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On the equivalence of some eternal additive coalescents*

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Abstract. In this paper, we study additive coalescents. Using their representation as fragmentation processes, we prove that the law of a large class of eternal additive coalescents is absolutely continuous with respect to the law of the standard additive coalescent on any bounded time interval.

Résumé. Nous étudions dans ce papier les coalescents additifs. En utilisant leur représentation en tant que processus de fragmentation, nous prouvons que certains coalescents additifs éternels ont une loi absolument continue par rapport à la loi du coalescent additif standard sur n'importe quel intervalle de temps borné inférieurement.

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1. Introduction

This paper deals with additive coalescent processes, a class of Markov processes first introduced by Evans and Pitman [12]. In the simple situation of a system initially composed of a finite number k of clusters with masses $m_1, m_2, \dots, m_k$, the dynamic is such that each pair of clusters (m_i, m_j) merges into a unique cluster with mass $m_i + m_j$ at rate $m_i + m_j$, independently of the other pairs. In the sequel, we always assume that the initial total mass is equal to 1 (i.e. $m_1 + \cdots + m_k = 1$). This assumption induces no loss of generality since we can then deduce the law of any additive coalescent process through a linear time-change. Hence, an additive coalescent takes values on the compact set

$$\mathcal{S}^{\downarrow} = \left\{ x = (x_i)_{i \geq 1}, x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i \geq 1} x_i \leq 1 \right\},$$

endowed with the topology of uniform convergence.

Evans and Pitman [12] proved that one can define an additive coalescent on the whole real line for a system starting at time $t = -\infty$ with an infinite number of infinitesimally small clusters. Such a process is called an *eternal additive coalescent process*. More precisely, if we denote by $(C^n(t), t \geq 0)$ the additive coalescent starting from the configuration $(1/n, 1/n, \dots, 1/n)$, they proved that the sequence of processes $(C^n(t + \frac{1}{2}\ln n), t \geq -\frac{1}{2}\ln n)$ converges in distribution on the space of càdlàg paths with values in the set $\mathcal{S}^{\downarrow}$ toward some process $(C^{\infty}(t), t \in \mathbb{R})$, called the standard additive coalescent. It should be noted that this process is defined for all time $t \in \mathbb{R}$. Remarkably, the standard additive coalescent becomes, through time-reversal, a fragmentation process. Namely, the process $(F(t), t \geq 0)$

*Sur l'équivalence de certains coalescents additifs éternels defined by $F(t) = C^\infty(-\ln t)$ is a self-similar fragmentation process with index of self similarity $\alpha = 1/2$, without erosion and with dislocation measure ν given by

$$\nu(x_1 \in \mathrm{d}y) = \big(2\pi y^3(1-y)^3\big)^{-1/2}\,\mathrm{d}y \quad \text{for } y \in \left(\frac{1}{2}, 1\right), \qquad \nu(x_3 > 0) = 0.$$

We refer to [8,9] for the definition of erosion, dislocation measure, and index of self similarity of a fragmentation process and a proof of the property mentioned above. Let us just recall that, in a fragmentation process, distinct fragments evolve independently of each other.

Aldous and Pitman [1] constructed such a fragmentation process F by cutting the skeleton of the Brownian continuum random tree according to a Poisson point process. In another paper [2], they gave a generalization of this result: consider for each $n \in \mathbb{N}$ a decreasing sequence $r_{n,1} \ge \cdots \ge r_{n,n} \ge 0$ with sum 1, set $\sigma_n^2 = \sum_{i=1}^n r_{n,i}^2$ and suppose that

$$\lim_{n\to\infty} \sigma_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{n,i}}{\sigma_n} = \theta_i \quad \text{for all } i \in \mathbb{N}.$$

Assume further that either $\sum_i \theta_i^2 < 1$ or $\sum_i \theta_i = \infty$. Then, according to [2], if $M^n = (M^n(t), t \ge 0)$ denotes the additive coalescent process starting with n clusters with masses $r_{n,1} \ge \cdots \ge r_{n,n}$, then $(M^{(n)}(t - \ln\sigma_n), t \ge \ln\sigma_n)$ has a limit distribution as $n \to \infty$, which can be obtained by cutting a specific inhomogeneous random tree with a point Poisson process. Furthermore, any extreme eternal additive coalescent may be obtained this way up to a deterministic time translation (i.e. any eternal additive coalescent can be obtained as a mixture of such additive coalescents).

Bertoin [5] gave another construction of the limit of the process $(M^{(n)}(t - \ln\sigma_n), t \ge \ln\sigma_n)$ in the following way: let b_θ be the bridge with exchangeable increments defined, for $s \in [0, 1]$, by

$$b_\theta(s) = \sigma b_s + \sum_{i=1}^\infty \theta_i(\mathbb{1}_{\{s \ge V_i\}} - s),$$

where $(b_s, s \in [0, 1])$ is a standard Brownian bridge, $(V_i)_{i\ge1}$ is an i.i.d. sequence of uniform random variables on $[0, 1]$ independent of b and $\sigma = 1 - \sum_i \theta_i^2$. Let $\varepsilon_\theta = (\varepsilon_\theta(s), s \in [0, 1])$ be the excursion obtained from b_θ by Vervaat's transform, i.e. $\varepsilon_\theta(s) = b_\theta(s + m \bmod 1) - b_\theta(m)$, where m is the point of $[0, 1]$ where b_θ reaches its minimum. For all $t \ge 0$, consider

$$\varepsilon_\theta^{(t)}(s) = ts - \varepsilon_\theta(s), \qquad S_\theta^{(t)}(s) = \sup_{0\le u\le s} \varepsilon_\theta^{(t)}(u),$$

and define $F^\theta(t)$ as the sequence of the lengths of constancy intervals of the process $(S_\theta^{(t)}(s), 0 \le s \le 1)$. Then, the limit of the process $(M^{(n)}(t - \ln\sigma_n), t \ge \ln\sigma_n)$ has the same law as $(F^\theta(\mathrm{e}^{-t}), t \in \mathbb{R})$.

Miermont [15] studied the same process in the special case where ε_θ is the normalized excursion below the supremum of a spectrally negative Lévy process. More precisely, let $(X_t, t \ge 0)$ be a Lévy process with no positive jump, with unbounded variation and with positive, finite mean. Let $\overline{X}(t) = \sup_{0\le s\le t} X_t$ and let $\varepsilon_X = (\varepsilon_X(s), s \in [0, 1])$ denote the normalized excursion with duration 1 of the reflected process $\overline{X} - X$. We now define the processes $\varepsilon_X^{(t)}(s)$, $S_X^{(t)}(s)$ and $F^X(t)$ in the same way as for b_θ. Then, the process $(F^X(\mathrm{e}^{-t}), t \in \mathbb{R})$ is a mixture of some eternal additive coalescents (see [15] for more details). Furthermore, $(F^X(t), t \ge 0)$ is a fragmentation process in the sense that distinct fragments evolve independently of each other (but is not necessarily homogeneous in time). It is quite remarkable that the Lévy property of X ensures the branching property of F^X. We emphasize that there exist other eternal additive coalescents for which this property fails. Notice that when the Lévy process X is the standard Brownian motion B, the process $(F^B(\mathrm{e}^{-t}), t \in \mathbb{R})$ is the standard additive coalescent and $(F^B(t), t \ge 0)$ is a self-similar and time-homogeneous fragmentation process.

In this paper, we study the relationship between the laws of F^X and F^B. From now on, we denote by F the canonical process on the space of functions $\mathbb{R} \to \mathcal{S}^\downarrow$ and $(\mathcal{F}_t, t \ge 0)$ the natural filtration, i.e. $\mathcal{F}_t = \sigma(F(s), 0 \le s \le t)$. We also define $\mathbb{P}^{(X)}$ (resp. $\mathbb{P}^{(B)}$) on the space of functions $\mathbb{R} \to \mathcal{S}^\downarrow$ such that F under $\mathbb{P}^{(X)}$ (resp. $\mathbb{P}^{(B)}$) has the law of F^X (resp. F^B).

We prove that, for certain Lévy processes $(X_t, t \ge 0)$, the law $\mathbb{P}^{(X)}_{|\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{P}^{(B)}_{|\mathcal{F}_t}$ and we explicitly compute its density.

Theorem 1.1. *Let $(\Gamma(t), t \geq 0)$ be a subordinator without drift. Assume that $\mathbb{E}(\Gamma_1) < \infty$ and choose any $c \geq \mathbb{E}(\Gamma_1)$. We define $X_t = B_t - \Gamma_t + ct$, where B denotes a Brownian motion independent of Γ. Let $(p_t(u), u \in \mathbb{R})$ and $(q_t(u), u \in \mathbb{R})$ stand for the respective densities of B_t and X_t. In particular $p_t(u) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{u^2}{2t})$. Let $\mathcal{S}_1$ be the space of positive sequences with sum* 1. *We consider the function $\mathbf{g}: \mathbb{R}_+ \times \mathcal{S}_1 \to \mathbb{R}$ defined by*

$$\begin{cases} \mathbf{g}(0, \mathbf{1}) = \frac{q_1(0)}{p_1(0)} & \text{with } \mathbf{1} \stackrel{def}{=} (1, 0, 0, \ldots), \\ \mathbf{g}(t, \mathbf{x}) = \mathrm{e}^{tc} \prod_{i=1}^{\infty} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} & \text{with } \mathbf{x} = (x_i)_{i\geq 1}. \end{cases} \tag{1}$$

Then, for all $t \geq 0$, the function $\mathbf{g}(t, \cdot)$ is bounded on $\mathcal{S}_1$ and has the following properties:

1. *Under $\mathbb{P}^{(B)}$, $\mathbf{g}(t, F(t))$ is an $\mathcal{F}_t$-martingale.*
2. *For every $t \geq 0$, the law of $\mathbb{P}^{(X)}_{|\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{P}^{(B)}_{|\mathcal{F}_t}$ with density $\mathbf{g}(t, F(t))/\mathbf{g}(0, \mathbf{1})$.*

Let us notice that the first part of the theorem is a direct consequence of the second part. In terms of coalescent processes, this theorem yields the following corollary.

