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Riemann surfaces and spin structures

RIEMANN SURFACES AND SPIN STRUCTURES

By Michael F. ATIYAH.

INTRODUCTION.

The purpose of this paper is to reprove some classical results on compact Riemann surfaces using elliptic operators, and to show how these results fit naturally into the general context of spin-manifolds. This programme was stimulated by discussion with D. Mumford and I am greatly indebted to him for information about the classical theory. Moreover, in [8], Mumford gives new algebraic proofs of the theorems in question, rather in the spirit of this paper.

The theorems we have in mind concern the division of the canonical divisor class K by 2 (or, multiplicatively, taking the square root of the canonical line bundle). If g is the genus of the Riemann surface then there are 2^{2g} solutions of the equation $2D = K$ — in the divisor class group. These solutions are by no means equivalent to one another. For example the complete linear systems $|D|$ may have quite different dimensions, for the different choices of D — as is already seen for the case $g = 1$, when $D = 0$ differs from the other three solutions.

If the complex structure on our Riemann surface varies continuously with some parameter t then the 2^{2g} divisor classes above will also vary continuously. The dimensions of the linear systems $|D_t|$ are not in general constant functions of t — they can jump. In fact this is related to the fact mentioned above (that dim $|D|$ depends on the choice of D) because dim $|D_t|$ is a multi-valued function of t and, as we go round a closed path in t-space, we may take one choice of D into another.

If we reduce all dimensions modulo two then there are some remarkable classical theorems concerning this situation. In the first place we have stability under deformation :

THEOREM 1. — *Let* X_t *be a holomorphic family of compact Riemann surfaces* $(t \in C, |t| < 1)$ *and let* D_t *be a holomorphic family of divisor classes on* X_t *such that* $2D_t = K_t$ *(where* K_t *is the canonical divisor class). Then* $\dim |D_t| \bmod 2$ *is independent of* t.

Remarks. — A holomorphic family means that X_t is the fibre of a proper holomorphic map $f : Y \to C$ where Y is a complex analytic surface and $df \neq 0$. To define a holomorphic family of divisor classes it is best to pass to the point of view of line-bundles. Thus if (by abuse of notation) K_t now denotes the canonical line of X_t, then we have a holomorphic line-bundle K on Y (with $K_t = K | X_t$), and a holomorphic line-bundle L on Y with $L^2 \cong K$ defines a family L_t of line-bundles corresponding to the divisor classes D_t. It is uniquely determined by the choice of L_0. With this notation $|D_t|$ is the projective space associated to the vector space $\Gamma(L_t)$ of holomorphic sections of L_t, so that $\dim |D_t| = \dim \Gamma(L_t) - 1$. From now on we shall use the line-bundle terminology.

To explain the second result let us denote by $\mathcal{S}(X)$ the set of line-bundles (up to isomorphism) which are square roots of the canonical line bundle of X. This set has 2^{2g} elements. In fact it is clearly a principal homogeneous space for the group of line bundles of order 2. This group is naturally isomorphic with $H^1(X, \mathbf{F}_2)$ which is a vector space over $\mathbf{F}_2$ (the integers mod 2). Thus $\mathcal{S}(X)$ has a natural structure of *affine space* over $\mathbf{F}_2$ with $H^1(X, \mathbf{F}_2)$ as its group of translations. Then the second classical result is

THEOREM 2. — *The function* $\varphi : \mathcal{S}(X) \to \mathbf{F}_2$ *defined by* $\varphi(L) = \dim \Gamma(L)$ $\bmod 2$ *is a quadratic function whose associated bilinear form is the cup-product* [1] *on* $H^1(X, \mathbf{F}_2)$.

Remarks. — If A is an affine space over $\mathbf{F}_2$ a function $\varphi : A \to \mathbf{F}_2$ is called quadratic if, for all $a \in A$ and $x, y \in T(A)$ (the vector space space of translations of A),

$$H_a(x, y) = \varphi(a + x + y) - \varphi(a + x) - \varphi(a + y) + \varphi(a)$$

is a bilinear form on $T(A)$. It then follows that H_a is independent of a : it is called the associated bilinear form. If $\varphi(a) = 0$ and we identify A

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[1] We identify $H^2(X, \mathbf{F}_2)$ with $\mathbf{F}_2$.

with T(A) by $a \leftrightarrow a + x$ then φ becomes a quadratic function on T(A), in the usual sense, with associated form H. If H is non-degenerate there are essentially just two possible quadratic functions. These are distinguished by their Arf invariant or alternatively by the number of zeros of φ. Arf invariant zero is equivalent to the existence of an isotropic subspace of dimension g (where $\dim A = 2g$) : in this case φ has $2^{g-1}(2^g + 1)$ zeros. The function $\varphi + 1$ on A then has $2^{g-1}(2^g - 1)$ zeros and corresponds to a quadratic form on T(A) of Arf invariant one. The identification of the function φ in Theorem 2 is then completed by

THEOREM 3. — *The function φ in Theorem 2 has $2^{g-1}(2^g + 1)$ zeros.*

For example, if $g = 1$, the three non-trivial square roots L of K ($= 1$ in this case) have $\varphi(L) = 0$, whereas $\varphi(1) = 1$. For $g = 3$ there are 28 values of L with $\varphi(L) = 1$: these correspond to the famous 28 bitangents of a plane quartic.

