L_f = G_f \cap L$. From this we conclude, that there is a homomorphism ψ, of the indicated sort, of G_π, such that $\psi(a) = \det(\text{Ad}(a^{-1}) \mid \mathfrak{d}/d)$ $(a \in G_f)$ and $\psi \mid L \equiv 1$. We recall finally [*cf.* 4 (*e*)], that χ_f is the holomorphic character of $H = \exp \mathfrak{h} \subset L_{\mathbf{C}}$ determined by

$$\chi_f(\exp(l)) \equiv (\exp[i.(l, f)]) \qquad (l \in \mathfrak{h}).$$

With a continued use of the notations of 4 (*e*) we have

LEMMA 5.2. — *For $a \in G$ and $g \in \mathcal{H}$ let us put* $(\rho(a)\, g)(x) \equiv (\psi(a))^{1/2} g(t^{-1} xa)$ $[t = s(a) \in S]$. *Then we have*: *a.* $\rho(a) g \in \mathcal{H}$ and $\mu_{\rho(a) g}(L/D) = \mu_g(L/D)$ [*cf.* 4 (*a*)]; *b. Denoting by $\pi^e(a)$ the operator corresponding to $\rho(a)$ in $\mathbf{H}(\pi)$, the map $a \mapsto \pi^e(a)$ defines a continuous projective representation of G_π, such that $\pi^e(a)\, \pi^e(b) = \omega(a, b)\, \pi^e(ab)$ $(a, b \in G_\pi)$ where $\omega(a, b) = \overline{\chi_f(l(rt))}$ $[r = s(a),\ t = s(b)]$, and $\pi^e \mid L = \pi$.*

Proof. — *a.* Let us put $g'(x) \equiv g(t^{-1} xa)$. If $\delta \in D$, we have $g'(\delta x) \equiv \chi_f(\delta)\, g'(x)$ $(x \in L)$. In fact, since $t^{-1} \delta t \in D$, we can conclude, that $g'(\delta x) = g(t^{-1} \delta t . t^{-1} xa) = \chi_f(t^{-1} \delta t) g'(x)$; but evidently $\chi_f(b^{-1} \delta b) \equiv \chi_f(\delta)$ on D for any fixed b in G_f. By what we have just seen, the expression $\chi_f(h)\, g'(kl_0)$ $(h \in H,\ k \in E;\ l_0$ fix in L) depends only on the product hk; hence we can write it as $K(z)$ $(z \in HE)$. We claim, that the map $z \mapsto K(z)$ is holomorphic on HE. In fact,

$$K(hk) = \chi_f(h)\, g'(kl_0) = \chi_f(t^{-1} ht)\, g(t^{-1} kt . l_1), \qquad \text{where} \quad l_1 = t^{-1} l_0\, t . l(a) \in L.$$

In this fashion the desired conclusion follows from the fact, implied by $g \in \mathcal{H}$, that $hk \mapsto \chi_f(h)\, g(kl_1)$ is holomorphic, along with the observation that the map $z \mapsto t^{-1} zt$ of HE into itself is also holomorphic. To establish part 1 of our lemma it remains to be shown, that $\mu_{\rho(a) g}(L/D) = \mu_g(L/D)$.

Since evidently $\Delta_L \equiv 1$, $\Delta_D \equiv 1$, there is an L invariant measure $dv(p)$ on L/D, and by what we saw at the end of 4 (*a*), we have

$$\mu_g(L/D) = \int_{L/D} |g(x)|^2 \, dv(p).$$

Since G_f normalizes D, G_f operates on the right on L/D; if $p = Dx$ ($x \in L$) and b is some element of G_f we have $pb = Db^{-1}xb$. We claim, that $dv(pb) = \psi(b)\, dv(p)$. In fact, let dx and $d\delta$ be elements of the invariant measure on L and D resp., h some element of C (L) and let us put, as in 4, $H(p) = \int_D h(\delta x)\, d\delta$. We have

$$H(pb) = \int_D h(\delta\, b^{-1}\, xb)\, d\delta = \det(\mathrm{Ad}(b^{-1}) \mid d) \int_D h(b^{-1}\, \delta\, xb)\, d\delta.$$

Therefore, if $d\delta$ is appropriately normalized

$$\int_{L/D} H(pb)\, dv(p) = \det(\mathrm{Ad}(b^{-1}) \mid d) \int_L h(b^{-1}xb)\, dx$$
$$= \psi(b^{-1}) \int_D h(x)\, dx = \psi(b^{-1}) \int_{L/D} H(p)\, dv(p)$$

proving our assertion. Thus finally, since $\pi(a) = \pi(t)$:

$$\mu_{\rho(a)g}(L/D) = \psi(a) \int_{L/D} |g(t^{-1}\; xt\, l(a))|^2\, dv(p) = \int_{L/D} |g(xl(a))|^2\, dv(p)$$
$$= \int_{L/D} |g(x)|^2\, dv(p) = \mu_g(L/D)$$

proving the first part of Lemma 5.2.

b. Let a, b elements of G_π and g as above. Putting $r = s(a)$, $m = l(a)$, $t = s(b)$, $n = l(b)$ we have, that $ab = s(rt)\,[l(rt)\, t^{-1}\, mtn]$ and therefore $s(ab) = s(rt)$. In this fashion

$$(\rho(ab)\, g)(x) \equiv (\psi(s(rt)))^{1/2}\, g((s(rt))^{-1}\, xab) = (\psi(a))^{1/2}\, (\psi(b))^{1/2}\, g(l(rt)\, (rt)^{-1}\, xab)$$

which, since $l(rt) \in L_f \subset D$, is the same as $\chi_f(l(rt))\, (\psi(a))^{1/2} (\psi(b))^{1/2}\, g((rt)^{-1}xab)$. On the other hand, $(\rho(b)\, g)(x) \equiv (\psi\, b))^{1/2}\, g(t^{-1}.xb)$, and thus

$$(\rho(a)\, (\rho(b)\, g))(x) \equiv (\psi(a))^{1/2}\, (\psi(b))^{1/2}\, (g(t^{-1}\, r^{-1}\, xab),$$

implying

$$\rho(a)\, \rho(b) = \overline{\chi_f(l(rt))}\, \rho(ab) \qquad \text{and} \qquad \pi^e(a)\, \pi^e(b) = \overline{\chi_f(l(rt))}\, \pi^e(ab).$$

Q. E. D.

6. Given an element g of $\mathfrak{g}'$ we shall denote by B_g the skewsymmetric bilinear form defined on $\mathfrak{g}\times\mathfrak{g}$ by $B_g(x, y) = ([x, y], g)$ $(x, y\in\mathfrak{g})$ [*cf.* also 4 (*b*)]. If g is specified by the context, we shall often omit indicating it. Given a subspace $\mathfrak{a}$ of the underlying space of $\mathfrak{g}$, we shall write $\mathfrak{a}^{\perp}$ for its orthogonal complement in $\mathfrak{g}'$ with respect to the canonical bilinear form on $\mathfrak{g}\times\mathfrak{g}'$, and denote by $\mathfrak{a}_B^{\perp}$ its orthogonal complement, with respect to B, in $\mathfrak{g}$. Let us observe, that with these notations we have $\mathfrak{g}_B^{\perp} = \mathfrak{g}_g =$ Lie algebra of the stabilizer of g in G [*cf.* 4 (*c*)]. Let us put $f = \mathfrak{g}\mid\mathfrak{d}$ and $(G_f)_0 = \exp(\mathfrak{g}_f)$; then $\mathfrak{g}_f = \mathfrak{d}_B^{\perp}$ $(\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}])$. The following three lemmas were used by B. Kostant in his lectures referred to at the start of the last section.

LEMMA 6.1. — *Assume, that* $g\in\mathfrak{g}$, $f = g\mid\mathfrak{d}$ *and let us put* $\mathfrak{g}_\pi = \mathfrak{d} + \mathfrak{g}_f$. *If* $g_1\in\mathfrak{g}'$ *satisfies* $g\mid\mathfrak{g}_\pi = g_1\mid\mathfrak{g}_\pi$ *then there is an element* a *in* L_f *such that* $ag = g_1$.

Proof. — *a*. Let us observe first, that if l is any element in $\mathfrak{g}_f$, we have

$$\exp(l)\,g = g - (\operatorname{ad}(l))'\,g.$$

In fact, since

$$\exp(l)\,g = \sum_{j=0}^{\infty}\frac{(-(\operatorname{ad}(l))')^j}{j\,!}\,g$$

to obtain the desired conclusion, it suffices to show, that $[(\operatorname{ad}(l))']^j = 0$ for $j \geqq 2$. If k is an arbitrary element in $\mathfrak{g}$, we have

$$(k, [(\operatorname{ad}(l))']^j g) = B\,(l\,[l, k_1]) \qquad \text{with} \quad k_1 = [(\operatorname{ad}(l))]^{j-2}\,k;$$

but the right hand side vanishes, since $l\in\mathfrak{g}_f = \mathfrak{d}_B^{\perp}$, and $[l, k_1]\in\mathfrak{d}$.

From this we conclude, that if a is any element in $(G_f)_0$, there is an l in $\mathfrak{g}_f$ such that $ag = g - (\operatorname{ad}(l))'g$. If, however, a lies in $L_f = (G_f)_0\cap L$, we can assume, that l belongs to $\mathfrak{d}_f = \mathfrak{g}_f\cap\mathfrak{d}$.

b. If l and k are elements of $\mathfrak{g}_f$ and $\mathfrak{d}_f$ resp., we have $B(l, k) = 0$ implying, that $(\operatorname{ad}(\mathfrak{d}_f))'g\subset\mathfrak{g}_f$; since evidently $(\operatorname{ad}(\mathfrak{d}_f))'\,g\subset\mathfrak{d}^{\perp}$, we conclude, that $(\operatorname{ad}(\mathfrak{d}_f))'\,g\subset\mathfrak{g}_\pi^{\perp}$. In this fashion, to complete the proof of our lemma it will be enough to show, that $\dim[(\operatorname{ad}(\mathfrak{d}_f))'g] = \dim\mathfrak{g}_\pi^{\perp}$. Evidently, the left hand side is equal to $\dim\mathfrak{d}_f - \dim(\mathfrak{g}_g\cap\mathfrak{d})$. On the other hand, we have

$$\dim\mathfrak{g}_\pi^{\perp} = \dim\mathfrak{g} - \dim\mathfrak{g}_\pi = \dim\mathfrak{g} - \dim\mathfrak{d} - \dim\mathfrak{d}_B^{\perp} + \dim\mathfrak{d}_f;$$

since, however,

$$\dim\mathfrak{d}_B^{\perp} = \dim\mathfrak{g} - \dim\mathfrak{d} + \dim(\mathfrak{g}_g\cap\mathfrak{d})$$

we obtain finally, that

$$\dim \mathfrak{g}_{\pi}^{\perp} = \dim \mathfrak{d}_f - \dim (\mathfrak{g}_g \cap \mathfrak{d}) = \dim [(\mathrm{ad}\,(\mathfrak{d}_f))' g].$$

Q. E. D.

The following lemma will be of much use in subsequent parts of this paper.

LEMMA 6.2. — *For* $g \in \mathfrak{g}'$ *let us put* $\mathfrak{k}_\pi = \mathfrak{d} + \mathfrak{g}_g$. *If* $g_1 \in \mathfrak{g}'$ *satisfies* $g \mid \mathfrak{k}_\pi = g_1 \mid \mathfrak{k}_\pi$, *there is an element* a *of* $(G_f)_0$ $(f = g \mid \mathfrak{d})$ *such that* $ag = g_1$.

Proof. — By (a) of the previous lemma, for any a in $(G_f)_0$ we have $ag = g - (\mathrm{ad}\,(l))' g$, where l is in $\mathfrak{g}_f$. On the other hand, $(\mathrm{ad}\,\mathfrak{g}_f)' g$ is orthogonal to $\mathfrak{k}_\pi$. In this fashion to prove our lemma, it will be enough to establish, that

$$\dim [(\mathrm{ad}\,(\mathfrak{g}_f))' g] = \dim \mathfrak{g}_f - \dim \mathfrak{g}_g$$

is the same as $\dim \mathfrak{k}_\pi^{\perp}$; but this is clear, since

$$\dim \mathfrak{k}_\pi^{\perp} = \dim \mathfrak{g} - \dim \mathfrak{k}_\pi = \dim \mathfrak{g} - \dim \mathfrak{d} + \dim (\mathfrak{g}_g \cap \mathfrak{d}) - \dim \mathfrak{g}_g = \dim \mathfrak{d}_{\mathfrak{g}}^{\perp} - \dim \mathfrak{g}_g$$
$$\text{and} \qquad \mathfrak{g}_f = \mathfrak{d}_{\mathfrak{g}}^{\perp}.$$

Q. E. D.

REMARK 6.1. — If g and g_1 are elements of $\mathfrak{g}'$, such that $g \mid \mathfrak{d} = g_1 \mid \mathfrak{d}$, we have $G_g = G_{g_1}$ and $\overline{G}_g = \overline{G}_{g_1}$ [*cf.* 4 (c)]. In fact, by virtue of our assumption $g_1 = g + u$ with $u \in \mathfrak{d}^{\perp}$. On the other hand, clearly $au \equiv u$ for all a in G implying $G_{g_1} = G_g$. Let φ be the character of G determined by the condition $\varphi(\exp(l)) = \exp[i(l, u)]$ $(l \in \mathfrak{g})$. We have evidently $\chi_{g_1}(a) \equiv \varphi(a) \chi_g(a)$ on $G_{g_1} = G_g$, and thus for any a, $b \in G_g$:

$$\chi_{g_1}(aba^{-1}\,b^{-1}) = \chi_g(aba^{-1}\,b^{-1}),$$

completing the proof of our assertion.

LEMMA 6.3. — *With the notations of Lemma* 6.1 *let us put*

$$\mathfrak{G} = \{ a;\, a \in G_f,\, ag \mid \mathfrak{g}_f = g \mid \mathfrak{g}_f \}.$$

Then $\mathfrak{G} = G_g L_f$.

Proof. — We have evidently $G_g \subset \mathfrak{G}$ and by (b) in the proof of Lemma 6.1, $L_f \subset \mathfrak{G}$ and hence $G_g L_f \subset \mathfrak{G}$. If, on the other hand, a in G_f is such, that $ag \mid \mathfrak{g}_f = g \mid \mathfrak{g}_f$, since $ag \mid \mathfrak{d} = g \mid \mathfrak{d}$, we have also $ag \mid \mathfrak{g}_\pi = g \mid \mathfrak{g}_\pi$ and thus, by Lemma 6.1, there is an element b in L_f with $ag = bg$ implying $a \in G_g L_f$ and therefore $\mathfrak{G} \subset G_g L_f$.

Q. E. D.

Let χ_f be the character of L_f determined by

$$\chi_f(\exp(l)) = \exp[i(l, f)] \qquad (l \in \mathfrak{d}_f).$$

We denote by $\mathring{L}_f$ the kernel of χ_f. Since for any a in G_f we have

$$(a\chi_f)(\delta) \equiv \chi_f(a^{-1}\delta a) \equiv \chi_f(\delta) \qquad (\delta \in L_f),$$

$\mathring{L}_f$ is invariant in G_f.

Lemma 6.4. — *Let us put* $\overline{\mathfrak{G}} = \{a; a \in \mathfrak{G}, aba^{-1}b^{-1} \in \mathring{L}_f$ for all b in $\mathfrak{G}\}$. *Then* $\overline{\mathfrak{G}} = \overline{G}_g L_f$.

Proof. — Let us form $\Lambda = G_f/\mathring{L}_f$; we shall denote by σ the canonical homomorphism from G_f onto Λ. Since $a\chi_f \equiv \chi_f$ on L_f for all $a \in G_g$, we have $\sigma(L_f) \subset \Lambda^\natural$ and thus $L_f \subset \overline{\mathfrak{G}}$. In this fashion the element $bl\,(b \in G_g, l \in L_f)$ of $\mathfrak{G}$ will belong to $\overline{\mathfrak{G}}$ if and only if $aba^{-1}b^{-1}$ lies in $\mathring{L}_f$ for all a in G_g. Hence to complete the proof of our lemma it is enough to show, that if for $a, b \in G_g$ the element $aba^{-1}b^{-1}$ belongs to $\mathring{L}_f$, then it lies in $\mathring{G}_g$ too, and conversely. But we have $[G_g, G_g] \subset G_g \cap L_f$, the right hand side being the connected subgroup $\exp(\mathfrak{g}_g \cap \mathfrak{d}_f) \subset (G_g)_0 \cap L_f$, and evidently $\chi_f \mid G_g \cap L_f \equiv \chi_g \mid G_g \cap L_f$.

Q. E. D.

In the following, for the convenience of the reader, we repeat several things already touched upon in Section 2 (*cf.* in particular Remark 2.2). Let us assume, that Γ is a central extension, through the circle group $\mathbf{T}$, of Z, itself isomorphic to $\mathbf{R}^a \times \mathbf{Z}^b$. We denote by Γ_1 and U the centralizer of the connected component Γ_0 of Γ and of the center $\Gamma_0^\natural$ of Γ_0 resp.

Lemma 6.5. — *With the previous notations we have* $\Gamma_1^\natural = U^\natural$ *and* $U/\Gamma_0 U^\natural = \Gamma/\Gamma_1^\natural$.

Proof. — *a.* We observe (*cf.* [2], p. 188), that there is a continuous realvalued bilinear form B on $Z \times Z$ such that, putting

$$\alpha(z, z') = \exp[(i/2)B(z, z')] \qquad (z, z' \in Z),$$

Γ is isomorphic to the group defined on the set of pairs (z, u) $(z \in Z, u \in \mathbf{T})$ by the law of multiplication $(z, u)(w, v) = (z + w, \alpha(z, w).uv)$. We shall denote by $\mathbf{T}$ also the subgroup $\{(0, u); u \in \mathbf{T}\}$ of Γ. Given a subset S of Z, S will also stand for $\{(s, 1); s \in S\}$. We write $S_B^\perp$ for the subset $\{t; t \in Z, B(t, s) = 0$ for all s in $S\}$. Let us put $Z_1 = Z_0 \cap (Z_0)_B^\perp$, where Z_0 is the connected component of the neutral element in Z. Since

$(z, u)(w, v)(z, u)^{-1} = (w, (\alpha(z, w))^2 . v)$, we find readily, that $U = (Z_1)^{\perp}_{B} . \mathbf{T}$. On the other hand we have $(Z_1)_B = Z_0 + (Z_0)^{\perp}_{B}$ and therefore, since $\Gamma_1 = (Z_0)^{\perp}_{B} . \mathbf{T}$, we conclude that $U = \Gamma_0 \Gamma_1$ and hence also $\Gamma_1^{\natural} \subset U^{\natural}$. Finally, if a is some element in $U^{\natural} \subset \Gamma_0 \Gamma_1$ we can write $a = bc$ $(b \in \Gamma_0, c \in \Gamma_1)$. Since a commutes, in particular, with Γ_0, we have $b \in \Gamma_0^{\natural} \subset \Gamma_1$, and thus $a \in \Gamma_1$ and even $a \in \Gamma_1^{\natural}$, proving $U^{\natural} = \Gamma_1^{\natural}$.

b. To establish $U/\Gamma_0 U^{\natural} = \Gamma_1/\Gamma_1^{\natural}$ we observe that, by what preceeds, the left hand side is the same as $\Gamma_0 \Gamma_1 / \Gamma_0 . \Gamma_1^{\natural}$. From here the desired conclusion follows by noting, that $\Gamma_0^{\natural} = \Gamma_0 \cap \Gamma_1 = \Gamma_0 \cap \Gamma_1^{\natural}$.

Q. E. D.

LEMMA 6.6. — *With the previous notations let us put* $\Gamma_0 = \exp \gamma$. Let v be an element of γ such that $\exp(\mathbf{R} v) = \mathbf{T} \subset \Gamma_0$, *and assume, that* $d \in \gamma'$ *is such, that* $(v, d) \neq 0$. *Then* $\Gamma_1 = \Gamma_d$.

Proof. — We can identify the underlying space of γ to $\mathbf{R} . v + Z_0$ such that

$$\exp(z_0 + cv) = (z_0, \exp(ic)) \qquad (z_0 \in Z_0, c \in \mathbf{R}).$$

We have then for any $(z, u) \in \Gamma$:

$$(z, u)(z_0 + c . v) = z_0 + (c + B(z, z_0))\, v.$$

In order, that (z, u) belong to Γ_d, the expression

$$(z_0 + c . v, (z, u)^{-1} d) = (z_0 + cv, d) + B(z, z_0)(v, d)$$

must be the same as $(z_0 + cv, d)$ for all $z_0 + cv$ which, by virtue of our assumption $(v, d) \neq 0$ means, that $z \in (Z_0)^{\perp}_{B}$. In this fashion $\Gamma_d = (Z_0)^{\perp}_{B} . \mathbf{T} = \Gamma_1$, proving our lemma.

Q. E. D.

Let us observe, that $[G, G] = L$ is a closed, connected and invariant subgroup of G (*cf.* [20], Theorem 3.1, p. 138). It belongs to the subalgebra $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ which, since $\mathfrak{g}$ is solvable, is nilpotent and thus L is of type I (*cf.* [8], Théorème 3, p. 161). From this we conclude, that we can substitute [G, G] in place of L in Section 3.

PROPOSITION 6.1. — *Let* π *be an element of* $\hat{L}$, *which belongs to the orbit* $Lf \subset d'$ [*cf.* 4 (*f*)]. *Let* K *and* $\mathfrak{U}$ *be as in Lemma* 3.6 *and assume, that* g *is an arbitrary element of* $\mathfrak{g}'$ *such that* $g \mid \mathfrak{d}' = f$. *Then* $K = \overline{G}_g L$ *and* $\mathfrak{U}/(G_\pi)_0 K$ *is isomorphic to* $G_g/\overline{G}_g$.

Proof. — *a*. The statements made being clear if $f = 0$, in the following we shall assume, that $f \neq 0$. By Lemma 5.1, $G_\pi = G_f L$. Let us form, as at the start of Section 3, the groups G_π^e and $\mathbf{M}_\pi = G_\pi^e/L$ by using in place of the cocycle α *loc. cit.*, the cocycle ω of Lemma 5.2. Employing the notations of the latter one verifies easily, that $\mathbf{M}_\pi$ can be realized as the group defined on the set of pairs $\{(r, u);\ r \in S,\ u \in \mathbf{T}\}$ by the law of multiplication $(r, u)(t, v) = (s(rt), \omega(r, t).uv)$ and if Φ is the canonical homomorphism from G_π^e onto $\mathbf{M}_\pi$, we have

$$\Phi((a, u)) = (s(a), u) \qquad (a \in G_\pi).$$

b. Let us put $\Gamma = \mathbf{M}_\pi$ and assume, that $U^\natural$ is as in Lemma 6.5. We denote by $\overline{S}$ the subset of $S \subset G_f$ (*cf.* Lemma 5.2) such that $\overline{G}_g L_f = \overline{S} L_f$. With these notations *we claim, that*

$$U^\natural = \{(r, u);\ s \in \overline{S},\ u \in \mathbf{T}\}.$$

In fact,

1° Let us define the map Ψ from G_f into Γ by

$$\Psi(a) = (s(a), \overline{\chi_f(l(a))}) \qquad (a \in G_f).$$

We claim, that Ψ is a surjective homomorphism, the kernel of which coincides with $\mathring{L}_f$. In fact, if $r,\ t \in S$, and $m,\ n \in L_f$ we have

$$rm.tn = s(rt)[l(rt).t^{-1}mt.n]$$

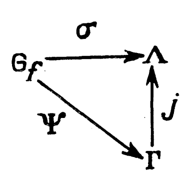
and hence, since

$$\chi_f(t^{-1}mt) = \chi_f(m),$$

$$\Psi(rm.tn) = (s(rt), \overline{\chi_f(l(rt))\chi_f(m)\chi_f(n)}) = (r, \overline{\chi_f(m)})(t, \overline{\chi_f(n)}) = \Psi(rm)\Psi(tn).$$

In order to establish, that Ψ is surjective, it is enough to show, that $\chi_f \not\equiv 1$ on $L_f = \exp(\mathfrak{d}_f)$, or that $f \mid \mathfrak{d}_f \neq 0$. We have $\mathfrak{d}_f^\perp = (\mathrm{ad}\,\mathfrak{d})'f$ and hence if $f \mid \mathfrak{d}_f = 0$ there is an l in $\mathfrak{d}$ such that $f = (\mathrm{ad}(l))'f$. This, however, is impossible, if $f \neq 0$, since $\mathrm{ad}(l)$ is nilpotent. Finally it is clear, that $\ker \Psi = \ker(\chi_f \mid L_f) = \mathring{L}_f$.

2° Let us form, as in the proof of Lemma 6.4, the group $\Lambda = G_f/\mathring{L}_f$. We denote by σ the canonical homomorphism from G_f onto Λ, and by j the isomorphism from Γ onto Λ such that the diagramm



be commutative. Let Λ_1 be the centralizer of Λ_0 in Λ. We have by Lemma 6.5, $U^\natural = \Gamma_1^\natural$ and hence $j(U^\natural) = \Lambda_1^\natural$.

3° We denote by λ the Lie algebra of Λ_0, and by d the element of λ' such that $\delta\sigma(d) = g \mid \mathfrak{g}_f$. We observe, that if $w \in \lambda$ is such, that $\exp(\mathbf{R}w) = j(\mathbf{T})$, we have $(w, d) \neq 0$. In fact, evidently $j(\mathbf{T}) = \sigma(L_f)$. Let l be an element of $\mathfrak{d}_f$ such that $d\sigma(l) = w$. Then

$$(w, d) = (d\sigma(l), d) = (l, \delta\sigma(d)) = (l, f) \neq 0.$$

Applying Lemma 6.6 with Λ in place of Γ *loc. cit.* we conclude, that $\Lambda_1 = \Lambda_d$.

4° Let us observe next, that with the above choice of d we have $\overset{-1}{\sigma}(\Lambda_d^\natural) = \overline{G}_f L_f$. To this end we note, that if $\mathfrak{T}$ and $\overline{\mathfrak{T}}$ are as in Lemmas 6.3 and 6.4 resp., we have clearly $\mathfrak{T} = \overset{-1}{\sigma}(\Lambda_d)$; thus $\sigma(\mathfrak{T}) = \Lambda_d$ and $\overline{\mathfrak{T}} = \overset{-1}{\sigma}(\Lambda_d^\natural)$. In this fashion the desired conclusion follows from Lemma 6.4.

5° Summing up, we have $j(U^\natural) = \Lambda_d^\natural$ and hence $\Psi(\overline{G}_g L_f) = U^\natural$, from where it is clear, if we put $\overline{G}_g L_f = \overline{S} L_f$ ($\overline{S} \subset S$) that $U^\natural = \{(r, u); r \in \overline{S}, u \in \mathbf{T}\}$ as stated at the start of (*b*).

c. Given a subset M of G_π let us write, as in Section 3, M^e for $\{(m, u); m \in M, u \in \mathbf{T}\} \subset G_\pi^e$. By what we saw above we have $\overset{-1}{\Phi}(U^\natural) = \overline{G}_g^e L$. But, by the proof of Lemma 3.5, the left hand side is the same as K^e, proving, that $K = \overline{G}_g L$.

d. To establish the second assertion of our proposition we recall (*cf.* Lemma 3.3), that $\mathfrak{U}^e = \overset{-1}{\Phi}(U)$ and $(G_\pi^e)_0 = \overset{-1}{\Phi}(\Gamma_0)$. In this fashion $\mathfrak{U}/(G_\pi)_0 K$ is isomorphic to $U/\Gamma_0 U^\natural$ and hence, by Lemma 6.5, to $\Gamma_1/\Gamma_1^\natural$. By 3° and 4° in (*b*) above the last group is isomorphic to $G_g L_f/\overline{G}_g L_f$. Since $G_g \cap L_f = \overline{G}_g \cap L_f$ this implies finally, that $\mathfrak{U}/(G_\pi)_0 K$ is isomorphic to $G_g/\overline{G}_g$, completing the proof of our proposition.

Q. E. D.

REMARK 6.2. — For later use we note, that the above reasonings imply easily, that $\mathfrak{U} = (G_f)_0 G_g L$.

7. Let g be an element of $\mathfrak{g}'$ such that $f = g \mid \mathfrak{d} \neq 0$. We assume, that $\mathfrak{h} = \text{pol}(f)$ [*cf.* 4 (*b*)] is such, that $\overline{G}_g \mathfrak{h} \subset \mathfrak{h}$ [*cf.* 4 (*e*)]. Let us form the representation $\pi = \text{Ind}(\mathfrak{h}, f)$ of $L = [G, G]$ [*cf.* 4 (*f*)]; we shall denote also by π its image in $\hat{L}$. If π and K are related as in Lemma 3.6, by

Proposition 6.1 we have $K = \overline{G}_g L$. Putting $d = \mathfrak{h} \cap \mathfrak{d}$, we have $\overline{G}_g d \subset d$, and the homomorphism $a \to \det(\mathrm{Ad}(a^{-1}) \mid \mathfrak{d}/d)$ of $\overline{G}_g$ admits an extension ψ to K such that $\psi \mid L \equiv 1$. Let us denote by $\mathcal{H}_0$ the linear variety of function on L, formed as $\mathcal{H}$ in Lemma 5.2, but with $\mathfrak{h} = \mathrm{pol}(f)$ as specified above. Given $e(x) \in \mathcal{H}_0$ and $a \in K$, we set $(\tau'(a)e)(x) \equiv (\psi(a))^{1/2} e(t^{-1}xa)$ $[t = s(a)]$. We show as *loc. cit.* that : 1° $\tau'(a)$ transforms $\mathcal{H}_0$ into itself and gives rise on $\mathbf{H}(\pi)$ to a unitary operator $\tau(a)$; 2° the map $a \mapsto \tau(a)$ $(a \in K)$ is a continuous projective representation of K, such that

$$\tau(a)\tau(b) = \overline{\chi_f(l(rt))}\,\tau(ab) \qquad [a, b \in K; r = s(a), t = s(b)].$$

Let χ be some element of $\hat{\overline{G}}_g$ [*cf.* 4 (*c*)], and let us put

$$\tau_\chi(a) = \chi(s(a))\,\tau(a) \qquad (a \in K).$$

We claim, that τ_χ *is a unitary representation of* K *on* $\mathbf{H}(\pi)$. To this end, with the notations just used let us note, that

$$\tau_\chi(a)\tau_\chi(b) = \chi(rt)\,\tau(a)\tau(b) = \chi(rt)\,\overline{\chi_f(l(rt))}\,\tau(ab) = \chi(s(rt))\,\tau(ab),$$

whence the desired conclusion follows by observing, that $s(ab) = s(rt)$.

LEMMA 7.1. — *With the above notations the representations* τ_χ *and* ind $(\mathfrak{h}, \chi g;\ K)$ [*cf.* 4 (*e*)] *are unitarily equivalent.*

Proof. — For notations unexplained below the reader is referred to 4 (*a*) and 4 (*e*) resp..

a. Putting $A = \overline{G}_g D$ we get $A_0 = \exp(\mathfrak{g}_g + d)$; also $K_0 = \exp(\mathfrak{g}_g + \mathfrak{d})$. In this fashion, if a belongs to A we have

$$\Delta_A(a) = \det(\mathrm{Ad}(a) \mid (\mathfrak{g}_g + d)) \qquad \text{and} \qquad \Delta_K(a) = \det(\mathrm{Ad}(a) \mid (\mathfrak{g}_g + \mathfrak{d}))$$

and thus

$$\eta(a) = \Delta_A(a)/\Delta_K(a) = \det(\mathrm{Ad}(a^{-1}) \mid \mathfrak{d}/d) = \psi(a).$$

Hence there is a Borel measure $dw(q)$ on K/A satisfying $dw(qk) = \psi(k)\,dw(q)$, and if h is some element of $\mathcal{H}$ we have

$$\mu_h(K/A) = \int_{K/A} (|h(x)|^2/\psi(x))\,dw(q).$$

b. We write again $\overline{G}_g L_f = \bar{S}L_f$ [*cf.* (*b*) in the proof of Proposition 6.1]. Let $dv(p)$ be an element of the invariant measure on L/D.

For g in $\mathcal{H}_0$ we define the function G (x) on K by

$$G(x) \equiv (\psi(t))^{1/2} \chi(t)\, g(y) \qquad (x = ty;\, t \in \bar{S},\, y \in L).$$

To establish our Lemma, it will be enough to prove, that : A. G belongs to $\mathcal{H}$ and, if $dw(q)$ is properly normalized, $\mu_G(K/A) = \mu_g(L/D)$ for all g in $\mathcal{H}_0$; B. If $a = rn$ $(r \in \bar{S},\, n \in L)$ is a fixed element of K and we put

$$g'(y) \equiv \chi(r)(\tau'(a)\, g)(y) \equiv \chi(r)\, \psi(a))^{1/2}\, g(r^{-1} ya) \qquad (y \in L),$$

then $G'(x) \equiv G(xa)$ $(x \in K)$.

c. Let us prove first assertion (A) formulated above.

1° We claim, that if a is some element of A, we have

$$G(ax) \equiv (\psi(a))^{1/2}\, \varphi(a)\, G(x)$$

on K. In fact, if $a = rm$, $x = ty$ $(r,\ t \in \bar{S};\ m \in D,\ y \in L)$ then $ax = s(rt)[l(rt)\, t^{-1}\, mt.y]$; in this fashion

$$G(ax) = (\psi(s(rt)))^{1/2} \chi(s(rt))\, g(l(rt)\, t^{-1}\, mt.y).$$

Since g belongs to $\mathcal{H}_0$ we have

$$g(l(rt)\, t^{-1}\, mt.y) = \chi_f(l(rt))\, \chi_f(m)\, g(y)$$

and thus

$$G(ax) = (\psi(r))^{1/2} \chi(r)\, \chi_f(m)\, (\psi(t))^{1/2} \chi(t)\, g(y).$$

Since $\psi(r) = \psi(a)$ and $\chi(r)\, \chi_f(m) = \varphi(a)$ [*cf.* 4 (*e*)] we conclude finally, that

$$G(ax) = (\psi(a))^{1/2}\, \varphi(a)\, G(x).$$

2° We show next that if, for given $dv(p)$, the measure $dw(q)$ on K/A is appropriately normalized, then we have

$$\mu_G(K/A) = \int_{K/A} (|G(x)|^2/\psi(x))\, dw(q) = \int_{L/D} |g(y)|^2\, dv(p) = \mu_g(L/D).$$

Since $\bar{G}_g$ normalizes D, $\bar{G}_g$ operates on L/D by the rule

$$pb = Db^{-1} xb = b^{-1} Dxb \qquad (p = Dx,\, b \in \bar{G}_g).$$

Reasoning as in (*a*) of the proof of Lemma 5.2 we show, that $dv(pb) = \psi(b)\, dv(p)$ $(b \in \bar{G}_g)$. Let us denote by γ the homeomophism from L/D onto K/A which assigns A x to D x $(x \in L)$. One sees at once, that if b and l are elements of $\overline{G}_g$ and L resp., we have

$$\gamma(pb) = \gamma(p)\, b \qquad \text{and} \qquad \gamma(pl) = \gamma(p)\, l.$$

Therefore the image $dv'(q)$ of $dv(p)$ under γ satisfies $dv'(qk) = \psi(k)\, dv'(q)$ for all k in K, and hence we can assume, that it coincides with $dw(q)$. To obtain the desired conclusion it is enough to remark that, writing $H(q)$ and $h(p)$ for the functions $|G(x)|^2/\psi(x)$ and $|g(y)|^2$ on K/A and L/D resp., we have $H(\gamma p) \equiv h(p)\ (p \in L/D)$.

3° To complete the proof of assertion (A) in (b), it suffices now to show that, for any fixed a_0 in K the map $hk \mapsto \chi_f(h)\, G(ka_0)$ $(h \in H,\ k \in E)$ is holomorphic on $HE \subset E_{\mathbf{C}}$. Assuming $a_0 = rm$ $(r \in \overline{S},\ m \in L)$ we get

$$ka_0 = krm = r(r^{-1}kr)\, m,$$

and thus

$$G(ka_0) = (\psi(r))^{1/2}\, \chi(r)\, G(r^{-1}kr.m)$$

and

$$\chi_f(h)\, G(ka_0) = (\psi(r))^{1/2}\, \chi(r)\, \chi_f(r^{-1}hr)\, G(r^{-1}kr.m)$$

from where the conclusion follows as in (a), Lemma 5.2.

d. We complete our proof of Lemma 7.1 by establishing assertion (B) in (b) above. If $a = rm$ and $x = ty$ $(r, t \in \overline{S};\ m, y \in L)$ we have

$$xa = s(tr)\, [l(tr)\, t^{-1} ya];$$

hence

$$\begin{aligned} G(xa) &= (\psi(s(tr)))^{1/2}\, \chi(s(tr))\, g(l(tr)\, r^{-1} ya) \\ &= (\psi(t))^{1/2}\, (\psi(a))^{1/2}\, \chi(s(tr))\, \chi_f(l(tr))\, g(r^{-1} ya) \\ &= (\psi(t))^{1/2}\, \chi(t)\, [\chi(r)\, (\psi(a))^{1/2}\, g(r^{-1} ya)] = (\psi(t))^{1/2}\, \chi(t)\, g'(y) = G'(x). \end{aligned}$$

Q. E. D.

LEMMA 7.2. — *With the above notations assume, that* $\mathfrak{h}_j = \mathrm{pol}(f)$ *is such that* $\overline{G}_g\, \mathfrak{h}_j \subset \mathfrak{h}_j$ $(j = 1, 2)$. *Then the unitary representations* ind $(\mathfrak{h}_1, \chi, g;\ K)$ *and* ind $(\mathfrak{h}_2, \chi, g;\ K)$ *are unitarily equivalent.*

Proof. — Let us denotes by M the Lie group defined on the set $\{(b, l);\ b \in \overline{G}_g,\ l \in L\}$ by the law of multiplication

$$(b, l)(b_1, l_1) = (bb_1,\ b_1^{-1}\, lb_1\, l_1).$$

We assume first, that $\mathcal{H}_0$ and π are as in the proof of Lemma 7.1. For g in $\mathcal{H}_0$ and $m = (b, l) \in M$ let us write $(U'(m)\, g)(x) \equiv (\psi(b))^{1/2}\, g(b^{-1} xbl)$.

Proceeding as in (*a*) of the proof of Lemma 5.2 one shows easily, that $(U'(m)\,g)(x)$ lies again in $\mathcal{H}_0$ such that its norm is equal to that of g. Denoting by $U(m)$ the corresponding unitary operator on $\mathbf{H}(\pi)$, a simple computation proves, that the map $m \mapsto U(m)$ is a continuous unitary representation of M. We observe finally, that if τ is as at the start of this section we have for $a = rm$ $(r \in \bar{S},\ m \in L)$ that $\tau(a) = U(r, m)$.

Let us repeat the above construction by substituting in place of $\mathfrak{h}$ the subalgebras $\mathfrak{h}_j$ $(j = 1, 2)$ of our lemma; we denote by U_j $(j = 1, 2)$ the representations of M arising in this fashion. Next we make use of the crucial fact, established in [1] (*cf.* Theorem III.3.1, *loc. cit.*) that U_1 and U_2 are unitarily equivalent. Bearing in mind the connection, just pointed out, between τ and U we conclude from this, that the unitary equivalence class of the projective representation τ of K is not affected by a change of $\mathfrak{h} = \mathrm{pol}(f)$ employed in its construction. Since by definition $\tau_\chi(a) = \chi(t)\,\tau(a)$ $[t = s(a)]$ the same observation applies to τ_χ, and hence the assertion of our lemma is implied by Lemma 7.1.

Q. E. D.

Corollary 7.1. — *With the previous notations, the representations* $\mathrm{ind}(\mathfrak{h}_1, \chi, g)$ *and* $\mathrm{ind}(\mathfrak{h}_2, \chi, g)$ *are unitarily equivalent.*

Proof. — By virtue of Lemma 7.2, it suffices to observe, that

$$\mathrm{ind}(\mathfrak{h}_j, \chi, g) = \underset{K \uparrow G}{\mathrm{ind}}\,(\mathrm{ind}(\mathfrak{h}_j, \chi, g;\ K)) \qquad (j = 1, 2) \quad [cf. 4(g)].$$

Remark 7.1. — Before proceeding we summarize some notations and results of Section 3. As above (*cf.* the observations preceeding Proposition 6.1) we assume, that *loc. cit.* we have $L = [G, G]$. Given $\pi \in \hat{L}$ we denote by K_π and $\mathcal{U}_\pi$ the group K and $\mathcal{U}$ resp. as in Lemma 3.6. We set $\mathfrak{G}(\pi) = \{\rho;\ \rho \in \hat{K}_\pi,\ \rho \mid K_\pi = \pi\}$; $\mathfrak{G}(\pi)$ is nonempty and if ζ is a concrete representation of the class $\rho \in \hat{K}_\pi$, $\underset{K_\pi \uparrow G}{\mathrm{ind}}\,\zeta$ is a semifinite factor representation, the type of which is I if and only if the group $\mathcal{U}_\pi/(G_\pi)_0\,K$ is finite (*cf.* Lemma 3.7). We put $\mathfrak{G} = \cup_{\pi \in \hat{L}}\, \mathfrak{G}(\pi)$ and define the map $\eta : \mathfrak{G} \mapsto \overset{\frown}{G}$ (= set of all quasi-equivalence classes of factor representations of G; *cf.* [12], 18.6.2, p. 323) by $\eta(\rho)$ = quasi-equivalence class of $\underset{K_\pi \uparrow G}{\mathrm{ind}}\,\zeta$ $[\zeta \in (\{\rho\})_c,\ \rho \in \mathfrak{G}(\pi)]$. $\mathfrak{G}$ is a transformation space of G and we have $\eta(\rho_1) = \eta(\rho_2)$ $(\rho_j \in \mathfrak{G},\ j = 1, 2)$ if and only if there is an element a in G such that $a\,\rho_1 = \rho_2$ (*cf.* Proposition 3.1).

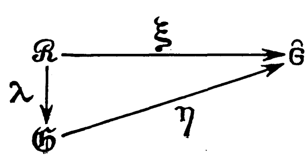
If π is the unitary equivalence class of Ind $(\mathfrak{h}, f)$ [*cf.* 4 (f)] then, by Proposition 6.1, if g is an arbitrary element of $\mathfrak{g}'$ with $\mathfrak{g} \mid \mathfrak{d} = f$, we have $K_\pi = \overline{G}_g\, L$ and $\mathfrak{U}_\pi/(G_\pi)_0\, K_\pi = G_g/\overline{G}_g$. Given $g \in \mathfrak{g}'$, unless specified otherwise, π will stand for the unitary equivalence class corresponding to $L f$ [*cf.* 4 (f); $f = g \mid \mathfrak{d}$]; in particular usually *we shall not explicitely indicate the relation between* g *and* π.

Lemma 7.3. — *Let* g *be an element of* $\mathfrak{g}'$ *with* $f = g \mid \mathfrak{d} \neq 0$, $\mathfrak{h} = \mathrm{pol}\,(f)$ *satisfying* $\overline{G}_g\, \mathfrak{h} \subset \mathfrak{h}$ *and* $\chi \in \hat{\hat{G}}_g$. *Then* ind $(\mathfrak{h}, \chi, g)$ *is a semifinite factor representation. It is of type* I *if and only if the group* $G_g/\overline{G}_g$ *is finite.*

Proof. — Since, if ind $(\mathfrak{h}, \chi, g;\, K_\pi) = \zeta$, we have $\underset{K_\pi \uparrow G}{\mathrm{ind}}\, \zeta = \mathrm{ind}\,(\mathfrak{h}, \chi, g)$ [*cf.* 4 (g)], by virtue of Remark 7.1 it is enough to establish, that $\zeta \mid L \in (\{\pi\})_c$. This, however, is implied at once by Lemma 7.1.

Q. E. D.

Remark 7.2. — Let us write $\mathcal{R}$ for set $\bigcup_{g \in \mathfrak{g}'} \hat{\hat{G}}_g$; an element p in $\mathcal{R}$ is determined by a pair (g, χ) $\left(g \in \mathfrak{g}', \chi \in \hat{\hat{G}}_g\right)$. For a complex subalgebra $\mathfrak{h}$ of $\mathfrak{d}_{\mathbf{C}}$ we put $\mathfrak{h} = \mathrm{pol}\,(p)$, if $\mathfrak{h} = \mathrm{pol}\,(g \mid \mathfrak{d})$ and $\overline{G}_g\, \mathfrak{h} \subset \mathfrak{h}$. Also, we shall write, with such an $\mathfrak{h}$, ind $(\mathfrak{h}, p)$ and ind $(\mathfrak{h}, p;\, K_\pi)$ in place of ind $(\mathfrak{h}, \chi, g)$ and ind $(\mathfrak{h}, \chi, g;\, K_\pi)$ resp.. By virtue of Corollary 7.1 and Lemma 7.3, the quasi-equivalence class of ind $(\mathfrak{h}, p)$ is an element, well determined by $p \in \mathcal{R}$, of $\hat{G}$; we shall denote it by $\xi\,(p)$. Similarly, ind $(\mathfrak{h}, p;\, K_\pi)$ is an element of $\left(\hat{\hat{K}}_\pi\right)_c$, the unitary equivalence class of which belongs to $\mathfrak{G}\,(\pi)$ and depends on p only (*cf.* the proof of Lemma 7.3); we denote it by $\lambda\,(p)$. If $g \mid \mathfrak{d} = 0$, $K_\pi = G$ and we define $\lambda\,(p)$ as the element, corresponding to χ_g, of G. For $p = (g, \chi) \in \mathcal{R}$ and $a \in G$ we put $ap = (ag,\ a\chi)$ $\left[a\chi \in \hat{\hat{G}}_{ag}\right.$ is defined by $(a\chi)(b) = \chi\,(a^{-1}ba)$, $b \in \overline{G}_{ag}$; *cf.* 4 $\left.(h)\right]$; we have obviously $a\,(bp) = (ab)\,p$ $(a, b \in G)$. With these notations we have $\lambda\,(ap) = a\,\lambda\,(p)$. In fact, this follows at once from Remark 4.1 substituting the inner automorphism $b \mapsto a.ba^{-1}$ $(b \in G)$ in place of α *loc. cit.* Let us observe finally, that by what we saw above, the diagramm



is commutative.

LEMMA 7.4. — *For* $\rho \in \mathfrak{G}(\pi)$, *the set* $\overset{-1}{\lambda}(\rho)$ *is an orbit of* $\mathcal{U}_\pi$.

Proof. — Suppose, that $p_j = (g_j, \chi_j)$ are elements in $\overset{-1}{\lambda}(\rho)$ $(j = 1, 2)$; we are going to show the existence of an element u in $\mathcal{U}_\pi$ such that $up_1 = p_2$. Let us assume first, that $g_1 \mid \mathfrak{d} = g_2 \mid \mathfrak{d}$ $(= f$, say). Since, if $f = 0$ our statement is trivial, we shall assume, that $f \neq 0$. We claim, that in this case $\chi_1 = \chi_2$. In fact, we have $G_{g_1} = G_{g_2}$ $(= B$, say; *cf.* Remark 6.1). Let ω' be a character of B such that $\chi_2 = \omega' \chi_1$. We have then $\omega' \mid (L \cap G_{g_1}) \equiv 1$; let us denote by ω the character of $K_\pi = BL$ such that $\omega \mid B \equiv \omega'$ and $\omega \mid L \equiv 1$. We infer easily from the proof of Lemma 7.1, that if $\mathfrak{h} = \text{pol}(p_j)$ $(j = 1, 2)$ then $\text{ind}(\mathfrak{h}, p_2; K_\pi)$ is unitarily equivalent to $\omega . \text{ind}(\mathfrak{h}, p_1; K_\pi)$. Since however, by $\lambda(p_1) = \lambda(p_2)$, it is also unitarily equivalent to $\text{ind}(\mathfrak{h}, p_1; K_\pi)$ we conclude, that $\omega' = 1$ and thus $\chi_1 = \chi_2$. We write $\mathfrak{k}_\pi = \mathfrak{d} + \mathfrak{g}_{g_1} = \mathfrak{d} + \mathfrak{g}_{g_2}$ and observe, that $(K_\pi)_0 = \exp(\mathfrak{k}_\pi)$. We have $g_1 \mid \mathfrak{k}_\pi = g_2 \mid \mathfrak{k}_\pi$ since

$$i(g_1 \mid \mathfrak{k}_\pi) = d(\chi_1 \mid (K_\pi)_0) = d(\chi_2 \mid (K_\pi)_0) = i(g_2 \mid \mathfrak{k}_\pi).$$

Hence, by virtue of Lemma 6.2, there is an element b of $(G_f)_0$ with $bg_1 = g_2$. Let us add, that in this case also $b\chi_1 = \chi_1 = \chi_2$. In fact, we have first $b\,\overline{G}_{g_1}\,b^{-1} = \overline{G}_{bg_1} = \overline{G}_{g_2} = \overline{G}_{g_1}$. In this fashion it is enough to establish, that for all δ in G_g : $b\,\delta b^{-1}\,\delta^{-1} \in \mathring{G}_{g_1} = \ker(\chi_{g_1} \mid (G_g)_0)$, which is true, if we can show $b\,\delta b^{-1}\,\delta^{-1} \in \mathring{L}_f = \ker(\chi_f \mid L_f)$ (*cf.* the proof of Lemma 6.4). But with notations as in (*b*) of the proof of Proposition 6.1 we have $\sigma(G_g) = \Lambda_0$, and $\sigma(\overline{G}_g) \subset \sigma(G_g L_f) = \Lambda_1$ whence we conlude, that $[(G_f)_0, \overline{G}_{g_1}] \subset \ker \sigma = \mathring{L}_f$. We recall (*cf.* Remark 6.2), that $\mathcal{U}_\pi = (G_f)_0\, G_g\, L$. Hence, summing up, we have shown, if $g_1 \mid \mathfrak{d} = g_2 \mid \mathfrak{d}$, that p_1 and p_2 lie on the same $\mathcal{U}_\pi$ orbit. From here we settle the general case as follows. Writing again $\sim$ to indicate unitary equivalence, if $\mathfrak{h}_j = \text{pol}(p_j)$ $(j = 1, 2)$ we have by assumption $\text{Ind}(\mathfrak{h}_1, f_1) \sim \text{ind}(\mathfrak{h}_1, p_1; K_\pi) \mid L \sim \text{ind}(\mathfrak{h}_2, p_2; K_\pi) \mid L \sim \text{Ind}(\mathfrak{h}_2, f_2)$, and thus $\text{Ind}(\mathfrak{h}_1, f_1) \sim \text{Ind}(\mathfrak{h}_2, f_2)$. Hence, by 4 (*f*), there is an element l of L such that replacing, if necessary, p_1 by lp_1, we have $g_1 \mid \mathfrak{d} = g_2 \mid \mathfrak{d}$ and therefore we can complete our proof as above.

Q. E. D.

LEMMA 7.5. — *With the previous notations we have* $\xi(p_1) = \xi(p_2)$ *if and only if* p_1 *and* p_2 *lie on the same orbit of* $\mathcal{R}$.

Proof. — The condition being evidently sufficient, let us prove its necessity. If $\xi(p_1) = \xi(p_2)$ we have $\eta(\lambda(p_1)) = \eta(\lambda(p_2))$ and hence, by Propositon 3.1, there is an element a of G with $a\lambda(p_1) = \lambda(p_2)$ in $\mathfrak{G}$ But $a\lambda(p_1) = \lambda(ap_1)$ and thus our statement is implied by lemma 7.4.

Q. E. D.

We sum up the main conclusions of the previous considerations as follows.

