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THETA CHARACTERISTICS OF AN ALGEBRAIC CURVE

BY DAVID MUMFORD.

Let X be a non-singular complete algebraic curve over an algebraically closed ground field k of char $\neq 2$. We are interested here in vector bundles E over X such that there exists a quadratic form (1) :

$$Q: \quad E \to \Omega_X^1$$

which is everywhere non-degenerate, i. e. for all $x \in X$, choosing a differential ω which is non-zero at x, Q induces a k-valued quadratic form in the fibre

$$\frac{Q}{\omega}(x): \quad E(x) \to k$$

which is to be non-degenerate. Our first result is that for such E, $\dim \Gamma(E) \bmod 2$ *is stable under deformations of* X *and* E. If E is a line bundle L, then the existence of Q just means that $L^2 \simeq \Omega_X^1$. The set of such L is called classically the set of *theta-characteristics* S (X) of X (*cf.* Krazer [K]). S (X) is a principal homogeneous space over J_2, the group of line bundles L such that $L^2 \simeq \mathcal{O}_X$. Now on S (X) we have the function

$$e_*(L) = \dim \Gamma(L) \bmod 2 \in \mathbf{Z}/2\mathbf{Z}$$

and on J_2 we have the well-known skew-symmetric bilinear form

$$e_2 : J_2 \times J_2 \to \{\pm 1\}$$

(*cf.* Weil [W], Lang ([L 1], p. 173 and p. 189), or Mumford ([M 1], p. 183)).

(1) By this we mean the composition of (i) $E \to E \otimes E$, $s \mapsto s \otimes s$, and (ii) a symmetric linear homomorphism $B : E \otimes E \to \Omega_X^1$.

Our second theorem is that e_* is a quadratic function whose associated bilinear form is e_2 (considered additively); more precisely, this means

$$(\star) \qquad \begin{cases} e_*(L) + e_*(L\otimes\alpha) + e_*(L\otimes\beta) + e_*(L\otimes\alpha\otimes\beta) = \ln e_2(\alpha, \beta), \\ \qquad L \in S(X), \qquad \alpha, \beta \in J_2, \end{cases}$$

where

$$\ln(-1) = 1, \qquad \ln(+1) = 0.$$

This is proved by applying the first result to bundles $L\otimes\mathfrak{A}$, where $L\in S(X)$ and $\mathfrak{A}$ is a quaternionic Azumaya algebra over X. Given J_2 and e_2, it is easy to check that, up to isomorphism there are exactly two pairs (S, e_*) consisting of a principal homogeneous space S under J_2 and a function e_* on S satisfying ($\star$). These two possibilities are distinguished by the Arf invariant of e_*, or more simply by whether e_* takes the values 0 and 1 at $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$ points of S respectively, or whether the opposite happens. The third result whose proof we will only sketch is that the former happens, i. e. e_* is more often 0 than 1.

All these results in the case E = L were proven by Riemann over the complex ground field using his theta function (*cf.* [R], p. 212 and 487). In the general case, they follow easily by the results of ([M 2], § 2) on abstract theta functions and by Riemann's theorem that the multiplicity of the theta divisor $\Theta\subset J^{g-1}$ at a point x equals $\dim\Gamma(L_x)$, where L_x is the line bundle of degree $g-1$ corresponding to x. This last has been proven by me in all characteristics (unpublished) and is a special case of the results in the thesis of G. Kempf, soon to be published. The inspiration of this paper came from several conversations with M. Atiyah in which he asked whether there was a simple direct proof of these results not involving the theory of theta-functions. In particular, it was his suggestion to look at all vector bundles E admitting a Q rather than only at the line bundles.

