Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2008, Vol. 44, No. 6, 1162–1172
DOI: 10.1214/07-AIHP176

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Determinantal transition kernels for some interacting particles on the line

A. B. Dieker[a,*] and J. Warren[b]

[a] *H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: ton.dieker@isye.gatech.edu*
[b] *University of Warwick, Department of Statistics, Coventry, CV4 7AL, United Kingdom. E-mail: j.warren@warwick.ac.uk*

Received 25 July 2007; accepted 12 November 2007

Abstract. We find the transition kernels for four Markovian interacting particle systems on the line, by proving that each of these kernels is intertwined with a Karlin–McGregor-type kernel. The resulting kernels all inherit the determinantal structure from the Karlin–McGregor formula, and have a similar form to Schütz's kernel for the totally asymmetric simple exclusion process.

Résumé. Nous trouvons les noyaux de transition de quatre systèmes markoviens de particules en interaction sur une ligne, en prouvant que chacun de ces noyaux s'entrelace avec un noyau du type de Karlin–McGregor. Tous les noyaux résultants héritent de la structure de déterminant de la formule de Karlin–McGregor et ont une forme similaire à celle du noyau de Schütz pour le processus d'exclusion simple totalement asymétrique.

MSC: Primary 60J05; 60K35; 05E10; secondary 05E05; 15A52

Keywords: Interacting particle system; intertwining; Karlin–McGregor theorem; Markov transition kernel; Robinson–Schensted–Knuth correspondence; Schütz theorem; stochastic recursion; symmetric functions

1. Introduction

Non-colliding Markov processes are canonical examples of stochastic processes with a determinantal transition kernel, given by the Karlin–McGregor formula. A determinantal transition kernel of a different form, yet similar to the Karlin–McGregor kernel, was encountered by Schütz [17] in his study of the totally asymmetric simple exclusion process. This work has stimulated much recent research, e.g., [4,10,14,16,21,22].

In this note we explicitly connect Schütz type formulae for particle systems and Karlin–McGregor-type formulae for non-colliding processes. Our approach builds upon deep connections between particle processes and non-colliding processes (or random-matrix theory), which have been recently discovered [2,6,9,13]. A combinatorial correspondence known as the Robinson–Schensted–Knuth (RSK) correspondence links these processes. This RSK correspondence gives a *coupling* of the non-colliding process and the particle process. Our key contributions are that this coupling implies an *intertwining* of their transition semigroups, and that this intertwining is enough to find the transition kernel of the particle process.

We give a number of new formulae by applying this method to four variants of the RSK correspondence. Augmented with systems arising from suitable limiting procedures, the particle systems we treat in this way are known to play pivotal roles in a wide range of interesting applied problems. For instance, they appear in the context of queues in series, last-passage percolation, growth models, and fragmentation models.

---

*Work was done at University College Cork.

This note is organized as follows: In Section 2, we introduce four interacting particle systems and we present the associated transition kernels. Section 3 describes the four variants of the RSK correspondence we use in our analysis, and derives the aforementioned intertwining of the semigroups. Finally, it is the topic of Section 4 to use this intertwining for finding the transition mechanism of the interacting particles.

2. Interacting particles on the line; main results

We are concerned with a system of N particles, each with a position in the integer lattice $\mathbf{Z}$, evolving in discrete time. We will consider four possible cases. Particles will move from the left to the right making either Bernoulli or geometrically sized jumps, with one of two possible interactions that maintain their relative orderings (blocking or pushing). We begin by describing these four processes more precisely. In each case $Y_i(n)$ denotes the position of particle number i at time n. We order the particles, so that Y takes values in either W^N or $\hat{W}^N$, where

$$W^N = \bigl(z \in \mathbf{Z}^N; z_N \le z_{N-1} \le \cdots \le z_1\bigr),$$
$$\hat{W}^N = \bigl(z \in \mathbf{Z}^N; z_1 \le z_2 \le \cdots \le z_N\bigr).$$

Throughout, we let $p = (p_1, p_2, \ldots, p_N)$ be a vector with each $p_k \in (0, 1)$.

Case A: Geometric jumps with pushing. Particles are labelled from left to right, so $Y_1(n) \le Y_2(n) \le \cdots \le Y_N(n)$. Between time $n-1$ and n, each of the particles moves to the right according to some geometrically distributed jump, having parameter p_i for particle i. The order in which the particles jump is given by their labels, so the leftmost particle jumps first. Overtaken particles (if any) are moved to the same position as the jumping particle, a position from which the next particle subsequently makes its own jump. One can thus think of particles 'pushing' other particles to maintain their relative orderings.

This leads to the following stochastic recursion. The evolution is generated from a family $(\xi(k,n); k \in \{1,2,\ldots,N\}, n \in \mathbf{N})$ of independent geometric random variables satisfying $\mathbf{P}(\xi(k,n) = r) = (1-p_k)p_k^r$ for $r = 0, 1, 2, \ldots,$ via the recursions $Y_1(n) = Y_1(n-1) + \xi(1,n)$, and for $k = 2, 3, \ldots, N$,

$$Y_k(n) = \max\bigl(Y_k(n-1), Y_{k-1}(n)\bigr) + \xi(k,n).$$

Note that $Y = (Y(n); n \ge 0)$ is a Markov chain on $\hat{W}^N$.

One application area where this recursion arises is the theory of queueing networks. Indeed, the particle system with exponentially distributed jumps, which is obtained after a suitable limiting procedure, relates to a series Jackson network. Here $Y(n)$ corresponds to the departure instants of the nth customer from each of N queues in series. These networks are investigated further in our companion paper [5].

The vector Y also plays an important role in the context of directed last-passage percolation with geometrically distributed travel times and 'origin' $(1,1)$, where $Y(n)$ can be interpreted as the vector of maximal travel times to the sites $(n+1,1), \ldots, (n+1,N)$. Very recently, Johansson [10] has derived the transition kernel of Y in the case of equal rates $p_1 = \cdots = p_N$, with different methods than presented here.

