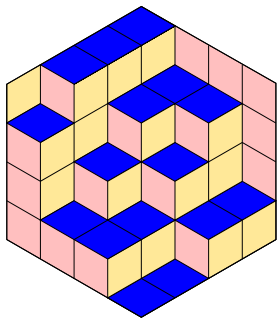

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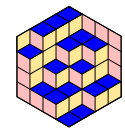

Principal specializations of Schubert polynomials in classical types

Eric Marberg & Brendan Pawlowski

Abstract There is a remarkable formula for the principal specialization of a type A Schubert polynomial as a weighted sum over reduced words. Taking appropriate limits transforms this to an identity for the backstable Schubert polynomials recently introduced by Lam, Lee, and Shimozono. This note identifies some analogues of the latter formula for principal specializations of Schubert polynomials in classical types B, C, and D. We also describe some more general identities for Grothendieck polynomials. As a related application, we derive a simple proof of a pipe dream formula for involution Grothendieck polynomials.

1. Introduction

There is a remarkable formula for the principal specialization $\mathfrak{S}_w(1, q, q^2, \ldots, q^{n-1})$ of a (type A) Schubert polynomial as a weighted sum over reduced words. Originally a conjecture of Macdonald [11], this identity was first proved algebraically by Fomin and Stanley [6]. Billey, Holroyd, and Young [2, 16] have recently found the first bijective proof of Macdonald's conjecture.

In this note we identify some apparently new analogues of Macdonald's identity for the principal specializations of Schubert polynomials in other classical types. Our methods are based on the algebraic techniques of Fomin and Stanley and will also lead to a simple proof of (a K-theoretic generalization of) the main result of [8].

To state our main theorems we need to recall a few definitions. Throughout, we let x_i for $i \in \mathbb{Z}$ be commuting indeterminates. We use the term *word* to mean a finite sequence $a_1 a_2 \cdots a_p$ whose letters belong to some totally ordered alphabet. This alphabet will usually consist of the integers $\mathbb{Z}$ with their usual ordering, and in any case will always contain $(\mathbb{Z}, <)$ as a subposet.

Definition 1.1. *A* bounded compatible sequence *for a word $a = a_1 a_2 \cdots a_p$ is a weakly increasing sequence of integers $\mathbf{i} = (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p)$ with the property that*

$$i_j < i_{j+1} \text{ whenever } a_j \leqslant a_{j+1} \quad \text{and} \quad i_j \leqslant a_j \text{ whenever } 0 < i_j.$$

Let $\mathsf{Compatible}(a)$ *denote the set of all such sequences. Given $\mathbf{i} = (i_1 \leqslant \cdots \leqslant i_p) \in \mathsf{Compatible}(a)$, define $x_{\mathbf{i}} = x_{i_1} \cdots x_{i_p}$ and write $0 < \mathbf{i}$ if the numbers $i_1, \ldots, i_p$ are all positive.*

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Let $s_i = (i, i+1)$ denote the permutation of $\mathbb{Z}$ interchanging i and $i+1$. Fix a positive integer n and let $S_n := \langle s_1, s_2, \dots, s_{n-1} \rangle \subset S_{\mathbb{Z}} := \langle s_i : i \in \mathbb{Z} \rangle$. Both S_n and $S_{\mathbb{Z}}$ are Coxeter groups with respect to their given generating sets. A *reduced word* for $w \in S_{\mathbb{Z}}$ is a word $a_1 a_2 \cdots a_p$ of shortest possible length such that $w = s_{a_1} s_{a_2} \cdots s_{a_p}$. Let $\mathsf{Reduced}(w)$ denote the set of all such words.

Definition 1.2. *The* Schubert polynomial *of $w \in S_n$ is*

$$\mathfrak{S}_w := \sum_{a \in \mathsf{Reduced}(w)} \sum_{0 < \mathbf{i} \in \mathsf{Compatible}(a)} x_{\mathbf{i}} \in \mathbb{Z}[x_1, x_2, \dots, x_{n-1}].$$

Schubert polynomials are often defined inductively using divided difference operators, following the approach of Lascoux and Schützenberger. The formula that we have given is [3, Thm. 1.1]. The identity of Macdonald [11] mentioned at the start of this introduction is as follows.

Theorem 1.3 (Fomin and Stanley [6, Thm. 2.4]). *If $w \in S_n$ then*

$$\mathfrak{S}_w(1, q, q^2, \dots, q^{n-1}) = \sum_{a = a_1 a_2 \cdots a_p \in \mathsf{Reduced}(w)} \frac{[a_1]_q [a_2]_q \cdots [a_p]_q}{[p]_q!} q^{\mathrm{comaj}(a)},$$

where $\mathrm{comaj}(a) := \sum_{a_i < a_{i+1}} i$ *and* $[a]_q := \frac{1-q^a}{1-q}$ *and* $[p]_q! := [p]_q \cdots [2]_q [1]_q$.

Taking appropriate limits transforms the preceding formula into an identity for the *backstable Schubert polynomials*, which may be defined as follows.

Definition 1.4. *The* backstable Schubert polynomial *of $w \in S_n$ is*

$$\overleftarrow{\mathfrak{S}}_w := \sum_{a \in \mathsf{Reduced}(w)} \sum_{\mathbf{i} \in \mathsf{Compatible}(a)} x_{\mathbf{i}} \in \mathbb{Z}[\![\dots, x_{-1}, x_0, x_1, \dots, x_{n-1}]\!].$$

This is the same as the formula for $\mathfrak{S}_w$ except now $\mathbf{i} = (i_1 \leqslant i_2 \cdots \leqslant i_p)$ may contain non-positive integers. If $w \in S_n$ then $\overleftarrow{\mathfrak{S}}_w(\dots, 0, 0, x_1, x_2, \dots, x_{n-1}) = \mathfrak{S}_w$, while $\overleftarrow{\mathfrak{S}}_w(\dots, x_{-2}, x_{-1}, x_0, 0, 0, \dots, 0)$ is the *Stanley symmetric function* of w in the variables x_i for $i \leqslant 0$ [10, Thm. 3.2].

Note that $\overleftarrow{\mathfrak{S}}_w$ is usually not a polynomial. These power series were introduced by Lam, Lee, and Shimozono [10] in connection with Schubert calculus on infinite flag varieties. They also arise as cohomology classes of degeneracy loci in products of flag varieties [15].

If $F \in \mathbb{Z}[\![\dots, x_{-1}, x_0, x_1, \dots, x_{n-1}]\!]$ is homogeneous then the formal power series $F(x_i \mapsto q^{i-1})$ obtained by setting $x_i = q^{i-1}$ for all integers $i < n$ is well-defined. The following result is easy to derive from Theorem 1.3 and is also a special case of Theorem 3.3. In this statement, for a word $a = a_1 a_2 \cdots a_p$ we write $\Sigma a := \sum_{i=1}^{p} a_i$ and $\ell(a) := p$.

Theorem 1.5. *If $w \in S_n$ then* $\overleftarrow{\mathfrak{S}}_w(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{Reduced}(w)} \frac{q^{\Sigma a + \mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)}$ *where the right hand expression is interpreted as a Laurent series in q^{-1}.*

Example 1.6. Setting $x_i = q^{i-1}$ in the definition of $\overleftarrow{\mathfrak{S}}_w$ gives another formula for $\overleftarrow{\mathfrak{S}}_w(x_i \mapsto q^{i-1})$ as a sum over the reduced words for w. The corresponding terms in these two summations need not agree, however: for a given word $a = a_1 a_2 \cdots a_p \in \mathsf{Reduced}(w)$, it can happen that

$$\sum_{\mathbf{i} \in \mathsf{Compatible}(a)} q^{(i_1-1)+(i_2-1)+\cdots+(i_p-1)} \neq \frac{q^{\Sigma a + \mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^p-1)}.$$

For example, if $w = (1,2)(3,4)$ and $a = 1,3$ then $\sum_{\mathbf{i}\in\mathsf{Compatible}(a)} q^{(i_1-1)+\cdots+(i_p-1)}$ is

$$\sum_{1\geqslant i_1<i_2\leqslant 3} q^{(i_1-1)+(i_2-1)} \in q^2+2q+2+q^{-1}\mathbb{Z}[\![q^{-1}]\!]$$

while $\frac{q^{\Sigma a+\mathrm{comaj}(a)}}{(q-1)(q^2-1)\cdots(q^p-1)} = \frac{q^5}{(q-1)(q^2-1)}$ expands into the Laurent series

$$q^5(q^{-1}+q^{-2}+q^{-3}+\cdots)(q^{-2}+q^{-4}+\cdots) \in q^2+q+2+q^{-1}\mathbb{Z}[\![q^{-1}]\!].$$

For $w = (1,2)(3,4)$ there are only two reduced words and one has

$$\begin{aligned}
\overleftarrow{\mathfrak{S}}_{(1,2)(3,4)} &= \overleftarrow{e}_1^2 + (2x_1+x_2+x_3)\overleftarrow{e}_1 + x_1^2 + x_1x_2 + x_1x_3 \\
&= \cdots + x_0^2 + 2x_{-1}x_1 + x_{-2}x_2 + x_{-3}x_3 \\
&\quad + 2x_0x_1 + x_{-1}x_2 + x_{-2}x_3 \\
&\quad + x_1^2 + x_0x_2 + x_{-1}x_3 \\
&\quad + x_1x_2 + x_0x_3 \\
&\quad + x_1x_3
\end{aligned}$$

where $\overleftarrow{e}_d$ is the symmetric function $\sum_{i_1<i_2<\cdots<i_d\leqslant 0} x_{i_1}x_{i_2}\cdots x_{i_d}$. One computes

$$\overleftarrow{\mathfrak{S}}_{(1,2)(3,4)}(x_i \mapsto q^{i-1}) = \frac{q^4}{(q-1)^2} = \cdots + 7q^{-4} + 6q^{-3} + 5q^{-2} + 4q^{-1} + 3 + 2q + q^2$$

using either Theorem 1.5 or the formula $\overleftarrow{e}_d(q^{-1},q^{-2},\ldots) = \frac{1}{(q-1)(q^2-1)\cdots(q^d-1)}$.