Theorem 1 shows that the function φ in Theorem 2 does not depend on the complex structure of X (since the space of moduli is connected). To prove Theorems 2 and 3 therefore we could choose a particular complex structure which simplified the calculations. In fact hyperelliptic curves (double coverings of the projective line) provide convenient models and Theorems 2 and 3 reduce to combinatorial calculations involving the branch points of the double covering $X \to P_1$. However our methods will give natural proofs of all three theorems.

We shall begin, in paragraph 1, with a simple direct analytical proof of Theorem 1. These methods, with a little extra topology, will then be extended in paragraph 2 to prove Theorem 2. In paragraph 3 we shall show how the problems and methods of the first two sections fit into the general theory of spin-manifolds. Using this general theory we shall then in paragraph 4 prove Theorem 3. We shall also give a natural generalization of Theorem 2 to spin-manifolds.

Finally in paragraph 5, following a suggestion of J.-P. Serre, we shall deduce the following theorem as a purely algebraic consequence of Theorems 1-3 :

THEOREM 4. — *Let X_t be a holomorphic family of compact Riemann surfaces parametrized by the punctured unit disc : $0 < |t| < 1$. Then there is a holomorphic family of line-bundles L_t on X_t such that $L_t^2 \cong K_t$.*

Remarks. — 1. We shall in fact prove a slightly stronger differentiable version of this theorem : this also applies to Theorem 1.

2. Theorem 4 seems not to have been observed before, although a formal analogue for curves over finite fields was known (*see* § 5).

1. Mod 2 STABILITY. — The proof of Theorem 1 depends on the following simple lemma of functional analysis :

LEMMA (1.1). — *Let* H *be a complex Hilbert space and denote by* $\mathcal{H}$ *the space of all continuous* R-*linear operators* T *on* H *such that* ([2]) :

(i) $T(ih) = -iT(h)$ $(h \in H)$;

(ii) $T^\star = -T$;

(iii) o *is an isolated point of finite multiplicity in the spectrum of* T.

Then $T \mapsto \dim_{\mathbf{C}} \mathrm{Ker}\,T \bmod 2$ *defines a continuous (and therefore locally constant) map* $\mathcal{H} \to \mathbf{F}_2$, *where* $\mathcal{H}$ *is given the uniform (operator norm) topology.*

This lemma is proved in ([6], (5.1)) as part of a more general result about spaces $\mathscr{F}^k$ (the $\mathcal{H}$ of Lemma (1.1) is $\mathscr{F}^2$ of [6]). Roughly speaking the proof goes as follows.

For S sufficiently close to T in $\mathcal{H}$ we have

$$\dim_{\mathbf{R}} \mathrm{Ker}\,T = \sum_{0 \leq \lambda < \varepsilon} \dim_{\mathbf{R}} E_\lambda,$$

where E_λ is the real subspace of H which complexifies to give the $\pm\, i\, \sqrt{\lambda}$-eigenspaces of S. By (i) all E_λ admit multiplication by i and so are complex subspaces of H. Since $E_0 = \mathrm{Ker}\,S$ the lemma will be proved if we can show that every other $E_\lambda(\lambda > o)$ has a quarternion structure. To do this for E_λ define $j = \dfrac{S}{\sqrt{\lambda}}$ and we have $j^2 = \dfrac{S^2}{\lambda} = -\, 1$ and $ji = -\, ij$ [by (i)].

Let us return now to a compact Riemann surface X with canonical line bundle K and let L be a square root of K. Consider the $\bar{\partial}$-operator on L

$$\bar{\partial}_{\mathrm{L}} :\quad C^\infty(\mathrm{L}) \to C^\infty(\mathrm{L} \otimes \bar{\mathrm{K}}).$$

The holomorphic sections of L are the solutions of $\bar{\partial}_{\mathrm{L}} u = \mathrm{o}$. For any u, $v \in C^\infty(\mathrm{L})$ we have a product $uv \in C^\infty(\mathrm{L}^2) = C^\infty(\mathrm{K})$ and so

$$(1.2) \qquad \int_{\mathrm{X}} \bar{\partial}u.v + \int_{\mathrm{X}} u.\bar{\partial}v = \int_{\mathrm{X}} \bar{\partial}(uv) = \int_{\mathrm{X}} d(uv) = \mathrm{o}.$$

([2]) In (ii), (iii) we consider H as a real Hilbert space.

Now choose a hermitian metric on X : this induces one on K and L and gives rise to an anti-linear isomorphism

$$h: \quad C^\infty(L \otimes \overline{K}) \to C^\infty(L)$$

defined by

$$\langle u, h(w) \rangle = \int_X uw, \qquad u \in C^\infty(L), \quad w \in C^\infty(L \otimes \overline{K}),$$

where $\langle\,,\,\rangle$ denotes the hermitian inner product on $C^\infty(L)$ induced by the metric. The composition $P = h\bar{\partial}_L$ is therefore an anti-linear map $C^\infty(L) \to C^\infty(L)$ and, for $u,\, v \in C^\infty(L)$, we have

$$\langle u, Pv \rangle + \langle v, Pu \rangle = 0.$$

Taking real parts this gives

$$\mathrm{Re}\langle u, Pv \rangle = - \mathrm{Re}\langle v, Pu \rangle = - \mathrm{Re}\langle Pu, v \rangle.$$

Since $\mathrm{Re}\langle u, v \rangle$ is the euclidean inner product on H (considered as real space) this implies that $P^* = - P$, where P^* is the formal adjoint of the unbounded operator P.