1. Stability of $\dim\Gamma(E) \bmod 2$. — Given (X, E, Q), the idea is to represent $\Gamma(E)$ as the intersection of two maximal isotropic subspaces W_1, W_2 of a big even-dimensional vector space V with non-degenerate quadratic form q, where (V, q) obviously varies continuously when (X, E, Q) vary continuously. But then it is well-known that in such a case $\dim W_1\cap W_2 \bmod 2$ is invariant under continuous deformation (*cf.* Bourbaki [B], vol. 24, *Fromes sesquilinéaries*, § 6, ex. 18 *d*). We carry this out as follows : let $\mathfrak{a}$ be a cycle on X of the form $\sum_{i=1}^{N} P_i$, where $N \gg 0$

and the P_i are distinct points. Look at the commutative diagram :

$$
\begin{array}{ccccccccc}
 & & & & o & & o & & \\
 & & & & \downarrow & & \downarrow & & \\
o & \longrightarrow & E(-\mathfrak{a}) & \longrightarrow & E & \longrightarrow & E/E(-\mathfrak{a}) & \longrightarrow & o \\
 & & \| & & \downarrow & & \downarrow & & \\
o & \longrightarrow & E(-\mathfrak{a}) & \longrightarrow & E(\mathfrak{a}) & \longrightarrow & E(\mathfrak{a})/E(-\mathfrak{a}) & \longrightarrow & o \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E(\mathfrak{a})/E & = & E(\mathfrak{a})/E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & o & & o & &
\end{array}
$$

Since $\Gamma(E(-\mathfrak{a})) = (o)$ and $H^1(E(\mathfrak{a}) = (o)$ for $\mathfrak{a}$ sufficiently positive, this gives rise to the diagram :

$$
\begin{array}{ccccccccc}
 & & o & & o & & & & \\
 & & \downarrow & & \downarrow & & & & \\
o & \longrightarrow & \Gamma(E) & \longrightarrow & \Gamma(E/E(-\mathfrak{a})) & \longrightarrow & H^1(E(-\mathfrak{a})) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
o & \longrightarrow & \Gamma(E(\mathfrak{a})) & \longrightarrow & \Gamma(E(\mathfrak{a})/E(-\mathfrak{a})) & \longrightarrow & H^1(E(-\mathfrak{a})) & \longrightarrow & o \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Gamma(E(\mathfrak{a})/E)) & = & \Gamma(E(\mathfrak{a})/E) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & o & & & &
\end{array}
$$

from which it follows immediately that $\Gamma(E)$ is the intersection of the subspaces :

$$W_1 = \Gamma(E(\mathfrak{a})),$$
$$W_2 = \Gamma(E/E(-\mathfrak{a}))$$

of the vector space :

$$V = \Gamma(E(\mathfrak{a})/E(-\mathfrak{a})).$$

Next note that polarizing Q defines a non-degenerate bilinear form

$$B : \quad E \otimes E \rightarrow \Omega^1,$$

hence $E \cong \mathrm{Hom}(E, \Omega^1)$. It follows immediately by Serre duality that

$$\dim H^0(E) = \dim H^0(\mathrm{Hom}(E, \Omega^1)) = \dim H^1(E),$$

so $\chi(E) = o$. Therefore $\chi(E(\mathfrak{a})) = Nr$, and since $H^1(E(\mathfrak{a})) = (o)$, $\dim W_1 = Nr$, where $r = \mathrm{rank}\ (E)$. Obviously $\dim W_2 = Nr$ and

$\dim V = 2Nr$. Next define a quadratic form q on V as follows : if $a_i \in E(\mathfrak{a})_{P_i}$, $1 \leq i \leq N$, define the section $\bar{a}$ of $E(\mathfrak{a})/E(-\mathfrak{a})$, then set

$$q(\bar{a}) = \sum_{i=1}^{N} \operatorname{Res}_{P_i} Q(a_i),$$

where Q is here extended to a quadratic map $E(\mathfrak{a}) \to \Omega^1_X(2\mathfrak{a})$. If $\bar{a} \in W_1$, then the a_i all come from one global section a of $E(\mathfrak{a})$, and $q(\bar{a}) = 0$ since the sum of the residues of a rational differential on X is zero. If $\bar{a} \in W_2$, then $a_i \in E_{P_i}$, so $Q(a_i) \in (\Omega^1_X)_{P_i}$ so again $q(\bar{a}) = 0$. Thus W_1 and W_2 are isotropic subspaces of V of half the dimension, i. e. are maximal isotropic subspaces.