Our first new results are versions of the preceding theorem for Schubert polynomials in other classical types. We begin with type B/C. Given $0 < i < n$, define $t_i = t_{-i} \coloneqq (i,i+1)(-i,-i-1)$ and $t_0 \coloneqq (-1,1)$. Define $W_n^{\mathsf{BC}} \coloneqq \langle t_0, t_1, \ldots, t_{n-1}\rangle$ to be the Coxeter group consisting of the permutations w of $\mathbb{Z}$ with $w(i) = i$ for $|i| > n$ and $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$.

A *signed reduced word of type B* for an element $w \in W_n^{\mathsf{BC}}$ is a word $a_1a_2\cdots a_p$ with letters in the set $\{-n+1,\ldots,-1,0,1,\ldots,n-1\}$ of shortest possible length such that $w = t_{a_1}t_{a_2}\cdots t_{a_p}$. Let -0 denote a formal symbol distinct from 0 that satisfies $-1 < -0 < 0 < 1$ and set $t_{-0} \coloneqq t_0$. A *signed reduced word of type C* for $w \in W_n^{\mathsf{BC}}$ is a word $a_1a_2\cdots a_p$ with letters in $\{-n+1,\ldots,-1,-0,0,1,\ldots,n-1\}$ of shortest possible length such that $w = t_{a_1}t_{a_2}\cdots t_{a_p}$. Let $\mathsf{Reduced}_B^{\pm}(w)$ and $\mathsf{Reduced}_C^{\pm}(w)$ denote the respective sets of signed reduced words for w.

Definition 1.7. *The* type B/C Schubert polynomials *of* $w \in W_n^{\mathsf{BC}}$ *are*

$$\mathfrak{S}_w^{\mathsf{B}} \coloneqq \sum_{\substack{a\in\mathsf{Reduced}_B^{\pm}(w)\\ \mathbf{i}\in\mathsf{Compatible}(a)}} x_{\mathbf{i}} \quad \textit{and} \quad \mathfrak{S}_w^{\mathsf{C}} \coloneqq \sum_{\substack{a\in\mathsf{Reduced}_C^{\pm}(w)\\ \mathbf{i}\in\mathsf{Compatible}(a)}} x_{\mathbf{i}} = 2^{\ell_0(w)}\mathfrak{S}_w^{\mathsf{B}}$$

where $\ell_0(w) \coloneqq |\{i \in \mathbb{Z} : w(i) < 0 < i\}|$.

Both of the "polynomials" $\mathfrak{S}_w^{\mathsf{B}}$ and $\mathfrak{S}_w^{\mathsf{C}}$ are formal power series in the ring $\mathbb{Z}[\![\ldots,x_{-1},x_0,x_1,\ldots,x_{n-1}]\!]$. If we substitute $x_i \mapsto z_i$ for $i > 0$ and $x_i \mapsto x_{1-i}$ for $i \leqslant 0$, then $\mathfrak{S}_w^{\mathsf{B}}$ and $\mathfrak{S}_w^{\mathsf{C}}$ specialize to the Schubert polynomials of types B and C defined by Billey and Haiman in [1]; compare our definition with [1, Thm. 3].

Let $\mathsf{Reduced}_C(w)$ for $w \in W_n^{\mathsf{BC}}$ denote the subset of words in $\mathsf{Reduced}_C^{\pm}(w)$ whose letters all belong to $\{0,1,\ldots,n-1\}$. In Section 2.2 we prove the following analogue of Theorem 1.5.

Theorem 1.8. *If $w \in W_n^{\mathsf{BC}}$ then*

$$\mathfrak{S}_w^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \sum_{a=a_1a_2\cdots a_p \in \mathsf{Reduced}_C(w)} \frac{(q^{a_1}+1)(q^{a_2}+1)\cdots(q^{a_p}+1)}{(q-1)(q^2-1)\cdots(q^p-1)} q^{\mathrm{comaj}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

Example 1.9. If $w = (1,-2)(2,-1) \in W_n^{\mathsf{BC}}$ then the set $\mathsf{Reduced}_C^{\pm}(w)$ has 8 elements, formed by adding arbitrary signs to the letters in $a_1a_2a_3 = 0,1,0$. One can show that

$$\begin{aligned}\mathfrak{S}_{(1,-2)(2,-1)}^{\mathsf{C}} &= 4\overleftarrow{e}_2\overleftarrow{e}_1 - 4\overleftarrow{e}_3 \\ &= \cdots + 4x_{-2}x_{-1}^2 + 4x_{-2}^2x_0 + 8x_{-3}x_{-1}x_0 + 4x_{-4}x_0^2 \\ &\quad + 4x_{-3}x_0^2 + 8x_{-2}x_{-1}x_0 \\ &\quad + 4x_{-2}x_0^2 + 4x_{-1}^2x_0 \\ &\quad + 4x_{-1}x_0^2\end{aligned}$$

where $\overleftarrow{e}_d := \sum_{i_1<i_2<\cdots<i_d\leqslant 0} x_{i_1}x_{i_2}\cdots x_{i_d}$ as in Example 1.6. It follows that

$$\begin{aligned}\mathfrak{S}_{(1,-2)(2,-1)}^{\mathsf{C}}(x_i \mapsto q^{i-1}) &= \frac{4q}{(q-1)^2(q^3-1)} \\ &= \cdots + 36q^{-9} + 28q^{-8} + 20q^{-7} + 12q^{-6} + 8q^{-5} + 4q^{-4}.\end{aligned}$$

We turn next to type D. For $1 < i < n$, let

$$r_i = r_{-i} := (i,i+1)(-i,-i-1) = t_i$$

but define

$$r_1 := (1,2)(-1,-2) = t_1 \quad \text{and} \quad r_{-1} := (1,-2)(-1,2) = t_0t_1t_0.$$

Define $W_n^{\mathsf{D}} := \langle r_{-1}, r_1, r_2, \ldots, r_{n-1}\rangle$ to be the Coxeter group of permutations $w \in W_n^{\mathsf{BC}}$ for which the number of positive integers i with $w(i) < 0$ is even. A *signed reduced word* for $w \in W_n^{\mathsf{D}}$ is a word $a_1a_2\cdots a_p$ with letters in the set $\{-n+1,\ldots,-2,-1,1,2,\ldots,n-1\}$ of shortest possible length such that $w = r_{a_1}r_{a_2}\cdots r_{a_p}$. Let $\mathsf{Reduced}_D^{\pm}(w)$ denote the set of such words.

Definition 1.10. *The* type D Schubert polynomial *of $w \in W_n^{\mathsf{D}}$ is*

$$\mathfrak{S}_w^{\mathsf{D}} = \sum_{a \in \mathsf{Reduced}_D^{\pm}(w)} \sum_{\mathbf{i} \in \mathsf{Compatible}(a)} x_{\mathbf{i}} \in \mathbb{Z}[\![\ldots, x_{-1}, x_0, x_1, \ldots, x_{n-1}]\!].$$

If we again substitute $x_i \mapsto z_i$ for $i > 0$ and $x_i \mapsto x_{1-i}$ for $i \leqslant 0$, then our definition of the power series $\mathfrak{S}_w^{\mathsf{D}}$ specializes to Billey and Haiman's formula for the Schubert polynomial of type D given in [1, Thm. 4].

Suppose $a = a_1a_2\cdots a_p$ is a sequence with $a_i \in \{\pm1, \pm2, \pm3, \ldots, \pm(n-1)\}$. Define

$$\mathrm{comaj}_{\mathsf{D}}(a) := |\{i : a_i > 0\}| + \sum_{a_i \prec a_{i+1}} 2i \tag{1}$$

where $\prec$ is the order $-1 \prec -2 \prec \cdots \prec -n \prec 1 \prec 2 \prec \cdots \prec n$. For example, if $a = a_1a_2a_3a_4 = -1,-2,3,1$ then $\mathrm{comaj}_{\mathsf{D}}(a) = 2+(2+4) = 8$. We prove the following in Section 2.3.