Corollary 1.2. *Let C^X be a mixture of additive coalescents associated with the Lévy process X. Since we have $(C^X(t), t \in \mathbb{R}) \stackrel{law}{=} (F^X(\mathrm{e}^{-t}), t \in \mathbb{R})$, the law of $(C^X(s), s \geq t)$ is absolutely continuous with respect to the law of the standard additive coalescent started at time $t \in \mathbb{R}$, with density given by $\mathbf{g}(\mathrm{e}^{-t}, C(t))/\mathbf{g}(0, \mathbf{1})$, C being the canonical process on the space of càdlàg functions $\mathbb{R} \to \mathcal{S}^{\downarrow}$.*

Remark 1.3. *It is known that, if Γ is a subordinator, the density $(q_t(u), u \in \mathbb{R})$ of the Lévy process $X_t = B_t - \Gamma_t + ct$ is well defined for $t > 0$. Moreover, the function $(t, u) \mapsto q_t(u)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathbb{R}$ (see Chapter* 5 *of* [18]).

Let us notice that $\mathbf{g}(t, \cdot)$ is a multiplicative function, i.e. it can be written as a product of functions, each of them depending only on the size of a single fragment. In the sequel we use the notation

$$g(t, x) = \mathrm{e}^{tcx} \frac{q_x(-tx)}{p_x(-tx)} \qquad \text{for } x \in (0, 1] \text{ and } t \geq 0,$$

so that $\mathbf{g}(t, \mathbf{x}) = \prod_i g(t, x_i)$. As a consequence of the branching property of F^B and the multiplicative form of $\mathbf{g}$, the process F^X also has the branching property. Indeed, for every multiplicative bounded continuous function $\mathbf{f}: \mathcal{S}^{\downarrow} \mapsto \mathbb{R}_+$, for all $t' > t > 0$ and $\mathbf{x} \in \mathcal{S}^{\downarrow}$, since $\mathbf{g}(t, F(t))$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^{(B)}$, we have

$$\mathbb{E}^{(X)}\big(\mathbf{f}\big(F(t')\big)|F(t) = \mathbf{x}\big) = \frac{1}{\mathbf{g}(t, \mathbf{x})} \mathbb{E}^{(B)}\big(\mathbf{g}\big(t', F(t')\big)\mathbf{f}\big(F(t')\big)|F(t) = \mathbf{x}\big).$$

Using the branching property of F under $\mathbb{P}^{(B)}$ and the multiplicative form of $\mathbf{g}(t, \cdot)$, we get

$$\mathbb{E}^{(X)}\big(\mathbf{f}\big(F(t')\big)|F(t) = \mathbf{x}\big) = \frac{1}{\mathbf{g}(t, \mathbf{x})} \prod_i \mathbb{E}^{(B)}\big(\mathbf{g}\big(t', F(t')\big)\mathbf{f}\big(F(t')\big)|F(t) = (x_i, 0, \ldots)\big).$$

And we deduce that

$$\begin{aligned} \mathbb{E}^{(X)}\big(\mathbf{f}\big(F(t')\big)|F(t) = \mathbf{x}\big) &= \frac{1}{\mathbf{g}(t, \mathbf{x})} \prod_i g(t, x_i) \mathbb{E}^{(X)}\big(\mathbf{f}\big(F(t')\big)|F(t) = (x_i, 0, \ldots)\big) \\ &= \prod_i \mathbb{E}^{(X)}\big(\mathbf{f}\big(F(t')\big)|F(t) = (x_i, 0, \ldots)\big). \end{aligned}$$

Let $M_{\mathbf{x}}$ (resp. M_{x_i}) be the random measure on $(0,1)$ defined by $M_{\mathbf{x}} = \sum_i \delta_{s_i}$ where the (random) sequence $(s_i)_{i\geq 1}$ has the law of $F(t')$ conditioned on $F(t)=\mathbf{x}$ (resp. $F(t)=(x_i,0,\ldots)$). Then, for every bounded continuous function $k:\mathbb{R}\to\mathbb{R}$, we have

$$\mathbb{E}^{(X)}\big(\exp\big(-\langle k, M_{\mathbf{x}}\rangle\big)\big) = \prod_{i=1}^{\infty} \mathbb{E}^{(X)}\big(\exp\big(-\langle k, M_{x_i}\rangle\big)\big),$$

which shows that $M_{\mathbf{x}}$ has the same law as $\sum_i M_{x_i}$ where the random measures $(M_{x_i})_{i\geq 1}$ are independent. Hence the process F has the branching property under $\mathbb{P}^{(X)}$. Let us note that other multiplicative martingales have already been studied in the case of branching random walks [10,11,14,16].

This paper is divided in two sections. The first section is devoted to the proof of Theorem 1.1. In the second section, we give an integro-differential equation solved by g using the fact that $\mathbf{g}(t, F(t))$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^{(B)}$.

2. Proof of Theorem 1.1

The assumptions and notations of Theorem 1.1 are implicitly enforced throughout this section.

2.1. *Absolute continuity*

In order to prove Theorem 1.1, we first prove the absolute continuity of the law of $F(t)$ under $\mathbb{P}^{(X)}$ with respect to the law of $F(t)$ under $\mathbb{P}^{(B)}$ for any fixed time $t>0$ and for any finite number of fragments. We start with:

Definition 2.1. *Let $\mathbf{x}=(x_1,x_2,\ldots)$ be a sequence of positive numbers with sum 1. We call the random variable $\tilde{\mathbf{x}}=(x_{j_1},x_{j_2},\ldots)$ a size-biased rearrangement of $\mathbf{x}$ if we have*:

$$\forall i\in\mathbb{N},\quad \mathbb{P}(j_1=i)=x_i,$$

and, by induction,

$$\forall i\in\mathbb{N}\setminus\{i_1,\ldots,i_k\},\quad \mathbb{P}(j_{k+1}=i\,|\,j_1=i_1,\ldots,j_k=i_k)=\frac{x_i}{1-\sum_{l=1}^{k} x_{i_l}}.$$

Of course, when the sequence $\mathbf{x}$ is itself random, the same construction can be done for each realization of $\mathbf{x}$ and thus we obtain another random variable $\tilde{\mathbf{x}}$ called a size-biased rearrangement of $\mathbf{x}$. In particular, for any $t\geq 0$, we have $\sum_{i=1}^{\infty} F_i(t)=1\,\mathbb{P}^{(X)}$-a.s. (this follows from the construction from an excursion of X since the Lévy process has unbounded variation, cf. [15], Section 3.2). Thus, in what follows, considering a possibly enlarged probability space, we shall denote by $\tilde{F}(t)=(\tilde{F}_1(t),\tilde{F}_2(t),\ldots)$ a size-biased rearrangement of $F(t)$ (well defined $\mathbb{P}^{(X)}$ almost surely).

The following lemma gives the distribution of the n first fragments of $F(t)$ under $\mathbb{P}^{(X)}$, chosen with a size-biased pick.

Lemma 2.2. *For all $n\in\mathbb{N}$, for all $x_1,\ldots,x_n\in\mathbb{R}_+$ such that $S=\sum_{i=1}^{n} x_i<1$, we have*

$$\mathbb{P}^{(X)}\big(\tilde{F}_1(t)\in \mathrm{d}x_1,\ldots,\tilde{F}_n(t)\in \mathrm{d}x_n\big)=\frac{t^n}{q_1(0)}q_{1-S}(St)\prod_{i=1}^{n}\frac{q_{x_i}(-tx_i)}{1-\sum_{k=1}^{i} x_k}\,\mathrm{d}x_1\cdots \mathrm{d}x_n.$$

Proof. On the one hand, Miermont [15] gave a description of the law of $F(t)$ under $\mathbb{P}^{(X)}$: let $T^{(t)}$ be a subordinator with Lévy measure $z^{-1}q_z(-tz)\mathbb{1}_{z>0}\,\mathrm{d}z$. Then, $F(t)$ has the law of the sequence of the jumps of $T^{(t)}$ before time t conditioned on $T_t^{(t)}=1$.