THEOREM 1. — *Let* G *be a connected and simply connected solvable Lie group with the Lie algebra* $\mathfrak{g}$. *Let* $p = (g, \chi)$ *be an element of* $\mathcal{R}$ (*cf.* Remark 7.2) *such that* $g \mid \mathfrak{d} \neq 0$ ($\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$). *Then the unitary representation* ind $(\mathfrak{h}, p)$ *of* G, *for any choice of* $\mathfrak{h} = \mathrm{pol}\,(p)$ (cf. *loc. cit.*) *is a semifinite factor representation. It is of type* I *if and only if the index of the reduced stabilizer* $\overline{G}_g$ [cf. 4 (*c*)] *in the stabilizer* G_g *of* g *is finite. For* $p_j \in \mathcal{R}$ *and* $\mathfrak{h}_j = \mathrm{pol}\,(p_j)$ ($j = 1, 2$) *the representation* ind $(\mathfrak{h}_1, p_1)$ *is quasi-equivalent to* ind $(\mathfrak{h}_2, p_2)$ *if and only if* p_1 *and* p_2 *lie on the same orbit in* $\mathcal{R}$; *in this case they are also unitarily equivalent. Finally,* ind $(\mathfrak{h}, p)$ *on* $L = [G, G]$ *restricts to the transitive quasi-orbit corresponding to* $L\,(g \mid \mathfrak{d})$ *in* $\hat{L}$.

CHAPTER II.

GENERALIZED ORBITS OF THE COADJOINT REPRESENTATION.

SUMMARY. — The factor representations obtained in Chapter I provide a central decomposition of the regular representation only if sufficiently many orbits of G on $\mathfrak{g}'$ are locally closed. This is certainly so, if G is of type I, but the group of Dixmier (*cf.* Summary, Chapter I) shows, that this can very well be the case even if the representation, belonging to an element of $\mathfrak{g}'$ in the general position, of the transitive theory is not of type I. On the other hand, for the group of Mautner, which is the connected and simply connected solvable Lie group corresponding to the Lie algebra spanned over the reals by the elements $\{ e_j;\ 1 \leq j \leq 5 \}$ with the monvanishing brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_1, e_4] = \theta e_5, \quad [e_1, e_5] = -\theta e_4, \quad (\theta = \text{irrational}).$$

all representations as in Chapter I are irreducible, but, disregarding a variety of lower dimension in $\mathfrak{g}'$, no orbit is locally closed. A closer inspection of the central decomposition of the regular representation for this group strongly suggests, that in the general case one might obtain the " central components " by forming continuous direct sums of appropriate groups of the representations of the transitive theory. The purpose of this and the next chapter is the verification of this conjecture. More specifically, in the present chapter we define the geometrical principles of this grouping, which will be done by introducing an appropriate G invariant equivalence relation on $\mathcal{R} = \cup_{g \in \mathfrak{g}'} \hat{G}_g$ (for the latter cf. *loc. cit.* or Section 7, Chapter I). We recall, incidentally, that the factor representations of the previous chapter are parametrized by $\mathcal{R}/G$. In Section 2 we establish the existence of an equivalence relation $\mathfrak{R}$ on $\mathfrak{g}'$, uniquely determined by the conditions, that any of its orbit be G invariant and locally closed, and that any G orbit be dense in it. For a type I

group the $\mathfrak{R}$ orbits are simply the orbits of the coadjoint representations. In the case of the group of Mautner, the orbits of $\mathfrak{R}$ are the closures of the orbits of the coadjoint representation. Let $\mathfrak{O}$ be an orbit of $\mathfrak{R}$ and let us put $\mathfrak{B}(\mathfrak{O}) = \cup_{g \in \mathfrak{O}} \hat{G}_g$. In Section 5 we show, that $\mathfrak{B}(\mathfrak{O})$ can in a natural fashion be endowed with a topology, which makes it a principal bundle, with a structure group isomorphic to a torus of the dimension of the rank of $\overline{G}_g/(G_g)_0$ (g arbitrary in $\mathfrak{O}$), over $\mathfrak{O}$, and G acts on $\mathfrak{B}(\mathfrak{O})$ as a group, the action of which commutes with that of the structure group. We also show, that the torus bundle $\mathfrak{B}(\mathfrak{O})$ is trivial. Let $p_0(g) : \mathfrak{O} \to \mathfrak{B}(\mathfrak{O})$ be a cross section. If a is some element in G, there is an element $\mu(a, g)$ of the structure group, such that we have $ap_0(g) = \mu(a, g)p_0(ag)$ for all $g \in \mathfrak{O}$. In Section 6 we show, that for an appropriate choice of our cross section $\mu(a, g)$ is independent of g. We use this in Section 7 to prove, that the collection of the closures of G orbits defines an equivalence relation $\mathfrak{S}$ on $\mathfrak{B}(\mathfrak{O})$. It will be the purpose of the next chapter to establish, that the equivalence relation $\mathfrak{S}$ defined on $\mathcal{R} = \cup_{g \in \mathfrak{g}'} \hat{G}_g$ by the union of all $\mathfrak{S}$ orbits for all possible choice of $\mathfrak{O}$ in $\mathfrak{g}'/\mathfrak{R}$, will have the property indicated before. Let us observe, incidentally, that the $\mathfrak{S}$ orbits are homogeneous spaces of connected solvable Lie groups. Let τ be the canonical projection from $\mathfrak{B}(\mathfrak{O})$ onto $\mathfrak{O}$. If $\mathfrak{O}$ is acted upon transitively by G, which is always the case if G is of type I, then for any $\mathfrak{S}$ orbit O in $\mathfrak{B}(\mathfrak{O})$, (O, τ) is a simple convering of $\mathfrak{O}$. (This is so also for the group of Mautner, but has for reason the triviality of the structure groups) In Section 8 we show on the example of a group of twelve dimensions, that in the general case the situation is completely different. We construct examples of $\mathfrak{B}(\mathfrak{O})$, such that the structure group is onedimensional and that either $\mathfrak{B}(\mathfrak{O})$ itself is a $\mathfrak{S}$ orbit or, for any $\mathfrak{S}$ orbit O in $\mathfrak{B}(\mathfrak{O})$, (O, τ) is a finite covering of $\mathfrak{O}$, and the degree of the covering can be prescribed.

1. In the following V will denote a finite dimensional vector space over the field of the real or complex numbers. We shall write $\mathfrak{g}$ for a nilpotent Lie algebra over the same ground field; $G = \exp \mathfrak{g}$ will stand for the corresponding connected and simply connected Lie group.

We recall, that a linear representation of G on V is called unipotent, if the range of its differential is composed of nilpotent operators only. Let F be a subset of V and $a(p)$ some compex valued function on F. We shall say, that $a(p)$ is locally rational on F, if for any point p_0 of F there are polynonials $P(x, p_0)$ and $Q(x, p_0)$ on V, such that $Q(p_0, p_0) \neq 0$, and for some neighborhood U of p_0 in V, where $Q \mid U \neq 0$, on $U \cap F$ $a(p)$ coincides with the rational function $P(x; p_0)/Q(x; p_0)$. Note that, in particular, a locally rational function on F is continuous in the relative topology of the latter.

With the previous terminology we have

PROPOSITION 1.1. — *Suppose, that* G *acts via a unipotent representation on* V. *Let* $V = V_0 \supset V_1 \supset \ldots \supset V_m = (0)$ *be a Jordan-Hölder sequence for* G, *and assume, that* $v_j \in V_{j-1} - V_j$ $(1 \leq j \leq m)$. *Then there is a sequence of subsets* $V = F_0 \supset F_1 \supset \ldots \supset F_M \neq (0)$ *with the following properties. For any* j $(1 \leq j \leq M)$ F_j *is* G *invariant and the dimension of any*

G *orbit in* $F_{j-1} - F_j$ *is constant;* F_M *is the collection of all fixed points of* G. *Let* j *be fixed as above, and let us denote by* d *the* (positive) *dimension of some* G *orbit in* $F_{j-1} - F_j$. *There is a subset* $0 < j_1 < \ldots < j_d < m$ of $\{1, 2, \ldots, m\}$ *and a system of* m *functions* $\{P_j(z; x); 1 \leq j \leq m\}$ *on* $\mathbf{K}^d \times (F_{j-1} - F_j)$ ($\mathbf{K}$ being the ground field under consideration) *with the following properties :* 1° *For any fixed* x *in* $F_{j-1} - F_j$, $P_j(z, x)$ *is a polymonial in* $z = (z_1, z_2, \ldots, z_d) \in \mathbf{K}^d$, *and the coefficients are locally rational functions on* $F_{j-1} - F_j$; 2° *We have* $P_{j_k}(z; x) \equiv z_k$ $(1 \leq k \leq d)$; 3° *If* j *is some integer between* 1 *and* m *and* $t = \sup_{j_k \leq j} k$, $P_j(z, x)$ *depends on* $\{z_h; h \leq t\}$ *only;* 4° *For any* x *in* $F_{j-1} - F_j$ *we have*

$$G\,x = \left\{ v;\ v = \sum_{j=1}^{m} P_j(z; x).v_j,\ z \in \mathbf{K}^d. \right\}.$$

For each x, *the functions* $\{P_j(z; x); 1 \leq j \leq m\}$ *are uniquely determined by the condition, that they be polynonials in* $z \in \mathbf{K}^d$ *satisfying conditions* 2°, 3° *and* 4°.

Remark. — We are going to make the description of the above situation much more precise in the case, when the representation in question is the coadjoint representation of G (*cf.* Section 4 below).

Proof. — Given a in G and v in V, we shall write av for the action of a on v. In the following, to take a definite case, we shall assume, that $\mathbf{K} = \mathbf{R}$; the case $\mathbf{K} = \mathbf{C}$ can be settled similarly.

a. Given l in $\mathfrak{g}$, let us put $lv = d/dt\,[\exp(tl)]\,v\,|_{t=0}$ $(v \in V)$. We denote by π_j the canonical projection from V onto V/V_j $(0 \leq j \leq m)$. Let us write $\mathfrak{g}_{\pi_j(x)}$ for the Lie algebra of the stable group of $\pi_j(x)$ $(x \in V)$ with respect to the action of G on V/V_j. We have

$$\mathfrak{g}_{\pi_j(x)} = \{l; lx \equiv 0\ (V_j)\}, \qquad \mathfrak{g} = \mathfrak{g}_{\pi_0(x)} \supseteq \mathfrak{g}_{\pi_1(x)} \supseteq \ldots \supseteq \mathfrak{g}_{\pi_m(x)} = \mathfrak{g}_x$$

and $\dim \mathfrak{g}_{\pi_{j-1}(x)}/\mathfrak{g}_{\pi_j(x)} = 1$ or 0.

We denote by $f(x)$ the function from V into the collection of all subsets of $\{1, 2, \ldots, m\}$ (empty subset included) defined in the following fashion : j belongs to $f(x)$ if and only if $\mathfrak{g}_{\pi_{j-1}(x)} \supsetneq \mathfrak{g}_{\pi_j(x)}$. Let $\mathcal{E}$ be the range of f. For $e \in \mathcal{E}$, we shall write $d(e)$ for the number of elements (≥ 0) in e. Observe, that $d(f(x)) = \dim(\mathfrak{g}/\mathfrak{g}_x)$, and thus we have $d(f(x)) = \dim o(x)$, where we set $o(x) = G\,x$. Let us put, for some e in $\mathcal{E}$, $\mathfrak{O}_e = \{x; f(x) = e\}$; then $\mathfrak{O}_e$ is invariant under G. In fact, if we write, for a in G and l in $\mathfrak{g}$, $al = \mathrm{Ad}(a).l$, then $a \exp(l)\,a^{-1} = \exp(al)$, and $\mathfrak{g}_{\pi_j(ax)} = a\mathfrak{g}_{\pi_j(x)}$, proving our statement.

b. Let us fix an element e of $\mathcal{E}$. Our next objective will be to establish the existence of a joint parametrization for the G orbits in $\mathfrak{O}_e$, as indicated in the above proposition for $F_{j-1} - F_j$. Evidently, we can assume $d(e) > 0$.

Let $\{ l_j;\ 1 \leqq j \leqq L = \dim \mathfrak{g} \}$ be a basis in $\mathfrak{g}$, $\{ v'_j;\ 1 \leqq j \leqq m \}$ a basis in V′, such that $(v_i, v'_j) = \delta_{ij}\ (1 \leqq i, j \leqq m)$. Let us put $a_{ik}(x) = (l_k x, v'_i)$, and $A_j(x)$ for the $j \times L$ matrix $\{ a_{ik}(x);\ 1 \leqq i \leqq j,\ 1 \leqq k \leqq L \}$ $(x \in \mathfrak{O}_e$ fixed). We have by definition $j \in e = f(x)$ if and only if there is l in $\mathfrak{g}$ such that $lx \equiv v_j\ (V_j)$ implying, that j belongs to e if and only if

$$\operatorname{rank} A_j(x) = \operatorname{rank}(A_{j-1}(x)) + 1.$$

Assuming $e = \{ 0 < j_1 < j_2 < \ldots < j_d \leqq m \}\ [d = d(e) > 0]$ we have in this fashion $\operatorname{rank}(A_{j_r}(x)) = r\ (1 \leqq r \leqq d)$. Let

$$M_r(r) = \{ b_{ik}^{(r)}(x);\ 1 \leqq i, k \leqq r \}$$

be an $r \times r$ nonsingular submatrix of $A_{j_r}(x)$, $\{ y_j^{(r)}(v);\ 1 \leqq j \leqq r \}$ be rational functions on V such that $\sum_j b_{ij}^{(r)}(v)\, y_j^{(r)}(v) \equiv \delta_{ir}\ (v \in V)$, and let us put

$$L_r(x) = \sum_j y_j^{(r)}(x)\, l_{\alpha_j},$$

where we have denoted by $0 < \alpha_1 < \ldots < \alpha_r \leqq L$ the coloumn indices of $A_{j_r}(x)$, corresponding to $M_r(x)$. Observe, that by virtue of our construction, $L_r(x)\, x \equiv v_{j_r}\ (V_{j_r})$ and $\{ L_{r+1}(x), \ldots, L_d(x) \}$ is a supplementary basis in $\mathfrak{g}_{\pi_{j_r}(x)}$ to $\mathfrak{g}_x$.

c. Let us put

$$g_j(t; x) = \exp(t\, L_j(x)) \qquad [t \in \mathbf{R};\ 1 \leqq j \leqq d],$$
$$T = (t_1, t_2, \ldots, t_d) \in \mathbf{R}^d \qquad \text{and} \qquad g(T; x) = g_1(t_1; x)\, g_2(t_2; x) \ldots g_d(t_d; x).$$

With these notations we have

$$G\, x = o(x) = \{ y;\ y = g(T, x)\, x,\ T \in \mathbf{R}^d \}.$$

To this end it is enough to show, that if $\bar{\mathfrak{g}}$ is a subalgebra of codimension 1 of $\mathfrak{g}$, and if $l \in \mathfrak{g} - \bar{\mathfrak{g}}$ and $\overline{G} = \exp \bar{\mathfrak{g}}$, the map $\Phi : \mathbf{R} \times \overline{G} \to G$ defined by $\Phi(t, \bar{g}) = \exp(tl)\, \bar{g}$ is a homeomorphism. Through a repeated application of the said assertion we can then conclude, thet the map $\Phi_1 : \mathbf{R}^d \times G_1 \to G$ $(G_1 = \exp \mathfrak{g}_x)$ defined by $\Phi_1(T, g) = g(T; x)\, g$ is a homeomorphism. Let $\mathfrak{g} = \mathfrak{g}_L \supset \mathfrak{g}_{L-1} \supset \ldots \supset \mathfrak{g}_0 = 0$ be a Jordan-Hölder sequence for $\mathfrak{g}$

such that $\mathfrak{g}_{L-1} = \bar{\mathfrak{g}}$, and assume, that $l_j \in \mathfrak{g}_j - \mathfrak{g}_{j-1}$ $(l_L = l)$. Writing $xy = \log(\exp x . \exp y)$,

$$x = \sum_{j=1}^{m} x_j l_j, \qquad \ldots \qquad (x, y \in \mathfrak{g}),$$

a simple inspection of the Hausdorff-Campbell formula yields that

$$\begin{aligned} (xy)_L &= x_L + y_L \\ &\vdots \\ (xy)_j &= x_j + y_j + \psi_j(x_{j+1}, \ldots, x_L; y_{j+1}, \ldots, y_L) \qquad (1 \leqq j \leqq L - 1) \end{aligned}$$

where the functions $\{\psi_j\}$ are polymonials. Therefore, to obtain the desired conclusion it is enough to observe, that for a given system $\{g_j; 1 \leqq j \leqq L\}$ in $\mathbf{R}$, the set of equations

$$g_L = x_L + 0, \qquad g_{L-1} = 0 + y_{L-1} + \psi_{L-1}(x_L; 0).$$
$$g_{L-2} = 0 + y_{L-2} + \psi_{L-2}(0, x_L; y_{L-1}, 0), \qquad \ldots$$

admits a unique solution in x_L and $\{y_j; 1 \leqq j \leqq L - 1\}$.

d. Let us write $(x \in \mathfrak{O}_e$, fixed) :

$$g(T; x)\, x = \sum_{j=1}^{m} Q_j(T, x)\, v_j.$$

Evidently, the functions $\{Q_j(T; x)\}$ are polynomials in $T \in \mathbf{R}^d$. For a j $(1 \leqq j \leqq m)$ let us put $h = \sup_{j_k \leqq j} k$. We have

$$\pi_j(g(T; x)\, x) = g_1(t_1; x) \ldots g_h(t_h; x)\, \pi_j(x) = \sum_{k=1}^{j} Q_k(T; x)\, \pi_j(v_k).$$

Hence $Q_j(t; x)$ depends only on $(t_1, t_2, \ldots, t_h)$. If $j = j_k$, we observe, that since $L_k(x)\, x \equiv v_{j_k}\, (V_{j_k})$ we obtain

$$g(t_k; x)\, \pi_{j_k}(x) = t_k\, \pi_{j_k}(v_{j_k}) + \pi_{j_k}(x);$$

thus $Q_{j_k}(T; x)$ is of the form $t_k + R_k(t_1, t_2, \ldots, t_{k-1}; x)$, where R_k is a polynomial. Let us set $z_k = Q_{j_k}(T; x)$ $(1 \leqq k \leqq d)$; there is a system of polynomials $\{\psi_k(z_1, z_2, \ldots, z_{k-1}; x); 1 \leqq k \leqq d\}$ such that

$$t_k = z_k + \psi_k(z_1, z_2, \ldots, z_{k-1}; x)$$

and the coefficients of ψ_k are polynomials in those of $\{Q_{j_k}\}$. Substituting these expressions for t_k in the remaining members of the family $\{Q_j\}$, we obtain a system of functions $\{P_j(z; x); 1 \leqq j \leqq m\}$ having the following properties : 1° For each j, $P_j(z;x)$ is a polynomial in $z=(z_1,z_2,\dots z_d)\in\mathbf{R}^d$; 2° We have $P_{j_k}(z;x)\equiv z_k$ $(1\leq k\leq d)$; 3° If $t=\sup_{j_k\leq j} k$, $P_j(z;x)$ depends on $\{z_k; 1\leq k\leq t\}$ only; 4° We have

$$o(x)=\left\{v;\, v=\sum_{j=1}^{m} P_j(z;x)\, v_j,\, z\in\mathbf{R}^d\right\}.$$

In addition, it is clear from our construction, that any coefficient of a product of powers of the z's in P_j, as function of the components of x, is of the form P/Q, where P and Q are polynomials on V $[Q(x)\neq 0]$. If U is a neighborhood of x in V, such that on U all denominators are different from zero, then we have analogous statements, with the same P_j's, for any other element x in $U\cap\mathfrak{O}_e$.

e. Let us assume now, that for a given element x of V there are two sets of polynomials $\{P_j(z); 1\leq j\leq m\}$ and $\{Q_j(z); 1\leq j\leq m\}$ leading to a parametrization of $o(x)$ as above and such that $P_{j_k}(z)\equiv z_k$, $Q_{l_k}(z)\equiv z_k$ $(1\leq k\leq d)$. We claim, that $P_j\equiv Q_j$ $(1\leq j\leq m)$. To this end it evidently suffices to show, that $j_k=l_k$ $(1\leq k\leq d)$. But this follows at once from the observation, that if $t=\sup_{j_k\leq j} k$, we have

$$t=\dim \pi_j(o(x)) \qquad (0\leq j\leq m).$$

To complete the proof of Proposition 1.1, let us assume, that the number of elements in $\mathcal{E}$ [*cf.* (*a*)] is M + 1. To obtain the sets $\{F_j; 1\leq j\leq M\}$ it will be enough to take $F_M=\mathfrak{O}_{e_0}$, e_0 being the empty set in $\mathcal{E}$; otherwise let $\{\mathfrak{O}_k; 1\leq k\leq M\}$ be the family of sets $\{\mathfrak{O}_e; e\neq e_0\}$ arranged in some order, and let us define

$$F_j=(\cup_{k>j}\mathfrak{O}_k)\cup F_M \qquad (0\leq j\leq M-1).$$

Q. E. D.

REMARK 1.1. — Let us assume, that G as in Proposition 1.1 is given as an invariant subgroup of the group A, such that the representation of G considered above arises by resticting to G a representation of A on V. *Let us suppose in addition, that we have* $AV_j\subseteq V_j$ $(0\leq j\leq m)$. *Then we have also* $AF_j\subseteq F_j$ $(0\leq j\leq M)$. In fact, to prove this it suffices to establish, that $f(ax)\equiv f(x)$ for all x in V and a in A. For $a\in A$ and $l\in\mathfrak{g}$ we define al by $a\exp(l)\,a^{-1}=\exp(al)$. Then the desired conclusion is implied by the observation, that $\mathfrak{g}_{\pi_j(ax)}=a\,\mathfrak{g}_{\pi_j(x)}$ $[0\leq j\leq m$; *cf.* $(a)]$.

REMARK 1.2. — Given a Jordan-Hölder sequence $\{V_j\}$, a sequence of elements $\{v_j; 1\leq j\leq m\}$, satisfying $v_j\in V_{j-1}-V_j$, will be referred to as a Jordan-Hölder basis in the sequel. This, by virtue of Proposition 1.1, determines a unique parametrization, with the properties as specified *loc. cit.*, of any orbit G x; we shall call it simply the canonical parametrization and the indices in $f(x)$ [*cf.* (*a*)] the indices of this parametrization.

Assertions, similar to the following Proposition 1.2 have been widely used in the literature but, as far as the present author can ascertain it, these have never been proved.

PROPOSITION 1.2. — *Let* V *be a finite dimensional real vector space,* A *a closed connected subgroup of* GL (V). *Assume, that* A *can be written as* LM, *where* M *is an invariant unipotent subgroup,* L *a closed abelian subgroup of the form* HT, *where* H *and* T *are connected groups of semisimple endomorphism having real and complex eigenvalues of absolute value one resp.. Let x be a fixed element of* V, A_x *the stabilizer of x in* A, *and* $\{a_n\}$ *a sequence of elements in* A *such that* $a_n x \to ax$ $(a \in A)$. *Then* $a_n \to a \bmod (A_x)$

Proof. — *a.* LEMMA 1.1. — *The assertion of Proposition* 1.2 *is valid, if* A *itself is a unipotent group* (that is A = M).

Proof. — With the notations of (*c*) in the proof of Proposition 1.1, if y is in A x, we have $y = g(T, x)\, x$ $(\mathbf{T} \in \mathbf{R}^d)$, and by what we saw in (*d*) *loc. cit.*, T is uniquely determined by y and depends continuously on it.

Q. E. D.

b. LEMMA 1.2. — *Let* $M_{\mathbf{C}}$ *be the complexification of* M *acting on* $V_{\mathbf{C}}$. *Then we have* $(M_{\mathbf{C}}\, x) \cap V = M\, x$.

Proof. — Let $V = V_0 \supset V_1 \supset \ldots \supset V_m = (0)$ be a Jordan-Hölder sequence for M and $v_j \in V_{j-1} - V_j$ $(1 \leq j \leq m)$. Then $\{(V_j)_{\mathbf{C}};\ 0 \leq j \leq m\}$ is a Jordan-Hölder sequence for $M_{\mathbf{C}}$ in $V_{\mathbf{C}}$. Let

$$M\, x = \left\{ y;\ y = \sum_{j=1}^{m} P_j(z)\, v_j,\ z \in \mathbf{R}^d \right\} \quad \text{and} \quad M_{\mathbf{C}}\, x = \left\{ v;\ v = \sum_{j=1}^{m} Q_j(u)\, v_j,\ u \in \mathbf{C}^d \right\}$$

be the corresponding canonical parametrizations (*cf.* Remark 1.2 above). To establish our lemma, it evidently suffices to show, that $P_j(u) \equiv Q_j(u)$ for all $u \in \mathbf{C}^d$ and j. To this end it is enough to prove that the indices of these two parametrizations (cf. *loc. cit.*) coincide. Let π_j be the canonical projection from V onto V/V_j; we denote by the same symbol the canonical projection from $V_{\mathbf{C}}$ onto $V_{\mathbf{C}}/(V_j)_{\mathbf{C}}$. To obtain the desired conclusion, writing $M = \exp \mathfrak{m}$ and remembering what we saw in (*a*) of the proof of Proposition 1.1 it suffices to observe, that evidently

$$(\mathfrak{m}_{\mathbf{C}})_{\pi_j(x)} = (\mathfrak{m}_{\pi_j(x)})_{\mathbf{C}} \qquad (0 \leq j \leq m).$$

Q. E. D.

c. Let $V_{\mathbf{C}} = W_0 \supset W_1 \supset \ldots \supset W_m = (0)$ be a Jordan-Hölder sequence for $LM_{\mathbf{C}}$. Since, by assumption, L consists of semisimple endomorphisms for each j $(1 \leq j \leq m)$ we can determine $w_j \in W_{j-1} - W_j$ such that $lw_j \equiv \varphi_j(l)\, w_j$ $(l \in L)$. Let $\mathfrak{O}_k = F_{k-1} - F_k$ be as in Proposition 1.1, belonging to $M_{\mathbf{C}}$ and $V_{\mathbf{C}}$ in place of G and V *loc. cit.*; we recall (*cf.* Remark 1.1), that $A\,\mathfrak{O}_k \subseteq \mathfrak{O}_k$. We consider the corresponding canonical parametrization $M_{\mathbf{C}}\, x = \left\{ \sum_{j=1}^{m} P_j(u; x)\, w_j;\, u \in \mathbf{C}^d \right\}$ for $x \in \mathfrak{O}_k$ with the indices $0 < j_1 < j_2 < \ldots < j_d \leq m$. Let E be the complement of this set in $\{1, 2, \ldots, m\}$ and for j in E let us put $\lambda_j(x) \equiv P_j(0; x)$. With these notations we have

LEMMA 1.3. — *For any l in* L *and m in* $M_{\mathbf{C}}$ *we have* $\lambda_j(lmx) \equiv \varphi_j(l)\, \lambda_j(x)$ $(x \in \mathfrak{O}_k,\, j \in E)$.

Proof. — Since $M_{\mathbf{C}}\, mx = M_{\mathbf{C}}\, x$, we have by the uniqueness of the canonical parametrization $P_j(z; x) \equiv P_j(z; mx)$ and thus, in particular, $\lambda_j(mx) \equiv \lambda_j(x)$ $(x \in \mathfrak{O}_k, j \in E)$. On the other hand, we have $M_{\mathbf{C}}\, lx = l\, M_{\mathbf{C}}\, x$ and in this fashion for each $u \in \mathbf{C}^d$ there is a $u' \in \mathbf{C}^d$ such that

$$P_j(u; x)\, \varphi_j(l) \equiv P_j(u'; lx) \qquad (1 \leq j \leq m)$$

from where, putting $j = j_k$ $(1 \leq k \leq d)$ we get, that $u'_k = \varphi_j(l)\, u_k$. Therefore finally

$$\lambda_j(lx) \equiv P_j(0; lx) \equiv \varphi_j(l)\, P_j(0; x) \equiv \varphi_j(l)\, \lambda_j(x),$$

completing the proof of our lemma.

Q. E. D.

d. Using the preceeding observations, we can establish Proposition 1.2 in the following fashion. Let us put $L_M = \{ l;\, l \in L,\, lx \in M\,x \}$ (x as in the statement of our proposition); since $M\,x$ is closed in V, L_M is a closed subgroup of L. Assuming $\{a_n\}$, a in A such that $a_n\, x \to ax$ in V, we write $a_n = l_n\, m_n$, $a = lm$ $(m_n,\ m \in M;\ l_n,\ l \in L)$ and observe, that to obtain the conclusion $a_n \to a$ (mod A_x) it suffices to prove, that $l_n \to l$ (mod L_M). In fact, if this is the case we can write $l_n = k_n\, r_n$, $l = kr$ with k_n, k in L and r_n, r in L_M. There are elements p_n and p in M such that $r_n\, m_n\, x = p_n\, x$

and $rmx = px$ and we have $p_n x \to px$ in V. Therefore to arrive at the desired conclusion it is enough to observe, that by Lemma 1.1 $p_n \to p \pmod{M_x}$.

Assuming, that with the notations of (*c*) above, $\mathfrak{O}_k$ contains x, let us denote by E′ the subset of E for which $\lambda_j(x) \neq 0$; we are going to prove, that $L_M = \bigcap_{j \in E'} \ker \varphi_j$. If $l \in L_M$ we have $lx = mx$ $(m \in M)$ and thus by Lemma 1.3, $\lambda_j(lx) = \varphi_j(l)\lambda_j(x) = \lambda_j(x)$ implying $\lambda_j(l) = 1$ $(j \in E')$. Conversely, if l satisfies the last condition we have $\lambda_j(lx) = \varphi_j(l)\lambda_j(x) = \lambda_j(x)$ for all j in E. On the other hand, if y is arbitrary in $\mathfrak{O}_k$, the orbit $M_{\mathbf{C}}y$ intersects the hyperplane $\left\{\sum_{j \in E} u_j w_j\right\}$ in the single point $\left\{\sum_{j \in E} \lambda_j(y) w_j\right\}$. Therefore if $\lambda_j(y_1) = \lambda_j(y_2)$ $(j \in E)$ for a pair of elements y_1 and y_2 in $\mathfrak{O}_k$, they must lie on the same $M_{\mathbf{C}}$ orbit. Hence, in particular, there is an m in $M_{\mathbf{C}}$ such that $lx = mx$. But by virtue of Lemma 1.2 m can be chosen in M, proving $l \in L_M$. Let j be a fixed element in E′. Since $\lambda_j(u)$ is locally rational on $\mathfrak{O}_k \supset A\,x$ (*cf.* the begin of this section and Proposition 1.1) we have

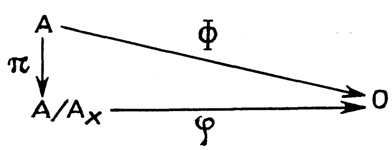
$$\varphi_j(l_n)\lambda_j(x) = \varphi_j(l_n)\lambda_j(m_n x) = \lambda_j(a_n x) \to \lambda_j(ax) = \varphi_j(l)\lambda_j(x)$$

and hence $\varphi_j(l_n) \to \varphi_j(l)$ if $n \to +\infty$. If $l_n = h_n t_n$, $l = ht$ $(h_n, h \in H;$ $t_n, t \in T)$ we have also $\varphi_j(h_n) \to \varphi_j(h)$ and $\varphi_j(t_n) \to \varphi_j(t)$ for all j in E′. But then also $h_n \to h \pmod{H \cap L_M}$ and $t_n \to t \pmod{T \cap L_M}$ proving, that $l_n = h_n t_n \to l = ht \pmod{L_M}$.

Q. E. D.

Corollary 1.1. — *Let* A, V *and* x *be as in Proposition* 1.2. *Then* $O = A\,x \subset V$ *is locally closed.*

Proof. — We define the map Φ from A onto O by $\Phi(a) = ax$ $(a \in A)$. Let π be the canonical map from A onto A/A_x; then there is a bijection φ between A/A_x and O such that



be commutative. To prove our statement it suffices to establish, that φ is a homeomorphism between A/A_x and O, the latter being taken in the topology it inherits from V. We infer from Proposition 1.2, that φ^{-1} is continuous. On the other hand, if U is open in O, $\overset{-1}{\pi}(\overset{-1}{\varphi}\,\mathrm{U})) = \overset{-1}{\Phi}(\mathrm{U})$ is open in A and hence so is $\overset{-1}{\varphi}(\mathrm{U})$ in A/A_x.

Q. E. D.

Corollary 1.2. — *Let* V *be a finite dimensional real vector space and* B *a linear solvable algebraic group in* V. *Then each orbit of* B *is locally closed in* V.

Proof. — It suffices to take into consideration, that the connected component B_0 of B satisfies the conditions of A in Proposition 1.1 and that B/B_0 is finite (*cf.* [30], p. 439).

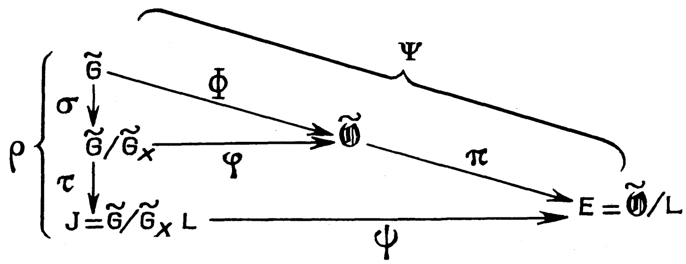
Q. E. D.

2. Let $\mathfrak{g}$ be a real solvable Lie algebra and $\mathrm{G} = \exp \mathfrak{g}$ the corresponding connected and simply connected Lie group. We denote by $\tilde{\mathfrak{g}}$ a fixed Lie algebra with the following properties : $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, $[\mathfrak{g}, \mathfrak{g}] = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$, and $\tilde{\mathfrak{g}}$ admits a faithful linear representation ρ, such that $\rho(\tilde{\mathfrak{g}})$ is an algebraic Lie algebra. To obtain $\tilde{\mathfrak{g}}$ with the indicated properties we can take, for instance, a faithful linear representation of $\mathfrak{g}$, and take the algebraic closure of its image (*cf.* [6 *a*], Theorem. 13, p. 173). Let us consider the connected and simply connected solvable group $\tilde{\mathrm{G}}$ determined by $\tilde{\mathfrak{g}}$. $\tilde{\mathrm{G}}$ acts on $\mathfrak{g}$ by inner automorphisms, and its range through this representation is the connected component of a linear algebraic group $\subset \mathrm{GL}(\mathfrak{g})$; the same observation applies to the contragredient representation of $\tilde{\mathrm{G}}$ on $\mathfrak{g}'$. Therefore, if x is any element in $\mathfrak{g}'$, by virtue of Corollary 1.2 above, $\tilde{\mathrm{G}}x$ is locally closed in $\mathfrak{g}'$. We are going to make use of these observations to prove the following

Proposition 2.1. — *Let* $\mathfrak{g}$ *be a real solvable Lie algebra,* G *the corresponding connected and simply connected group; we assume, that* G *acts on* $\mathfrak{g}'$ *through the coadjoint representation. There exists an equivalence relation* $\mathfrak{R}$ *on* $\mathfrak{g}'$, *uniquely determined by the following properties* : 1° *Any orbit* $\mathfrak{O}$ *of* $\mathfrak{R}$ *is locally closed in* $\mathfrak{g}'$ *and is* G *invariant*; 2° *For any* $p \in \mathfrak{O}$, $\mathrm{G}p$ *is dense in* $\mathfrak{O}$. *In addition we have, that* (α) $\mathrm{G}p = \mathfrak{O}$ *if and only if* $\mathrm{G}p$ *is locally closed,* (β) *For each orbit* $\mathfrak{O}$ *of* $\mathfrak{R}$ *there is a connected and simply connected Lie group* G_1 *such that* $\mathrm{G} \subset \mathrm{G}_1$, $[\mathrm{G}_1, \mathrm{G}_1] = [\mathrm{G}, \mathrm{G}]$ *and, for any* p *in* $\mathfrak{O}$, $\mathfrak{O} = \mathrm{G}_1 p$.

Proof. — *a.* Let x be a fixed element of $\mathfrak{g}'$ and let us write $\tilde{\mathfrak{O}} = \tilde{\mathrm{G}}x$. We denote by Φ the map from $\tilde{\mathrm{G}}$ onto $\tilde{\mathfrak{O}}$ defined by $\Phi(a) = ax$ $(a \in \tilde{\mathrm{G}})$, by π, σ, τ and ρ the canonical maps from $\tilde{\mathfrak{O}}$ onto $\mathrm{E} = \tilde{\mathfrak{O}}/\mathrm{L}$, from $\tilde{\mathrm{G}}$ onto

$\tilde{G}/\tilde{G}_x$, from $\tilde{G}/\tilde{G}_x$ onto $J = \tilde{G}/\tilde{G}_x L$ and from $\tilde{G}$ onto J resp. (we recall, that $L = [G, G] = [\tilde{G}, \tilde{G}]$). We put $\Psi = \pi \circ \Phi$ and define φ and ψ such that the following diagramm be commutative



b. Using the above notations we prove first, that ψ is a homeomorphims between J and E. If U is an open set in E, Ψ being evidently continuous, $\overset{-1}{\Psi}(U) = \overset{-1}{\rho}(\overset{-1}{\psi}(U))$ is open in $\tilde{G}$ and hence $\overset{-1}{\psi}(U)$ is open in J. On the other hand, we know from Proposition 1.2, that $\overset{-1}{\varphi}$ is continuous and therefore, if W is open in J, $\psi(W) = \pi(\varphi(\overset{-1}{\tau}(W)))$ is open in $E = \tilde{\mathfrak{O}}/L$, completing the proof of our statement.

c. Let us denote by $\mathfrak{O}$ the relative closure of $G\,x$ in $\tilde{\mathfrak{O}}$. Our next objective is to show the existence of a closed, connected subgroup G_1 of $\tilde{G}$, containing G, such that $G_1\,x = \mathfrak{O}$. To this end we write B for the closure of the connected subgroup $\rho(G)$ in the connected abelian group $J = \tilde{G}/\tilde{G}_x L$ and show, that the connected component of the identity G_1 in $\overset{-1}{\rho}(B)$ satisfies the requirement. Since both $G\,x$ and $\mathfrak{O}$ are L invariant, it is enough to establish, that

$$\pi(G_1\,x) = \pi(\Phi(G_1)) = \pi(\mathfrak{O}).$$

But by the diagramm of (*a*) we have

$$\pi(\Phi(G_1)) = \Psi(G_1) = \psi(\rho(G_1)) = \psi(\overline{\rho(G)}) = \overline{\psi(\rho(G))}$$

since, by (*b*) above, ψ is a homeomorphism, and thus $\pi(G_1\,x) = \overline{\Psi(G)}$. Let F be a subset of $\tilde{\mathfrak{O}}$; we shall denote by $\overline{F}$ its relative closure in $\tilde{\mathfrak{O}}$. If F is such, that $LF = F$, we have $\pi(\overline{F}) = \overline{\pi(F)}$, and hence

$$\pi(\mathfrak{O}) = \pi(\overline{G\,x}) = \overline{\pi(G\,x)} = \overline{\pi(\Phi(G))} = \overline{\Psi(G)},$$

proving, that $\pi(G_1\,x) = \pi(\mathfrak{O})$ and thus also $G_1\,x = \mathfrak{O}\ (= \overline{G\,x} \subset \tilde{\mathfrak{O}})$.

d. Let us observe now, that, if we replace x by another element y of $\tilde{\mathfrak{O}}$, the construction of (*c*) leads to the same group G_1, and thus, in particular, $\overline{G\,y} = G_1\,y$. From this we conclude, that defining $y \sim x$ provided $y \in \overline{G\,x}$, we obtain on $\mathfrak{g}'$ an equivalence relation of the desired sort.

e. To show uniqueness, it will suffice to establish the following statement. *Let* O_1 *and* O_2 *be* G *invariant, locally closed subsets in* $\mathfrak{g}'$, *such that any* G *orbit contained in them is dense. Then, if* $O_1 \cap O_2 \neq \emptyset$, *we have* $O_1 = O_2$. Let us put $O_0 = O_1 \cap O_2$; O_0 is locally closed in O_1, and therefore there is an open set U and a closed set F in O_1, such that $O_0 = F \cap U$. If p is some point in O_0, we have $G\,p \subset O_0 \subset F \subset O_1$, but since $G\,p$ is dense in O_1, $F = O_1$ and thus O_0 is open in O_1. But if $O_0 \subsetneq O_1$ and q lies in $O_1 - O_0$, evidently $G\,q$ cannot be dense in O_1. In this fashion $O_1 \subseteqq O_2$, and thus by symmetry $O_1 = O_2$.

f. To complete the proof of Proposition 2.1, it is now enough to show, that if $G\,x$ is locally closed in $\mathfrak{g}'$, then $G\,x = \mathfrak{O}$ [*cf*. (*c*)]. For this, however, it suffices to repeat the above reasoning with $O_0 = G\,x$ and $O_1 = \mathfrak{O}$.

Q. E. D.

3. The purpose of this section is to collect a few elementary facts, which will be employed in an essential fashion in Section 5.

a. In the following $G = \exp \mathfrak{g}$ and $\tilde{G} = \exp \tilde{\mathfrak{g}}$ will have the meaning as in the previous section.

b. Let x be a nonzero element of $\mathfrak{g}'$, which we shall keep fixed. Let us put $\tilde{\mathfrak{O}} = \tilde{G}\,x \subset \mathfrak{g}'$.

If g is some element of $\mathfrak{g}'$ and $f = g \mid \mathfrak{d}$, the subgroup $G_f \subset G$ normalizes $(L_{\mathbf{C}})_f \subset G_{\mathbf{C}}$ and hence, since $\overline{G}_g \subset G_f$ [for $\overline{G}_g$, *cf*. I.4 (*c*)], $\overline{G}_g\,(L_{\mathbf{C}})_f$ is a subgroup, to be denoted by $\overline{H}_g$, of $G_{\mathbf{C}} = \exp \mathfrak{g}_{\mathbf{C}}$. Since

$$\overline{G}_g \cap (L_{\mathbf{C}})_f = \exp (\mathfrak{d} \cap \mathfrak{g}_g)$$

is connected, $\overline{H}_g$ is closed in $G_{\mathbf{C}}$ (*cf*. Lemma 4.2, Chapter I).

If g and g_1 are elements in $\mathfrak{g}'$, such that $g \mid \mathfrak{d} = g_1 \mid \mathfrak{d} = f$, we have $\overline{G}_g = \overline{G}_{g_1}$ (*cf*. Remark 6.1, Chapter I) and hence also $\overline{H}_g = \overline{H}_{g_1}$.

c. If l and l_1 are elements in $(\mathfrak{d}_{\mathbf{C}})_f$, we have $([l, l_1], f) = 0$, and hence, since $(L_{\mathbf{C}})_f$ is connected, there is a continuous homomorphism φ_f of $(L_{\mathbf{C}})_f$ into the group $\mathbf{C}^*$ of nonzero complex numbers uniquely determined by $d\varphi_f = i\,(f \mid (\mathfrak{d}_{\mathbf{C}})_f)$. If χ is arbitrary in $\overset{A}{G}_g$ [*cf*. I.4 (*c*)], there is a

$\chi' \in \mathrm{Hom}(\overline{\mathrm{H}}_g, \mathbf{C}^*)$, uniquely determined by $\chi' \mid \overline{\mathrm{G}}_g \equiv \chi$, $\chi' \mid (\mathrm{L}_{\mathbf{C}})_f \equiv \varphi_f$. In fact, any element of G_f leaves φ_f invariant, and in addition since

$$\chi \mid (\mathrm{G}_g)_0 \equiv \chi_g \quad [d\chi_g = i\,(g \mid \mathfrak{g}_g)] \qquad \text{and} \qquad \overline{\mathrm{G}}_g \cap (\mathrm{L}_{\mathbf{C}})_f = (\overline{\mathrm{G}}_g)_0 \cap \mathrm{L}_f = \exp(\mathfrak{g}_g \cap \mathfrak{d}),$$

we have evidently

$$\chi \left| \left(\overline{\mathrm{G}}_g \cap (\mathrm{L}_{\mathbf{C}})_f\right) = \varphi_f \right| \left(\overline{\mathrm{G}}_g \cap (\mathrm{L}_{\mathbf{C}})_f\right).$$

Let us put

$$\overset{\wedge}{\mathrm{H}}_g = \left\{ \psi;\ \psi \in \mathrm{Hom}(\overline{\mathrm{H}}_g, \mathbf{C}^*),\ \psi \mid \overline{\mathrm{G}}_g \in \overset{\wedge}{\mathrm{G}}_g,\ \psi \mid (\mathrm{L}_{\mathbf{C}})_f \equiv \varphi_f \right\}.$$

We also write

$$\overset{\ominus}{\mathrm{H}}_g = \{ \psi;\ \psi \in \mathrm{Hom}(\overline{\mathrm{H}}_g, \mathbf{T}),\ \psi \mid (\overline{\mathrm{H}}_g)_0 \equiv 1 \}.$$

$\overset{\ominus}{\mathrm{H}}_g$ is isomorphic to the dual of the free abelian group $\overline{\mathrm{H}}_g/(\overline{\mathrm{H}}_g)_0 = \overline{\mathrm{G}}_g/(\mathrm{G}_g)_0$, and $\overset{\ominus}{\mathrm{H}}_g$ acts, through multiplication, simply transitively on the set $\overset{\wedge}{\mathrm{H}}_g$.

d. Let g and g_1 be elements of $\tilde{\mathfrak{O}} = \tilde{\mathrm{G}}\,x$ and assume, that $g_1 = ag$ $(a \in \tilde{\mathrm{G}})$. We are going to show, that $\overline{\mathrm{H}}_{g_1} = a\,\overline{\mathrm{H}}_g\,a^{-1}$. We put $\omega_a(b) = aba^{-1}$ $(b \in \mathrm{G}_{\mathbf{C}})$. The indicated assertion is implied by the following series of observations, the verification of which we leave to the reader [*cf.* also I.4 (*h*)]. 1° $\omega_a(\mathrm{G}_g) = \mathrm{G}_{ag}$, whence also $\omega_a((\mathrm{G}_g)_0) = (\mathrm{G}_{ag})_0$; 2° $\chi_{ag} \circ \omega_a \equiv \chi_g$ on $(\mathrm{G}_g)_0$; 3° From 1° and 2° we conclude easily, that $\omega_a(\overline{\mathrm{G}}_g) = \overline{\mathrm{G}}_{ag}$; 4° $\omega_a((\mathrm{L}_{\mathbf{C}})_f) = (\mathrm{L}_{\mathbf{C}})_{af}$. By aid of 3° and 4° we obtain finally

$$a\,\overline{\mathrm{H}}_g\,a^{-1} = \omega_a(\overline{\mathrm{H}}_g) = \overline{\mathrm{H}}_{ag}.$$

From this relation we derive at once, that if $g, g_1 \in \tilde{\mathfrak{O}}$, we have $\overline{\mathrm{G}}_g \mathrm{L}_{\mathbf{C}} = \overline{\mathrm{G}}_{g_1} \mathrm{L}_{\mathbf{C}}$. Let us observe, incidentally, that by virtue of what we saw in (*b*) above, the same conclusion holds true if we only know, that $g \mid \mathfrak{d}$ and $g_1 \mid \mathfrak{d}$ lie on the same G orbit in $\mathfrak{d}'$.

In the following we shall denote by $\overline{\mathrm{H}}$ the closed subgroup $\overline{\mathrm{G}}_g \mathrm{L}_{\mathbf{C}}$ of $\mathrm{G}_{\mathbf{C}}$ $(g =$ arbitrary element in $\tilde{\mathfrak{O}})$.

Let us observe also, that the above remarks, along with $\varphi_{af} \circ \omega_a = \varphi_f$, imply, that $\overset{\wedge}{\mathrm{H}}_{g_1} \circ \omega_a = \overset{\wedge}{\mathrm{H}}_g$.

e. We set $\mathrm{J} = \overline{\mathrm{H}}/(\overline{\mathrm{H}})_0$, and write Φ for the canonical homomorphism of $\overline{\mathrm{H}}$ onto J. Let us put

$$\overset{\ominus}{\mathrm{H}} = \hat{\mathrm{J}} \circ \Phi = \{ \chi;\ \chi \in \mathrm{Hom}(\overline{\mathrm{H}}, \mathbf{T}),\ \chi \mid (\overline{\mathrm{H}})_0 \equiv 1 \}.$$

Given $g \in \tilde{\mathfrak{G}}$ and $\psi \in \overset{\ominus}{\mathrm{H}}$ write $\psi_g \equiv \psi \mid \overline{\mathrm{H}}_g$; we observe, that the map $\psi \to \psi_g \left(\psi \in \overset{\ominus}{\mathrm{H}}\right)$ is an isomorphism of $\overset{\ominus}{\mathrm{H}}$ onto $\overset{\ominus}{\mathrm{H}}_g$ [*cf.* (*c*)].

f. We denote by $\mathfrak{B}(\tilde{\mathfrak{G}})$ the set $\bigcup_{g \in \tilde{\mathfrak{G}}} \overset{\wedge}{\mathrm{H}}_g$. We let the group $\overset{\ominus}{\mathrm{H}}$ act on $\mathfrak{B}(\tilde{\mathfrak{G}})$ according to the following rule : if $p = (g, \chi)$ $\left(g \in \tilde{\mathfrak{G}},\ \chi \in \overset{\wedge}{\mathrm{H}}_g\right)$ is some element of $\mathfrak{B}(\tilde{\mathfrak{G}})$, and $\varphi \in \overset{\ominus}{\mathrm{H}}$, we put $\varphi p = (g, \varphi_g \chi)$. If we define the map τ from $\mathfrak{B}(\tilde{\mathfrak{G}})$ onto $\tilde{\mathfrak{G}}$ by $\tau p = g$ we have $\tau\varphi p = \tau p$. Also, by what we saw in (*c*) and (*e*), $\overset{\ominus}{\mathrm{H}}$ acts simply transitively on $\overset{-1}{\tau}(g)$ for each g in $\tilde{\mathfrak{G}}$. We let $\tilde{\mathrm{G}}$, too, act on $\mathfrak{B}(\tilde{\mathfrak{G}})$ by setting $ap = (ag, a\chi)$ $[p = (g, \chi)]$ [*cf.* the end of (*d*) above]. We observe, that if φ is some element of $\overset{\ominus}{\mathrm{H}}$ we have evidently $a\varphi = \varphi$, and thus $a(\varphi p) = \varphi(ap)$, or the actions of $\tilde{\mathrm{G}}$ and $\overset{\ominus}{\mathrm{H}}$ commute with each other.

4. The purpose of this section is to complete Proposition 1.1 in the special case, when $\mathrm{V} = \mathfrak{g}'$ and G acts on $\mathfrak{g}'$ *via* the coadjoint representation.

Proposition 4.1. — *Let us assume, that $\mathfrak{g}$ is a nilpotent Lie algebra over the real or complex field* (denoted by $\mathbf{K}$). *Then, assuming in Proposition* 1.1 *that* $\mathrm{V} = \mathfrak{g}'$ *and that* G *acts on* V *via the coadjoint representation, there is a collection of homogeneous polynomials* $\{ \mathrm{Q}_j(x);\ 1 \leqq j \leqq \mathrm{M} \}$ *on* V, *such that* $\mathrm{F}_j = \{ x; x \in \mathrm{V},\ \mathrm{Q}_k(x) = 0 \text{ for } k \leqq j \}$ *and that for a sufficiently large integer* N, $(\mathrm{Q}_j(x))^{\mathrm{N}}\, \mathrm{P}_k(z; x)$ $(1 \leqq k \leqq m)$ *is the restriction of a polynomial function on* $\mathbf{K}^d \times \mathrm{V}$ *to* $\mathbf{K}^d \times (\mathrm{F}_{j-1} - \mathrm{F}_j)$.