Now say (X, E, Q) vary in an algebraic family, i. e. we are given

(i) $\pi : \mathcal{X} \to S$, proper smooth family of curves of genus g;

(ii) $\mathcal{E}$ on $\mathcal{X}$: a vector bundle of rkr;

(iii) $Q : \mathcal{E} \to \Omega^1_{\mathcal{X}/S}$ a non-degenerate quadratic form.

Theorem. — *The function* $S \to \mathbf{Z}/2\mathbf{Z}$ *defined by*

$$s \mapsto \dim \Gamma(X_s, E_s) \bmod 2$$

is constant on connected components of S.

Proof. — After an étale base change $S' \to S$, we may assume that locally $\mathcal{X}/S$ admits N disjoint sections $\sigma_i : S \to \mathcal{X}$. If $\mathfrak{A}$ is the relative Cartier divisor $\Sigma \sigma_i(S)$ on $\mathcal{X}$ over S, then as above, we find three locally free sheaves on S :

$$\begin{array}{l} \mathcal{W}_1 = \pi_* \mathcal{E}(\mathfrak{A}) \hookrightarrow \\ \qquad\qquad\qquad\qquad \pi_*[\mathcal{E}(\mathfrak{A})/\mathcal{E}(-\mathfrak{A})] = \mathcal{V} \\ \mathcal{W}_2 = \pi_*[\mathcal{E}/\mathcal{E}(-\mathfrak{A})] \hookrightarrow \end{array}$$

and a non-degenerate quadratic form $q : \mathcal{V} \to \mathcal{O}_S$. Moreover, for each $s \in S$, $(\mathcal{V}, \mathcal{W}_1, \mathcal{W}_2, q)$ induce, after tensoring with $k(s)$, the previous quadruple (V, W, W_2, q). In particular,

$$\Gamma(X_s, E_s) \cong [\mathcal{W}_1 \otimes_{\mathcal{O}_s} k(s)] \cap [\mathcal{W}_2 \otimes_{\mathcal{O}_s} k(s)]$$

[$\cap$ inside $\mathcal{V} \otimes_{\mathcal{O}_s} k(s)$]. But using the fact that $q \equiv 0$ on $\mathcal{W}_1$ and $\mathcal{W}_2$, by the standard constructions in the theory of quadratic forms, one finds that $\exists$ locally isomorphisms :

$$\mathcal{V} \cong \mathcal{W}_1 \oplus \mathcal{W}_1^*$$
$$\mathcal{V} \cong \mathcal{W}_2 \oplus \mathcal{W}_2^*$$

taking q into the hyperbolic quadratic form on the right. Therefore $\exists$ locally an isomorphism $\varphi : \mathcal{V} \to \mathcal{V}$ such that $\varphi(\mathcal{W}_1) = \mathcal{W}_2$ and $q \circ \varphi = q$. It follows that $\det\varphi \in \Gamma(\mathcal{O}_S^*)$ satisfies $(\det\varphi)^2 = 1$, hence $\det\varphi = +1$ or -1 on each connected component of S. But it is easy to check that for all $s \in S$,

$$(\det\varphi)(s) = (-1)^{Nr + \dim \Gamma(E_s)}$$

(*cf.* the exercise in Bourbaki referred to above).