On the other hand, consider a subordinator T on the time interval $[0,u]$ conditioned by $T_u=y$ and pick a jump of T by size-biased sampling. Its distribution has density

$$\frac{zuh(z)f_u(y-z)}{yf_u(y)}\,\mathrm{d}z,$$

where h is the density of the Lévy measure of T and f_u is the density of T_u (see Theorem 2.1 of [17]). In the present case, we have

$$u = t, \qquad y = 1, \qquad h(z) = z^{-1} q_z(-tz), \qquad f_u(z) = \frac{u}{z} q_z(u - zt) \quad \text{(cf. Lemma 9 of [15])}.$$

Hence, we get

$$\mathbb{P}^{(X)}\big(\tilde{F}_1(t) \in \mathrm{d}z\big) = \frac{t q_z(-tz) q_{1-z}(zt)}{(1-z) q_1(0)} \,\mathrm{d}z.$$

This proves the lemma for $n = 1$. We prove the general case by induction. Assume that the result holds for $n - 1$. We have

$$\begin{aligned}&\mathbb{P}^{(X)}\big(\tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_n(t) \in \mathrm{d}x_n\big)\\ &\quad = \mathbb{P}^{(X)}\big(\tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_{n-1}(t) \in \mathrm{d}x_{n-1}\big)\mathbb{P}^{(X)}\big(\tilde{F}_n(t) \in \mathrm{d}x_n | \tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_{n-1}(t) \in \mathrm{d}x_{n-1}\big).\end{aligned}$$

Furthermore, Perman, Pitman and Yor [17] proved that the nth size-biased picked jump Δ_n of a subordinator before time u, conditioned by $T_u = y$ and $\Delta_1 = x_1, \ldots, \Delta_{n-1} = x_{n-1}$, has the law of a size-biased picked jump of the subordinator T before time u conditioned by $T_u = y - x_1 - \cdots - x_{n-1}$. Hence we get

$$\begin{aligned}&\mathbb{P}^{(X)}\big(\tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_n(t) \in \mathrm{d}x_n\big)\\ &\quad = \left(\frac{t^{n-1}}{q_1(0)} q_{1-S_{n-1}}(S_{n-1}t) \prod_{i=1}^{n-1} \frac{q_{x_i}(-tx_i)}{1 - S_i}\right) \frac{t q_{x_n}(-tx_n) q_{1-S_n}(S_n t)}{(1 - S_n) q_{1-S_{n-1}}(S_{n-1}t)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_n,\end{aligned}$$

where $S_i = \sum_{k=1}^{i} x_k$. This completes the proof of the lemma. □

Of course, the lemma also holds for $\mathbb{P}^{(B)}$ (take $\Gamma = c = 0$), thus we obtain:

Corollary 2.3. *For all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $S = \sum_{i=1}^{n} x_i < 1$, we have*

$$\frac{\mathbb{P}^{(X)}(\tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_n(t) \in \mathrm{d}x_n)}{\mathbb{P}^{(B)}(\tilde{F}_1(t) \in \mathrm{d}x_1, \ldots, \tilde{F}_n(t) \in \mathrm{d}x_n)} = \frac{g_n(t, x_1, \ldots, x_n)}{\mathbf{g}(0, \mathbf{1})},$$

with

$$g_n(t, x_1, \ldots, x_n) = \frac{q_{1-S}(St)}{p_{1-S}(St)} \prod_{i=1}^{n} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}.$$

In order to establish that the law of $F(t)$ under $\mathbb{P}^{(X)}$ is absolutely continuous with respect to the law of $F(t)$ under $\mathbb{P}^{(B)}$ with density $\mathbf{g}(t, \cdot)/\mathbf{g}(0, \mathbf{1})$, it remains to check that the function g_n converges, as n tends to infinity, to $\mathbf{g}$ $\mathbb{P}^{(B)}$-a.s. and in $L^1(\mathbb{P}^{(B)})$. To this aim, we first prove two lemmas:

Lemma 2.4. *We have $\frac{q_y(-ty)}{p_y(-ty)} < 1$ for all $y > 0$ sufficiently small. As a consequence, if $(x_i)_{i \ge 1}$ is a sequence of positive numbers with $\lim_{i \to \infty} x_i = 0$, then the product $\prod_{i=1}^{n} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ converges as n tends to infinity.*

Proof. Recall that X has the form $X_t = B_t - \Gamma_t + tc$, thus we have

$$\forall s > 0,\ \forall u \in \mathbb{R}, \quad q_s(u) = \mathbb{E}\big(p_s(u + \Gamma_s - cs)\big).$$

Replacing $p_s(u)$ by its expression $\frac{1}{\sqrt{2\pi s}}\exp(-\frac{u^2}{2s})$, we get

$$\frac{q_s(u)}{p_s(u)} = \exp\left(cu - \frac{c^2 s}{2}\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_s^2}{2s} - \Gamma_s\left(\frac{u}{s} - c\right)\right)\right], \tag{2}$$

i.e., for all $y > 0$, for all $t \geq 0$,

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\left(-y\left(ct + \frac{c^2}{2}\right)\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\right)\right].$$

In view of the inequality $(\gamma-\alpha)(\gamma-\beta) \geq -(\frac{\beta-\alpha}{2})^2$, we have

$$-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c) \leq \frac{y(t+c)^2}{2}$$

so we deduce that

$$\frac{q_y(-ty)}{p_y(-ty)} \leq \mathrm{e}^{t^2 y/2}.$$

Fix $c' \in (0,c)$ and let $f(y) = \mathbb{P}(\Gamma_y \leq c'y)$. Since Γ_t is a subordinator without drift, we have $\lim_{y\to 0} f(y) = 1$ (indeed, $\Gamma_y = \mathrm{o}(y)$ a.s., *cf.* [4]). On the event $\{\Gamma_y \leq c'y\}$, we have

$$\begin{aligned}\exp\left(-y\left(ct + \frac{c^2}{2}\right)\right)\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\right) &\leq \exp\left(-y\left(\frac{1}{2}(c-c')^2 + t(c-c')\right)\right)\\ &\leq \exp(-\varepsilon y),\end{aligned}$$

with $\varepsilon = \frac{1}{2}(c-c')^2$. This gives us the upper bound

$$\frac{q_y(-ty)}{p_y(-ty)} \leq \mathrm{e}^{-\varepsilon y} f(y) + \left(1 - f(y)\right)\mathrm{e}^{yt^2/2}.$$

Since $f(y) \to 1$ as $y \to 0$, we obtain

$$\mathrm{e}^{-\varepsilon y} f(y) + \left(1 - f(y)\right)\mathrm{e}^{yt^2/2} = 1 - \varepsilon y + \mathrm{o}(y).$$

Thus, we have $\frac{q_y(-ty)}{p_y(-ty)} < 1$ for y small enough, and so the product converges for every sequence $(x_i)_{i\geq 0}$ which tends to 0. □

Lemma 2.5. *We have*

$$\lim_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} = \mathrm{e}^{tc}.$$

Proof. In view of equation (2), we have:

$$\frac{q_{1-s}(st)}{p_{1-s}(st)} = \exp\left(tsc - \frac{c^2}{2}(1-s)\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}\left(\frac{ts}{1-s} - c\right)\right)\right].$$

For s close enough to 1, we have $\frac{ts}{1-s} - c \geq 0$, hence

$$\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}\left(\frac{ts}{1-s} - c\right)\right) \leq 1.$$

Since Γ_t is a subordinator without drift, we have $\lim_{u\to 0}\frac{\Gamma_u}{u}=0$ a.s., and so

$$\lim_{s\to 1}\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)}-\Gamma_{1-s}\left(\frac{ts}{1-s}-c\right)\right)=1 \quad \text{a.s.}$$

Using the dominated convergence theorem, we get

$$\lim_{s\to 1^-}\frac{q_{1-s}(st)}{p_{1-s}(st)}=\mathrm{e}^{tc}.$$

□

We can now prove the absolute continuity of the law of $F(t)$ under $\mathbb{P}^{(X)}$ with respect to that of $F(t)$ under $\mathbb{P}^{(B)}$. Since $S_n=\sum_{i=1}^n F_i(t)$ converges $\mathbb{P}^{(B)}$-a.s. to 1, Lemmas 2.4 and 2.5 imply that $G_n=g_n(t,\tilde{F}_1(t),\dots,\tilde{F}_n(t))$ converges to $G=\mathbf{g}(t,F(t))$ $\mathbb{P}^{(B)}$-a.s.