Proof. — *a.* We can obviously assume, that the Jordan-Hölder sequence $\{ \mathrm{V}_j; 0 \leqq j \leqq m \}$ of *loc. cit.* arises by considering a Jordan-Hölder sequence $\mathfrak{g} = \mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \ldots \supset \mathfrak{g}_0 = (0)$ in $\mathfrak{g}$ and by taking $\mathrm{V}_j = \mathfrak{g}_j^{\perp} \subset \mathfrak{g}'$. Let $l_j \in \mathfrak{g}_j - \mathfrak{g}_{j-1}$ $(1 \leqq j \leqq m)$ and $(l_i, l'_j) = \delta_{ij}$; then $l'_j \in \mathfrak{g}_{j-1}^{\perp} - \mathfrak{g}_j^{\perp}$ and we can suppose, that $v_j = l'_j$.

b. Given an element x of $\mathfrak{g}'$, we denote again by B_x the skew-symmetric bilinear form $\mathrm{B}_x(l_1, l_2) = ([l_1, l_2], x)$ $(l_1,\ l_2 \in \mathfrak{g})$ on $\mathfrak{g} \times \mathfrak{g}$ [*cf.* I.4 (*b*)]. Given a subspace $\mathfrak{h}$ of $\mathfrak{g}$, we write $\mathfrak{h}^{\perp}_{\mathrm{B}_x}$ for its orthogonal complement in $\mathfrak{g}$ with respect to B_x.

Let us put $\mathrm{R}(x) = (\mathfrak{g})^{\perp}_{\mathrm{B}_x}$ and $\mathfrak{g}_j(x) = \mathfrak{g}_j + \mathrm{R}(x)$ $(0 \leqq j \leqq m)$. Since obviously $\mathfrak{g}_{\pi_j(x)} = (\mathfrak{g}_j(x))^{\perp}_{\mathrm{B}_x}$ [*cf.* (*a*) in the proof of Proposition 1.1] we have $j \in f(x)$ if and only if $\mathfrak{g}_{j-1}(x) \subsetneq \mathfrak{g}_j(x)$.

c. Let e and g be different elements of $\mathcal{E}$ (cf. *loc. cit.*); we shall define an order relation between them as follows. We set $g < e$ if $e = \emptyset$ (empty set); otherwise we define $g < e$ if either $d(g) > d(e)$, or

$$d(e) = d(g) = d, \qquad e = \{0 < j_1 < j_2 < \ldots < j_d \leqq m\},$$
$$g = \{0 < k_1 < k_2 < \ldots < k_d \leqq m\}$$

and $\alpha = \sup \beta$ with $j_\beta = k_\beta$ implies, that $k_{\alpha+1} < j_{\alpha+1}$.

We set $Q_e(x) \equiv 1$ if $e = \emptyset$; otherwise let us define

$$Q_e(x) = \det\{([l_i, l_j], x);\ i, j \in e\}.$$

With the above notations we shall now prove the following assertion. Assume $e = f(x)$; then e is the smallest element in $\mathcal{E}$, for which $Q_g(x) \neq 0$. To this end let us observe first, that $Q_e(x) \neq 0$. In fact, since obviously $l_j \in \mathfrak{g}_j(x) - \mathfrak{g}_{j-1}(x)$ $(j \in e)$, the system $\{l_j;\ j \in e\}$ is a basis in $\mathfrak{g} \bmod (R(x))$. Hence to complete the proof of the above statement it is enough to establish, that $Q_g(x) = 0$ if $g < e$. This is obvious if $d(g) > d(e)$. Otherwise, with notations as above we have, that $\{l_{k_1}, l_{k_2}, \ldots, l_{k_\alpha}\} \subset \mathfrak{g}_{j_\alpha}(x)$; since $\dim(\mathfrak{g}_{j_\alpha}(x)/R(x)) = \alpha$, this implies at once, that the system $\{l_j;\ j \in g\}$ is linearly dependent in $\mathfrak{g} \bmod (R(x))$ and thus $Q_g(x) = 0$.

Note, that in this fashion we can conclude, that

$$\mathfrak{O}_e = \{x;\ Q_g(x) = 0 \text{ for } g < e \text{ and } Q_e(x) \neq 0\}.$$

d. Let e be an element of $\mathcal{E}$ different from $\emptyset$. Assuming

$$e = \{0 < j_1 < j_2 < \ldots < j_d \leqq m\}$$

we put

$$e_k = l_{j_k} \qquad (1 \leqq k \leqq d).$$

For x in $\mathfrak{O}_e$, let us define the system $\{l_k(x);\ 1 \leqq k \leqq d\}$ by the condition, that

$$B_x(e_i, l_k(x)) = \delta_{ik} \qquad \text{and} \qquad l_k(x) = \sum_{t=1}^{d} a_{kt}(x)\, e_t.$$

Then evidently $l_k(x) \in (\mathfrak{g}_{j_{k-1}}(x))^{\perp}_{B_x} - (\mathfrak{g}_{j_k}(x))^{\perp}_{B_x}$ and $Q_e(x)\, l_k(x)$ $(1 \leqq k \leqq d)$ is the restriction to $\mathfrak{O}_e$ of a polynomial map $\mathfrak{g}' \to \mathfrak{g}$. Therefore, to complete the proof or Proposition 4.1, it suffices to substitute the system $\{l_k(x);\ 1 \leqq k \leqq d\}$, constructed above, in place of the system denoted in the same fashion at the end of (*b*) in the proof of Proposition 1.1, and carry out the construction of the canonical parametrization of Gx as *loc. cit.*

Q. E. D.

Corollary 4.1. — *With the assumptions of Proposition* 4.1, *putting* $\mathfrak{O}_j = F_{j-1} - F_j$ *for each* j $(1 \leq j \leq M)$ *there is a map* h *of* $\mathfrak{O}_j$ *into itself, such that, for a sufficiently large integer* N, $(Q_j(x))^N h(x)$ *is the restriction to* $\mathfrak{O}_j$ *of a polynomial map of* $\mathfrak{g}'$ *into itself, such that for any* x *in* $\mathfrak{O}_j$: 1° $h(ax) \equiv h(x)$ $(a \in G)$; 2° $h(x) \in G\,x$.

Proof. — We assume, that $\mathfrak{O}_j = \mathfrak{O}_e$ $(e \in \mathcal{E})$ and write E for the complement of e in $\{1, 2, \ldots, m\}$. Then, by virtue of Lemma 1.3, and of the above proposition it suffices to define

$$h(x) = \sum_{j \in E} \lambda_j(x)\, l'_j \quad (x \in \mathfrak{O}_e).$$

Q. E. D.

Remark 4.1. — For later use we observe the following. Assuming $d(e) > 0$, let us write P_e for the hyperplane $\{l'; (l_j, l') = 0, j \in e\}$ in $\mathfrak{g}'$. Then, putting $V_e = P_e \cap \mathfrak{O}_e$, we have evidently $\lambda_j(x) \equiv x_j$ if $x \in V_e$ $(j \in E)$. In particular, as $\sum_{j \in E} y_j l'_j$ describes the Zariski relatively open set V_e, we obtain each orbit in $\mathfrak{O}_e$ precisely once by considering the varieties of the form $\{x; x \in \mathfrak{g}', \lambda_j(x) = y_j, j \in E\}$.

Proposition 4.2. — *Let* $\mathfrak{O}_j$ *be as in Corollary* 4.1, *and let us assume, that* $d = \dim o(x)$ *for* $x \in \mathfrak{O}_j$ $[o(x) = G\,x]$. *Then there exists a positive integer* N, *a map* l *from* $\mathbf{K}^d \times \mathfrak{O}_j$ *into* $\mathfrak{g}$, *and a map* R *from* $\mathfrak{O}_j \times \mathfrak{O}_j$ *into* $\mathbf{K}^d$ *such that* $(Q_j(x))^N l(T, x)$ $[(Q_j(x))^N R(y, x)$ resp.$]$ *is the restriction to* $\mathbf{K}^d \times \mathfrak{O}_j$ [to $\mathfrak{O}_j \times \mathfrak{O}_j$ resp.] *of a polynomial map on* $\mathbf{K}^d \times \mathfrak{g}'$ (on $\mathfrak{g}' \times \mathfrak{g}'$ resp.) *such that :* 1° *For each fixed* x *in* $\mathfrak{O}_j$, *the map* $[T \to \exp[l(T, x)]\,x]$ *is a bijection between* $\mathbf{K}^d$ *and* $G\,x$; 2° *If* $y \in G\,x$ *and* $y = \exp[l(T, x)]\,x$, *we have* $T = R(y, x)$.

Proof. — We suppose again, that $\mathfrak{O}_j = \mathfrak{O}_e$. By virtue of (c) in the proof of Proposition 1.1, to obtain $l(T, x)$ it suffices to consider the system $\{l_k(x); 1 \leq k \leq d\}$ determined in (d) above, and write down the product $\exp[t_1 l_1(x)] \exp[t_2 l_2(x)] \ldots \exp[t_d l_d(x)]$, through a repeated application of the Hausdorff-Campbell formula, as

$$\exp[l(T, x)] \qquad [T = (t_1, t_2, \ldots, t_d) \in \mathbf{K}^d].$$

Let us assume, that

$$e = \{0 < j_1 < j_2 < \ldots < j_d \leq m\}.$$

Putting for $\mathfrak{O}_e \ni y = \sum_{j=1}^{m} y_j l'_j$,

$$R_k(y, x) = y_{j_k} + \psi_k(y_{j_1}, y_{j_2}, \ldots, y_{j_{k-1}}; x)$$

[*cf.* (*d*), proof of Proposition 1.1], to obtain the map R with the properties specified above, it is enough to define

$$R(y, x) = (R_1(y, x), \ldots, R_d(y, x)) \qquad (x, y \in \mathfrak{O}_e).$$

Q. E. D.

The following result will be used in Chapter IV only.

LEMMA 4.1. — *There is a bound* K (m) *depending on the dimension* m *of* $\mathfrak{g}$ *only such that, with the notations of Proposition* 1.1 *and* 4.1, N *can be chosen not to exceed* K (m), *and then the degree in* x *of* $(Q_e(x))^N P_j(z; x)$ *does not exceed* $(2m + 1) K(m)$.

Proof. — Given a polynomial P in the groups of variables $x, y, \ldots$, we shall write $\deg_x P$, etc. for its degree in the components of x, y, etc. resp. We fix e in $\mathcal{E}$ such that $d = d(e) > 0$. Let us observe, that this implies $m = \dim \mathfrak{g} > 2$.

a. We observe, that by virtue of (*d*) in the proof of Proposition 1.1, for each j, $1 \leq j \leq m$, there is a polynomial $F_j(T; a, x)$ on $\mathbf{K}^d \times \mathbf{K}^{d^2} \times \mathfrak{g}'$, such that $\deg_T F_j$, $\deg_a F_j \leq m - 1$, $\deg_x F_j \leq 1$, an that putting

$$a(x) = \{ a_{ik}(x); 1 \leq i, k \leq d \} \in \mathbf{K}^{d^2}$$

[*cf.* (*d*), Proposition 4.1], we have

$$Q_j(T; x) \equiv F_j(T; a(x), x).$$

b. Let us set $z_k = F_{j_k}(T; a, x)$. Then we get $t_k = G_k(z; a, x)$ $(1 \leq k \leq d)$, where G_k is a polynomial on $\mathbf{K}^d \times \mathbf{K}^{d^2} \times \mathfrak{g}'$. Let us show, that for each k, $\deg_a G_k \leq m^m$. To this end we observe first, that if L_k is such, that $\deg_a G_j \leq L_k$ for $1 \leq j \leq k$, then $\deg_a G_{k+1} \leq (m - 1)(L_k + 1)$. In fact, we can write $z_{k+1} = t_{k+1} + H_{k+1}(T; a, x)$ where, by virtue of (*a*) above $\deg_a H_{k+1} \leq m - 1$ and $\deg_T H_k \leq m - 1$. The desired conclusion follows by taking into account, that G_k arises upon replacing the variable t_j $(1 \leq j \leq k)$ in $z_{k+1} - H_{k+1}(T; a, x)$ through G_j. Next we note, that $\deg_a G_1 = 0 < m - 1$. Hence we obtain, that $\deg_a G_k \leq m^m$ $(1 \leq k \leq d)$ by observing, that if we set $L_1 = m - 1$ and $L_{j+1} = (m - 1)(L_j + 1)$, then $L_j \leq m^j$ $(j = 1, 2, \ldots)$. Since $d < m$ we get finally, that $L_k \leq L_d < L_m = m^m$ $(1 \leq k \leq d)$. One proves similarly, that $\deg_x G_k \leq m^m$.

c. Upon replacing in F_j [*cf.* (*a*)] t_k through G_k, we obtain a polynomial $P_j(z; a, x)$ such that $P_j(z; a(x), x) \equiv P_j(z; x)$. Since

$$\deg_x F_j = 1 < m - 1, \qquad \deg_a F_j \leq m - 1, \qquad \text{and} \qquad \deg_T F_j \leq m - 1,$$

using (*b*) we conclude, that $\deg_a P_j$, $\deg_x P_j \leq (m-1)(m^m+1)$.

d. Let $F(a, x)$ be the coefficient of some power of z in $P_j(z; a, x)$. By (*c*) it is a sum of terms $a^r x^s$ with $|r|$, $|s| \leq K(m)$, where we put $K(m) = (m-1)(m^m+1)$. We recall, that we have $a_{ik}(x) = b_{ik}(x)/Q_e(x)$, where $b_{ik}(x)$ and $Q_e(x)$ are homogeneous of degree $< m$ $(1 \leq i, k \leq m)$. Taking $a = a(x)$ [*cf.* (*a*)] we conclude therefore, that $a^r x^s$ is of the form $h(x)/(Q_e(x))^{K(m)}$, where $h(x)$ is a polynomial, the degree of which is not larger than $|s| + m|r| + m(K(m) - |r|) < (2m+1)K(m)$ thus we get, that $(Q_e(x))^{K(m)} P_j(z; x)$ is a polynomial in x, the degree of which does not exceed $(2m+1)K(m)$ $(1 \leq j \leq m)$.

Q. E. D.

5. The purpose of this section is to define on $\mathfrak{B}(\tilde{\mathfrak{O}})$ [*cf.* 3 (*f*)] the structure of a differentiable manifold, which turns it into a principal bundle over $\tilde{\mathfrak{O}}$, with the structure group $\overset{\circ}{H}$, acted upon smoothly by $\tilde{G}$, such that the actions of these groups commute.

In the following we shall assume, that for $g \in \tilde{\mathfrak{O}} : \overline{G}_g \supsetneq (G_g)_0$, and write m for the rank of $\overline{G}_g/(G_g)_0$. Observe, that $\overset{\circ}{H}$ is isomorphic to T^m. By a smooth map from a C^∞ manifold into another we shall mean a C^∞ map.

5.1. PROPOSITION 5.1. — *Let a be a fixed element of* J [*cf.* 3 (*e*)]. *There is a smooth map σ from $\tilde{\mathfrak{O}}$ into $\overline{H}$ such that :* 1°. $\Phi(\sigma(g)) \equiv a$; 2° *For any g in $\tilde{\mathfrak{O}}$, $\sigma(g)$ lies in $\overline{H}_g$.*

Proof. — *a*. We denote by π the canonical projection from $\mathfrak{g}'$ onto $\mathfrak{d}' \subset (\mathfrak{d}_{\mathbf{C}})'$. Let us choose a Jordan-Hölder sequence for the action of $\tilde{G}$ on $(\mathfrak{d}_{\mathbf{C}})'$. By virtue of Remark 1.1, replacing G *loc. cit* by $L_{\mathbf{C}}$ and V by $(\mathfrak{d}_{\mathbf{C}})'$, we can conclude, that there is a j $(1 \leq j \leq M)$ such that $\pi(\tilde{\mathfrak{O}}) \subset \mathfrak{O}_j = F_{j-1} - F_j$.

b. Let b be a fixed element of $\overline{H}$ such that $\Phi(b) = a$. We have for any $f \in \pi(\tilde{\mathfrak{O}}) \subset \mathfrak{O}_j \subset (\mathfrak{d}_{\mathbf{C}})'$: $b^{-1} f \in L_{\mathbf{C}} f$. With the notations of Proposition 4.2 (with $G = L_{\mathbf{C}}$, $K = \mathbf{C}$) let us form the function

$$l(f) = l(R(b^{-1} f, f), f);$$

for a sufficiently large integer N $(Q_j(f))^N l(f)$ is the restriction of a polynomial map, from $(\mathfrak{d}_{\mathbf{C}})'$ into $\mathfrak{d}_{\mathbf{C}}$, to $\pi(\tilde{\mathfrak{O}})$. Putting $d(f) \equiv \exp[l(f)]$ we have $d(f) = b^{-1} f$, and therefore setting $\sigma(f) \equiv bd(f)$, the smooth map $\sigma(f)$ from $\pi(\tilde{\mathfrak{O}})$ into $\overline{H}$ will have the following two properties : 1° $\Phi(\sigma(f)) \equiv a$; 2° $\sigma(f) f = f$ for all f in $\pi(\tilde{\mathfrak{O}})$.

c. Let us define now the smooth map σ from $\tilde{\mathfrak{O}}$ into $\overline{H}$ by $\sigma(g) = \sigma(f)$ if $f = g \mid \mathfrak{d}$, and $\sigma(f)$ as in (*b*) above; we show next, that it satisfies the conditions of Proposition 5.1. We have evidently $\Phi(\sigma(g)) \equiv a$ on $\tilde{\mathfrak{O}}$. On the other hand, since $\sigma(g) \in \overline{H} = L_{\mathbf{C}} \overline{G}_g$ and $\sigma(g) f = f$, we have also $\sigma(g) \in \overline{H}_g = \overline{G}_g (L_{\mathbf{C}})_f$.

Q. E. D.

REMARK 5.1. — Observe, that we have actually proved, that $\sigma(g)$ in Proposition 5.1 can be chosen in such a fashion, that it depend only on the projection of g onto $\mathfrak{d}'$.

5.2. *a*. Let $\{ a_j; 1 \leq j \leq m \}$ $[m = \text{rank of } \overline{G}_g/(G_g)_0 \text{ for } g \in \tilde{\mathfrak{O}}]$ be a basis in $J = \overline{H}/(\overline{H})_0 \sim \overline{G}_g/(G_g)_0$. For each j we denote by σ_j a smooth map from $\tilde{\mathfrak{O}}$ into $\overline{H}$, related to a_j as is σ to a in Proposition 5.1. Putting, for ω in $\mathbf{C}^*$, $\arg \omega = \omega / |\omega|$, we define a map α from $\mathfrak{B}(\tilde{\mathfrak{O}})$ into $\tilde{\mathfrak{O}} \times \mathbf{T}^m$ by setting, for $p = (g, \chi) \in \mathfrak{B}(\tilde{\mathfrak{O}})$:

$$\alpha p = (g; \arg(\chi(\sigma_1(g))), \arg(\chi(\sigma_2(g))), \ldots, \arg(\chi(\sigma_m(g)))).$$

We observe, that α is a bijection between the underlying sets of $\mathfrak{B}(\tilde{\mathfrak{O}})$ and $\tilde{\mathfrak{O}} \times \mathbf{T}^m$ resp. In fact, if we have for $p = (g, \chi)$, $p' = (g', \chi')$ in $\mathfrak{B}(\tilde{\mathfrak{O}})$: $\alpha(p) = \alpha(p')$, then, by definition, $g = g'$, $\chi, \chi' \in \hat{\overline{H}}_g$ and

$$\arg(\chi(\sigma_j(g))) = \arg(\chi'(\sigma_j(g))) \qquad (1 \leq j \leq m).$$

By 3 (*f*) there is a φ in $\hat{J}$ such that $\chi' = (\varphi \circ \Phi) \chi$; since $\Phi(\sigma_j(g)) \equiv a_j$, this implies at once, that $\varphi(a_j) = 1$ for all j, and hence $\varphi \equiv 1$, $\chi \equiv \chi'$ and finally $p = p'$. In this fashion, to establish, that α is a bijection, it is enough to show, that it is surjective. But with the above notations

$$\arg(((\varphi \circ \Phi) \chi)(\sigma_j(g))) = \varphi(a_j) \arg(\chi(\sigma_j(g))) \qquad (1 \leq j \leq m)$$

and hence the desired conclusion follows from the fact, that $\varphi(a_j)$, for each j, can arbitrarily be prescribed in $\mathbf{T}$.

b. Let $\{ a'_j; 1 \leq j \leq m \}$ be a second basis in J. Distinguishing notions, introduced above, relative to this new basis, by a prime, we show, that the map $\alpha' \circ \overset{-1}{\alpha}$ is a diffeomorphism of $\tilde{\mathfrak{O}} \times \mathbf{T}^m$ with itself. To this end let us write for each $j : a'_j = a^{n_{1j}}.a^{n_{2j}}\ldots a_m^{n_{mj}}$. Then we have also

$$\sigma'_j(g) = (\sigma_1(g))^{n_{1j}}.(\sigma_2(g))^{n_{2j}}\ldots(\sigma_m(g))^{n_{mj}}.h_j(g),$$

where $h_j(g)$ is a smooth map from $\tilde{\mathfrak{O}}$ into $\overline{\mathrm{H}}$, such that, for each g, its value lies in $(\overline{\mathrm{H}}_g)_0 = (\mathrm{G}_g)_0(\mathrm{L}_{\mathbf{C}})_f$. Let us write χ'_g for the element of $\mathrm{Hom}\left((\overline{\mathrm{H}}_g)_0, \mathbf{C}^*\right)$ determined by $\chi'_g \mid (\mathrm{G}_g)_0 \equiv \chi_g$ and $\chi'_g \mid (\mathrm{L}_{\mathbf{C}})_f \equiv \varphi_f$ (*cf.* 3°). Putting

$$\varphi_j(g) = \chi'_g(h_j(g)) \quad \text{and} \quad \omega = (\omega_1, \omega_2, \ldots, \omega_m) \in \mathbf{T}^m, \quad \ldots,$$

we conclude, that

$$\left(\alpha^1 \circ \overset{-1}{\alpha}\right)(g, \omega) = (g, \omega'), \quad \text{where} \quad \omega'_j = \omega_1^{n_{1j}}\ldots\omega_m^{n_{mj}} \arg(\varphi_j(g)).$$

In this fashion, to complete our proof, it suffices to prove, that for each j, the function φ_j is smooth. Let g_0 be a fixed element of $\tilde{\mathfrak{O}}$ and let us choose a basis $\{l_j ; 1 \leqq j \leqq \mathrm{N}\}$ which is supplementary to $\tilde{\mathfrak{g}}_{g_0}$ in $\tilde{\mathfrak{g}}$. We set

$$g_j(t) = \exp(tl_j), \quad \mathrm{T} = (t_1, t_2, \ldots, t_{\mathrm{N}}) \in \mathrm{R}^{\mathrm{N}} \quad \text{and} \quad g(\mathrm{T}) = g_1(t_1)\, g_2(t_2)\ldots g_{\mathrm{N}}(t_{\mathrm{N}}).$$

There is an open sphere O around the neutral element in $\mathbf{R}^{\mathrm{N}}$, such that the map $[\mathrm{T} \mapsto a(\mathrm{T})\, g_0]$ is a diffeomorphism between O and some neighborhood of g_0 on $\tilde{\mathfrak{O}}$. In this fashion it is enough to establish, that $\varphi_j(g(\mathrm{T})\, g_0)$ is smooth on O. But this is clear from the observation, that

$$\varphi_j(g(\mathrm{T})\, g_0) \equiv \chi'_{g(\mathrm{T})g_0}(h_j(g(\mathrm{T})\, g_0)) \equiv \chi'_{g_0}((g(\mathrm{T}))^{-1}\, h_j(g(\mathrm{T})\, g_0)\, g(\mathrm{T})).$$

c. Using the above remarks, we can now define the structure of a differentiable manifold on $\mathfrak{B}(\tilde{\mathfrak{O}})$ by the condition, that the map α [*cf.* (*a*)] be a diffeomorphism between $\mathfrak{B}(\tilde{\mathfrak{O}})$ and $\tilde{\mathfrak{O}} \times \mathbf{T}^m$.

d. Let us show finally, the the map from $\tilde{\mathrm{G}} \times \mathfrak{B}(\tilde{\mathfrak{O}})$ onto $\mathfrak{B}(\tilde{\mathfrak{O}})$, which assigns ap to (a, p), is smooth. To this end it is enough to establish, that the map $(a, q) \mapsto \alpha\left(a\overset{-1}{\alpha}(q)\right)\left(q \in \tilde{\mathfrak{O}} \times \mathbf{T}^m\right)$ from $\tilde{\mathrm{G}} \times \tilde{\mathfrak{O}} \times \mathbf{T}^m$ onto $\tilde{\mathfrak{O}} \times \mathbf{T}^m$ is smooth. If $q = (g, \omega)$ we have $\overset{-1}{\alpha}(q) = (g, \chi)$, where $\chi \in \overset{\wedge}{\mathrm{H}}_g$ is determined by $\omega_j = \arg \chi(\sigma_j(g))$ $(1 \leqq j \leqq m)$. We have then

$$\alpha\left(a\overset{-1}{\alpha}\right)(q) = (ag, \omega'), \quad \text{where} \quad \omega'_j = \arg(a\,\chi(\sigma_j(ag))).$$

But

$$a\,\chi(\sigma_j(ag)) = \chi(a^{-1}\,\sigma_j(ag)\,a) \quad \text{and} \quad a^{-1}\,\sigma_j(ag)\,a \equiv \sigma_j(g)\,k_j(a, g),$$

k_j being a smooth function from $\tilde{G} \times \tilde{\mathfrak{O}}$ into $\overline{H}$, such that for each g, $k_j(a, g)$ lies in $(\overline{H}_g)_0$. From here we can complete the proof as at the end of (*b*) above.

6. — 6.1. *a.* A smooth map $p(g)$ from $\tilde{\mathfrak{O}}$ into $\mathfrak{B}(\tilde{\mathfrak{O}})$ is called a cross section, if $\tau\, p(g) \equiv g\, [g \in \tilde{\mathfrak{O}}$; *cf.* 3 (*e*)]. Each cross section determines a smooth map μ of $\tilde{G} \times \tilde{\mathfrak{O}}$ into $\overset{\circ}{H}$ by the condition, that $ap(g) \equiv \mu(a, g)\, p(ag)$. One verifies by an easy computation the identity $\mu(a, bg)\, \mu(b, g) \equiv \mu(ab, g)$ $(a, b \in \tilde{G};\ g \in \tilde{\mathfrak{O}})$. Furthermore, if $p_1(g)$ is another cross section, there is a smooth map $\varphi(g)$ from $\tilde{\mathfrak{O}}$ into $\overset{\circ}{H}$, such that $\varphi(g)\, p_1(g) \equiv p(g)$ $(g \in \tilde{\mathfrak{O}})$. Putting, similarly as above, $ap_1(g) \equiv \mu_1(a, g)\, p_1(a, g)$, we find, that $\mu_1(a, g) \equiv \mu(a, g)\, [\varphi(ag)/\varphi(g)]$. Finally, if a belongs to G_g we have $\mu(a, g) = 1$ $(=$ unity in $\overset{\circ}{H})$. In fact, to see this it suffices to recall, that $\psi \in \hat{H}_g$ means, that $\psi \,|\, \overline{G}_g \in \hat{G}_g$ and $\psi \,|\, (L_{\mathbf{C}})_f \equiv \varphi_f$ [*cf.* 3 (*c*)] and therefore, if a belongs to G_g we have $a\,\psi = \psi$. Hence, writing $p(g) = (g, \psi)$ we get

$$ap(g) = (g, a\,\psi) = (g, \psi) = p(g) = \mu(a, g)\, p(g)$$

and thus $\mu(a, g) = 1$, proving our assertion.

b. In the following we shall consider some special examples of cross sections, which will be of interest later. Suppose, that $\{\sigma_j(g);\ 1 \leq j \leq m\}$ and α have the same meaning as in 5.2 (*a*). Let us observe, that for each $g \in \tilde{\mathfrak{O}}$ there is an element $\chi_0(g)$ of $\hat{H}_g$, uniquely determined by the condition, that $\chi_0(g)\,(\sigma_j(g)) > 0$ $(1 \leq j \leq m)$. Setting $p_0(g) \equiv (g, \chi_0(g))$, we get a cross section. In fact, the only thing which requires verification is that the map $[g \mapsto p_0(g)]$ is smooth which, however, follows at once from $\alpha(p_0(g)) \equiv (g, 1) \in \tilde{\mathfrak{O}} \times \mathbf{T}^m$ $(1 =$ unity in $\mathbf{T}^m)$. Let us put

$$ap_0(g) \equiv \mu_0(a, g)\, p_0(g).$$

If φ is some element of $\overset{\circ}{H}$, we shall write $\hat{\varphi}$ for the element of $\hat{J}$, such that $\hat{\varphi} \circ \Phi \equiv \varphi$ [*cf.* 3 (*e*)]. We recall, that χ'_g is defined on $(\overline{H}_g)_0 = (G_g)_0\,(L_{\mathbf{C}})_f$ by the condition, that $\chi'_g \,|\, (G_g)_0 \equiv \chi_g$ and $\chi'_g \,|\, (L_{\mathbf{C}})_f \equiv \varphi_f$ [*cf.* 5.2 (*b*)]. With these notations we find, that

$$\hat{\mu}_0(a, g) \equiv \arg\,(\chi_g\,(a^{-1}\,\sigma_j(ag)\, a\,(\sigma_j(g))^{-1})) \qquad (\Phi(\sigma_j(g)) \equiv a_j \in J \text{ for } 1 \leq j \leq m).$$

We have, in fact, $a\,\chi_0(g) \equiv (\mu_0(a, g))_g\, \chi_0(ag)$, whence

$$\chi_0(g)\,(a^{-1}\,\sigma_j(ag)\, a) \equiv \hat{\mu}_0(a, g)\,(a_j)\, \chi_0(ag)\,(\sigma_j(ag)).$$

By virtue of $\chi_0(g)(\sigma_j(g)) > 0$, we infer, that

$$\hat{\mu}_0(a, g)(a_j) \equiv \arg \chi_0(g)(a^{-1}\sigma_j(ag)\,a) \equiv \arg \chi_0(g)(a^{-1}\sigma_j(ag)\,a\,(\sigma_j(g))^{-1}).$$

We observe that the argument inside $\chi_0(g)$ belongs to $(\ker \Phi) \cap \overline{H}_g = (\overline{H}_g)_0$; hence we obtain finally, that

$$\hat{\mu}_0(a, g)(a_j) \equiv \arg \chi'_g(a^{-1}\sigma_j(ag)\,a\,(\sigma_j(g))^{-1}) \qquad (a \in \tilde{G},\ g \in \tilde{\mathfrak{O}};\ 1 \leqq j \leqq m).$$

Let us suppose, as we can (*cf.* Remark 5.1) that $\sigma_j(g)$ depends on $f = \pi(g) = g \mid \mathfrak{d}$ only; then, if a belongs to $(G_f)_0$, we have $\mu_0(a, g) = 1$. In fact, as above

$$\chi_0(g)(a^{-1}\sigma_j(ag).a) \equiv \hat{\mu}_0(a, g)(a_j)\,\chi_0(ag)(\sigma_j(ag)).$$

But the left hand side is equal to $\chi_0(g)(\sigma_j(g)) > 0$ since, if $a \in (G_f)_0$ and $b \in \overline{G}_f L_f$ we have $aba^{-1} \equiv b \bmod (\mathring{L}_f)$ (*cf.* the proof of Lemma 7.4, Chapter I), whence the desired conclusion is clear. If we assume finally, that $\sigma_j(g)$ if of the form $b_j\, d_j(g)$ $[b_j \in \overline{H},\ d_j(g) \in L_{\mathbf{C}}$; *cf.* the proof of Proposition 5.1$]$ then we can even infer, that

$$\hat{\mu}_0(a, g)(a_j) \equiv \arg \varphi_f(a^{-1}\sigma_j(ag)\,a\,(\sigma_j(g))^{-1}),$$

and thus $\mu_0(a, g)$ depends on $f = \pi(g)$ only. In fact, to see this it suffices to verify, that

$$a^{-1}\sigma_j(ag)\,a\,(\sigma_j(g))^{-1} = a^{-1}\sigma_j(ag)\,a\,(\sigma_j(ag))^{-1}\,\sigma_j(ag)\,(\sigma_j(g))^{-1}$$

belongs to $L_{\mathbf{C}} = [\tilde{G}_{\mathbf{C}}, \tilde{G}_{\mathbf{C}}]$; but this is clear, since now

$$\sigma_j(ag)\,(\sigma_j(g))^{-1} \equiv b_j\, d_j(ag)\,(d_j(g))^{-1}\, b_j^{-1} \in L_{\mathbf{C}} \qquad (1 \leqq j \leqq m).$$

c. We shall call the function $\mu(a, g)$, determined by the cross section $p(g)$ [*cf.* (a)] the obstruction cocycle belonging to $p(g)$. Let us denote the group of all smooth functions from $\tilde{G} \times \tilde{\mathfrak{O}}$ into $\overset{\circ}{H}$, satisfying $\mu(a, bg)\,\mu(b, g) \equiv \mu(ab, g)$ $(a, b \in \tilde{G};\ g \in \tilde{\mathfrak{O}})$, by $Z^1(\tilde{\mathfrak{O}})$, the subgroup of all elements of the form $\varphi(ag)/\varphi(g)$, where $\varphi(g)$ is a smooth function from $\tilde{\mathfrak{O}}$ into $\overset{\circ}{H}$, of $Z^1(\tilde{\mathfrak{O}})$, by $B^1(\tilde{\mathfrak{O}})$, and let us put

$$H^1(\tilde{\mathfrak{O}}) = Z^1(\tilde{\mathfrak{O}})/B^1(\tilde{\mathfrak{O}}).$$

Then, by (a), the set of all obstruction cocycles on $\tilde{\mathfrak{O}}$ determines an element of $H^1(\overline{\mathfrak{O}})$, which we shall denote sometimes by $[\tilde{\mathfrak{O}}]$.

6.2. PROPOSITION 6.2. — *There is a cross section from* $\tilde{\mathfrak{O}}$ *into* $\mathfrak{B}(\tilde{\mathfrak{O}})$ *such that the corresponding obstruction cocycle is independent of the component, in* $\tilde{\mathfrak{O}}$, *of its argument, and thus gives rise to an element* $\omega \in \mathrm{Hom}(\tilde{G}, \overset{\circ}{\overline{H}})$. *Through an appropriate choice of our cross section we can arrange, that the kernel of* ω *contain the closed subgroup* $(G_f)_0\, G_g\, L$ *of* G (g = arbitrary in $\tilde{\mathfrak{O}}$ and $f = g \mid \mathfrak{d}$).

Proof. — *a.* Let us put $\tilde{\mathfrak{O}}_1 = \pi(\tilde{\mathfrak{O}}) \subset \mathfrak{d}'$. According to what we saw in 6.1 (*a*) and (*b*), to establish our proposition, it suffices to prove the following. For a in G and f in $\tilde{\mathfrak{O}}_1$ let us define

$$\nu(a, f) = \varphi_f(a^{-1}\sigma(af)\, a(\sigma(f))^{-1}),$$

where $\sigma(f) \equiv bd(f)$ $(b \in \overline{H})$ is as in the proof of Proposition 5.1. Then there is a smooth function ψ from $\tilde{\mathfrak{O}}_1$ into $\mathbf{C}^*$ and an element ω of $\mathrm{Hom}(\tilde{G}, \mathbf{C}^*)$, such that

$$\nu(a, f) \equiv \omega(a)(\psi(af)/\psi(f)) \qquad (a \in \tilde{G}, f \in \tilde{\mathfrak{O}}).$$

b. To establish the assertion just formulated, we prove first, that there is a smooth function $\mu(f)$ from $\tilde{\mathfrak{O}}_1$ into $\mathbf{C}^*$, such that

$$\nu(l, f) \equiv \mu(lf)/\mu(f) \qquad (l \in L, f \in \tilde{\mathfrak{O}}_1)..$$

Let us put $\tilde{\mathfrak{O}}_{\mathbf{C}} = \tilde{G}_{\mathbf{C}}\, x$ ($x \in \tilde{\mathfrak{O}}$ fixed) and $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1 = \pi(\tilde{\mathfrak{O}}_{\mathbf{C}}) \subset \mathfrak{d}'_{\mathbf{C}}$. We show now, that we can find a smooth function ρ from $L_{\mathbf{C}} \times (\tilde{\mathfrak{O}}_{\mathbf{C}})_1$ into $\mathbf{C}^*$, such that $\rho(a, bf)\,\rho(b, f) \equiv \rho(ab, f)$ $[a, b \in L_{\mathbf{C}}; f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1]$, $\rho(a, f) = 1$ if $a \in (L_{\mathbf{C}})_f$ and ρ, when restricted to $L \times \tilde{\mathfrak{O}}_1$, coincides with ν [as in (*a*)]. To this end we observe, that the map $d(f)$ from $\tilde{\mathfrak{O}}_1$ into $L_{\mathbf{C}}$, introduced in 5.1 (*b*) and which satisfies $d(f)\, f \equiv b^{-1} f$ $(f \in \tilde{\mathfrak{O}})$, can be viewed as the restriction to $\tilde{\mathfrak{O}}_1 \subset (\tilde{\mathfrak{O}}_{\mathbf{C}})_1$ of a map [to be denoted again by $d(f)$] defined and having similar properties on $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1$. In fact, we remark first, that any Jordan-Hölder sequence for the action of $\tilde{G} = \exp \tilde{\mathfrak{g}}$ on $\mathfrak{d}'_{\mathbf{C}}$ is invariant under the action of $\tilde{G}_{\mathbf{C}} = \exp \tilde{\mathfrak{g}}_{\mathbf{C}}$, and hence, with the notations of 5.1 (*a*), we can conclude, that $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1 \subset \mathfrak{O}_j$. Our second remark concerns the fact, that if b is a fixed element of $\overline{H}$, then for any f in $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1$, $bf \subset L_{\mathbf{C}} f$. In fact, if $f = \pi(g)$, $g = ax$ $(a \in G_{\mathbf{C}}, x \in \mathfrak{g}')$ then, since

$$\overline{H} = \overline{G}_x\, L_{\mathbf{C}} \subset G_x\, L_{\mathbf{C}} \subset (G_{\mathbf{C}})_{ax}\, L_{\mathbf{C}} = L_{\mathbf{C}}\,(G_{\mathbf{C}})_{ax}$$

we have $bg \in L_{\mathbf{C}}\, g$, and hence also $bf \in L_{\mathbf{C}}\, f$. In this fashion, to obtain the desired extension we can proceed as in 5.1 (b) and put

$$d(f) = \exp[l(\mathrm{R}(b^{-1}f, f), f)] \qquad [f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1].$$

Defining $\sigma(f)$ by $bd(f)$ $[f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1]$ we set on $L_{\mathbf{C}} \times (\tilde{\mathfrak{O}}_{\mathbf{C}})_1$:

$$\rho(a, f) \equiv \varphi_f(a^{-1}\sigma(af)\, a\, (\sigma(f))^{-1}).$$

It is clear, that ρ on $L \times (\tilde{\mathfrak{O}})_1$ coincides with ν, and also, that it satisfies

$$\rho(a, bf)\,\rho(b, f) = \rho(ab, f) \qquad [a, b \in L_{\mathbf{C}},\, f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1].$$

Therefore, to complete our construction, it suffices to show, that $\rho(a, f) = 1$ if $a \in (L_{\mathbf{C}})_f$. But then we have

$$\rho(a, f) = \varphi_f(a^{-1})\,\varphi_f(\sigma(f)\, a\, (\sigma(f))^{-1}) = \varphi_f(a^{-1})\,\varphi_f(a) = 1$$

since $\sigma(f).f \equiv f$ $[f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1]$.

Our next objective is to establish the existence of a smooth function μ from $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1$ to $\mathbf{C}^*$, such that $\rho(a, f) \equiv \mu(af)/\mu(f)$ $[a \in L_{\mathbf{C}}, f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1]$. Then, restricting the last relation to $L \times \tilde{\mathfrak{O}}$ we shall have obtained the analogous conclusion for ν, announced at the start of this point (b). To this end we recall first, that by virtue of Corollary 4.1 there is a map h of $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1 \subset \mathfrak{O}_j$ into itself, such that, for a sufficiently large integer N, $(Q_j)^{N} h$ is the restriction to $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1$ of a polynomial map of $\mathfrak{d}'_{\mathbf{C}}$ into itself, and such that : 1° $h(af) \equiv h(f)$ $(a \in L_{\mathbf{C}})$, 2° $h(f) \in L_{\mathbf{C}}\, f$ $[f \in (\tilde{\mathfrak{O}}_{\mathbf{C}})_1]$. Using the notations of Proposition 4.2 let us put $\delta(f) \equiv \exp[l(\mathrm{R}(f, h(f)), h(f))]$; this is a smooth map of $(\tilde{\mathfrak{O}}_{\mathbf{C}})_1$ into $L_{\mathbf{C}}$ and we have $\delta(f)\, h(f) \equiv f$. Next we show, that $\mu(f) \equiv \mu(\delta(f), h(f))$ satisfies $\rho(a, f) \equiv \mu(af)/\mu(f)$ on $L_{\mathbf{C}} \times (\tilde{\mathfrak{O}}_{\mathbf{C}})_1$. In fact, since

$$\delta(af)\, h(af) = af = a\,\delta(f)\, h(f)$$

there is an element d_0 in $L_{\mathbf{C}}$ such that

$$d_0\, h(f) = h(f) \qquad \text{and} \qquad \delta(af) = a\,\delta(f)\, d_0.$$

In this fashion, using that $\rho(a, f) = 1$ if $a \in (L_{\mathbf{C}})_f$ we get that

$$\begin{aligned}\mu(af) &= \rho(\delta(af), h(af)) = \rho(a\,\delta(f)\, d_0, h(f)) = \rho(a, \delta(f)\, d_0\, h(f))\,\rho(\delta(f)\, d_0, h(f)) \\ &= \rho(a, f)\,\rho(\delta(f), d_0\, h(f))\,\rho(d_0, h(f)) = \rho(a, f)\,\mu(f),\end{aligned}$$

and thus $\rho(a, f) \equiv \mu(af)/\mu(f)$, completing the proof of our statement.

c. Using the previous observations, we shall finish the proof of Proposition 6.2 as follows. Let us put

$$\nu_1(a, f) \equiv \nu(a, f)[\mu(af)/\mu(f)]^{-1} \qquad (a \in \tilde{G}, f \in \tilde{\mathfrak{O}}_1)$$

where μ is as above in (*b*). We have on $L \times \tilde{\mathfrak{O}}_1$ that $\nu_1(a, f) \equiv 1$, from where we conclude easily, that for any choice of d, d' and d'' fix in L : $\nu_1(dad', d''f) \equiv \nu_1(a, f)$ on $\tilde{G} \times \tilde{\mathfrak{O}}_1$. To complete our proof it evidently suffices to establish the existence of a smooth map ψ from $\tilde{\mathfrak{O}}_1$ into $\mathbf{C}^*$ and of a $\omega \in \mathrm{Hom}(\tilde{G}, \mathbf{C}^*)$, such that $\nu_1(a, f) \equiv \omega(a)[\psi(af)/\psi(f)]$ [*cf.* the end of (*a*)].

Let us denote by A the group $\tilde{G}/L$, and by λ the canonical homomorphism from $\tilde{G}$ onto A. Since $\tilde{G}$ is simply connected and $L = [\tilde{G}, \tilde{G}]$ (*cf.* the start of Section 2), A is isomorphic to a vector group. Also, by what we saw above, there is a smooth function H from $A \times A$ into $\mathbf{C}^*$ such that $\nu_1(a, bf_0) = H(\lambda(a), \lambda(b))$ $(a, b \in \tilde{G}; f_0$ fixed in $\tilde{\mathfrak{O}}_1)$.

We observe, that to obtain the necessary conclusion, it is enough to show, that $H(a, k) \equiv \eta(a)(\varphi(k + a)/\varphi(k))$, where $\eta \in \mathrm{Hom}(A, \mathbf{C}^*)$ and φ is a smooth function from A into $\mathbf{C}^*$ satisfying $\varphi(k + b) \equiv \varphi(k)$ for each k in A and b in the closed subgroup $B = \lambda(\tilde{G}_{f_0}) = \tilde{G}_{f_0} L/L$ of A. In fact, defining then $\psi(f) = \varphi(\lambda(a))$ for $f = af_0$ $(a \in \tilde{G})$, and putting $\omega = \eta \circ \lambda$, we obtain with ψ and ω so defined the desired relation for ν_1.

We have obviously $H(a + b, k) \equiv H(a, b + k)\, H(b, k)$ for any a, b, k in A. Also $H(a, b + k) \equiv H(a, k)$ on $A \times A$ for any fixed b in B. From this we conclude, that putting $\gamma(b) \equiv H(b, 0)$ $(b \in B)$ we have $\gamma(b + b') \equiv \gamma(b)\gamma(b')$ on $B \times B$.

We finish our proof by showing, that $\eta \in \mathrm{Hom}(A, \mathbf{C}^*)$ satisfies the above relation for H, if and only if its restriction to B coincides with γ. In fact, the only point to be noted then is that γ is obviously extendible to an element of $\mathrm{Hom}(A, \mathbf{C}^*)$. Suppose, that we have

$$H(a, k) \equiv \eta(a)[\varphi(a + k)/\varphi(k)];$$

putting $a = b \in B$, $k = 0$, we obtain

$$H(b, 0) = \eta(b)[\varphi(b)/\varphi(0)] = \eta(b),$$

since $\varphi(b + k) \equiv \varphi(k)$ $(b \in B, k \in A)$. Conversely, let us assume, that $\eta(b) \equiv H(b, 0)$. We put $\varphi(k) \equiv H(k, 0)/\eta(k)$, and observe, that

$$\varphi(k + b) \equiv \varphi(k) \qquad (k \in A,\ b \in B).$$

In fact,

$$\varphi(k+b) = \mathrm{H}(k+b, 0)/\eta(k+b)$$
$$= (\mathrm{H}(k, b)/\eta(k))(\mathrm{H}(b, 0)/\eta(b)) = \mathrm{H}(k, 0)/\eta(k) = \varphi(k).$$

Finally we have

$$\eta(a)\varphi(k+a) = \eta(a)[\mathrm{H}(k+a, 0)/\eta(a+k)]$$
$$= \mathrm{H}(a, k)[\mathrm{H}(k, 0)/\eta(k)] = \mathrm{H}(a, k)\varphi(k)$$

for all a and k in A, completing the proof of Proposition 6.2.

Q. E. D.

6.3. Let $\mathfrak{O}$ be an orbit of $\mathfrak{N}$ on $\mathfrak{g}'$ (*cf.* Proposition 2.1); we recall (cf. *loc. cit*), that there is a connected and simply connected group $\mathrm{G}_1 \supset \mathrm{G}$, such that $[\mathrm{G}_1, \mathrm{G}_1] = [\mathrm{G}, \mathrm{G}] = \mathrm{L}$ and $\mathrm{G}_1 x = \mathfrak{O}$ for any x in $\mathfrak{O}$. Imitating the procedure of Sections 3 and 5 above, we can define a bundle $\mathfrak{B}(\mathfrak{O})$ over $\mathfrak{O}$, which is similar to the bundle $\mathfrak{B}(\tilde{\mathfrak{O}})$ over $\tilde{\mathfrak{O}}$, through the following steps.

a. First we observe, that the closed subgroup $\overline{\mathrm{G}}_g \mathrm{L}$ of G does not depend on the particular choice of g in $\mathfrak{O}$. Setting $\mathrm{K} = \overline{\mathrm{G}}_g \mathrm{L}$ $(g \in \mathfrak{O})$, we denote by Λ the canonical homomorphism from K onto

$$\mathrm{I} = \mathrm{K}/(\mathrm{K})_0 = \overline{\mathrm{G}}_g/(\mathrm{G}_g)_0 \qquad (g \in \mathfrak{O}).$$

We put

$$\overset{\multimap}{\mathrm{K}} = \hat{\mathrm{I}} \circ \Lambda \subset \hat{\mathrm{K}} \qquad \text{and} \qquad \overset{\multimap}{\mathrm{G}}_g = \{\chi; \chi = \text{character of } \overline{\mathrm{G}}_g, \chi \mid (\mathrm{G}_g)_0 = 1\}.$$

One verifies easily, that for any g in $\mathfrak{O}$, the map $\mathrm{K} \ni \psi \mapsto \psi_g = \psi \mid \overline{\mathrm{G}}_g$ is an isomorphism between $\overset{\multimap}{\mathrm{K}}$ and $\overset{\multimap}{\mathrm{G}}_g$. Let us form the group $\overline{\mathrm{H}}$ for $\tilde{\mathfrak{O}} = \tilde{\mathrm{G}}\, x$ $(x \in \mathfrak{O})$ [*cf.* 3 (*d*)]; we have $\overline{\mathrm{H}} = \overline{\mathrm{G}}_x \mathrm{L}_{\mathbf{C}}$. We note, that $\overline{\mathrm{H}} \supset \mathrm{K}$, and obviously the map $\overset{\multimap}{\mathrm{H}} \ni \psi \mapsto \psi' = \psi \mid \mathrm{K}$ is an isomorphism between $\overset{\multimap}{\mathrm{H}}$ and $\overset{\multimap}{\mathrm{K}}$ [for $\overset{\multimap}{\mathrm{H}}$, *cf.* 3 (*e*)]. Similarly, the map $\overset{\mathbb{A}}{\mathrm{H}}_g \ni \psi \mapsto \psi' = \psi \mid \overline{\mathrm{G}}_g$ [for $\overset{\mathbb{A}}{\mathrm{H}}_g$ *cf.* 3 (*c*)] is a bijection between $\overset{\mathbb{A}}{\mathrm{H}}_g$ and $\overset{\mathbb{A}}{\mathrm{G}}_g$.

We write now $\mathfrak{B}(\mathfrak{O})$ for the set $\bigcup_{g \in \mathfrak{O}} \overset{\mathbb{A}}{\mathrm{G}}_g$. If $\varphi \in \overset{\multimap}{\mathrm{K}}$ and $p = (g, \chi) \in \mathfrak{B}(\mathfrak{O})$ we define φp by $(g, \varphi_g \chi)$; given a in G (or in $\tilde{\mathrm{G}}$) we put $ap = (ag, a\chi)$. One sees at once, that $a(\varphi p) = \varphi(ap)$.

b. We employ next a local version of the construction of Section 5 to define a differentiable structure on $\mathfrak{B}(\mathfrak{O})$, which turns it into a fiber bundle over $\mathfrak{O}$ with the structure group $\overset{-\circ-}{\mathrm{K}}$, such that G acts on $\mathfrak{B}(\mathfrak{O})$ as a group of smooth transformations, commuting with the action of $\overset{-\circ-}{\mathrm{K}}$. We start by observing, that given an element a in I and a fixed element x of $\mathfrak{O}$, there is a neighborhood U of the latter in $\mathfrak{O}$, and a smooth function $\sigma : \mathrm{U} \to \mathrm{K}$, such that : 1° $\Lambda(\sigma(g)) \equiv a$, 2° $\sigma(g) \in \overline{\mathrm{G}}_g$, for all g in U. In fact, for all g in $\mathfrak{O}$ we have, if b is any element in K, $bg \in \mathrm{L}\, g$. Assume now, that $b \in \mathrm{K}$ satisfies $\Lambda(b) = a$. To attain our goal it suffices to establish the existence of a U as above, and of a smooth map $d : \mathrm{U} \to \mathrm{L}$, such that $d(g)\, g \equiv b^{-1} g$ on U. Let $\{ l_j ; 1 \leqq j \leqq \mathrm{N} \}$ be a supplementary basis to $(\mathfrak{g}_1)_x$ in $\mathfrak{g}_1$ ($\mathrm{G}_1 = \exp \mathfrak{g}_1$), and let us put $\mathrm{T} = (t_1, t_2, \ldots, t_\mathrm{N}) \in \mathbf{R}^\mathrm{N}$ and $a(\mathrm{T}) = \exp(t_1 l_1) \exp(t_2 l_2) \ldots \exp(t_\mathrm{N} l_\mathrm{N})$. The map $\mathrm{T} \mapsto a(\mathrm{T})\, x$ is a diffeomorphism between an open sphere S around zero in $\mathbf{R}^\mathrm{N}$, and some neighborhood U of x on $\mathfrak{O}$. In this fashion it will be enough to determine a smooth map $d' : \mathrm{S} \to \mathrm{L}$, such that $d'(\mathrm{T})\, x \equiv f(\mathrm{T})\, x$, where

$$f(\mathrm{T}) \equiv (a(\mathrm{T}))^{-1}\, b^{-1}\, a(\mathrm{T})\, x.$$

But to obtain this it suffices to take in the proof of Proposition 1.1, $\mathrm{V} = \mathfrak{g}'$, $\mathrm{G} = \mathrm{L}$ and define $d'(\mathrm{T}) \equiv g(\mathrm{T}; x)$ [*cf.* (*d*), *loc. cit*] where

$$t_k \equiv z_k + \psi_k(z_1, z_2, \ldots, z_{k-1}; x), \qquad z_\alpha = (f(\mathrm{T}))_{j_\alpha} \qquad [1 \leqq \alpha \leqq \dim(\mathrm{L}\, x)]$$

provided $f(\mathrm{T}) = \sum_{j=1}^{m} (f(\mathrm{T}))_j\, \varphi_j$ $(\mathrm{T} \in \mathrm{S})$.