Case B: Bernoulli jumps with blocking. Particles are labelled from right to left, so $Y_N(n) \le Y_{N-1}(n) \le \cdots \le Y_1(n)$. Between time $n-1$ and n each particle attempts to move one step to the right, but it is constrained not to overtake the particle to its right. Particle i moves with probability p_i. The particles are now updated from right to left, so it is the updated position of the particle to the right that acts as a block.

The evolution is generated from a family $(\xi(k,n); k \in \{1,2,\ldots,N\}, n \in \mathbf{N})$ of independent Bernoulli random variables satisfying $\mathbf{P}(\xi(k,n) = +1) = 1 - \mathbf{P}(\xi(k,n) = 0) = p_k$, via the recursions $Y_1(n) = Y_1(n-1) + \xi(1,n)$, and for $k = 2, 3, \ldots, N$,

$$Y_k(n) = \min\bigl(Y_k(n-1) + \xi(k,n), Y_{k-1}(n)\bigr).$$

In particular, Y is a Markov chain on W^N.

The process Y has been investigated by Rákos and Schütz [15] in the context of a fragmentation model. On shifting the ith particle i positions to the left, Y corresponds to the discrete-time totally asymmetric simple exclusion process (TASEP) with sequential updating. Moreover, the process arises in a directed first-passage percolation model known as the Seppäläinen model [18] with 'origin' $(0, 0)$, where $Y(n)$ corresponds to the vector of instants at which the sites $(n, 0), \ldots, (n, N-1)$ become wet.

An important process arises if we scale Y as in the law of small numbers, i.e., by setting $p_i = \alpha_i/M$ and considering $Y(\lfloor Mt \rfloor)$ for $t \in \mathbf{R}_+$ as $M \to \infty$. In the case of equal rates, the resulting continuous-time Markov process describes, after a deterministic shift, the positions of N particles in the (continuous-time) TASEP. This is the framework originally studied by Schütz [17], who derives the transition kernel of this process. It has recently been extended to particles hopping at different rates by Rákos and Schütz [16].

The same continuous-time process is also of significant importance for series Jackson queueing networks as well as for a corner-growth model. In the queueing context, it represents the cumulative number of departures from each of the queues; see [5]. In the corner-growth model, it represents the height of the first N columns. We refer to König's survey paper [11] for these and further connections, such as the relation between the Nth component of this process and directed last-passage percolation with exponentially distributed travel times.

Case C: Geometric jumps with blocking. Once again particles are labelled from right to left, so $Y_N(n) \le Y_{N-1}(n) \le \cdots \le Y_1(n)$. Between time $n-1$ and n each particle attempts to move a geometrically distributed number of steps to the right, starting with the leftmost particle. As in case B, a particle is blocked by the closest particle to its right if it tries to overtake another particle, but this time it is the old position of that particle that acts as a block.

The evolution is generated from a family $(\xi(k, n); k \in \{1, 2, \ldots, N\}, n \in \mathbf{N})$ of independent geometric random variables satisfying $\mathbf{P}(\xi(k, n) = r) = (1 - p_k) p_k^r$ for $r = 0, 1, 2, \ldots,$ via the recursions $Y_1(n) = Y_1(n-1) + \xi(1, n)$, and for $k = 2, 3, \ldots, N$,

$$Y_k(n) = \min\big(Y_k(n-1) + \xi(k, n), Y_{k-1}(n-1)\big).$$

As laid out by Draief et al. [6], this recursion arises in the study of so-called tandem stores in series. The random vector Y represents the cumulative demand met at each of the stores when the first store is saturated. Alternatively, Y can be interpreted as the minimum-weight vector of certain weighted lattice paths.

Case D: Bernoulli jumps with pushing. Now it is natural to label particles left to right, so $Y_1(n) \le Y_2(n) \le \cdots \le Y_N(n)$. We update from right to left, and preserve the ordering by pushing particles to the right. The evolution is generated from a family $(\xi(k, n); k \in \{1, 2, \ldots, N\}, n \in \mathbf{N})$ of independent Bernoulli random variables satisfying $\mathbf{P}(\xi(k, n) = +1) = 1 - \mathbf{P}(\xi(k, n) = 0) = p_k$, via the recursions $Y_1(n) = Y_1(n-1) + \xi(1, n)$, and for $k = 2, 3, \ldots, N$,

$$Y_k(n) = \max\big(Y_k(n-1) + \xi(k, n), Y_{k-1}(n)\big).$$

The process Y is the discrete-time analogue of a particle system studied by Alimohammadi et al. [1], which has been studied further by Borodin and Ferrari [3]. It also plays an important role in the directed *last*-passage analogue of the Seppäläinen model.

We summarize the description of the four cases in Table 1.

Table 1
Summary of the four cases

Case	A	B	C	D
Jump distribution	geometric	Bernoulli	geometric	Bernoulli
Interaction	pushing	blocking	blocking	pushing
Updating	from left	from right	from left	from right

We need some well-known symmetric functions in order to present the Markov transition kernel of Y in each of the four cases. The rth complete homogeneous symmetric polynomials in the indeterminates $\alpha_1, \dots, \alpha_N$ are given by

$$h_r(\alpha) = \sum_{k_1 \geq 0, \dots, k_N \geq 0; k_1 + k_2 + \dots + k_N = r} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N}.$$

By convention $h_0 = 1$ and $h_r = 0$ for $r < 0$. Now for $1 \leq i < j \leq N$, let $h_r^{(ij)}(\alpha) = h_r(\alpha^{(ij)})$ where $\alpha^{(ij)}$ is the N-vector $(0, \dots, 0, \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j, 0, \dots 0)$ obtained from α by setting the first i weights, and the last $N - j$ weights equal to 0. Equivalently it is the rth complete homogeneous symmetric polynomial in the indeterminates $\alpha_{i+1}, \dots, \alpha_j$. We set $h_r^{(jj)}(\alpha) = \mathbf{1}(r = 0)$.