Theorem 1.11. *If $w \in W_n^{\mathsf{D}}$ then*

$$\mathfrak{S}_w^{\mathsf{D}}(x_i \mapsto q^{i-1}) = \sum_{a=a_1a_2\cdots a_p \in \mathsf{Reduced}_D^{\pm}(w)} \frac{(q^{|a_1|+1})(q^{|a_2|}+1)\cdots(q^{|a_p|}+1)}{(q^2-1)(q^4-1)\cdots(q^{2p}-1)} q^{\mathrm{comaj}_{\mathsf{D}}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

EXAMPLE 1.12. If $w = (1,-1)(4,-4) \in W_n^{\mathsf{D}}$ then the set $\mathsf{Reduced}_D^{\pm}(w)$ has 32 elements, formed by adding signs to the letters in $a_1a_2a_3a_4a_5a_6 = 3,2,1,1,2,3$ in all ways that give opposite signs to the two entries with absolute value one. One can compute that

$$\begin{aligned}\mathfrak{S}^{\mathsf{D}}_{(1,-1)(4,-4)} &= x_1x_2x_3\overleftarrow{P}_3 + (x_1x_2 + x_1x_3 + x_2x_3)\overleftarrow{P}_4 + (x_1+x_2+x_3)\overleftarrow{P}_5 + \overleftarrow{P}_6 \\ &= \cdots + x_0^4x_1x_3 + 2x_{-1}x_0^3x_2x_3 + 2x_{-1}^2x_0x_1x_2x_3 + 2x_{-2}x_0^2x_1x_2x_3 \\ &\qquad + x_0^4x_2x_3 + 2x_{-1}x_0^2x_1x_2x_3 \\ &\qquad + x_0^3x_1x_2x_3\end{aligned}$$

where $\overleftarrow{P}_d$ for $d>0$ is the Schur P-function $\frac{1}{2}\sum_{a=0}^{d} e_a(x_0,x_{-1},\dots)h_{d-a}(x_0,x_{-1},\dots)$. Using the formula $\overleftarrow{P}_d(q^{-1},q^{-2},\dots) = \frac{(q+1)(q^2+1)\cdots(q^{d-1}+1)}{(q-1)(q^2-1)\cdots(q^d-1)}$ one can check that

$$\begin{aligned}\mathfrak{S}^{\mathsf{D}}_{(1,-1)(4,-4)}(x_i \mapsto q^{i-1}) &= \frac{q^{12}(q^2+1)}{(q-1)^3(q^3-1)(q^5-1)} \\ &= \cdots + 46q^{-5} + 27q^{-4} + 15q^{-3} + 7q^{-2} + 3q^{-1} + 1,\end{aligned}$$

which agrees with Theorem 1.11.

Setting $q=1$ in Theorem 1.5 leads to surprising enumerative formulas involving reduced words, compatible sequences, and plane partitions [5]. By contrast, the power series $\overleftarrow{\mathfrak{S}}_w$, $\mathfrak{S}^{\mathsf{B}}_w$, $\mathfrak{S}^{\mathsf{C}}_w$, and $\mathfrak{S}^{\mathsf{D}}_w$ do not converge upon specializing $x_i \mapsto 1$ for all i. It would be interesting to find variations of our formulas with clearer enumerative content.

The second half of this note contains a few other related results. In Section 3, we extend Theorems 1.5, 1.8, and 1.11 to identities for *Grothendieck polynomials*. Our proofs of these formulas are fairly straightforward adaptations of the algebraic methods in [6, 9]. It is an interesting open problem to find bijective proofs of these identities along the lines of [2].

Our approach has one other notable application, which we discuss in Section 4. There, we develop a simple alternate proof of the main result of [8], which gives a pipe dream formula for certain *involution Schubert polynomials*. We are able to prove a more general K-theoretic formula, partially resolving an open question from [8, § 6].

2. Principal specializations of Schubert polynomials

This section contains our proofs of Theorems 1.8 and 1.11. Throughout, we fix a positive integer n and let R be an arbitrary commutative ring containing the ring of formal power series $\mathbb{Z}[\![x_i : i<n]\!]$.

2.1. Nil-Coxeter algebras.

The algebra introduced in this section figures prominently in [6] and in several of our arguments. Let (W,S) be a Coxeter system with length function ℓ. Let $\mathsf{NilCox} = \mathsf{NilCox}(W)$ be the R-module of all formal R-linear combinations of the symbols u_w for $w \in W$. This module has a unique R-algebra structure with bilinear multiplication satisfying

$$u_v u_w = \begin{cases} u_{vw} & \text{if } \ell(vw) = \ell(v)+\ell(w), \\ 0 & \text{if } \ell(vw) < \ell(v)+\ell(w), \end{cases} \qquad \text{for } v,w \in W.$$

Following [6, § 2], we refer to NilCox as the *nil-Coxeter algebra* of (W,S). Choose $x,y \in R$. Given $s \in S$, define $h_s(x) \coloneqq 1 + xu_s \in \mathsf{NilCox}$. One checks that if $s,t \in S$ and $st = ts$ then

$$h_s(x)h_s(y) = h_s(x+y) \quad \text{and} \quad h_s(x)h_t(y) = h_t(y)h_s(x).$$

We will also need the following general identity, which is equivalent to [6, Lem. 5.4] after some minor changes of variables:

Lemma 2.1 ([6, Lem. 5.4]). *Let $\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_N$ be some elements of an R-algebra with identity 1, and suppose q, z_1, z_2, $\dots z_N$ are formal variables. Then*

$$\prod_{j=-\infty}^{0}\prod_{i=1}^{N}(1+q^{j-1}z_i\mathfrak{t}_i) = \sum_{p\geqslant 0}\sum_{a_1,a_2,\dots,a_p}\frac{z_{a_1}z_{a_2}\cdots z_{a_p}}{(q-1)(q^2-1)\cdots(q^p-1)}q^{\operatorname{comaj}(a)}\mathfrak{t}_{a_1}\mathfrak{t}_{a_2}\cdots\mathfrak{t}_{a_p}$$

where $\operatorname{comaj}(a) \coloneqq \sum_{a_i<a_{i+1}} i$ *and the coefficients on the right are viewed as Laurent series in* q^{-1}.

2.2. Type B/C. Here, let $\mathsf{NilCox} = \mathsf{NilCox}(W_n^{\mathsf{BC}})$ denote the nil-Coxeter algebra of type B/C Coxeter system $(W,S) = (W_n^{\mathsf{BC}}, \{t_0, t_1, \dots, t_{n-1}\})$ and define $h_i(x) \coloneqq 1 + xu_{t_i} \in \mathsf{NilCox}$ for integers $-n < i < n$ and $x \in R$. Recall that $t_i = t_{-i}$ so we always have $h_i(x) = h_{-i}(x)$. Let

$$\begin{aligned} A_i(x) &\coloneqq h_{n-1}(x)h_{n-2}(x)\cdots h_i(x),\\ B(x) &\coloneqq h_{n-1}(x)\cdots h_1(x)h_0(x)h_{-1}(x)\cdots h_{-n+1}(x),\\ C(x) &\coloneqq h_{n-1}(x)\cdots h_1(x)h_0(x)h_0(x)h_{-1}(x)\cdots h_{-n+1}(x), \end{aligned} \tag{2}$$

and note that $h_0(x)h_0(x) = h_0(2x)$. Finally consider the infinite products in NilCox given by

$$\mathfrak{S}^{\mathsf{B}} \coloneqq \prod_{i=-\infty}^{0} B(x_i)\prod_{i=1}^{n-1}A_i(x_i) \quad\text{and}\quad \mathfrak{S}^{\mathsf{C}} \coloneqq \prod_{i=-\infty}^{0} C(x_i)\prod_{i=1}^{n-1}A_i(x_i). \tag{3}$$

It straightforward to see that $\mathfrak{S}^{\mathsf{B}} = \sum_{w\in W_n^{\mathsf{BC}}}\mathfrak{S}^{\mathsf{B}}_w \cdot u_w$ and $\mathfrak{S}^{\mathsf{C}} = \sum_{w\in W_n^{\mathsf{BC}}}\mathfrak{S}^{\mathsf{C}}_w \cdot u_w$. Less trivially:

Proposition 2.2. *It holds that*

$$\mathfrak{S}^{\mathsf{B}} = \prod_{j=-\infty}^{0}\left(h_0(x_j)\prod_{i=1}^{n-1}h_i(x_{i+j}+x_j)\right) \quad \textit{and} \quad \mathfrak{S}^{\mathsf{C}} = \prod_{j=-\infty}^{0}\prod_{i=0}^{n-1}h_i(x_{i+j}+x_j).$$

Proof. We will just prove the formula for $\mathfrak{S}^{\mathsf{C}}$ since the other case is similar. Let

$$\widetilde{A}_i(x) \coloneqq h_i(x)h_{i+1}(x)\cdots h_{n-1}(x).$$

Since $A_i(x) = A_{i+1}(x)h_i(x)$ and $C(x) = A_1(x)h_0(x+x)\widetilde{A}_1(x)$, we have

$$\mathfrak{S}^{\mathsf{C}} = \prod_{i=-\infty}^{-1} C(x_i)\cdot A_1(x_0)h_0(x_0+x_0)\widetilde{A}_1(x_0)A_1(x_1)A_2(x_2)\cdots A_{n-1}(x_{n-1}).$$

The elements $h_{i-2}(x)$, $A_i(y)$, and $\widetilde{A}_i(z)$ all commute by [6, Lem. 4.1]. Using this fact and the identities $A_i(x) = A_{i+1}(x)h_i(x)$ and $\widetilde{A}_i(x) = h_i(x)\widetilde{A}_{i+1}(x)$, one can show

$$h_0(x_0+x_0)\widetilde{A}_1(x_0)A_1(x_1)A_2(x_2)\cdots A_{n-1}(x_{n-1}) = \prod_{i=2}^{n-1}A_i(x_{i-1})\cdot\prod_{i=0}^{n-1}h_i(x_i+x_0).$$

Substituting this into our formula above gives $\mathfrak{S}^{\mathsf{C}} = \mathfrak{S}^{\mathsf{C}}(x_i \mapsto x_{i-1})\prod_{i=0}^{n-1}h_i(x_i+x_0)$ so by induction we have $\mathfrak{S}^{\mathsf{C}} = \mathfrak{S}^{\mathsf{C}}(x_i \mapsto x_{i-N})\prod_{j=-N+1}^{0}\prod_{i=0}^{n-1}h_i(x_{i+j}+x_j)$ for all $N \geqslant 0$. But it is easy to see that $\lim_{N\to\infty}\mathfrak{S}^{\mathsf{C}}(x_i \mapsto x_{i-N}) = 1$ as a limit of power series, so the result follows by sending $N \to \infty$. □

We can now prove Theorem 1.8.