Let us choose a basis $\{ a_j ; 1 \leqq j \leqq m \}$ in the free abelian group $\mathrm{I}\left(\sim \overline{\mathrm{G}}_x/(\mathrm{G}_x)_0\right)$; we denote by σ_j $(1 \leqq j \leqq m)$ maps corresponding to a_j as σ above to $a \in \mathrm{I}$, all defined on a neighborhood U of x on $\mathfrak{O}$. We denote by τ the map from $\mathfrak{B}(\mathfrak{O})$ onto $\mathfrak{O}$ defined by $\tau\, p = g$ $[p = (g, \chi)]$. Let us define the map β from $\overset{-1}{\tau}(\mathrm{U})$ onto $\mathrm{U} \times \mathbf{T}^m$ by $\beta(p) = (g, \omega)$ where, if $p = (g, \chi) \in \overset{-1}{\tau}(\mathrm{U}) \subset \mathfrak{B}(\mathfrak{O})$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in \mathbf{T}^m$ we have $\omega_j = \chi(\sigma_j(g))$ $(1 \leqq j \leqq m)$. We leave to the reader to verify, that by requiring, that β be a diffeomorphism between $\overset{-1}{\tau}(\mathrm{U})$ and $\mathrm{U} \times \mathbf{T}^m$ for all possible choice of U and $\{ \sigma_j \}$ as above, we obtain the differentiable structure on $\mathfrak{B}(\mathfrak{O})$ with the properties specified at the begin of (*b*) (*cf.* Section 5 for similar reasonings).

c. Let us denote by $\mathfrak{B}'(\mathfrak{O})$ the portion of $\mathfrak{B}(\tilde{\mathfrak{O}})$ over $\mathfrak{O} \subset \tilde{\mathfrak{O}}$, that is $\mathfrak{B}'(\mathfrak{O}) = \overset{-1}{\tau}(\mathfrak{O}) \subset \mathfrak{B}(\tilde{\mathfrak{O}})$ [τ being the canonical projection from $\mathfrak{B}(\tilde{\mathfrak{O}})$ onto $\tilde{\mathfrak{O}}$] with the induced structure. One verifies easily, that the map $\delta : \mathfrak{B}'(\mathfrak{O}) \mapsto \mathfrak{B}(\mathfrak{O})$ defined by $\delta(g, \chi) = (g, \chi')$ [*cf.* (*a*)] is a diffeomorphism satisfying $\delta(\varphi p) = \varphi' \delta(p)$ and $\delta(ap) = a\,\delta(p)$ for all p in $\mathfrak{B}'(\mathfrak{O})$, φ in $\overset{\circ}{\mathrm{H}}$ and a in G.

We conclude from this, using the result of 6.2 above, that the bundle $\mathfrak{B}(\mathfrak{O})$ defined in (*b*) is trivial. More specifically, there is a cross section p_0, such that $ap_0(g) \equiv \omega(a)\, p_0(ag)$ $(a \in \mathrm{G},\ g \in \mathfrak{O})$, where ω is a continuous homomorphism of G in $\overset{\circ}{\mathrm{K}}$, of which we can assume, that $\ker \omega \supset \mathrm{G}_g (\mathrm{G}_f)_0\, \mathrm{L}$ $(g \in \mathfrak{O},\ f = g \mid \mathfrak{d})$.

We can define, similarly as at the end of 6.1, the groups $\mathrm{Z}^1(\mathfrak{O})$, $\mathrm{B}^1(\mathfrak{O})$ and $\mathrm{H}^1(\mathfrak{O})$ with respect to G and $\mathfrak{O}$ in place of $\tilde{\mathrm{G}}$ and $\tilde{\mathfrak{O}}$ as *loc. cit.* We shall denote the image of ω [considered as an element of $\mathrm{Z}^1(\mathfrak{O})$] in $\mathrm{H}^1(\mathfrak{O})$ by $[\mathfrak{O}]$, and call it the obstruction to a G invariant cross section. One sees at once, that if $\mathfrak{O}$ is acted upon transitively by G, $[\mathfrak{O}]$ is equal to the identity in $\mathrm{H}^1(\mathfrak{O})$. We shall, however, show later (*cf.* Section 8 below) that this is by no means so in the general case.

7. — 7.1. In the following $\mathfrak{B}(\mathfrak{O})$ will stand for the bundle defined in 6.3; we shall assume, that $\dim \overset{\circ}{\mathrm{K}} > 0$.

PROPOSITION 7.1. — *There is an equivalence relation* $\mathbf{S}$ *on* $\mathfrak{B}(\mathfrak{O})$, *uniquely determined by the property, that if* $\mathrm{O} \in \mathfrak{B}(\mathfrak{O})/\mathbf{S}$ *and* p *is arbitrary in* O, *we have* $\overline{\mathrm{G}\,p} = \mathrm{O}$. *Furthermore, there is a connected solvable group* $\mathfrak{G}$ *operating on* $\mathfrak{B}(\mathfrak{O})$ *through an action commuting with that of the structure group, such that* $\mathfrak{G} \supset \mathrm{G}$, $[\mathfrak{G}, \mathfrak{G}] = [\mathrm{G}, \mathrm{G}]$, *and the orbits of* $\mathfrak{G}$ *coincide with the orbits of* $\mathbf{S}$ *on* $\mathfrak{B}(\mathfrak{O})$.

Proof. — *a.* By what we saw in 6.3 there is a homeomorphism η from $\mathfrak{B}(\mathfrak{O})$ onto $\mathfrak{O} \times \mathbf{T}^m$ and an isomorphism ε from $\overset{\circ}{\mathrm{K}}$ onto $\mathbf{T}^m$ with the following properties. If $p \in \mathfrak{B}(\mathfrak{O})$, $\varphi \in \overset{\circ}{\mathrm{K}}$ and $\eta\ p) = (g, \omega)$ $(g \in \mathfrak{O},\ \omega \in \mathbf{T}^m)$ we have $\eta(\varphi p) = (g, \varepsilon(\varphi)\,\omega)$, and if a is any element in G, then

$$\eta(a \overset{-1}{\eta}(q)) = (g, \omega(a)\,\omega) \qquad [q = (g, \omega)],$$

where $\omega(a)$ is a continuous homomorphism from G into $\mathbf{T}^m$ such that $\mathrm{G}_x \subset \ker \omega$ for any x in $\mathfrak{O}$. Therefore to establish the truth of our proposition it suffices to prove the analogous statement for the action just described, of G on $\mathfrak{O} \times \mathbf{T}^m$.

b. Let x be a fixed element in $\mathfrak{O}$ and G_1 as in Proposition 2.1. We denote by E_1 the quotient space $\mathfrak{O}/\mathrm{L}$ and by π_1 the canonical map from $\mathfrak{O}$ onto E_1. If $a \in \mathrm{G}_1$ we have $a\,\pi_1(y)\, [= \pi_1(ay)] = \pi_1(y)$ for all y in $\mathfrak{O}$

if and only if a belongs to $(G_1)_x L$. Therefore G_1 acts on E_1 as the abelian group $J_1 = G_1/(G_1)_x L$.

Let us put $\Psi_1(a) = \pi_1(ax)$ $(a \in G_1)$ and let us write ρ_1 for the canonical homomorphism from G_1 onto J_1. We denote finally by ψ_1 the map from J_1 onto E_1 defined such that the diagramm

$$\begin{array}{ccc} G_1 & \xrightarrow{\quad\Psi_1\quad} & \\ \rho_1 \downarrow & & \searrow \\ J_1 = G_1/(G_1)_x L & \xrightarrow{\quad\psi_1\quad} & E_1 = \mathcal{O}/L \end{array}$$

be commutative. We claim, that ψ_1 is a homeomorphism. For this it obviously suffices to prove, that ψ_1 is an open map. If U is an open set in J_1, we have $\psi_1(U) = \Psi_1(\overset{-1}{\rho_1}(U))$. In this fashion to obtain the desired conclusion it is enough to show, that Ψ_1 is an open map. Let $\tilde{G}$ be as in the proof of Proposition 2.1. We shall have attained our goal by proving, that for a sufficiently small neighborhood V of the unity in $\tilde{G}$, $\Psi_1(V \cap G_1)$ is open. Let us put, as *loc. cit.*, $\Phi(a) = ax$ $(a \in \tilde{G})$; we recall, that Φ is an open map from $\tilde{G}$ onto $\tilde{\mathcal{O}} = \tilde{G}x$. Since $\Psi_1(V \cap G_1) = \pi_1(\Phi(V \cap G_1))$, it will therefore be enough to prove, that $\Phi(V \cap G_1) = \Phi(V) \cap \Phi(G_1)$. But if $ax = bx$ for a in V and b in G_1, we have $a = bc$ with c in $\tilde{G}_x$. We recall now from 2 (*c*), that G_1 is the connected component of the identity in the closed subgroup $G_1 . \tilde{G}_x$ of $\tilde{G}$. Hence if V is a sufficiently small neighborhood of the identity in $\tilde{G}$ we have, that a $V \cap G_1 . \tilde{G}_x = V \cap G_1$, and thus $\Phi(V \cap G_1) = \Phi(V) \cap \Phi(G_1)$, completing the proof of our statement.

c. Let us denote by A the dense subgroup $\rho_1(G)$ of J_1. We note, that there is a continuous homomorphism ω_1 from A into $\mathbf{T}^m$, such that $\omega \equiv \omega_1 \circ \rho_1$ on G [*cf*. (*a*)]. In fact, we have $\rho_1(a) =$ unity for a in G if and only if it belongs to $G \cap (G_1)_x L = G_x L$; but we know, that $\ker \omega \supset G_x L$.

We put $\tilde{J}_1 = J_1 \times \mathbf{T}^m$ (direct product of abelian groups). Let $\tilde{A}$ be the subgroup $\{(a, \omega_1(a)); a \in A\}$ of $\tilde{J}_1$; we denote by B the closure of $\tilde{A}$ in $\tilde{J}_1$. We put $\tilde{E}_1 = E_1 \times \mathbf{T}^m = (\mathcal{O} \times \mathbf{T}^m)/L$, and write $\tilde{\pi}_1$ for the canonical projection from $\mathcal{O} \times \mathbf{T}^m$ onto $\tilde{E}_1$. For (j, ω) in $\tilde{J}_1$ we put

$$\tilde{\psi}_1((j, \omega)) = (\psi_1(j), \omega) \in \tilde{E}_1.$$

According to what we saw in (*b*) above, $\tilde{\psi}_1$ is a homeomorphism between $\tilde{J}_1$ and $\tilde{E}_1$.

d. Let y be the element $(x, 1)$ (1 = unity in $\mathbf{T}^m$) of $\mathfrak{D} \times \mathbf{T}^m$. To prove the existence of an equivalence relation as in Proposition 7.1, it will be enough to show, that if u is in $\overline{\mathrm{G}\, y} \subset \mathfrak{D} \times \mathbf{T}^m$, we have $\overline{\mathrm{G}\, u} = \overline{\mathrm{G}\, y}$. For this it suffices to establish $\overline{\tilde{\pi}_1 (\mathrm{G}\, u)} = \overline{\tilde{\pi}_1 (\mathrm{G}\, y)}$, since $\tilde{\pi}_1 (\overline{\mathrm{G}\, u}) = \overline{\tilde{\pi}_1 (\mathrm{G}\, u)}$ and $\tilde{\pi}_1 (\overline{\mathrm{G}\, y}) = \overline{\tilde{\pi}_1 (\mathrm{G}\, y)}$. But taking the image of both sides under the inverse of $\tilde{\psi}_1$, the desired conclusion is implied by the fact, that if b is any element in B, we have $\overline{b + \tilde{\mathrm{A}}} = \mathrm{B}$.

The uniqueness of the equivalence relation $\mathbf{S}$ is evident.

e. Finally, to obtain the group $\mathfrak{G}$, we consider first the direct product of groups $\tilde{\mathrm{G}}_1 = \mathrm{G}_1 \times \mathbf{T}^m$. It operates on $\mathfrak{D} \times \mathbf{T}^m$ by the rule $(a, \omega)(g, \omega') = (ag, \omega\omega')$. For $(a, \omega) \in \tilde{\mathrm{G}}_1$ let us put

$$\tilde{\rho}_1 ((a, \omega)) = (\rho_1 (a), \omega) \in \tilde{\mathrm{J}}_1.$$

One shows easily, that the connected component of the identity in the complete inverse image of B under $\tilde{\rho}_1$ has all the properties of $\mathfrak{G}$. For later use, let us observe, that $\mathfrak{G}\,(\mathrm{G}_1)_x$ is closed in $\tilde{\mathrm{G}}_1$, and that $\mathfrak{G} = (\mathfrak{G}\,(\mathrm{G}_1)_x)_0$.

Q. E. D.

REMARK 7.1. — Let O be an orbit of $\mathbf{S}$ and $p \in \mathrm{O}$. Let us observe, that O, as a subset of $\mathfrak{B}(\tilde{\mathfrak{D}})$ [*cf.* 6.3 (*c*)], coincides with the closure of G p in $\mathfrak{B}(\tilde{\mathfrak{D}})$.

7.2. We close this Section 7 by quoting two statements, which shed some light on the structure of the orbits of $\mathbf{S}$ and their position in $\mathfrak{B}(\mathfrak{D})$.

PROPOSITION 7.2.1. — *There is a unique closed, connected subgroup* Γ *of* $\overset{\ominus}{\mathrm{K}}$, *such that the projection of any orbit* O *of* $\mathbf{S}$ *onto* $\mathfrak{B}(\mathfrak{D})/\Gamma$, *along with the canonical projection from the latter* (considered as a $\overset{\ominus}{\mathrm{K}}/\Gamma$ bundle) *onto* $\mathfrak{D}$ *is a finite covering of* $\mathfrak{D}$.

PROPOSITION 7.2.2. — *The following three conditions are equivalent :* 1° *For any* O *in* $\mathfrak{B}(\mathfrak{D})/\mathbf{S}$, dim O = dim $\mathfrak{D}$; 2° (O, τ) *is a finite covering of* $\mathfrak{D}$; 3° [$\mathfrak{D}$] *is of a finite order in* $\mathrm{H}^1(\mathfrak{D})$ [*cf.* 6.3 (*c*)].

8. The purpose of this closing section of Chapter II is to show by an example, that the obstruction to a G invariant cross section, in general, is different from the unity in $\mathrm{H}^1(\mathfrak{D})$.

a. In the following, given a real vector space V and a finite subset v_1, v_2, ..., v_M of V, we shall denote the subspace, spanned by these elements, of V, by $[v_j; 1 \leq j \leq M]$.

Let us denote by $\mathfrak{g}$ the twelve dimensional real Lie algebra, spanned by the elements $e_1, e_2, \ldots, e_{12}$ with the following nonvanishing brackets

$$\begin{array}{lll} [e_1, e_2] = e_3, & [e_4, e_5] = e_6, & \\ [e_1, e_5] = e_6, & [e_2, e_5] = e_6, & \\ [e_1, e_7] = e_8, & [e_1, e_8] = - e_7, & \\ [e_2, e_9] = \lambda e_{10}, & [e_2, e_{10}] = - \lambda e_9 & (\lambda = \text{irrational}), \\ [e_4, e_{11}] = e_{12}, & [e_4, e_{12}] = - e_{11}. & \end{array}$$

One sees at once, that $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}] = [e_j; j = 3, j \geq 6]$ is abelian, and so is $\mathfrak{g}/\mathfrak{d}$, and therefore $\mathfrak{g}$ is solvable. We have also $\mathfrak{g}^\natural = \mathbf{R}\, e_3 + \mathbf{R}\, e_6$.

Let us put $\mathcal{V} = [e_j; 1 \leq j \leq 6]$ and $\mathcal{F} = [e_k; 7 \leq k \leq 12]$. $\mathcal{V}$ is a nilpotent, $\mathcal{F}$ an abelian subalgebra of $\mathfrak{g}$ and $[\mathcal{V}, \mathcal{F}] = \mathcal{F}$.

Writing $G = \exp \mathfrak{g}$, $V = \exp \mathcal{V}$ and $F = \exp \mathcal{F}$, any element a of G can uniquely be represented as a product $f\nu$ ($f \in F$, $\nu \in V$).

b. Let us consider a basis $\{ e'_j; 1 \leq j \leq 12 \}$, dual to the basis $\{ e_k; 1 \leq k \leq 12 \}$, in $\mathfrak{g}'$. Given y in $\mathfrak{g}$, we shall write $y = \sum_{j=1}^{12} y_j e_j$ etc. Let us put

$$W = \{ x; x_6 \neq 0, x_7^2 + x_8^2 \neq 0, x_9^2 + x_{10}^2 \neq 0, x_{11}^2 + x_{12}^2 \neq 0 \}.$$

An easy computation, the details of which we leave to the reader, shows, that if x is in W, we have

$$\begin{aligned} o(x) = G\, x = \{ y;\ & y_1, y_2, y_4 = \text{arbitrary}, y_3 = x_3, y_6 = x_6, y_5 = x_5 + (t_1 + t_2 + t_3)\, x_6, \\ & y_7 + i y_8 = e^{-it_1} (x_7 + i x_8), y_9 + i y_{10} = e^{-i\lambda t_2} (x_9 + i x_{10}), \\ & y_{11} + i y_{12} = e^{-it_3} (x_{11} + i x_{12}), (t_1, t_2, t_3) = \text{arbitrary in } \mathbf{R}^3 \}. \end{aligned}$$

Hence, in particular, for $x \in W$ we have $\dim o(x) = 6$.

c. We put $\sigma = \exp [2\pi (e_1 - e_4)]$, and write Σ for the subgroup generated by σ. One sees at once, that σ commutes with any element in $L = [G, G] = \exp \mathfrak{d}$, and that, if g belongs to W, we have $G_g L = \Sigma L$. Therefore, in particular, G_g is abelian, implying $G_g = \overline{G}_g$, and $G_g/(G_g)_0 \sim \mathbf{Z}$.

d. Assuming again $x \in W$, we have for the orbit $\mathfrak{O}$ of $\mathfrak{R}$ (cf. Proposition 2.1) containing x :

$$\begin{aligned} \mathfrak{O} = \{ y;\ & y_1, y_2, y_4, y_5 = \text{arbitrary}, y_3 = x_3, y_6 = x_6, \\ & y_7^2 + y_8^2 = x_7^2 + x_8^2, y_9^2 + y_{10}^2 = x_9^2 + x_{10}^2, y_{11}^2 + y_{12}^2 = x_{11}^2 + x_{12}^2 \} \subset W. \end{aligned}$$

e. Writing K for $\bar{G}_g L = \Sigma L$ [*cf.* 6.3 (*a*) and (*c*) above] with g in $\mathfrak{O}$, we have $K/(K)_0 \sim \mathbf{Z}$, and $\Lambda(\sigma)$ is a generator of the latter. Let us put

$$f(y) \equiv \exp\left[\frac{2\pi x_3}{\lambda(x_9^2 + x_{10}^2)}(y_{10} e_9 - y_9 e_{10})\right] \in L$$

and $\sigma(y) \equiv \sigma f(y) \equiv f(y)\sigma$. Then we have for all y in $\mathfrak{O}$: 1° $\Lambda(\sigma(y)) \equiv \Lambda(\sigma)$; 2° $\sigma(y) \in \bar{G}_y = G_y$. For $a = fv$ [$f \in F$, $v \in V$; *cf.* (*a*)] let us put

$$\omega(a) \equiv \omega(v) \equiv \exp[2\pi i v_2 x_3] \qquad \left[v = \exp\left[\sum_{j=1}^{6} v_j e_j\right]\right].$$

Obviously ω is a character of G, and an easy computation shows, that $\chi_y(a^{-1}\sigma(ay)\,a(\sigma(y))^{-1}) \equiv \omega(a)\,(a \in G)$. Writing, similarly as in 6.3 (*c*), δ for the diffeomorphism between $\mathfrak{B}(\mathfrak{O})$ and $\mathfrak{O} \times \mathbf{T}$ defined by $\delta(g, \chi) = (g, \chi(\sigma(g)))\ \left(g \in \mathfrak{O},\ \chi \in \hat{G}_g\right)$ and putting, for q in $\mathfrak{O} \times \mathbf{T}$, aq for $\delta\left(a \overset{-1}{\delta}(q)\right)$ we obtain $a(g, \omega) = (ag, \omega(a)\,\omega)\ (a \in G)$. In particular, ω is an obstruction cocycle.

f. Using the preceeding remarks, it is now easy to show, that $[\mathfrak{O}]$ is trivial if and only if $2\pi x_3$ belongs to the subgroup $\mathbf{Z} + \lambda\mathbf{Z}$ of $\mathbf{R}^1$. By the same token, we can conclude, that $[\mathfrak{O}]$ is of a finite order in $H^1[\mathfrak{O}]$ if and only if $2\pi x_3$ belongs to $\mathbf{Q} + \lambda\mathbf{Q}$ ($\mathbf{Q}$ = field of rational numbers). Hence, in particular, upon removing from W [*cf.* (*a*)] a sequence of $\mathfrak{R}$ invariant hyperplanes, we can arrange, that for the remainder $[\mathfrak{O}]$ be always of infinite order. The subsequent reasoning will show, that in the latter case $\mathbf{S}$ (*cf.* Proposition 7.1) contains but one orbit (*cf.* also Proposition 7.2.1).

To establish the above statement let us assume, that $[\mathfrak{O}]$ is trivial [that is, it equals to the unity in $H^1(\mathfrak{O})$]. We denote by φ a smooth function on $\mathfrak{O}$, such that $\omega(a) \equiv \varphi(ay)/\varphi(y)\ (a \in G,\ y \in \mathfrak{O})$, and $|\varphi(y)| \equiv 1$. Since $F \subset \ker \omega$, we conclude at once, that $\varphi(fy) \equiv \varphi(y)$ on $\mathfrak{O}$ for all f in F, and hence $\varphi(y)$ does not depend on y_1, y_2, y_4. Let us put $\mathfrak{O}_1 = \mathfrak{O}/F$; it can naturally be identified to the subset $\{y; y_5 = \text{arbitrary}, y_7^2 + y_8^2 = x_7^2 + x_8^2, y_9^2 + y_{10}^2 = x_9^2 + x_{10}^2, y_{11}^2 + y_{12}^2 = x_{11}^2 + x_{12}^2\}$ of $\mathbf{R}^1 \times \mathbf{T}^3$. If $a = fv$ is some element in G, we have

$$ay = (y_5 - (v_1 + v_2 + v_4)\,x_6,\ e^{iv_1}(y_7 + iy_8),\ e^{i\lambda v_2}(y_9 + iy_{10}),\ e^{iv_4}(y_{11} + iy_{12})).$$

Let us define the map ε from $\mathfrak{O}$ onto $\Gamma = \mathbf{R}^1 \times \mathbf{T}^3$ by

$$\varepsilon(y) = (-y_5/x_6,\ (y_7 + iy_8)/|y_7 + iy_8|,\ (y_9 + iy_{10})/|y_9 + iy_{10}|,\ (y_{11} + iy_{12})/|y_{11} + iy_{12}|).$$

Writing $\varepsilon(y) = (u, \omega_1, \omega_2, \omega_3)$ ($u \in \mathbf{R}$; $\omega_j \in \mathbf{T}$ for $j = 1, 2, 3$) we have

$$\varepsilon(ay) = (u + v_1 + v_2 + v_4, e^{iv_1}\omega_1, e^{i\lambda v_2}\omega_2, e^{iv_4}\omega_3).$$

We denote by Γ' the dense subgroup $\{v; v = (v_1 + v_2 + v_4, e^{iv_1}, e^{i\lambda v_2}, e^{iv_4}), (v_1, v_1, v_4)$ arbitrary in $\mathbf{R}^3\}$ of Γ. Let φ' be the function corresponding to φ on Γ; we can obviously assume, that $\varphi'(e) = 1$ (e = unity in Γ). Then φ' (on Γ') is a character of Γ', and since it is continuous, it belongs to the character group of Γ. We must have therefore

$$\varphi'(u, \omega_1, \omega_2, \omega_3) \equiv \exp[icu]\,\omega_1^{n_1}\,\omega_2^{n_2}\,\omega_3^{n_3}$$

on Γ ($c \in \mathbf{R}$, $n_j \in \mathbf{Z}$, $j = 1, 2, 3$ properly chosen). On the other hand, by assumption, on Γ' : $\varphi'(v) \equiv \omega(v) \equiv \exp[2\pi i v_2 x_3]$, which implies at once, that $2\pi x_3 \in \mathbf{Z} + \lambda\mathbf{Z}$. The converse statement follows easily from the previous reasonings.

CHAPTER III

The nontransitive theory

Summary. — We start this chapter by showing, that each orbit O of the equivalence relation $\mathfrak{S}$ on $\mathcal{R}$ (*cf.* Summary of Chapter II). carries an, up to a positive multiplicative constant uniquely determined, G invariant Borel measure μ. Using this, in Section 2 we assign to each orbit O of $\mathfrak{S}$ a factor representation as follows. We recall first (*cf.* Section 7, Chapter I), that the procedure of the transitive theory assigns to each point p of $\mathcal{R}$ a unitary equivalence class $\mathcal{F}(p)$ of concrete factor representations. We have $\mathcal{F}(p) = \mathcal{F}(p')$ if and only if p and p' lie on the same G orbit. One can easily show, that there is a field $\{T(p); p \in O\}$ of concrete unitary representations, such that $T(p)$ belongs to $\mathcal{F}(p)$, and that we can form $\int_O \oplus T(p)\, d\mu(p)$. We show (*cf.* Theorem 2, Section 2), that this integral defines a factor representation, the unitary equivalence class of which is independent of the particular choice of the field used in its construction. It is of type I, if and only if O is a G orbit, and if for some (and hence for all) p in O, $\mathcal{F}(p)$ is composed of type I factors. We know, that if $p = (g, \chi)$ $\left(\chi \in \hat{G}_g\right)$ the latter condition is fulfilled if and only if the reduced stabilizer of g is of a finite index in the stabilizer of g (*cf.* Theorem 1, Chapter I). Let $\mathfrak{O}$ be the projection of O into $\mathfrak{g}'$; then $\mathfrak{O}$ is an $\mathfrak{R}$ orbit, and O is an $\mathfrak{S}$ orbit in $\mathfrak{B}(\mathfrak{O})$ (*cf.* e. g. Summary, Chapter II). Thus, in particular, $\mathfrak{O}$ must be locally closed in $\mathfrak{g}'$. We conclude therefore, that if the $\mathfrak{R}$ orbit $\mathfrak{O}$ is not a G orbit, then no $\mathfrak{S}$ orbit of $\mathfrak{B}(\mathfrak{O})$ can give rise to a type I factor. Also, in order, that G be of type I, any orbit of the coadjoint representation must be locally closed, or $\mathfrak{g}'/G = \mathfrak{g}'/\mathfrak{R}$. One can show, that if O_j are $\mathfrak{S}$ orbits, such that their projection into the dual of $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ coincide, and T_j are representations corresponding to O_j ($j = 1, 2$), then there is a character φ of G, such that $T_2 = \varphi T_1$, and

conversely. Hence, in particular, two $\mathfrak{S}$ orbits in $\mathfrak{B}(\mathfrak{O})$ give rise to factor representations of the same type. We could not decide, if the factors of Theorem 2 are all semifinite or not. It will, however, follow from the results of Chapter IV, that the collection of $\mathfrak{R}$ orbits, giving rise to a factor representation of type III, at the worst can be enclosed in a set of Lebesgue measure zero, of $\mathfrak{g}'$. To explain the motivation behind Theorem 3 (*cf.* Section 3), let us consider again the first derived group $L = [G, G]$. Denoting by $\mathcal{L}_L$ and $\mathcal{L}_G$ the left regular representation of L and G resp., we have $\mathcal{L}_G = \operatorname{ind}_{L \uparrow G} \mathcal{L}_L$. Since L is nilpotent, it is unimodular and of type I. Assume, that $\mathcal{L}_L = \int_{\hat{L}} \oplus T(\zeta)\, d\mu(\zeta)$, where $d\mu(\zeta)$ is the Plancherel measure, and $T(\zeta)$ is an appropriate multiple of a concrete irreducible representation of the unitary equivalence class $\zeta \in \hat{L}$. Then this decomposition is central; in other words, the von Neumann algebra $\mathbf{R}(\mathcal{L}_L)$ generated by $\mathcal{L}_L$ (that is, the left ring of L) contains the ring of all diagonalisable operators. In fact, the latter coincides with the center of $\mathbf{R}(\mathcal{L}_L)$. We have also

$$\mathcal{L}_G = \int_{\hat{L}} \oplus W(\zeta)\, d\mu(\zeta), \qquad \text{where} \quad W(\zeta) = \operatorname{ind}_{L \uparrow G} T(\zeta).$$

In general, neither are the $W(\zeta)$'s factor representations, nor is the last decomposition central. One obtains a decomposition with the latter property by appropriately « grouping » the « summands » on the right hand side. To this end we can proceed, for instance, as follows. Let $\tilde{G}$ be a connected and simply connected Lie group with the Lie algebra $\tilde{\mathfrak{g}}$, such that $\tilde{G} \supset G$, $[\tilde{G}, \tilde{G}] = [G, G]$ and $\tilde{\mathfrak{g}}$ be isomorphic to an algebraic Lie algebra (*cf.* Section 2, Chapter II). Then $\tilde{G}$ operates on $\hat{L}$, such that $\hat{L}/\tilde{G}$ is countably separated, and there is a measure τ on $S = \hat{L}/\tilde{G}$ such that denoting, for $s \in S$, the corresponding $\tilde{G}$ orbit by $O(s)$, and by $d\nu_s(\zeta)$ a suitably chosen measure, which is quasi-invariant under $\tilde{G}$, on $O(s)$, the Plancherel measure μ is a continuous direct sum, with respect to τ, of all these measures. Let us put

$$Z(s) = \int_{O(s)} \oplus W(\zeta)\, d\nu_s(\zeta).$$

We have

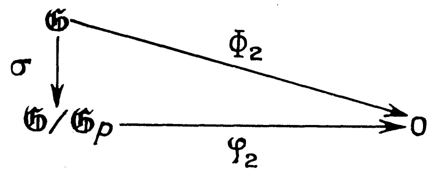
$$\mathcal{L}_G = \int_S \oplus Z(s)\, d\tau(s)$$

and this decomposition is already central (*cf.* for all this Section 9, Chapter IV). This being so, Theorem 3 (*cf.* Section 3 below) asserts, that any of these representations $Z(s)$ ($s \in S$) is quasi-equivalent to a central continuous direct sum of an appropriate subcollection of the factor representations defined by Theorem 2. The results of this chapter will be used in an essential fashion at the end of the next chapter to analyze the structure of the regular representation of G.

1. Below we shall employ the notations of II.7.1.

PROPOSITION 1.1. — *Let* O *be any orbit of* $\mathbf{S}$ *on* $\mathfrak{B}(\mathfrak{O})$. *There is a* (up to a multiplicative constant) *unique nonzero positive* G *invariant Borel measure on* O.

Proof. — *a*. Let p be a fixed element of O. For a in $\mathfrak{G}$ let us put $\Phi_2(a) = ap$. We denote by σ the canonical map from $\mathfrak{G}$ onto $\mathfrak{G}/\mathfrak{G}_p$, and define the map φ_2 from $\mathfrak{G}/\mathfrak{G}_p$ onto $\mathrm{O} = \mathfrak{G}\, p$, such that the diagramm



be commutative. *We claim, that* φ_2 *is a homeomorphism.* To show this, it evidently suffices to prove, that Φ_2 is open. We identify, as in 7.1 (*a*), $\mathfrak{B}(\mathfrak{D})$ to $\mathfrak{D} \times \mathbf{T}^m$. Assuming, that $p = (x, \omega_0)$ we put $\Phi_1(a) = ax\ (a \in \mathrm{G}_1)$. Let us write, for b in $\tilde{\mathrm{G}}_1$, $\tilde{\Phi}_1(b) = bp$. If $b = (a, \omega)$, we have

$$\tilde{\Phi}_1(b) = (ax, \omega\omega_0) = (\Phi_1(a), \omega\omega_0).$$

We proved in 7.1 (*b*) that Φ_1 is open, which implies, that $\tilde{\Phi}_1$ too, is open. In this fashion to establish, that $\Phi_2 = \tilde{\Phi}_1 | \mathfrak{G}$ is open, it is enough to show, that for a sufficiently small neighborhood V of the identity in $\tilde{\mathrm{G}}_1$, we have $\tilde{\Phi}_1(\mathrm{V} \cap \mathfrak{G}) = \tilde{\Phi}_1(\mathrm{V}) \cap \tilde{\Phi}_1(\mathfrak{G})$ (cf. *loc. cit.* for a similar reasoning). If $a \in \mathrm{V}$ and $b \in \mathfrak{G}$ are such that $ap = bp$, we have, since $(\tilde{\mathrm{G}}_1)_p = (\mathrm{G}_1)_x$, $a \in \mathrm{V} \cap \mathfrak{G}.(\mathrm{G}_1)_x$. But we have $\mathfrak{G} = (\mathfrak{G}(\mathrm{G}_1)_x)_0$ [*cf.* 7.1 (*e*)] and hence, if V is sufficiently small, $a \in \mathrm{V} \cap \mathfrak{G}$, and thus

$$\tilde{\Phi}_1(\mathrm{V}) \cap \tilde{\Phi}_1(\mathrm{G}) \subset \tilde{\Phi}_1(\mathrm{V} \cap \mathfrak{G}),$$

proving our statement.

We conclude from the preceding reasoning, that we shall have proved our proposition at once we can show the existence and uniqueness of a positive nontrivial G invariant measure on $\mathfrak{G}/\mathfrak{G}_p$.

b. Our next objective is to establish, that $\mathfrak{G}/\mathfrak{G}_p$ carries a $\mathfrak{G}$ invariant measure of the indicated sort. Let us write $\mathfrak{G} = \exp \mathfrak{g}_0$, and $((\mathrm{G}_1)_x)_0 = \exp[(\mathfrak{g}_1)_x] = (\mathfrak{G}_p)_0$ [since $\mathfrak{G}_p = \mathfrak{G} \cap (\mathrm{G}_1)_x$ and $((\mathrm{G}_1)_x)_0 \subset \mathrm{G}$]. Hence to arrive at our goal it is enough to show that

$$\det(\mathrm{Ad}(a) \mid \mathfrak{g}_0) = \det(\mathrm{Ad}(a) \mid (\mathfrak{g}_1)_x) \quad \text{for all } a \text{ in } (\mathrm{G}_1)_x.$$

To this end let us observe once more, that $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{d}$ (*cf.* for the second relation Proposition 2.1). In fact, if $\tilde{\mathrm{G}}_1 = \exp \tilde{\mathfrak{g}}_1$, we have $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \times \mathfrak{a}$, where $\mathfrak{a}$ is abelian, and thus $[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{g}}_1] = [\mathfrak{g}_1, \mathfrak{g}_1]$. On the other hand, if a is any element of G, we have $(a, \omega(a)) \in \mathfrak{G}$, and hence

$(l, 1) \in [\mathfrak{G}, \mathfrak{G}]$ $(l \in \mathrm{L})$ implying $[\mathfrak{g}_1, \mathfrak{g}_1] \subset [\mathfrak{g}_0, \mathfrak{g}_0]$. Using this observation we can conclude, that

$$\det(\mathrm{Ad}(a) \mid \mathfrak{g}_0) = \det(\mathrm{Ad}(a) \mid \mathfrak{d}) = \det(\mathrm{Ad}(a) \mid \mathfrak{g}),$$

and

$$\det(\mathrm{Ad}(a) \mid (\mathfrak{g}_1)_x) = \det(\mathrm{Ad}(a) \mid (\mathfrak{g}_1)_x \cap \mathfrak{d}) = \det(\mathrm{Ad}(a) \mid \mathfrak{g}_x),$$

and therefore it is enough to show, that

$$\det(\mathrm{Ad}(a) \mid \mathfrak{g}/\mathfrak{g}_x) = 1 \qquad [a \in (\mathrm{G}_1)_x].$$

For l_1, l_2 in $\mathfrak{g}$ let us put $\mathrm{B}(l_1, l_2) = ([l_1, l_2], x)$; B gives rise to a nondegenerate skew symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_x$. Putting $al = \mathrm{Ad}(a)\, l$ and $ax = (\mathrm{Ad}(a^{-1}))'\, x$, we have

$$\mathrm{B}(al_1, al_2) = ([al, al_2], x) = (a[l, l_2], x) = ([l_1, l_2], a^{-1}x) = ([l_1, l_2], x) = \mathrm{B}(l_1, l_2)$$
$$(l_1, l_2 \in \mathfrak{g}, \quad \text{for all } a \in (\mathrm{G}_1)_x);$$

consequently $\mathrm{Ad}(a) \mid \mathfrak{g}/\mathfrak{g}_x$ leaves B invariant, implying $\det(\mathrm{Ad}(a) \mid \mathfrak{g}/\mathfrak{g}_x) = 1$.

c. Let μ be a Borel measure on $\mathfrak{G}/\mathfrak{G}_p$, and let us suppose, that it is G invariant. To establish the uniqueness statement of our proposition it will suffice to show, that μ is also $\mathfrak{G}$ invariant. Let us denote by dk an element of the L invariant measure on $\mathfrak{G}/\mathfrak{G}_p \supset \sigma(\mathrm{L}) = \mathrm{L}/\mathrm{L}_x$; dk is invariant also under the action of $\mathfrak{G}_p$. Therefore, if f is some continuous function of compact support on $\mathfrak{G}/\mathfrak{G}_p$, the function $\mathrm{F}(a) = \int_{\sigma(\mathrm{L})} f(ak)\, dk$ satisfies $\mathrm{F}(a_0\, a) \equiv \mathrm{F}(a)$ for any $a_0 \in \mathfrak{G}_p \mathrm{L}$. Let us put $\mathfrak{A} = \mathfrak{G}/\mathfrak{G}_p \mathrm{L}$; we denote by λ the canonical homomorphism from $\mathfrak{G}$ onto $\mathfrak{A}$. Using the same letter to indicate the function corresponding to F, as above, on $\mathfrak{A}$, and putting $\bar{a} = \lambda(a)$, we can conclude, that $\mathrm{F}(\bar{a})$ is continuous and of a compact support on $\mathfrak{A}$. It is known furthermore, that any function of the said sort can be obtained in this fashion.

Since, by assumption, μ is G invariant, it is, in particular, L invariant. Hence there is a positive Borel measure ν on $\mathfrak{A}$, such that for any f as above we have

$$\int_{\mathfrak{A}} \left(\int_{\sigma(\mathrm{L})} f(ak)\, dk \right) d\nu = \int_{\mathfrak{G}/\mathfrak{G}_p} f(g)\, d\mu.$$

The G invariance of μ implies the invariance of ν by translations of $\lambda(\mathrm{G}) \subset \mathfrak{A}$. But since $\lambda(\mathrm{G})$ is dense in $\mathfrak{A}$ [*cf.* 7.1 (*e*); observe, that with the notation as *loc. cit.*, $\lambda = \tilde{\rho}_1 \mid \mathrm{G}$], ν is also $\mathfrak{A}$ invariant, implying the $\mathfrak{G}$ invariance of μ.

Q. E. D.

2. — 2.1. Before proceeding, let us recall the following facts from Chapter I. Assume, that $p \in \mathcal{R} = \cup_{g \in \mathfrak{g}'} G_g$. For an $\mathfrak{h} \subset \mathfrak{d}_{\mathbf{C}}$ with $\mathfrak{h} = \mathrm{pol}(p)$ we form ind $(\mathfrak{h}, p)$ (*cf.* Remark 7.2, Chapter I). By Theorem 1 ind $(\mathfrak{h}, p)$ is a factor representations of type I or II, uniquely determined, up to unitary equivalence, by p. We denote by $\mathcal{F}(p)$ the unitary equivalence class of representations determined in this fashion. $\mathcal{F}(p)$ is of type I if and only if, assuming $p = (g, \chi)$, the order of $G_g/\overline{G}_g$ is finite. For p, $p' \in \mathcal{R}$ we have $\mathcal{F}(p) = \mathcal{F}(p')$ if and only if $p' = ap$ for some a in G. Otherwise any pair of representations, with members from $\mathcal{F}(p)$ and $\mathcal{F}(p')$ resp., is disjoint. By virtue of Lemma 4.3, Chapter I, we have a ind $(\mathfrak{h}, p) =$ ind $(a\,\mathfrak{h}, ap)$, in the sense of unitary equivalence, for all a in G. We shall also use the following relation, the easy verification of which, by aid of the reasonings of Lemma 7.1, Chapter I, we leave to the reader. Let us suppose, that ξ is a character of G, such that $d\xi = ic$ $(c \in \mathfrak{g}')$. Then

$$\xi \operatorname{ind}(\mathfrak{h}, p; K) = \operatorname{ind}(\mathfrak{h}, \xi \mid (\overline{G}_g)\, \chi, g + c; K)$$

in the sense of unitary equivalence.

2.2. Let O be a fixed orbit of $\mathbf{S}$ in $\mathfrak{B}(\mathfrak{D})$, and μ a G invariant Borel measure on O (*cf* Proposition 1.1). In the following the notions of measurability, summability etc. will be understood with respect to the measure space derived from the field of Borel sets on O by aid of μ.

By a *field of polarizations* we shall mean a rule, which assigns to each point p of O a polarization $\mathfrak{h}_p$ with respect to p [or $\mathfrak{h}_p = \mathrm{pol}(p)$]. One can construct a special class of such objects in the following fashion. Let p_0 be a fixed point of O. Assuming $p_0 = (g_0, \chi_0)$ and putting $f_0 = g_0 \mid \mathfrak{d}$, the contragredient action on $\mathfrak{d}'$ of $K = \mathfrak{G}_p \mid \mathfrak{d}$ leaves f_0 invariant. Since $[\mathfrak{G}, \mathfrak{G}] = L$ (*cf.* Proposition 7.1, Chapter I), we have also $[K, K] \subset \mathrm{Ad}(L)$. Therefore [*cf.* the end of 1.4 (*b*)] there is a polarization with respect to f_0 ($= \mathfrak{h}$, say), which is invariant under the action of $\mathfrak{G}_p$ in $\mathfrak{d}_{\mathbf{C}}$. Thus, if $ap_0 = a' p_0$ $(a, a' \in \mathfrak{G})$ we have $a\,\mathfrak{h} = a'\,\mathfrak{h}$, and hence we can define $\mathfrak{h}_p = a\,\mathfrak{h}$, if $p = a\,p_0$. We have $\mathfrak{h}_p = \mathrm{pol}(p)$ and also $a\,\mathfrak{h}_p = \mathfrak{h}_{ap}$ $(p \in O,\ a \in \mathfrak{G})$.

Let $\{ \mathfrak{h}_p;\ p \in O \}$ be a field of polarizations and let us put

$$T(p) = \operatorname{ind}(\mathfrak{h}_p, p) \qquad \text{and} \qquad H(p) = \mathbf{H}(T(p)).$$

We are going to show, that on the field of Hilbert spaces $\{ H(p);\ p \in O \}$ we can define a measurable structure, such that the field $\{ T(p);\ p \in O \}$ of concrete representations turns out to be measurable. We recall (*cf.* [13], Proposition 4, Chapter II, § 1, and [12], 18.7.1), that to accomplish this we have to construct a sequence $\{ f_n(p); n = 1, 2, \ldots \}$ of fields of vectors, such that : 1° For each fixed p, the set $\{ f_n(p) \}$ is total in $H(p)$; 2° For each pair n, m of positive integers and a in G, the function $p \mapsto (T(p)(a) f_n(p), f_m(p))$ is measurable. Assuming, that all this can be done for a special choice of our field of polarizations it is clear, that the same can be done at least in one fashion for any other choice, too. The result, however, is essentially uniquely determined. In fact (*cf* [12], 8.2.3, Proposition and 18.7.6), if $\{ H(p), T(p) \}$ and $\{ H'(p), T'(p) \}$ are measurable fields of Hilbert spaces and of unitary representations resp., such that $T(p)$, $T'(p) \in \mathcal{F}(p)$ $(p \in O)$, then for each $p \in O$ there is a unitary map $V(p) : H(p) \to H'(p)$, which makes $T(p)$ correspond to $T'(p)$, and which has the property, that if $\{ f(p); p \in O \}$ is a field of vectors, measurable with respect to $\{ H(p); p \in O \}$, then $\{ f'(p) \equiv V(p) f(p); p \in O \}$ is measurable with respect to $\{ H'(p); p \in O \}$. For later use let us observe, that in this case, in particular, the unitary representations

$$T = \int_O \oplus T(p)\, d\mu \quad \text{and} \quad T' = \int_O \oplus T'(p)\, d\mu$$

are unitarily equivalent. To establish the statement formulated above, let us fix a point p_0 of O. To attain our goal it obviously suffices to exhibit a neighborhood U of p_0, a measurable field of Hilbert spaces $\{ H(p) \}$ and of representations resp. over U, such that $T(p) \in \mathcal{F}(p)$. Let us choose U such, that there exist a continuous map $p \mapsto a(p)$ from U into $\mathfrak{G}$ satisfying $p \equiv a(p)\, p_0$. We denote by $\mathfrak{h}$ a polarization with respect to p_0. Let us define

$$H(p) = \mathbf{H}(\operatorname{ind}(\mathfrak{h}, p_0)) \quad \text{and} \quad T(p) = a(p) \operatorname{ind}(\mathfrak{h}, p_0) \quad (p \in U).$$

Since $T(p)$ is unitarily equivalent to $\operatorname{ind}(a(p)\,\mathfrak{h},\ a(p)\, p_0 = p)$, and since $a(p)\,\mathfrak{h} = \operatorname{pol}(p)$, we have $T(p) \in \mathcal{F}(p)$. Let $\{ f_n; n = 1, 2, \ldots \}$ be a total sequence in $H(p_0)$. Putting $f_n(p) \equiv f_n$ $(p \in U,\ n = 1, 2, \ldots)$ obviously all conditions will be met.

Summing up, given a field of polarizations $\{ \mathfrak{h}_p, p \in O \}$, and writing, as above, $T(p) = \operatorname{ind}(\mathfrak{h}_p, p)$ we can form the unitary representation $\int_O \oplus T(p)\, d\mu$. Its unitary equivalence class depends on O only; we shall denote it by $\mathcal{F}(O)$.

2.3. The following two lemmas are close to a result of E. Effros (*cf.* [16], Proposition 8.6).

LEMMA 2.3.1. — *The representations of* $\mathcal{F}(O)$ *are factor representations of* G.

Proof. — With notations as above, let us denote by $\mathfrak{T}$ the ring of the decomposition $T = \int_0 \oplus T(p)\, d\mu$ (= ring of diagonalisable operators). Since, for any a in G, $T(a) \in \mathfrak{T}'$ (= commutant of $\mathfrak{T}$), we have $\mathbf{R}(T) \subset \mathfrak{T}'$. Therefore, if A belongs to the center of $\mathbf{R}(T)$, we have $A \in \mathfrak{T}'$. Hence A is decomposable; we shall write $A = \int_0 \oplus A(p)\, d\mu$. If a is fix in G, we have $T(a) A = AT(a)$ and thus $T(p)(a) A(p) = A(p) T(p)(a)$ almost everywhere with respect to μ. Hence the same conclusion holds true if a varies over a countable dense subset of G, from where we derive, that changing, if necessary, $\{A(p)\}$ on a set of μ measure zero, we can assume $T(p)(a) A(p) = A(p) T(p)(a)$ for all p in O and a in G. In this fashion we have $A(p) \in (\mathbf{R}(T(p)))'$ for all $p \in O$. Let $\{A_n;\ n = 1, 2, \ldots\}$ be a sequence of linear combinations of the operators $\{T(a);\ a \in G\}$ such that $A_n \to A$ in $\mathbf{H}(T)$ strongly. Replacing, if necessary, $\{A_n\}$ by a suitable subsequence, we have then $A_n(p) \to A(p)$ strongly for all p, which do not belong to a set E of μ measure zero. We obtain in this fashion, that for all p in O-E, $A(p)$ belongs to the center of $\mathbf{R}(T(p))$. Since $\mathbf{R}(T(p))$ is a factor, we conclude, that there is a bounded measurable function $\varphi(p)$ on O, such that $A(p) \equiv \varphi(p) I_p$ [I_p = identity operator on $\mathbf{H}(p)$] almost everywhere. We can obviously assume that, with the above notations, $A_n(p) \to \varphi(p) I_p$ with the possible exception of a G invariant set F of μ measure zero. Let a be a fixed element in G; for a p in O-F let us put $p' = ap$. Let U be a unitary map from $H(p)$ onto $H(p')$ such that $T(p') = UT(p) U^{-1}$. Then we have for each $n = 1, 2, \ldots$,

$$A_n(p') = UA_n(p) U^{-1},$$

whence, passing to the limit, we conclude, that $\varphi(ap) \equiv \varphi(p)$ for all a in G and all p in O-F. Let us consider the Borel measure $d\rho = \varphi(p)\, d\mu$ on O; according to what we have just seen, it is G invariant. From this, reasoning as in 1 (c) we derive, that $d\rho$ is also $\mathfrak{G}$ invariant implying, that φ coincides almost everywhere with a constant α, and thus $A = \alpha I$. Summing up, if the bounded operator A on $\mathbf{H}(T)$ belongs to the center of $\mathbf{R}(T)$, it is scalar multiple of the identity proving, that T is a factor representation of G. Q. E. D.

LEMMA 2.3.2. — *Let* O *be an orbit of* $\mathfrak{H}$ *on* $\mathfrak{B}(\mathfrak{O})$. *The* (factor) *representations of* $\mathcal{F}(O)$ *are of type* I *if and only if* O, *and hence also* $\mathfrak{O}$ *is a* G *orbit, and if for some* $p \in O$ [and thus for all $p \in \mathfrak{B}(\mathfrak{O})$] $\mathcal{F}(p)$ *consists of type* I *representations.*

Proof. — The sufficiency of the above condition follows from $\mathcal{F}(ap) = \mathcal{F}(p)$ $(a \in G,\ p \in O)$ along with standard facts of reduction theory (*cf.* [12], 8.1.7 and 18.7.6). Let us observe immediately, that in this case, if $U \in \mathcal{F}(p_0)$ (p_0 arbitrary in O), and $T \in \mathcal{F}(O)$, T is unitarily equivalent to a multiple of U.

