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Stochastic fractional partial differential equations driven by Poisson white noise

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Abstract

We study a stochastic fractional partial differential equations of order $\alpha > 1$ driven by a compensated Poisson measure. We prove existence and uniqueness of the solution and we study the regularity of its trajectories.

Équations aux dérivées partielles fractionnaires stochastiques dirigées par un bruit poissonnien

Résumé

On étudie une équation aux dérivées partielles stochastiques fractionnaires d'ordre $\alpha > 1$ dirigée par une mesure de Poisson compensée. On montre l'existence et l'unicité de la solution et on étudie la régularité de ses trajectoires.

1. Introduction

In recent years fractional calculus knew a great progress considering their applications in different fields of science, including numerical analysis, physics, engineering, biology, economics and finance.
Partial differential equations is one of the mathematical domains where the fractional calculus is strongly used (see Podlubny [8]). However, few publications treat stochastic partial differential equations involving fractional derivatives. Most of them investigate evolution type equations, driven by a fractional power of the Laplacian (see [3], [10]). These operators generate symmetric stable semigroups when the order of derivation is less than 2.
Our idea is inspired by the paper of Debbi and Dozzi [5] in which the authors studied a nonlinear stochastic fractional partial differential equations of high order containing also derivatives of entire order and perturbed by

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Gaussian white noise.
The aim of this paper is to generalize the result of [1] to nonlinear stochastic fractional partial differential equations. We study the existence, uniqueness and regularity of the solution of the following one-dimensional stochastic fractional partial differential equation, formally given by

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x,\omega) &= D_\alpha^\delta u(t,x,\omega) + f(t,u(t,x)) + \int_{\mathcal{U}} g(t,u(t,x),z)\tilde{N}(dt,dx,dz)\\ u(0,x) &= u_0(x)\end{aligned} \tag{1.1}$$

for $t \in (0,+\infty)$, $x \in \mathbb{R}$, where $(\mathcal{U}, \mathcal{B}(\mathcal{U}), q)$ is a σ finite measure space, D_δ^α is the fractional differential operator with respect to the spacial variable, $f : (0,+\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ and $g : (0,+\infty)\times\mathbb{R}\times\mathcal{U} \longrightarrow \mathbb{R}$ are measurable.
One rigorous formulations of the equation (1.1) can be given by the following integral equation

$$\begin{aligned}u(t,x) = &\int_{\mathbb{R}} G_\alpha(t,x-y)u_0(y)dy\\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)f(s,u(s,y))dyds\\ &+ \int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} G_\alpha(t-s,x-y)g(s,u(s-,y),z)\tilde{N}(ds,dy,dz)\end{aligned} \tag{1.2}$$

for $t \in [0,+\infty)$ and $x \in \mathbb{R}$, where $\{G_\alpha(t,x), x \in \mathbb{R},\ t \in [0,+\infty)\}$ stands for the fundamental solution of the operator D_δ^α.
Some equations similar to equation (1.1) are studied also in [6] and [2], but with the Laplacian operator in place of the operator D_δ^α.

We assume that the functions f and g satisfy the following growth and Lipschitz conditions:
(H)
For all $T > 0$, there exists a constant $K = K_T$, such that for all $0 \le t \le T,\ x, y \in \mathbb{R},\ z \in \mathcal{U}$

- $|f(t,x)|^2 + \int_{\mathcal{U}} |g(t,x,z)|^2 q(dz) \le K_T(1+|x|^2)$
- $|f(t,x)-f(t,y)|^2 + \int_{\mathcal{U}} |g(t,x,z)-g(t,y,z)|^2 q(dz) \le K_T|x-y|^2$.

The remaining of the paper is structured as follows. In Section 2, we give the definitions of stochastic integral with respect to $\tilde{N}(ds,dy,dz)$ and the derivative operator D_δ^α, then we recall some inequalities satisfied by the fundamental solution G_α and we precise the notion of solutions

of the equation (1.1). Section 3 is dedicated to the proof of existence and uniqueness of the solution. In Section 4 we prove the spatial Hölder continuity of the solution.

2. **Preliminaries and definitions**

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let $(\mathcal{U}, \mathcal{B}(\mathcal{U}), q)$ be a σ finite measure space and let N be a Poisson measure on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{U}$ with intensity measure $\nu(ds, dx, dz) = dsdxq(dz)$. The compensated Poisson measure is denoted by $\tilde{N} = N - \nu$.
Now, we define the stochastic integrals we will use.