Q. E. D.

2. Applications. — Suppose we choose one line bundle L_0 such that $L_0^2 \cong \Omega_X^1$. Then the map

$$E \mapsto E \otimes L_0$$

gives a bijection between vector bundles with non-degenerate quadratic forms $Q : E \to \mathcal{O}_X$, and vector bundles with non-degenerate quadratic forms $Q : E \to \Omega_X^1$. Now it is easy to see that locally in the étale topology, any (E, Q), where Q is $\mathcal{O}_X$-valued, is isomorphic to $(\mathcal{O}_X^r, Q_0)$, where

$$Q_0(a_1, \ldots, a_r) = \Sigma a_i^2.$$

Therefore, if $\mathbf{O}(r)$ is the full orthogonal group, the pairs (E, Q) are classified by the cohomology set

$$H^1(X_{\text{ét}}, \mathbf{O}(r)).$$

Using the structure : $\mathbf{O}(r) \cong \mathbf{Z}/2\mathbf{Z}.\ \mathbf{SO}(r)$ (the product being direct or semi-direct according as r is odd or even), and the standard isomorphisms of $\mathbf{SO}(r)$ with $\mathbf{G}_m (r = 2)$, $\mathbf{SL}(2)/(-I)$ $(r = 3)$ and $\mathbf{SL}(2) \times \mathbf{SL}(2)/(-I, -I)$ $(r = 4)$ it is easy to determine all (E, Q)'s if rank E = 1, 2, 3 or 4. We get the following bundles :

(1) $E = L$ a line bundle, $L^2 \cong \mathcal{O}_X$.

(2) $E = L \oplus L^{-1}$, L any line bundle, Q hyperbolic.

(3) If $\pi : X' \to X$ is an étale double covering, then for all line bundles L′ on X′ such that [2]

$$Nm_{X'/X}(L') \cong \mathcal{O}_X.$$

Let $E = \pi_* L'$. Then the norm defines $Q : E \to \mathcal{O}_X$.

(4) $E = L \otimes S^2 F$, where

F is any vector bundle of rank 2; L is a line bundle such that

$$(\star) \qquad (L \otimes \Lambda^2 F)^2 \cong \mathcal{O}_X.$$

---

(2) Nm of a line bundle L′ means apply Nm to transition functions defining L′ for an open cover of the type $U_\alpha = \pi^{-1}(V_\alpha)$.

There is a canonical quadratic function

$$\delta: \quad S^2F \to (\Lambda^2 F)^2$$

which is the classical discriminant of a quadratic form, and δ plus the isomorphism ($\star$) defines Q.

(5) $E = F_1 \otimes F_2$, where

F_1, F_2 are rank 2 vector bundles and

$$\Lambda^2 F_1 \otimes \Lambda^2 F_2 \cong \mathcal{O}_X.$$

The quadratic form comes from the composition

$$(F_1 \otimes F_2) \otimes (F_1 \otimes F_2) \to \Lambda^2 F_1 \otimes \Lambda^2 F_2 \xrightarrow{\sim} \mathcal{O}_X.$$

(6) $E =$ a quaternion algebra over $\mathcal{O}_X$, i. e. a locally free sheaf of rank 4, plus a multiplication $E \otimes E \to E$, making the fibres into isomorphic copies of $M_2(k)$. The quadratic form is just the *reduced* norm

$$Nm: \quad E \to \mathcal{O}_X.$$

There are other rank 4 E's whose cohomology classes are not killed by the map $H^1(o(4)) \xrightarrow{\det} H^1(\mathbf{Z}/2\mathbf{Z})$, but we will not write all these down.

In example 2, it follows easily by the Riemann-Roch theorem that

$$\dim H^0(E \otimes L_0) \equiv \deg L \pmod 2$$

so the stability is obvious in this case. In example 3, the stability is a classical theorem of Wirtinger [Wi]. In fact let $\iota: X' \to X'$ be the involution interchanging the sheets and let J, J' be the Jacobians of X, X' respectively, then Nm defines a homomorphism

$$Nm: \quad J' \to J$$

and it turns out that the kernel of Nm consists of exactly two components :

$$P_0 = \text{locus of the line bundles } M \otimes \iota^* M^{-1}, \quad \deg M = 0$$

and

$$P_1 = \text{locus of line bundles } M \otimes \iota^* M^{-1}, \quad \deg M = 1.$$

The result of Wirtinger is that the map

$$P_0 \cup P_1 \to \mathbf{Z}/2\mathbf{Z}$$
$$L \mapsto \dim H^0(L \otimes \pi^* L_0) \bmod 2$$

is constant on P_0 and P_1 and take different values on the two components. We can prove this as follows :

LEMMA 1. — *If* L *is a line bundle on* X′ *such that* $\mathrm{Nm}\,L \cong \mathcal{O}_X$, *then* $L \cong M \otimes \iota^* M^{-1}$ *for some line bundle* M *on* X′. *Moreover,* M *can be chosen of degree* 0 *or* 1.