To show the necessity, we employ the previous notations and assume, that T is of type I. Then we have a representation of $\mathbf{H}(T)$ in the form $H_1 \otimes H_2$, and an irreductible unitary representation T′ of G on H_1, such that $T = T' \otimes I_2$ (I_2 = unit operator on H_2). Let again $\mathfrak{T}$ be the ring of decomposition. Since we have $\mathfrak{T} \subset (\mathbf{R}(T))' = I \otimes B(H_2)$, there is an abelian von Neumann algebra Q on H_2, such that $\mathfrak{T} = I \otimes Q.\mathfrak{T}$, and thus also Q, is $\ast$-isomorphic to $L^\infty_\mu(O)$ acting by multiplications on $L^2_\mu(O)$. Therefore, there is a subdivision of O into a sequence of pairwise disjoint measurable sets $\{O_n;\ n = 1, 2, \ldots\}$, such that H_2 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus (H_n \otimes L^2_\mu(O_n))$ (H_n = n-dimensional unitary space), and if $A(\varphi)$ is the operator, corresponding in Q to $\varphi \in L^\infty_\mu(O)$, and if we put $\varphi_n \equiv \varphi \mid O_n$, and $L(\varphi_n)$ for the multiplication operator by φ_n on $L^2_\mu(O_n)$, we have

$$A(\varphi) = \sum_{n=1}^{\infty} \oplus (I_n \otimes L(\varphi_n))$$

under the above unitary correspondence (I_n = unity on H_n). We conclude from all this, that if E_n is the subset of O, where $T(p)$ is unitarily equivalent to an n-fold multiple of T′, then E_n is measurable; in fact, it differs from O_n by a set of measure zero. Also, the complement of $\bigcup_{n=1}^{\infty} E_n$ in O is of measure zero. Each E_n is evidently G invariant and most importantly, since $\mathcal{F}(p') = \mathcal{F}(p)$ $(p,\ p' \in O)$ implies, that $p' = ap$ for some a in G (*cf.* 2.1), E_n is a G orbit.

To prove, that O (and hence also $\mathfrak{O}$) is a G orbit, it suffices to establish that, in the notations of 1 above, $\lambda(G) = \mathfrak{A}$. If $\lambda(G) \subsetneq \mathfrak{A}$, any G orbit in O of is μ measure zero. But we have just shown, that if T is of type I, there is a G orbit of positive measure, and hence O itself must be a G orbit.

Q. E. D.

2.4. We sum up the result of the previous discussion in the following fashion.

Theorem 2. — *Let* O *be an orbit of* $\mathfrak{S}$, *and* μ *a* G *invariant positive Borel measure on* O. *If* $\{\mathfrak{h}_p;\ p \in O\}$ *is a field of polarizations, we can form the unitary representation* $T = \int_O \oplus \operatorname{ind}(\mathfrak{h}_p, p)\, d\mu$ *which, up to unitary equi-*

valence, is well determined by O. T *is a factor representation. It is of type* I *if and only if* O *is a* G *orbit, anf for some* $p = (g, \chi) \in$ O (and hence for all) *the index of the reduced stabilizer of g in the stabilizer is finite.*

REMARK 4.3.1. — One can show, that, if G is of type I, the representations of $\mathcal{F}$ (O) are multiples of the irreducible representations assigned to O by Auslander and Kostant in [1].

3. In the following $\tilde{G}$ will have the same meaning as set forth at the start of II.2. Let x_0 an element of $\mathfrak{g}'$, which will be kept fixed in the sequel. Denoting again by π the canonical projection from $\mathfrak{g}'$ onto $\mathfrak{d}'$, we put

$$\tilde{\Omega} = \pi(\tilde{G} x_0) \quad \text{and} \quad \Omega = \overset{-1}{\pi}(\tilde{\Omega}) \subset \mathfrak{g}'.$$

3.1. *a.* Let us write Σ for the orthogonal complement of $\mathfrak{d}$ in $\mathfrak{g}'$. Since, if $\tilde{G} = \exp \tilde{\mathfrak{g}}$, we have $[\tilde{\mathfrak{g}}, \mathfrak{g}] \subset [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \mathfrak{d}$, we conclude, that for $a \in G$ and $\sigma \in \Sigma$, $a\,\sigma = \sigma$, and hence the direct product $\tilde{G} \times \Sigma$ acts transitively on Ω. Let us observe next, that Ω carries a Borel measure, invariant with respect to $\tilde{G} \times \Sigma$. In fact, we start by showing, that $\tilde{\Omega}$ has a $\tilde{G}$ invariant Borel measure dv. Since $\tilde{\Omega} = \tilde{G}\,\pi(x_0)$ is homeomorphic, under the natural identification, to $\tilde{G}/\tilde{G}_{\pi(x_0)}$ (*cf.* Proposition 1.2, Chapter II), to attain our goal, it is enough to establish the existence of a measure of the said sort on the latter. To this end, it suffices to verifiy, that for any a in $\tilde{G}_{f_0}$ $[f_0 = \pi(x_0)]$ we have

$$\det(\mathrm{Ad}(a) \mid \tilde{\mathfrak{g}}) = \det(\mathrm{Ad}(a) \mid \tilde{\mathfrak{g}}_{f_0}).$$

This is so, if we can show, that

$$\det(\mathrm{Ad}(a) \mid \mathfrak{d}) = \det(\mathrm{Ad}(a) \mid \mathfrak{d}_{f_0}) \quad \text{or} \quad \det(\mathrm{Ad}(a) \mid \mathfrak{d}/\mathfrak{d}_{f_0}) = 1 \qquad (a \in \tilde{G}_{f_0}),$$

but this follows at once from the fact, that Ad (a) leaves invariant the nondegenerate skew-symmetric form, corresponding to $(l_1, l_2) \mapsto ([l_1, l_2], f_0)$ $(l_1, l_2 \in \mathfrak{d})$ on $\mathfrak{d}/\mathfrak{d}_{f_0} \times \mathfrak{d}/\mathfrak{d}_{f_0}$ [*cf.* 1 (b) for a similar reasoning].

If $f(x)$ is continuous and of a compact support on Ω, $F(\bar{x}) = \int_\Sigma f(x+\sigma)\,d\sigma$, where $d\sigma$ is a positive translation invariant measure on Σ, will be of the same kind on $\tilde{\Omega}$ $[\bar{x} = \pi(x)]$. Therefore there is a Borel measure $d\mu$ on Ω determined by

$$\int_\Omega f(x)\,d\mu = \int_{\tilde{\Omega}} \left(\int_\Sigma f(x+\sigma)\,d\sigma \right) dv.$$

The invariance of $d\mu$ under the action of Σ is clear; that it is also $\tilde{G}$ invariant follows from $F(\overline{ax}) = F(a\overline{x})$ $(a \in \tilde{G})$.

b. If g and g_1 are elements in Ω, we have $\mathfrak{g}_{g_1} + \mathfrak{d} = \mathfrak{g}_g + \mathfrak{d}$. In fact, to this end it is enough to observe, that : 1° $\mathfrak{g}_g$ depends on $g \mid \mathfrak{d}$ only; 2° If a is in G, then $\mathfrak{g}_{ag} = a\,\mathfrak{g}_g$. We put $\mathfrak{k} = \mathfrak{g}_g + \mathfrak{d}$ $(g \in \Omega)$ and denote by ρ the canonical projection from $\mathfrak{k}'$ onto $\mathfrak{d}'$. We write $\Omega_k = \overset{-1}{\rho}(\tilde{\Omega})$. Reasoning as in (*a*) we conclude, that if Λ is the orthogonal complement of $\mathfrak{d}$ in $\mathfrak{k}'$, there is a Λ and $\tilde{G}$ invariant Borel measure $d\mu_k$ on Ω_k, such that

$$\int_{\Omega_k} f(y)\, d\mu_k = \int_{\tilde{\Omega}} \left(\int_{\Lambda} f(x + \lambda)\, d\lambda \right) dv$$

where $d\lambda$ is translation invariant on Λ. Let Σ_1 be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}'$; for $d\sigma_1$ translation invariant and appropriately normalized, we have

$$\int_{\Omega} f(x)\, d\mu = \int_{\Omega_k} \left(\int_{\Sigma_1} f(x + \sigma_1)\, d\sigma_1 \right) d\mu_k.$$

3.2. LEMMA 3.2.1. — *$\tilde{G}\,x$ is closed in Ω for all x in Ω.*

Proof. — Let $\{a_n;\ n = 1, 2, \ldots\} \subset \tilde{G}$ be such, that $\lim_{n \to +\infty} a_n x = g$ belongs to Ω; we have to show, that g is in $\tilde{G}\,x$. Since also $a_n \pi(x) \to \pi(g)$, by virtue of Proposition 1.2, Chapter II we have $a_n = b_n c_n$, where $b_n \to b \in G$, and c_n is in $\tilde{G}_{\pi(x)}$. In this fashion, $c_n \to b^{-1} g$, and to complete our proof it is enough to show, that there is a c in $\tilde{G}_{\pi(x)}$ such that $\lim_{n \to \infty} c_n x = cx$. If $a \in \tilde{G}_{\pi(x)}$, we have $ax = x + \gamma(a)$, and $\gamma(ab) = \gamma(a) + \gamma(b)$, therefore it suffices to establish, that $\Gamma = \gamma(\tilde{G}_{\pi(x)})$ is closed. But by Corollary 1.1, Chapter II, $x + \Gamma = (x + \Sigma) \cap \tilde{G}\,x$ is locally closed in $\mathfrak{g}'$, and thus Γ is closed. Q. E. D.

3.3. *a*. Observe, that the groups $\overline{G}_g L$ and $\overline{G}_g L_{\mathbf{C}}$ do not depend on the choice of g in Ω; we shall denote them by K and $\overline{H}$ resp.

With the notations of II.3 (*f*) let us consider the set $\mathfrak{B}(\Omega) = \bigcup_{g \in \Omega} \overset{A}{H}_g$; we define actions of $\tilde{G}$ and $\overset{\circ}{H}$ on $\mathfrak{B}(\Omega)$ as *loc. cit.*

If a is some element in $J = H/(\overline{H}_0)$, there is a smooth map σ from Ω into $\overline{H}$ such that : 1° $\Phi(\sigma(g)) \equiv a$; 2° $\sigma(g) \in \overline{H}_g$. In fact, by Remark 5.1 in Chapter II, there is a map σ' from $\tilde{G}\,x$ into $\overline{H}$ (x arbitrarily fixed in Ω) having the properties stated, such that $\sigma'(g)$ depends on $g \mid \mathfrak{d}$ only.

Therefore it suffices to define for $g \in \Omega : \sigma(g) = \sigma'(g_1)$ where $g_1 \in \tilde{G}x$ is such, that $g \mid \dot{\mathfrak{d}} = g_1 \mid \dot{\mathfrak{d}}$. Let $\{a_j; 1 \leqq j \leqq m\}$ be a basis in J; we denote by σ_j maps, as above, corresponding to a_j $(1 \leqq j \leqq m)$. If $p = (g, \chi) \in \mathfrak{B}(\Omega)$ $\left(\chi \in \hat{H}_g\right)$ we put

$$\alpha p = (g; \arg \chi(\sigma_1(g)), \ldots, \arg \chi(\sigma_m(g))) \in \Omega \times \mathbf{T}^m.$$

We show, as in II.5.2 (*a*) that α is a bijection, and define, as *loc. cit.*, on $\mathfrak{B}(\Omega)$ the structure of a differentiable manifolds by requiring that α be a diffeomorphism.

We know [*cf.* II.6.3 (*a*)] that there is a canonical identification between $\bigcup_{g \in \Omega} \hat{H}_g$ and $\bigcup_{g \in \Omega} \hat{G}_g$. We define on the latter the structure of a bundle over the base space Ω with the structure group $\overset{\circ}{K}$ by transfer of structure; this, too, will be denoted by $\mathfrak{B}(\Omega)$.

Let x be an element in Ω. We showed above in 3.2, that $\tilde{O} = \tilde{G}x$ is closed in Ω, and we see at once, that $\mathfrak{B}(\tilde{\mathfrak{O}})$, as defined in II.5.2, is just the portion of $\mathfrak{B}(\Omega)$ over $\tilde{\mathfrak{O}} \subset \Omega$. Therefore, in particular, we can speak of the orbits of $\mathfrak{S}$ in $\mathfrak{B}(\Omega)$. Let us observe, that by Remark 7.1, Chapter II, if $p \in \mathfrak{B}(\Omega)$, the orbit of $\mathfrak{S}$ containing p is just the closure of $G\,p$ in $\mathfrak{B}(\Omega)$.

b. The closed subgroup $(G_f)_0 K$ of G is independent of the particular choice of f in $\tilde{\Omega}$; we shall denote it by M in the sequel. Bearing in mind what we saw in II.6.2 (*a*) we can conclude, that there is a cross section p_0 from Ω into $\mathfrak{B}(\Omega)$ satisfying $ap_0(g) \equiv \omega(a)\, p_0(ag)$ $(a \in G,\ g \in \Omega)$ where $\omega \in \mathrm{Hom}\left(G, \overset{\circ}{K}\right)$ is such, that $M \subset \ker \omega$.

c. If $p(g)$ is any cross section, the map $(g, \varphi) \mapsto \varphi\, p(g)$ from $\Omega \times \overset{\circ}{K}$ onto $\mathfrak{B}(\Omega)$ is a homeomorphism. Let $d\varphi$ be an element of the invariant measure on $\overset{\circ}{K}$ $(\sim \mathbf{T}^m)$. If h is continuous and of a compact support on $\mathfrak{B}(\Omega)$, the value of the integral

$$\int_{\Omega \times \overset{\circ}{K}} h(\varphi\, p(g))\, d\varphi\, d\mu$$

where $d\mu$ is as in 3.1 (*a*), does not depend on the particular choice on p. We denote by $d\eta$ the corresponding measure on $\mathfrak{B}(\Omega)$.

d. Let x be an element of Ω, and $\mathfrak{h} \subset \mathfrak{d}_{\mathbf{C}}$ such that $\mathfrak{h} = \mathrm{pol}(\pi(x))$ and $\tilde{G}_{\pi(x)}\, \mathfrak{h} \subseteqq \mathfrak{h}$. Since $\overline{G}_x \subseteqq G_{\pi(x)} \subseteqq \tilde{G}_{\pi(x)}$, this implies in particular, that $\overline{G}_x\, \mathfrak{h} \subseteqq \mathfrak{h}$. We define $\mathfrak{h}_g$ $(g \in \Omega)$ by $a\,\mathfrak{h}$ if $\pi(g) = a\,\pi(x)$ and set $\mathfrak{h}_p = \mathfrak{h}_g$ for $p = (g, \chi) \in \mathfrak{B}(\Omega)$. Observe, that $\mathfrak{h}_p = \mathrm{pol}(p)$ (*cf.* Remark 7.2, Chapter I). Let us put $U(p) = \mathrm{ind}(\mathfrak{h}_p, p; K)$ and $H(p) = \mathbf{H}(U(p))$ (cf. *loc. cit.*). We have the following rules of computation : 1° If $\varphi \in \overset{\circ}{K}$, $\varphi\, U(p) = U(\varphi\, p)$; 2° If $a \in \tilde{G}$ then $a\, U(p) = U(ap)$, always in the sense of unitary equivalence. Bearing this in mind, we can easily define a measurable structure on $\{U(p), H(p); p \in \mathfrak{B}(\Omega)\}$ (*cf.* 2.2). We proceed analogously as *loc. cit.* Let $p_0 = (x_0, \chi_0) \in \mathfrak{B}(\Omega)$ be fixed, and U a neighborhood of x_0 on $\tilde{G}\, x_0$, such that there is a continuous function $a(g)$ from U into $\tilde{G}$ with $a(g)\, x_0 \equiv g$ $(g \in U)$. If $\Gamma \subset \mathfrak{d}^{\perp}$ is as in 3.2 above (with x_0 in place of x *loc. cit.*) and Γ_1 a supplementary subspace to the connected component of zero of Γ, in $\mathfrak{d}^{\perp}$, there is a small neighborhood V of zero in Γ_1, such that the map $\eta(g, \gamma) = a(g)\, x_0 + \gamma = g + \gamma$ be a homeomorphism between $U \times V$ and some neighborhood of x_0 in Ω; we shall denote the latter by W. For $\gamma \in \Gamma_1$ we write ψ_γ for the character of K_0 determined by $d\psi_\gamma = i(\gamma \mid (\mathfrak{d} + \mathfrak{g}_{x_0}))$. Let ξ be a character of K such that $\xi \mid K_0 \equiv \psi_\gamma$. As a slight extension of the rules 1-2 given above one shows easily (*cf.* 2.1) that $a\, \mathrm{ind}(\mathfrak{h}_g, \chi, g; K)$ is unitarily equivalent to $\mathrm{ind}(\mathfrak{h}_{ag+\gamma}, \xi'.a\,\chi, ag + \gamma; K)$ $(\xi' = \xi \mid \overline{G}_{ag})$. We denote again by τ the canonical projection from $\mathfrak{B}(\Omega)$ onto Ω. If $p = (g, \chi) \in \overset{-1}{\tau}(W)$, let us define the representation $V(p)$ on the space $H(p_0)$ by $\xi\, a(g_1)\, U(p_0)$, where $g_1 \in U$ and ξ are such that $g = g_1 + \gamma\ (\gamma \in V)$ and $\chi = \xi a(g_1) \chi_0\ (\xi \mid K_0 \equiv \psi_\gamma)$. The field $\{V(p), H(p_0); p \in \overset{-1}{\tau}(W)\}$ is obviously measurable, and $V(p) = U(p)$ if $p \in \overset{-1}{\tau}(W)$ in the sense of unitary equivalence, completing the proof of our statement.

Let us note, that by what we have just seen, $\{U(p)\}$ is actually Borel measurable.

3.4. In the following we shall often indicate the unitary equivalence of the representations U and V by writing $U \sim V$. Let P be a von Neumann algebra on which the group A acts by $\star$-automorphisms; we shall denote by P^A the collection of all fixed points. With these and the preceeding notations we have

LEMMA 3.4.1. — *Let us put* $U = \int_{\mathfrak{B}(\Omega)} \oplus\, U(p)\, d\eta$, *and denote by* Q *the ring of this decomposition on* $\mathbf{H}(U)$. *Then we have* $Q^M \subset \mathbf{R}(U)$ $[M = (G_f)_0 K;$ $f = g \mid \mathfrak{d}$, $g \in \Omega$; *cf.* 3.3 (*b*)].

Proof. — In the following often, when a direct integral of Hilbert spaces over the measure space (X, μ) is specified by the context, we shall write $L^\infty_\mu(X)$ for the ring of all diagonalisable operators (= ring of decomposition).

a. Let $p_0(g)$ be a cross section as in 3.3 (*b*). We set

$$V(g) = U(p_0(g)) \quad (g \in \Omega), \qquad V = \int_\Omega \oplus V(g)\, d\mu, \qquad \text{and} \qquad X = \int_{\overset{\ominus}{K}} \oplus \varphi\, d\varphi.$$

Then, since $\varphi U(p) \sim U(\varphi p)$, there is a unitary map from $\mathbf{H}(U)$ onto $\mathbf{H}(V) \otimes \mathbf{H}(X)$, which maps U onto $V \otimes X$ and Q onto $L^\infty_\mu(\Omega) \otimes L^\infty_\varphi(K)$. Recalling, that $\ker \omega \supset M$, to prove our lemma, it will be sufficient to show, that $\mathbf{R}(V \otimes X) \supset (L^\infty_\mu(\Omega))^M \otimes L^\infty_\varphi(\overset{\ominus}{K})$.

b. Let us observe, that if a is any element in M we have $a\, U(p) \sim U(p)$ (*cf*. Remark 6.2, Chapter I). In fact, assuming $p = (g, \chi)$, we can obviously confine ourselves to the case, when $a \in (G_f)_0$ $(f = g \mid \mathfrak{d})$. But then, $a\, U(p) \sim \operatorname{ind}(a\, \mathfrak{h}_g, a\, \chi, ag; K)$; by definition $a\, \mathfrak{h}_g = \mathfrak{h}_g$, and we infer from the proof of Lemma 6.1 and of Lemma 7.4, Chapter I, that $a\chi = \chi$ and $ag \mid \mathfrak{k} = g \mid \mathfrak{k}$ resp. $[\mathfrak{k} = \mathfrak{d} + \mathfrak{g}_g$; *cf*. 3.1 (*b*) above]. Hence to complete our proof it suffices to observe, that if $g \mid \mathfrak{k} = g_1 \mid \mathfrak{k}$, then

$$\operatorname{ind}(\mathfrak{h}, \chi, g; K) = \operatorname{ind}(\mathfrak{h}, \chi, g_1; K).$$

We can conclude from this, that $V(ag) \sim V(g)$. In fact,

$$V(ag) = U(p_0(ag)) = U(ap_0(g)) \sim U(p_0(g)) = V(g) \qquad (a \in M).$$

c. Let us put $W = V \mid K_0$ and $W(g) = V(g) \mid K_0$; we have $W = \int_\Omega \oplus W(g)\, d\mu$ and $W(ag) \sim W(g)$ $(a \in M)$. Let A be some decomposable operator on $\mathbf{H}(W)$, and $\{A(g);\ g \in \Omega\}$ the corresponding field of operators $\left[\text{that is } A = \int_\Omega \oplus A(g)\, d\mu\right]$. We shall write $A = (s)$ if for $a \in M$ we have $A(ag) = V A(g) V^{-1}$, where V is a unitary map from $\mathbf{H}(W(g))$ onto $\mathbf{H}(W(ag))$, such that $VW(g)V^{-1} = W(ag)$. We observe now, that to prove the statement at the end of (*a*) above, and hence Lemma 3.4.1, it is enough to show, that $A = (s)$ implies, that V belongs to $\mathbf{R}(W)$. In fact to see this we take into account, that for any a in K we have $V(a) = (s)$, since $W(g) \mid L \sim \operatorname{Ind}(\mathfrak{h}_g, g \mid \mathfrak{d})$ is irreducible, and thus $W(g') = VW(g)V^{-1}$ $(g' = bg,\ b \in M)$ necessarily implies $V(g') = VV(g)V^{-1}$. Hence, by assumption, $\mathbf{R}(W) \supset \{V(a);\ a \in K\}$ and therefore, by virtue of

$$\mathbf{R}(V \otimes X) \supset \mathbf{R}((V \otimes X) \mid K_0) = \mathbf{R}(W) \otimes I,$$

we get $\mathbf{R}(V \otimes X) \supseteqq \mathbf{R}(W) \otimes R(X) \left[= \mathbf{R}(V) \otimes L^\infty_\varphi(\overset{\ominus}{K})\right]$. In this fashion to attain our goal it suffices to remark, that if $\zeta(g)$ is a bounded measurable and M invariant function on Ω we have trivially $\int_{\Omega} \oplus \zeta(g)\, I_g\, d\mu = (s)$ [I_g = unit operator on $\mathbf{H}(W(g))$] and therefore

$$\mathbf{R}(V \otimes X) \supset (L^{\infty}_{\mu}(\Omega))^{M} \otimes L^{\infty}_{\varphi}(\overset{\ominus}{K}).$$

d. Let us denote by ξ the canonical projection from $\mathfrak{g}'$ onto $\mathfrak{k}'$; then we have $\Omega_k = \xi(\Omega)$ [*cf.* 3.1 (*b*)]. Given $(g, \chi) \in \mathfrak{B}(\Omega)$, let us form, as at the start of Section 7, Chapter I, the representation τ_χ, by taking $\mathfrak{h}_g$ in place of $\mathfrak{h}$ *loc. cit.* Then the concrete representation $\tau_\chi \mid K_0$ depends on $\xi(g)$ only; we shall denote it by $S(h)$ [$h = \xi(g)$]. Putting $\bar{S}(g) = S(\xi(g))$, $W(g)$ is unitarily equivalent to $\bar{S}(g)$ and hence $W \sim \int_{\Omega} \oplus \bar{S}(g)\, d\mu = \bar{S}$, say. Let $A = (s)$ be some operator on $\mathbf{H}(W)$, $\bar{A}$ its image on $\mathbf{H}(\bar{S})$ under the said unitary equivalence, and assume $\bar{A} = \int_{\Omega} \oplus \bar{A}(g)\, d\mu$. Taking into account, that by virtue of Lemma 6.2, Chapter I, 1° If $g, g' \in \Omega$ and $\xi(g) = \xi(g')$ there is an element a in $(G_f)_0$ [$f = \pi(g) = \pi(g')$], such that $g' = ag$; 2° Evidently on Ω_k the M and L orbits resp. coincide, one concludes that $\bar{A}(g)$ depends on $\xi(g)$ only and that, if $l \in L$, $\bar{A}(lg) = V\bar{A}(g)V^{-1}$, provided $\bar{S}(lg) = V\bar{S}(g)V^{-1}$. Let us put $S = \int_{\Omega_k} \oplus S(h)\, d\mu_k$ [*cf.* 3.1 (*b*)]. For a decomposable A on $\mathbf{H}(S)$ we shall write $A = (t)$, if $A = \int_{\Omega_k} \oplus A(h)\, d\mu_k$, and $A(lh) = VA(h)V^{-1}$ if $S(hl) = VS(h)V^{-1}$ $(l \in L)$. According to what we have just seen, to prove Lemma 3.4.1 it is enough to establish that $A = (t)$ implies $A \in \mathbf{R}(S)$.

In the following, if S is replaced by a unitarily equivalent representation, $A = (t)$ will stand for the correspondingly modified condition.

e. We denote again by ρ the canonical map from $\mathfrak{k}'$ onto $\mathfrak{d}'$ and recall, that $\rho(\Omega_k) = \tilde{\Omega}$.

Lemma 3.4.2. — *There is a Borel map ψ from $\tilde{\Omega}$ into Ω_k, such that :* 1° $\psi(f) \mid \mathfrak{d} \equiv f$; 2° $\psi(lf) = l\,\psi(f)$ $(l \in L, f \in \tilde{\Omega})$.

Proof. — Let $\mathfrak{k} = \mathfrak{k}_r \supset \mathfrak{k}_{r-1} \supset \ldots \supset \mathfrak{k}_m = \mathfrak{d} \supset \ldots \supset \mathfrak{k}_0 = (0)$ be such, that $\{\mathfrak{k}_j;\ 0 \leq j \leq m\}$ is a Jordan-Hölder sequence in $\mathfrak{d}$. Let us assume, that $l_j \in \mathfrak{k}_j - \mathfrak{k}_{j-1}$ and that $(l_i, l'_j) = \delta_{ij}$ $(1 \leq i, j \leq r)$; we put $\bar{l}'_j = \rho(l'_j)$ $(1 \leq j \leq m)$. With the notations of the proof of Proposition 4.1, Chapter II, we write $\mathcal{E}'$ for the subset of elements of $\mathcal{E}$, such that $\Omega^e = \tilde{\Omega} \cap \mathfrak{O}_e \neq 0$; it obviously suffices to define ψ on Ω^e $(e \in \mathcal{E}')$. We assume, that $h(x)$ $(x \in \mathfrak{O}_e)$ is as in Corollary 4.1, Chapter II, and put, with the notations of Proposition 4.2, Chapter II, $d(x) = \exp[l(\mathrm{R}(x, h(x)), h(x))]$. Then $h(x)$ and $d(x)$ are continuous maps on $\mathfrak{O}_e$, such that $x \equiv d(x) f(x)$ $(x \in \mathfrak{O}_e)$. If $y = \sum_{j=1}^{m} y_j \bar{l}'_j$ is some element of $\mathfrak{d}'$, we shall write $\bar{y}$ for the element $\sum_{j=1}^{m} y_j\, l'_j$ of $\mathfrak{k}'$. With these notations we set $\psi(x) = d(x)\, \overline{h(x)}$, and claim, that on Ω^e we have the required properties. In fact, first

$$\rho(\psi(x)) = d(x)\, \rho\left(\overline{h(x)}\right) = d(x)\, h(x) = x.$$

Second, if l is some element in L, we have $l\, \psi(x) = l\, d(x) . h(x)$ but also $ld(x) . h(x) = d(lx) . h(x)$, and thus $ld(x) = d(lx)\, d_0$, where d_0 satisfies $d_0\, h(x) = h(x)$. But if $l\rho(h) = \rho(h)$ $(h \in \Omega_k,\ l \in \mathrm{L})$ we have also $lh = h$, and therefore $d_0\, \overline{h(x)} = \overline{h(x)}$. In this fashion we get $l\psi(x) = d(lx)\, \overline{h(lx)} = \psi(lx)$ [since $h(lx) \equiv h(x)$] for all l in L and x in Ω^e completing the proof of our lemma. Q. E. D.

f. We write again Λ for the orthogonal complement of $\mathfrak{d}$ in $\mathfrak{k}'$. The map ψ' of $\tilde{\Omega} \times \Lambda$ into Ω_k defined by $\psi'(f, \lambda) = \psi(f) + \lambda$ [ψ as in (*e*)] is a Borel isomorphism, and it makes correspond to $d\mu_k$ the measure $dv\, d\lambda$ on $\tilde{\Omega} \times \Lambda$ [*cf* 3.1 (*b*)]. Let us observe, that if l is any element of L, we have $l\psi'(f, \lambda) = \psi'(lf, \lambda)$. Given λ in Λ, we write φ_λ for the character of $\mathrm{K}_0 = \exp \mathfrak{k}$ determined by $d\varphi_\lambda = i\, \lambda$. We have $\ker \varphi_\lambda \supset \mathrm{L}$ and for any h in Ω_k, $\mathrm{S}(h + \lambda) \sim \varphi_\lambda\, \mathrm{S}(h)$. Let us put $\mathrm{T}(f) = \mathrm{S}(\psi(f))$ $(f \in \tilde{\Omega})$; we have $\mathrm{T}(lf) \sim \mathrm{T}(f)$ $(l \in \mathrm{L})$, since $\mathrm{S}(lh) \sim \mathrm{S}(h)$ [*cf.* (*d*)] implies that

$$\mathrm{T}(lf) = \mathrm{S}(\psi(lf)) = \mathrm{S}(l\, \psi(f)) \sim \mathrm{S}(\psi(f)) = \mathrm{T}(f).$$

Summing up, $\mathrm{S} = \int_{\overline{\Omega}_k} \oplus\, \mathrm{S}(h)\, d\mu_k$ is unitarily equivalent to the direct integral,

$$\int_{\tilde{\Omega} \times \Lambda} \oplus\, \varphi_\lambda\, \mathrm{T}(f)\, dv\, d\lambda,$$

and thus also to $\mathrm{T} \otimes \Phi$, where $\mathrm{T} = \int_{\tilde{\Omega}} \oplus\, \mathrm{T}(f)\, dv$, $\Phi = \int_{\Lambda} \oplus\, \varphi_\lambda\, d\lambda$. If $\mathrm{A} = (t)$ [*cf.* the end of (*d*)] and $\mathrm{A} = \int_{\tilde{\Omega} \times \Lambda} \oplus\, \mathrm{A}(f, \lambda)\, dv\, d\lambda$ we have

$$\mathrm{A}(lf, \lambda) = \mathrm{VA}(f, \lambda)\, \mathrm{V}^{-1}, \quad \text{if} \quad \mathrm{T}(lf) = \mathrm{VT}(f)\, \mathrm{V}^{-1} \qquad (l \in \mathrm{L}).$$

g. Let us consider now a Borel transversal for the action of L on $\tilde{\Omega}$. To obtain this, we can proceed, for instance, as follows. Putting E for the complement of $e \in \mathcal{E}'$ [*cf.* (*e*)] in $\{1, 2, \ldots, m\}$, and writing H_e for the hyperplane spanned in $\mathfrak{d}'$ by $\{\bar{l}_j;\ j \in E\}$, we take $\mathcal{O} = \bigcup_{e \in \mathcal{E}'} (H_e \cap \Omega^e)$. Let f_0 be a fixed element of $\tilde{\Omega}$, and let us identify $\tilde{\Omega}$ to the homogeneous space $\mathfrak{H} = \tilde{G}/\tilde{G}_{f_0}$ (*cf.* Proposition 1.2, Chapter II). We put $\mathfrak{A} = \tilde{G}/\tilde{G}_{f_0} L$, and denote by ζ the canonical projection from $\mathfrak{H}$ onto $\mathfrak{A}$. Retaining the same notation for the image of $\mathcal{O}$ in $\mathfrak{H}$, we observe, that the restriction of ζ to $\mathcal{O}$ is a Borel isomorphism with $\mathfrak{A}$. Hence there is a field of representations $\{P(a);\ a \in \mathfrak{A}\}$ of K_0, such that $P(\zeta(q)) \equiv T(q)$ $(q \in \mathcal{O} \subset \mathfrak{H})$. Let us define $T'(q) = P(\zeta(q))$ $(q \in \mathfrak{H})$. Then $T \otimes \Phi$ is unitarily equivalent to $\int_{\mathfrak{H} \times \Lambda} \oplus\, \varphi_\lambda T'(q)\, dv\, d\lambda$, and we have $A = (t)$ if and only if

$$A = \int_{\mathfrak{H} \times \Lambda} \oplus\, A(q, \lambda)\, dv\, d\lambda$$

and $A(q, \lambda) = A(q', \lambda)$ if $q' = lq$ $(l \in L)$. Let σ be the canonical map from $\tilde{G}$ onto $\mathfrak{H}$. Proceeding as in 1 (*c*) we can find an invariant measure dk on $\sigma(L) = L/L_{f_0}$, and a measure $d\nu$ on $\mathfrak{A} = \mathfrak{H}/L$ such that we have for any F, which is continuous and of a compact support on $\mathfrak{H}$,

$$\int_{\mathfrak{A}} \left(\int_{\sigma(L)} F(ak)\, dk \right) d\nu = \int_{\mathfrak{H}} F(y)\, dv.$$

Let us form the representation $P = \int_{\mathfrak{A}} \oplus\, P(a)\, d\nu$; we denote by $\mathcal{B}$ the corresponding ring of decomposition. To prove Lemma 3.4.1 we have to establish, that $A = (t)$ implies, that A belongs to $\mathbf{R}(T \otimes \Phi)$ [*cf.* (*f*) above]. Reasoning as in (*d*) above we conclude, that this is certainly so, if we can prove, that $\mathbf{R}(P \otimes \Phi) = \mathcal{B}' \otimes \mathbf{R}(\Phi)$, for which it is enough to show, that if $\overline{P} = P \mid L$, we have $R(\overline{P}) = \mathcal{B}'$. To establish the last relation, let us observe first, that by virtue of our construction $\overline{P}(a) \sim \overline{P}(a')$ implies $a = a'$ $[\overline{P}(a) = P(a) \mid L]$. Next we recall, that the map from $\mathfrak{d}'/L$ into $\hat{L}$, which assigns to the orbit Lf in $\mathfrak{d}'$ the equivalence class of Ind $(\mathfrak{h}, f)$ [*cf.* I.4 (*f*)] is continuous (*cf.* [31], Proposition 2, p. 89) and hence it gives rise to a Borel isomorphism between $\mathfrak{d}'/L$ and $\hat{L}$ (*cf.* [2], Proposition 2.5, p. 7; observe, that L being of type I, $\hat{L}$ is standard and thus, in particular, is countably generated). Now we have a canonical map τ from $\mathfrak{A}$, *via* $\tilde{\Omega}/L$, into $\hat{L}$, which establishes a Borel isomorphism between $\mathfrak{A}$ and $\tau(\mathfrak{A}) \subset \hat{L}$. Hence we can conclude, that there is a measure $d\bar{\nu}$ on $\hat{L}$, which is concentrated on a $\tilde{G}$ orbit, such that $\overline{P}$ is unitarily equivalent to $\int_{\hat{L}} \oplus \pi(\zeta)\, d\bar{\nu}(\zeta)$ where $\{\pi(\zeta); \zeta \in \hat{L}\}$ is a Borel measurable field of representations on $\hat{L}$, such that $\pi(\zeta) \in \{\zeta\}_c$ (*cf.* [12], 4.6.2, Proposition, p. 95). $\mathcal{B}$ goes over into the ring of the diagonalisable operators of the last decomposition. In this fashion the desired conclusion follows from the fact, that $\mathbf{R}\left(\int_{\hat{L}} \oplus \pi\, d\bar{\nu}\right)$ is the commuting ring of the ring of decomposition (*cf.* [12], 8.6.4, Proposition, p. 155 and 18.7.6, p. 325).

Q. E. D.

Let us put, for $p = (g, \chi) \in \mathfrak{B}(\Omega)$, $T(p) = \operatorname{ind}(\mathfrak{h}_g, p)$ (*cf.* 2.2). Then we have [*cf.* I.4(*g*)], $T(p) = \operatorname{ind}_{K \uparrow G} U(p)$. Hence along with the field of representations $\{U(p); p \in \mathfrak{B}(\Omega)\}$, the field $\{T(p); p \in \mathfrak{B}(\Omega)\}$, too, is measurable and, putting

$$T = \int_{\mathfrak{B}(\Omega)} \oplus T(p)\, d\eta \qquad \text{and} \qquad U = \int_{\mathfrak{B}(\Omega)} \oplus U(p)\, d\eta$$

we get $T = \operatorname{ind}_{K \uparrow G} U$ (*cf.* [23], Theorem 10.1, p. 123).

Lemma 3.4.3. — *Let* P *be the ring of all diagonalisable operators of the decomposition* $T = \int_{\mathfrak{B}(\Omega)} \oplus T(p)\, d\eta$. *We have* $P^G \subset \mathbf{R}(T)$.

Proof. — *a.* Let A be a Borel fundamental domain of G mod K. The restriction of the canonical map, from G onto G/K, to A is a Borel isomorphism with its image. Let $d\zeta$ be the inverse image on A of the invariant measure on G/K. We have

$$T(p) \mid K = \int_A \oplus a\, U(p)\, d\zeta(a) \qquad \text{and} \qquad T \mid K = \int_A \oplus a\, U\, d\zeta(a).$$

Hence, in particular, we have an identification of $\mathbf{H}(T)$ to $L^2_\zeta(A) \otimes \mathbf{H}(U)$, such that P corresponds to $L^\infty_\zeta(A) \otimes Q$, Q having the same meaning as in Lemma 3.4.1. Given ψ in $L^\infty_\eta(\mathfrak{B}(\Omega))$, we denote by $L(\psi)$ the corresponding operator in Q. For a in G, we put $(a\psi)(p) \equiv \psi(a^{-1}p)$. In this fashion, to prove our lemma, it will be enough to show, that $a\psi = \psi$ for all a in G implies $I \otimes L(\psi) \in P$.

b. Since $a\, U(p) \sim U(ap)$, and $d\eta$ is G invariant, we conclude at once, that $a\, U \sim U$ $(a \in G)$. Let φ_a be the unique $\star$-automorphism of $\mathbf{R}(U)$, such that $\varphi_a(U) = a\, U$. Then, if ψ is such, that $L(\psi) \in \mathbf{R}(U)$, we have $\varphi_a(L(\psi)) = L(a\psi)$.

c. Let B be an operator in $\mathbf{R}$ (U); then the field of operators $\{\varphi_a(B); a\in A\}$ is measurable, and we have $\int_A \oplus \varphi_a(B)\, d\zeta(a)\in \mathbf{R}(T)$. In fact, this is certainly true, when B is a finite linear combination of the operators $\{U(b); b\in G\}$. The general case follows by choosing a sequence of operators $\{B_n\}$ of the said form, such that $B=\lim_{n\to+\infty} B_n$ strongly.

d. Assume now, that $\psi\in L^{\infty}_{\eta}(B(\Omega))$ is such, that $a\psi=\psi$ for all a in G. Then we have, by virtue of Lemma 3.4.1, $L(\psi)\in Q^G\subset Q^M\subset \mathbf{R}(U)$. On the other hand [*cf.* (*b*)], $\varphi_a(L(\psi))\equiv L(\psi)$ $(a\in G)$. In this fashion finally, by (*c*) :

$$\mathbf{R}(T)\ni \int_A \oplus \varphi_a(L(\psi))\, d\zeta(a) = I\otimes L(\psi)$$

completing the proof of our statement. Q. E. D.

We recall, that the decomposition $V=\int_X \oplus V(x)\, d\mu$ of the representation V is called central, if the von Neumann algebra $\mathbf{R}$ (V) generated by the operators of V contains the ring of all diagonalisable operators.

LEMMA 3.4.4. — *With the previous notations, there is a Borel measure* $d\varepsilon$ *on* $E=\mathfrak{B}(\Omega)/\mathfrak{S}$, *such that* $T=\int_E \oplus T(\varepsilon)\, d\varepsilon$, *where* $T(\varepsilon)\in\mathcal{F}(\varepsilon)$ (*cf.* 2.2), *the decomposition being central.*

Proof. — *a*. Choosing a cross section $p_0(g)$ from Ω into $\mathfrak{B}(\Omega)$ as in 3.3 (*b*), we identify $\mathfrak{B}(\Omega)$ to $\Omega\times\mathbf{T}^m$. Putting again $\Sigma=\mathfrak{d}^{\perp}$, let us form the direct product of groups $H=\tilde{G}\times\Sigma\times\mathbf{T}^m$, and let us define an action of H on $\mathfrak{B}(\Omega)$ by setting

$$(a,\sigma,\omega)(g,\omega')=(ag+\sigma,\omega\omega')\qquad (a\in\tilde{G};\sigma\in\Sigma;\omega,\omega'\in\mathbf{T}^m; g\in\Omega).$$

Evidently, H acts transitively on $\mathfrak{B}(\Omega)$. We write $\mathfrak{H}=\tilde{G}\times\mathbf{T}^m$ $(H=\mathfrak{H}\times\Sigma)$ and infer from II.7.1, that there is a closed, connected subgroup $\mathfrak{G}$ of $\mathfrak{H}$, satisfying $[\mathfrak{G},\mathfrak{G}]=[H,H]$, such that for any p in $\mathfrak{B}(\Omega)$, $\mathfrak{G}p$ coincides with the orbit of $\mathfrak{S}$ containing p. Hence, in particular, we have

$$E=\mathfrak{B}(\Omega)/\mathfrak{S}=\mathfrak{B}(\Omega)/\mathfrak{G}.$$

b. Let $p_0=(x_0,\omega_0)$ be a fixed point of $\mathfrak{B}(\Omega)$; we have just seen, that $\mathfrak{B}(\Omega)=H_{p_0}$. Our next objective is to show, that the natural identification between $\mathfrak{B}(\Omega)$ and H/H_{p_0} is a homeomorphism. To this end it will suffice to establish, that if $\{h_n\}$ and h are elements in H, such that $h_n p_0 \to hp$, then we have also $h_n \to h \bmod \mathrm{H}_{p_0}$. Let us assume $h_n = (a_n, \sigma_n, \omega_n)$ and $h = (a, \sigma, \omega)$. We have clearly $\omega_n \to \omega$, and hence, putting $\mathrm{H}' = \tilde{\mathrm{G}} \times \Sigma$, it is enough to prove, that $a_n x_0 + \sigma_n \to ax_0 + \sigma$ implies that $(a_n, \sigma_n) \to (a, \sigma)$ $(\bmod \mathrm{H}'_{x_0})$. The assumption yields

$$a_n \pi(x_0) \to a \pi(x_0)$$

and hence, as in the proof of Lemma 3.2.1, we can write $a_n = b_n . c_n$, $a = bc$, such that $b_n \to b$ and $c_n, c \in \tilde{\mathrm{G}}_{\pi(x_0)}$. Thus it will suffice to show, that $(c_n, \sigma_n) \to (c, \sigma) \bmod \mathrm{H}'_{x_0}$, if $c_n x_0 + \sigma_n \to cx_0 + \sigma$. But, with notations as *loc. cit.* we have

$$c_n x_0 + \sigma_n = x_0 + \gamma(c_n) + \sigma_n = cx_0 + \sigma'_n,$$

where we put

$$\sigma'_n = \gamma(c_n) + \sigma_n - \gamma(c).$$

Hence $(c_n, \sigma_n) = (c, \sigma'_n) \bmod \mathrm{H}'_{x_0}$ and evidently $(c, \sigma_n) \to (c, \sigma)$ in H', proving our statement.

c. Given ε in $\mathfrak{B}(\Omega)/\mathfrak{H}$ $[= \mathfrak{B}(\Omega)/\mathfrak{G}]$, we denote by $\mathrm{O}(\varepsilon)$ the corresponding $\mathfrak{H}$ $(\mathfrak{G})$ orbit in $\mathfrak{B}(\Omega)$. We are going to show now, that there is a Borel measure $d\varepsilon$ on $\mathrm{E} = \mathfrak{B}(\Omega)/\mathfrak{H}$, such that if μ_ε is an appropriately normalized G invariant measure on $\mathrm{O}(\varepsilon)$ (*cf.* Proposition 1.1), the H invariant measure η on $\mathfrak{B}(\Omega)$ [*cf.* 3.3 (*c*)] turn out to be the continuous sum of the measures $\{\mu_\varepsilon; \varepsilon \in \mathrm{E}\}$ with respect to $d\varepsilon$. We denote by the same letter the image of η on $\mathrm{H}/\mathrm{H}_{p_0}$ [*cf.* (*b*)]. We put $\mathfrak{E} = \mathrm{H}/(\mathrm{H}_{p_0} . \mathfrak{G})$ and write σ for the canonical map from H onto $\mathrm{H}/\mathrm{H}_{p_0}$. Let $d\mu$ be a $\mathfrak{G}$ invariant measure on $\sigma(\mathfrak{G}) \sim \mathfrak{G}/\mathfrak{G}_{p_0}$. By virtue of what we saw above, we shall attain our goal by showing, that if $f(q)$ is continuous and of a compact support on $\mathrm{H}/\mathrm{H}_{p_0}$, then

$$\int_{\sigma(\mathfrak{G})} f(ak)\, d\mu \equiv \mathrm{F}(a)$$

satisfies $\mathrm{F}(aa_0) = \mathrm{F}(a)$ for all a_0 in $\mathrm{H}_{p_0} \mathfrak{G}$. In fact, in this case, as in III.1, we can conclude, that with an appropriately chosen invariant measure $d\varepsilon$ on $\mathfrak{E}$, we have

$$\int_{\mathrm{H}/\mathrm{H}_{p_0}} f(q)\, d\eta = \int_{\mathfrak{E}} \mathrm{F}(\varepsilon)\, d\varepsilon.$$

To verify the indicated statement, we observe, that it suffices to prove, that if $\mathfrak{G} = \exp \mathfrak{g}_0$ and $h \in H_{p_0}$ $[h = (a, \sigma, \omega)]$ we have

$$\det (\mathrm{Ad}\,(h) \mid \mathfrak{g}_0) = \det (\mathrm{Ad}\,(h) \mid (\mathfrak{g}_0)_{p_0}).$$

Reasoning analogously as *loc. cit.*, we observe, that to this end it is enough to show

$$\det (\mathrm{Ad}\,(a) \mid \mathfrak{g}) = \det (\mathrm{Ad}\,(a) \mid \mathfrak{g}_{x_0}) \quad \text{for all } a \in \tilde{G}_{\pi(x_0)}.$$

But this is implied at once by the relation $([al_1, al_2], x_0) = ([l_1\, l_2], x_0)$ valid for all $l_1, l_2 \in \mathfrak{g}$ and $a \in \tilde{G}_{\pi(x_0)}$.

d. Using the previous results, we can complete the proof of Lemma 3.4.4 as follows. Let us put $T(\varepsilon) = \int_{O(\varepsilon)} \oplus\, T(p)\, d\mu_\varepsilon$; then we have $T(\varepsilon) \in \mathcal{F}(\varepsilon)$ (*cf.* the end of 2.2). By virtue of what we saw in (*c*) also

$$T = \int_{\mathfrak{B}(\Omega)} \oplus\, T(p)\, d\eta = \int_{\mathfrak{E}} \oplus\, T(\varepsilon)\, d\varepsilon$$

(*cf.* [24], Theorem 2.11, p. 204). But by Lemma 3.4.3 this decomposition is central.

Q. E. D.

LEMMA 3.4.5. — *With the previous notations, there is a* $\tilde{G}$ *orbit* $\mathfrak{D}$ *on* $\hat{L}$, *and a* $\tilde{G}$ *invariant measure* $d\nu$ *on* $\mathfrak{D}$, *such that* T *is unitarily equivalent to*

$$M.\operatorname*{ind}_{L \uparrow G}\left(\int_{\mathfrak{D}} \oplus\, \pi(\zeta)\, d\nu(\zeta)\right)$$

where M *is either one or infinite.*

Proof. — *a.* Taking into account, in particular, what we saw in 3.1 (*a*) and 3.1 (*b*), we conclude, that if $f(p)$ is continuous and of a compact support on $\mathfrak{B}(\Omega)$, then we have

$$\int_{\mathfrak{B}(\Omega)} f(p)\, d\eta = \int_{\overset{\circ}{K} \times \Omega} f(\varphi\, p_0(g))\, d\varphi\, d\mu$$
$$= \int_\Omega \left(\int_{\overset{\circ}{K} \times \Lambda} \left(\int_{\Sigma_1} f(\varphi\, p_0(g + \sigma + \sigma_1))\, d\sigma_1 \right) d\varphi\, d\lambda \right) dv.$$

b. Since we have $T = \operatorname{ind}_{K \uparrow G} U$ (*cf.* Lemma 3.4.3) it will suffice to establish, that

$$U \sim M . \operatorname{ind}_{L \uparrow K} \left(\int_{\mathfrak{D}} \oplus \pi(\zeta)\, d\nu(\zeta) \right)$$

where $\mathfrak{D}$, ν and M are as in our lemma.

c. Let g be a fixed element in $\mathfrak{g}'$. Then $U(\varphi\, p_0(g + \sigma_1))$ does not depend on $\sigma_1 \in \Sigma_1$, and therefore we have

$$\int_{\Sigma_1} \oplus U(\varphi\, p_0(g + \sigma_1))\, d\sigma_1 = M_1 . U(\varphi\, p_0(g))$$

where M_1 is one or infinite.

d. Let λ be some element of Λ and assume, that $\lambda_1 \in \mathfrak{g}'$ is such, that $\lambda_1 \mid \mathfrak{k} = \lambda$. Then, if $p_0(g) = (g, \chi_0(g))$, the expression $\chi_0(g + \lambda_1)\overline{\chi_0(g)}$ depends on λ only. Denoting it by φ'_λ, we observe that φ'_λ is a character of $\overline{G}_g$, such that $d(\varphi'_\lambda \mid (G_g)_0) = i(\lambda \mid \mathfrak{g}_g)$. Writing φ_λ for the uniquely determined character of $K = \overline{G}_g L$ with $\varphi_\lambda \mid \overline{G}_g \equiv \varphi'_\lambda$, $\varphi_\lambda \mid L \equiv 1$, we obtain, that

$$U(\varphi\, p_0(g + \lambda_1)) \sim \varphi \operatorname{ind}(\mathfrak{h}_g, \varphi'_\lambda \chi_0(g), g + \lambda_1; K) \sim \varphi\varphi_\lambda \operatorname{ind}(\mathfrak{h}_g, \chi_0(g), g; K).$$

In this fashion we conclude that

$$\int_{\hat{K} \times \Lambda} \oplus \left(\int_{\Sigma_1} \oplus U(\varphi\, p_0(g + \sigma + \sigma_1))\, d\sigma_1 \right) d\varphi\, d\lambda$$
$$\sim M_1 . \operatorname{ind}(\mathfrak{h}_g, \chi_0(g), g; K) \otimes \left(\int_{\hat{K} \times \Lambda} \oplus\, \varphi\varphi_\lambda\, d\varphi\, d\varphi_\lambda \right).$$

Let $A = K/L$ and denote by α the canonical homomorphism from K onto A. We have $A = A_0 \times B$, where B is isomorphic to $K/K_0 \sim \mathbf{Z}^m$. Let $A_0^{\perp}$ and $B^{\perp}$ be the annihilator of A_0 and B resp. in $\hat{A}$. Assuming, that da_0, db and da are appropriately normalized invariant measures on $A_0^{\perp}$, $B^{\perp}$ and $\hat{A}$ resp., we obtain, that

$$\int_{\hat{K} \times \Lambda} \oplus\, \varphi\varphi_\lambda\, d\varphi\, d\lambda \sim \left[\int_{A_0^{\perp} \times B^{\perp}} \oplus (\psi\chi)\, da_0(\psi)\, db(\chi) \right] \circ \alpha \sim \left(\int_{\hat{A}} \oplus\, \chi\, da(\chi) \right) \circ \alpha.$$

In this fashion the left hand side is unitarily equivalent to the regular representation of $A = K/L$ lifted to K. Let us set $f = g \mid \mathfrak{d}$ and $\mathfrak{h}_f = \mathfrak{h}_g$ [*cf.* 3.3 (*d*)]. Since $\operatorname{ind}(\mathfrak{h}_g, \chi_0(g), g; K) \mid L \sim \operatorname{Ind}(\mathfrak{h}_f, f)$ we conclude (*cf.* [26], Lemma 1, p. 325) that

$$\text{ind}\,(\mathfrak{h}_g, \chi_0(g), g; \text{K}) \otimes \left(\int_{\overset{\circ}{\text{K}} \times \Lambda} \oplus\, \varphi\varphi_\lambda \, d\varphi \, d\lambda \right)$$

is unitarily equivalent to $\underset{\text{L} \uparrow \text{K}}{\text{ind}}\,(\text{Ind}\,(\mathfrak{h}_f, f))$. Thus finally

$$\text{U} = \int_{\mathfrak{B}(\Omega)} \oplus\, \text{U}(p)\, d\eta \sim \text{M}_1 . \underset{\text{L} \uparrow \text{K}}{\text{ind}} \left(\int_{\tilde{\Omega}} \oplus\, \text{Ind}\,(\mathfrak{h}_f, f)\, dv\,(f) \right).$$

e. Reasoning as in (*g*) of the proof of Lemma 3.4.2 we show, that putting, with the notations as *loc. cit.*, $\mathfrak{D} = \tau(\mathfrak{A})$ and $\nu = \bar{\nu} \mid \mathfrak{D}$ we have

$$\int_{\tilde{\Omega}} \oplus \text{Ind}\,(\mathfrak{h}_f, f)\, dv\,(f) \sim \text{M}_2 \int_{\mathfrak{D}} \oplus\, \pi(\zeta)\, d\nu\,(\zeta)$$

where M_2 is one or infinite. Hence, writing $\text{M} = \text{M}_1 . \text{M}_2$, we get by virtue of (*b*) above

$$\text{T} \sim \text{M}. \underset{\text{L} \uparrow \text{G}}{\text{ind}} \left(\int_{\mathfrak{D}} \oplus\, \pi(\zeta)\, d\nu\,(\zeta) \right).$$

Q. E. D.

We sum up the previous results in the following.