It suffices to check that G_n is uniformly bounded, which implies the L^1 convergence. Recall that there exists $\varepsilon>0$ such that:

$$\forall x\in(0,\varepsilon),\quad \frac{q_x(-tx)}{p_x(-tx)}\le 1.$$

Besides, the density function $(t,u)\to q_t(u)$ is continuous on $\mathbb{R}_+^*\times\mathbb{R}$, so that the function $x\to\frac{q_x(-tx)}{p_x(-tx)}$ is also continuous on $[\varepsilon,1]$, therefore, bounded on this interval by some constant A. Since there are at most $\frac{1}{\varepsilon}$ fragments of $F(t)$ larger than ε, we deduce the upper bound:

$$\prod_{i=1}^{\infty}\frac{q_{F_i(t)}(-tF_i(t))}{p_{F_i(t)}(-tF_i(t))}\le A^{1/\varepsilon}.$$

Likewise, the function $S\to\frac{q_{1-S}(St)}{p_{1-S}(St)}$ is continuous on $[0,1)$ and has a finite limit at 1^-, so it is bounded by some constant $D>0$ on $[0,1]$. Therefore, we get

$$G_n\le A^{1/\varepsilon}D,\quad \mathbb{P}^{(B)}\text{-a.s.}$$

In consequence, G_n converges to G $\mathbb{P}^{(B)}$-a.s. and in $L^1(\mathbb{P}^{(B)})$. Furthermore, by construction, G_n is a martingale under $\mathbb{P}^{(B)}$ for the filtration $\mathcal{G}_n\stackrel{\text{def}}{=}\sigma(\tilde{F}_1(t),\dots,\tilde{F}_n(t))$. Hence, for all $n\in\mathbb{N}$,

$$\mathbb{E}^{(B)}\big(G|\tilde{F}_1(t),\dots,\tilde{F}_n(t)\big)=G_n.$$

This implies, for every bounded continuous function $f:\mathcal{S}_1\to\mathbb{R}$, that

$$\mathbb{E}^{(X)}\big[f\big(F(t)\big)\big]=\frac{1}{\mathbf{g}(0,\mathbf{1})}\mathbb{E}^{(B)}\big[f\big(F(t)\big)\mathbf{g}\big(t,F(t)\big)\big].$$

We have proved that, for any fixed time $t\ge 0$, the law of $F(t)$ under $\mathbb{P}^{(X)}$ is absolutely continuous with respect to that of $F(t)$ under $\mathbb{P}^{(B)}$ with density $\mathbf{g}(t,F(t))/\mathbf{g}(0,\mathbf{1})$. On the other hand, Miermont [15] proved that the process $(F(\mathrm{e}^{-t}),t\in\mathbb{R})$ is an eternal additive coalescent under $\mathbb{P}^{(X)}$ as well as under $\mathbb{P}^{(B)}$ (but with different entrance laws). Therefore, the semi-groups are the same, which implies the absolute continuity of $\mathbb{P}^{(X)}_{|\mathcal{F}_t}$ with respect to $\mathbb{P}^{(B)}_{|\mathcal{F}_t}$ with density $\mathbf{g}(t,F(t))/\mathbf{g}(0,\mathbf{1})$.

2.2. *Sufficient condition for equivalence*

In this section, we give a sufficient condition for the equivalence of the measures $\mathbb{P}^{(B)}_{|\mathcal{F}_t}$ and $\mathbb{P}^{(X)}_{|\mathcal{F}_t}$, i.e. a sufficient condition for the strict positivity of $\mathbf{g}(t,F(t))$, $\mathbb{P}^{(B)}$-a.s.

Proposition 2.6. *Let ϕ be the Laplace exponent of the subordinator Γ, i.e.*

$$\forall s \geq 0,\ \forall q \geq 0, \quad \mathbb{E}\big(\exp(-q\Gamma_s)\big) = \exp\big(-s\phi(q)\big).$$

Assume that there exists $\delta > 0$ such that

$$\lim_{x\to\infty} \phi(x)x^{\delta-1} = 0, \tag{3}$$

then the function $\mathbf{g}(t, F(t))$ defined in Theorem 1.1 *is strictly positive $\mathbb{P}^{(B)}$-a.s.*

Let us note that condition (3) is quite weak. Indeed, if Π is the Lévy measure of the subordinator and $I(x) = \int_0^x \overline{\Pi}(t)\,\mathrm{d}t$ where $\overline{\Pi}(t)$ denotes $\Pi((t,\infty))$, it is well known that $\phi(x)$ behaves like $xI(1/x)$ as x tends to infinity (see [4], Section III). Thus, condition (3) is equivalent to $I(x) = \mathrm{o}(x^\delta)$ as x tends to 0 (recall that we always have $I(x) = \mathrm{o}(1)$).

Proof of Proposition 2.6. Let $t > 0$ and let $\mathbb{P}_t^{(B)}$ denote the law of $F(t)$ under $\mathbb{P}^{(B)}$. We must check that $\prod_{i=1}^\infty \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ is strictly positive $\mathbb{P}_t^{(B)}(\mathrm{d}\mathbf{x})$-almost surely. Using (2), we have:

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\biggl(-y\biggl(ct + \frac{c^2}{2}\biggr)\biggr)\mathbb{E}\biggl[\exp\biggl(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\biggr)\biggr].$$

Since we have $\sum_{i=1}^\infty x_i = 1$, $\mathbb{P}_t^{(B)}$-a.s., we get

$$\prod_{i=1}^\infty \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \geq \exp\biggl(-ct + \frac{c^2}{2}\biggr)\prod_{i=1}^\infty \mathbb{E}\biggl[\exp\biggl(-\frac{\Gamma_{x_i}^2}{2x_i} + c\Gamma_{x_i}\biggr)\biggr].$$

Let us now obtain a lower bound for $\mathbb{E}[\exp(-\frac{\Gamma_y^2}{2y} + c\Gamma_y)]$. Since $c \geq \mathbb{E}(\Gamma_1)$, we have

$$\mathbb{E}\biggl[\exp\biggl(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\biggr)\biggr] \geq \mathbb{E}\biggl[\exp\biggl(\frac{\Gamma_y}{y}\biggl(\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2}\biggr)\biggr)\biggr].$$

Set $A = \mathbb{E}(\Gamma_1)$ and let $K > 0$. Notice that the inequality $\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2} \geq -Ky$ may be rewritten $\Gamma_y \leq 2(A+K)y$. Markov's inequality yields

$$\mathbb{P}\big(\Gamma_y \geq 2(A+K)y\big) \leq \frac{A}{2(A+K)}.$$

Therefore, we get

$$\begin{aligned}
\mathbb{E}\biggl[\exp\biggl(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\biggr)\biggr] &\geq \mathbb{E}\biggl[\exp\biggl(\frac{\Gamma_y}{y}\biggl(\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2}\biggr)\mathbb{1}_{\{\Gamma_y \leq 2(A+K)y\}}\biggr)\biggr] \\
&\geq \mathbb{E}\big(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y \leq 2(A+K)y\}}\big) \\
&\geq \mathbb{E}\big(\exp(-K\Gamma_y)\big) - \mathbb{E}\big(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y > 2(A+K)y\}}\big) \\
&\geq \exp\big(-\phi(K)y\big) - \frac{A}{2(A+K)}.
\end{aligned}$$

This inequality holds for all $K > 0$. Hence, choosing $K = y^{-1/2-\varepsilon}$ with $\varepsilon > 0$, we get

$$\mathbb{E}\biggl[\exp\biggl(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\biggr)\biggr] \geq \exp\big(-\phi\big(y^{-1/2-\varepsilon}\big)y\big) - Ay^{1/2+\varepsilon}.$$

Furthermore, the product $\prod_{i=1}^{\infty}\mathbb{E}[\exp(-\frac{\Gamma_{x_i}^2}{2x_i}+c\Gamma_{x_i})]$ is strictly positive whenever

$$\sum_{i=1}^{\infty}\left(1-\mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i}+c\Gamma_{x_i}\right)\right]\right)$$

converges. Hence, we obtain the following sufficient condition:

$$\exists\varepsilon>0\quad\text{such that}\ \sum_{i=1}^{\infty}\left(1-\exp\left(-\phi\left(x_i^{-1/2-\varepsilon}\right)x_i\right)+Ax_i^{1/2+\varepsilon}\right)<\infty,\quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Recall that the distribution of the Brownian fragmentation at time t is equal to the distribution of the jumps of a stable subordinator T with index $1/2$ before time t conditioned on $T_t=1$ (see [1]). In particular, for all $\varepsilon>0$

$$\sum_{i=1}^{\infty}x_i^{1/2+\varepsilon}<\infty,\quad \mathbb{P}_t^{(B)}\text{-a.s.}\quad\text{(see Formula (9) of [1]).}$$

Thus, the measures $\mathbb{P}_{|\mathcal{F}_t}^{(B)}$ and $\mathbb{P}_{|\mathcal{F}_t}^{(X)}$ are equivalent whenever there exist two strictly positive numbers ε,ε' such that, for x small enough,

$$\phi\left(x^{-1/2-\varepsilon}\right)x\le x^{1/2+\varepsilon'}.$$

One can easily check that this condition is equivalent to (3). □

In Theorem 1.1, we assumed that X_t is of the form $B_t+\Gamma_t-ct$, with $c\ge\mathbb{E}(\Gamma_1)$ and where Γ_t is subordinator. It is natural to ask whether the theorem holds for a wider class of Lévy processes. Let us first notice that the process X must fulfill the conditions of Miermont's paper [15] recalled in the Introduction, i.e. X has no positive jumps, unbounded variation and finite positive mean. A simple example would be to choose $X_t=\sigma^2B_t+\Gamma_t-ct$, with $\sigma>0$, $\sigma\ne1$. However, it is clear that Theorem 1.1 fails in this case. Indeed, if we choose $X_t=2B_t$, using Proposition 3 of [15], we get

$$\left(F^X(2t),t\ge0\right)\overset{\text{law}}{=}\left(F^B(t),t\ge0\right).$$

But, it is well known that we have

$$\lim_{n\to\infty}n^2F_n(t)=t\sqrt{\frac{2}{\pi}},\quad \mathbb{P}^{(B)}\text{-a.s.}\quad\text{(see [7]).}$$

Hence, the laws $\mathbb{P}_t^{(B)}$ and $\mathbb{P}_{2t}^{(B)}$ are mutually singular.