2.1. **Stochastic integral with respect to** $\tilde{N}(ds, dy, dz)$

Let us introduce the following class:

$$F_{pred} := \{h(t, y, z, \omega) : h \text{ is } \mathcal{F}_t \text{ predictable and } \forall t > 0, \\ E \int_0^t \int_{\mathbb{R}} \int_{\mathcal{U}} |h(s, y, z)|^2 \nu(ds, dy, dz) < \infty\Big\}$$

it is known (see Ikeda and Watanabe [7]) that for any $t > 0$, the stochastic integral $\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} h(s, y, z)\tilde{N}(ds, dy, dz)$, for $h \in F_{pred}$, can be well defined. The stochastic integral has the following isometry property

$$E\left\{\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} h(s, y, z)\tilde{N}(ds, dy, dz)\right\}^2 = \\ E \int_0^t \int_{\mathbb{R}} \int_{\mathcal{U}} |h(s, y, z)|^2 \nu(ds, dy, dz)\,.$$

Thus for any $t > 0$, $\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} h(s, y, z)\tilde{N}(ds, dy, dz) \in L^2(\Omega)$.

This integral can be extended to a more general class F of integrands without the predictable property, in the following manner :

A function h is said to be of class F if is $\mathcal{F}_t$ adapted, and there exists a sequence $h_n \in F_{pred}$ such that for any $t > 0$

$$E \int_0^t \int_{\mathbb{R}} \int_{\mathcal{U}} |h_n(s, y, z) - h(s, y, z)|^2 \nu(ds, dy, dz) \to 0 \quad \text{as } n \to \infty$$

For $h \in F$, for any $t > 0$, $\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} h(s,y,z)\tilde{N}(ds,dy,dz)$ is defined as the $L^2(\Omega)$-limit of the following Cauchy sequence

$$\{\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} h_n(s,y,z)\tilde{N}(ds,dy,dz)\}_{n\in\mathbb{N}}$$

(see Albeverio *et al.* [1] for more details).

2.2. **Definition of the operator** D_δ^α

The fractional differential operator D_δ^α is an extension of the inverse of the generalized Riesz-Feller Potential when $\alpha > 2$. It is given for $\alpha > 0$ by

Definition 2.1. The fractional differential D_δ^α is given by

$$D_\delta^\alpha \varphi = \mathcal{F}^{-1}\{\psi_\alpha . \mathcal{F}\varphi\}$$

where

$$\psi_\alpha(\lambda) = -|\lambda|^\alpha e^{-i\delta\frac{\pi}{2}sgn(\lambda)}.$$

$|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, $[\alpha]_2$ is the largest even integer less or equal to α, and $\delta = 0$ when $\alpha \in 2\mathbb{N}+1$. $\mathcal{F}$ (respectively $\mathcal{F}^{-1}$) is the Fourier (respectively Fourier inverse) transform.
The operator D_δ^α is a closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. It is selfadjoint only when $\delta = 0$ and in this case, it coincides with the fractional power of the Laplacian. When $\alpha = 2$ it is the Laplacian itself (see [4, 5]).

The Green function $G_\alpha(t,x)$ associated to the equation (1.1) is the fundamental solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}G(t,x) = D_\alpha^\delta G(t,x), \;\; t > 0, \; x \in \mathbb{R}, \\ G(0,x) = \delta_0(x). \end{cases}$$

Using Fourier's calculus we obtain

$$G_\alpha(t,x) = \mathcal{F}^{-1}\{e^{\delta\psi_\alpha(\lambda)t}; x\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\lambda x - t|\lambda|^\alpha e^{-i\delta\frac{\pi}{2}sgn(\lambda)})d\lambda.$$

Let us recall some well-known properties, (see [5]), of the Green kernel G_α which will be used later.

Lemma 2.2. *For* $\alpha \in (0, +\infty)$

(1) $\int_{\mathbb{R}} G_\alpha(t,x)dx = 1$.