We also need e_r, the rth elementary symmetric function defined as

$$e_r(\alpha) = \sum_{k_1 < k_2 < \dots < k_r} \alpha_{k_1} \cdots \alpha_{k_r}.$$

In analogy with the complete homogeneous symmetric functions, we use the conventions $e_r^{(jj)}(\alpha) = \mathbf{1}(r = 0)$ and $e_0 = 1$. We also set $e_r^{(ij)}(\alpha) = e_r(\alpha^{(ij)})$ and $e_r = 0$ for $r < 0$.

Given a function f on $\mathbf{Z}$ and a vector $\alpha \in \mathbf{R}_+^N$, we write

$$f_\alpha^{(ij)}(k) = \begin{cases} \sum_{\ell=0}^{i-j} (-1)^\ell e_\ell^{(ji)}(\alpha) f(k+\ell) & \text{if } j \leq i, \\ \sum_{\ell=0}^{\infty} h_\ell^{(ij)}(\alpha) f(k+\ell) & \text{if } i \leq j, \end{cases} \tag{1}$$

and

$$\hat{f}_\alpha^{(ij)}(k) = \begin{cases} \sum_{\ell=0}^{i-j} (-1)^\ell e_\ell^{(ji)}(\alpha) f(k-\ell) & \text{if } j \leq i, \\ \sum_{\ell=0}^{\infty} h_\ell^{(ij)}(\alpha) f(k-\ell) & \text{if } i \leq j, \end{cases}$$

provided all series converge absolutely. Our main theorem uses this notation for functions f belonging to two different families, $(w_n; n \in \mathbf{Z})$ and $(v_n; n \in \mathbf{Z})$, for which the desired convergence holds. These families are defined through

$$w_n(k) = \binom{n-1+k}{k} \mathbf{1}(k \geq 0; n \geq 0) \quad \text{and} \quad v_n(k) = \binom{n}{k} \mathbf{1}(0 \leq k \leq n; n \geq 0).$$

We write $w_{n,\alpha}^{(ij)}$ for $f_\alpha^{(ij)}$ with $f = w_n$, and define $\hat{w}_{n,\alpha}^{(ij)}$, $v_{n,\alpha}^{(ij)}$, and $\hat{v}_{n,\alpha}^{(ij)}$ similarly. Moreover, we abbreviate the vector $(p_1^{-1}, \dots, p_N^{-1})$ by p^{-1}, and define the vector π through $\pi_i = p_i/(1 - p_i)$.

Theorem 1. *The transition kernel Q_n of the process Y is given by the following expressions*:

Case A: Geometric jumps with pushing. *We have for $y, y' \in \hat{W}^N$*,

$$Q_n(y, y') = \prod_{k=1}^N \left[(1-p_k)^n p_k^{y'_k - y_k}\right] \det\left\{\hat{w}_{n,p^{-1}}^{(ij)}\left(y'_i - y_j + i - j\right)\right\}.$$

Case B: Bernoulli jumps with blocking. *We have for $y, y' \in W^N$*,

$$Q_n(y, y') = \prod_{k=1}^N \left[(1-p_k)^n \pi_k^{y'_k - y_k}\right] \det\left\{v_{n,\pi}^{(ij)}\left(y'_i - y_j - i + j\right)\right\}.$$

Case C: Geometric jumps with blocking. *We have for $y, y' \in W^N$*,

$$Q_n(y, y') = \prod_{k=1}^N \left[(1-p_k)^n p_k^{y'_k - y_k}\right] \det\left\{w_{n,p}^{(ij)}\left(y'_i - y_j - i + j\right)\right\}.$$

Case D: Bernoulli jumps with pushing. *We have for* $y, y' \in \hat{W}^N$,

$$Q_n(y, y') = \prod_{k=1}^{N} \left[(1 - p_k)^n \pi_k^{y'_k - y_k}\right] \det\left\{\hat{v}^{(ij)}_{n,\pi^{-1}}\left(y'_i - y_j + i - j\right)\right\}.$$

The remainder of this paper is devoted to a proof of this theorem.

3. The RSK correspondence and its variants

In each of the four cases considered in the previous section, the Markov process of interest $(Y(n); n \in \mathbf{N})$ is constructed from a family $(\xi(k, n); k \in \{1, 2, \ldots, N\}, n \in \mathbf{N})$ of random innovations. In this section we will construct a second Markov process Z from the same innovations data, using the RSK algorithm or one of its variants. In each case, we will be able to find the (Karlin–McGregor-type) transition semigroup of Z and show that it is intertwined with the transition semigroup of Y.

We give some definitions in order to describe the RSK-type algorithms in the form we need. A partition λ with k parts is an integer vector $\lambda_1, \ldots, \lambda_k$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Consider an array of strictly positive integers $T = (T_{ij}; 1 \leq i \leq k, 1 \leq j \leq \lambda_i)$ of shape λ satisfying $T_{ij} \leq T_{i+1,j}$ and $T_{ij} \leq T_{i,j+1}$ (interpret T_{ij} as infinity if it is undefined). We write $T \in \mathbf{T}_N^{n,\wedge}$ if the integers in T do not exceed n and increase *strictly* down the columns, while λ consists of at most N parts. In the terminology of enumerative combinatorics, $\mathbf{T}_N^{n,\wedge}$ consists of semi-standard Young tableaux (SSYT) with at most N rows and content $\{1, \ldots, n\}$. Similarly, we write $T \in \mathbf{T}_N^{n,<}$ if the integers in T do not exceed n and increase strictly along the rows, while λ consists of at most N parts. We write $\mathrm{sh}(T)$ for the shape of T, which we consider to be an element of W^N by padding the vector with zeros if necessary (N is fixed throughout).