3. An integro-differential equation

Let $\mathcal{L}$ be the infinitesimal generator of the Brownian fragmentation. Assuming that $\mathbf{g}$ belongs to its domain and using the fact that $\mathbf{g}(t,F(t))$ is a $\mathbb{P}^{(B)}$-martingale, we would have $\partial_t\mathbf{g}+\mathcal{L}\mathbf{g}=0$. In this section, we first compute the infinitesimal generator of a fragmentation process, which enables us to deduce an integro-differential equation solved by the function g.

3.1. *The infinitesimal generator of a fragmentation process*

We start by recalling an unpublished result obtained by Bertoin and Rouault [9].

Let $\mathcal{D}$ denote the space of functions $f:[0,1]\mapsto(0,1]$ of class $\mathcal{C}^1$ with $f(0)=1$. Given $f\in\mathcal{D}$ and $\mathbf{x}\in\mathcal{S}^{\downarrow}$, we set

$$\mathbf{f}(\mathbf{x})=\prod_{i=1}^{\infty}f(x_i).$$

For $\alpha\in\mathbb{R}_+$ and ν measure on $\mathcal{S}^{\downarrow}$ such that $\int_{\mathcal{S}^{\downarrow}}(1-x_1)\nu(\mathrm{d}\mathbf{x})<\infty$, we define the operator

$$\mathcal{L}_\alpha\mathbf{f}(\mathbf{x})=\mathbf{f}(\mathbf{x})\sum_{i=1}^{\infty}x_i^{\alpha}\int\nu(\mathrm{d}\mathbf{y})\left(\frac{\mathbf{f}(x_i\mathbf{y})}{f(x_i)}-1\right)\quad\text{for } f\in\mathcal{D}\text{ and }\mathbf{x}\in\mathcal{S}^{\downarrow}.$$

Proposition 3.1. *Let $(Y(t),t\ge 0)$ be a self-similar fragmentation with index of self-similarity $\alpha\ge 0$, with dislocation measure ν and without erosion. Then, for every function $f\in\mathcal{D}$, the process*

$$\mathbf{f}\big(Y(t)\big)-\int_0^t\mathcal{L}_\alpha\mathbf{f}\big(Y(s)\big)\,\mathrm{d}s$$

is a martingale.

We shall use the following lemma.

Lemma 3.2. *For $f\in\mathcal{D}$, $\mathbf{y}\in\mathcal{S}^{\downarrow}$, $r\in[0,1]$, we have*

$$\left|\frac{\mathbf{f}(r\mathbf{y})}{f(r)}-1\right|\le 2C_f\mathrm{e}^{C_f}r(1-y_1),$$

with $C_f=\|\frac{f'}{f^2}\|_\infty$.

Notice that, since f is $\mathcal{C}^1$ on $[0,1]$ and strictly positive, C_f is always finite.

Proof of Lemma 3.2. We first write

$$\big|\ln f(ry_1)-\ln f(r)\big|\le\left\|\frac{f'}{f}\right\|_\infty(1-y_1)r\le C_f(1-y_1)r,$$

from which we deduce

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)}-1\le\frac{f(ry_1)}{f(r)}-1\le\mathrm{e}^{C_f(1-y_1)r}-1\le C_f\mathrm{e}^{C_f}(1-y_1)r.$$

Besides, we have

$$\ln\frac{1}{f(x_i)}\le\frac{1}{f(x_i)}-1\le C_fx_i,\quad\text{which implies }\mathbf{f}(\mathbf{x})\ge f(x_1)\exp\left(-C_f\sum_{i=2}^{\infty}x_i\right).$$

Hence, we get

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)}\ge\frac{f(ry_1)}{f(r)}\exp\big(-C_f(1-y_1)r\big)\ge\exp\big(-2C_f(1-y_1)r\big),$$

and we deduce

$$1-\frac{\mathbf{f}(r\mathbf{y})}{f(r)}\le 2C_f(1-y_1)r.$$

□

Proof of Proposition 3.1. Assume first that $f \equiv 1$ in a neighbourhood of 0. In consequence, $\mathbf{f}(\mathbf{x})$ depends only on a finite number of terms of the sequence $\mathbf{x}$. Thus, we can write

$$\mathbf{f}(\mathbf{x}) = b(x_1, x_1 + x_2, \dots, x_1 + \cdots + x_n),$$

where b is a function from $[0, 1]^n$ to $[0, 1]$ of class C^1. Furthermore, Berestycki [3] proved that, if Y is a fragmentation process, then, for any $k \in \mathbb{N}$, $(Y_1(t) + \cdots + Y_k(t), t \geq 0)$ is a pure jump process. Considering only the jumps of Y larger than ε and letting ε tend to 0 with the help of a dominated convergence theorem, it is easily checked that $(\mathbf{f}(Y(t)), t \geq 0)$ is also a pure jump process with finite variation. Therefore, if $\mathcal{T}$ denotes the set of times where some dislocation occurs (it is a countable set), we have

$$\mathbf{f}\big(Y(t)\big) - \mathbf{f}\big(Y(0)\big) = \sum_{s \in [0,t] \cap \mathcal{T}} \big(\mathbf{f}\big(Y(s)\big) - \mathbf{f}\big(Y(s-)\big)\big).$$

Let, for $s \in \mathcal{T}$, k_s denote the index of the fragment which has been broken at time s, and let Δ_s be the element of $\mathcal{S}^{\downarrow}$ according to which $Y_{k_s}(s-)$ has been broken. With these notations, we can write

$$\sum_{s \in [0,t] \cap \mathcal{T}} \mathbf{f}\big(Y(s)\big) - \mathbf{f}\big(Y(s-)\big) = \sum_{s \in [0,t] \cap \mathcal{T}} \mathbf{f}\big(Y(s-)\big) \left(\sum_{i=1}^{\infty} \mathbb{1}_{k_s = i} \frac{\mathbf{f}(Y_i(s-)\Delta_s)}{f(Y_i(s-))} - 1 \right).$$

Recall that a dislocation of a fragment of mass r occurs with rate $\nu_r(\mathrm{d}\mathbf{y}) = r^{\alpha} \nu(\mathrm{d}\mathbf{y})$, thus the predictable compensator of this quantity is (see Section I.3 of [13] or [7])

$$\int_0^t \mathrm{d}s\, \mathbf{f}\big(Y(s-)\big) \int_{\mathcal{S}^{\downarrow}} \nu(\mathrm{d}\mathbf{y}) \sum_{i=1}^{\infty} Y_i^{\alpha}(s-) \left(\frac{\mathbf{f}(Y_i(s-)\mathbf{y})}{f(Y_i(s-))} - 1 \right) = \int_0^t \mathcal{L}_{\alpha} \mathbf{f}\big(Y(s)\big)\, \mathrm{d}s.$$

This completes the proof when f is constant on some neighbourhood of 0. We now turn our attention to the general case. Given a function $f \in \mathcal{D}$, we consider a sequence $(f_n)_{n \in \mathbb{N}}$ of functions of $\mathcal{D}$ such that:

- f_n converges uniformly to f.
- $f_n \equiv 1$ on $[0, 1/n]$.
- $\sup_n \|f_n'\|_{\infty} < \infty$.
- For all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, $f_n(x) \geq f(x)$.

Such a sequence clearly exists since f is $\mathcal{C}^1$ and $f(0) = 1$. Moreover, with these hypotheses, we have, for all $x \in [0, 1]$, $\ln f_n(x) \to \ln f(x)$ and $|\ln f_n(x)| \leq |\ln f(x)|$. Hence, using the dominated convergence theorem, we get $\mathbf{f}_n(\mathbf{x}) \to \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}^{\downarrow}$. This implies, for all $i \in \mathbb{N}$ and $\mathbf{y} \in \mathcal{S}^{\downarrow}$,

$$\left(\frac{\mathbf{f_n}(x_i \mathbf{y})}{f_n(x_i)} - 1 \right) \underset{n \to \infty}{\longrightarrow} \left(\frac{\mathbf{f}(x_i \mathbf{y})}{f(x_i)} - 1 \right).$$

Thanks to the previous lemma, these quantities are uniformly bounded by $C x_i (1 - y_1)$, thus $\mathcal{L}_{\alpha} \mathbf{f_n}(\mathbf{x})$ converges to $\mathcal{L}_{\alpha} \mathbf{f}(\mathbf{x})$ and we have $|\mathcal{L}_{\alpha} \mathbf{f_n}(\mathbf{x})| \leq C \int_{\mathcal{S}^{\downarrow}} (1 - y_1) \nu(\mathrm{d}y)$. We conclude that $\mathbf{f}(X(t)) - \int_0^t \mathcal{L}_{\alpha} \mathbf{f}(X(s))\, \mathrm{d}s$ is also a martingale. □

The next lemma is a generalization of Proposition 3.1.