Proof. — In terms of divisor classes, let L be represented by a divisor $\mathfrak{a}$. Then $\mathrm{Nm}\,L \cong \mathcal{O}_X$ means that $\pi_*\mathfrak{a} = (f)$, for some function f on X. According to Tsen's theorem, the function field $k(X)$ is C^1 (*cf.* Lang [L 2]), hence $f = \mathrm{Nm}\,g$, some $g \in k(X')$. Therefore $\pi_*(\mathfrak{a} - (g)) = (f) - (\mathrm{Nm}\,g) = 0$. Therefore the divisor $\mathfrak{a} - (g)$ is a linear combination of the divisors $x - \iota(x)$, $x \in X'$. In other words, $\mathfrak{a} - (g)$ can be written $\mathfrak{b} - \iota^{-1}\mathfrak{b}$, for some divisor $\mathfrak{b}$. So if $M = \mathcal{O}_X(\mathfrak{b})$, L is isomorphic to $M \otimes \iota^* M^{-1}$. Finally, we may replace M by $M \otimes \pi^* N$ for any line bundle N on X without destroying the property $L \cong M \otimes \iota_* M^{-1}$. In this way, we can normalize the degree of M to be 0 or 1.

Q. E. D.

This proves that $\mathrm{Ker}(\mathrm{Nm})$ is the union of the two sets P_0, P_1 described above. Clearly P_0 and P_1 are irreducible. By our general result, the number

$$\dim H^0(L \otimes \pi^* L_0) = \dim H^0(\pi_* L \otimes L_0)$$

is constant mod 2 on P_0 and P_1. It remains to check that it has *opposite* parity on the two varieties P_0 and P_1. We can see this as follows :

Step I : For some $L \in \mathrm{Ker}\,\mathrm{Nm}$, $H^0(L \otimes \pi^* L_0) \neq (0)$.

Proof. — Choose $\omega \in \Gamma(\Omega^1_X)$ and let $\mathfrak{a} = (\omega)$.

For each point x occuring in $\mathfrak{a}$, choose an $x' \in X'$ over x, and thus find a *positive* divisor $\mathfrak{a}'$ on X′ such that $\pi_*\mathfrak{a}' = \mathfrak{a}$. If $L' = \mathcal{O}_X(\mathfrak{a}')$, then $H^0(L') \neq (0)$ and $\mathrm{Nm}\,L' \cong \mathcal{O}_X(\mathfrak{a}) \cong \Omega^1_X$. Therefore

$$\mathrm{Nm}(L' \otimes \pi^* L_0^{-1}) \cong \mathrm{Nm}\,L' \otimes L_0^{-2} \cong \mathcal{O}_X,$$

and hence $L = L' \otimes \pi_* L_0^{-1}$ has the required properties.

Step II : If $L \in \mathrm{Ker}\,\mathrm{Nm}$ is such that $H^0(L \otimes \pi^* L_0) \neq (0)$, then for almost all $x \in X'$,

$$\dim H^0(L(x - \iota x) \otimes \pi^* L_0) = \dim H^0(L \otimes \pi^* L_0) - 1.$$