We study four different ways to associate a $\mathbf{T}_N^{N,\wedge}$-valued process $(\mathcal{P}(n); n \geq 0)$ to the data $(\xi(k, n); k \in \{1, \ldots, N\}, n \in \mathbf{N})$, each corresponding to a different variant of RSK. More details on the different variants can for instance be found in [7,8].

We begin by noting two methods for constructing a two-line array of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{pmatrix}$$

from the innovation data, where the a_i are nondecreasing. Both methods have the property that the column $\binom{a}{b}$ appears $\xi(b, a)$ times in the array. The first method, *lexicographic array construction*, requires that $b_i \leq b_{i+1}$ if $a_i = a_{i+1}$. The second method requires that $b_i \geq b_{i+1}$ if $a_i = a_{i+1}$, and we therefore call it the *anti-lexicographic array construction*.

Next we describe two methods for constructing a sequence $(\mathcal{P}(n); n \geq 0)$ of SSYT in $\mathbf{T}_N^{N,\wedge}$ from the given two-line array. Both constructions are inductive, and start with an empty SSYT $\mathcal{P}(0)$. Given $\mathcal{P}(n)$, the SSYT $\mathcal{P}(n+1)$ is found by inserting the elements b_i for which $a_i = n + 1$. If there are M such elements, the methods construct a sequence $\mathcal{P}^1(n), \ldots, \mathcal{P}^M(n)$ such that $\mathcal{P}^1(n) = \mathcal{P}(n)$ and $\mathcal{P}(n+1) = \mathcal{P}^M(n)$. The SSYT $\mathcal{P}^{i+1}(n)$ is constructed from $\mathcal{P}^i(n)$ by inserting the next unused b-element of the two-line array. The first method, *row insertion*, inserts an element b into $\mathcal{P}^{i+1}(n)$ using the following rules:

- If every entry in the first row of $\mathcal{P}^i(n)$ is *smaller than or equal* to b, then b is appended to the end of the row.
- Otherwise, b is used to replace the leftmost entry in the row which is *strictly larger* than b.

The entry replaced is inserted in the same manner into the second row, and this process continues until an entry is either placed at the end of a row or it is placed in the first position of an empty row. The second method, *column insertion*, is a modification of the above procedure. Instead of inserting entries along the rows, the entries are now inserted down the columns. Now the following rules are followed:

- If every entry in the column is *strictly smaller* than b, then b is appended to the end of the column.
- Otherwise, b is used to replace the uppermost entry in the column which is *greater than or equal* to b.

Four combinations of ξ-values, array constructions, and insertion algorithms are of special interest, as there is a combinatorial correspondence underlying the above construction of the process $\mathcal{P}$. To explain this, let $\mathcal{Q}(n)$ be

Table 2
The four correspondences

Correspondence	RSK	Dual RSK	Burge	Dual Burge
Innovations data	$0, 1, 2, \dots$	$0, 1$	$0, 1, 2, \dots$	$0, 1$
Range $(\mathcal{P}(n), \mathcal{Q}(n))$	(2)	(3)	(2)	(3)
Insertion algorithm	row	column	column	row
Array construction	lexicographic	lexicographic	anti-lexicographic	anti-lexicographic

the unique array of integers for which the entries $1, \dots, m$ form an array with the same shape as $\mathcal{P}(m)$ for $m \le n$. Depending on the RSK variant chosen, the pair $(\mathcal{P}(n), \mathcal{Q}(n))$ belongs either to

$$\bigl\{(S, T) \in \mathbf{T}_N^{N,\wedge} \times \mathbf{T}_N^{n,\wedge}\colon \operatorname{sh}(S) = \operatorname{sh}(T)\bigr\} \tag{2}$$

or to

$$\bigl\{(S, T) \in \mathbf{T}_N^{N,\wedge} \times \mathbf{T}_N^{n,<}\colon \operatorname{sh}(S) = \operatorname{sh}(T)\bigr\}. \tag{3}$$

The four combinatorial correspondences map the sequence $(\xi(k, m); k \in \{1, 2, \dots, N\}, m \in \{1, 2, \dots, n\})$ *bijectively* to $(\mathcal{P}(n), \mathcal{Q}(n))$ for any $n \ge 1$. We present these four combinations in Table 2, along with the name under which the resulting bijection (correspondence) is known.

It is our next aim to relate the four correspondences to the interacting-particle framework of Section 2. Further analysis is facilitated by a second process $Z = (Z(n); n \ge 0)$ arising from the process $\mathcal{P}$. We define this process by letting $Z(n)$ be the shape of $\mathcal{P}(n)$, so Z takes values in W^N. After giving the transition kernel of Z, we prove that the transition semigroup of Y is intertwined with the transition semigroup of Z.

The semi-standard Young tableaux in $\mathbf{T}_N^{N,\wedge}$ play an important role in our analysis since the process $\mathcal{P}$ is $\mathbf{T}_N^{N,\wedge}$-valued, and we need some further definitions for such tableaux. For $T \in \mathbf{T}_N^{N,\wedge}$, we let $\operatorname{ledge}(T)$ be the N-vector for which element i is the number of i's in row i (the terminology 'ledge' is motivated in the next section). This vector may contain zeros and we always have $\operatorname{ledge}(T) \in W^N$. Similarly, we let $\operatorname{redge}(T)$ be the N-vector for which the ith entry is the number of elements in the first row that do not exceed i; we always have $\operatorname{redge}(T) \in \hat{W}^N$. For a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of weights, we define the weight α^T of $T \in \mathbf{T}_N^{N,\wedge}$ by

$$\alpha^T = \prod_{i=1}^N \alpha_i^{n_i(T)},$$

where $n_i(T)$ is the number of i's in T. This is the usual definition for the weight of a SSYT.