Lemma 3.3. *Let $f : \mathbb{R}_+ \times [0, 1] \to (0, 1]$ be a C^1 function equal to 1 on $\mathbb{R}_+ \times \{0\}$ such that*

- $\sum_i |\partial_t f(t, x_i)| < \infty$, *for every* $\mathbf{x}$, *uniformly for* t *on any compact interval*,
- $\frac{\partial^2 f(t,x)}{\partial t\, \partial x}$ *exists and is bounded on any compact interval.*

For $\mathbf{x}\in\mathcal{S}^{\downarrow}$, *define* $\mathbf{f}(t,\mathbf{x})=\prod_i f(t,x_i)$. *Let* $(Y(t),t\geq 0)$ *be a self-similar fragmentation with index of self-similarity* $\alpha\geq 0$, *with dislocation measure* ν *and without erosion. Then, the process*

$$\mathbf{f}\big(t,Y(t)\big)-\int_0^t \mathcal{L}_\alpha\mathbf{f}\big(s,Y(s)\big)+\partial_t\mathbf{f}\big(s,Y(s)\big)\,\mathrm{d}s$$

is a martingale.

Proof. Let us first notice that $\partial_t\mathbf{f}(t,\mathbf{x})$ exists since $\sum_i|\partial_t f(t,x_i)|<\infty$ uniformly for t on any compact interval. Indeed, given $T>0$, the function $f(t,x)$ is strictly positive and continuous on $[0,T]\times[0,1]$ so we can find a constant η such that

$$\forall(t,x)\in[0,T]\times[0,1],\qquad f(t,x)>\eta.$$

Thus $\sum_{i=1}^{\infty}|\frac{\partial_t f(t,x_i)}{f(t,x_i)}|$ converges uniformly on $[0,T]$. Using the dominated convergence theorem, we get

$$\partial_t\mathbf{f}(t,\mathbf{x})=\mathbf{f}(t,\mathbf{x})\sum_{i=1}^{\infty}\frac{\partial_t f(t,x_i)}{f(t,x_i)}.$$

Furthermore, for any $\mathbf{x},\mathbf{y}\in\mathcal{S}^{\downarrow}$, we have

$$\big|\ln\mathbf{f}(t,\mathbf{x})-\ln\mathbf{f}(t,\mathbf{y})\big|\leq\sup\left\{x\in[0,1],\frac{\partial_x f(t,x)}{f(t,x)}\right\}\|\mathbf{x}-\mathbf{y}\|_1$$

and

$$\left|\sum_i\frac{\partial_t f(t,x_i)}{f(t,x_i)}-\sum_i\frac{\partial_t f(t,y_i)}{f(t,y_i)}\right|\leq\sup\left\{x\in[0,1],\partial_x\left(\frac{\partial_t f(t,x)}{f(t,x)}\right)\right\}\|\mathbf{x}-\mathbf{y}\|_1,$$

where $\|\mathbf{x}-\mathbf{y}\|_1=\sum_i|x_i-y_i|$ denotes the ℓ^1 norm. Using the fact that f has bounded derivatives and that $\frac{\partial^2 f(t,x)}{\partial t\,\partial x}$ is also bounded, we deduce that the function $\mathbf{x}\to\partial_t\mathbf{f}(t,\mathbf{x})$ is continuous on $\mathcal{S}^{\downarrow}$ (for either the ℓ^1 norm or the ℓ^∞ norm which are equivalent on $\mathcal{S}^{\downarrow}$). Let us now fix $t_2>t_1\geq 0$. We have

$$\mathbb{E}\big[\mathbf{f}\big(t_1,Y(t_2)\big)-\mathbf{f}\big(t_1,Y(t_1)\big)|\mathcal{F}_{t_1}\big]=\mathbb{E}\left[\int_{t_1}^{t_2}\mathcal{L}_\alpha\mathbf{f}\big(t_2,Y(s)\big)\,\mathrm{d}s\bigg|\mathcal{F}_{t_1}\right]$$

and

$$\mathbb{E}\big[\mathbf{f}\big(t_2,Y(t_2)\big)-\mathbf{f}\big(t_1,Y(t_2)\big)|\mathcal{F}_{t_1}\big]=\mathbb{E}\left[\int_{t_1}^{t_2}\partial_t\mathbf{f}\big(s,Y(t_2)\big)\,\mathrm{d}s\bigg|\mathcal{F}_{t_1}\right].$$

Therefore, for any partition $t_1=s_0<s_1<\cdots<s_n=t_2$, we get

$$\mathbb{E}\big[\mathbf{f}\big(t_2,Y(t_2)\big)-\mathbf{f}\big(t_1,Y(t_1)\big)|\mathcal{F}_{t_1}\big]=\mathbb{E}\left[\int_{t_1}^{t_2}\mathcal{L}_\alpha\mathbf{f}\big(s'',Y(s)\big)+\partial_t\mathbf{f}\big(s,Y(s')\big)\,\mathrm{d}s\bigg|\mathcal{F}_{t_1}\right],\tag{4}$$

where $s'=s_k$ and $s''=s_{k-1}$ if $s\in]s_{k-1},s_k]$. Using Lemma 3.2, we can get an upper bound for $|\mathcal{L}_\alpha\mathbf{f}(s'',Y(s))|$. Moreover, the process $(Y(s),s\geq 0)$ is continuous almost everywhere. Therefore, in view of the dominated convergence theorem and letting $\max|s_k-s_{k-1}|$ tend to 0, we deduce from (4) that

$$\mathbb{E}\big[\mathbf{f}\big(t_2,Y(t_2)\big)-\mathbf{f}\big(t_1,Y(t_1)\big)|\mathcal{F}_{t_1}\big]=\mathbb{E}\left[\int_{t_1}^{t_2}\mathcal{L}_\alpha\mathbf{f}\big(s,Y(s)\big)+\partial_t\mathbf{f}\big(s,Y(s)\big)\,\mathrm{d}s\bigg|\mathcal{F}_{t_1}\right],$$

which concludes the proof of the lemma. □

3.2. *Application to* $\mathbf{g}(t, F(t))$

Recall that $q_t(x)$ stands for the density of the Lévy process X (fulfilling the hypotheses of Theorem 1.1) and $p_t(x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})$ is the density of a Brownian motion. We proved in the first section that the function

$$G_t = \mathbf{g}\big(t, F(t)\big) = \mathrm{e}^{tc}\prod_{i=1}^{\infty}\frac{q_{F_i(t)}(-tF_i(t))}{p_{F_i(t)}(-tF_i(t))}$$

is a $\mathbb{P}^{(B)}$-martingale. Recall also the notation

$$g(t,x) = \mathrm{e}^{tcx}\frac{q_x(-tx)}{p_x(-tx)} \quad \text{for } x \in (0,1], t \geq 0 \quad \text{and} \quad g(t,0) = 1.$$

So that $\mathbf{g}$ may be written in the form

$$\mathbf{g}(t,\mathbf{x}) = \prod_{i=1}^{\infty} g(t,x_i) \quad \text{for } \mathbf{x} \in \mathcal{S}^{\downarrow}, t \geq 0.$$

It certainly seems interesting to have a better understanding of the density function $\mathbf{g}$. To this end, we establish, in the last section of this paper, an integro-differential equation satisfied by g when the Lévy measure of the subordinator Γ appearing in the decomposition of the Lévy process X is finite.

Proposition 3.4. *Assume that the Lévy* Π *measure of the subordinator* Γ *is finite* (*i.e.* $\int_0^\infty \Pi(\mathrm{d}x) < \infty$), *then the function* g *solves the equation*:

$$\begin{cases} \partial_t g(t,x) + \sqrt{x}\int_0^1 \frac{\mathrm{d}y}{\sqrt{8\pi y^3(1-y)^3}}\big(g(t,xy)g\big(t,x(1-y)\big) - g(t,x)\big) = 0 \\ g(0,x) = \frac{q_x(0)}{p_x(0)}. \end{cases}$$

In order to prove Proposition 3.4, we shall apply Lemma 3.3 to the function $\mathbf{g}$. Since, $\mathbf{g}(t, F(t))$ is already a martingale, this will give

$$\mathcal{L}_{1/2}\mathbf{g}\big(s, F(s)\big) + \partial_t \mathbf{g}\big(s, F(s)\big) = 0, \quad \mathbb{P}^{(B)}\text{-a.s.}$$

To this end, we must first establish some regularity results concerning $\mathbf{g}$. Let us first note that the function $(t,u) \mapsto q_t(u)$, density of a Lévy process X_t, is $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathbb{R}$, thus $(t,x) \mapsto g(t,x)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+ \times (0,1]$. In particular, for all $x \in [0,1]$, the function $t \to g(t,x)$ is $\mathcal{C}^1$ and so $\partial_t g(t,x)$ is well defined. The following lemma gives additional information about $\mathbf{g}$ when the Lévy measure of the subordinator Γ is finite.