Proof. — Let $r = \dim H^0(L \otimes \pi^* L_0)$. Then provided ιx is not a common zero of all the sections of $L \otimes \pi^* L_0$,

$$\dim H^0(L(-\iota x) \otimes \pi^* L_0) = r - 1,$$

hence

$$\dim H^0(L(x - \iota x) \otimes \pi^* L_0) = r \quad \text{or} \quad r - 1.$$

If it equals r, then

$$\dim H^1(L(x-\iota x)\otimes \pi^* L_0) = \dim H^1(L(-\iota x)\otimes \pi^* L_0),$$

hence by Serre duality

$$\dim H^0(L^{-1}(\iota x - x)\otimes \pi^* L_0) = \dim H^0(L^{-1}(\iota x)\otimes \pi^* L_0),$$

hence x is a common zero of all the sections of $L^{-1}(\iota x)\otimes \pi^* L_0$, hence x is a common zero of all the sections of $L^{-1}\otimes \pi^* L_0$. But by Riemann-Roch

$$\begin{aligned}\dim H^0(L^{-1}\otimes \pi^* L_0) &= \dim H^1(L^{-1}\otimes \pi^* L_0)\\ &= \dim H^0(L\otimes \pi^* L_0) > 0\end{aligned}$$

so for almost all x, this is false.

Q. E. D.

Step III : If $L \in P_i$, then $L(x-\iota x)\in P_{1-i}$.

Proof. — Clear.

This proves all of Wirtinger's results in all char. $\neq 2$.

3. THE IDENTITY BETWEEN e_* AND e_2. — We begin by the observation that in view of Tsen's theorem all Azumaya algebras [3] over X are *split*, i. e. equal Hom (E, E) for some vector bundle E over X (*cf.* Grothendieck [G]). In particular, this means that example 6 of quadratic bundles in paragraph 2, the quaternion bundles with reduced norm, are special cases of example 5. Now if $L, M \in J_2$, let

$$A = \mathcal{O}_X + L + M + L\otimes M$$

and make A into a quaternion bundle by fixing isomorphisms $L^2 \cong \mathcal{O}_X$, $M^2 \cong \mathcal{O}_X$ to define $l_1.l_2$, and $m_1.m_2$ [if $l_i \in \Gamma(U, L)$, $m_i \in \Gamma(U, M)$] and by the rule

$$l.m = -m.l,\qquad l\in\Gamma(U, L),\qquad m\in\Gamma(U, M).$$

It follows that

$$A \cong \mathrm{Hom}(E, E)$$

for some vector bundle E of rank 2.

LEMMA 2 : $e_2(L, M) = (-1)^{\deg \Lambda^2 E}$.

Proof. — Let $L = \mathcal{O}(\mathfrak{a})$, $M = \mathcal{O}(\mathfrak{b})$ for suitable divisors $\mathfrak{a}$ and $\mathfrak{b}$ with disjoint support. Then $2\mathfrak{a} = (f)$ and $2\mathfrak{b} = (g)$ for some $f, g \in k(X)$ and

---

(3) This means a locally free sheaf of $\mathcal{O}_X$-algebras whose fibres are isomorphic to $M_n(\mathcal{O}_X)$.

by definition $e_2(\mathrm{L}, \mathrm{M}) = f(\mathfrak{b})/g(\mathfrak{a})$. Let $\pi : \mathrm{X}' \to \mathrm{X}$ be the double covering defined by L and let $\iota : \mathrm{X}' \to \mathrm{X}'$ be the involution, so that $\sqrt{f} \in k(\mathrm{X}')$. Now the vector bundle E is unique up to a substitution $\mathrm{E} \mapsto \mathrm{E} \otimes \mathrm{N}$, N a line bundle on X and this substitution does not affect the truth of the lemma. We will prove the lemma by constructing one of the possible E's. By lemma 1 in the previous section, since

$$\mathrm{N}m\,(\pi^* \mathrm{M}) \cong \mathrm{M}^2 \cong \mathcal{O}_{\mathrm{X}},$$

it follows that there exists a line bundle P on X′ such that

$$\iota^* \mathrm{P} \underset{\alpha}{\overset{\approx}{\to}} \mathrm{P} \otimes \pi^* \mathrm{M}.$$