Now P is a first order elliptic differential operator. As usual [3] we associate to it the bounded operator $T = (1 + P^*P)^{-\frac{1}{2}}P$. Then $T = - T^*$ and is an anti-linear Fredholm operator [4], which is equivalent to saying that T lies in the space $\mathscr{H}$ of Lemma (1.1). Moreover $\mathrm{Ker}\,T = \mathrm{Ker}\,P = \Gamma(L)$ (since all solutions of $Pu = 0$ are C^∞).

Given now a holomorphic family of Riemann surfaces X_t as in Theorem 1 with a family of line-bundles L_t (such that $L_t^2 = K_t$) we obtain a family of bounded operators T_t. Moreover T_t is continuous in t (see [3]). Theorem 1 now follows by applying Lemma (1.1). Actually this requires an extra argument because the Hilbert space H_t on which T_t acts also varies continuously [5] with t. However this is taken care of by the observation that, given a continuous family of hermitian operators A_t (on a Hilbert space H) we can, for small t, find a continuous family of invertible operators P_t such that $A_t = P_t A_0 P_t^{-1}$: the details are left to the reader.

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[3] *See* [3] for a general summary of elliptic operator theory.

[4] i. e. T has closed range and dim KerT = dim KerT* < ∞.

[5] Since the family X_t is locally a C^∞ product the space H_t is fixed (independent of t) but the inner product varies.

Note that, in this proof of Theorem 2, it is enough that the X_t should form a differentiable family of (complex) Riemann surfaces : it is not necessary to have the family holomorphic.

2. THE QUADRATIC FUNCTION ([6]). — In this section we shall investigate the function $\varphi : \mathcal{S}(X) \to \mathbf{F}_2$ of Theorem 2. By definition, for any $L \in \mathcal{S}(X)$ (i. e. $L^2 \cong K$) we have $\varphi(L) = \dim \Gamma(L) \bmod 2$. In the preceeding section we saw that a hermitian metric on X defines an anti-linear bounded operator $T = T_L$ and that

$$\varphi(L) = \operatorname{Ker} T_L \bmod 2.$$

T_L is an operator on the square-integrable sections of L. To study φ we shall first generalize it slightly.

Suppose now that E is any real vector bundle over X. Then using a partition of unity (or a connection in E) we can define an extended $\bar{\partial}$-operator

$$\bar{\partial}_L(E) : \quad C^\infty(L \otimes_R E) \to C^\infty(L \otimes_C \overline{K} \otimes_R E).$$

Using-metrics on X and E this defines, as in paragraph 1, an anti-linear operator $T_L(E)$ on the space of square-integrable sections of $L \otimes_R E$. Since $\bar{\partial}_L$ is skew-symmetric [see (1.2)] $\bar{\partial}_L(E)$ will be skew-symmetric modulo o-order terms, hence $T_L(E)$ is skew-adjoint modulo compact operators. Replacing $T_L(E)$ by $\frac{1}{2}(T_L(E) - T_L^*(E))$ — which is still Fredholm — we may therefore assume that $T_L(E)$ is strictly skew-adjoint. Hence it belongs to the space $\mathcal{H}$ of Lemma (1.1) (for H the space of square-integrable sections of $L \otimes_R E$). Because of Lemma (1.1) the function

$$E \mapsto \varphi_L(E) = \dim_\mathbf{C} \operatorname{Ker} T_L(E) \bmod 2$$

is independent of the various choices made (metrics and connections). Clearly

$$\varphi_L(E \oplus E') = \varphi_L(E) + \varphi_L(E')$$

and so φ_L extends by linearity to a group homomorphism

$$\varphi_L : \quad KO(X) \to \mathbf{F}_2,$$

where $KO(X)$ is the Grothendieck group of real vector bundles on X (see [1]). From its definition we have $\varphi_L(1) = \varphi(L)$.

([6]) It is interesting to compare this section with the corresponding section (§ 3) in [8].

If in particular we take E to be a real *line-bundle* then $E^2 \cong 1$ (the trivial line-bundle) and so $L' = L \otimes_R E$ is another square root of K. Moreover since the coordinate transformations of E can be taken as constants (in fact ± 1) we do not need a partition of unity to define $\bar{\partial}_L(E)$: in fact $\bar{\partial}_L(E) = \bar{\partial}_{L'}$. Hence $T_L(E) = T_{L'}$ and so $\varphi_L(E) = \varphi(L')$. Thus we have proved

PROPOSITION (2.1). — *We have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Pic}_R(X) & \xrightarrow{\otimes L} & \mathscr{S}(X) \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathrm{KO}(X) & \xrightarrow{\varphi_L} & \mathbf{F}_2
\end{array}
$$

where $\mathrm{Pic}_R(X)$ *is the multiplicative group of real line-bundles over* X *(isomorphic to the additive group* $H^1(X, \mathbf{F}_2)$*) and* $\mathrm{Pic}_R(X) \to \mathrm{KO}(X)$ *is the natural inclusion into the multiplicative group of units of the ring* $\mathrm{KO}(X)$.

Since φ_L is an additive homomorphism while $\mathrm{Pic}_R(X) \to \mathrm{KO}(X)$ is multiplicative this Proposition really explains the algebraic nature of φ. We proceed to elucidate this further.