Case A: Geometric jumps with pushing. Consider the RSK algorithm, i.e., lexicographic array construction and row insertion. The row-insertion algorithm shows that the vector-valued process $(\operatorname{redge}(\mathcal{P}(n)); n \ge 0)$ is *exactly the same* as the process $(Y(n); n \ge 0)$. Note also that Y is Markov relative to the filtration of the Markov process $\mathcal{P}$.

The dynamics of the RSK algorithm show that $\mathcal{Q}(n)$ is a SSYT. Therefore, using the bijective property of RSK we obtain for $S \in \mathbf{T}_N^{N,\wedge}$, $T \in \mathbf{T}_N^{n,\wedge}$,

$$\mathbf{P}\bigl(\mathcal{P}(n) = S, \mathcal{Q}(n) = T\bigr) = \prod_{k=1}^N [1 - p_k]^n p^S \mathbf{1}\bigl(\operatorname{sh}(S) = \operatorname{sh}(T)\bigr). \tag{4}$$

Using the fact that $\mathcal{Q}(n)$ encodes the shapes $(Z(m); m \le n)$, as argued in [12], Section 3.2, we can use (4) to compute the law of the shape process Z. Indeed, summing (4) over appropriate $S \in \mathbf{T}_N^{N,\wedge}$, $T \in \mathbf{T}_N^{n,\wedge}$ we obtain for $m \le n$,

$$\mathbf{P}\bigl(Z(n) = z(n), Z(m) = z(m), \dots, Z(1) = z(1)\bigr) = \prod_{k=1}^N [1 - p_k]^n s_{z(n)}(p) g_{z(n)/z(m)}^{n-m},$$

provided the left-hand side is nonzero. Here $s_z(p) = \sum_{T \in \mathbf{T}_N^{N,\wedge}; \mathrm{sh}(T)=z} p^T$ is a symmetric function (in p) known as a Schur polynomial. Also, $g^k_{\lambda/\mu}$ is the number of skew SSYT with shape λ/μ and entries from $\{1, \ldots, k\}$ [20], Section 7.10. The Jacobi–Trudi identity [20], Theorem 7.16.1, implies

$$g^k_{\lambda/\mu} = \det\{w_k(\lambda_i - \mu_j - i + j)\},$$

and we thus find that Z is a Markov chain on W^N with transition kernel given by

$$P_n(z, z') = \prod_{k=1}^N [1 - p_k]^n \frac{s_{z'}(p)}{s_z(p)} \det\{w_n(z'_i - z_j - i + j)\}.$$

A similar reasoning, now summing (4) over $S \in \mathbf{T}_N^{N,\wedge}$ with $\mathrm{redge}(S) = y \in \hat{W}^N$ and $\mathrm{sh}(S) = Z(n)$, shows that

$$\mathbf{P}(Y(n) = y | Z(m), m \le n) = \hat{K}_p(Z(n), y),$$

where

$$\hat{K}_\alpha(z, y) = \frac{1}{s_z(\alpha)} \sum_{T \in \mathbf{T}_N^{N,\wedge}; \mathrm{sh}(T)=z,\ \mathrm{redge}(T)=y} \alpha^T.$$

Note that $\mathbf{P}(Y(n) = y | Z(m), m \le n)$ depends on $(Z(m); m \le n)$ only through $Z(n)$. This yields for $y \in \hat{W}^N$ and $m \le n$,

$$\begin{aligned}
\mathbf{P}(Y(n) = y | Z(k), k \le m) &= \mathbf{P}[\mathbf{P}(Y(n) = y | Z(k), k \le n) | Z(k), k \le m] \\
&= \mathbf{P}[\hat{K}_p(Z(n), y) | Z(k), k \le m] \\
&= \sum_{z \in W^N} P_{n-m}(Z(m), z) \hat{K}_p(z, y).
\end{aligned}$$

On the other hand, the left-hand side can be written as

$$\begin{aligned}
\mathbf{P}(Y(n) = y | Z(k), k \le m) &= \mathbf{P}[\mathbf{P}(Y(n) = y | \mathcal{P}(k), k \le m) | Z(k), k \le m] \\
&= \mathbf{P}[Q_{n-m}(Y(m), y) | Z(k), k \le m] \\
&= \sum_{y' \in \hat{W}^N} \hat{K}_p(Z(m), y') Q_{n-m}(y', y).
\end{aligned}$$

On combining these two displays, we deduce the intertwining relationship $P_n \hat{K}_p = \hat{K}_p Q_n$, where the product $P\hat{K}$ of the kernel P with domain $W^N \times W^N$ and the kernel $\hat{K}$ with domain $W^N \times \hat{W}^N$ is defined as $P\hat{K}(z, y) = \sum_{z' \in W^N} P(z, z') \hat{K}(z', y)$. The next section investigates the intertwining relationship in detail to find the kernel Q_n.

Case B: Bernoulli jumps with blocking. Under lexicographic array construction and column insertion, $(\mathrm{ledge}(\mathcal{P}(n)); n \ge 0)$ is exactly the same as the process $(Y(n); n \ge 0)$. The bijective property of the dual RSK shows that for $S \in \mathbf{T}_N^{N,\wedge}$, $T \in \mathbf{T}_N^{n,<}$,

$$\mathbf{P}(\mathcal{P}(n) = S, \mathcal{Q}(n) = T) = \prod_{k=1}^N [1 - p_k]^n \pi^S \mathbf{1}(\mathrm{sh}(S) = \mathrm{sh}(T)).$$

As in [12], Section 3.3 on combining this with the dual Jacobi–Trudi identity, we find that Z is a Markov chain on W^N with transition kernel

$$P_n\big(z,z'\big)=\prod_{k=1}^{N}[1-p_k]^n\frac{s_{z'}(\pi)}{s_z(\pi)}\det\big\{v_n\big(z'_i-z_j-i+j\big)\big\}.$$

After setting

$$K_\alpha(z,y)=\frac{1}{s_z(\alpha)}\sum_{T\in\mathbf{T}^{N,\wedge};\mathrm{sh}(T)=z,\ \mathrm{ledge}(T)=y}\alpha^T,$$

we may derive the intertwining $P_nK_\pi=K_\pi Q_n$ along the lines of case A.