(2) $G_\alpha(t,x)$ *satisfy the semigroup property, i.e. for* $0 < s < t$

$$G_\alpha(t+s,x) = \int_{\mathbb{R}} G_\alpha(t,y)G_\alpha(s,x-y)dy.$$

(3) *For* $0 < \alpha \leq 2$, *the function* $G_\alpha(t,.)$ *is the density of a Lévy stable process in time* t.

(4) *For fixed* t, $G_\alpha(t,.) \in S^\infty = \{f \in C^\infty$ *and* $\frac{\partial^k}{\partial x^k} f$ *is bounded and tends to zero when* $|x|$ *tend to* ∞, $\forall k \in \mathbb{N}\}$.

(5) $\frac{\partial^n G_\alpha}{\partial x^n}(t,x) = t^{-\frac{n+1}{\alpha}} \frac{\partial^n G_\alpha}{\partial \xi^n}(1,\xi)\big|_{\xi = t^{-\frac{1}{\alpha}}x}$, *for all* $n \geq 0$.

(6) *Let* $\alpha \in (1,+\infty)$. *Then there exists a constant* K_α *such that*

- $|G_\alpha(1,x)| \leq K_\alpha(1+|x|^{1+\alpha})^{-1}$,
- $|G^n_\alpha(1,x)| \leq K_\alpha \frac{1+|x|^{\alpha+n-1}}{(1+|x|^{\alpha+n})^2}$.

(7) *For* $\alpha \in (1,+\infty)$, *for any fixed* $n \in \mathbb{N}$, *and* $T \geq 0$, *for* γ *such that* $\frac{1}{\alpha+n+1} < \gamma < \frac{\alpha+1}{n+1}$,

$$\int_0^T \int_{-\infty}^{+\infty} |\frac{\partial^n G_\alpha}{\partial y^n}(s,y)|^\gamma dyds < \infty.$$

In this paper, we restrict ourselves to the case $\alpha > 1$ in order that $\int_0^T \int_{\mathbb{R}} G^2_\alpha(t,x)dxds < \infty$ (see Lemma 2.2, property (7)).
Let us now give a precise formulation of solutions for Equation (1.1).

Definition 2.3. Let $0 < T < \infty$. A measurable stochastic field

$$\{u(t,x),\ t \in [0,T], x \in \mathbb{R}\}$$

is said to be a mild solution of the equation (1.1) on $[0,T]$ if

- $u(t,x)$ is adapted for every x.
- $\{u(t,x)\}$, as a family of $L^2(\Omega,\mathcal{F},P)$-valued random variables, is right continuous and has left limits in the variable $t \in [0,T]$, namely,

$$u(t-,x,.) = L^2(\Omega) - \lim_{s \uparrow t} u(s,x,.), \quad t \in [0,T].$$

In this paper we simply call such a u *modified càdlàg* in t (after the French acronym).

- Equation (1.2) holds a.s. for all $t \in [0,T]$ and $x \in \mathbb{R}$.

A solution u on $[0,T]$ is said to be $L^p(\Omega)$- bounded if

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} E|u(t,x)|^p < \infty.$$

All positive constants appearing in this paper are called C. They may change from one line to the next one.

3. **Existence and uniqueness**

The main result of this section is the following

Theorem 3.1. *Let $p \geq 2$ and $T > 0$ be fixed. Under condition (H) and the assumption that the initial condition u_0 is $L^p(\Omega)$ bounded, equation (1.1) has a unique mild solution u on $[0,T]$ which is $L^p(\Omega)$ bounded.*

For the proof we need the following lemma.
Fix $T > 0$ and let

$$\begin{aligned}\mathcal{E} \;:=\; \Big\{\{v(t,x), t\in[0,T], x\in\mathbb{R}\} : v \text{ is measurable }, \mathcal{F}_t \text{ adapted,}\\ \text{modified càdlàg in } t \text{ such that, } \sup_{t\leq T}\sup_{x\in\mathbb{R}} E|v(t,x)|^p < \infty\Big\}.\end{aligned}$$

Lemma 3.2. *Let $v \in \mathcal{E}$ and set*

$$X(t,x) := \int_0^{t^+}\int_{\mathbb{R}}\int_{\mathcal{U}} G_\alpha(t-s,x-y)g(s,v(s-,y),z)\tilde{N}(ds,dy,dz),$$

$$Y(t,x) := \int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)f(s,v(s,y))dyds.$$

Then X and Y belong to $\mathcal{E}$.