Proof of Theorem 1.8. To obtain the desired formula, set $x_i = q^{i-1}$ in Proposition 2.2, apply Lemma 2.1 with $N = n$, $z_i = 1 + q^{i-1}$, and $\mathfrak{t}_i = u_{t_{i-1}}$, and then extract the coefficient of u_w. □

2.3. Type D. Now, let $\mathsf{NilCox} = \mathsf{NilCox}(W_n^{\mathsf{D}})$ denote the nil-Coxeter algebra of $(W, S) = (W_n^{\mathsf{D}}, \{r_{-1}, r_1, \ldots, r_{n-1}\})$ and define $h_i(x) := 1 + xu_{t_i} \in \mathsf{NilCox}$ for all $i \in \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$ and $x \in R$. Let

$$\begin{aligned} A_i(x) &:= h_{n-1}(x)h_{n-2}(x)\cdots h_i(x), \\ \widetilde{A}_i(x) &:= h_i(x)h_{i+1}(x)\cdots h_{n-1}(x), \\ D(x) &:= h_{n-1}(x)\cdots h_1(x)h_{-1}(x)\cdots h_{-n+1}(x). \end{aligned} \tag{4}$$

The Coxeter group W_n^{D} has a unique automorphism $w \mapsto w^*$ that maps $r_i \mapsto r_{-i}$ for $1 \leqslant i < n$. This map extends by linearity to an R-algebra automorphism of NilCox with $u_w^* := u_{w^*}$. We have $A_i(x)^* = A_i(x)$ for $1 < i < n$ and $D(x)^* = D(x)$, while $A_1(x)^* = h_{n-1}(x)h_{n-2}(x)\cdots h_2(x)h_{-1}(x)$. Consider the infinite products in NilCox

$$\mathfrak{S}^{\mathsf{D}} := \prod_{i=-\infty}^{0} D(x_i) \prod_{i=1}^{n-1} A_i(x_i) \quad \text{and} \quad (\mathfrak{S}^{\mathsf{D}})^* := \prod_{i=-\infty}^{0} D(x_i) \prod_{i=1}^{n-1} A_i(x_i)^*. \tag{5}$$

We have $\mathfrak{S}^{\mathsf{D}} = \sum_{w \in W_n^{\mathsf{D}}} \mathfrak{S}_w^{\mathsf{D}} \cdot u_w$ and $(\mathfrak{S}^{\mathsf{D}})^* = \sum_{w \in W_n^{\mathsf{D}}} \mathfrak{S}_w^{\mathsf{D}} \cdot u_{w^*}$. In addition:

Proposition 2.3. *It holds that*

$$\mathfrak{S}^{\mathsf{D}} = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} h_{-i}(x_{i+2j-1} + x_{2j-1}) \prod_{i=1}^{n-1} h_i(x_{i+2j} + x_{2j}) \right).$$

Proof. Since $A_i(x) = A_{i+1}(x)h_i(x)$ and $D(x) = A_1(x)^* \widetilde{A}_1(x)$, we have

$$\mathfrak{S}^{\mathsf{D}} = \prod_{i=-\infty}^{-1} D(x_i) \cdot A_1(x_0)^* \widetilde{A}_1(x_0) A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}).$$

Repeating the argument in the proof of Proposition 2.2, we deduce that $\mathfrak{S}^{\mathsf{D}} = (\mathfrak{S}^{\mathsf{D}})^*(x_i \mapsto x_{i-1}) \prod_{i=1}^{n-1} h_i(x_i + x_0)$. An analogous identity holds for $(\mathfrak{S}^{\mathsf{D}})^*$. Alternating these formulas gives

$$\mathfrak{S}^{\mathsf{D}} = \mathfrak{S}^{\mathsf{D}}(x_i \mapsto x_{i-2N}) \prod_{j=-N+1}^{0} \left(\prod_{i=1}^{n-1} h_{-i}(x_{i+2j-1} + x_{2j-1}) \prod_{i=1}^{n-1} h_i(x_{i+2j} + x_{2j}) \right)$$

for all $N \geqslant 0$. It is again easy to see that $\lim_{N \to \infty} \mathfrak{S}^{\mathsf{D}}(x_i \mapsto x_{i-2N}) = 1$ as a limit of formal power series, so the result follows by sending $N \to \infty$. □

We can now also prove Theorem 1.11.

Proof of Theorem 1.11. By Proposition 2.3 we have

$$\begin{aligned} &\mathfrak{S}^{\mathsf{D}}(x_i \mapsto q^{i-1}) \\ &\quad = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} \left(1 + q^{2(j-1)} \cdot (1+q^i) \cdot u_{r_{-i}}\right) \cdot \prod_{i=1}^{n-1} \left(1 + q^{2(j-1)} \cdot q(1+q^i) \cdot u_{r_i}\right) \right). \end{aligned}$$

To get the desired expression for $\mathfrak{S}_w^{\mathsf{D}}$, apply Lemma 2.1 with q replaced by q^2 and $N = 2n-2$ to the right side of the preceding identity, using the parameters $z_i = 1+q^i$, $z_{n-1+i} = q(1+q^i)$, $\mathfrak{t}_i = u_{r_{-i}}$, and $\mathfrak{t}_{n-1+i} = u_{r_i}$ for $1 \leqslant i < n$. Then extract the coefficient of u_w. □

3. Principal specializations of Grothendieck polynomials

In this section we describe some extensions of Theorems 1.5, 1.8, and 1.11 for *Grothendieck polynomials* in classical types. The identities proved here are more general but also more technical than the formulas sketched in the introduction.

3.1. Id-Coxeter algebras. Again let (W,S) be an arbitrary Coxeter system with length function ℓ. For the results in this section, we work in a generalization of the algebra $\mathsf{NilCox}(W)$. Recall that R is an arbitrary commutative ring containing $\mathbb{Z}[\![x_i : i<n]\!]$. From this point on, we fix an element $\beta \in R$.

Let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W)$ be the R-module of all formal R-linear combinations of the symbols π_w for $w \in W$. This module has a unique R-algebra structure with bilinear multiplication satisfying

$$\pi_v\pi_w = \pi_{vw} \text{ if } \ell(vw)=\ell(v)+\ell(w) \quad \text{and} \quad \pi_s^2 = \beta\pi_s$$

for $v,w \in W$ and $s \in S$ [9, Def. 1], which we refer to as the *id-Coxeter algebra* of (W,S). For $x,y \in R$ and $s \in S$, define

$$x \oplus y \coloneqq x+y+\beta xy \quad \text{and} \quad h_s^{(\beta)}(x) \coloneqq 1+x\pi_s. \tag{6}$$

Then $h_s^{(\beta)}(x)h_s^{(\beta)}(y) = h_s^{(\beta)}(x\oplus y)$, and if $st=ts$ then $h_s^{(\beta)}(x)h_t^{(\beta)}(y) = h_t^{(\beta)}(y)h_s^{(\beta)}(x)$ [9, Lem. 1].

3.2. Type A. Let $\overleftarrow{S}_n \coloneqq \langle s_i : i<n\rangle$ be the Coxeter group of permutations $w \in S_{\mathbb{Z}}$ with $w(i)=i$ for all $i>n$. In this section we write $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(\overleftarrow{S}_n)$ and set $\pi_i \coloneqq \pi_{s_i} \in \mathsf{IdCox}_\beta$ for integers $i<n$. Define $\mathsf{Hecke}(w)$ for $w \in \overleftarrow{S}_n$ to be the set of words $a_1a_2\cdots a_N$ such that $\pi_w = \beta^{N-\ell(w)}\pi_{a_1}\pi_{a_2}\cdots\pi_{a_N}$. Recall the set $\mathsf{Compatible}(a)$ from Definition 1.1.

Definition 3.1. *The* backstable Grothendieck polynomial *of* $w \in S_n \subsetneq \overleftarrow{S}_n$ *is*

$$\overleftarrow{\mathfrak{G}}_w \coloneqq \sum_{a\in\mathsf{Hecke}(w)} \sum_{\mathbf{i}\in\mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}} \in \mathbb{Z}[\beta][\![\ldots,x_{-1},x_0,x_1,\ldots,x_{n-1}]\!].$$

The function $\mathfrak{G}_w \coloneqq \overleftarrow{\mathfrak{G}}_w(\ldots,0,0,x_1,x_2,\ldots,x_{n-1})$ is the ordinary *Grothendieck polynomial* of $w \in S_n$. The power series $G_w \coloneqq \overleftarrow{\mathfrak{G}}_w(\ldots,x_3,x_2,x_1,0,0,\ldots,0)$ given by setting $x_i \mapsto 0$ for $i>0$ and $x_i \mapsto x_{1-i}$ for $i \leqslant 0$ is a symmetric function in the x_i variables, which is usually called the *stable Grothendieck polynomial* of $w \in S_n$.

Specializing $\beta \mapsto 0$ transforms $\overleftarrow{\mathfrak{G}}_w \mapsto \overleftarrow{\mathfrak{S}}_w$ from Section 1. The Grothendieck polynomials $\mathfrak{G}_w$ are closely related to the K-theory of flag varieties and Grassmannians [4, 13]. We do not know of a similar geometric interpretation for the backstable Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w$.

For $i<n$ and $x \in R$, let

$$h_i^{(\beta)}(x) \coloneqq 1+x\pi_i \quad \text{and} \quad A_i^{(\beta)}(x) \coloneqq h_{n-1}^{(\beta)}(x)h_{n-2}^{(\beta)}(x)\cdots h_i^{(\beta)}(x).$$

Define

$$\overleftarrow{\mathfrak{G}} \coloneqq \cdots A_{n-3}^{(\beta)}(x_{n-3})A_{n-2}^{(\beta)}(x_{n-2})A_{n-1}^{(\beta)}(x_{n-1}) = \prod_{i=-\infty}^{n-1} A_i^{(\beta)}(x_i) \in \mathsf{IdCox}_\beta. \tag{7}$$

If $w \in S_n$ then the coefficient of π_w in this expression is $\overleftarrow{\mathfrak{G}}_w$.