Case C: Geometric jumps with blocking. The dynamics of the column insertion algorithm show that (ledge($\mathcal{P}(n)$); $n\ge 0$) is exactly the same as the process $(Y(n);n\ge 0)$. Moreover, since the law of the process Z is invariant under the choice of the insertion algorithm, Z is Markovian with the same kernel as in case A. The intertwining is $P_nK_p=K_pQ_n$.

Case D: Bernoulli jumps with pushing. Now (redge($\mathcal{P}(n)$); $n\ge 0$) is exactly the same as the process $(Y(n);n\ge 0)$. The process Z is Markovian with the same kernel as in case B, and we have the intertwining $P_n\hat{K}_\pi=\hat{K}_\pi Q_n$.

4. Determinantal intertwining kernels

It is the aim of this section to show how the kernel Q_n of Y can be recovered from any of the intertwining identities $P_n\hat{K}_\alpha=\hat{K}_\alpha Q_n$ and $P_nK_\alpha=K_\alpha Q_n$, where P_n is the (known) Karlin–McGregor-type kernel of the process Z. In fact, we prove the stronger assertion that the intertwining kernels K_α and $\hat{K}_\alpha$ are invertible.

In what follows, it is convenient to embed $\mathbf{T}_N^{N,\wedge}$ in a space parametrized by an array of variables $\mathbf{x}=(x^1,\dots,x^N)$ with $x^k=(x_1^k,x_2^k,\dots,x_k^k)\in\mathbf{Z}^k$, such that the coordinates satisfy the inequalities

$$x_k^k\le x_{k-1}^{k-1}\le x_{k-1}^k\le x_{k-2}^{k-1}\le\cdots\le x_2^k\le x_1^{k-1}\le x_1^k$$

for $k=2,\dots,N$. The pattern $\mathbf{x}$ corresponding to a given SSYT in $\mathbf{T}_N^{N,\wedge}$ is found by letting x_i^j mark the position of the last entry in row i whose label does not exceed j. We write $\mathbf{K}^N$ for the set of all $\mathbf{x}$ satisfying the above constraint, and say that any $\mathbf{x}\in\mathbf{K}^N$ is a *Gelfand–Tsetlin* (*GT*) *pattern*. In contrast to the tableau setting, it is not required that $\mathbf{x}$ has non-negative entries. For $\mathbf{x}\in\mathbf{K}^N$, we set $\mathrm{sh}(\mathbf{x})=(x_1^N,x_2^N,\dots,x_N^N)$, $\mathrm{ledge}(\mathbf{x})=(x_1^1,\dots,x_N^N)$, and $\mathrm{redge}(\mathbf{x})=(x_1^1,\dots,x_1^N)$; these definitions are consistent with those given for a SSYT in $\mathbf{T}_N^{N,\wedge}$. For a vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)$ of weights, we define the weight $\alpha^{\mathbf{x}}$ of a GT pattern $\mathbf{x}$ by

$$\alpha^{\mathbf{x}}=\alpha_1^{x_1^1}\prod_{k=2}^{N}\alpha_k^{\sum x_i^k-\sum x_i^{k-1}},$$

in accordance with our previous definition for tableaux in $\mathbf{T}_N^{N,\wedge}$.

Instead of studying K_α and $\hat{K}_\alpha$, it is equivalent but more convenient to work with modified versions which do not involve Schur functions but have a polynomial prefactor. It is natural to start with the kernel K_α, since this is a 'square' $W^N\times W^N$ matrix. We define, suppressing the dependence on α,

$$\Lambda(z,y)=\alpha_1^{-y_1}\cdots\alpha_N^{-y_N}\sum_{\mathbf{x}\in\mathbf{K}^N;\mathrm{sh}(\mathbf{x})=z,\ \mathrm{ledge}(\mathbf{x})=y}\alpha^{\mathbf{x}}.\tag{5}$$

We first show that $\Lambda(z,y)$ can be written as a determinant.

Proposition 2. *For $y, z \in W^N$, we have*

$$\Lambda(z, y) = \det\{h^{(jN)}_{z_i - y_j - i + j}(\alpha)\}.$$

Proof. The proof is a variant of a well-known argument using non-intersecting lattice paths to derive the Jacobi–Trudi formulae, see for example [20]. Fix z and y belonging to W^N with $z_N = y_N$. Each Gelfand–Tsetlin pattern $\mathbf{x} \in \mathbf{K}^N$ having $\mathrm{sh}(\mathbf{x}) = z$ and $\mathrm{ledge}(\mathbf{x}) = y$ can be encoded as a non-intersecting $(N-1)$-tuple of paths $(P_1, P_2, \dots, P_{N-1})$ on the edges of the square lattice with vertex set $\mathbf{Z}^2$. Paths always traverse edges in the direction of increasing coordinates: either 'upwards' or 'rightwards'. The path P_k begins at the vertex $(y_k - k, k+1)$, ends at $(z_k - k, N)$, and contains the 'horizontal' edges from $(x_k^{r-1} - k, r)$ to $(x_k^r - k, r)$, for $r = k+1, k+2, \dots, N$. This correspondence between patterns and paths is a bijection and consequently we may write the sum defining $\Lambda(z, y)$ as

$$\sum_{(P_1, \dots, P_{N-1})} w(P_1) \cdots w(P_{N-1}),$$

where the sum is over all non-intersecting paths which have starting points and end points given in terms of y and z as above, and where the weight $w(P_k)$ is defined to be