Since $\dim X = 2$ the augmentation ideal $\widetilde{\mathrm{KO}}(X)$ has cube equal to zero. This implies that the composite map

$$\alpha : H^1(X, \mathbf{F}_2) \cong \mathrm{Pic}_R(X) \to \mathrm{KO}(X)$$

is (affine) quadratic. Since φ_L is additive (and $\otimes L$ is bijective) this implies that φ is quadratic, which is the first assertion of Theorem 2. To prove the second part of Theorem 2 we have to show that the bilinear form on $H^1(X, \mathbf{F}_2)$ given by

$$(x, y) \mapsto \varphi_L[\alpha(x+y) - \alpha(x) - \alpha(y) + \alpha(o)]$$

coincides with the cup-product [identifying $H^2(X, \mathbf{F}_2)$ with $\mathbf{F}_2$]. This will follow from two lemmas.

LEMMA (2.2). — *Let* $u \in \mathrm{KO}(X)$ *be the pull back of the generator of* $\widetilde{\mathrm{KO}}(S^2) \cong \mathbf{F}_2$ *by a map* $X \to S^2$ *of degree one. Then*

$$\alpha(x+y) - \alpha(x) - \alpha(y) + \alpha(o) = (xy)u,$$

where $\alpha : H^1(X, \mathbf{F}_2) \to \mathrm{KO}(X)$ *and* $H^2(X, \mathbf{F}_2)$ *is identified with* $\mathbf{F}_2$.

LEMMA (2.3). — *Let* u *be the element defined in* (2.2). *Then* $\varphi_L(u) = 1$.

To prove (2.2) let $f : X \to S^1 \times S^1$ be a map such that $x = f^\star(a)$, $y = f^\star(b)$ where a, b come from generators of $H^1(S^1, \mathbf{F}_2)$ by the two projections. By naturality it is then enough to prove (2.2) with (X, x, y) replaced by $(S^1 \times S^1, a, b)$, but this is equivalent to saying that

$$\widetilde{KO}(S^1) \otimes \widetilde{KO}(S^1) \to \widetilde{KO}(S^2)$$

is an isomorphism (all groups being $\cong \mathbf{F}_2$), which is well-known (*see* for example [2]).

Remark. — It is not difficult to prove that the total Stiefel-Whitney class

$$\omega : \quad \widetilde{KO}(X) \to 1 \oplus H^1(X, \mathbf{F}_2) \oplus H^2(X, \mathbf{F}_2)$$

is in this case an isomorphism $\big[$of the additive group $\widetilde{KO}(X)$ onto the multiplicative group of the cohomology ring$\big]$. This implies (2.2).

To prove (2.3) we give another description of the element u. Let P be a holomorphic line-bundle on X corresponding to a point divisor. Regarded simply as continuous line-bundle P is then induced from the corresponding line-bundle Q on the 2-sphere S^2 by a map $X \to S^2$ of degree one. Since $Q - 2$ generates $\widetilde{KO}(S^2)$ — where we regard Q as a 2-dimensional real bundle — it follows that $u = P - 2 \in \widetilde{KO}(X)$. Since

$$\varphi_L(P - 2) = \varphi_L(P) - \varphi_L(2) = \varphi_L(P) \in \mathbf{F}_2$$

we have just to show that $\varphi_L(P) = 1$. Now since P is holomorphic $\bar{\partial}_L$ has a natural extension to $L \otimes_R P \cong L \otimes_C P \oplus L \otimes_C P^\star$: it coincides with the $\bar{\partial}$-operator of the holomorphic bundle $L \otimes_C P \oplus L \otimes_C P^\star$. Hence

$$\varphi_L(P) = \dim \Gamma(L \otimes_C P) + \dim \Gamma(L \otimes_C P^\star) \bmod 2.$$

By Serre duality we have (since $L^\star \otimes K \cong L$)

$$\dim \Gamma(L \otimes_C P^\star) = \dim H^1(X, L \otimes_C P)$$

and so

$$\varphi_L(P) \equiv \dim H^0(X, L \otimes_C P) - \dim H^1(X, L \otimes_C P) \bmod 2$$
$$\equiv \deg(L \otimes P) - g + 1 \quad \text{by Riemann-Roch}$$
$$\equiv g - g + 1 \equiv 1$$

as required. This completes the proof of Lemma (2.3) and hence of Theorem 2.

3. RELATION WITH SPIN-MANIFOLDS. — In this section we shall set the preceeding discussion of Riemann surfaces into the general context of spin-manifolds. We begin by recalling a few basic facts (*see* [7]).

Let $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the standard double covering and let $g \in \mathrm{H}^1(\mathrm{SO}(n), \mathbf{F}_2)$ be its corresponding cohomology class. Let X be an oriented Riemannian n-dimensional manifold, and let P be its principal tangent $\mathrm{SO}(n)$-bundle. Then a spin-structure on X is a double covering $Q \to P$ whose restriction to each fibre of P is the standard covering (of course isomorphic double coverings are regarded as defining the same spin-structure). Q is then a principal $\mathrm{Spin}(n)$-bundle over X. Considering the exact sequence

$$(3.1) \qquad 0 \to \mathrm{H}^1(X, \mathbf{F}_2) \to \mathrm{H}^1(P, \mathbf{F}_2) \to \mathrm{H}^1(\mathrm{SO}(n), \mathbf{F}_2) \xrightarrow{\delta} \mathrm{H}^2(X, \mathbf{F}_2)$$

arising from the fibration $\mathrm{SO}(n) \to P \to X$, we see that

(i) a spin-structure exists $\Leftrightarrow \delta(g) = 0$;

(ii) if $\delta(g) = 0$ the spin-structures are classified by a coset of $\mathrm{H}^1(X, \mathbf{F}_2)$ in $\mathrm{H}^1(P, \mathbf{F}_2)$.