THEOREM 3. — *Suppose, that* $\tilde{\text{G}} \supset \text{G}$ *is connected, simply connected and such that* $[\tilde{\text{G}}, \tilde{\text{G}}] = [\text{G}, \text{G}]$, *and if* $\tilde{\text{G}} = \exp \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}$ *admits a faithful algebraic representation. Let* $\mathfrak{D}$ *be an orbit of* $\tilde{\text{G}}$ *on* $\hat{\text{L}}$, *and* $d\nu$ *a* $\tilde{\text{G}}$ *invariant measure on* $\mathfrak{D}$. *Then an appropriate multiple of the representation*

$$\underset{\text{L} \uparrow \text{G}}{\text{ind}} \left(\int_{\mathfrak{D}} \oplus\, \pi(\zeta)\, d\nu\,(\zeta) \right) \qquad [\pi(\zeta) \in (\{\zeta\})_c, \zeta \in \hat{\text{L}}]$$

is unitarily equivalent to a central continuous direct sum of the factor representations described in Theorem 2.

Proof. — This is evident from Lemmas 3.4.4 and 3.4.5.

Q. E. D.

REMARK 3.4.1. — It follows from the previous reasonings, that to form the said central decomposition, we can confine ourselves to orbits of $\mathfrak{H}$, the projections of which, to $\mathfrak{g}'$, lie in the inverse image (in $\mathfrak{g}'$), of the $\tilde{\text{G}}$ orbit, corresponding to $\mathfrak{D} \subset \hat{\text{L}}$ on $\mathfrak{d}'/\text{L}$ [*cf.* I.4 (*f*)].

REMARK 3.4.2. — Observe, that we have $T = \operatorname{ind}_{L \uparrow G}\left(\int_{\mathfrak{D}} \oplus \pi(\zeta)\, d\nu(\zeta)\right)$ if and only if : 1° $\mathfrak{k} = \mathfrak{g}$ (on the relevant part of $\mathfrak{g}$; *cf.* above); 2° The representations in $\mathfrak{D} \subset \hat{L}$ are one dimensional.

CHAPTER IV

STRUCTURE OF THE REGULAR REPRESENTATION

SUMMARY. — Let **M** be a von Neumann algebra. Below by a trace on **M** we shall mean a trace on the set of all positive operators of **M**, which is faithful, semi-finite and normal (*cf.* [13], p. 81-82). An operator $A \in \mathbf{M}$ satisfying $\varphi(A^* A) < +\infty$ will be called a generalized Hilbert-Schmidt operator with respect to φ (the reference to which will be omitted, if specified by the context). Let G be a separable locally compact group. We denote by dx an element of the right invariant Haar measure. If T is a unitary representation of G and $f \in L^1(G)$ we put

$$T(f) = \int_G f(x)\, T(x)\, dx.$$

We denote by $\mathcal{R}$ ($\mathcal{L}$) the right (left) regular representation resp. of G, and by $\mathbf{R}(G)$ ($\mathbf{L}(G)$) the right (left) ring resp. of G.

It was shown by I. E. Segal in 1950 (*cf.* [34]), that if G is unimodular, $\mathbf{R}(G)$ is semi-finite. More precisely, there is a trace ψ on $\mathbf{R}(G)$, uniquely determined by the condition, that we have for any $f \in L^1(G) \cap L^2(G)$:

$$\psi(\mathcal{R}(f)[\mathcal{R}(f)]^*) = (f, f) \qquad \left[= \int_G |f(x)|^2\, dx\right]. \tag{1}$$

Hence, in particular, generalized Hilbert-Schmidt operators [even those of the form $\mathcal{R}(f)$ ($f \in L^1(G)$)] generate $\mathbf{R}(G)$. If on the von Neumann algebra generated by the unitary representation T there is a trace with the analogous property, we shall call T a trace class representation. Assume now, that G is not unimodular, and define the function Δ on G by $d(ax) = \Delta(a)\, dx$ ($a \in G$); thus $\Delta \not\equiv 1$. It is easy to see, that in this case no formula of the type (1) can hold true. In fact, upon replacing f by $f_s(x) = f(s^{-1}x)$ (s fixed in G), the left and right hand sides get multiplied with $(\Delta(s))^2$ and $\Delta(s)$ resp. This in itself, of course, does not exclude, that $\mathbf{R}(G)$ be semi-finite, but R. Godement showed by an example, that for a properly chosen G, $\mathbf{R}(G)$ may turn out a factor of type III. We can add also, that even if $\mathbf{R}(G)$ is semi-finite and hence carries a trace, this does not necessarily make $\mathcal{R}$ a trace class representation in the sense of the definition given above. The purpose of Section 1-7 of this chapter is to prove, that if G is a solvable and connected (but not necessarily simply connected) Lie group, the right (or left) regular representation is of trace class (*cf.* Theorem 4, Section 7). Hence, in particular, $\mathbf{R}(G)$ is semi-finite, which corollary has already been extended by J. Dixmier to an arbitrary connected group (*cf.* [14]). The starting point of our proof is the following observation. Assume, that $f \in L^2(G)$ is such, that by convolution on the right it gives rise to a bounded operator V_f on $L^2(G)$. It is easy to see, that in the unimodular case (1) is equivalent to

$$\psi(V_f (V_f)^*) = (f, f) \tag{2}$$

for any such f. In general we have, as above

$$\psi (V_{f_s} (V_{f_s})^*) = (\Delta (s))^2 \psi (V_f (V_f)^*) \qquad \text{and} \qquad (f_s, f_s) = \Delta (s) (f, f).$$

One is tempted to rectify this assymetry through replacing the right hand side of (2) by $(M' f, M' f)$, where M' is a self adjoint, positive, non singular operator, all bounded functions of which lie in $\mathbf{L}(G)$, and which satisfies $(M' f_s, M' f_s) = (\Delta (s))^2 (M' f, M' f)$ for all s in G. Since the left hand side here can be written as $\Delta (s) (M' L (s) f, M' L (s) f)$, to this end it suffices to determine M' such that we have in addition

$$L (s) M' L (s^{-1}) = (\Delta (s))^{-1/2} M' \qquad (s \in G).$$

That following up this lead we, in fact, arrive at the desired result, this we infer from the theory of quasi-unitary algebras by J. Dixmier (*cf.* for all this Section 7 below, in particular Lemma 7.1 and 7.2). In this fashion, our task is reduced to finding an operator M' with the indicated properties. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{n}$ the greatest nilpotent ideal of $\mathfrak{g}$ and N the connected subgroup, determined by $\mathfrak{n}$, of G. Since, if N is not simply connected, additional complications may occur (*cf.* Section 3), here we confine ourselves to sketching what happens, if N is simply connected. In Section 4 we show the existence of nonzero elements p and q in the center of the universal enveloping algebra of $\mathfrak{n}$ (acted upon by G *via* interior automorphisms) such that $ap = \varepsilon (a) p$, $aq = \eta (a) q$ and $\eta (a)/\varepsilon (a) \equiv \det (\mathrm{Ad} (a)) \equiv \Delta (a)$ for all a in G (*cf.* in particular Corollary 4.1). The right invariant differential operators, corresponding to p and q, on G give rise to commuting non singular selfadjoint operators P and Q on $L^2 (G)$. To obtain M' as above, it suffices to consider the minimal closed extension of $| P |^{1/2}.| Q |^{-1/2}$. Using the fact, that by virtue of our construction, for any $f \in C_c^\infty (G)$, $Q f$ lies in the domain of M', one derives easily the existence of a left invariant differential operator D, such that, for $f \in C_c^\infty (G)$, $\mathcal{R} (D f)$ is a generalized Hilbert-Schmidt operator and that these operators generate $\mathbf{R}(G)$, implying, that the right regular representation is of trace class. Let us observe, that p and q as above, and hence also the corresponding trace, are not uniquely determined, the degree of nonuniqueness depending on the " size " of the field of G invariant rational functions on the dual of the underlying space of $\mathfrak{n}'$. As a partial substitute for this lack of uniqueness we show, that p and q can be chosen such, that their degrees do not exceed a constant depending on the dimension of $\mathfrak{n}$ only. The essence of the above construction can be well illustrated by the following simple example. Let G be the connected component of the identity of the group of all affine transformations of the real line. We can realize G as the collection of all pairs $\{ (t, x) ; t, x \in \mathbf{R} \}$ with the multiplication $(t, x) (t', x') = (t + t', x + e^t.x')$. It is known, that G has altogether two infinite dimensional irreducible representations. One can show furthermore, that if T is such a representation and $\varphi \in C_c^\infty (G)$, then the operator $T (\varphi)$ is completely continuous if and only if we have $\int_{\mathbf{R}} \varphi (t, x)\, dx \equiv 0$ identically in t. Let us define the operator D by $(D \varphi) (t, x) \equiv e^t (\partial\varphi/\partial x) (t, x)$; D is the left invariant vector field determined by a generator of the one parameter subgroup of translations. By what we have just said, $T (D\varphi)$ is completely continuous (even of Hilbert-Schmidt class) if $\varphi \in C_c^\infty (G)$.

The previous results, in particular, imply, that if G is any connected solvable Lie group, the type III component of $\mathbf{R}(G)$ (say) is always trivial. Our last major result (*cf.* Theorem 5, Section 9) asserts, that if G is simply connected, then either the type I or type II components are trivial. In other words, the right regular representation admits a central continuous direct sum decomposition of factor representations, consisting of type I or of type II constituents only. By virtue of the results of Chapter III (*cf.* Summary, or Theorem 2 and 3, *loc. cit.*), to establish this we need essentially two propositions. First, that either $G_g = \overline{G}_g$, or $G_g/\overline{G}_g$ is infinite almost always with respect to the Lebesgue measure on $\mathfrak{g}'$ [*cf.* Proposition 8.1; for $\overline{G}_g$ *cf.* Summary, Chapter I or I.4 (*c*)]. Second, that either the collection of the locally closed orbits of the coadjoint representation or its complement is of measure zero in $\mathfrak{g}'$. We show finally, that in general Theorem 5 is false, if G is not simply connected.

1. The purpose of this part is to summarize several results of the theory of unitary representations of connected and simply connected nilpotent Lie groups, to be used below.

Let $\mathfrak{n}$ be a nilpotent Lie algebra over the reals, and N the corresponding connected and simply connected Lie group. We recall, that the exponential map establishes an analytic isomorphism between the underlying spaces of $\mathfrak{n}$ and N resp. Given some function $\varphi(n)$ on N, we write $\varphi(l)$ for the function on $\mathfrak{n}$, which is determined by $\varphi(l) \equiv \varphi(\exp(l))$ $(l \in \mathfrak{n})$.

a. Given a (necessarily biinvariant) Haar measure dn on N, and $\varphi \in L^1(N)$, we put for any unitary representation T of N :

$$T(\varphi) = \int_N \varphi(n)\, T(n)\, dn.$$

Then, if $\varphi \in C_c^\infty(N)$ *and* T *is irreducible, the operator* $T(\varphi)$ *is of trace class* (*cf.* e. g. [29], Théorème, p. 108).

b. We recall, that the measure dl, which is the inverse image of dn under the exponential map, on $\mathfrak{n}$ is translation invariant. [In fact, this follows at once from the form of the law of composition on $N = \exp \mathfrak{n}$, given in (*c*) of the proof of Proposition 1.1, Chapter II; replace $\mathfrak{g}$ *loc. cit.* by $\mathfrak{n}$.] Assuming again $\varphi \in C_c^\infty(N)$ we set

$$\hat{\varphi}(l') = \int_{\mathfrak{n}} \varphi(l) \langle l, l' \rangle\, dl, \qquad \text{where} \quad \langle l, l' \rangle \equiv \exp[i(l, l')],$$

(l, l') being the value of the canonical bilinear form on $\mathfrak{n} \times \mathfrak{n}'$ at $l \in \mathfrak{n}$, $l' \in \mathfrak{n}'$. *Then there is an orbit* O *of the coadjoint representation in* $\mathfrak{n}'$ *such that, with* $T(\varphi)$ *as in* (*a*) *above*

$$\text{(1)} \qquad \operatorname{Tr}(T(\varphi)) = \int_O \hat{\varphi}(l')\, dv$$

where dv *is a positive invariant measure on* O. *The integral on the right hand side converges absolutely, and* O *and* dv *are uniquely determined by the class of* T *in* $\hat{N}$. *Conversely, to any orbit* O *there corresponds an irreducible representation by virtue of* (1) (*cf.* [29], Théorème, p. 111).

c. If dim O is not zero, the measure dv, called the canonical measure, is obtained as follows. O carries a nondegenerate invariant 2-form ω; setting $d = \dim O$, dv is the positive measure which corresponds to the d form $\omega^{d/2}/C(d)$ $[C(d) = (d/2)!\,\pi^{d/2}.2^d]$ on O (*cf.* [30], Theorem, p. 271).

d. One can show, that the orbit $O \subset \mathfrak{n}'$ of (1) coincides with the orbit corresponding to T as in I.4 (*f*). We shall call canonical the map, sending the equivalence class of T into O, of $\hat{N}$ onto $\mathfrak{n}'/N$.

2. The purpose of this section is to discuss several aspects of the theory of irreducible unitary representations of a not necessarily simply connected, connected nilpotent Lie group. Our main goal is to derive the property of the Plancherel measure described in Lemma 2.3.

Let $\mathfrak{n}$ be as above and N a corresponding connected group. Then N is of the form N_1/Σ, where N_1 is a connected and simply connected group belonging to $\mathfrak{n}$, and Σ a discrete subgroup of the center of N_1. We denote by $\mathfrak{n}^\natural$ the center of $\mathfrak{n}$, and by Γ the discrete subgroup of $\mathfrak{n}^\natural$, which is the inverse image of Σ under the exponential map. Let us write $\mathfrak{N}$ for the locally compact abelian group $\mathfrak{n}/\Gamma$ and $\hat{\mathfrak{N}}$ for its dual. $\hat{\mathfrak{N}}$ can canonically be identified with the annihilator, in the sense of the duality between the underlying groups of $\mathfrak{n}$ and $\mathfrak{n}'$ resp., of Γ in $\mathfrak{n}'$. Observe, that since Γ is left invariant by the adjoint representation of N_1, we have, by duality, an action of N_1 on $\hat{\mathfrak{N}}$.

LEMMA 2.1. — *The restriction of the canonical map from* $\hat{N}_1$ *onto* $\mathfrak{n}'/N_1$, *to* $\hat{N} \subset N_1$ *establishes a bijection between* $\hat{N}$ *and* $\hat{\mathfrak{N}}/N_1$.

Proof. — We write Φ for the canonical map in question, and we show, that $\Phi(\hat{N}) = \hat{\mathfrak{N}}/N_1$ $(\subset \mathfrak{n}'/N_1)$. If T is an irreducible representation of N_1, we have for all c in $\mathfrak{n}^\natural$: $T(\exp(c)) \equiv \chi_T(c).I$, where χ_T is a character of $\mathfrak{n}^\natural$. T belongs to $\hat{N}$ if and only if the kernel of χ_T contains Γ. Let O be the image of the equivalence class of T under Φ. One sees at once, that if l' is fixed in O, the linear form $c \mapsto (c, l')$ on $\mathfrak{n}^\natural$ is independent of the particular choice of l' in O. Thus for any such l' the map

$$c \mapsto \langle c, l' \rangle = \exp[i(c, l')]$$

defines a character χ_0 of $\mathfrak{n}^\natural$, and we have $O \in \hat{\mathfrak{N}}_0/N_1$ if and only if χ_0 is identically one on Γ. Therefore to complete our proof of Lemma 2.1 it is enough to show, that $\chi_0 = \chi_T$. Let c be a fixed element of $\mathfrak{n}^\natural$, and let us replace in (1) (Section 1) $\varphi(l)$ by $\varphi(l-c)$ $(l \in \mathfrak{n})$. One sees at once, that in this case the left and right hand side *loc. cit.* gets multiplied with $\chi_T(c)$ and $\chi_O(c)$ respectively. Thus, because of the arbitrariness of φ we can conclude, that $\chi_T \equiv \chi_O$.

Q. E. D.

Our next objective is to derive the analogue of the trace formula (1) in the non simply connected case. Let us denote by Ψ and ψ the canonical homomorphism from N_1 onto $N = N_1/\Sigma$, and from the underlying group of $\mathfrak{n}$ onto $\mathfrak{N} = \mathfrak{n}/\Gamma$ respectively. Using

$$\exp(l + c) = \exp(l)\exp(c)$$

for all l in $\mathfrak{n}$ and c in $\mathfrak{n}^{\natural}$, one verifies easily, that there is an isomorphism ω of the underlying manifold of $\mathfrak{N}$ onto that of N, such that the diagramm

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{\exp} & N_1 \\ \psi \downarrow & & \downarrow \Psi \\ \mathfrak{N} & \xrightarrow{\omega} & N \end{array}$$

be commutative. Similarly as in the simply connected case, given any function $\varphi(n)$ on N, we write $\varphi(l)$ for the function defined on $\mathfrak{N}$, and uniquely determined by $\varphi(\omega(l)) \equiv \varphi(l)$ $(l \in \mathfrak{N})$. With these notations we have to following

LEMMA 2.2. — *Let* T *be an irreducible representation of* N, φ *a* C_c^∞ *function on* N, *dn a Haar measure on* N, *and dl the measure, corresponding on* $\mathfrak{N}$, *via* ω, *to dn. Then the operator*

$$\text{(1)} \qquad T(\varphi) = \int_{\mathfrak{N}} \varphi(n)\, T(n)\, dn$$

is of trace class, and we have

$$\text{(2)} \qquad \operatorname{Tr}(T(\varphi)) = \int_O \hat{\varphi}(l')\, dv.$$

Here O *is the element, corresponding to* T *in the sense of Lemma* 2.1, *of* $\hat{\mathfrak{N}}/N_1$, $\hat{\varphi}(l')$ $(l' \in \hat{\mathfrak{N}})$ *is the Fourier transform of* $\varphi(l)$ *on* $\mathfrak{N}$; *in other words*

$$\hat{\varphi}(l') = \int_{\mathfrak{N}} \varphi(l) \langle l, l' \rangle_0\, dl$$

where $\langle l, l' \rangle_0$ *is the canonical bilinear form on* $\mathfrak{N} \times \hat{\mathfrak{N}}$; *finally, dv is a positive invariant measure on* O, *which can be computed according to the algo-*

rithm given in 1 (*c*) *above, and the integral on the right hand side converges obsolutely.*

Proof. — *a*. Let $d\sigma$ be the translation invariant measure on Σ, for which the measure of the neutral element is one. Let φ_0 be a function in $C_c^\infty(N_1)$, such that

$$\varphi(n) = \int_\Sigma \varphi_0(\sigma n_0)\, d\sigma \qquad [n = \Psi(n_0)].$$

We denote by dn_1 the Haar measure on N_1, with which we have

$$\int_N \varphi(n)\, dn = \int_{N_1} \varphi_0(n)\, dn_1$$

whenever φ and φ_0 are connected as above. We form the representation $T' = T \circ \Psi$ of N_1; then, with dn and dn_1 as before, we have $T(\varphi) = T'(\varphi_0)$, showing, that $T(\varphi)$ is of trace class.

b. To prove (2) we recall first, that by virtue of (1) [*cf.* 1 (*b*)] we have

$$\mathrm{Tr}\,(T'(\varphi_0)) = \int_O \hat{\varphi}_0(l'_1)\, dv$$

where

$$\hat{\varphi}_0(l'_1) = \int_{\mathfrak{n}} \varphi_0(l_1) \langle l_1, l'_1 \rangle\, dl_1 \qquad (l'_1 \in \mathfrak{n}')$$

and dl_1 is the measure, corresponding to dn_1 *via* the exponential map, on $\mathfrak{n}$. Hence to obtain the desired conclusion it suffices to show, that the restriction of $\hat{\varphi}_0$ to $\hat{\mathfrak{N}} \subset \mathfrak{n}'$ is identical to $\hat{\varphi}$. To this end we observe, that, if $d\gamma$ is the translation invariant measure on $\Gamma \subset \mathfrak{n}$, for which the measure of the zero element is equal to one, we have

$$\varphi(l) = \int_\Gamma \varphi_0(\gamma + l_1)\, d\gamma \qquad [l = \psi(l_1) \in \mathfrak{N}].$$

On the other hand, if $l' \in \hat{\mathfrak{N}}$ and $\gamma \in \Gamma$, $\langle l + \gamma, l' \rangle = \langle l, l' \rangle$, and therefore

$$\hat{\varphi}(l') = \int_{\mathfrak{N}} \varphi(l) \langle l, l' \rangle_0\, dl = \int_{\mathfrak{N}} \left(\int_\Gamma \varphi_0(l_1 + \gamma)\, d\gamma \right) \langle l, l' \rangle_0\, dl = \int_{\mathfrak{n}} \varphi_0(l) \langle l, l' \rangle\, dl = \hat{\varphi}_0(l')$$

proving our statement.

Q. E. D.

Let $\{\pi(\zeta); \zeta \in \hat{N}\}$ be a Borel field of unitary representations, such that $\pi(\zeta)$ is of the class ζ (*cf.* [12], 4.6.2, p. 95 and 18.7.3, p. 324). Let us recall, that the Plancherel measure μ for N is a positive measure on $\hat{N}$, such that for any φ in $C_c^\infty(N)$:

$$\varphi(e) = \int_{\hat{N}} \operatorname{Tr}(\pi(\zeta)(\varphi))\, d\mu(\zeta).$$

Having fixed the Haar measure dn, utilized in forming $\pi(\zeta)(\varphi)$, on N [*cf.* (1)], $d\mu$ is uniquely determined by this relation (*cf.* [12], 18.8, 1-2, p. 327-328). We also observe, that the map, sending the orbit $O \subset \hat{\mathfrak{n}}$ into the equivalence class of a corresponding irreducible representation (*cf.* Lemma 2.1) is continuous (*cf.* [31], Proposition 2, p. 89) and hence it establishes an isomorphism betweeen the Borel structure of $\hat{\mathfrak{n}}/N$ and $\hat{N}$ (*cf.* [2], Proposition 2.5, p. 7). In view of this fact, in the following, whenever speaking of the Plancherel measure of a connected nilpotent group, we shall mean the corresponding measure on $\hat{\mathfrak{n}}/N$. Let us denote by $\{\gamma_j; 1 \leqq j \leqq s\}$ a basis for the lattice Γ in $\mathfrak{n}^\natural$ $(s > 0)$. Let $Z(s)$ be the collection of all s-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ with integral components. Putting $H_\alpha = \{l'; l' \in \mathfrak{n}', (\gamma_j, l') = 2\pi\alpha_j;\ 1 \leqq j \leqq s\}$, we have $NH_\alpha \subseteq H_\alpha$ and $\hat{\mathfrak{n}} = \bigcup_{\alpha \in Z(s)} H_\alpha$.

LEMMA 2.3. — *Let* P *be a polynomial function on* $\mathfrak{n}'$, *such that for* α *in* $Z(s)$, *the restriction of* P *to* H_α *is* N *invariant and not identically zero. Then the direct image, in* $\hat{\mathfrak{n}}/N$, *of the set of zeros of* P *on* H_α *is of Plancherel measure zero.*

Proof. — Let dl be as in Lemma 2.2. We denote by dl' the measure, dual to dl, on $\hat{\mathfrak{n}}$ and write dl'_α for the part of dl' on H_α $[\alpha \in Z(s)]$. For an element λ of H_α/N, we denote by O_λ and dv_λ the corresponding N orbit and canonical measure [*cf.* 1 (*c*)] resp. By virtue of Lemma 2.2, to obtain the Plancherel measure $d\mu$, it suffices to determine, for each α in $Z(s)$, a positive measure $d\mu_\alpha$ on H_α/N, such that

$$\int_{H_\alpha} g(l')\, dl'_\alpha = \int_{H_\alpha/N} \left(\int_{O_\lambda} g(l')\, dv_\lambda \right) d\mu_\alpha$$

holds for any rapidly decreasing function $g(l')$ on $\mathfrak{n}'$. In fact, in this case, by virtue of the Plancherel formula for the abelian group $\mathfrak{n}$, it will be enough to define a positive measure $d\mu$ on $\hat{\mathfrak{n}}/N$ by the condition, that its part on H_α/N be $d\mu_\alpha$. From now on we keep α fixed, and omit it from our notations. Using the definitions and notations of the proof of Proposition 4.1, Chapter II, let e be the smallest element of $\mathcal{E}$, for which the restriction of Q_e to H is not identically zero. Below we shall suppose $d(e) > 0$, and leave to the reader the easy modifications necessary, when $d(e) = 0$. We can obviously assume, that the Jordan-Hölder sequence, fixed as *loc. cit.* (replace $\mathfrak{g}$ by $\mathfrak{n}$) is such, that $\mathfrak{n}_s$ is the smallest subspace, containing Γ, of $\mathfrak{n}^\natural$, and that $l_j = \gamma_j$ $(1 \leq j \leq s)$. We observe, that for any e in $\mathcal{E}$ we have $\{1, 2, \ldots, s\} \subset E$. In fact, let us recall, that j belongs to $f(x)$ if and only if $\mathfrak{n}_{j-1}(x) \subsetneq \mathfrak{n}_j(x)$. But since, for $j \leq s$, $\mathfrak{n}_j \subseteq \mathfrak{n}^\natural \subset R(x)$, we can conclude, that $\mathfrak{n}_j(x) \equiv R(x)$ for each j not exceeding s. In this fashion we get (*cf.* Remark 4.1, Chapter II) that

$$P_e \cap H = \left\{ x;\, x = \sum_{j=1}^{m} x_j\, l'_j,\ x_j = 2\,\pi\alpha_j \text{ for } 1 \leq j \leq s, \text{ and } x_j = 0 \text{ for } j \in e \right\}.$$

Let us show now, that $\mathfrak{O}_e \cap P_e \cap H$ is Zariski open in $P_e \cap H$. To this end it is enough to see, that the restriction of Q_e to $P_e \cap H$ is not identically zero. If x is any element in $\mathfrak{O}_e \cap H$, then $y = \sum_{j \in E} \lambda_j(x)\, l'_j$ lies in $P_e \cap \mathfrak{O}_e \cap H$ and $0 \neq Q_e(x) = Q_e(y)$, proving our statement. The map $x \mapsto \sum_{j \in E} \lambda_j(x)\, l'_j$ $(x \in \mathfrak{O}_e \cap H)$ gives rise to a diffeomorphism between $(\mathfrak{O}_e \cap H)/N$, and $P_e \cap \mathfrak{O}_e \cap H$. Let us write E' for the complement of $\{1, 2, \ldots, s\}$ in E. Using Remark 4.1, Chapter II, we see at once, that if $S(x)$ is a polynomial on $\mathfrak{n}'$, such that its restriction to H is N invariant, then there is a polynomial $U(\lambda)$ in the indeterminates $\{\lambda_j;\, j \in E'\}$, such that, on H, $S(x) \equiv U(\lambda(x))$ where the right hand side is obtained by replacing λ_j through $\lambda_j(x)$ $(x \in \mathfrak{O}_e \cap H)$. Bearing in mind all this, by imitating the argument of [30], p. 278-279, we arrive at the following description of $d\mu_\alpha$. Let us denote by L_e a polynomial, satisfying $L_e^2 \equiv Q_e$, on $\mathfrak{n}'$. Along with Q_e, its restriction to H is N invariant, and, as we have just seen it, this can be written as $R_e(\lambda(x))$. Then $d\mu_\alpha$ *is carried by* $(\mathfrak{O}_e \cap H)/N$, *and there it corresponds to the measure* $|R_e(\lambda)|\, d\lambda$ *on* $P_e \cap \mathfrak{O}_e \cap H$; $d\lambda$ stands for the product of the differentials of the variables $\{\lambda_j;\, j \in E'\}$. Let now P be as in the statement of Lemma 2.3. To conclude our proof it is enough to observe, that if we write the restriction of P to H in the form $V(\lambda(x))$, then the subset

$$\left\{x; x = \sum_{j=1}^{s} 2\pi\alpha_j l'_j + \sum_{j\in E'} \lambda_j l'_j, V(\lambda) = 0\right\}$$

of $P_e \cap \mathfrak{O}_e \cap H$ is of measure zero with respect to $|R_e(\lambda)|\, d\lambda$.

Q. E. D.

3. Let G be a connected but not necessarily simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{n}$ the greatest nilpotent ideal of $\mathfrak{g}$ and by N the connected subgroup, belonging to $\mathfrak{n}$, of G. Let G_1 be a connected and simply connected Lie group belonging to $\mathfrak{g}$, and N_1 the connected subgroup determined by $\mathfrak{n}$. We denote by M a discrete central subgroup of G_1, such that G_1/M is isomorphic to G. We put $\Sigma = M \cap N_1$ and Γ for the complete inverse image of Γ under the exponential map from $\mathfrak{n}$ onto N_1. We let G act on $\mathfrak{n}$ through inner automorphisms and on $\mathfrak{n}'$ by the corresponding contragredient representation. Given $a \in G$ and $l \in \mathfrak{n}$ ($l' \in \mathfrak{n}'$ resp.) we shall write al (al' resp.) for the action of a on l (l' resp.). We observe, that G leaves Γ invariant, and thus G operates on $\mathfrak{N} = \mathfrak{n}/\Gamma$, and by duality, also on $\hat{\mathfrak{N}}$ (*cf.* 2 above). In the next sections important role will be played by nonzero rational functions R, defined on one of the connected components of $\hat{\mathfrak{N}} \subset \mathfrak{n}'$, and verifying $R(ax) \equiv \Delta(a) R(x)$ where $\Delta(a) \equiv \det(\mathrm{Ad}(a))$ $(a \in G)$. Our next main objective is to establish the existence of such functions (*cf.* Proposition 4.1 below). Our problem, of course, would be simpler, if it were possible to take the restriction of a rational function, defined over $\mathfrak{n}'$ and having the indicated transformation properties, to $\hat{\mathfrak{N}} \subset \mathfrak{n}'$. That this is not necessarily feasible is shown by the following example. Let us denote by $\mathfrak{g}$ the solvable Lie algebra, spanned over the reals by the elements $\{e_0, e_1, \ldots, e_5\}$ satisfying the following commutation relations :

$$[e_0, e_1] = e_1, \quad [e_0, e_2] = -e_2, \quad [e_0, e_3] = -e_3, \quad [e_1, e_2] = e_4, \quad [e_1, e_3] = e_5,$$

all other brackets having the value zero. Here $\mathfrak{n}$ is spanned by $\{e_j; 1 \leq j \leq 5\}$, and we have $\mathfrak{n}^{\natural} = \mathbf{R}\, e_4 + \mathbf{R}\, e_5 = \mathfrak{g}^{\natural}$. Let G_1 and N_1 be as above with respect to the Lie algebra just defined. We denote by H the orthogonal complement of $\mathfrak{n}^{\natural}$ in $\mathfrak{n}'$. To prove our point, it will be enough to show, that if $S(x)$ is a rational function on $\mathfrak{n}'$, such that $S(ax) \equiv [\Delta(a)]^{-1} S(x)$ $(a \in G_1)$, and if the restriction of S to H is defined, then it is identically zero. We denote by $\{x_j; 1 \leqq j \leqq 5\}$ coordinates with respect to a basis, dual to $\{e_j; 1 \leqq j \leqq 5\}$, in $\mathfrak{n}'$. Let us observe next, that the system of polynomials $\{x_4, x_5, x_2 x_5 - x_3 x_4\}$ constitutes a system of algebraically independent generators of the ring of all N_1 invariant polynomials on $\mathfrak{n}'$ (*cf.* [9], p. 326). We set $R(x) \equiv x_2 x_5 - x_3 x_4$ and show, that $R(ax) \equiv [\Delta(a)]^{-1} R(x)$ $(a \in G_1)$. To this end let us write $T_t \equiv (\mathrm{Ad}(\exp(-te_0)))'$ and remark, that it suffices to prove, that $R(T_t x) \equiv \exp(t) R(x)$ for all t, which is immediately clear. Thus S/R is invariant with respect to G_1 and hence it can be written as p/q, where p and q are relatively prime polynomials satisfying $p(ax) \equiv \alpha(a) p(x)$ and $q(ax) \equiv \alpha(a) q(x)$ for all a in G_1. Since α is identically one on N_1, we can write p as $\sum_{j=0}^{n} \lambda_j(x_4, x_5) R^j$ $(\lambda_n \not\equiv 0)$, where the λ_j's are polynomials in x_4 and x_5, and thus invariants of G_1. We conclude therefore, that p is of the form $\lambda(x_4, x_5) R^n$, and similarly, q can be written as $\mu(x_4, x_5) R^k$ $(k \geqq 0)$. But since p and q are relatively prime, we have $n = k = 0$, and λ and μ are relatively prime polynomials in x_4 and x_5. Summing up, S is identical to $R\lambda/\mu$, and $\mu \not\equiv 0$ on $H = \{x; x \in \mathfrak{n}', x_4 = x_5 = 0\}$, but then $S \equiv 0$ on H.

4. Proposition 4.1. — *With the previous notations, let* s *be the rank of* Γ, $\mathfrak{n}_s$ *the subspace, spanned by elements of* Γ, *of* $\mathfrak{n}$, *and* π *the canonical projection from* $\mathfrak{n}'$ *onto* $\mathfrak{n}'_s = \mathfrak{n}'/\mathfrak{n}_s^{\perp}$. *Given* λ *in* $\mathfrak{n}'_s$, *we put* $H_\lambda = \overset{-1}{\pi}(\lambda)$. *Then there exist polynomials* p *and* q *on* $\mathfrak{n}'$, *such that their degrees do not exceed a bound* $B(m)$ *depending on* $m = \dim \mathfrak{n}$ *only, and such that* p *and* q, *when restricted to* H_λ, *are not identically zero and satisfy* $p(ax) \equiv \alpha(a) p(x)$, $q(ax) \equiv \beta(a) q(x)$ *for all* x *in* H *and* a *in* G; α *and* β *are positive characters of* G, *such that* $\alpha(a)/\beta(a) \equiv \Delta(a)$, *where* $\Delta(a) = \det(\mathrm{Ad}(a))$.

Proof. — In the following we shall assume, that $s > 0$, and that $\mathfrak{n}$ is nonabelian. The easy modifications, necessary to settle the remaining cases, will be left to the reader.

a. We consider G as acting on the complexification $\mathfrak{n}'_{\mathbf{C}}$ of $\mathfrak{n}'$ and denote by $\mathfrak{G}$ the smallest algebraic subgroup, containing $(\mathrm{Ad}(G) \mid \mathfrak{n})'$ of $GL(\mathfrak{n}'_{\mathbf{C}})$. (Observe, that $\mathfrak{G}$ is irreducible, and hence connected.) Let $\mathfrak{n}_s^{\perp}$ be the orthogonal complement of $\mathfrak{n}_s \subset \mathfrak{n}$ in $\mathfrak{n}'$. Keeping λ fixed we set $\mathcal{H} = H_\lambda + i\,\mathfrak{n}_s^{\perp}$ and observe, that $\mathfrak{G}$ leaves $\mathcal{H}$ invariant. With these notations we claim, that to prove our proposition *it suffices to find two polynomials* P *and* Q *on* $\mathfrak{n}'_{\mathbf{C}}$, *such that their restrictions to* $\mathcal{H}$ *are not identically zero, and such that* $P(gx) \equiv \alpha_1(g) P(x)$, $Q(gx) \equiv \beta_1(g) Q(x)$ *for all* g *in* $\mathfrak{G}$ *and* x *in* $\mathcal{H}$, *where* $\alpha_1(g)/\beta_1(g) \equiv \det(g)$, *and such that their degrees do not exceed a bound* $B_1(m)$ *depending on* m *only.* In fact, let us write $r = r_1 + ir_2$ and $s = s_1 + is_2$ respectively for the restrictions of P and Q to $\mathfrak{n}' \subset \mathfrak{n}'_{\mathbf{C}}$. By virtue of our assumptions r and s, when restricted to H_λ, are not identically zero and thus, for instance, the real part $F(x)$ of r/s is defined and not identically zero on H_λ. We observe next, that for all g in G : $\det(\mathrm{Ad}(g^{-1}) \mid \mathfrak{n}) = \det(\mathrm{Ad}(g^{-1})) = \Delta(g^{-1})$ and therefore $F(gx) \equiv \Delta(g^{-1}) F(x)$ $(x \in H_\lambda,\ g \in G)$. On the other hand, $F(x) \equiv g(x)/\mid s(x) \mid^2$ where

$$g(x) \equiv r_1(x)\, s_1(x) - r_2(x)\, s_2(x)$$

and, putting $\beta_2(g) \equiv \beta_1((\mathrm{Ad}(g^{-1}) \mid \mathfrak{n})')$, we have $\mid s(gx) \mid^2 \equiv \mid \beta_2(g) \mid^2 \mid s(x) \mid^2$. Hence we conclude finally, that to satisfy the conditions of our proposition, it suffices to take $p(x) \equiv \mid s(x) \mid^2$, $q(x) \equiv g(x)$, $\alpha(g) \equiv \mid \beta_2(g) \mid^2$, $\beta(g) \equiv \mid \beta_2(g) \mid^2 \Delta(g^{-1})$ $(g \in G)$, and $B(m) = 2\, B_1(m)$.

b. Let $\bar{\mathfrak{g}}$ be the Lie algebra of $\mathfrak{G} \subset \mathrm{GL}(\mathfrak{n}'_{\mathbf{C}})$. By virtue of our choice of the latter, $\bar{\mathfrak{g}}$ is the smallest algebraic Lie algebra of endomorphisms of $\mathfrak{n}'_{\mathbf{C}}$ containing $(\mathrm{ad}\, \mathfrak{g}_{\mathbf{C}} \mid \mathfrak{n}_{\mathbf{C}})'$. Let us denote by $\mathfrak{n}_1$ the collection of all nilpotent elements in $\bar{\mathfrak{g}}$; then there exists an abelian algebraic Lie algebra $\mathfrak{h}$ of semi-simple endomorphisms in $\bar{\mathfrak{g}}$, such that $\bar{\mathfrak{g}}$, as a vector space over the complex field, is direct sum of the underlying spaces of $\mathfrak{n}_1$ and $\mathfrak{h}$ respectively (*cf.* [6], vol. III, Proposition 20, p. 130). We denote by H and N_1 the connected subgroups, belonging to $\mathfrak{h}$ and $\mathfrak{n}_1$ resp. of $\mathfrak{G}$. If g is any element of $\mathfrak{G}$, it can uniquely be written as hn, where $h \in H$, $n \in N_1$ (*cf.* [6], vol. III, Proposition 21, p. 131). Let us denote by $\mathfrak{m}$ the subalgebra $(\mathrm{ad}(\mathfrak{n}_{\mathbf{C}}))'$ of $\mathfrak{n}_1$, and by $\mathfrak{M}$ the corresponding connected subgroup of $\mathfrak{G}$. Observe, that $\mathfrak{m}$ is an ideal in $\bar{\mathfrak{g}}$. In fact, putting $\mathfrak{g}_0 = (\mathrm{ad}(\mathfrak{g}_{\mathbf{C}}) \mid \mathfrak{n}_{\mathbf{C}})'$, we have $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = [\mathfrak{g}_0, \mathfrak{g}_0]$ (*cf.* [6], vol. II, Theorem 13, p. 173); but since $\mathfrak{g}$ is solvable we also have

$$[\mathfrak{g}_0, \mathfrak{g}_0] = (\mathrm{ad}\,[\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}] \mid \mathfrak{n}_{\mathbf{C}})' \subset (\mathrm{ad}\, \mathfrak{n}_{\mathbf{C}} \mid \mathfrak{n}_{\mathbf{C}})' = \mathfrak{m},$$

proving our statement. Furthermore, since $\bar{\mathfrak{g}}/\mathfrak{m}$ is abelian, the same is valid for $\mathfrak{G}/\mathfrak{M}$ (*cf.* [6], vol. III, Proposition 12, p. 120).

c. Let us put $\mathfrak{g}_1 = \mathfrak{h} + \mathfrak{m}$; $\mathfrak{g}_1$ is a subalgebra of $\bar{\mathfrak{g}}$. We write $\mathfrak{G}_1$ for the corresponding connected subgroup of $\mathfrak{G}$ and observe, that $\mathfrak{G}_1 = H\mathfrak{M}$. We claim, that to prove our proposition, *it suffices to show, that there exist polynomial functions* S *and* T *on* $\mathfrak{n}'_{\mathbf{C}}$, *such that* $S \mid \mathcal{H} \not\equiv 0$, $T \mid \mathcal{H} \not\equiv 0$, $S(gx) \equiv \gamma(g)\, S(x)$, $T(gx) \equiv \delta(g)\, T(x)$ *for all* g *in* $\mathfrak{G}_1$ *and* x *in* $\mathcal{H}$, *where* $\gamma(g)/\delta(g) \equiv \det(g)$, *and such that the degrees of* S *and* T *do not exceed a bound* $L(m)$ *depending on* $m = \dim \mathfrak{n}$ *only.* To show this we observe first, that there is a homomorphism α_1 (β_1) of $\mathfrak{G} = HN_1$ into the multiplicative group of all nonzero complex numbers, uniquely determined by the condition, that $\alpha_1(h) \equiv \gamma(h)$ $[\beta_1(h) \equiv \delta(h)]$ for all h in H and such that $\alpha_1 \equiv 1$ ($\beta_1 \equiv 1$ resp.) on N_1. We have evidently $\alpha_1(g)/\beta_1(g) \equiv \det g$ on $\mathfrak{G}$. In this fashion, by virtue of (*a*), to prove our assertion, it is enough to establish the following statement. *Let* $S(x)$ *be a polynomial on* $\mathfrak{n}'_{\mathbf{C}}$ *such that* $S(gx) \equiv \gamma(g) S(x)$ *for all* g *in* $\mathfrak{G}_1$ *and* x *in* $\mathcal{H}$, *and* $S \mid \mathcal{H} \not\equiv 0$. *Let* α *be the homomorphism of* $\mathfrak{G}$ *into the group of all nonzero complex numbers, such that* $\alpha \mid \mathfrak{G}_1 \equiv \gamma$ *and* $\alpha \mid N_1 \equiv 1$. *Then there is a polynomial function* $R(x)$ *on* $\mathfrak{n}'_{\mathbf{C}}$, *such that* $R \mid \mathcal{H} \not\equiv 0$, $\deg R \leqq \deg S$ *and* $R(gx) \equiv \alpha(g) R(x)$ *for all* g *in* $\mathfrak{G}$ *and* x *in* $\mathcal{H}$. Let us denote by W the vector space, over the complex numbers, composed of all polynomials P on $\mathfrak{n}'_{\mathbf{C}}$, satisfying $P(gx) \equiv \gamma(g) P(x)$ for all g in $\mathfrak{G}_1$ and x in $\mathcal{H}$, and $\deg P \leqq \deg S$. We write W_0 for the subspace of all elements, vanishing identically on $\mathcal{H}$, of W, and set $V = W/W_0$. We have evidently $\dim W < +\infty$ and hence also $\dim V < +\infty$. On the other hand, by virtue of the existence of $S(x)$ as above, W properly contains W_0 and thus $\dim V > 0$. Given any element g of $\mathfrak{G}$ and P in W, let us put $(T(g) P)(x) \equiv P(g^{-1} x)$. We observe, that the linear map $P \mapsto T(g) P$ $(P \in W)$ transforms W into itself. To see this it suffices to take into account, that $\mathfrak{G}_1 = H\mathfrak{M}$ is invariant in $\mathfrak{G}$, since $\mathfrak{G}/\mathfrak{M}$ is abelian [*cf.* the end of (*b*) above]. On the other hand, it is evident, that $T(g)$ leaves W_0 invariant. We write $U(g)$ for the operator, induced by $T(g)$ on $V = W/W_0$; the map $g \mapsto U(g)$ is a linear representation of $\mathfrak{G}$ on V. Since the image of any element l of $\mathfrak{n}_1$ in the differential of U is obviously a nilpotent operator, by virtue of the theorem of Engel, applied to $U(N_1)$ as acting on V, we can conclude, that there is a nonzero element r in V, such that $U(n) r \equiv r$ for all n in N_1. Let R be an element of W lying over r. We have $R \mid \mathcal{H} \not\equiv 0$ and also $R(gx) = \alpha(g) R(x)$ for all g in $\mathfrak{G}$ and x in $\mathcal{H}$. To prove the last statement let us assume, that $g = hn$ $(h \in H,\ n \in N_1)$. Then if x is in $\mathcal{H}$, by virtue of the definition of W and α :

$$R(gx) = \gamma(h) R(nx) = \alpha(h) R(nx) = \alpha(g) R(nx);$$

but in view of our choice of R, $R(nx) \equiv R(x)$ on $\mathcal{H}$, and thus

$$R(gx) = \alpha(g) R(x),$$

completing the proof of the statement made at the start of (*c*).

d. Summing up once more, *to establish our proposition, it suffices to construct polynomials* P *and* Q *on* $\mathfrak{n}'_{\mathbf{C}}$, *such that* $P \mid \mathcal{H} \not\equiv 0$, $Q \mid \mathcal{H} \not\equiv 0$,

$P(gx) = \gamma(g) P(x)$, $Q(gx) = \delta(g) Q(x)$ *for all* g *in* $\mathfrak{G}_1 = H\mathfrak{M}$ *and* x *in* $\mathcal{H}$, *where* $\gamma(g)/\delta(g) \equiv \det(g)$, *and* $\deg(P)$, $\deg(Q) \leqq L(m)$, *where* $L(m)$ *depends on* $\dim \mathfrak{n} = m$ *only.*

e. In the following, given an endomorphism A of a vector space V, A′ will stand for the transpose of A, operating on the dual V′ of V. If $\mathfrak{A}$ is a family of endomorphisms of V, we write $\mathfrak{A}'$ for $\{B; B = A', A \in \mathfrak{A}\}$. Let $\overline{\mathfrak{G}}$ be the smallest algebraic group of endomorphisms of $\mathfrak{n}_{\mathbf{C}}$, which contains $\mathrm{Ad}(G) | \mathfrak{n}$. Since the group of all automorphisms of $\mathfrak{n}_{\mathbf{C}}$ is algebraic, we conclude, that any element of $\overline{\mathfrak{G}}$ is an automorphism of $\mathfrak{n}_{\mathbf{C}}$. We have evidently $\overline{\mathfrak{G}}' \supset \mathfrak{G}$ [*cf.* (*a*)] implying, that $\mathfrak{G}'$ consists of automorphisms of $\mathfrak{n}_{\mathbf{C}}$. Let us observe, that $\overline{\mathfrak{G}}$ leaves the center of $\mathfrak{g}_{\mathbf{C}}$ $(\subset \mathfrak{n}_{\mathbf{C}})$ elementwise fixed, and hence the same is valid for $\mathfrak{G}'$.