Proposition 3.2. *It holds that* $\overleftarrow{\mathfrak{G}} = \prod_{j=-\infty}^{0}\prod_{i=-\infty}^{n-1} h_i^{(\beta)}(x_{i+j})$.

Proof. We have $\overleftarrow{\mathfrak{G}} = \cdots A_1^{(\beta)}(x_0)h_0^{(\beta)}(x_0)A_2^{(\beta)}(x_1)h_1^{(\beta)}(x_1)\cdots A_{n-1}^{(\beta)}(x_{n-2})h_{n-1}^{(\beta)}(x_{n-1})$ by definition. As $h_i^{(\beta)}(x)$ and $A_{i+2}^{(\beta)}(y)$ commute, it follows that $\overleftarrow{\mathfrak{G}} = \overleftarrow{\mathfrak{G}}(x_i \mapsto x_{i-1})\prod_{i=-\infty}^{n-1} h_i^{(\beta)}(x_i)$ so by induction $\overleftarrow{\mathfrak{G}} = \overleftarrow{\mathfrak{G}}(x_i \mapsto x_{i-N})\prod_{j=-N+1}^{0}\prod_{i=-\infty}^{n-1} h_i^{(\beta)}(x_{i+j})$ for all $N \geqslant 0$. But we have $\lim_{N\to\infty} \overleftarrow{\mathfrak{G}}(x_i \mapsto x_{i-N}) = 1$ as a limit of formal power series, so the result follows by sending $N \to \infty$. □

We can now prove the following generalization of Theorem 1.5.

THEOREM 3.3. *If $w \in S_n \subsetneq \overleftarrow{S}_n$ then*

$$\overleftarrow{\mathfrak{G}}_w(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{Hecke}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\Sigma a + \mathrm{comaj}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

Proof. If $w \in S_n$ then the coefficient of π_w in $\overleftarrow{\mathfrak{G}}$ is the same as the coefficient of π_w in the truncated product $\prod_{j=-\infty}^{0}\prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+j})$. This coefficient is $\overleftarrow{\mathfrak{G}}_w$, and the theorem follows by applying Lemma 2.1 with $N = n-1$ and $z_i\mathfrak{t}_i = q^i\pi_{s_i}$ to the latter expression. □

There are *Grothendieck polynomials* in the other classical types [9] which generalize $\mathfrak{G}_w^{\mathsf{B}}$, $\mathfrak{G}_w^{\mathsf{C}}$, and $\mathfrak{G}_w^{\mathsf{D}}$ in the same way that $\overleftarrow{\mathfrak{G}}_w$ generalizes $\overleftarrow{\mathfrak{S}}_w$. We discuss these formal power series next.

3.3. TYPE B/C. In this section let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}})$ and write $\pi_i := \pi_{t_i} \in \mathsf{IdCox}_\beta$ for $-n < i < n$. Given a permutation $w \in W_n^{\mathsf{BC}}$, define $\mathsf{Hecke}_B^{\pm}(w)$ and $\mathsf{Hecke}_C^{\pm}(w)$ to be the sets of words $a_1a_2\cdots a_N$, with letters in $\{-n+1,\ldots,-1,0,1,\ldots,n-1\}$ and $\{-n+1 < \cdots < -1 < -0 < 0 < 1 < \cdots < n-1\}$, respectively, such that $\pi_w = \beta^{N-\ell(w)}\pi_{a_1}\pi_{a_2}\cdots\pi_{a_N} \in \mathsf{IdCox}_\beta$, where $\ell(w)$ denotes the usual Coxeter length of w and $\pi_{-0} := \pi_0 \in \mathsf{IdCox}_\beta$. Recall that we view -0 as a symbol distinct from 0.

DEFINITION 3.4. *The* type B/C Grothendieck polynomials *of $w \in W_n^{\mathsf{BC}}$ are*

$$\mathfrak{G}_w^{\mathsf{B}} := \sum_{\substack{a \in \mathsf{Hecke}_B^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}} \quad \textit{and} \quad \mathfrak{G}_w^{\mathsf{C}} := \sum_{\substack{a \in \mathsf{Hecke}_C^{\pm}(w) \\ \mathbf{i} \in \mathsf{Compatible}(a)}} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}}.$$

We may consider the finite sums

$$\mathfrak{G}^{\mathsf{B}} := \sum_{w \in W_n^{\mathsf{BC}}} \mathfrak{G}_w^{\mathsf{B}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}) \quad \text{and} \quad \mathfrak{G}^{\mathsf{C}} := \sum_{w \in W_n^{\mathsf{BC}}} \mathfrak{G}_w^{\mathsf{C}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}).$$

Define $A_i^{(\beta)}(x)$, $B^{(\beta)}(x)$, and $C^{(\beta)}(x)$ as in (2) but with $h_i(x)$ replaced by

$$h_i^{(\beta)}(x) := 1 + x\pi_i \in \mathsf{IdCox}_\beta(W_n^{\mathsf{BC}}) \qquad \text{for } -n < i < n \text{ and } x \in R.$$

Then $\mathfrak{G}^{\mathsf{B}}$ and $\mathfrak{G}^{\mathsf{C}}$ are given by the formulas in (3) with A_i, B, C replaced by $A_i^{(\beta)}$, $B^{(\beta)}$, $C^{(\beta)}$. Comparing with [9, Def. 9] shows that $\mathfrak{G}_w^{\mathsf{B}}$ and $\mathfrak{G}_w^{\mathsf{C}}$ are obtained from Kirillov and Naruse's *double Grothendieck polynomials* $\mathcal{G}_w^{\mathsf{B}}(a,b;x)$ and $\mathcal{G}_w^{\mathsf{C}}(a,b;x)$ by setting $a_i \mapsto x_i$, $b_i \mapsto 0$, and $x_i \mapsto x_{1-i}$.

PROPOSITION 3.5. *It holds that*

$$\mathfrak{G}^{\mathsf{B}} = \prod_{j=-\infty}^{0}\left(h_0^{(\beta)}(x_j)\prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+j} \oplus x_j)\right) \quad \textit{and} \quad \mathfrak{G}^{\mathsf{C}} = \prod_{j=-\infty}^{0}\prod_{i=0}^{n-1} h_i^{(\beta)}(x_{i+j} \oplus x_j).$$

Proof. Since $A_i^{(\beta)}(x)$ commutes with $\widetilde{A}_i^{(\beta)}(x) \coloneqq h_i^{(\beta)}(x)h_{i+1}^{(\beta)}(x)\cdots h_{n-1}^{(\beta)}(x)$ by [9, Lem. 3], the result follows by the same proof as Proposition 2.2, *mutatis mutandis.* □

Given a word $a = a_1a_2\cdots a_p$ with $a_i \in \{-n+1 < \cdots < -1 < -0 < 0 < 1 < \cdots < n-1\}$, let $I(a)$ be the set of indices $i \in [p]$ with $a_i \in \{1, 2, \ldots, n-1\}$ and define

$$\Sigma_{\mathsf{BC}}(a) \coloneqq \sum_{i\in I(a)} a_i \quad \text{and} \quad \mathrm{comaj}_{\mathsf{BC}}(a) \coloneqq \sum_{a_i \prec a_{i+1}} i \tag{8}$$

where $\prec$ is the order $-0 \prec 0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots$. For example, if $a = -1, 1, -2, 1$ then $\Sigma_{\mathsf{BC}}(a) = 1 + 1 = 2$ and $\mathrm{comaj}_{\mathsf{BC}}(a) = 1 + 2 = 3$.

Theorem 3.6. *If $w \in W_n^{\mathsf{BC}}$ then the following identities hold:*

(1) $\mathfrak{G}_w^{\mathsf{B}}(x_i \mapsto q^{i-1}) = \sum_{a\in \mathsf{Hecke}_B^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\Sigma_{\mathsf{BC}}(a)+\mathrm{comaj}_{\mathsf{BC}}(a)}$.

(2) $\mathfrak{G}_w^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \sum_{a\in \mathsf{Hecke}_C^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q-1)(q^2-1)\cdots(q^{\ell(a)}-1)} q^{\Sigma_{\mathsf{BC}}(a)+\mathrm{comaj}_{\mathsf{BC}}(a)}$.

The right hand expressions in both parts are interpreted as Laurent series in q^{-1}.

The second identity reduces to Theorem 1.8 when $\beta = 0$ since the sum $\sum_a q^{\Sigma_{\mathsf{BC}}(a)+\mathrm{comaj}_{\mathsf{BC}}(a)}$ over all words $a = a_1a_2\cdots a_p \in \mathsf{Reduced}_C^{\pm}(w)$ with the same unsigned form is exactly the product $(q^{|a_1|}+1)(q^{|a_2|}+1)\cdots(q^{|a_p|}+1)q^{\mathrm{comaj}(|a_1||a_2|\cdots|a_p|)}$.