Choose α such that the composition

$$\mathrm{P} = \iota^* \iota^* \mathrm{P} \underset{\iota^*\alpha}{\overset{\approx}{\to}} \iota^* \mathrm{P} \otimes \iota^* \pi^* \mathrm{M} \underset{\alpha}{\overset{\approx}{\to}} (\mathrm{P} \otimes \pi^* \mathrm{M}) \otimes \pi^* \mathrm{M} = \mathrm{P} \otimes \pi^* \mathrm{M}^2 \cong \mathrm{P}$$

is the identity. Let $\mathrm{E} = \pi_* \mathrm{P}$. Then (*a*) E is a $\pi^* \mathcal{O}_{\mathrm{X}'}$-algebra, and since $\pi_* \mathcal{O}_{\mathrm{X}'} \cong \mathcal{O}_{\mathrm{X}} + \mathrm{L}$, we are given an action of L on E; and (*b*) the isomorphism α defines an action of M on E. It is easy to check that altogether these actions make E into an A-module. Then it follows automatically [since $\mathrm{M}_2(k)$ has a unique module of dimension 2 over k, up to isomorphism] that $\mathrm{A} \cong \mathrm{Hom}(\mathrm{E}, \mathrm{E})$.

But now

$$\pi^* \mathrm{E} \cong \pi^* \pi_* \mathrm{P} \cong \mathrm{P} + \iota^* \mathrm{P},$$

hence

$$\pi^* (\Lambda^2 \mathrm{E}) \cong \mathrm{P} \otimes \iota^* \mathrm{P},$$

hence

$$2 \deg \Lambda^2 \mathrm{E} = \deg \pi^* (\Lambda^2 \mathrm{E}) = \deg \mathrm{P} + \deg \iota^* \mathrm{P} = 2 \deg \mathrm{P},$$

or $\deg \Lambda^2 \mathrm{E} = \deg \mathrm{P}$. Moreover, if $\mathrm{P} = \mathcal{O}(\mathfrak{c})$ for some divisor $\mathfrak{c}$ on X′ disjoint from $\pi^{-1}(\mathfrak{a})$ and $\pi^{-1}(\mathfrak{b})$, then the existence of α means that there is an $h \in k(\mathrm{X}')$ such that

$$(h) + \iota^{-1} \mathfrak{c} - \mathfrak{c} = \pi^{-1} \mathfrak{b}.$$

Then

$$(\mathrm{N}mh) = 2\mathfrak{b} = (g),$$

so $\mathrm{N}mh = \lambda . g$, $\lambda \in k^*$. Therefore :

$$\begin{aligned} g(\mathfrak{a}) &= \mathrm{N}mh(\mathfrak{a}) = h(\pi^{-1}\mathfrak{a}) = h((\sqrt{f})) \\ &= \sqrt{f}((h)), \quad \text{by reciprocity} \\ &= \sqrt{f}(\pi^{-1}\mathfrak{b}) . \frac{\sqrt{f}(\mathfrak{c})}{\sqrt{f}(\iota^{-1}\mathfrak{c})} = f(\mathfrak{b}) . (-1)^{\deg \mathfrak{c}} \\ &= f(\mathfrak{b}) . (-1)^{\deg \Lambda^2 \mathrm{E}}. \end{aligned}$$

Q. E. D.

Now fix L_0 such that $L_0^2 \simeq \Omega_X^1$. Then

$$e_*(L_0) + e_*(L_0 \otimes L) + e_*(L_0 \otimes M) + e_*(L_0 \otimes L \otimes M)$$
$$\equiv \dim H^0(L_0 \otimes A)$$
$$= \dim H^0(L_0 \otimes \mathrm{Hom}(E, E)).$$