Let us choose now a Jordan-Hölder sequence

$$\mathfrak{n}_{\mathbf{C}} = \mathfrak{m}_m \supset \mathfrak{m}_{m-1} \supset \ldots \supset \mathfrak{m}_0 = (0) \qquad (\dim \mathfrak{m}_j = j,\ j = 1, \ldots, m)$$

for the action of the solvable Lie algebra $\mathfrak{g}_1' = \mathfrak{h}' + \mathfrak{m}'$ [*cf.* the start of (*c*)] of derivations of $\mathfrak{n}_{\mathbf{C}}$. By what we have just seen, we can assume, that $\mathfrak{m}_s$ is the complexification of $\mathfrak{n}_s \subset \mathfrak{g}^{\natural}$, where $\mathfrak{n}_s$ is as in the statement of Proposition 4.1. Since along with H, H′, too, consists of semi-simple endomorphisms, for each $j = 1, 2, \ldots, m$ we can find a nonzero element l_j in $\mathfrak{m}_j - \mathfrak{m}_{j-1}$, such that $hl_j = \mu_j(h)\, l_j$ for all h in H. Observe, that we have $\mu_j \equiv 1$ for $1 \leqq j \leqq s$. With the notations of Proposition 4.1, Chapter II, taking *loc. cit.* $\mathbf{C}$, $\mathfrak{m}$, $\mathfrak{M}$ and $\{\mathfrak{m}_j\}$ in place of $\mathbf{K}$, $\mathfrak{g}$, G and $\{\mathfrak{g}_j\}$ resp., let us denote by e the smallest element of $\mathcal{E}$, such that $Q_e | \mathcal{H} \not\equiv 0$. Assume first, that $d(e) = 0$; $\mathfrak{M}$ acts then trivially on $\mathcal{H}$. Let $\{x_j; 1 \leqq j \leqq m\}$ be coordinates with respect to a basis, dual to $\{l_j; 1 \leqq j \leqq m\}$, on $\mathfrak{n}_{\mathbf{C}}$. We claim, that by setting $P(x) = \prod_{j=s+1}^{m} x_j$, $Q(x) \equiv 1$, $\gamma(g) \equiv \det(g)$, $\delta(g) \equiv 1$ $(g \in \mathfrak{G}_1)$ the conditions of (*d*) above are statified. In fact, all what we have to show is that $P(hx) \equiv (\det(h))\, P(x)$ for all x in $\mathcal{H}$ and h in H; but this is certainly so, since $\det(h) = \prod_{j=s+1}^{m} \mu_j(h)$. Let us assume now, that $d(e) > 0$, an let us consider, as in Lemma 1.3, Chapter II, the system $\{\lambda_j(x); j \in E\}$. Evidently $\{1, 2, \ldots, s\} \subset E'$. Writing E′ for the difference of these two sets, the points $\{\lambda_j(x); j \in E'\}$ of $\mathbf{C}^k$ [$k = m - d(e) - s$; coordinates arranged according to increasing j], if x varies over the Zariski open set $\mathcal{H} \cap \mathfrak{O}_e$ in $\mathcal{H}$, describes a set of the same kind (*cf.* Remark 4.1, Chapter II), and thus, in particular, $\lambda_j(x) \not\equiv 0$ if j belongs to E'. Let us show now, that $\lambda_j(hx) \equiv \mu_j(h)\lambda_j(x)$ for all h in $\mathcal{H}$, x in $\mathfrak{O}_e$ and j in E. To this end, by virtue of Lemma 1.3, Chapter II, it is enough to observe, that if h belongs to H, then its transpose h' is an automorphism of $\mathfrak{n}_{\mathbf{C}}$ satisfying $h'.l_j = \mu_j(h)\, l_j$ $(1 \leqq j \leqq m)$. We recall, that by definition

$$Q_e(x) \equiv \det\{([l_i, l_j], x);\, i, j \in e\}.$$

Observing, that for h in H :

$$([l_i, l_j], hx) = (h'[l_i, l_j], x) = ([h' l_i, h' l_j], x) = \mu_i(h)\,\mu_j(h)\,([l_i, l_j], x) \qquad (i, j \in e),$$

we conclude, that for all x in $\mathfrak{n}'_{\mathbf{C}}$ we have $Q_e(hx) \equiv (\mu(h))^2\, Q_e(x)$, where

$$\mu(h) = \prod_{j \in e} \mu_j(h).$$

We denote by P_e a polynomial, homogeneous of degree $d(e)/2$, over $\mathfrak{n}'_{\mathbf{C}}$, such that $P_e^2 \equiv Q_e$. Then we have $P_e(hx) \equiv \mu(h)\, P_e(x)$ for all h in H. We set

$$R(x) = P_e(x) \prod_{j \in E'} \lambda_j(x).$$

R (x) is a rational function, the restriction of which to $\mathcal{H}$ is not identically zero. Since

$$\det(h) = \prod_{j=s+1}^{m} \mu_j(h) = \mu(h) \prod_{j \in E'} \mu_j(h),$$

we have $R(hx) \equiv (\det(h))\, R(x)$ on $\mathcal{H}$ for all h in H. In addition, by virtue of our construction, $R \mid \mathcal{H}$ is invariant under $\mathfrak{M}$ and thus, since $\mathfrak{G}_1 = H\,\mathfrak{M}$ and $\det(n) \equiv 1$ on $\mathfrak{M}$, also $R(gx) \equiv (\det(g))\, R(x)$ $(x \in \mathcal{H}, g \in \mathfrak{G}_1)$. By virtue of Lemma 4.1, Chapter II, since $\lambda_j(x) \equiv P_j(0; x)$, with notations as *loc. cit.* we can write $\lambda_j(x) \equiv \Lambda_j(x)/(Q_e(x))^{K(m)}$, where $\Lambda_j(x)$ is a polynomial satisfying $\deg(\Lambda_j(x)) \leqq (2m+1)\, K(m)$ $(j \in E')$. Thus putting

$$P(x) \equiv P_e(x) \prod_{j \in E'} \Lambda_j(x) \qquad \text{and} \qquad Q(x) \equiv (Q_e(x))^{kK(m)} \qquad [k = m - d(e) - s],$$

the polynomials P and Q satisfy $\deg(P)$, $\deg(Q) \leqq L(m)$, where $L(m) = m^2(2m+1)\, K(m)$. Let us define finally $\mu(g) = \mu(h)$ if $g = hn$ $(h \in H, n \in \mathfrak{M})$. We have on $\mathcal{H}$: $Q(gx) \equiv \nu(g)\, Q(x)$, where $\nu(g) \equiv [\mu(g)]^{2kK(m)}$

$(g \in \mathfrak{G}_1)$. Summing up, it is clear from the previous discussion, that choosing P, Q and L (m) as above, and setting $\gamma(g) \equiv (\det(g)) . \nu(g)$, $\delta(g) \equiv \nu(g)$ $(g \in \mathfrak{G}_1)$, all the conditions of (d) above are met, and thus the proof of Proposition 4.1 is complete.

Q. E. D.

Given a finite dimensional Lie algebra $\mathfrak{g}$ over the reals, we denote by U $(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let φ be an automorphism of $\mathfrak{g}$; we shall write φ also for the corresponding automorphism of U $(\mathfrak{g})$.

Corollary 4.1. — *Let* G *be a connected solvable Lie group with the Lie algebra* $\mathfrak{g}$. *We denote by* $\mathfrak{n}$ *the greatest nilpotent ideal of* $\mathfrak{g}$ *and by* $\mathcal{V}$ *the center of* U $(\mathfrak{n})$ $[\subset \mathrm{U}(\mathfrak{g})]$. *There are nonzero elements p and q in* $\mathcal{V}$, *such that* $gp \equiv \varepsilon(g)\, p$, $gq \equiv \eta(g)\, q$, *and* $\eta(g)/\varepsilon(g) \equiv \det(\mathrm{Ad}(g))$ $(g \in \mathrm{G})$.

Proof. — We denote by S $(\mathfrak{g})$ the symmetric algebra over the underlying space of $\mathfrak{g}$. We recall, that there is an isomorphism Φ of the underlying space of S $(\mathfrak{g})$ onto that of U $(\mathfrak{g})$, such that for any finite collection $\{x_1, x_2, \ldots, x_M\}$ of elements of $\mathfrak{g}$ the image, under Φ, of the product $x_1 x_2 \ldots x_M$, computed in S $(\mathfrak{g})$, be the same as

$$\frac{1}{\mathrm{M}!}\left(\sum_{\pi \in \Pi_M} x_{\pi(1)}\, x_{\pi(2)} \ldots x_{\pi(M)}\right)$$

computed in U $(\mathfrak{g})$. If φ is any automorphism of $\mathfrak{g}$, we have $\Phi(\varphi\, v) = \varphi\, \Phi(v)$ for all v in S $(\mathfrak{g})$. Let $\mathcal{F}(\mathfrak{g}')$ be the algebra of all polynomial functions on the dual $\mathfrak{g}'$ of the underlying space of $\mathfrak{g}$. There is an isomorphism Ψ from S $(\mathfrak{g})$ onto $\mathcal{F}(\mathfrak{g}')$ such that

$$\Psi(x_1 x_2 \ldots x_M)(l') \equiv \prod_{j=1}^{M} (x_j, l') \qquad (l' \in \mathfrak{g}).$$

If A is some endomorphism of the underlying space of $\mathfrak{g}$, we have $\Psi(\mathrm{A}\, v)(l') \equiv \Psi(v)(\mathrm{A}'\, l')$ $[v \in \mathrm{S}(\mathfrak{g}),\ l' \in \mathfrak{g}']$. We replace above $\mathfrak{g}$ by $\mathfrak{n}$, and assuming, that p, q in $\mathcal{F}(\mathfrak{n}')$ are as in Proposition 4.1, form their images in U $(\mathfrak{n})$ under $\Phi \circ \overset{-1}{\Psi}$. Denoting these again by the same letters, by virtue of what we saw above, we have $gp \equiv (1/\alpha(g))\, p$, $gq \equiv (1/\beta(g))\, q$ $(g \in \mathrm{G})$. Since the restriction of α and β to N is identically one, p and q lie in $\mathcal{V}$, and thus to complete the proof of Corollary 4.1 it suffices to set $\varepsilon(g) \equiv 1/\alpha(g)$, $\eta(g) \equiv 1/\beta(g)$ $(g \in \mathrm{G})$.

Q. E. D.

5. Before proceeding we wish to recall the following facts.

a. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. Denoting by $C_c^\infty(G)$ the family of all complex valued C^∞ functions of a compact support on G let us put, for a given a in G, $(R(a)f)(x) \equiv f(xa)$ $[f \in C_c^\infty(G)]$. The map $a \mapsto R(a)$ defines a representation of G on $C_c^\infty(G)$, considered as a vector space over the complex numbers. Let $U(\mathfrak{g})$ be the universal envelopping algebra of $\mathfrak{g}$. For p in $U(\mathfrak{g})$, $dR(p)$ is a left invariant differential operator on G. We put $p(f) = (dR(p)f)(e)$ $[f \in C_c^\infty(G)$, $e =$ unity of G$]$; the linear form $f \mapsto p(f)$ is a distribution, the support of which is $\{e\}$. *The map assigning to* $p \in U(\mathfrak{g})$ *this functional is an isomorphism between* $U(\mathfrak{g})$ *and the algebra of all distributions with support at* e, *the product in the latter being defined by convolution.* In fact, let $\{l_j; 1 \leqq j \leqq m\}$ be a basis in $\mathfrak{g}$, $r = (r_1, r_2, \ldots, r_m)$ ($r_j \geqq 0$ integer for $j = 1, 2, \ldots, m$). With notations as in Corollary 4.1 we put

$$l(r) = \Phi(l_1^{r_1} l_2^{r_2} \ldots l_m^{r_m}) \in U(\mathfrak{g}).$$

The desired conclusion is implied by the fact, that the collection of all these elements span $U(\mathfrak{g})$ as a vector space over $\mathbf{R}$, along with the observation (*cf.* [19], p. 98) that for $f \in C_c^\infty(G)$ we have

$$(1) \qquad l(r)(f) = \frac{\partial^{|r|}}{\partial t_1^{r_1} \partial t_2^{r_2} \ldots \partial t_m^{r_m}} f(\exp(l(T)))\,|_{T=0}$$

where $|r| = r_1 + r_2 + \ldots + r_m$, and for $T = (t_1, t_2, \ldots, t_m)$ we have set $l(T) = \sum_{j=1}^{m} t_j l_j$.

b. There is an isomorphism φ from $S(\mathfrak{g})$ into the algebra of all complex-valued polynomial functions on $\mathfrak{g}'$, uniquely determined by the condition, that

$$\varphi(l_1^{r_1} l_2^{r_2} \ldots l_m^{r_m})(l') \equiv i^{|r|} \prod_{j=1}^{m} (l_j, l')^{r_j} \qquad (l' \in \mathfrak{g}').$$

Let us set $\varepsilon = \varphi \circ \overset{-1}{\Phi}$; ε is an isomorphism of the underlying space of $U(\mathfrak{g})$ with its image such that, for any a in G we have $\varepsilon(ap)(l') \equiv \varepsilon(p)(a^{-1} l')$ $[p \in U(\mathfrak{g}),\ l' \in \mathfrak{g}']$. This map admits a unique extension, to be denoted again by ε, to an isomorphism between the underlying space (over $\mathbf{C}$) of $[U(\mathfrak{g})]_{\mathbf{C}}$ and that of the collection of all complex valued polymonial fonctions over $\mathfrak{g}'$. Let $\mathfrak{U}$ be the center of $U(\mathfrak{g})$; an element p belongs to $\mathfrak{U}_{\mathbf{C}}$ if and only if $\varepsilon(p)$ is G invariant.

c. Let T be some continuous unitary representation of G. If p is any distribution of compact support on G, there is an operator $T(p)$ on the variety of all indefinitely many times differentiable vectors of $\mathbf{H}$ (T), uniquely determined by the condition, that we have for any pair f, g of vectors of the said type : $(\mathrm{T}(p) f, g) = p_x (\mathrm{T}(x) f, g)$. If p is central, T (p) is a scalar multiple of the identity map I. In this case we shall write (with a slight abuse of notations) $\mathrm{T}(p) = \omega \mathrm{I}$ ($\omega \in \mathbf{C}$).

d. We assume now, that $\mathfrak{g}$ is nilpotent and G connected and simply connected. Let T be an irreducible unitary representation of G belonging to the orbit O in $\mathfrak{g}'$ (*cf.* 1). If $p \in \mathfrak{U}$, $\varepsilon(p)$ is G invariant [*cf.* (*b*)]; we shall write $\varepsilon(p)(\mathrm{O})$ for its value on O. With these notations A. A. Kirillov proved (*cf.* [22], Theorem 7.2), that $\mathrm{T}(p) = \varepsilon(p)(\mathrm{O})\,\mathrm{I}$.

e. Identifying the underlying manifold of G to that of $\mathfrak{g}$ by means of the exponential map let us observe, that for any $p \in \mathrm{U}(\mathfrak{g})$ its Fourier transform $\hat{p}$, formed with respect to the underlying abelian group of $\mathfrak{g}$, coincides with $\varepsilon(p)$. In fact, putting for $\varphi \in \mathrm{C}_c^\infty(\mathfrak{g})$, $\varphi(-l) \equiv \varphi_-(l)$, we have by definition $p(\varphi_-) = \hat{p}(\hat{\varphi})$, where $\hat{\varphi}$ is as in 1 (*b*). Let us choose a basis $\{ l_j ; 1 \leq j \leq m \}$ in $\mathfrak{g}$. With notations as in (*a*) above, to establish our statement, it is enough to consider the case of $f(l(r))$. Let $\{ l'_j ; 1 \leq j \leq m \}$ be a basis in $\mathfrak{g}'$ such that $(l_i, l'_j) = \delta_{ij}$ and denote by $\{ x_i \}$ and $\{ y_j \}$ the corresponding coordinates on $\mathfrak{g}$ and $\mathfrak{g}'$ resp. We have by (1),

$$p(\varphi_-) = (-1)^{|r|} \frac{\partial^{|r|}}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_m^{r_m}} \varphi(x) \Big|_{x=o} = (-1)^{|r|} \frac{\partial^{|r|}}{\partial x^r} \varphi(x) \Big|_{x=o}.$$

On the other hand $\varepsilon(l(r))(y) \equiv i^{|r|} y_1^{r_1} y_2^{r_2} \ldots y_m^{r_m} \equiv i^{|r|} y^r$. Therefore to complete our proof it is enough to recall, that if dx is an element of the Lebesgue measure on $\mathbf{R}^m$, we have

$$i^{|r|} y^r \int_{\mathbf{R}^m} \varphi(x)\, e^{ixy}\, dx = (-1)^{|r|} \int_{\mathbf{R}^m} \frac{\partial^{|r|}}{\partial x^r} \varphi(x)\, e^{ixy}\, dx.$$

6. We denote again by $\mathfrak{n}$ a nilpotent Lie algebra and by N a corresponding connected, but not necessarily simply connected nilpotent group. Using the notations of Section 2, we identify the underlying manifold of N to that of the abelian group $\mathfrak{N} = \mathfrak{n}/\Gamma$ by means of ω (cf. *loc. cit.*). Let $\mathcal{D}$ be the family of all central distributions of compact support on N. If $\mu \in \mathcal{D}$, we write $\hat{\mu}$ for the Fourier transform, in the sense of the abelian group $\mathfrak{N}$, of μ on $\hat{\mathfrak{N}}$. Observe, that evidently $\hat{\mu}$ is N invariant. Given an N orbit O in $\hat{\mathfrak{N}}$, we shall write $\hat{\mu}(\mathrm{O})$ for the value of $\hat{\mu}$ on O.

By virtue of what we said in (*e*) of the previous section, the following lemma is a slight extension of the result of Kirillov quoted in (*d*) *loc. cit.*

LEMMA 6.1. — *With the previous notations assume, that* T *is an irreducible unitary representation, belonging to the orbit* $O \subset \hat{\mathfrak{n}}$, *of* N (*cf.* Lemma 2.1). *Then, if* $\mu \in \mathcal{O}$, *we have* $T(\mu) = \hat{\mu}(O)\, I$.

Proof. — *a.* If φ is in $C_c^\infty(N)$, we have

$$T(\mu \times \varphi) = T(\mu)\, T(\varphi) \qquad (\mu \times \varphi = \text{convolution on } N).$$

Forming the trace of both sides, by virtue of Lemma 2.2 we conclude, that the assertion to be proved implies

$$(1) \qquad \int_O \widehat{\mu \times \varphi}\, dv = \hat{\mu}(O) \int_O \hat{\varphi}\, dv.$$

But the inverse implication, too, is valid. In fact, replacing in (1), φ by $\varphi \times \psi$ $[\psi \in C_c^\infty(N)]$ we conclude, that $\mathrm{Tr}\left(\left[T(\mu \times \varphi) - \hat{\mu}(O)\, T(\varphi)\right] T(\psi)\right) = 0$, whence, varying first ψ and then φ we get easily, that $T(\mu) = \hat{\mu}(O)\, I$. In this fashion, to prove our lemma, it suffices to establish (1).

b. Let us observe, that if O is a zerodimensional orbit, we have certainly $T(\mu) = \hat{\mu}(O)\, I$. In fact, let us assume $O = \{k_0\}$; then T coincides with the character φ of N defined by $\varphi(\omega(l)) \equiv \langle l, k_0 \rangle$ $(l \in \mathfrak{n})$. In this fashion $T(\mu) = \mu(\langle l, k_0 \rangle) = \hat{\mu}(k_0)$, proving our statement.

c. We proceed now to prove (1) by induction according to the dimension of N. By (*b*) our statement is valid if $\dim N \leqq 2$, since in this case any orbit is zerodimensional.

Let $\mathfrak{n}'$ be the image of $\mathfrak{n}^\natural$ in $\mathfrak{n}$. Since $\langle nl, l \rangle \equiv \langle l, n^{-1} l \rangle$ $\left(n \in N, l \in \mathfrak{n}, l' \in \hat{\mathfrak{n}}\right)$, putting, with k_0 in $O : f(l) \equiv \langle l, k_0 \rangle$ $(l \in \mathfrak{n}')$, the character f on $\mathfrak{n}'$ depends only on O. We denote by Δ the kernel of f, and assume, that the dimension of the component Δ_0 of its neutral element is positive. Let us put $M = N/\omega(\Delta)$; we have $\dim(M) < \dim(N)$. We write $\mathfrak{M} = \mathfrak{n}/\Delta$ and observe, that the relation of M to $\mathfrak{M}$ is analogous to that of N to $\mathfrak{n}$ (*cf.* Section 2). The dual of $\mathfrak{M}$ is identifiable to the annihilator of Δ in $\hat{\mathfrak{n}}$. Hence, by virtue of our construction : $O \subset \hat{\mathfrak{M}} \subset \hat{\mathfrak{n}}$. Given a distribution α, invariant under translations by Δ, on $\mathfrak{n}$, we denote by α_1 the corresponding distribution on $\mathfrak{M}$. Let $d\delta$ be an element of the invariant measure on Δ. For $\varphi \in C_c^\infty(\mathfrak{n})$ we put $\nu(\varphi) = \int_\Delta \varphi(\delta)\, d\delta$. Denoting by $\tilde{}$ the Fourier transform, with an appropriately chosen invariant measure, on $\mathfrak{M}$, we have

$\hat{\varphi} | \hat{\mathfrak{M}} \equiv (\widetilde{\varphi \times \nu})_1$ [*cf.* (*b*) in Lemma 2.2]. $(\mu \times \nu)_1$ is a distribution of compact support, which is central with respect to M, and

$$(\varphi \times \mu \times \nu)_1 \equiv (\varphi \times \nu)_1 \times (\mu \times \nu)_1.$$

Hence, by virtue of the assumption of our inductive procedure

$$\int_0 (\widehat{\mu \times \varphi})\, dv = \int_0 (\widetilde{\varphi \times \mu \times \nu})_1\, dv = (\widetilde{\mu \times \nu})_1 (0) \int_0 (\widetilde{\varphi \times \nu})_1\, dv = \hat{\mu}(0) \int_0 \hat{\varphi}\, dv$$

which is the desired conclusion.

d. If $\dim \Delta_0 = 0$ we have $\dim (\mathfrak{n}^\natural) = 1$ and O is not orthogonal to $\mathfrak{n}^\natural$. We recall now the following results of Kirillov (*cf.* [29], p. 130-136). Let $0 \neq z \in \mathfrak{n}^\natural$, $I = \{z, y\}$ a 2-dimensional ideal. There is $\varphi \neq 0$ in $\mathfrak{n}'$ such that $\mathrm{ad}\,(l)\, y \equiv \varphi(l)\, z$ $(l \in \mathfrak{n})$, and $\mathfrak{n}_0 = \ker(\varphi)$ is the centralizer of I. Let us put $N' = \exp(\mathfrak{n}_0)$; N′ is a closed subgroup, of codimension 1, of N. This being so, *any* N *orbit* O in $\mathfrak{n}'$, *which is not orthogonal to* $\mathfrak{n}^\natural$, *satisfies* $O + \mathfrak{n}_0^\perp = O$. Let $\mathfrak{N}'$ be the image of $\mathfrak{n}_0$ *in* $\mathfrak{N}$; we have $\hat{\mathfrak{N}}' = \hat{\mathfrak{N}}/\mathfrak{n}_0^\perp$. Assume now, that $O \subset \hat{\mathfrak{N}}$, and let O_0 be an arbitrary N′ orbit in its projection into $\hat{\mathfrak{N}}'$. Let x be an element of $\mathfrak{n}$ with $\varphi(x) = 1$, and denote by O_t the image of O_0 under $\exp(tx)$ $(t \in \mathbf{R})$. Fot φ in $C_c^\infty(\mathfrak{N})$, we write φ_0 for its restriction to $\mathfrak{N}'$. *Then, if* $\tilde{\varphi}_0$ *is the Fourier transform of* φ_0 *on* $\mathfrak{N}'$ (with a suitably chosen invariant measure) *we have*

$$\int_0 \hat{\varphi}(l')\, dv = \int_{\mathbf{R}} \left(\int_{O_t} \tilde{\varphi}_0(l)\, dv_t \right) dt$$

where dv_t is the canonical measure [*cf.* 1 (*c*)] on O_t. Let μ be in $\mathcal{D}$. Since $\hat{\mu}$ is N invariant we conclude, that it is invariant under translations by elements of $\mathfrak{n}_0^\perp \subset \hat{\mathfrak{N}}$, and thus the support of μ is contained in $\mathfrak{N}'$. Assume now, that O is as in our lemma. We have, for each fixed t, by virtue of our inductive procedure

$$\int_{O_t} (\widetilde{\mu \times \varphi})_0 = \int_{O_t} \widetilde{\mu \times \varphi_0}\, dv_t = \tilde{\mu}(O_t) \int_{O_t} \tilde{\varphi}_0\, dv_t.$$

But since $\tilde{\mu}(O_t) \equiv \hat{\mu}(O)$, we obtain finally, that

$$\int_0 (\widehat{\mu \times \varphi})\, dv = \hat{\mu}(0) \int_{\mathbf{R}} \left(\int_{O_t} \tilde{\varphi}_0\, dv_t \right) dt = \hat{\mu}(0) \int_0 \hat{\varphi}(l')\, dv.$$

Q. E. D.

In the following we continue to identify, whenever convenient, the underlying manifolds of N and $\mathfrak{N}$ resp. We shall call a complex valued function P on $\hat{N}$ a polymonial function, if the function corresponding, by virtue of Lemma 2.1, to P on $\hat{\mathfrak{N}}/N$, on each component of the latter, arises out of a polymonial function of $\mathfrak{n}'$. We say, that the polymonial function P on $\hat{N}$ is of bounded degree, if the said polymonials can be chosen such, that their degrees are uniformly bounded. The polynomial functions P and Q will be called proportional, if the corresponding functions on $\hat{\mathfrak{N}}$ are proportional on each connected component (the factor of proportionality being permitted to vary). Assuming, that N is not simply connected, let us denote by $\mathfrak{T}$ the maximal torus in the centre of N; we write s ($s > 0$) for its dimension. We say, that a distribution on N is of a bounded degree, if it can be written as a finite sum of distributions of the form $\nu \times k$, where k is in U ($\mathfrak{n}$) [*cf.* 5 (*a*)] and ν is a complex valued measure carried by $\mathfrak{T}$ and there of the form $f\,dt$, where $f \in C^\infty(\mathfrak{T})$ and dt is the element of the normalized invariant measure on $\mathfrak{T}$. Observe, incidentally, that the convolution of ν and k with respect to N and the underlying group of $\mathfrak{N}$ resp. coincide. We denote the collection of all central distributions of bounded degree by $\mathcal{D}_0$. Any element of $\mathcal{D}_0$ is carried by $\mathfrak{T}$. If N is simply connected, we define $\mathcal{D}_0$ by $\mathfrak{U}_{\mathbf{C}}$ [*cf.* 5 (*a*), *b*)].

LEMMA 6.2. — *Let* P *be a polynomial function of bounded degree on* $\hat{N}$. *Then there is an element* μ *of* $\mathcal{D}_0$ *such that, with the usual identifications,* $\hat{\mu}$ *and* P *are proportional.*

Proof. — Assume first, that N is simply connected. Then there is a μ in $\mathfrak{U}_{\mathbf{C}}$ such that $P = \varepsilon(\mu)$ [*cf.* 5 (*b*)] and we have also $P \equiv \hat{\mu}$ by 5 (*e*). Let us suppose now, that N is not simply connected. With notations as above and in Lemma 2.3, let $\{ P_\alpha;\ \alpha \in Z(s) \}$ be a sequence of polynomial functions, such that $P_\alpha \mid H_\alpha$ is N invariant, and that the corresponding function on H_α/N coincides with the restrictions of P to $H_\alpha/N \subset \hat{\mathfrak{N}}/N$. By virtue of our assumption on P, we can suppose the existence of a constant $K > 0$, not depending on α, such that $\deg(P_\alpha) < K$ [$\alpha \in Z(s)$]. Using coordinates $\{ t_j;\ 1 \leqq j \leqq s \}$ corresponding to the basis $\{ \gamma_j;\ 1 \leqq j \leqq s \}$ of Γ, on $\mathfrak{T}$ (identified here to $\mathfrak{n}_s/\Gamma$), let us put $\chi_\alpha(t) \equiv \exp(-2\pi i t \alpha)$ $\left[\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in Z(s),\ t\alpha = \sum_{j=1}^{s} t_j \alpha_j\right]$. We write also χ_α for the measure on N, which is carried by $\mathfrak{T}$ and there coincides with $\chi_\alpha\,dt$ ($dt = dt_1\,d_2\ldots dt_s$).

If p is some element of $U(\mathfrak{n})$, we have $\widehat{p\times\chi_\alpha} \mid H_\alpha \equiv \varepsilon(p) \mid H_\alpha$ and $\widehat{p\times\chi_\alpha} \mid H_\gamma \equiv 0$ if $\gamma \neq \alpha$. If $P_\alpha = \varepsilon(p_\alpha)\,[\alpha \in Z(s)]$ we can write $p_\alpha = \sum_{|r| \leq K} c_r^{(\alpha)}\, l(r)$ [*cf.* 5 (*a*)]. Let us put $\delta_\alpha = \sup |c_r^{(\alpha)}|$ and

$$\gamma_\alpha = \begin{cases} 1 & \text{if } \delta_\alpha = 0, \\ \exp(-|\alpha|)\,\delta_\alpha^{-1} & \text{otherwise } \left(|\alpha| = \sum_{j=1}^{s} \alpha_j\right). \end{cases}$$

Then for each fixed r ($|r| \leq K$) the sequence $\{\gamma_\alpha c_r^{(\alpha)};\ \alpha \in Z(s)\}$ is rapidly decreasing, and hence the function $f_r \equiv \sum_\alpha \gamma_\alpha c_r^{(\alpha)} \chi_\alpha$ is C^∞ on $\mathfrak{T}$. Writing ν_r for the measure on $\mathfrak{N}$, which is concentrated on $\mathfrak{T}$, and there coincides with $f_r\, dt$, one sees at once, that if we set $\mu = \sum_{|r|\leq K} \nu_r \times l(r)$, we have for all α in $Z(s)$: $\hat{\mu} \mid H_\alpha \equiv \gamma_\alpha P_\alpha \mid H_\alpha$. Since $P_\alpha \mid H_\alpha$ is N invariant, μ, too, is N invariant. Thus μ belongs to $\mathcal{D}_0$, and it satisfies all the requirements of Lemma 6.2. Q. E. D.

7. Let G be a separable locally compact group, dx an element of the right invariant measure on G and let us put $d(ax) = \Delta(a)\, dx$ $(a \in G)$. We recall, that the right regular representation $a \mapsto \mathcal{R}(a)$ $(a \in G)$ of G is the continuous unitary representation, corresponding to the map $f(x) \mapsto f(xa)$ on the Hilbert space $L^2(G)$ of the equivalence classes of all complex valued functions, square integrable with respect to dx. The right ring $\mathbf{R}(G)$ of G is the von Neumann algebra generated by the operators $\{\mathcal{R}(a);\, a \in G\}$. The left regular representation $a \to \mathcal{L}(a)$ is the unitary representation, corresponding to $f(x) \mapsto (\Delta(a))^{-1/2} f(a^{-1}x)$ $(a \in G)$ on $L^2(G)$; the left ring $\mathbf{L}(G)$ is the von Neumann algebra generated by $\{\mathcal{L}(a);\, a \in G\}$. Putting for $f \in L^2(G)$:

$$(Sf)(x) \equiv f(x^{-1})/(\Delta(x))^{1/2},$$

we have $\|Sf\| = \|f\|$, $S\,\mathcal{L}(a)\,S = \mathcal{R}(a)$ $(a \in G)$ and thus also $S\mathbf{L}(G)S = \mathbf{R}(G)$. We recall also, that $\mathbf{R}(G)$ is the commutant of $\mathbf{L}(G)$, that is $\mathbf{R}(G) = (\mathbf{L}(G))'$ (*cf.* the proof of Lemma 7.1 below or [13], 5, p. 80). We shall write J for the selfadjoint operator on $L^2(G)$, which is the minimal closed extension of the map $f(x) \mapsto \sqrt{\Delta(x)}\, f(x)$ of $C(G) \subset L^2(G)$ onto itself. Let A and B be, not necessarily bounded, selfadjoint operators on a Hilbert space. We say, that A and B commute, if any bounded function of A commutes with any bounded function of B. We recall, that in this case the product AB is densely defined and admits a selfadjoint minimal closed extension, which we shall denote by [AB]. Given a von Neumann algebra **M**, we say, that the (possibly unbounded) selfadjoint operator A is affiliated with **M**, in symbols A η **M**, if A commutes with any operator of the commutant **M**′ of **M** (cf. e. g. [13], 10, p. 15). In the following, whenever speaking of a trace on **M**, we shall mean a trace on the set of all positive operators of **M** in the sense of [13], Definition 1 (p. 81), which is in addition faithful, semi-finite and normal (*cf.* p. 82, *loc. cit.*). We recall, that in this case **M** is semi-finite (*cf.* [13], Proposition 8, p. 99). If φ is a trace on **M**, we shall say that an operator A in **M** is generalized Hilbert-Schmidt operator, if $\varphi(A^*A) < +\infty$ (*cf.* [12], A. 32, p. 338). If f, $g \in L^2(G)$ we shall put

$$(f \times g)(x) = \int_G f(xy^{-1})\, g(y)\, dy.$$

We say, that the element $f \in L^2(G)$ is right bounded, if the map $g \mapsto g \times f$ $[g \in L^2(G)]$ gives rise to a bounded operator V_f on $L^2(G)$. Since V_f commutes with $\mathcal{L}(a)$ $(a \in G)$, by what we saw above, V_f belongs to **R** (G).

With these notations and terminology we have

LEMMA 7.1. — *Suppose, that there is a selfadjoint, positive and non singular operator* M′ *affiliated with* **L** (G), *such that, putting* $M = SM'S$ *we have* $J = [M'M^{-1}]$. *Then there is a trace* φ *on* **R** (G), *uniquely determined by the property, that for any right bounded* f *in* $L^2(G)$ *lying in the domain of* M′, V_f *be a generalized Hilbert-Schmidt operator and*

$$\varphi(V_f^* . V_f) = \| M' f \|^2. \tag{1}$$

Proof. — Our assertion is a simple consequence of a result of J. Dixmier (*cf.* [7], Théorème 2, p. 287). We recall (cf. *loc. cit.*), that the quasi-unitary algebra A is an algebra over the complex numbers, on which an involutive antiautomorphism $x \mapsto x^s$, an automorphism $x \mapsto x^j$ and an inner product (x, y) are defined, such that with respect to the latter A becomes a pre-Hilbert space satisfying the following axioms : (i) $(x^s, x^s) = (x, x)$; (ii) $(x, x^j) \geq 0$; (iii) $(xy, z) = (y, x^{sj} z)$; (iv) the mapping $x \mapsto xy$ (y fixed) is continuous; (v) the linear combinations of the elements of the form $xy + (xy)^j$ are dense in A (x, y, z arbitrary in A). One verifies easily, that one obtains the structure of a quasi-unitary algebra on C (G) by defining

$$(f, g) = \int_G f(x)\overline{g(x)}\, dx, \qquad (f.g)(x) \equiv (f \times g)(x), \qquad f^s(x) \equiv \overline{f(x^{-1})}/(\Delta(x))^{1/2},$$
$$f^j(x) \equiv \sqrt{\Delta(x)} . f(x) \qquad [f, g \in C(G)].$$

Given a quasi-unitary algebra A, we denote by $\mathbf{H}_A$ the Hilbert space, which is its completion. By virtue of (iv), for each $x \in A$ there exists a bounded operator U_x (V_x) on $\mathbf{H}_A$, satisfying $U_x y = xy$ ($V_x y = yx$ resp.) for every $y \in A \subset \mathbf{H}_A$. One can show, that the weak closure $\mathbf{R}^g$ ($\mathbf{R}^d$ resp.) of $\{U_x;\ x \in A\}$ ($\{V_x;\ x \in A\}$ resp.) is a von Neumann algebra on $\mathbf{H}_A$, and $(\mathbf{R}^g)' = \mathbf{R}^d$. In the case of $A = C(G)$ one has : $\mathbf{H}_A = L^2(G)$, $\mathbf{R}^g = \mathbf{L}(G)$ and $R^d = \mathbf{R}(G)$. Hence, in particular, $\mathbf{R}(G) = (\mathbf{L}(G))'$ (as stated above). The minimal closed extension J of the map $x \mapsto x^j$ ($x \in A$) is selfadjoint, positive and non singular. We denote by S the involution, arising by extending the map $x \mapsto x^s$ ($x \in A$) to $\mathbf{H}_A$. We have $\mathbf{R}^d = S\,\mathbf{R}^g\,S$. One sees easily, that for $A = C(G)$ the operators J and S are defined as before. An element f of $\mathbf{H}_A$ is called right bounded, if there is a bounded operator V_f on $\mathbf{H}_A$, such that we have $V_f x = U_x f$ for all x in A; observe, that if $A = C(G)$, this coincides with our previous definition. Now we are in position to state the result of Dixmier referred to at the start, of which our lemma is the special case for $A = C(G)$. With the above notations *let us suppose, that* $M' \eta \mathbf{R}^g$ *has the properties enumerated in our lemma and in particular, putting* $M = SM'.S$, *assume, that* $J = [M'.M^{-1}]$. *Then there is a trace* φ *on* $\mathbf{R}^d$, *uniquely determined by the property, that if* f *is right bounded and lies in the domain of* M', $V_f^*.V_f$ *is a generalized Hilbert-Schmidt operator and* $\varphi(V_f^*.V_f) = \|M' f\|^2$.

Q. E. D.

Lemma 7.2. — *With the previous notations assume, that* M' *is a selfadjoint, positive and invertible operator such that* $M' \eta \mathbf{L}(G)$, M' *commutes with* J *and for any* a *in* G : $\mathcal{L}(a) M' \mathcal{L}(a^{-1}) = (\Delta(a))^{-1/2} M'$. *Then, putting* $M = SM'.S$, *we have* $J = [M'.M^{-1}]$.

Proof. — We have for all a in G : $\mathcal{L}(a)\, J\, \mathcal{L}(a^{-1}) = (\Delta(a))^{1/2} J$, from where we conclude, that $K = [JM^{-1}]$ commutes with $\mathcal{L}(a)$ ($a \in G$) and hence, by virtue of $\mathbf{R}(G) = (\mathbf{L}(G))'$ we have $K \eta \mathbf{R}(G)$. Let us put $M_1 = K^{-1}$; by what preceeds, $J = [M'.M_1^{-1}]$. Let us set $SM'.S = M \eta \mathbf{R}(G)$ and $SM_1 S = M'_1 \eta \mathbf{L}(G)$. Next we observe, that $SJS = J^{-1}$, whence we infer, that also $J = [M'_1 M^{-1}]$. Let us note, that M commutes with M_1, since $M = SM'.S$ commutes with J^{-1}. Therefore M' and M'_1, too, commute. From all this we conclude, that there is a selfadjoint, positive, invertible operator C affiliated with $\mathbf{L}(G) \cap \mathbf{R}(G)$ [= center of $\mathbf{L}(G)$ and $\mathbf{R}(G)$], satisfying $SCS = C$ and commuting with J, such that $M = [CM_1]$ and $M'_1 = [CM']$. In this fashion we have $[M'.M^{-1}] = [C^{-1}.J]$. Writing L for the left hand side, we have $SLS = L^{-1}$. Hence the same holds true for $[C^{-1}.J]$, whence we conclude, that C^2, and thus also C is equal to the unit operator and that $J = [M'.M^{-1}]$. Q. E. D.

LEMMA 7.3. — *Let* G *a be connected solvable group. Then there is an operator* $M' \eta\, \mathbf{L}(G)$ *with the properties of Lemma* 7.2.

Proof. — Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{n}$ the greatest nilpotent ideal of $\mathfrak{g}$. We put $N = \exp(\mathfrak{n}) \subset G$; N is a closed invariant subgroup of G.

a. Let dn and dy be elements of the invariant measures on N and G/N resp., such that we have for all $f \in C(G)$:

$$\int_G f(x)\, dx = \int_{G/N} \left(\int_N f(nx)\, dn \right) dy.$$

Let S be a Borel subset of G, such that for any a in G we have a representation $a = ns$ with uniquely determined factors in N and S resp. The restriction of the canonical map from G onto G/N, to S is a Borel isomorphisn with its image. Let $d\tau$ be the measure, corresponding to dy on S. We denote by $\mathcal{L}_N$ the left regular representation of N. There is a unitary map from $L^2(G)$ onto $L^2(N) \otimes L^2_\tau(S)$, which carries $\mathcal{L}(n)$ into $\mathcal{L}_N(n) \otimes I$ $(n \in N)$; to simplify notations we shall write $\mathcal{L}(n) = \mathcal{L}_N(n) \otimes I$. Let a be some element of G and let us denote by ψ_a the unique $*$-automorphism of $\mathbf{R}(\mathcal{L}_N) = \mathbf{L}(N)$ with $\psi_a(\mathcal{L}_N(n)) = \mathcal{L}_N(a^{-1}na)$ $(n \in N)$. If H is a selfadjoint operator such that $H \eta\, \mathbf{L}(N)$ and $H = \int_{-\infty}^{+\infty} \lambda\, dE_\lambda$, we define $\psi_a(H)$ by $\int_{-\infty}^{+\infty} \lambda\, d\psi_a(E_\lambda)$. This being so, let us assume, that H is positive, invertible, and that for all $a \in G$ it satisfies $\psi_a(H) = (\Delta(a))^{-1/2} H$. We claim, that in this case $M' = H \otimes I$ satisfies the conditions of our lemma. To this end we have to prove, that M' commutes with J, and $\mathcal{L}(a) M' . \mathcal{L}(a^{-1}) = (\Delta(a))^{-1/2} M$ for all a in G. The first assertion is clear, since by virtue of $\Delta \mid N \equiv 1$, we have under the above identification $J = I \otimes K$, where K corresponds to multiplication by $(\Delta(s))^{1/2}$ on $L^2_\tau(S)$ $(s \in S)$. To establish the second we observe, that

$$\mathcal{L}(a) M' . \mathcal{L}(a^{-1}) = \psi_a(H) \otimes I = (\Delta(a))^{-1/2} . (H \otimes I) = (\Delta(a))^{-1/2} . M' \qquad (a \in G).$$

b. We conclude from the previous remarks, that to prove our Lemma, it is enough, in particular, to find an H as above with $H \eta\, (\mathbf{L}(N))^\natural$.

We recall [*cf.* 1 (*a*)], that if $T \in (\hat{N})_c$ and $\varphi \in C_c^\infty(N)$, $T(\varphi)$ is of trace class, and thus N is of type I (*cf.* [12], 13.9.4, p. 271). Let E_∞ (E_1) be the set $\{\zeta;\ \zeta \in \hat{N},\ \dim \zeta = +\infty\}$ $(\{\zeta;\ \zeta \in \hat{N},\ \dim \zeta = 1\}$ resp.$)$. By virtue of the theorem of Lie we have $\hat{N} = E_1 \cup E_\infty$. Let H_∞ be an infinite dimensional unitary space, μ the Plancherel measure of N (*cf.* [12], 18.8.3, Définition,

p. 328), and $\nu = \mu \mid E_\infty$. We denote by $\mathfrak{H}_\infty$ the Hilbert space of all ν measurable functions $\{A(\zeta)\}$ on E_∞, with values in the set of Hilbert-Schmidt operators on H_∞, satisfying

$$\int_{E_\infty} \mathrm{Tr}\,(A(\zeta)\,[A(\zeta)]^*)\, d\nu(\zeta) < +\infty.$$

We form analogously $\mathfrak{H}_1$ by aid of a one-dimensional space H_1 and $\mu \mid E_1$. Let $\{\pi(\zeta);\, \zeta \in \hat{N}\}$ be a Borel measurable field of unitary representations of N on H_∞ and H_1 resp. such that $\pi(\zeta) \in (\{\zeta\})_c\,(\zeta \in \hat{N})$. We put $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_\infty$ and recall (*cf.* [12], p. 327-328), that there is a unitary map from $L^2(N)$ onto $\mathfrak{H}$, which sends $f \in C_c^\infty(N) \subset L^2(N)$ into $\{\pi(\zeta)(f);\, \zeta \in \hat{N}\} \in \mathfrak{H}$ and makes correspond the center $(\mathbf{L}(N))^\natural$ of $\mathbf{L}(N)$ to the ring of multiplications by μ measurable bounded complex valued functions on $\hat{N}$. We shall write for the latter simply $L_\mu^\infty(\hat{N})$.

Let a be a fixed element of G and let us determine the $*$-automorphism of $L_\mu^\infty(N)$ corresponding to $\psi_a \mid (L(N))^\natural$ [*cf.* (a)]. To this end we denote by V the unitary operator on $L^2(N)$ such that $(Vf)(n) \equiv (\Delta(a))^{-1/2} f(a^{-1} na)$ $[f \in L^2(N)]$. Since $VL_N(n)V^{-1} = \psi_a(L_N(n))$, also $VAV^{-1} = \psi_a(A)$ for all A in $\mathbf{L}(N)$. Let W be the unitary operator corresponding to V on $\mathfrak{H}$. If $f \in C_c^\infty(N)$ we have

$$W\{\pi(\zeta)(f)\} = \{\pi(\zeta)(Vf)\}.$$

On the other hand, for each fixed $\zeta \in \hat{N}$:

$$\pi(\zeta)(Vf) = (\Delta(a))^{-1/2} \int_N f(a^{-1} na)\, \pi(\zeta)(n)\, dn = (\Delta(a))^{1/2}\,[a^{-1}\pi(\zeta)](f).$$

Let $U(\zeta)$ be a unitary operator with $a^{-1}\pi(\zeta) = U(\zeta)\,\pi(a^{-1}\zeta)\,[U(\zeta)]^*$; then

$$W\{\pi(\zeta)(f)\} = \{(\Delta(a))^{1/2}\, U(\zeta).\pi(a^{-1}\zeta)(f)\,(U(\zeta))^*\}.$$

If F is an element of $L_\mu^\infty(\hat{N})$, we write M (F) for the corresponding multiplication operator on $\mathfrak{H}$; we put also $F_a(\zeta) \equiv F(a^{-1}\zeta)$ $(a \in G)$. With these notations what we have just seen shows, that $WM(F)W^{-1} = M(F_a)$.

By virtue of the above considerations, to prove our lemma it suffices to find a Borel measurable function F on $\hat{N}$, such that $0 < F < +\infty$ almost everywhere with respect to μ, and $F_a = (\Delta(a))^{-1/2} F$ for all a in G.

c. Let $\mathfrak{n}$ and $\hat{\mathfrak{n}}$ be as at the start of 3, and let ψ be the map which assigns to $\zeta \in \hat{N}$ the corresponding orbit in $\hat{\mathfrak{n}}/N$. We recall (*cf.* 2), that ψ is a Borel isomorphism between the underlying Borel structures of these spaces. Let us add, that ψ is equivariant with respect to the action of G on $\hat{N}$ and $\hat{\mathfrak{n}}/N$ resp.; we leave the easy verification to the reader (*cf.* Remark 4.1, Chapter I).

d. In the following we shall assume, that the rank s of Γ (*cf.* 3) is positive, and leave to the reader the modifications necessary to settle the remaining case. With the notation of Lemma 2.3 we have $\hat{\mathfrak{n}} = \cup_{\alpha \in Z(s)} H_\alpha$. We apply Proposition 4.1 to H_α [α fix in Z (s)] in place of H_λ *loc. cit.* As a result we obtain polynomials $p_\alpha(x)$ and $q_\alpha(x)$ on $\mathfrak{n}'$, such that $p \mid H_\alpha \not\equiv 0$, $q \mid H_\alpha \not\equiv 0$ and $p_\alpha(ax) \equiv \mu_\alpha(a) p_\alpha(x)$, $q_\alpha(ax) \equiv \nu_\alpha(a) q_\alpha(x)$ on H_α for all a in G, and $\mu_\alpha(a)/\nu_\alpha(a) \equiv \Delta(a)$. Repeating the same construction for all $\alpha \in Z(s)$, we define the p (q) on $\hat{\mathfrak{n}}/N$ by the condition, that its restriction to H_α/N, when lifted to H_α, coincide with p_α (q_α resp.). Let us put $P \equiv p \circ \psi$, $Q \equiv q \circ \psi$. These are Borel functions on $\hat{N}$. and, by virtue of Lemma 2.3, the G-invariant sets [*cf.* (*c*) above] $\{\zeta;\ P(\zeta) = 0\}$ and $\{\zeta;\ Q(\zeta) = 0\}$ are of Plancherel measure zero. Writing $G(\zeta) \equiv P(\zeta)/Q(\zeta)$ we have by our construction $G(a\zeta) \equiv \Delta(a)\, G(\zeta)$ [whenever $G(\zeta)$ is defined]. In this fashion the function $F \equiv |G|^{1/2}$ satisfies all the requirements formulated at the end of (*b*). Q. E. D.

LEMMA 7.4. — *We can make a choice of* M′ *in Lemma* 7.3 *with the following property. There is a distribution* ρ *on* N *of a degree not exceeding* B (m) (m = dim $\mathfrak{n}$; *cf.* Proposition 4.1 and Section 6), *such that* $\rho \times f$ *lies in the domain of* M′ *for all* f *in* C_c^∞ (G) *and* $\rho \times f \equiv 0$ *implies* $f \equiv 0$. *If* N *is simply connected, the support of* ρ *is the unity.*

Proof. — We continue to use the notations of the proof of Lemma 7.3. By virtue of Proposition 4.1 we can assume, that in (*d*) above deg (p_α), deg (q_α) $\leqq$ B (m) [$\alpha \in Z(s)$]. By Lemma 6.2 we can also suppose, that there are elements ρ, $\sigma \in \mathcal{O}_0$ such that $P(\zeta) \equiv \hat{\sigma}(\zeta)$, $Q(\zeta) \equiv \hat{\rho}(\zeta)$ $(\zeta \in \hat{N})$, and deg (ρ), deg (σ) $\leqq$ B (m). We claim, that M′ corresponding to P, Q as just specified, along with ρ, satisfy the conditions of our lemma. Let H_0 be the operator, corresponding to M (P/Q), on L^2 (N) [*cf.* (*b*) in the proof of Lemma 7.3] and let us put $M_0 = H_0 \otimes I$ [*cf.* (*a*), *loc. cit.*]; we have $(M')^2 = M_0$. If T is some operator, we shall denote its domain by D (T). Since

$$D(M') \supset D((M')^2) = D(|M_0|) = D(M_0),$$

to prove our lemma, it suffices to show, that $\rho \times f \in D(M_0)$ for all $f \in C_c^\infty$ (G). To this end it is enough to establish, that if $g \in C_c^\infty$ (N) we have $\rho \times g \in D(H_0)$

and $H_0(\rho\times g)=\sigma\times g$. In fact, given $k\in C_c^\infty(G)$ and a fixed $s\in S$, let us put $k_s(n)\equiv k(ns)$ $(n\in N)$. Then, if $h=\rho\times f$ $[f\in C_c^\infty(G)]$, we have $h_s(n)\equiv(\rho\times f_s)(n)$, and thus $h_s\in D(H_0)$ and

$$(H_0 h_s)(n)\equiv(\sigma\times f_s)(n)\equiv(\sigma\times f)(ns) \qquad (n\in N);$$

finally, $(\sigma\times f)(ns)\in L^2(N)\otimes L_\tau^2(S)$, proving our assertion. Since $\hat{\rho}(\zeta)\equiv Q(\zeta)$ and $\hat{\sigma}(\zeta)\equiv P(\zeta)$, by Lemma 6.1 we have $\pi(\zeta)(\rho\times g)\equiv Q(\zeta)\pi(\zeta)(g)$ and $\pi(\zeta)(\sigma\times g)\equiv P(\zeta)\pi(\zeta)(g)$ $\left(\zeta\in\hat{N}\right)$. From this we see, that the element $\{\pi(\zeta)(\rho\times g)\}$ of $\mathfrak{H}$ [*cf.* (*b*), *loc. cit.*] lies indeed in D (M (P/Q)), and that $M(P/Q)\{\pi(\zeta)(\rho\times g)\}=\{\pi(\zeta)(\sigma\times g)\}$ $[g\in C_c^\infty(N)]$ completing the proof of our lemma.

Q. E. D.

LEMMA 7.5. — *With* ρ *as in Lemma* 7.4, *the linear manifold* $\{\rho\times f;\ f\in C_c^\infty(G)\}$ *is dense in* $L^2(G)$.

Proof. — One verifies easily, that concerning S [*cf.* (*a*), proof of Lemma 7.3] we can make the following assumptions : 1° Denoting by S_0 the interior of S, S_0 is a submanifold of G; 2° Let $\mathcal{F}$ be the collection of all functions on S, vanishing outside S_0 and the restriction of which to S_0 belongs to $C_c^\infty(S_0)$. *Then* $\mathcal{F}$ *is dense in* $L^2(S)$, *and for any* $g\in\mathcal{F}$ *and* $h\in C_c^\infty(N)$ *the function f on* G *defined by* $f(ns)\equiv h(n)\,g(s)$ $(n\in N,\ s\in S)$ *belongs to* $C_c^\infty(G)$. This being so, since $L^2(G)=L^2(N)\otimes L_\tau^2(S)$, it suffices to show, that $\mathcal{G}=\{\rho\times f;\ f\in C_c^\infty(N)\}$ is dense in $L^2(N)$. Let $\mathcal{R}_N$ be the right regular representation of N. If the said assertion is false, there is a $h\in L^2(N)$, $h\neq 0$, such that $(\mathcal{R}_N(n)\,h,g)\equiv 0$ for all n in N and g in $\mathcal{G}$. From this we conclude, that $(C\,h,g)=0$, for all $C\in(\mathbf{L}(N))^\natural$ and $g\in\mathcal{G}$. Let $\{A(\zeta)\}$ be the element of $\mathfrak{H}$ corresponding to h. Since the unitary correspondence between $L^2(N)$ and $\mathfrak{H}$ maps $(\mathbf{L}(N))^\natural$ onto the ring of multiplications by all bounded measurable functions on $\hat{N}$ [*cf.* (*b*), *loc. cit.*], bearing in mind, that $\hat{\rho}(\zeta)\equiv Q(\zeta)$, we conclude, that for any $f\in C_c^\infty(N)$ and a bounded measurable function $a(\zeta)$ we have

$$\int_N a(\zeta)\,Q(\zeta)\,\mathrm{Tr}\,(\pi(\zeta)(f)\,[A(\zeta)]^*)\,d\mu(\zeta)=0.$$

But since $Q(\zeta)\neq 0$ almost everywhere [*cf.* (*d*), *loc cit.*] in view of the arbitrariness of $a(\zeta)\in L_\mu^\infty(N)$ we conclude, that $\mathrm{Tr}\,(\pi(\zeta)(f)\,[A(\zeta)]^*)=0$ almost everywhere, and thus h is orthogonal to $C_c^\infty(N)$, contradicting $h\neq 0$.