Proof. Part (1) is similar so we just prove (2). As $h_i^{(\beta)}(x_{i+j} \oplus x_j) = h_i^{(\beta)}(x_j)h_i^{(\beta)}(x_{i+j})$, we have

$$\mathfrak{G}^{\mathsf{C}}(x_i \mapsto q^{i-1}) = \prod_{j=-\infty}^{0} \prod_{i=0}^{n-1} (1 + q^{j-1}\cdot\pi_i)(1 + q^{j-1}\cdot q^i\cdot\pi_i)$$

by Proposition 3.5. The identity for $\mathfrak{G}_w^{\mathsf{C}}$ follows by extracting the coefficient of π_w from the right side after applying Lemma 2.1 with $N = 2n$ and with the parameters $z_1, z_2, \ldots, z_{2n}$ and $\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_{2n}$ replaced by $1, 1, 1, q, 1, q^2, \ldots, 1, q^{n-1}$ and $\pi_0, \pi_0, \pi_1, \pi_1, \ldots, \pi_{n-1}, \pi_{n-1}$, respectively. □

3.4. Type D. In this section let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(W_n^{\mathsf{D}})$ and $\pi_i \coloneqq \pi_{r_i} \in \mathsf{IdCox}_\beta$. Given $w \in W_n^{\mathsf{D}}$, let $\mathsf{Hecke}_D^{\pm}(w)$ be the set of words $a_1a_2\cdots a_N$ with letters in $[\pm(n-1)] \coloneqq \{\pm1, \pm2, \ldots, \pm(n-1)\}$ such that $\pi_w = \beta^{N-\ell(w)}\pi_{a_1}\pi_{a_2}\cdots\pi_{a_N} \in \mathsf{IdCox}_\beta$, where $\ell(w)$ is the usual Coxeter length.

Definition 3.7. *The* type D Grothendieck polynomial *of $w \in W_n^{\mathsf{D}}$ is*

$$\mathfrak{G}_w^{\mathsf{D}} \coloneqq \sum_{a\in \mathsf{Hecke}_D^{\pm}(w)} \sum_{\mathbf{i}\in \mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}}.$$

We consider the sum

$$\mathfrak{G}^{\mathsf{D}} \coloneqq \sum_{w\in W_n^{\mathsf{D}}} \mathfrak{G}_w^{\mathsf{D}} \cdot \pi_w \in \mathsf{IdCox}_\beta(W_n^{\mathsf{D}}).$$

If we define $A_i^{(\beta)}(x)$ and $D^{(\beta)}(x)$ as in (4) but with $h_i(x)$ replaced by

$$h_i^{(\beta)}(x) \coloneqq 1 + x\pi_i \in \mathsf{IdCox}_\beta(W_n^{\mathsf{D}}) \qquad \text{for } i \in [\pm(n-1)] \text{ and } x \in R,$$

then $\mathfrak{G}^{\mathsf{D}}$ is given by the formula in (5) with A_i and D replaced by $A_i^{(\beta)}$ and $D^{(\beta)}$. Comparing with [9, Def. 9] shows that $\mathfrak{G}_w^{\mathsf{D}}$ is obtained from Kirillov and Naruse's *double Grothendieck polynomial* $\mathcal{G}_w^{\mathsf{D}}(a, b; x)$ by making the substitutions $a_i \mapsto x_i$, $b_i \mapsto 0$, and $x_i \mapsto x_{1-i}$.

PROPOSITION 3.8. *It holds that*

$$\mathfrak{G}^{\mathsf{D}} = \prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} h_{-i}^{(\beta)}(x_{i+2j-1} \oplus x_{2j-1}) \prod_{i=1}^{n-1} h_i^{(\beta)}(x_{i+2j} \oplus x_{2j}) \right).$$

Proof. Similar to Proposition 3.5, the result follows by repeating the proof of Proposition 2.3 after adding "(β)" superscripts to all relevant symbols and substituting $h_i^{(\beta)}(x)h_i^{(\beta)}(y) = h_i^{(\beta)}(x \oplus y)$ wherever the identity $h_i(x)h_i(y) = h_i(x+y)$ is used. □

To state an analogue of Theorem 1.11 for $\mathfrak{G}_w^{\mathsf{D}}$, we consider the ordered alphabet

$$\{-1' \prec -1 \prec -2' \prec -2 \prec \cdots \prec -n' \prec -n \prec 1' \prec 1 \prec 2' \prec 2 \prec \cdots \prec n' \prec n\}.$$

If $w \in W_n^{\mathsf{D}}$ then let $\mathsf{PrimedHecke}_D^{\pm}(w)$ denote the set of words in this alphabet which become elements of $\mathsf{Hecke}_D^{\pm}(w)$ when all primes are removed from its letters. Given such a word $a = a_1 a_2 \cdots a_p$, let $J(a)$ be the set of indices $i \in [p]$ for which a_i is unprimed, and define

$$\Sigma_{\mathsf{D}}(a) \coloneqq \sum_{i \in J(a)} |a_i| \quad \text{and} \quad \mathrm{comaj}_{\mathsf{D}}(a) \coloneqq |\{i : a_i \in \{1', 1, 2', 2, \dots\}\}| + \sum_{a_i \prec a_{i+1}} 2i.$$

For example, if $a = 2', -1', -1, -3, 2$ then $\Sigma_{\mathsf{D}}(a) = 1 + 3 + 2 = 6$ and $\mathrm{comaj}_{\mathsf{D}}(a) = 2 + (4 + 6 + 8) = 20$.

THEOREM 3.9. *If $w \in W_n^{\mathsf{D}}$ then*

$$\mathfrak{G}_w^{\mathsf{D}}(x_i \mapsto q^{i-1}) = \sum_{a \in \mathsf{PrimedHecke}_D^{\pm}(w)} \frac{\beta^{\ell(a)-\ell(w)}}{(q^2-1)(q^4-1)\cdots(q^{2\ell(a)}-1)} q^{\Sigma_{\mathsf{D}}(a)+\mathrm{comaj}_{\mathsf{D}}(a)}$$

where the right hand expression is interpreted as a Laurent series in q^{-1}.

As with Theorem 3.6, this identity reduces to Theorem 1.11 when $\beta = 0$.

Proof. The proof is similar to Theorem 3.6. Proposition 3.8 implies that $\mathfrak{G}^{\mathsf{D}}(x_i \mapsto q^{i-1})$ is

$$\prod_{j=-\infty}^{0} \left(\prod_{i=1}^{n-1} (1 + q^{2(j-1)} \cdot \pi_{-i})(1 + q^{2(j-1)} \cdot q^i \cdot \pi_{-i}) \right.$$
$$\left. \prod_{i=1}^{n-1} (1 + q^{2(j-1)} \cdot q \cdot \pi_i)(1 + q^{2(j-1)} \cdot q^{i+1} \cdot \pi_i) \right).$$

The identity for $\mathfrak{G}_w^{\mathsf{D}}$ follows by extracting the coefficient of π_w from this expression after applying Lemma 2.1 with q replaced by q^2 and with $N = 4n - 4$. When applying the lemma, we set the parameters $z_1, z_2, \dots, z_{2n-2}$ (respectively, $z_{2n-1}, z_{2n}, \dots, z_{4n-4}$) to $1, q, 1, q^2, 1, q^3 \dots$ (respectively, $q, q^2, q, q^3, q, q^4 \dots$), while taking $\mathtt{t}_1, \mathtt{t}_2, \dots, \mathtt{t}_{2n-2}$ (respectively, $\mathtt{t}_{2n-1}, \mathtt{t}_{2n}, \dots, \mathtt{t}_{4n-4}$) to be $\pi_{-1}, \pi_{-1}, \pi_{-2}, \pi_{-2}, \dots$ (respectively, $\pi_1, \pi_1, \pi_2, \pi_2, \dots$). □

4. Involution Grothendieck polynomials

This final section is something of a digression. Here, we reuse the techniques introduced above to give a simple proof of a new formula for certain *involution Grothendieck polynomials*.

In this section, we let $\mathsf{IdCox}_\beta = \mathsf{IdCox}_\beta(S_n)$ be the id-Coxeter algebra for the finite Coxeter system $(W, S) = (S_n, \{s_1, s_2, \dots, s_{n-1}\})$, and write $\pi_i \coloneqq \pi_{s_i} \in \mathsf{IdCox}_\beta$. Let

$$\mathcal{I}_n \coloneqq \left\{ w \in S_n : w = w^{-1} \right\} \quad \text{and} \quad \mathcal{I}_n^{\mathsf{FPF}} \coloneqq \left\{ w^{-1} 1^{\mathsf{FPF}} w : w \in S_n \right\}$$

where $1^{\mathsf{FPF}} = \cdots(1,2)(3,4)(5,6)\cdots$ denotes the permutation of $\mathbb{Z}$ mapping $i \mapsto i-(-1)^i$. The sets $\mathcal{I}_n$ and $\mathcal{I}_n^{\mathsf{FPF}}$ are always disjoint, although when n is even the elements of $\mathcal{I}_n^{\mathsf{FPF}}$ are naturally in bijection with the fixed-point-free elements of $\mathcal{I}_n$.