The class $\delta(g)$ is the second Stiefel-Whitney class $\omega_2(X)$. Note that the Riemannian metric is not really essential in these considerations : we can consider the double covering of $\mathrm{GL}^+(n, \mathrm{R})$ instead of $\mathrm{Spin}(n)$.

Suppose now that X is an almost complex manifold so that the structure group of its principal $\mathrm{SO}(2n)$-bundle P reduces to $\mathrm{U}(n)$. Since the two homomorphisms

$$\mathrm{U}(n) \to \mathrm{SO}(2n), \qquad \mathrm{U}(n) \xrightarrow{\det} \mathrm{U}(1)$$

both induce isomorphism in $\mathrm{H}^1(\ , \mathbf{F}_2)$ it follows [using (3.1) and analogous sequences for $\mathrm{U}(n)$ and $\mathrm{U}(1)$] that the spin-structures on X correspond bijectively to those double coverings of the $\mathrm{U}(1)$-bundle $\det P$ which restrict to the squaring map $\mathrm{U}(1) \to \mathrm{U}(1)$ on each fibre. If X is a *complex* manifold with canonical line-bundle K it follows that the spin-structures on X correspond bijectively to isomorphism classes of pairs (L, α) where L is a continuous line-bundle and $\alpha : L^2 \to K$ is a continuous isomorphism. Now given α, L inherits a holomorphic structure from K (making α a holomorphic isomorphism). Conversely, if X is *compact*, the holomorphic structure on L uniquely determines α up to a constant ([7]) (on each component of X). Replacing α by $c\alpha$, where c is

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([7]) Because the holomorphic automorphisms of a line-bundle are just the non-zero holomorphic functions.

a non-zero constant just corresponds to the isomorphism $c^{\frac{1}{2}} : L \to L$. Thus we have

PROPOSITION (3.2). — *The spin-structures on a compact complex manifold correspond bijectively to the isomorphism classes of holomorphic line bundles* L *with* $L^2 \cong K$, *where* K *is the canonical line-bundle.*

Remark. — It is easy to check that the bijection of (3.2) is compatible with the natural action of $H^1(X, \mathbf{F}_2)$.

Proposition (3.2) shows that the set $\mathcal{S}(X)$ of square roots of K (for X a Riemann surface) has a natural generalization as the set of spin-structures on any manifold X with $\omega_2 = 0$. We shall now show that the function $\varphi : \mathcal{S}(X) \to \mathbf{F}_2$ of Theorem 2 also has a natural generalization provided $\dim X \equiv 2 \bmod 8$.

We begin by recalling a few facts about the Clifford algebras C_n of the quadratic form $-\sum_1^n x_i^2$ on $\mathbf{R}^n$ (*see* [2]). For $n = 8k + 2$, C_n is a matrix algebra over the quarternions $\mathbf{H} = \mathbf{C} + j\mathbf{C}$, and the even part C_n^0 is a matrix algebra over $\mathbf{C}$. Let M be an irreducible C_n-module (so the commuting algebra is $\mathbf{H}$) and decompose it in the form $M = M^0 \oplus M^1$ where M^0, M^1 are the (± 1)-eigenspaces of the operator ej where $e \in \mathbf{R}^n \subset C_n$ is a unit vector $(ej)^2 = e^2 j^2 = -1 . -1 = +1)$. This grading is independent of e and makes M a $\mathbf{F}_2$-graded C_n-module. M^0 and M^1 are irreducible C_n^0-modules and $x \mapsto jx$ establishes a C_n^0-module isomorphism between them. The complex structure on M^0 and M^1 is given by the element $\omega = e_1 e_2 \ldots e_n$, where the e_i are an orthonormal basis of $\mathbf{R}^n$. Note that ω is in the centre of C_n^0 but anti-commutes with the e_i so that $x \mapsto e_i x$ is an anti-linear map $M^0 \to M^1$. Restricting to $\mathrm{Spin}(n) \subset C_n^0$ the modules M^0, M^1 become representations of $\mathrm{Spin}(n)$: they are isomorphic complex representations.

Suppose now that X is a (Riemannian) spin-manifold of dimension $n = 8k + 2$, and let P be its principal $\mathrm{Spin}(n)$-bundle. Then form the associated complex vector bundle

$$E = P \times_{\mathrm{Spin}(n)} M = E^0 \oplus E^1$$

where

$$E^i = P \times_{\mathrm{Spin}(n)} M^i \qquad (i = 0, 1).$$

As explained in [4] the Dirac operator D is then defined acting on $C^\infty(E)$. It is an elliptic first-order differential operator defined by

$$Ds = \sum e_i (\partial_i s),$$

where $\partial_i s$ is the covariant of s in the direction e_i, $e_i(\quad)$ denotes Clifford multiplication and the e_i are an orthonormal basis of tangent vectors. D is (formally) self-adjoint and interchanges E^0 and E^1. Moreover, since $e_i \omega = - \omega e_i$, D is anti-linear. Now define

$$P: \quad C^\infty(E^0) \to C^\infty(E^0)$$

to be the composition jD. Since $jD = Dj$ and $j^2 = -1$ (and $jj^\star = 1$) it follows that $P^\star = -P$. Since j is complex-linear P is still anti-linear and of course P is also elliptic.