$$\prod_{r=k+1}^{N} \alpha_r^{e(r)},$$

with $e(r)$ denoting the number of horizontal edges at height r contained in the path P_k. By the Gessel–Viennot formula (see Theorem 2.7.1 of [19]), this sum for $\Lambda(z, y)$ is equal to $\det(M)$ where M is an $(N-1) \times (N-1)$ matrix with (i, j)th entry given by $\sum_P w(P)$ where P runs through all paths connecting $(y_j - j, j+1)$ to $(z_i - i, N)$. This latter quantity is easily found to equal $h^{(jN)}_{z_i - y_j - i + j}(\alpha)$. The proposition follows on noting that $h^{(NN)}_{z_i - y_N - i + N}(\alpha) = 0$ for $i = 1, 2, 3, \dots, N-1$ and that $h^{(NN)}_{z_N - y_N}(\alpha) = \mathbf{1}(z_N = y_N)$. $\square$

We will use the following identity that is easily checked by means of generating functions,

$$\sum_{r \in \mathbf{Z}} (-1)^r e_r^{(iN)}(\alpha) h^{(jN)}_{n-r}(\alpha) = \begin{cases} h_n^{(ji)}(\alpha) & \text{if } j \le i, \\ (-1)^n e_n^{(ij)}(\alpha) & \text{if } i \le j. \end{cases} \tag{6}$$

Recall also the Cauchy–Binet formula

$$\sum_{z \in W^N} \det\{\phi_i(z_j - j)\} \det\{\psi_j(z_i - i)\} = \det\left\{\sum_{z \in \mathbf{Z}} \phi_i(z) \psi_j(z)\right\}. \tag{7}$$

Proposition 3. *The $W^N \times W^N$ matrix Λ is invertible. Its inverse is given by the $W^N \times W^N$ matrix Π defined as*

$$\Pi(y, z) = \det\{(-1)^{y_i - z_j - i + j} e^{(iN)}_{y_i - z_j - i + j}(\alpha)\}.$$

Proof. We first show that Π is a left inverse of Λ. From the Cauchy–Binet formula we deduce that

$$\Pi\Lambda(y, y') = \det\{m^{(ij)}_{y_i - y'_j - i + j}\}, \tag{8}$$

where

$$m_n^{(ij)} = \begin{cases} h_n^{(ji)} & \text{if } j \le i, \\ (-1)^n e_n^{(ij)} & \text{if } i \le j. \end{cases}$$

Observe that $m_r^{(ij)}$ is zero if either $i \le j$ and $r > j - i$ or if $i \ge j$ and $r < 0$. In particular the product of the diagonal elements appearing in the determinant on the right-hand side of (8) is $\prod_i \mathbf{1}(y_i' = y_i)$. We need to show that this is the only contribution to the determinant. Suppose that π is a permutation of $\{1, 2, \dots, N\}$ other than the identity, and consider the product of the $(i, \pi(i))$th entries. It is easy to see that there must be integers i_1 and i_2 such that $i_1 < i_2$, $\pi(i_1) > i_1$, $\pi(i_2) < i_2$ and $\pi(i_1) > \pi(i_2)$. Now the $(i_1, \pi(i_1))$th entry is zero if $y_{i_1} > y'_{\pi(i_1)}$, and the $(i_2, \pi(i_2))$th entry is zero if $y_{i_2} \le y'_{\pi(i_2)}$. But since $y_{i_1} \ge y_{i_2}$ and $y'_{\pi(i_1)} \le y'_{\pi(i_2)}$, at least one of these previous inequalities holds, and hence the product is zero.

The key ingredient for showing that Π is a right inverse of Λ is the trivial identity

$$\Pi(y, z') = \sum_{\ell_1, \dots, \ell_N \in \mathbf{Z}} (-1)^{\ell_1} e_{\ell_1}^{(0N)} \cdots (-1)^{\ell_N} e_{\ell_N}^{(NN)} \det\{\mathbf{1}(y_i - i = z_j' - j + \ell_i)\}.$$

Observe that the sum over the ℓ is finite since the summand is zero unless $0 \le \ell_i \le N - i$ for all i. We therefore have for any function f on $\mathbf{Z}^N$ and $0 \le \ell_i \le N - i$,

$$\begin{aligned}
&\sum_{y \in W^N} f(y_1 - 1, \dots, y_N - N) \det\{\mathbf{1}(y_i - i = z_j' - j + \ell_i)\} \\
&\quad = \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) f(z'_{\sigma(1)} + \ell_1, \dots, z'_{\sigma(N)} - \sigma(N) + \ell_N) \sum_{y \in W^N} \prod_{m=1}^{N} \mathbf{1}(y_m - m = z'_{\sigma(m)} - \sigma(m) + \ell_m) \\
&\quad = f(z_1' - 1 + \ell_1, \dots, z_N' - N + \ell_N),
\end{aligned}$$

where $\mathcal{S}_N$ is the set of permutations on $\{1, \dots, N\}$. After absorbing the sum over the ℓ in the determinant, we obtain $\Lambda\Pi(z, z') = \det\{m^{(ij)}_{z_i - z_j' - i + j}\} = \mathbf{1}(z = z')$. $\square$

By virtue of this proposition, the intertwining $\Lambda Q = P\Lambda$ yields $Q = \Pi P \Lambda$, and a straightforward computation using the Cauchy–Binet formula and (6) implies the following corollary: Recall the definition of $f_\alpha^{(ij)}$ in (1).