Let $\mathsf{InvolMod}_\beta$ and $\mathsf{FixedMod}_\beta$ denote the free R-modules consisting of all R-linear combinations of the symbols m_z for $z \in \mathcal{I}_n$ and $z \in \mathcal{I}_n^{\mathsf{FPF}}$, respectively. These sets have unique right IdCox_β-module structures (see [12, § 1.2 and § 1.3]) satisfying, for each integer $1 \leqslant i < n$,

$$m_z\pi_i = \begin{cases} m_{zs_i} & \text{if } z(i) < z(i+1) \text{ and } zs_i = s_iz, \\ m_{s_izs_i} & \text{if } z(i) < z(i+1) \text{ and } zs_i \neq s_iz, \\ \beta m_z & \text{if } z(i) > z(i+1), \end{cases} \qquad \text{for } z \in \mathcal{I}_n$$

and

$$m_z\pi_i = \begin{cases} m_{s_izs_i} & \text{if } z(i) < z(i+1), \\ \beta m_z & \text{if } i+1 \neq z(i) > z(i+1) \neq i, \\ 0 & \text{if } i+1 = z(i) > z(i+1) = i, \end{cases} \qquad \text{for } z \in \mathcal{I}_n^{\mathsf{FPF}}.$$

An *involution Hecke word* for $z \in \mathcal{I}_n$ is a word $a_1a_2\cdots a_p$ such that

$$m_1\pi_{a_1}\pi_{a_2}\cdots\pi_{a_p} = \beta^N m_z \in \mathsf{InvolMod}_\beta \quad \text{for some integer } N \geqslant 0.$$

To avoid excessive subscripts, define

$$m_1^{\mathsf{FPF}} := m_{1^{\mathsf{FPF}}} \in \mathsf{FixedMod}_\beta.$$

An *involution Hecke word* for $z \in \mathcal{I}_n^{\mathsf{FPF}}$ is a word $a_1a_2\cdots a_p$ such that

$$m_1^{\mathsf{FPF}}\pi_{a_1}\pi_{a_2}\cdots\pi_{a_p} = \beta^N m_z \in \mathsf{FixedMod}_\beta \quad \text{for some integer } N \geqslant 0,$$

assuming $\beta^N \neq 0$ for $N \geqslant 0$. Neither of these definitions depends on β, but in the fixed-point-free case we wish to exclude words $a_1a_2\cdots a_p$ for which $z := s_{a_{i-1}}\cdots s_{a_2}s_{a_1}1^{\mathsf{FPF}}s_{a_1}s_{a_2}\ldots s_{a_{i-1}}$ has $a_i+1 = z(a_i) > z(a_i+1) = a_i$ for some i.

Let $\mathsf{InvHecke}(z)$ denote the set of involution Hecke words for an element z in $\mathcal{I}_n$ or $\mathcal{I}_n^{\mathsf{FPF}}$. This set was denoted as either $\mathcal{H}^{\mathsf{O}}(z)$ for $z \in \mathcal{I}_n$ or $\mathcal{H}^{\mathsf{Sp}}(z)$ for $z \in \mathcal{I}_n^{\mathsf{FPF}}$ in [12]. Also define

$$\widehat{\ell}(z) = \min\{\ell(a) : a \in \mathsf{InvHecke}(z)\}.$$

For an explicit formula for $\widehat{\ell}$, see [12, Eq. (5.1)].

Example 4.1. If $y = s_3s_2s_3 = s_2s_3s_2 = (2,4) \in \mathcal{I}_n$, then $\mathsf{InvHecke}(y)$ is the set of all finite words on the alphabet $\{2,3\}$ in which 2 and 3 both appear. If $w = (2,3,4) = s_2s_3 \in S_n$ and $z = w^{-1}1^{\mathsf{FPF}}w = \cdots(-3,-2)(-1,0)(1,4)(2,3)(5,6)(7,8)\cdots \in \mathcal{I}_n^{\mathsf{FPF}}$, then $\mathsf{InvHecke}(z)$ is the set of words obtained by prepending 2 to a nonempty word on $\{1,3\}$. In either case $\widehat{\ell}(y) = \widehat{\ell}(z) = 2$.

Our final theorem concerns these analogues of $\mathfrak{G}_w$:

Definition 4.2. *The* involution Grothendieck polynomial *of* $z \in \mathcal{I}_n \sqcup \mathcal{I}_n^{\mathsf{FPF}}$ *is*

$$\widehat{\mathfrak{G}}_z := \sum_{a \in \mathsf{InvHecke}(z)} \sum_{0 < \mathbf{i} \in \mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i}) - \widehat{\ell}(z)} x_{\mathbf{i}} \in \mathbb{Z}[\beta][x_1, x_2, \ldots, x_{n-1}].$$

If n is even and $z \in \mathcal{I}_n^{\mathsf{FPF}}$ then $\widehat{\mathfrak{G}}_z$ coincides with the *symplectic Grothendieck polynomials* $\mathfrak{G}_z^{\mathsf{Sp}}$ studied in [13, 14]. The paper [13] also introduces certain *orthogonal Grothendieck polynomials* $\mathfrak{G}_z^{\mathsf{O}}$ indexed by $z \in \mathcal{I}_n$, but these are generally not the same as $\widehat{\mathfrak{G}}_z$. However, $\widehat{\mathfrak{G}}_z$ does specialize when $\beta = 0$ to both kinds of *involution Schubert polynomials* $\widehat{\mathfrak{S}}_z$ and $\widehat{\mathfrak{S}}_z^{\mathsf{FPF}}$ considered in [7, 8].

Because $\mathsf{InvolMod}_\beta$ and $\mathsf{FixedMod}_\beta$ are IdCox_β-modules, there exists for each $z \in \mathcal{I}_n \sqcup \mathcal{I}_n^{\mathsf{FPF}}$ a set $\mathsf{HeckeAtoms}(z) \subset S_n$ (see [12, § 2.1]) such that

$$\mathsf{InvHecke}(z) = \bigsqcup_{w \in \mathsf{HeckeAtoms}(z)} \mathsf{Hecke}(w) \quad \text{and} \quad \widehat{\mathfrak{G}}_z = \sum_{w \in \mathsf{HeckeAtoms}(z)} \beta^{\ell(w)-\widehat{\ell}(z)} \mathfrak{G}_w \tag{9}$$

where

$$\mathfrak{G}_w \coloneqq \overleftarrow{\mathfrak{G}}_w(\ldots, 0, 0, x_1, x_2, \ldots, x_{n-1}) = \sum_{a \in \mathsf{Hecke}(w)} \sum_{0<\mathbf{i} \in \mathsf{Compatible}(a)} \beta^{\ell(\mathbf{i})-\ell(w)} x_{\mathbf{i}}$$

for $w \in S_n$.

Again let $h_i^{(\beta)}(x) \coloneqq 1 + x\pi_i \in \mathsf{IdCox}_\beta$ and define

$$A_i^{(\beta)}(x) \coloneqq h_{n-1}^{(\beta)}(x) h_{n-2}^{(\beta)}(x) \cdots h_i^{(\beta)}(x) \tag{10}$$

and

$$\widetilde{A}_i^{(\beta)}(x) \coloneqq h_i^{(\beta)}(x) h_{i+1}^{(\beta)}(x) \cdots h_{n-1}^{(\beta)}(x) \tag{11}$$

for integers $1 \leqslant i < n$ and $x \in R$. Then consider the finite product

$$\mathfrak{G} \coloneqq A_1^{(\beta)}(x_1) A_2^{(\beta)}(x_2) \cdots A_{n-1}^{(\beta)}(x_{n-1}) = \sum_{w \in S_n} \mathfrak{G}_w \cdot \pi_w \in \mathsf{IdCox}_\beta. \tag{12}$$

Next let $\widehat{\mathfrak{G}} \coloneqq m_1 \mathfrak{G}$ and $\widehat{\mathfrak{G}}^{\mathsf{FPF}} \coloneqq m_1^{\mathsf{FPF}} \mathfrak{G}$. It is evident from (9) that

$$\widehat{\mathfrak{G}} = \sum_{z \in \mathcal{I}_n} \widehat{\mathfrak{G}}_z \cdot m_z \in \mathsf{InvolMod}_\beta \quad \text{and} \quad \widehat{\mathfrak{G}}^{\mathsf{FPF}} = \sum_{z \in \mathcal{I}_n^{\mathsf{FPF}}} \widehat{\mathfrak{G}}_z \cdot m_z \in \mathsf{FixedMod}_\beta.$$

Proposition 3.2 is inefficient for computing $\widehat{\mathfrak{G}}_z$ since while $\mathfrak{G}$ contains $\binom{n}{2}$ factors $h_i^{(\beta)}(x_i)$, it turns out that any m_z can be written in the form $m_z \pi_{a_1} \pi_{a_2} \cdots \pi_{a_p}$ where $p \leqslant \binom{n_1}{2} + \binom{n_2}{2}$ for $n_1 = \lceil \frac{n+1}{2} \rceil$ and $n_2 = \lfloor \frac{n+1}{2} \rfloor$. We can derive an involution version of Proposition 3.2, however.

Lemma 4.3. *For any integer $1 \leqslant i < n$ and elements $x_i, \ldots, x_{n-1}, y \in R$ it holds that*

$$\widetilde{A}_i^{(\beta)}(y) A_i^{(\beta)}(x_i) A_{i+1}^{(\beta)}(x_{i+1}) \cdots A_{n-1}^{(\beta)}(x_{n-1}) = \prod_{j=i+1}^{n-1} A_j^{(\beta)}(x_{j-1}) \cdot \prod_{j=i}^{n-1} h_j^{(\beta)}(x_j \oplus y).$$

Proof. Repeat the proof of [6, Lem. 4.1] with the symbols A_i, $\widetilde{A}_j$, h_k replaced by $A_i^{(\beta)}$, $\widetilde{A}_j^{(\beta)}$, $h_k^{(\beta)}$, and then apply the algebra anti-automorphism of IdCox_β that maps $\pi_w \mapsto \pi_{w^{-1}}$ to both sides. □

For $i > j > 0$, define $x_{i \oplus j} = x_{j \oplus i} \coloneqq x_i \oplus x_j = x_i + x_j + \beta x_i x_j$ and $x_{j \oplus j} \coloneqq x_j$.

Proposition 4.4. *The following identities hold:*

1. *We have* $\widehat{\mathfrak{G}} = \prod_{i=1}^{n-1} \prod_{j=\min(i,n-i)}^{1} h_{i+j-1}^{(\beta)}(x_{i \oplus j})$.
2. *If n is even then* $\widehat{\mathfrak{G}}^{\mathsf{FPF}} = \prod_{i=2}^{n-1} \prod_{j=\min(i-1,n-i)}^{1} h_{i+j-1}^{(\beta)}(x_{i \oplus j})$.