Proof. Let us first show that X is well defined.
For any fixed $t \in [0,T]$, let, for $(s,y,z) \in [0,t]\times\mathbb{R}\times\mathcal{U}$,

$$h_{t,x}(s,y,z) := G_\alpha(t-s,x-y)g(s,v(s-,y),z),$$

$$v^n(s,y) := v(0,y)1_{\{0\}}(s) + \sum_{k=0}^{2^n-1} v(\frac{kt}{2^n},y)1_{(\frac{kt}{2^n},\frac{(k+1)t}{2^n}]}(s),\ n\in\mathbb{N},$$

and

$$h^n_{t,x}(s,y,z) := G_\alpha(t-s,x-y)g(s,v^n(s,y),z).$$

Clearly v^n is $\mathcal{F}_s$ predictable, so h^n is also $\mathcal{F}_s$ predictable. Let us show that $h^n \in F_{pred}$.
By condition(H) we have :

$$\begin{aligned} &E\int_0^t\int_{\mathbb{R}}\int_{\mathcal{U}}|h^n_{t,x}(s,y,z)|^2\nu(ds,dy,dz)\\ &\leq K_T\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)^2(E|v^n(s,y)|^2+1)dyds<\infty. \end{aligned}$$

Thus $h^n_{t,x} \in F_{pred}$. On the other hand, by the condition (H) we have

$$\begin{aligned} &E\int_0^t\int_{\mathbb{R}}\int_{\mathcal{U}}|h^n_{t,x}(s,y,z)-h_{t,x}(s,y,z)|^2\nu(ds,dy,dz)\\ &\leq K_T\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)^2E|v^n(s,y)-v(s-,y)|^2dsdy \end{aligned}$$

By the definition of v^n and the fact that v is modified càdlàg we have

$$E|v^n(s,y)-v(s-,y)|^2 \underset{n\to\infty}{\longrightarrow} 0, \quad \text{for all } (s,y)\in[0,t]\times\mathbb{R}$$

and

$$E|v^n(s,y)-v(s-,y)|^2 \leq 4\sup_{r\leq T}\sup_{x\in\mathbb{R}}E|v(r,x)|^2,\ \forall(s,y,n)\in[0,t]\times\mathbb{R}\times\mathbb{N}.$$

Furthermore by Lemma 2.2, property (7), we have

$$\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)^2dsdy<\infty.$$

Then by the Lebesgue's dominated convergence theorem we obtain that

$$\int_0^t\int_{\mathbb{R}}G_\alpha(t-s,x-y)^2E|v^n(s,y)-v(s-,y)|^2dsdy \underset{n\to\infty}{\longrightarrow} 0\,.$$

Hence $h_{t,x}\in F$ for any fixed $t\in[0,T]$, so the integral
$\int_0^{t^+}\int_{\mathbb{R}}\int_{\mathcal{U}}G_\alpha(t-s,x-y)g(s,v(s-,y),z)\tilde{N}(ds,dy,dz)$ is well defined.
Using the same techniques, as in [1], we show that X is $\mathcal{F}_t$ adapted and modified càdlàg.

Further more, by Burkholder's inequality and condition (H), we have

$$\begin{aligned} E|X(t,x)|^p &\leq CE(\int_0^t \int_{\mathbb{R}} \int_{\mathcal{U}} G_\alpha^2(t-s,x-y) g^2(s,v(s-,y),z) dsdyq(dz))^{\frac{p}{2}} \\ &\leq CE(\int_0^t \int_{\mathbb{R}} G_\alpha^2(t-s,x-y)(1+v^2(s-,y)dsdy)^{\frac{p}{2}} \\ &\leq C(1+\sup_{s\leq T}\sup_{x\in\mathbb{R}} E|v(s,x)|^p) \int_0^T \int_{\mathbb{R}} G_\alpha^2(t-s,x-y)dsdy)^{\frac{p}{2}} \end{aligned}$$