Corollary 4. *Suppose that P is a $W^N \times W^N$ matrix of the form*

$$P(z, z') = \det\{f(z_i' - z_j - i + j)\}$$

for some function f on $\mathbf{Z}$. Suppose that Q is another $W^N \times W^N$ matrix and that the intertwining relation $P\Lambda = \Lambda Q$ holds. Then we have for $y, y' \in W^N$,

$$Q(y, y') = \det\{f_\alpha^{(ij)}(y_i' - y_j - i + j)\}.$$

We will also consider a second intertwining kernel that arises by replacing the left edge of the pattern with the right edge in (5): for $z \in W^N$, $y \in \hat{W}^N$, we set

$$\hat{\Lambda}(z, y) = \alpha_1^{-y_1} \cdots \alpha_N^{-y_N} \sum_{\mathbf{x} \in \mathbf{K}^N; \mathrm{sh}(\mathbf{x}) = z,\ \mathrm{redge}(\mathbf{x}) = y} \alpha^{\mathbf{x}}.$$

For $\mathbf{x} \in \mathbf{K}^N$, define $\hat{\mathbf{x}} \in \mathbf{K}^N$ by $\hat{x}_i^k = -x_{k-i+1}^k$. The correspondence $\mathbf{x} \mapsto \hat{\mathbf{x}}$ is bijective and it is easily verified that $\alpha^{\hat{\mathbf{x}}} = \beta^{\mathbf{x}}$ where $\beta = \alpha^{-1} = (\alpha_1^{-1}, \dots, \alpha_N^{-1})$. Using this correspondence and our results for Λ we obtain the following proposition.

Proposition 5. *For $z \in W^N$ and $y \in \hat{W}^N$, we have*

$$\hat{\Lambda}(z, y) = \det\{h^{(jN)}_{y_j - z_{N-i+1} - i + j}(\alpha^{-1})\}.$$

Moreover, $\hat{\Lambda}$ is invertible with inverse $\hat{\Pi}$ given by, for $y \in \hat{W}^N$ and $z \in W^N$,

$$\hat{\Pi}(y,z) = \det\left\{(-1)^{z_{N-j+1}-y_i-i+j} e^{(iN)}_{z_{N-j+1}-y_i-i+j}\left(\alpha^{-1}\right)\right\}.$$

The analogue of Corollary 4 follows immediately from this proposition.

Corollary 6. *Suppose that P is a $W^N \times W^N$ matrix of the form*

$$P\left(z,z'\right) = \det\left\{f\left(z'_i - z_j - i + j\right)\right\}$$

for some function f on $\mathbf{Z}$. Suppose that Q is a $\hat{W}^N \times \hat{W}^N$ matrix and that the intertwining relation $P\hat{\Lambda} = \hat{\Lambda}Q$ holds. Then we have for $y, y' \in \hat{W}^N$,

$$Q\left(y,y'\right) = \det\left\{\hat{f}^{(ij)}_{\alpha^{-1}}\left(y'_i - y_j + i - j\right)\right\}.$$

Acknowledgements

We thank Timo Seppäläinen for his helpful comments. The work of ABD was supported in part by the Science Foundation Ireland, grant number SFI04/RP1/I512.

References

[1] M. Alimohammadi, V. Karimipour and M. Khorrami. Exact solution of a one-parameter family of asymmetric exclusion processes. *Phys. Rev. E* **57** (1998) 6370–6376. MR1628226
[2] Yu. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields* **119** (2001) 256–274. MR1818248
[3] A. Borodin and P. L. Ferrari. Large time asymptotics of growth models on space-like paths I: PushASEP. Available at arXiv.org/abs/0707.2813, 2007.
[4] A. Borodin, P. L. Ferrari, M. Prähofer and T. Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.* **129** (2007) 1055–1080. MR2363389
[5] A. B. Dieker and J. Warren. Transition probabilities for series Jackson networks. Preprint, 2007.
[6] M. Draief, J. Mairesse and N. O'Connell. Queues, stores, and tableaux. *J. Appl. Probab.* **42** (2005) 1145–1167. MR2203829
[7] W. Fulton. *Young Tableaux.* Cambridge University Press, 1997. MR1464693
[8] E. R. Gansner. Matrix correspondences of plane partitions. *Pacific J. Math.* **92** (1981) 295–315. MR0618067
[9] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** (2000) 437–476. MR1737991
[10] K. Johansson. A multi-dimensional Markov chain and the Meixner ensemble. Available at arXiv.org/abs/0707.0098, 2007.
[11] W. König. Orthogonal polynomial ensembles in probability theory. *Probab. Surv.* **2** (2005) 385–447. MR2203677
[12] N. O'Connell. Conditioned random walks and the RSK correspondence. *J. Phys. A* **36** (2003) 3049–3066. MR1986407
[13] N. O'Connell. A path-transformation for random walks and the Robinson–Schensted correspondence. *Trans. Amer. Math. Soc.* **355** (2003) 3669–3697. MR1990168
[14] A. M. Povolotsky and V. B. Priezzhev. Determinant solution for the totally asymmetric exclusion process with parallel update. *J. Stat. Mech.* (2006) P07002.
[15] A. Rákos and G. Schütz. Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process. *J. Stat. Phys.* **118** (2005) 511–530. MR2123646
[16] A. Rákos and G. Schütz. Bethe ansatz and current distribution for the TASEP with particle-dependent hopping rates. *Markov Process. Related Fields* **12** (2006) 323–334. MR2249635
[17] G. M. Schütz. Exact solution of the master equation for the asymmetric exclusion process. *J. Stat. Phys.* **88** (1997) 427–445. MR1468391
[18] T. Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane. *Ann. Probab.* **26** (1998) 1232–1250. MR1640344
[19] R. P. Stanley. *Enumerative Combinatorics*, Vol. 1. Cambridge University Press, 1997. MR1442260
[20] R. P. Stanley. *Enumerative Combinatorics*, Vol. 2. Cambridge University Press, 1999. MR1676282
[21] C. A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. *Comm. Math. Phys.* **279** (2008) 815–844.
[22] J. Warren. Dyson's Brownian motions, intertwining and interlacing. *Electron. J. Probab.* **12** (2007) 573–590. MR2299928