It is easy to check that, in the case of Riemann surfaces, this operator P coincides with that defined in paragraph 1. Exactly as in the proof of Theorem 1 [i. e. using Lemma (1.1)] it follows that $\dim_{\mathbf{C}} \mathrm{Ker} P \bmod 2$ is independent of the choice of metric and depends only on the spin-structure. Note that

$$\mathrm{Ker} P \cong \mathrm{Ker} D^0 = \mathrm{Ker}(D^0)^\star D^0 = H,$$

where D^0 denotes the restriction of D to E^0 : H is the space of harmonic spinors on X. Thus we have

PROPOSITION (3.3). — *On a (Riemannian) spin-manifold of dimension $8k + 2$ the harmonic spinors H form a complex vector space and $\dim_{\mathbf{C}} H \bmod 2$ is independent of the Riemannian metric.*

Thus if $\mathcal{S}(X)$ denotes the set of spin-structures on X we have a function $\varphi: \mathcal{S}(X) \to \mathbf{F}_2$ defined by $s \mapsto \dim_{\mathbf{C}} H_s \bmod 2$, where H_s denotes the harmonic spinors for the spin-structure s (and some metric). This generalizes the function φ of Theorem 2.

Just as in paragraph 2 we can extend φ to define a homomorphism

$$\varphi_s: \quad KO(X) \to \mathbf{F}_2$$

for each $s \in \mathcal{S}(X)$ and we have the analogue of Proposition (2.1). In the next section we shall use the results of [5] to derive more information about φ_s. In particular we shall prove Theorem 3.

4. **APPLICATIONS OF THE INDEX THEOREM.** — The index theorem of [3] has been extended in [5] and enables us to compute " $\bmod 2$ indices " of elliptic operators in terms of K-theory. In particular the homomorphism φ_s defined at the end of paragraph 3 coincides with the direct image homomorphism $f_!$ for spin-manifolds ([5], Theorem (3.3)], where f is the map X $\to$ point and we identify KO^{-2} (point) with $\mathbf{F}_2$. In particular it follows that $\varphi_s(1)$ is an *invariant of spin-cobordism*. Now in dimen-

sion 2 the spin-cobordism group has just two elements [7]. Since $\varphi(s) = \varphi_s(1)$ is not identically zero (for example take X an elliptic curve) it follows that we have.

PROPOSITION (4.1). — *For a Riemann surface X with spin-structure s we have $\varphi(s) = 0$ if and only if (X, s) is a spin-boundary.*

Using (4.1) we will now prove Theorem 3. We take a standard embedding $X \subset \mathbf{R}^3$ as a sphere with g handles. Then $X = \partial Y$, Y the interior, and we have the standard symplectic basis $(x_1, \ldots, x_g, y_1, \ldots, y_g)$ of $H^1(X, \mathbf{F}_2)$ in which $x_1, \ldots, x_g$ extend to elements of $H^1(Y, \mathbf{F}_2)$. We give X the spin-structure s induced by the spin-structure of $\mathbf{R}^3$. Identifying $H^1(X, \mathbf{F}_2)$ with $\mathfrak{S}(X)$ by means of s the function $\varphi : \mathfrak{S}(X) \to \mathbf{F}_2$ gives a function $\psi : H^1(X, \mathbf{F}_2) \to \mathbf{F}_2$. By (4.1) we have $\psi(0) = \psi(x_i) = 0$ $(i = 1, \ldots, g)$. Hence the quadratic function ψ has Arf invariant

$$\mathrm{A}(\psi) = \sum \psi(x_i)\, \psi(y_i) = 0.$$

This implies that ψ, and so φ, has $2^{g-1}(2^g + 1)$ zeros, proving Theorem 3. In view of (4.1) this may be rephrased as

THEOREM 3'. — *On a compact orientable surface of genus g there are precisely $2^{g-1}(2^g + 1)$ spin-structures which bound.*

Returning to the general case of a spin-manifold of dimension $n = 8k + 2$ we observe that the function $\varphi : \mathfrak{S}(X) \to \mathbf{F}_2$ is not in general quadratic. Since $(\widetilde{KO}(X))^{n+1} = 0$ it follows that φ is a polynomial of degree $\leq n$. If however we assume that $H^1(X, \mathbf{Z}) \to H^1(X, \mathbf{F}_2)$ is surjective (as happens for Riemann surfaces) then we can again prove that φ is quadratic. In view of Proposition (2.1) it is enough to prove that for any three real line-bundles ξ, η, ζ on X we have