Lemma 3.5. *Assume that the Lévy measure of the subordinator* Γ *is finite*, *then*:

1. *For all* $t \geq 0$, $\partial_x g(t,0)$ *exists and the function* $(s,x) \to \partial_x g(s,x)$ *is continuous at* $(t,0)$.
2. *For all* $t \geq 0$, $\frac{\partial^2 g(t,0)}{\partial t\,\partial x}$ *exists and the function* $(s,x) \to \frac{\partial^2 g(s,x)}{\partial t\,\partial x}$ *is continuous at* $(t,0)$.
3. *For all* $\mathbf{x} \in \mathcal{S}^{\downarrow}$, $\sum_i |\partial_t g(t,x_i)| < \infty$ *uniformly for* t *in a compact interval.*

Proof. Recall that we have

$$g(t,x) = \exp\left(-x\frac{c^2}{2}\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right]. \tag{5}$$

Therefore, setting

$$u_t(x) = \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right],$$

a sufficient condition for g to satisfy 1 of Lemma 3.5 is that $(s,x)\to u_s'(x)$ exists and is continuous at $(t,0)$ for $t\in\mathbb{R}_+$. We write $u_t(x)=a_t(x,x)$ with

$$a_t(y,z)=\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2z}+\Gamma_y(t+c)\right)\right].$$

Since the function $(y,z)\to\frac{y^2}{2z^2}\exp(-\frac{y^2}{2z}+y(t+c))$ is bounded on $\mathbb{R}_+\times[\varepsilon,1]$ for any $\varepsilon>0$, we deduce that

$$\partial_z a_t(y,z)=\mathbb{E}\left[\frac{\Gamma_y^2}{2z^2}\exp\left(-\frac{\Gamma_y^2}{2z}+\Gamma_y(t+c)\right)\right]\quad\text{for } z\in(0,1].$$

Recall that the infinitesimal generator of a subordinator without drift and Lévy measure Π is given, for every bounded function f of class C^1 with bounded derivative, by

$$\forall y\in\mathbb{R}_+,\quad Lf(y)=\int_0^\infty\big(f(y+s)-f(y)\big)\Pi(\mathrm{d}s)\quad\text{(cf. Section 31 of [18]).}$$

Hence, we get for all $z_0>0$,

$$\partial_y a_t(y,z_0)=\mathbb{E}\left[\int_0^\infty\left(\exp\left(-\frac{(\Gamma_y+s)^2}{2z_0}+(\Gamma_y+s)(t+c)\right)-\exp\left(-\frac{\Gamma_y^2}{2z_0}+\Gamma_y(t+c)\right)\right)\Pi(\mathrm{d}s)\right],$$

and we deduce, for $t\geq 0$ and $x>0$

$$\begin{aligned}u_t'(x)&=\mathbb{E}\left[\frac{\Gamma_x^2}{2x^2}\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right]\\&\quad+\mathbb{E}\left[\int_0^\infty\left(\exp\left(-\frac{(\Gamma_x+y)^2}{2x}+(\Gamma_x+y)(t+c)\right)-\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right)\Pi(\mathrm{d}y)\right].\end{aligned}\tag{6}$$

For any $T>0$, it is easily seen that there exists a constant C_1 such that

$$\forall t\in[0,T],\ \forall y\in\mathbb{R}_+,\ \forall z\in(0,1],\quad \frac{y^2}{2z^2}\exp\left(-\frac{y^2}{2z}+y(t+c)\right)\leq\frac{C_1}{z}.$$

Furthermore, for z smaller than $z_0\stackrel{\text{def}}{=}\frac{1}{(2+T+c)^2}$, this function is decreasing in y for $y\geq z^{1/4}$ and we have

$$\forall t\in[0,T],\ \forall z\in(0,z_0],\ \forall y\geq z^{1/4},\quad \frac{y^2}{2z^2}\exp\left(-\frac{y^2}{2z}+y(t+c)\right)\leq\frac{C_2}{z^{3/2}}\mathrm{e}^{-1/(2\sqrt{z})}.$$

For $x\geq 0$, let A_x denote the event

$$A_x\stackrel{\text{def}}{=}\big\{\Gamma\text{ has a jump of size smaller than }x^{1/4}\text{ before time }x\big\}.$$

On the one hand, we have

$$\mathbb{E}\left[\frac{\Gamma_x^2}{2x^2}\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\mathbb{1}_{A_x}\right]\leq\frac{C_1}{x}\mathbb{P}(A_x).$$

Since we assumed the Lévy measure Π of Γ to be finite, the quantity $\frac{\mathbb{P}(A_x)}{x}\sim\Pi(0,x^{1/4}]$ tends to 0 as x tends to 0. On the other hand, we have

$$\mathbb{E}\left[\frac{\Gamma_x^2}{2x^2}\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\mathbb{1}_{A_x^c}\right]\leq\frac{C_2}{x^{3/2}}\mathrm{e}^{-1/(2\sqrt{x})},$$

which also tends to 0 as x tends to 0. This proves that the first term on the r.h.s. of (6) has limit 0 as (t', x) tends to $(t, 0)$. We now deal with the second term of (6). We notice that, for all $x \in (0, 1]$,

$$\left|\exp\left(-\frac{(\Gamma_x + y)^2}{2x} + (\Gamma_x + y)(t + c)\right) - \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right| \le 2\exp\left(\frac{(t + c)^2 x}{2}\right).$$

Notice also that, for all $y > 0$, $\exp(-\frac{(\Gamma_x+y)^2}{2x} + (\Gamma_x + y)(t + c)) - \exp(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c))$ converges almost surely to -1 as (t', x) tends to $(t, 0)$. We therefore conclude that, when the Lévy measure Π is finite, $\lim_{(t',x)\to(t,0)} u_t'(x)$ exists (and is equal to $-\Pi(\mathbb{R}_+)$).

The proof of 2 of Lemma 3.5 is very similar to the proof of 1. We feel free to omit the details. Simply note that

$$\begin{aligned}\frac{\partial^2 g(t, x)}{\partial t\,\partial x} = \partial_t u_t'(x) = {} & \mathbb{E}\left[\frac{\Gamma_x^3}{2x^2}\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right] \\ & + \mathbb{E}\left[\int_0^\infty \left((\Gamma_x + y)\exp\left(-\frac{(\Gamma_x + y)^2}{2x} + (\Gamma_x + y)(t + c)\right)\right.\right. \\ & \left.\left. - \Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right)\Pi(\mathrm{d}y)\right],\end{aligned}$$

so, using similar arguments as above, we may check that this quantity tends to 0 as x tends to 0.

We now prove 3 of Lemma 3.5. Let us fix $\mathbf{x} \in \mathcal{S}^\downarrow$. An easy application of the dominated convergence theorem shows that the function $t \to \mathbb{E}[\exp(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c))]$ is differentiable with derivative

$$\partial_t \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right] = \mathbb{E}\left[\Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right].$$

Hence, using (5), we get

$$\partial_t g(t, x) = \exp\left(-x\frac{c^2}{2}\right)\mathbb{E}\left[\Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right].$$

Thus, the function $t \to \partial_t g(t, x)$ is positive and increasing. Therefore, it remains to prove that, for any $t \ge 0$, for any $\mathbf{x} \in \mathcal{S}_1$, we have

$$\sum_{i=1}^\infty \partial_t g(t, x_i) < \infty. \tag{7}$$

To this end, we simply notice that

$$\mathbb{E}\left[\Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c)\right)\right] \le \mathbb{E}\left[\Gamma_x \exp\left(\frac{x(t + c)^2}{2}\right)\right] \le x\mathbb{E}[\Gamma_1]\exp\left(\frac{(t + c)^2}{2}\right).$$

Thus (7) follows readily from $\sum_i x_i \le 1$. □

We can now give the proof of Proposition 3.4.

Proof of Proposition 3.4. Thanks to Lemma 3.5, the function $\partial_x g(s, x)$ is continuous at $(t, 0)$, which implies that the integral

$$\int_0^1 \frac{\mathrm{d}y}{\sqrt{8\pi y^3(1 - y)^3}}\big(g(t, xy)g\big(t, x(1 - y)\big) - g(t, x)\big)$$

is well defined and is continuous in x and in t. Indeed, using a symmetry argument, this integral is equal to

$$2\int_0^{1/2} \frac{\mathrm{d}y}{\sqrt{8\pi y^3(1-y)^3}}\bigl(g(t,xy)g\bigl(t,x(1-y)\bigr)-g(t,x)\bigr).$$

Moreover, for all $y \in (0,1/2)$, $x \in (0,1]$, $t \in \mathbb{R}_+$, there exist $c, c' \in [0,x]$ such that

$$\frac{g(t,xy)g(t,x(1-y))-g(t,x)}{y} = x\bigl(g(t,x)\,\partial_x g(t,c) - g(t,xy)\,\partial_x g\bigl(t,c'\bigr)\bigr).$$

Therefore, since the function $(t,x) \to \partial_x g(t,x)$ is continuous on $\mathbb{R}_+ \times [0,1]$, the function $(t,x,y) \to |x(g(t,x)\,\partial_x g(t,c) - g(t,xy)\,\partial_x g(t,c'))|$ is uniformly bounded on $[0,T]\times[0,1]\times[0,\frac{1}{2}]$ for any $T>0$ so, by application of the dominated convergence theorem, the above mentioned integral is finite, continuous in t on $\mathbb{R}_+$ and continuous in x on $[0,1]$.