Q. E. D.

We shall say, that the unitary representation U of G is of trace class, if there is a trace on $\mathbf{R}$ (U), such that the set of all generalized Hilbert-Schmidt operators in U ($C^\star$ (G)) generate $\mathbf{R}$ (U) (*cf.* the start of this section, and [12], 6.6.7, p. 126 and 17.1.4, p. 305 resp.).

THEOREM 4. — *Let* G *be a connected solvable Lie group. Then its right regular representation is of trace class. More precisely, denoting by* N *the largest connected, nilpotent, invariant subgroup of* G, *there is a distribution* τ *on* N, *the degree of which does not exceed a constant depending on the dimension of* N *only, and a trace on* $\mathbf{R}$ (G), *such that* $\mathcal{R}(f \times \tau)$ $[f \in C_c^\infty(G)]$ *is a generalized Hilbert-Schmidt operator, and the collection of all operators of this form generates* $\mathbf{R}$ (G). *If* N *is simply connected, the support of* τ *is the unity.*

Proof.— Let M′ and ρ be as in Lemma 7.4. We denote by E the set of all generalized Hilbert-Schmidt operators in the sense of the trace determined by M′ (*cf* Lemma 7.1). Putting $\mathfrak{G} = \{ g;\ g = \rho \times f,\ f \in C_c^\infty(G) \}$ we have $\mathfrak{G} \subset C_c^\infty(G)$ and by Lemma 7.4 : $\mathfrak{G} \subset D(M')$, and hence $\{ V_g;\ g \in \mathfrak{G} \} \subset E$. Let K be the smallest weakly closed, $\star$-invariant subalgebra, containing the left hand side, of $\mathbf{R}$ (G). If $K \neq \mathbf{R}(G)$, there is a nonzero central projection P, such that $PA = 0$ for all A in K. Let $h \in L^2(G)$ be such, that $P\,h = h$. We have

$$(V_g^* h)(x) \equiv \int_G h(xy)\,\overline{g(y)}\,dy \equiv 0 \qquad (g \in \mathfrak{G}),$$

and hence, setting $x = e$, we see, that h is orthogonal to $\mathfrak{G}$. By virtue of Lemma 7.5 this implies, that $h = 0$ if $P\,h = h$, contradicting $P \neq 0$. In this fashion we conclude, that $\{ V_g;\ g \in \mathfrak{G} \}$ generates $\mathbf{R}$ (G). Let us define, for $f \in C(G)$, $f^+(x)$ by $f(x^{-1})/\Delta(x)$ $(x \in G)$; we have $V_f = \mathcal{R}(f^+)$. We denote by τ the distribution on N determined by the condition, that $\tau(h) \equiv \rho(h^+)$ $[h \in C_c^\infty(N),\ h^+(n) \equiv h(n^{-1})]$. Observe that τ, too, is a distribution of bounded degree; more precisely we have $\deg(\rho) = \deg(\tau)$. Also $(\rho \times k)^+ \equiv k^+ \times \tau$ $[k \in C_c^\infty(G)]$, whence we conclude, that if g is some element of $\mathfrak{G}$, g^+ is of the form $f \times \tau$ $[f \in C_c^\infty(G)]$ and conversely. In this fashion τ satisfies all the conditions of Theorem 4.

Q. E. D.

COROLLARY 7.1. — *Let* G *be a connected solvable Lie group. Then its right ring is semi-finite.*

Proof. — This follows at once from Theorem 4 and from the definition of the trace given at the begin of this section.

Q. E. D.

We shall call Plancherel measure of G the class of measures determined by $\mathcal{R}$ on $\hat{G}$ (*cf.* [12], p. 149-150 and 18.7.6, p. 325).

COROLLARY 7.2. — *There is a subset* E, *of Plancherel measure zero, of* $\hat{G}$, *such that any factor representation of* G, *the quasi-equivalence class of which does not belong to* E, *is of trace class. In particular, for any element* a, *different from the unity, of* G *there is a trace class factor representation* V, *such that* $V(a) \neq$ *unit operator.*

Proof. — This is an immediate consequence of 8.8.2, Théorème in [13] (p. 160; *cf.* also 18.7.6, p. 325).

Q. E. D.

REMARK 7.1. — Of course, in Theorem 4 and Corollary 4.1 the right regular representation can be replaced by the left regular representation.

8. Let $\tilde{G} = \exp(\tilde{\mathfrak{g}})$ be again as at the begin of Section 2, Chapter II. We recall, that if $\mathfrak{n}_1$ is the gratest nilpotent ideal of $\tilde{\mathfrak{g}}$, there is a subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{g}}$ such that $\tilde{\mathfrak{g}}$ is the direct sum of the underlying space of $\mathfrak{h}$ and $\mathfrak{n}_1$ resp., and, for any h in $\mathfrak{h}$, ad (h) is semi-simple (*cf.* [32], p. 439). Putting $H = \exp(\mathfrak{h})$ and $N_1 = \exp(\mathfrak{n}_1)$, we have $G = HN_1$. *In the following, whenever speaking of a Zariski open subset of a vector space, we shall assume, that it is non empty.*

LEMMA 8.1. — *There is a Zariski open subset* $\mathfrak{O}$ *of* $\mathfrak{g}'$, *and a closed subgroup* K *of* $\tilde{G}$, *such that for any* $x \in \mathfrak{O}$ *we have* $\tilde{G}_x N_1 = K$.

Proof. — Let us write $V = \mathfrak{g}'_{\mathbf{C}}$. We choose a Jordan-Hölder sequence $V = V_0 \supset V_1 \supset \ldots \supset V_M = (0)$ for the action of $\tilde{G}$ on V. If $v_j \in V_{j-1} - V_j$, we have $a v_j \equiv \varphi_j(a)\, v_j\, (V_j)$; since, if h is in H, the corresponding operator on V is semi-simple, we can assume, that $h v_j = \varphi_j(h)\, v_j$ for all $h \in H$ $(1 \leq j \leq M)$. Let $\{ l_j;\ 1 \leq j \leq J \}$ be a basis in $(\mathfrak{n}_1)_{\mathbf{C}}$. If $\{ v'_j;\ 1 \leq j \leq M \}$ are elements of a basis in V′, which is the dual of $\{ v_j \}$, we put $a_{ij}(x) = (l_j x, v'_i)$ and write $A_j(x)$ for the $j \times J$ matrix $\{ a_{ik}(x);\ i \leq j,\ 1 \leq k \leq J \}$. Let us put $m_0 = 0$ and $m_j = \sup_x \operatorname{rank}(A_j(x))$ $(1 \leq j \leq M)$. We denote by e the subset of $\{ 1, 2, \ldots, M \}$, such that $m_j > m_{j-1}$ if and only if j belongs to e. Similarly as in the proof of Proposition 1.1, Chapter II [G *loc. cit.* replaced by $(N_1)_{\mathbf{C}}$] let us form $\mathfrak{O}_e = \{ x;\ f(x) = e \}$; one sees at once, that $\mathfrak{O}_e$ is Zariski open in V. We write E for the complement of e in $\{ 1, 2, \ldots, M \}$ and infer from *loc. cit.*, that the functions $\lambda_j(x) \equiv P_j(0; x)$ $(j \in E,\ x \in \mathfrak{O}_e)$ are the restrictions to $\mathfrak{O}_e$ of some rational functions on V, all defined on $\mathfrak{O}_e$. Also, by Lemma 1.3, Chapter II we have $\lambda_j(ax) \equiv \varphi_j(a)\lambda_j(x)$ $[a \in \mathrm{H}(\mathrm{N}_1)_{\mathbf{C}},\ x \in \mathfrak{O}_e,\ j \in \mathrm{E}]$. Finally, if $x, x' \in \mathfrak{O}_e$ we have $\lambda_j(x) = \lambda_j(x')$ if and only if $x' \in (\mathrm{N}_1)_{\mathbf{C}}\, x$. Let us denote by $\mathfrak{O}'_e$ the subset of $\mathfrak{O}_e$, where none of the functions $\{\lambda_j(x);\ j \in \mathrm{E}\}$ vanishes. We put $\mathfrak{O} = \mathfrak{O}'_e \cap \mathfrak{g}'$; $\mathfrak{O}$ is Zariski open in $\mathfrak{g}'$. Let us form the subgroup $\mathrm{K} = \bigcap_{j \in \mathrm{E}} \ker(\varphi_j)$ of $\tilde{\mathrm{G}}$. We show now, that $\mathfrak{O}$ and K, as just defined, satisfy the conditions of our lemma. In fact, if $x \in \mathfrak{O}$, $a \in \tilde{\mathrm{G}}_x$ then

$$\lambda_j(ax) = \lambda_j(x) = \varphi_j(a)\lambda_j(x),$$

and thus, since $\lambda_j(x) \neq 0$, $\varphi_j(a) = 1$ $(j \in \mathrm{E})$ and hence $\tilde{\mathrm{G}}_x \subset \mathrm{K}$, proving, that $x \in \mathfrak{O}$ implies $\tilde{\mathrm{G}}_x \mathrm{N}_1 \subset \mathrm{K}$. To show the opposite inclusion, let k be some element of K and $x \in \mathfrak{O}$. We have

$$\lambda_j(kx) = \varphi_j(k)\lambda_j(x) = \lambda_j(x),$$

and therefore $kx \in (\mathrm{N}_1)_{\mathbf{C}}\, x \cap \mathfrak{g}' = \mathrm{N}_1 x$ (*cf.* Lemma 1.2, Chapter II), and $k \in \tilde{\mathrm{G}}_x \mathrm{N}_1$, completing the proof of Lemma 8.1.

Q. E. D.

The following lemma is a weaker version, for the action of an arbitrary unipotent group on a finite dimensional vector space, of Proposition 4.2, Chapter II.

LEMMA 8.2. — *Suppose, that the connected and simply connected nilpotent group* G *acts, via a unipotent representation, on a finite dimensional real vector space* V. *There is a Zariski open subset* $\mathfrak{O}$ *of* V, *a polynomial function* P *on* V, *which never vanishes on* $\mathfrak{O}$, *a map* l *from* $\mathbf{R}^d \times \mathfrak{O}$ *into* $\mathfrak{g}$ $[\mathrm{G} = \exp(\mathfrak{g})]$ *and a map* R *from* $\mathrm{G}\,\mathfrak{O} \times \mathfrak{O}$ *into* $\mathbf{R}^d$ *with the following properties :* 1° $\mathrm{P}(x)\, l(\mathrm{T}, x)$ $(\mathrm{T} \in \mathbf{R}^d,\ x \in \mathfrak{O})$ *is the restriction to* $\mathbf{R}^d \times \mathfrak{O}$ *of a polynomial map from* $\mathbf{R}^d \times \mathrm{V}$ *into* $\mathfrak{g}$, *and for each* x *in* $\mathfrak{O}$, *the map* $\mathbf{R}^d \ni \mathrm{T} \mapsto \exp[l(\mathrm{T}, x)]\, x$ *is a bijection between* $\mathbf{R}^d$ *and* $\mathrm{G}\,x$; 2° $\mathrm{P}(x)\,\mathrm{R}(y, x)$ *is the restriction to* $\mathrm{G}\,\mathfrak{O} \times \mathfrak{O}$ *of a polynomial map from* $\mathrm{V} \times \mathrm{V}$ *into* $\mathbf{R}^d$, *and for any* x *and* y *in* $\mathfrak{O}$ *and* $\mathrm{G}\,x$ *resp. we have* $y = \exp[l(\mathrm{R}(y, x), x)]\, x$.

Proof. — *a*. Let $\mathrm{V} = \mathrm{V}_0 \supset \mathrm{V}_1 \supset \ldots \supset \mathrm{V}_{\mathrm{M}} = (0)$ be a Jordan-Hölder sequence for the action of G on V, $v_j \in \mathrm{V}_{j-1} - \mathrm{V}_j$, $(v_i, v'_j) = \delta_{ij}$, and $\{l_j;\ 1 \leqq j \leqq m\}$ a basis in $\mathfrak{g}$. Let us form the matrices $\mathrm{A}_j(x)$ $(1 \leqq j \leqq \mathrm{M})$ and the subset $e = \{0 < j_1 < j_2 < \ldots < j_d \leqq \mathrm{M}\}$ of $\{1, 2, \ldots, \mathrm{M}\}$ as at the start of the proof of Lemma 8.1 above. For each $k = 1, 2, \ldots, d$, we denote by $\mu_k(x)$ a $k \times k$ submatrix of $\mathrm{A}_{j_k}(x)$, such that

$$\varphi_k(x) \equiv \det(\mu_k(x))$$

does not vanish identically on V. Reasoning as in (b) of the proof of Proposition 1.1, Chapter II, we show, that if $\mathfrak{O}$ is the Zariski open subset of V, formed of all points of V, where none of the polynomials $\{\varphi_k(x)\}$ vanishes, there are maps $\{l_k(x);\ k = 1, 2, \dots, d\}$ from $\mathfrak{O}$ into $\mathfrak{g}$, such that $\varphi_k(x)\, l_k(x)$ is the restriction to $\mathfrak{O}$ of a polynomial map from V into $\mathfrak{g}$, and that $l_k(x)\, x \equiv v_{j_k}(\mathrm{V}_{j_k})$ $(x \in \mathfrak{O})$. Let us form the function $l(\mathrm{T}, x)$, from $\mathbf{R}^d \times \mathfrak{O}$ into $\mathfrak{g}$, by the condition that we have, if $\mathrm{T} = (t_1, t_2, \dots, t_d) \in \mathbf{R}^d$ and $x \in \mathfrak{O}$:

$$\exp[l(\mathrm{T}, x)] = \exp[t_1 l_1(x)] \exp[t_2 l_2(x)] \dots \exp[t_d l_d(x)].$$

Using the reasonings as *loc. cit.* we conclude, that the map $l(\mathrm{T}, x)$ so defined satisfies the conditions of our lemma, provided for P we take a sufficiently high power of the product $\varphi_1(x)\, \varphi_2(x) \dots \varphi_d(x)$.

b. Let us put $(x \in \mathfrak{O})$:

$$\exp[l(\mathrm{T}, x)]\, x = \sum_{j=1}^{\mathrm{M}} \mathrm{Q}_j(\mathrm{T}; x)\, v_j.$$

We know [*cf.* (d), *loc. cit.*], that $\mathrm{Q}_{j_k}(\mathrm{T}; x)$ is of the form

$$t_k + \mathrm{R}_k(t_1, t_2, \dots, t_{k-1}; x),$$

and thus the set of equations $z_k = \mathrm{Q}_{j_k}(\mathrm{T}; x)$ $(1 \leq k \leq d)$ implies, that

$$t_k = z_k + \psi_k(z_1, z_2, \dots, z_{k-1}; x);$$

obviously we can assume, that $\mathrm{P}\,\psi_k$ $(1 \leq k \leq d)$ is a polynomial on $\mathbf{R}^d \times \mathrm{V}$ for each k. Hence finally, given $y = \sum_{j=1}^{\mathrm{M}} y_j v_j \in \mathrm{G}\, x$ $(x \in \mathfrak{O})$, it suffices to define

$$\mathrm{R}(y, x) = (\overline{\mathrm{R}}_1(y, x), \overline{\mathrm{R}}_2(y, x), \dots, \overline{\mathrm{R}}_d(y, x))$$

where

$$\overline{\mathrm{R}}_k(y, x) = y_{j_k} + \psi_k(y_{j_1}, \dots, y_{j_{k-1}}; x).$$

Q. E. D.

In the following G, $\tilde{\mathrm{G}}$, H and N_1 will have the same meaning as in Lemma 8.1. Also $\mathrm{G} = \exp(\mathfrak{g})$, $\mathrm{H} = \exp(\mathfrak{h})$, $\mathrm{N}_1 = \exp(\mathfrak{n}_1)$ and

$$\tilde{\mathrm{G}} = \exp(\tilde{\mathfrak{g}}) = \mathrm{HN}_1.$$

LEMMA 8.3. — *There is a Zariski open subset* $\mathfrak{O}$ *in* $\mathfrak{g}'$, *a polynomial function* P, *never vanishing on* $\mathfrak{O}$, *and a sequence of maps* $\{\tilde{g}_j(x);\ j = 1, 2, \dots\}$ *from* $\mathfrak{O}$ *into* $\tilde{\mathrm{G}}$ *with the following properties :* 1° *For each* x *in* $\mathfrak{O}$, *the* *sequence* $\{\tilde{g}_j(x)\}$ *forms a complete residue system in* $\tilde{G}_x$ *modulo* $(\tilde{G}_x)_0$; 2^o *For each fixed* j, $\tilde{g}_j(x)$ *is of the form* $k \exp[l(x)]$, *where* $P(x)\, l(x)$ *is the restriction to* $\mathfrak{O}$ *of a polynomial map from* $\mathfrak{g}'$ *into* $\mathfrak{n}_1$.

Proof. — *a.* Let $\mathfrak{O}_1$ and K be such as $\mathfrak{O}$ and K resp. are in Lemma 8.1. By taking in Lemma 8.2, $V = \mathfrak{g}'$ and $G = N_1$, we denote by $\mathfrak{O}_2$ the resulting Zariski open set in $\mathfrak{g}'$. Let us put $\mathfrak{O} = \mathfrak{O}_1 \cap \mathfrak{O}_2$, and let k be an arbitrary fixed element in K. We have, for any x in $\mathfrak{O}$, $k^{-1} x \in N_1 x$ and hence we can form, using the notations of Lemma 8.2, the map $l(x) = l(R(k^{-1} x, x), x)$ from $\mathfrak{O}$ into $\mathfrak{n}_1$. Let us put $\tilde{g}(x) = k \exp[l(x)]$. We have $\tilde{g}(x) \in \tilde{G}_x$ $(x \in \mathfrak{O})$, and replacing, if necessary, the polynomial function P of *loc. cit.* by a sufficiently high power of itself we can assume, that $P(x)\, l(x)$ is the restriction to $\mathfrak{O}$ of a polynomial map from $\mathfrak{g}'$ into $\mathfrak{n}_1$.

b. Let $\{k_j ; j = 1, 2, \ldots\}$ be a complete residue system in K modulo K_0. We denote by $\tilde{g}_j(x)$ the map, from $\mathfrak{O}$ into $\tilde{G}$, corresponding to k_j by virtue of the construction of (*a*) above. To complete the proof of our lemma, it suffices to show, that for each x in $\mathfrak{O}$, the sequence $\{\tilde{g}_j(x)\}$ is a complete residue system in $\tilde{G}_x$ modulo $(\tilde{G}_x)_0$. This, however, is implied by the relations $K_0 = (\tilde{G}_x)_0 N_1$ and $G_x \cap K_0 = (\tilde{G}_x)_0$. Q. E. D.

By a Zariski G_δ set in $\mathfrak{g}'$ we shall mean a non empty subset, which is intersection of a countable sequence of Zariski open sets.

LEMMA 8.4. — *There is a Zariski* G_δ *set* $\mathfrak{O} \subset \mathfrak{g}'$, *and a sequence of maps* $\{g_j(x) ; j = 1, 2, \ldots\}$ *from* $\mathfrak{O}$ *into* G *with the following properties :* 1^o *For each fixed* x *in* $\mathfrak{O}$, *the sequence* $\{g_j(x)\}$ *is a complete residue system in* G_x *modulo* $(G_x)_0$; 2^o *There is a polynomial function* P *on* $\mathfrak{g}'$, *which never vanishes on* $\mathfrak{O}$, *such that for each* j, $g_j(x)$ *is of the form* $k \exp[l_1(x)] \exp[l_2(x)]$, *where* $P(x)\, l_k(x)$ $(k = 1, 2)$ *are the restrictions to* $\mathfrak{O}$ *of polynomial maps from* $\mathfrak{g}'$ *into* $\tilde{\mathfrak{g}}$.

Proof. — In the following $\mathfrak{O}_1, \mathfrak{O}_2, \ldots$ will denote Zariski open subsets, specified by the context, in $\mathfrak{g}'$.

a. Let us show first, that there is an $\mathfrak{O}_1$ and a system of maps $\{v_j(x) ; 1 \leq j \leq s\}$ from $\mathfrak{O}_1$ into $\tilde{\mathfrak{g}}$, such that for each $x \in \mathfrak{O}_1$ its members form a basis in $\tilde{\mathfrak{g}}_x \bmod (\mathfrak{g}_x)$, and that there is a polynomial function P′ on $\mathfrak{g}'$, which never vanishes on $\mathfrak{O}_1$, such that the maps $\{P'(x)\, v_j(x)\}$ are the restrictions to $\mathfrak{O}_1$ of polynomial maps from $\mathfrak{g}'$ into $\tilde{\mathfrak{g}}$. Let $\{l_j ; 1 \leq j \leq M\}$ and $\{\bar{l}_k ; 1 \leq k \leq m\}$ be a basis in $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ resp. We write

$$a_{kj}(x) = (\bar{l}_k, l_j x), \qquad M(x) = \{a_{kj}(x) ; 1 \leq k \leq m,\ 1 \leq j \leq M\}$$

and observe, that $y = \sum_{j=1}^{M} y_j l_j$ belongs to $\tilde{\mathfrak{g}}_x$ if and only if we have

$$\sum_{j=1}^{M} a_{kj}(x) y_j = 0 \qquad (1 \leqq k \leqq m).$$

Let us put $r = \sup_x \operatorname{rank}(M(x))$. We denote by $\mu(x)$ an $r \times r$ submatrix of $M(x)$ such that $\varphi(x) \equiv \det(\mu(x)) \not\equiv 0$; obviously we can assume, that $\mu(x) = \{ a_{kj}(x);\ 1 \leqq k,\ j \leqq r \}$. Let $\mathfrak{O}_2$ be the set $\{ x;\ \varphi(x) \neq 0 \}$. There are functions $\{ y_j^{(k)}(x); 1 \leqq j \leqq r, 1 \leqq k \leqq M - r \}$ on $\mathfrak{O}_2$, such that

$$\sum_{j=1}^{r} a_{lj}(x) y_j^{(k)}(x) = a_{l,r+k}(x) \qquad (x \in \mathfrak{O}_2;\ l = 1, 2, \ldots, r)$$

and such that $\varphi(x) y_j^{(k)}(x)$ is the restriction of a polynomial to $\mathfrak{O}_2$. Putting $w_k(x) = \left(\sum_{j=1}^{r} y_j^{(k)}(x) l_j \right) - l_{r+k}$, we conclude, that if $x \in \mathfrak{O}_2$ the system $\{ w_k(x);\ 1 \leqq k \leqq M - r \}$ is a basis in $\tilde{\mathfrak{g}}_x$. Let us write $\mathfrak{a} = \tilde{\mathfrak{g}}/\mathfrak{g}$, and let us denote by Ψ the canonical homomorphism from $\tilde{\mathfrak{g}}$ onto $\mathfrak{a}$. We fix a point x_0 in $\mathfrak{O}_2$, such that the dimension of $\Psi(\tilde{\mathfrak{g}}_{x_0})$ be maximal, and denote by $\{ v_j(x);\ 1 \leqq j \leqq s \}$ a subsystem of $\{ w_k(x) \}$, such that $\{ \Psi(v_j(x_0)) \}$ is a basis in $\Psi(\tilde{\mathfrak{g}}_{x_0})$. Let $\mathfrak{O}_1 \subseteqq \mathfrak{O}_2$ be such, that the last relation holds true for all x in $\mathfrak{O}_1$. With this choice of $\mathfrak{O}_1$, $\{ v_j(x); 1 \leqq j \leqq s \}$ satisfies all requirements, provided we set $P' \equiv \varphi$.

b. One sees easily, that there is an $\mathfrak{O}_3 \subseteqq \mathfrak{O}_1$, and a system of constant vectors $\{ w_j;\ 1 \leqq j \leqq u \}$ in $\mathfrak{a}$, such that for any x in $\mathfrak{O}_3$ the set $\{ \Psi(v_j(x)),\ w_k;\ 1 \leqq j \leqq s,\ 1 \leqq k \leqq u \}$ forms a basis in $\mathfrak{a}$.

c. Let us put $\mathfrak{A} = \tilde{G}/G = \exp(\mathfrak{a})$; we denote by F the canonical homomorphism from $\tilde{G}$ onto $\mathfrak{A}$. By virtue of Lemma 8.3 there is an $\mathfrak{O}_4$ and a polynomial P'' not vanishing on $\mathfrak{O}_4$ such that, putting

$$f_j(x) = \log[F(\tilde{g}_j(x))]$$

(cf. *loc. cit.*), the maps $P''(x) f_j(x)$ from $\mathfrak{O}_4$ into $\mathfrak{a}$ are the restrictions of polynomial maps. We set $\mathfrak{O}_5 = \mathfrak{O}_3 \cap \mathfrak{O}_4$, and write for some x in $\mathfrak{O}_5$:

$$f_j(x) = \sum_{k=1}^{s} a_k^{(j)}(x) \Psi(v_k(x)) + \sum_{l=1}^{u} b_l^{(j)}(x) w_l.$$

It is clear, that there is a polynomial P_1 on $\mathfrak{g}'$, such that the functions $\{ P_1(x) a_k^{(j)}(x),\ P_1(x) b_l^{(j)}(x) \}$ are restrictions of polynomials to $\mathfrak{O}_5$.

Let us observe, that there exists a g in G_x ($x \in \mathfrak{O}_5$) such that $\tilde{g}_j(x) \equiv g$ $((G_x)_0)$ if and only if we have $b_l^{(j)}(x) = 0$ $(1 \leq l \leq u)$. We denote by J the set of all those positive integers j, for which $b_l^{(j)}(x) \equiv 0$ $(1 \leq l \leq u)$. Let J′ be the (possibly empty) complement of J. For $j \in J'$ we write

$$F_j = \left\{ x;\ x \in \mathfrak{O}_5,\ \sum_{l=1}^{u} (b_l^{(j)}(x))^2 = 0 \right\};$$

F_j is the intersection of $\mathfrak{O}_5$ with a Zariski closed set, which is different from $\mathfrak{O}_5$, and hence $\mathfrak{O} = \mathfrak{O}_5 - (\bigcup_{j \in J'} F_j)$ is a Zariski G_δ set. Let us denote by $\{ g_j(x);\ j = 1, 2, \ldots \}$ the sequence

$$\left\{ \tilde{g}_j(x) \exp\left(-\left(\sum_{k=1}^{s} a_k^{(j)}(x)\, v_k(x) \right) \right);\ j \in J \right\}$$

arranged in some order ($x \in \mathfrak{O}_5$). It is clear from our construction, that $g_j(x) \in G_x$ for all $x \in \mathfrak{O}$ (as above). Let us show, that $\{ g_j(x) \}$ is a complete residue system in G_x mod $((G_x)_0)$. To this end, however, it suffices to point out, that if $g_1, g_2 \in G_x$ are such, that $g_1 \equiv g_2$ $((\tilde{G}_x)_0)$, then also $g_1 \equiv g_2$ $((G_x)_0)$, since $G_x \cap (\tilde{G}_x)_0 = G \cap (\tilde{G}_x)_0 = (G_x)_0$. In this fashion, to complete the proof of Lemma 8.4 it is enough to show, that for each j, $g_j(x)$ is of the form indicated in its statement. But this is evident from our construction; for P *loc. cit.* we can take $P_1 . P' . P''$.

Q. E. D.

Remark 8.1. — Observe, that if $P(x) \neq 0$, $g_j(x)$ is defined and belongs to G_x for all j.

Given some element x of $\mathfrak{g}'$, we define the character χ_x of $(G_x)_0$, as in I.4 (*c*), by the condition, that $\chi_x[\exp(l)] = \exp[i(l, x)]$ $(l \in \mathfrak{g}_x)$. For a, b in G, we shall put $[a, b] = aba^{-1}.b^{-1}$. Bearing in mind, that $[G_x, G_x] \subset G_x \cap L \subset (G_x)_0$ $[L = [G, G]]$, for any pair of elements a, b in G_x we can form the expression $\varphi_x(a, b) = \chi_x([a, b])$.

Lemma 8.5. — *With the previous notations we have :*

1° $\varphi_x(aa', b) = \varphi_x(a, b)\, \varphi_x(a', b)$;

2° $\varphi_x(a, b) = \overline{\varphi_x(b, a)}$;

3° *a belongs to* $\overline{G}_x$ [*cf.* I.4 (*c*)] *if and only if* $\varphi_x(a, b) \equiv 1$ *for all* $b \in G_x$ $(a, a', b \in G_x)$.

Proof. — Ad (1) we have $[aa', b] = a[a', b]a^{-1}.[a, b]$, and hence the desired conclusion follows from the observation, that if $a \in G_x$ and

$d \in (G_x)_0$ we have $\chi_x(ada^{-1}) = \chi_x(d)$. Ad (2) This is implied at once by $([a, b])^{-1} = [b, a]$. Ad (3) This is immediat from the definition of $\overline{G}_x$.

Q. E. D.

In the following by μ we shall mean a positive translation invariant measure on $\mathfrak{g}'$.

Proposition 8.1. — *Denoting by $r(x)$ the index of $\overline{G}_x$ in G_x $(x \in \mathfrak{g}')$ let us form the sets $E_0 = \{x; r(x) = 0\}$, $E_f = \{x; 0 < r(x) < +\infty\}$ and $E_\infty = \{x; r(x) = +\infty\}$. Then E_f is of Lebesgue measure zero, and so is one of the sets E_0, E_∞.*

Proof. — *a.* Let the polynomial function P be as in Lemma 8.4, and let us form the set $\mathfrak{O}_0 = \{x; P(x) \neq 0\}$. We fix a pair (i, j) of positive integers and define for $x \in \mathfrak{O}_0$:

$$F_{ij}(x) = (\log([g_i(x), g_j(x)]), x).$$

We observe, that by virtue of the form, described in Lemma 8.4, of the maps $\{g_j(x); j = 1, 2, \ldots\}$, there is an entire function H on $\mathbf{C}^N$ and rational functions $\{r_j(x); 1 \leq j \leq N\}$ on $\mathfrak{g}'$, such that $P(x) r_j(x)$ is a polynomial, and for $x \in \mathfrak{O}_0$:

$$F_{ij}(x) = H(r_1(x), \ldots, r_N(x)).$$

From this we conclude, that if $F_{ij}(x) \equiv c$ (c = some constant) on a connected component of $\mathfrak{O}_0$, then we have $F_{ij}(x) \equiv c$ everywhere on $\mathfrak{O}_0$. Furthermore it is easy to see, that in the latter case the value of the constant c is zero. In fact, let x be some element of $\mathfrak{O}_0$, and δ a positive number, such that for $|t - 1| < \delta$ we have $tx \in \mathfrak{O}_0$. We have $g_i(tx) \equiv g_i(x)$ and $g_j(tx) \equiv g_j(x) \bmod (G_x)_0$ (*cf.* Remark 8.1). Since $(G_x)_0 \subset \overline{G}_x$ from this we conclude, that

$$\varphi_{tx}(g_i(tx), g_j(tx)) \equiv \varphi_{tx}(g_i(x), g_j(x)) \equiv \exp[itc] \equiv \exp[ic]$$

for $|t - 1| < \delta$, implying $c = 0$.

b. Let $\mathfrak{O}$ be as in Lemma 8.4; we have $\mathfrak{O} \subset \mathfrak{O}_0$. Let us assume now, that there is a pair (i, j) such that $F_{ij} | \mathfrak{O}_0 \not\equiv 0$. Then, by virtue of what we saw in (*a*) above, the set $E_r = \{x; x \in \mathfrak{O}_0, F_{ij}(x) = 2\pi r\}$ (r = real) is of Lebesgue measure zero. We observe now that, in consequence of Lemma 8.5, if x belongs to $\mathfrak{O} \cap (E_0 \cup E_f)$, there is a positive integer M such that $(\varphi_x(a, b))^M \equiv 1$ $(a, b \in G_x)$. Hence, with an appropriately chosen rational number r we have $x \in E_r$. Thus

$$\mathfrak{O} \cap (E_0 \cup E_f) \subset \bigcup_{r \in \mathbf{Q}} E_r$$

implying, that $\mu(\mathfrak{O} \cap (E_0 \cup E_f)) = 0$ and in this fashion, since $\mathfrak{O}$ is a Zariski G_δ set, $\mu(E_0 \cup E_f) = 0$. By virtue of (a), the only alternative to the hypothesis just made is that $F_{ij} \mid \mathfrak{O}_0 \equiv 0$ for all (i, j). In this case, however, we have $\mathfrak{O} \subset E_0$ and thus $\mu(E_f \cup E_\infty) = 0$, completing the proof of Proposition 8.1.

Q. E. D.

PROPOSITION 8.2. — *Let us denote by* E_c *the set of all those points in* $\mathfrak{g}'$, *for which the orbit* $G x$ *is locally closed. Then either* E_c *or its complement are of Lebesgue measure zero.*

Proof. — In the following we shall write $o(x) = G x$. Also, $\mathfrak{O}_1$, $\mathfrak{O}_2$, etc. will stand for Zariski open subsets, specified by the context, of $\mathfrak{g}'$.

a. We infer from the proof of Proposition 2.1, Chapter II, that $o(x)$ is locally closed if and only if $G\tilde{G}_x$ is closed, or what is the same, $G\tilde{G}_x/L \subset \tilde{G}/L$ is closed [*cf.* in particular (*e*), *loc. cit.*]. Let us denote by Ψ the canonical homomorphism from $\tilde{G}$ onto $\tilde{A} = \tilde{G}/L = \exp(A)$, and let us put $\Phi = \log \Psi$. Writing $B = \Phi(G)$ and $C_x = \Phi(\tilde{G}_x)$ we can conclude that x belongs to E_c if and only if the subgroup $B + C_x$ of the underlying group of A is closed in A.

b. Below we shall make use of the following elementary statement. Assume, that V is a finite dimensional real vector space, and $\mathcal{L}$ a discrete subgroup of V. *Then* $\mathcal{L}$ *is a free abelian group.* Denote by $\tilde{\mathcal{L}}$ the subspace, generated by the element of $\mathcal{L}$, of V, and let W be some subspace of V. *Then the subgroup* $W + \mathcal{L}$ *is closed if and only if the rank of* $\mathcal{L} \cap W$ *is the same as the dimension of* $\tilde{\mathcal{L}} \cap W$.

c. Reasoning as in (a) of the proof of Lemma 8.4 we show, that there is a system $\{ v_j(x) ; 1 \leq j \leq d \}$ of rational functions, all defined on $\mathfrak{O}_1 \subset \mathfrak{g}'$, with values in A, such that for each x in $\mathfrak{O}_1$, $\{ v_j(x) \}$ is a basis of $(C_x)_0$.

d. Let $\mathfrak{O}_2$ be as in Lemma 8.3, and let us fix a point $x_0 \in \mathfrak{O}_1 \cap \mathfrak{O}_2$. There is a system of integers $0 < j_1 < \ldots < j_r$ such that, putting

$$a_k(x) = \Phi(\tilde{g}_{j_k}(x)) \in C_x = \Phi(\tilde{G}_x) \qquad (1 \leq k \leq r; \text{ cf. } loc. cit.)$$

the set of $r + d$ vectors $\{ a_k(x_0), v_j(x_0) \}$ is a linearly independent one in A and generates C_x. Then there is an $\mathfrak{O}_3 \subset \mathfrak{O}_1 \cap \mathfrak{O}_2$, such that the system $\{a_k(x), v_j(x)\}$ is independent for $x \in \mathfrak{O}_3$, and in addition we can easily show, that putting $\Gamma_x = \mathbf{Z}.a_1(x) + \ldots + \mathbf{Z}.a_r(x)$, we have $C_x = \Gamma_x + (C_x)_0$ $(x \in \mathfrak{O}_3)$. In fact, to this end it suffices to observe, that if j is an arbitrary positive integer, there are rational functions $\{f_k(x), g_l(x)\}$, all defined on $\mathfrak{O}_3$, such that

$$\Phi(\tilde{g}_j(x)) = \sum_k f_k(x)\, a_k(x) + \sum_l g_l(x)\, v_l(x) \qquad (x \in \mathfrak{O}_3).$$

For any $x \in \mathfrak{O}_3$ the numbers $\{f_k(x)\}$ must be rational, and thus $f_k(x) \equiv m_k \in \mathbf{Z}$ on $\mathfrak{O}_3$, proving our statement.

e. We put $D_x = B + (C_x)_0$, and $E_x = D_x \cap \tilde{\Gamma}_x$ [*cf.* (*b*) above]. There is an integer e such that on an $\mathfrak{O}_4 \subset \mathfrak{O}_3$ appropriately chosen we have $\dim(E_x) \equiv e$. We can also assume, that there is a system of rational functions $\{\lambda_j(x);\ 1 \leqq j \leqq J\}$, all defined on $\mathfrak{O}_4$, with values in A′ (= dual of the underlying space of A), such that $v \in E_x$ if and only if we have $\lambda_j(x)(v) = 0$ $(1 \leqq j \leqq J)$.

f. Let us denote by $\mathcal{E}_e$ the set of all independent e tuples in $\mathbf{Z}^r$. If $\varepsilon \in \mathcal{E}_e$,

$$\varepsilon = \{f_1, f_2, \ldots, f_e\} \qquad \text{and} \qquad f_k = (m_1^{(k)}, m_2^{(k)}, \ldots, m_r^{(k)}) \qquad (m_i^{(k)} \in \mathbf{Z})$$

we write

$$E(\varepsilon) = \left\{x;\ x \in \mathfrak{O}_4, \sum_{j,k} (F_{jk}(x))^2 = 0\right\},$$

where we have put

$$F_{jk}(x) = \lambda_j(x)\left(\sum_\alpha m_\alpha^{(k)}\, a_\alpha(x)\right) \qquad (1 \leqq j \leqq J,\ 1 \leqq k \leqq e).$$

Let us observe, that we have only the following two possibilities : 1° $E(\varepsilon) = \mathfrak{O}_4$; 2° The complement of $E(\varepsilon)$ in $\mathfrak{O}_4$ is a Zariski open set in $\mathfrak{g}'$. Moreover we conclude by aid of (*b*) above, that $x \in E_c \cap \mathfrak{O}_4$ if and only if $x \in \bigcup_{\varepsilon \in \mathcal{E}_e} E(\varepsilon)$ (= E, say). If $E = \mathfrak{O}_4$, we have evidently $\mu(\mathfrak{g}' - E_c) = 0$. If, on the other hand, $E \subsetneq \mathfrak{O}_4$, the complement of E in $\mathfrak{O}_4$ is a Zariski G_δ set in $\mathfrak{g}'$, and thus $\mu(E_c) = 0$, completing the proof of Proposition 8.2.

Q. E. D.

9. Given a Lie group M specified by the context, in the following we shall denote by $\mathcal{L}_M$ its left regular representation. Let again G be a connected and simply connected solvable Lie group. Below we shall

assume, that $L = [G, G]$ is non abelian, and leave to the reader the easy modifications of the subsequent reasonings necessary, when L is abelian.

Let $\zeta \to \pi(\zeta)$ $(\zeta \in \hat{L})$ be a Borel measurable field of representations, such that $\pi(\zeta)$ is of the unitary equivalence class of $\zeta \in \hat{L}$ (*cf.* [12], 8.6.2, p. 154). Denoting by μ the Plancherel measure of L, we form the representation $\Pi = \int_{\hat{L}} \oplus \pi(\zeta)\, d\mu(\zeta)$; we have $\mathcal{L}_L = (+\infty)\Pi$ in the sense of unitary equivalence. Let us observe, that the action of $\tilde{G}$ (*cf.* the start of Section 2, Chapter II) on $\hat{L}$ is countably separated. In fact to see this we recall that the canonical map [*cf.* 1 (*d*)] from $\hat{L}$ onto $\mathfrak{d}'/L$ is a Borel isomorphism and equivariant with respect to the actions of $\tilde{G}$ on these spaces [*cf.* the remarks preceding Lemma 2.3 and (*c*) in Lemma 7.3]. Hence it suffices to establish, that $\mathfrak{d}'/\tilde{G}$ is countably separated. But this is implied by Corollary 1.1, Chapter II (applied to $A = \tilde{G} \mid \mathfrak{d}'$, $V = \mathfrak{d}'$; cf. *loc. cit.*) and [17], Theorem 1. From this we can conclude (*cf.* [18], Theorem 1, p. 390 and [23], Lemma 11.5, p. 126) that putting $S = \hat{L}/\tilde{G}$, there is a positive measure τ on S, such that μ is a continuous direct sum of measures concentrated on $\tilde{G}$ orbits, and which are quasi-invariant under $\tilde{G}$. Given a point s in S, we shall denote by $O(s)$ the corresponding $\tilde{G}$ orbit in $\hat{L}$, and by ν_s an appropriately chosen positive measure on $O(s)$, which is quasi-invariant under $\tilde{G}$. Let us put

$$U(s) = \int_{O(s)} \oplus \pi(\zeta)\, d\nu_s(\zeta).$$

Then we have $\Pi = \int_S \oplus U(s)\, d\nu_s(s)$ (*cf.* [24], Theorem 2.11, p. 204). We put $M = \operatorname{ind}_{L \uparrow G} \Pi$, $T(s) = \operatorname{ind}_{L \uparrow G} U(s)$ and observe (*cf.* [23], Theorem 10.1, p. 123), that

$$M = \int_S \oplus T(s)\, d\tau(s). \tag{1}$$

Lemma 9.1. — *The decomposition* (1) *is central.*

Proof. — We observe, that since L is of type I, we have

$$\mathbf{R}(\Pi) = \int_{\hat{L}} \oplus \mathbf{R}(\pi(\zeta))\, d\mu(\zeta) \qquad (cf.\ [12],\ 8.6.4,\ p.\ 155),$$

and thus, in particular, $L^\infty_\mu(\hat{L}) \subset \mathbf{R}(\Pi)$. From here we can complete the proof of our lemma as in Lemma 3.4.3, Chapter III, by substituting in place of T, $\mathfrak{B}(\Omega)$, η, $T(p)$, K and U as *loc. cit.* M, $\hat{L}$, μ, $\operatorname{ind}_{L\uparrow G} \pi(\zeta)$, L and Π resp.

Q. E. D.

Given a von Neumann algebra $\mathbf{N}$, we shall denote by $\mathbf{N}_I$, $\mathbf{N}_{II}$ and $\mathbf{N}_{III}$ its component of type I, II and III resp. (*cf.* [12], A 39, p. 339). We write again $\mathbf{L}(G) = \mathbf{R}(\mathcal{L}_G)$.

LEMMA 9.2. — *Suppose, that* $\mathbf{L}(G)_I \neq 0$. *Then the sets* E_∞ *and* $E_{nc} = \mathfrak{g}' - E_c$ (*cf.* Propositions 8.1 and 8.2 resp.) *are of Lebesgue measure zero in* $\mathfrak{g}'$.

Proof. — By virtue of our assumption, there is a nonzero abelian projection P in $\mathbf{L}(G)$ (*cf.* [13], p. 123). Conversely, if $\mathbf{N}$ is a von Neumann algebra containing a nonzero abelian projection, then $\mathbf{N}_I \neq 0$. Since $\mathcal{L}_G = \operatorname{ind}_{L\uparrow G} \mathcal{L}_L$ and $\mathcal{L}_L = (+\infty)\,\Pi$, we have $\mathcal{L}_G = (+\infty)\,M$, and hence $\mathbf{R}(M)$, too, contains a nonzero abelian projection Q. Let us write, by virtue of decomposition (1), $Q = \int_S \oplus\, Q(s)\, d\tau(s)$. Since Q is abelian if and only if $QABQ = QBAQ$ for all $A, B \in \mathbf{R}(M)$, we conclude, that there is a set $E \subset S$, such that $\tau(E) > 0$, and that, for $s \in E$, $Q(s)$ is a nonzero abelian projection in $\mathbf{R}(T(s))$. Let us denote by E' the complete inverse image of E in $\mathfrak{g}'$. Then E' cannot be a set of Lebesgue measure zero. In fact, in this case the Plancherel measure of the direct image of E' in $\hat{L}$ would be zero (*cf.* [30], p. 278-279) implying $\tau(E) = 0$. Let us suppose now that, for instance, E_∞ is not of measure zero. Then, by virtue of Proposition 8.1, its complement in $\mathfrak{g}'$ would have a measure zero, and therefore $E' \cap E_\infty$ would be nonempty. Let s be a point in E, such that the inverse image $O'(s)$, of $O(s)$, in $\mathfrak{g}'$ meets E_∞. Since E_∞ is invariant under $\tilde{G}$ and $\mathfrak{d}^\perp$ we have then $O'(s) \subset E_\infty$. In this fashion, by virtue of Theorem 3 [with $\mathfrak{D}$ *loc. cit.* replaced by $O(s)$; observe, that $d\nu_s$ is equivalent to a measure, invariant under $\tilde{G}$, on $O(s)$], Remark 3.4.1, Chapter III and Theorem 2, $\mathbf{R}(T(s))$ would admit a representation as a continuous direct sum of factors, none of which is of type I. This, however, contradicts the existence of a nonzero abelian projection in $\mathbf{R}(T(s))$.

Q. E. D.

We recall, that a von Neumann algebra N is uniform of type I_∞ if it is the tensor product of an abelian von Neumann algebra with the full ring of an infinite dimensional unitary space.

LEMMA 9.3. — *Suppose, that the von Neumann algebra* N *is the continuous direct sum of von Neumann algebras, uniform of type* I_∞, *over a standard measure space. Then* N *itself, too, is uniform of type* I_∞.

Proof. — *Cf.* [13], p. 243.

Q. E. D.

THEOREM 5. — *Let* G *be a connected and simply connected solvable Lie group. Then its left ring coincides with its type* I *or type* II *component.*

Proof. — By Corollary 7.1 and Remark 7.1 we have always $\mathbf{L}(G)_{III} = 0$. In this fashion it will suffice to establish, that if $\mathbf{L}(G)_I \neq 0$, then we have $\mathbf{L}(G)_{II} = 0$. By virtue of Lemma 9.2 our assumption implies, that the sets E_∞ and E_{nc} are of Lebesgue measure zero in $\mathfrak{g}'$. Therefore, by Theorem 2, Theorem 3 (*cf.* also Remark 3.4.1, Chapter III) and Lemma 9.3 there is a set $E \subset S$ of τ measure zero, such that, for $s \in S - E$, $\mathbf{R}(T(s))$ is uniform of type I_∞. Hence, by Lemmas 9.1 and 9.3 $\mathbf{L}(G)$, too, is uniform of type I_∞, and thus, in particular, $\mathbf{L}(G)_{II} = 0$.

Q. E. D.

REMARK 9.1. — It is known (*cf.* [26], p. 324), that the vanishing of the type II component of $\mathbf{L}(G)$ does not necessarily imply, that the group in question is of type I. An example, similar to that *loc. cit.* is as follows. Let us consider the six dimensional solvable Lie algebra $\mathfrak{g}$ spanned over the reals by the elements $\{e_j;\ 1 \leq j \leq 6\}$ with the following nonvanishing brackets :

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2, \qquad [e_1, e_4] = \theta e_5,$$
$$[e_1, e_5] = -\theta e_4 \quad (\theta = \text{irrational}), \qquad [e_2, e_3] = e_6, \qquad [e_4, e_5] = e_6.$$

Denoting by G the corresponding connected and simply connected group, we claim, that G has the property indicated above. In fact, let us observe first, that G is not of type I. This follows from the fact, that $\mathfrak{g}^\natural = \mathbf{R}\, e_6$, $G^\natural = \exp(\mathfrak{g}^\natural)$ and $G/G^\natural$ is isomorphic to the group of Mautner (*cf.* Summary, Chapter II). On the other hand, we have here $\mathbf{L}(G)_{II} = 0$. In fact, let us denote by $\mathfrak{n}$ the subalgebra generated by the elements $\{e_j;\ 2 \leq j \leq 6\}$, and let us put $N = \exp(\mathfrak{n})$. It is well known, that any unitary representation of N, the restriction of which to $G^\natural$ is a nontrivial character times the unit operator, is a multiple of an irreducible representation, uniquely determined by this character. From this, however, we can deduce at once, that any factor representation T of G, such that $T \mid G^{\natural}$ is not constant, is of type I. In this fashion any orbit, which is not locally closed is contained in the orthogonal complement of $\mathbf{R}\, e_6$, proving our statement.

Remark 9.2. — The above example makes it possible to show, that Theorem 5 fails, if G is not assumed to be simply connected. In fact, by virtue of the above discussion, to this end it suffices to consider the discrete central subgroup $\Gamma = \exp(\mathbf{Z}\, e_6)$, and form the quotient G/Γ.

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SOME NOTATIONAL CONVENTIONS.

(1) If G is a group, $G^{\natural}$ stands for its center; similarly for Lie algebras and von Neumann algebras.

(2) If W is a unitary representation, **H** (W) denotes the representation space, and **R** (W) the von Neumann algebra generated by the operators of W.

(3) If S is a set of unitary equivalence classes of unitary representations, S_c denotes the corresponding set of concrete representations.

(4) If $\mathfrak{G}$ is a group, G a subgroup of $\mathfrak{G}$ and ρ some representation of G, given a in $\mathfrak{G}$ we denote by $a\rho$ the representation of $a\,G\,a^{-1} \subset \mathfrak{G}$ defined by $(a\rho)(b) = \rho(a^{-1}ba)$ $(b \in a\,G\,a^{-1})$.

(5) If $\mathfrak{g}$ is a Lie algebra, exp ($\mathfrak{g}$) denotes a corresponding connected and simply connected Lie group, unless specified otherwise by the context. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, and $G = \exp(\mathfrak{g})$, $\exp(\mathfrak{h})$ denotes the connected subgroup, determined by $\mathfrak{h}$, of G.

(6) If G is a Lie group with the Lie algebra $\mathfrak{g}$, Ad (a) denotes the value of the adjoint representation of G at $a \in G$, and ad (l) the value assumed by the adjoint representation of $\mathfrak{g}$ at $l \in \mathfrak{g}$. Usually we shall write

$$al = \mathrm{Ad}\,(a)\,l\,(a \in G,\, l \in \mathfrak{g}) \qquad \text{and} \qquad ag = (\mathrm{Ad}\,(a^{-1}))'\,g\,(g \in \mathfrak{g}',\, a \in G).$$

Similarly, if $\mathfrak{n}$ is some ideal of $\mathfrak{g}$, and $f \in \mathfrak{n}'$, af will stand for the action of $(\mathrm{Ad}\,(a^{-1}) \mid \mathfrak{n})'$ on f.

(7) If G is a topological group, G_0 denotes the connected component of the identity of G.

(8) If T is a locally compact space, C (T) denotes the family of all continuous functions of a compact support on T.

(9) If G is a group acting as a group of transformations of the set X onto itself, and p a fixed element of X, we denote by G_p or Stab_p (G) the stable group of p in G.

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