(4.2) $(\xi - 1)(\eta - 1)(\zeta - 1) = 0.$

Now our hypothesis on $H^1(X, \mathbf{F}_2)$ implies that all line-bundles on X come from the line-bundles on S^1 by maps $X \to S^1$. Hence it is enough to prove that (4.2) holds when $X = S^1 \times S^1 \times S^1$ and ξ, η, ζ, come from S^1 by the three projections, i. e. that the product

$$\widetilde{KO}(S^1) \otimes \widetilde{KO}(S^1) \otimes \widetilde{KO}(S^1) \to \widetilde{KO}(S^3)$$

is zero — which is trivially true because $\widetilde{KO}(S^3) = 0$. Since φ is quadratic in this situation it is reasonable to ask for an explicit description

of it and in particular of its associated bilinear form, thus generalizing Theorems 2 and 3. To do this let us first write $\mathrm{Spin}(X) = f_!(1)$ for any spin-manifold of dimension n, where $f_! : \mathrm{KO}(X) \to \mathrm{KO}^{-n}(\text{point})$ is the direct-image homomorphism. This is zero unless $n \equiv 0, 1, 2$ or $4 \bmod 8$. For $n \equiv 2 \bmod 8$ it is the invariant we have been discussing. For $n = 1 \bmod 8$ it also has an interpretation as a $\bmod 2$ dimension of harmonic spinors ([5]; (3.1)]. For $n \equiv 0 \bmod 8$ it is equal to $\hat{A}(X)$, while for $n \equiv 4 \bmod 8$ it is equal to $\frac{1}{2}\hat{A}(X)$, where $\hat{A}$ is given by a certain polynomial in the Pontrjagin classes of X (see [4]). Then we have the following theorem :

THEOREM 5. — *Let* X *be a spin-manifold of dimension* $8k + 2$ *and assume that* $\mathrm{H}^1(X, \mathbf{Z}) \to \mathrm{H}^1(X, \mathbf{F}_2)$ *is surjective. Then the function* $\varphi : \mathcal{S}(X) \to \mathbf{F}_2$ *is quadratic. If we fix a spin-structure on X and use this to identify* $\mathcal{S}(X)$ *with* $\mathrm{H}^1(X, \mathbf{F}_2) = \mathrm{H}^1(X, \mathbf{Z}) \bmod 2$, *then in terms of a basis* $e_1, \ldots, e_m$ *of* $\mathrm{H}^1(X, \mathbf{Z})$ φ *is given by*

$$\varphi\left(\sum x_i e_i\right) = \mathrm{Spin}(X) + \sum x_i \mathrm{Spin}(Y_i) + \sum_{i<j} x_i x_j \mathrm{Spin}(Y_{ij}) \bmod 2,$$

where Y_i *is a submanifold dual to* e_i, $Y_{ij} = Y_i \cap Y_j$ *(assuming transversal intersections) and* Y_i, Y_{ij} *are given the induced spin-structure. The bilinear form* $\sum x_i x_j \mathrm{Spin}(Y_{ij})$ *can be given cohomologically by*

$$a, b \mapsto \hat{\mathcal{A}}_{8k}(X) \, ab[X], \qquad a, b \in \mathrm{H}^1(X, \mathbf{Z}).$$

Proof. — The quadratic nature of φ has already been proved. For the next part let $a, b \in \mathrm{H}^1(X, \mathbf{Z})$, let A, B be transversal submanifolds representing a, b and denote by α, β the line-bundles defined by a, b. If j^A denotes the inclusion map $A \to X$, then $\alpha - 1 = \eta j_!^A(1)$, where η is the generator of $\mathrm{KO}^{-1}(\text{point})$: this follows by considering the map $X \to S^1$ corresponding to a. Similarly $\beta - 1 = \eta j_!^B(1)$ and $(\alpha - 1)(\beta - 1) = \eta^2 j_!^{AB}(1)$, where $j^B : B \to X$, $j^{AB} : A \cap B \to X$ are the inclusions. Hence ([8]), if f^X denotes the map $X \to$ point, etc.

$$\begin{aligned}
\varphi(a+b) &= f_!^X(\alpha\beta) \\
&= f_!^X(1) + f_!^X(\alpha - 1) + f_!^X(\beta - 1) + f_!^X((\alpha - 1)(\beta - 1)) \\
&= f_!^X(1) + \eta f_!^A(1) + \eta f_!^B(1) + \eta^2 f_!^{AB}(1) \in \mathrm{KO}^{-2}(\text{point}) \\
&= \mathrm{Spin}(X) + \mathrm{Spin}(A) + \mathrm{Spin}(B) + \mathrm{Spin}(A \cap B) \in \mathbf{F}_2.
\end{aligned}$$

([8]) We use the functoriality of the direct image homomorphism, so that $f_!^A = f_!^X j_!^A$.

Since φ is quadratic this formula is equivalent to the one in terms of a basis. For the last part we have

$$\mathrm{Spin}\,(A \cap B) = \hat{\mathfrak{A}}_{8k}\,(A \cap B)\,[A \cap B]$$

$$= \hat{\mathfrak{A}}_{8k}\,(X)\,[A \cap B] \quad \text{since } A \cap B \text{ has trivial normal bundle}$$

$$= \hat{\mathfrak{A}}_{8k}\,(X)\,ab\,[X].$$

Remarks. — 1. The proof shows that the rational class $\hat{\mathfrak{A}}_{8k}(X)\,ab$ is always integral, so that it may be reduced modulo 2.

2. If $\dim X \equiv 0 \bmod 4$, then the cohomological formula shows that $\mathrm{Spin}\,X$ is independent of the choice of spin-structure. For $\dim X \equiv 2 \bmod 8$, Theorem 5 (and in particular Theorem 2) shows that this is not so.