In order to prove Proposition 3.4, we shall use Lemma 3.3, so we must check that $\mathbf{g}(t,\mathbf{x})$ is in the domain of the Brownian fragmentation generator. We already proved that $g(t,0)=1$. Moreover, Lemma 3.5 states that $x \to g(t,x)$ is of class C^1. Note that the hypotheses of Lemma 3.3 also state that $g(t,x)$ should be smaller than 1 for all x in $[0,1]$. However, this assumption can be dropped in the case of conservative fragmentation (i.e. for a fragmentation process Y such that $\sum_i Y_i(t) = 1$ almost surely). Indeed, given a function $f:[0,1]\to(0,\infty)$ of class C^1 with $f(0)=1$, we may pick $a \ge 0$ large enough such that the function $\tilde{f}(x) \stackrel{\text{def}}{=} \mathrm{e}^{-ax} f(x)$ satisfies all the hypotheses of Proposition 3.1. Applying the result for $\tilde{f}$ instead of f and using that $\sum_i Y_i(t) = 1$ a.s., we again deduce that

$$\mathbf{f}\bigl(Y(t)\bigr) - \int_0^t \mathcal{L}_\alpha \mathbf{f}\bigl(Y(s)\bigr)\,\mathrm{d}s$$

is a martingale.

We now deduce, using the expression of the dislocation measure of the Brownian fragmentation given in [6], that the infinitesimal generator is equal to

$$\mathcal{L}_{1/2}\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x})\sum_{i=1}^{\infty}\sqrt{x_i}\int \nu(\mathrm{d}\mathbf{y})\left(\frac{\mathbf{f}(x_i\mathbf{y})}{f(x_i)} - 1\right),$$

with

$$\nu(y_1 \in \mathrm{d}u) = \bigl(2\pi u^3(1-u)^3\bigr)^{-1/2}\,\mathrm{d}u \quad \text{for } u \in \left(\frac{1}{2},1\right), \qquad \nu(y_1+y_2 \ne 1) = 0.$$

Thus, applying Lemma 3.3 to $\mathbf{g}(t,\mathbf{x})$, we deduce that

$$M_t = \mathbf{g}\bigl(t,F(t)\bigr) - \mathbf{g}\bigl(0,F(0)\bigr) - \int_0^t \mathcal{L}_{1/2}\mathbf{g}\bigl(s,F(s)\bigr) + \partial_t \mathbf{g}\bigl(s,F(s)\bigr)\,\mathrm{d}s$$

is a $\mathbb{P}^{(B)}$-martingale. Since $\mathbf{g}(t,F(t))$ is already a $\mathbb{P}^{(B)}$-martingale, this implies

$$\mathcal{L}_{1/2}\mathbf{g}\bigl(s,F(s)\bigr) + \partial_t \mathbf{g}\bigl(s,F(s)\bigr) = 0, \quad \mathbb{P}^{(B)}\text{-a.s. for almost every } s>0,$$

i.e. for almost every $s>0$

$$\mathbf{g}\bigl(s,F(s)\bigr)\sum_{i=1}^{\infty}\left[F_i^{1/2}(s)\int_{\mathcal{S}^\downarrow}\nu(\mathrm{d}\mathbf{y})\left(\frac{\mathbf{g}(s,F_i(s)\mathbf{y})}{g(s,F_i(s))} - 1\right) + \frac{\partial_t g(s,F_i(s))}{g(s,F_i(s))}\right] = 0, \quad \mathbb{P}^{(B)}\text{-a.s.}$$

Let us fix $s \ge 0$. We use the notation $F(s) = (x_1,x_2,\ldots)$. Thus, since $\mathbf{g}(s,F(s))>0$,

$$\sum_{i=1}^{\infty}\left[x_i^{1/2}\int_{\mathcal{S}^\downarrow}\nu(\mathrm{d}\mathbf{y})\left(\frac{\mathbf{g}(s,x_i\mathbf{y})}{g(s,x_i)} - 1\right) + \frac{\partial_t g(s,x_i)}{g(s,x_i)}\right] = 0, \quad \mathbb{P}_s^{(B)}\text{-a.s.} \tag{8}$$

According to Lemma 3.2, we have

$$\left|x_i^{1/2}\int_{\mathcal{S}^\downarrow}\nu(\mathrm{d}\mathbf{y})\left(\frac{\mathbf{g}(s,x_i\mathbf{y})}{g(s,x_i)}-1\right)\right|\le C_{g,s}x_i\int_{\mathcal{S}^\downarrow}(1-y_1)\nu(\mathrm{d}\mathbf{y}),$$

where $C_{g,s}$ is a positive constant (which depends on g and s) and besides

$$\forall x_i\in(0,1),\ \forall s>0,\quad \frac{\partial_t g(s,x_i)}{g(s,x_i)}>0.$$

Therefore, the series (8) is absolutely convergent. Let us define

$$k(t,x)=\partial_t g(t,x)+\sqrt{x}\int_0^1\frac{\mathrm{d}y}{\sqrt{8\pi y^3(1-y)^3}}\big(g(t,xy)g\big(t,x(1-y)\big)-g(t,x)\big).\tag{9}$$

Hence, we have

$$\sum_{i=1}^{\infty}k(s,x_i)=0,\quad \mathbb{P}_s^{(B)}\text{-a.s. for almost every } s>0$$

and

$$\sum_{i=1}^{\infty}\big|k(s,x_i)\big|<\infty,\quad \mathbb{P}_s^{(B)}\text{-a.s. for almost every } s>0.$$

Furthermore, $x\to k(t,x)$ is continuous on $[0,1]$ hence, thanks to the following lemma, for almost every $s>0$, $k(s,x)=0$ for $x\in[0,1]$. Since $s\to k(s,x)$ is continuous on $\mathbb{R}_+$, we deduce $k\equiv 0$ on $\mathbb{R}_+\times[0,1]$, which, in view of (9), completes the proof of Proposition 3.4. □

Lemma 3.6. *Fix $t>0$. Let $\mathbb{P}_t^{(B)}$ denote the law of the Brownian fragmentation at time t. Let $k:[0,1]\mapsto\mathbb{R}$ be a continuous function, such that*

$$\sum_{i=1}^{\infty}k(x_i)=0\quad \mathbb{P}_t^{(B)}\text{-a.s.}\quad\text{and}\quad \sum_{i=1}^{\infty}\big|k(x_i)\big|<\infty,\quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Then $k\equiv 0$ on [0,1].

Proof. We first notice that, since $\sum_{i=1}^{\infty}|k(x_i)|<\infty\quad \mathbb{P}_t^{(B)}$-a.s., we have $k(0)=0$. Let $\mathbf{x}=(x_1,x_2,\ldots)$ be a random variable with law $\mathbb{P}_t^{(B)}$ ordered by a size-biased pick. Let also $\mathcal{S}$ denote the set of positive sequences with sum less than 1. Since $\mathbf{x}$ has the law of the size-biased reordering of the jumps of a stable subordinator T (with index 1/2) before time t, conditioned by $T_t=1$ (see [1]), it is obvious that

$$\forall u\in(0,1-S),\quad \mathbb{P}_t^{(B)}\big(x_1\in\mathrm{d}u|(x_i)_{i\ge 3}\big)/\mathrm{d}u>0,$$

where $S\overset{\text{def}}{=}\sum_{i\ge 3}x_i$. Let $\mathbb{Q}_t$ stand for the measure on $\mathcal{S}$ defined by

$$\forall A\subset\mathcal{S},\quad \mathbb{Q}_t(A)=\mathbb{P}_t^{(B)}\big((x_i)_{i\ge 3}\in A\big)$$

and let λ denote the Lebesgue measure on [0, 1]. We have, for $\mathbb{Q}_t$-almost every $y\in\mathcal{S}$

$$\forall u\in(0,1-S),\quad k(u)+k(1-S-u)+\sum_{i=1}^{\infty}k(y_i)=0,\quad \lambda(\mathrm{d}u)\text{-a.s.},$$

where $S=\sum_i y_i$. Choosing $y\in\mathcal{S}$ such that this equality holds for almost every $u\in(0,1-S)$, we deduce that there exists a constant $C=C(y)$ such that

$$k(u)+k(1-S-u)=C,\quad \text{for almost every } u\in(0,1-S).$$

Since k is continuous, this equality holds, in fact, for all $u\in[0,1-S]$. Furthermore, we have

$$\forall s\in(0,1),\quad \mathbb{Q}_t(S\in \mathrm{d}s)>0.$$

Hence, there exists, for almost every $s\in(0,1)$, a constant C_s such that

$$k(u)+k(1-s-u)=C_s\quad \text{for all } u\in(0,1-s).$$

In view of the continuity of k, this equality holds, in fact, for all $s\in[0,1]$. In particular, we get that

$$\forall x,y\in[0,1]^2,\ \text{such that } x+y\le 1,\quad k(x+y)=k(x)+k(y).$$

Therefore, k is a linear function. Finally, since $\sum_{i=1}^{\infty}x_i=1$, $\mathbb{P}_t^{(B)}$-a.s., we conclude that $k\equiv 0$ on $[0,1]$. □

Remark 3.7. *We would certainly like to drop the assumption that the Lévy measure of Γ is finite in Proposition* 3.4. *Let us note that this hypothesis is needed to prove that the function $(s,x)\to\partial_x g(s,x)$ is continuous at $(t,0)$. However, in the proof of Lemma* 3.5, *we showed that $\partial_x g(s,0)=-\frac{c^2}{2}-\Pi(0,\infty)$. This result indicates that, when the Lévy measure of Γ is infinite, $\partial_x g(s,0)$ is equal to $-\infty$ (assuming that we can justify the inversion of the limits), so g is not in $\mathcal{D}$ anymore.*

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