By Lemma 2.2, property (7), the last integral is finite. Then

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} E|X(t,x)|^p) < \infty$$

Finally $X \in \mathcal{E}$.
It is well known that Y is measurable $\mathcal{F}_t$ adapted and modified càdlàg.
By Hölder inequality with respect to the measure $|G_\alpha(t-s,x-y)|dsdy$ and the growth assumption on f, we have

$$\begin{aligned} E|Y(t,x)|^p &\leq E(\int_0^t \int_{\mathbb{R}} |G_\alpha(t-s,x-y)| f^p(s,v(s,y))dsdy) \\ &\quad \times \left\{ \int_0^t \int_{\mathbb{R}} |G_\alpha(t-s,x-y)|dsdy \right\}^{p-1} \\ &\leq CE \int_0^t \int_{\mathbb{R}} |G_\alpha(t-s,x-y)|(1+v^p(s,y))dsdy \\ &\leq C(1+\sup_{s\leq T}\sup_{x\in\mathbb{R}} E|v(s,x)|^p) \int_0^T \int_{\mathbb{R}} |G_\alpha(t-s,x-y)|dsdy. \end{aligned}$$

So

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} E|Y(t,x)|^p) < \infty.$$

and finally $Y \in \mathcal{E}$. □

Proof of Theorem 3.1.
• Proof of the existence: Let the sequence $(u_n, n \geq 1)$ be given by:

$$u_1(t,x) = \int_{\mathbb{R}} G_\alpha(t,x-y)u_0(y)dy$$

$$\begin{aligned} u_{n+1}(t,x) &= u_1(t,x) + \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) f(s,u_n(s,y)) dyds \\ &\quad + \int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} G_\alpha(t-s,x-y) g(s,u_n(s-,y),z) \tilde{N}(ds,dy,dz) \end{aligned}$$

It is easy to see that $u_1 \in \mathcal{E}$, so by Lemma 3.2, we have for all $n \geq 1, u_n \in \mathcal{E}$. Let

$$F_n(t,x) := E|u_{n+1}(t,x) - u_n(t,x)|^p,$$

and

$$H_n(t) := \sup_{s \leq t} \sup_{x \in \mathbb{R}} F_n(s,x).$$

we have $\forall t \in [0,T], \forall x \in \mathbb{R}$

$$F_n(t,x) \leq 2^{p-1}[A_n(t,x) + B_n(t,x)] \tag{3.1}$$

with

$$A_n(t,x) := E\left(\int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)(f(s,u_n(s,y)) - f(s,u_{n-1}(s,y)))dyds \right)^p$$

and

$$\begin{aligned} B_n(t,x) := E\Big\{ &\int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} G_\alpha(t-s,x-y) \\ &(g(s,u_n(s-,y),z) - g(s,u_{n-1}(s-,y),z)) \tilde{N}(ds,dy,dz) \Big\}^p. \end{aligned}$$

By condition (H) and Hölder inequality with respect to the measure $|G_\alpha(t-s,x-y)|dsdy$, we obtain

$$\begin{aligned} A_n(t,x) &\leq CE\Big(\int_0^t \int_{\mathbb{R}} |G_\alpha(t-s,x-y)||(u_n - u_{n-1})(s,y)|^p dyds\Big) \\ &\qquad \times \left(\int_0^t \int_{\mathbb{R}} |G_\alpha(t-s,x-y)| dyds \right)^{p-1}. \\ &\leq C \int_0^t \sup_{y \in \mathbb{R}} E|(u_n - u_{n-1})(s,y)|^p \Big(\int_{\mathbb{R}} |G_\alpha(t-s,x-y)| dy \Big) ds \\ &\leq C \int_0^t H_{n-1}(s) ds. \end{aligned} \tag{3.2}$$

By condition (H), Burkholder inequality and Hölder inequality with respect to the measure $G_\alpha^2(t-s,x-y)dsdy$ we obtain

$$\begin{aligned} B_n(t,x) &\leq CE\Big(\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)|u_n-u_{n-1}(s-,y)|^2dsdy\Big)^{\frac{p}{2}} \\ &\leq C\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)E|u_n-u_{n-1}(s-,y)|^p dsdy \\ &\qquad\qquad \times\Big(\int_0^t\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)dsdy\Big)^{\frac{p}{2}-1} \\ &\leq C\int_0^t \sup_{y\in\mathbb{R}}|(u_n-u_{n-1})(s,y)|^p\Big(\int_{\mathbb{R}} G_\alpha^2(t-s,x-y)dy\Big)ds \\ &\leq C\int_0^t (t-s)^{-\frac{1}{\alpha}}H_{n-1}(s)ds. \end{aligned} \tag{3.3}$$

Hence, by (3.1), (3.2), (3.3), we obtain that

$$H_n(t)\leq C\int_0^t (t-s)^{-\frac{1}{\alpha}}H_{n-1}(s)ds$$

for some positive constant C.