3. The bilinear form in Theorem 5 may be degenerate. An example is given by taking $X = Y \times Z$ where Z is a Riemann surface and Y is a spin-manifold of dimension $8k$ with $H^1(Y, \mathbf{F}_2) \neq 0$. In fact the spin number is multiplicative so that

$$\mathrm{Spin}\,(X) = \mathrm{Spin}\,(Y) \times \mathrm{Spin}\,(Z)$$

and hence this is independent of the choice of spin structure in Y (Remark 2). Thus rank $\varphi = 2$ genus $Z < \dim H^1(X, \mathbf{F}_2)$.

5. INVARIANT SPIN-STRUCTURES. — As we have seen the set $\mathcal{S}(X)$ of spin-structures on a Riemann surface is an affine space over $\mathbf{F}_2$ endowed with a non-degenerate quadratic function. Rather surprisingly such an algebraic structure has the fixed-point property, namely we have the following algebraic lemma [9] :

LEMMA (5.1). — *Let* V *be a finite-dimensional vector space over the field* $\mathbf{F}_2$ *and let* $\varphi : V \to \mathbf{F}_2$ *be a quadratic function whose associated bilinear form* $H(x, y)$ *is non-degenerate. Then any affine transformation* $x \mapsto Ax + B$ *of* V *which preserves the function* φ *has a fixed point.*

Proof. — By hypothesis we have

$$\varphi\,(x) = \varphi\,(Ax + B) = \varphi\,(Ax) + \varphi\,(B) + H\,(Ax, B).$$

Putting $x = 0$ we get $\varphi(B) = 0$ and so

(1) $$\varphi\,(x) = \varphi\,(Ax) + H\,(Ax, B).$$

This implies ([10]) $H(x, y) = H(Ax, Ay)$ and so $1 = A^{\star}A$, where $A^{\star}$ is the adjoint of A with respect to the non-degenerate inner product $H(x, y)$. Hence

$$A^{\star}x = x \;\Rightarrow\; Ax = x \;\Rightarrow\; H(x, B) = 0 \quad \text{by (i).}$$

Thus B is orthogonal to $\mathrm{Ker}(A - I)^{\star}$ and so lies in the image of $A - I$. Thus there exists $y \in V$ with $B = (A - I)y$ or equivalently $y = Ay + B$. This is the required fixed-point.

Applying this to Riemann surfaces we deduce

PROPOSITION (5.2). — *Any orientation preserving diffeomorphism of a compact oriented surface leaves fixed some spin-structure.*

Now a diffeomorphism of X corresponds to a differentiable fibration over the circle with X as fibre. Hence this Proposition — together with (3.2) — implies Theorem 4. In fact it implies a stronger version in which the family X_t need only vary differentiably with t.

Remarks. — 1. We can apply (5.1) to a spin manifold X of dimension $8k + 2$ provided it satisfies the hypothesis of Theorem 5 and provided also that the bilinear form of Theorem 5 is non-degenerate. We then obtain a result like (5.2).

2. Since Mumford in [8] has now established the analogue of Theorem 2 for algebraic curves over algebraically closed fields (of characteristic $\neq 2$), we can also apply Lemma (5.1) to that context. Thus, let k be a field (of characteristic $\neq 2$) such that every finite separable extension is cyclic and let X be an algebraic curve over k, then the canonical line bundle K of X has a square root invariant (up to isomorphism) by the Galois group of k. If the Brauer group of k is trivial (or if X has a rational point over k) this square root can be defined over k. This applies in particular to a finite field (in which case the result was already known, *cf.* [9]; p. 291, and to the field of power series $\displaystyle\sum_{-N}^{\infty} a_n t^n$ convergent near $t = 0$ (which is essentially the case of Theorem 4).

The case when $k = \mathbf{R}$ is interesting : this involves a Riemann surface with a complex conjugation, and the existence of an invariant square root of K can also be proved analytically by extending (5.2) to orientation reversing diffeomorphisms — the essential point being that the homomorphism $\varphi_L : KO(X) \to \mathbf{F}_2$ of paragraph 2 is independent of the

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([10]) This is just verifying (what we have already observed) that the bilinear form is canonically associated to the affine quadratic function.

orientation of X. Since the Brauer group of **R** is isomorphic to $\mathbf{F}_2$ an invariant square root of K, on a curve without real points, may not be definable over **R**. In fact it is not difficult to show that a line bundle L defined over **R** has, in this case, a square root defined also over **R** if and only if

$$\omega_2(L') = 0 \quad \text{in } H^2(X', \mathbf{F}_2),$$

where X' is the non-orientable surface representing the pairs of complex conjugate points of X, L' is the non-oriented $\mathbf{R}^2$-bundle over X' defined by L and ω_2 is the second Stiefel-Whitney class. For L = K, L' = K' is the tangent bundle of X', hence $\omega_2(K')$ is equal to the Euler number $e \bmod 2$ and so $\omega_2(K') = e(X') = \frac{1}{2} e(X) = 1 - g \bmod 2$, where g is the genus of X and we identify $H^2(X', \mathbf{F}_2)$ with $\mathbf{F}_2$. Thus, on a real curve X without real points the canonical line bundle K has a real square root if and only if g is *odd*. One can also show (by using a hyperelliptic curve or an embedding in $\mathbf{R}^3$) that, for odd g, there are 2^g real square roots of K and that the quadratic function φ of Theorem 2 takes the value zero for just half of these square roots. For example (taking $g = 3$) a real non-singular plane quartic with no real points has precisely four real bitangents.

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