But it is well-known that all vector bundles of given rank and degree form a connected set, i. e. any 2 can be found as fibres of a family of such vector bundles over a connected and even irreducible base S (*cf.* Seshadri [S]). Therefore by our stability theorem, if $x \in X$ is any point and $r = \deg \Lambda^2 E$, setting $E_r = \mathcal{O}_X \oplus \mathcal{O}_X(rx)$, then

$$\dim H^0(L_0 \otimes \mathrm{Hom}(E, E)) \equiv \dim H^0(L_0 \otimes \mathrm{Hom}(E_r, E_r))$$
$$= \dim H^0(L_0 \otimes \mathcal{O}_X(rx)) + 2 \dim H_0(L_0)$$
$$+ \dim H_0(L_0 \otimes \mathcal{O}_X(-rx))$$
$$\equiv \dim H^0(L_0 \otimes \mathcal{O}_X(rx)) - \dim H^0(L_0 \otimes \mathcal{O}_X(-rx))$$
$$= \chi(L_0 \otimes \mathcal{O}_X(rx)), \quad \text{by Serre duality}$$
$$= r$$
$$\equiv \ln e_2(L, M), \quad \text{by lemma 2.}$$

This completes the proof of identity ($\bigstar$) stated in the Introduction.

4. Final comments. — The third theorem mentioned in the introduction is that e_* takes the value 0 (resp. 1) $2^{g-1}(2^g + 1)$ [resp. $2^{g-1}(2^g - 1)$] times. This can be proven as follows : by the stability theorem in paragraph 1, if two curves X_1, X_2 lie in a family over a connected base S, then to prove it for X_1 it suffices to prove it for X_2. Since the moduli space of curves of genus of g is connected (Deligne-Mumford [D-M]), it therefore suffices to prove this for one X. Now for *hyperelliptic* X the result is very elementary. In fact, let

$$\pi : X \to P^1$$

be the double covering, with ramification points $P_1, \ldots, P_{2g+2} \in X$. Let $\mathfrak{a}$ be the divisor class π^{-1} (one point) on X. We have the relations

$$2P_1 \equiv \ldots \equiv 2P_{2g+2} \equiv \mathfrak{a},$$
$$P_1 + \ldots + P_{2g+2} \equiv (g+1)\mathfrak{a},$$
$$(g-1)\mathfrak{a} \equiv K_X, \quad \text{the canonical divisor class on X.}$$

Then the elements of S(X) are represented by the divisor classes

$$\mathfrak{d}_S^{(l)} = \sum_{\alpha \in S} P_\alpha + l.\mathfrak{a}, \qquad l = 0, 1, \ldots, \left[\frac{g-1}{2}\right],$$
$$S \subset \{1, 2, \ldots, 2g+2\},$$
$$\# S = g - 1 - 2l$$

and

$$\mathfrak{d}_S^{(-1)} = \sum_{\alpha \in S} P_\alpha - \mathfrak{a}, \qquad S \subset \{1, 2, \ldots, 2g+2\},$$
$$\# S = g + 1,$$

where $\mathfrak{d}_S^{(-1)} = \mathfrak{d}_T^{(-1)}$ if $\{1, 2, \ldots, 2g+2\} = S \cup T$. It is easy to check that

$$\dim H^0(X, \mathcal{O}_X(\mathfrak{d}_S^{(l)})) = l + 1$$

and then adding up the number of $\mathfrak{d}_S^{(l)}$ with l odd and l even, the result follows.

A natural question to ask is what happens in char. 2 ? Strangely, it turns out that in this case there is a natural line bundle L such that $L^2 \simeq \Omega_X^1$ In fact, for every $f \in k(X) - k(X)^2$, it follows immediately by expanding f locally as a power series that the differential df has only double zeros and double poles, i. e.

$$(df) = 2\mathfrak{a}, \qquad \text{for some divisor } \mathfrak{a}.$$

Moreover, if $f_1, f_2 \in k(X) - k(X)^2$, then $\{1, f_2\}$ is a 2-basis of $k(X)$, so

$$f_1 = a^2 f_2 + b^2, \quad \text{some } a, b \in k(X).$$

Therefore $df_1 = a^2 df_2$, hence if $(df_i) = 2\mathfrak{a}_i$, we find

$$\mathfrak{a}_1 = (a) + \mathfrak{a}_2$$

Therefore the divisor *class* $\mathfrak{a}$ is independent of f, and $L = \mathcal{O}_X(\mathfrak{a})$ is a canonical square root of Ω_X^1.

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