By Lemma 3.3 in [9], the series $\sum_{n\geq 1}(H_n(t))^{\frac{1}{p}}$ converges uniformly on $[0,T]$. Hence the sequence (u_n) converges in $L^p(\Omega)$ uniformly on $[0,T]\times\mathbb{R}$. Let

$$u(t,x)=L^p(\Omega)-\lim_n u_n(t,x)$$

It is easy to see that $u\in\mathcal{E}$ and moreover u satisfies equation (1.2). Thus u is a solution to equation (1.1) on $[0,T]$.

• Proof of the uniqueness: Let u and v be two mild solutions of the equation (1.1) on $[0,T]$ and set

$$H(t)=\sup_{s\leq t}\sup_{x\in\mathbb{R}} E|u(s,x)-v(s,x)|^2.$$

By the same argument as in the proof of existence, we get :

$$H(t)\leq C\int_0^t (t-s)^{-\frac{1}{\alpha}}H(s)ds.$$

Applying Gronwall Lemma, we obtain that $H(T)=0$. This clearly implies that $u(t,x,.)=v(t,x,.)$ a.s. □

4. Regularity of the solution

In this section we give the spatial regularity of the solution. This regularity is based on the following lemma proved in [5].

Lemma 4.1. *For $\frac{\alpha+1}{2} < \gamma < \alpha + 1$, we have*

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} |G_\alpha(v, 1+z) - G_\alpha(v,z)|^\gamma dz dv < \infty.$$

Theorem 4.2. *Suppose that u_0 is $L^p(\Omega)$ bounded for all $p \geq 2$ and let u denote the mild solution of the equation (1.1) on $[0,T]$, then we have: For $\alpha < 3$ and for fixed $t > 0$ the process $\{u(t,x), x \in \mathbb{R}\}$ has Hölder continuous trajectories with exponent $\frac{\alpha-1}{2} - \epsilon$, for any $\epsilon > 0$, P-a.s.*

Proof. The proof is similar to one given by Debbi and Dozzi in [5]. We have

$$\begin{aligned} u(t,x) = & \\ & \int_{\mathbb{R}} G_\alpha(t, x-y) u_0(y) dy + \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) f(s, u(s,y)) dy ds \\ & \quad + \int_0^{t^+} \int_{\mathbb{R}} \int_{\mathcal{U}} G_\alpha(t-s, x-y) g(s, u(s-,y), z) \tilde{N}(ds, dy, dz) \\ & = u_(t,x) + v(t,x) + w(t,x)\,. \end{aligned}$$

It is easy to see that u_1 is smooth function with respect to t and to x. By using Lemma 2.2, property (4), we can show that $x \to v(t,x)$ is a smooth function, so it is sufficient to estimate the spatial regularity of w. By Burkholder-Davis-Gundy inequality, condition (H) and the fact the solution is $L^p(\Omega)$ bounded, we have:

$$\begin{aligned} & E|w(t+h,x) - w(t,x)|^p \\ & \leq C(1 + \sup_{[0,T]\times\mathbb{R}} E|u(t,x)|^p) \times \{\int_0^t \int_{\mathbb{R}} |G_\alpha(v, y+h) - G_\alpha(v,y)|^2 dy dv\}^{\frac{p}{2}} \\ & \qquad \leq C h^{p\frac{\alpha-1}{2\alpha}} \{\int_0^{+\infty} \int_{-\infty}^{+\infty} |G_\alpha(v, 1+z) - G_\alpha(v,z)|^2\}^{\frac{p}{2}} dz dv\,. \end{aligned}$$

By Lemma 4.1, the last integral converge (we have taken $\alpha < 3$ in order that $\frac{\alpha+1}{2} < 2 < \alpha + 1$).
Hence,

$$E|w(t+h,x) - w(t,x)|^p \leq C h^{p\frac{\alpha-1}{2\alpha}}$$

we can choose p as large as we please, so that the result follows from Kolmogorov's Theorem. □

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