In part (2), the indices i and j always satisfy $i > j > 0$ so $x_{i \oplus j} = x_i \oplus x_j$.

Proof. We first prove part (1). The result is trivial when $n = 1$ so assume $n \geqslant 2$. For any $1 \leqslant i < n$ we have $m_1 \pi_i \pi_{i+1} = m_{s_{i+1} s_i s_{i+1}} = m_1 \pi_{i+1} \pi_i$ and consequently $m_1 h_i^{(\beta)}(x) h_j^{(\beta)}(y) = m_1 h_j^{(\beta)}(y) h_i^{(\beta)}(x)$ for all integers i, j and $x, y \in R$. Using this, one checks that $m_1 A_1^{(\beta)}(x) = m_1 \widetilde{A}_1^{(\beta)}(x)$, whence

$$\begin{aligned} \widehat{\mathfrak{G}} &= m_1 A_1^{(\beta)}(x_1) A_2^{(\beta)}(x_2) \cdots A_{n-1}^{(\beta)}(x_{n-1}) = m_1 \widetilde{A}_1^{(\beta)}(x_1) A_2^{(\beta)}(x_2) \cdots A_{n-1}^{(\beta)}(x_{n-1}) \\ &= m_1 h_1^{(\beta)}(x_1) \widetilde{A}_2^{(\beta)}(x_1) A_2^{(\beta)}(x_2) \cdots A_{n-1}^{(\beta)}(x_{n-1}). \end{aligned}$$

Applying Lemma 4.3 with $i=2$ and commuting $h_1^{(\beta)}(x_1)$ to the right gives

$$\widehat{\mathfrak{G}} = m_1 A_3^{(\beta)}(x_2)A_4^{(\beta)}(x_3)\cdots A_{n-1}^{(\beta)}(x_{n-2})h_1^{(\beta)}(x_{1\oplus 1})h_2^{(\beta)}(x_{1\oplus 2})\cdots h_{n-1}^{(\beta)}(x_{1\oplus(n-1)}).$$

We may assume by induction that

$$\begin{aligned} m_1 A_3^{(\beta)}(x_2)A_4^{(\beta)}(x_3)\cdots A_{n-1}^{(\beta)}(x_{n-2}) &= m_1 \prod_{i=1}^{n-3} \prod_{j=\min(i,n-2-i)}^{1} h_{i+j+1}^{(\beta)}(x_{(i+1)\oplus(j+1)}) \\ &= m_1 \prod_{i=2}^{n-2} \prod_{j=\min(i,n-i)}^{2} h_{i+j-1}^{(\beta)}(x_{i\oplus j}). \end{aligned}$$

This gives $\widehat{\mathfrak{G}} = m_1 \prod_{i=2}^{n-2}\prod_{j=\min(i,n-i)}^{2} h_{i+j-1}^{(\beta)}(x_{i\oplus j}) \cdot \prod_{k=1}^{n-1} h_k^{(\beta)}(x_{1\oplus i})$, and it is not hard to see that this formula can be transformed by appropriate commutations to the expression in part (1). For instance, if $n=8$ then what needs to be shown is equivalent to the claim that one can turn the reduced word $3\cdot 54\cdot 765\cdot 76\cdot 7\cdot 1234567$ into $1\cdot 32\cdot 543\cdot 7654\cdot 765\cdot 76\cdot 7$ using only relations of the form $ij \leftrightarrow ji$ for $|i-j|>1$.

The proof of part (2) is similar. Assume n is even and $1 \leqslant i < n$. If i is odd then $m_1^{\mathsf{FPF}}\pi_i = 0$ and $m_1^{\mathsf{FPF}}h_i^{(\beta)}(x) = m_1^{\mathsf{FPF}}$ for all $x \in R$. On the other hand, if i is even and $x, y \in R$ then

$$m_1^{\mathsf{FPF}}\pi_i\pi_{i+1} = m_1^{\mathsf{FPF}}\pi_i\pi_{i-1} \quad \text{and} \quad m_1^{\mathsf{FPF}}h_i^{(\beta)}(x)h_{i+1}^{(\beta)}(y) = m_1^{\mathsf{FPF}}h_i^{(\beta)}(x)h_{i-1}^{(\beta)}(y).$$

Using these relations repeatedly we deduce that $m_1^{\mathsf{FPF}}A_i^{(\beta)}(x) = m_1^{\mathsf{FPF}}\widetilde{A}_{i+1}^{(\beta)}(x)$ for any odd integer $1 \leqslant i < n$. By Lemma 4.3, we therefore have

$$\begin{aligned} \widehat{\mathfrak{G}}^{\mathsf{FPF}} &= m_1^{\mathsf{FPF}}A_1(x_1)A_2(x_2)\cdots A_{n-1}(x_{n-1}) = m_1^{\mathsf{FPF}}\widetilde{A}_2^{(\beta)}(x_1)A_2^{(\beta)}(x_2)\cdots A_{n-1}^{(\beta)}(x_{n-1}) \\ &= m_1^{\mathsf{FPF}}A_3^{(\beta)}(x_2)A_4^{(\beta)}(x_3)\cdots A_{n-1}^{(\beta)}(x_{n-2}) \\ &\qquad \cdot h_2^{(\beta)}(x_1\oplus x_2)h_3^{(\beta)}(x_1\oplus x_3)\cdots h_{n-1}^{(\beta)}(x_1\oplus x_{n-1}). \end{aligned}$$

From here, the result follows by induction as in the proof of part (1). □

Let $\diagdown_n := \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : i \geqslant j > 0\}$ and $\diagdown_n^{\neq} := \{(i,j)\in\mathbb{Z}\times\mathbb{Z} : i > j > 0\}$. Equip these sets with the total order defined by $(i,j) \prec (k,l)$ if $i<k$ or if $i=k$ and $j>l$. An *involution Hecke pipe dream* for $z\in\mathcal{I}_n$ (respectively, $z\in\mathcal{I}_n^{\mathsf{FPF}}$) is a finite subset D of $\diagdown_n$ (respectively, $\diagdown_n^{\neq}$) such that the word formed by listing the numbers $i+j-1$ as (i,j) runs over D in the order $\prec$ belongs to $\mathsf{InvHecke}(z)$. We write $\mathsf{InvDreams}(z)$ for the set of these subsets.

Theorem 4.5. *If $z\in\mathcal{I}_n$ or if n is even and $z\in\mathcal{I}_n^{\mathsf{FPF}}$ then*

$$\widehat{\mathfrak{G}}_z = \sum_{D\in\mathsf{InvDreams}(z)} \beta^{|D|-\widehat{\ell}(z)} \prod_{(i,j)\in D} x_{i\oplus j}$$

where we set $x_{i\oplus i} := x_i$ for $i>0$ and $x_{i\oplus j} := x_i + x_j + \beta x_i x_j$ for $i>j>0$.

When $\beta=0$ our result reduces to [8, Thm. 1.5], which was proved in a different way using somewhat involved recurrences. The methods here give a new and simpler proof. For generic β, Theorem 4.5 resolves the symplectic half of [8, Problem 6.9].

Proof. First assume $z\in\mathcal{I}_n$. Part (1) of Proposition 4.4 implies

$$\widehat{\mathfrak{G}}_z = \sum_{a=a_1\cdots a_N\in\mathsf{InvHecke}(z)} \beta^{N-\widehat{\ell}(z)} \sum_{\substack{0<\mathbf{i}=(i_1\leqslant\cdots\leqslant i_N)\in\mathsf{Compatible}(a) \\ i_j\leqslant a_j<2i_j\ \forall j}} x_{i_1\oplus(a_1-i_1+1)}\cdots x_{i_N\oplus(a_N-i_N+1)}.$$

One now checks that the map sending $(a,\mathbf{i})$ to $D=\{(i_j, a_j-i_j+1) : 1\leqslant j\leqslant \ell(a)\}$ is a bijection from the pairs indexing this double summation to the elements

of $\mathsf{InvDreams}(z)$. When n is even and $z \in \mathcal{I}_n^{\mathsf{FPF}}$, the same argument using part (2) of Proposition 4.4 gives the desired formula. □

EXAMPLE 4.6. Suppose $n = 4$. If $y = s_3s_2s_3 = s_2s_3s_2 = (2,4) \in \mathcal{I}_n$ as in Example 4.1, then the elements of $\mathsf{InvDreams}(y)$ are the sets of nonzero positions in the matrices

$$\begin{bmatrix} 0&0&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0 \\ 1&1&0 \\ 1&0&0 \end{bmatrix},$$

which are $\{(2,1),(2,2)\}$, $\{(2,1),(3,1)\}$, and $\{(2,1),(2,2),(3,1)\}$. By Theorem 4.5,

$$\widehat{\mathfrak{G}}_y = (x_2 \oplus x_1)x_2 + (x_2 \oplus x_1)(x_3 \oplus x_1) + \beta(x_2 \oplus x_1)x_2(x_3 \oplus x_1).$$

If $z = s_3 \cdot s_2 \cdot 1^{\mathsf{FPF}} \cdot s_2 \cdot s_3 \in \mathcal{I}_n^{\mathsf{FPF}}$ as in Example 4.1, then $\mathsf{InvDreams}(z)$ contains just one element $\{(2,1),(3,1)\}$, and Theorem 4.5 asserts $\widehat{\mathfrak{G}}_z = (x_2 \oplus x_1)(x_3 \oplus x_1)$.

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