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Exponential concentration for first passage percolation through modified Poincaré inequalities[1]

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Abstract. We provide a new exponential concentration inequality for first passage percolation valid for a wide class of edge times distributions. This improves and extends a result by Benjamini, Kalai and Schramm (*Ann. Probab.* **31** (2003)) which gave a variance bound for Bernoulli edge times. Our approach is based on some functional inequalities extending the work of Rossignol (*Ann. Probab.* **35** (2006)), Falik and Samorodnitsky (*Combin. Probab. Comput.* **16** (2007)).

Résumé. On obtient une nouvelle inégalité de concentration exponentielle pour la percolation de premier passage, valable pour une large classe de distributions des temps d'arêtes. Ceci améliore et étend un résultat de Benjamini, Kalai et Schramm (*Ann. Probab.* **31** (2003)) qui donnait une borne sur la variance pour des temps d'arêtes suivant une loi de Bernoulli. Notre approche se fonde sur des inégalités fonctionnelles étendant les travaux de Rossignol (*Ann. Probab.* **35** (2006)), Falik et Samorodnitsky (*Combin. Probab. Comput.* **16** (2007)).

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1. Introduction

First passage percolation was introduced by Hammersley and Welsh [10] to model the flow of a fluid in a randomly porous material (see [12] for a recent account on the subject). We will consider the following model of first passage percolation in $\mathbb{Z}^d$, where $d \geq 2$ is an integer. Let $E = E(\mathbb{Z}^d)$ denote the set of edges in $\mathbb{Z}^d$. The passage time of the fluid through the edge e is denoted by x_e and is supposed to be nonnegative. Randomness of the porosity is given by a product probability measure on $\mathbb{R}_+^E$. Thus, $\mathbb{R}_+^E$ is equipped with the measure $\mu = \nu^{\otimes E}$, where ν is a probability measure on $\mathbb{R}_+$ according to which each passage time is distributed, independently from the others. If u, v are two vertices of $\mathbb{Z}^d$, the notation α: $\{u, v\}$ means that α is a path with end points u and v. When $x \in \mathbb{R}_+^E$, $d_x(u, v)$ denotes the first passage time, or equivalently the distance from u to v in the metric induced by x,

$$d_x(u, v) = \inf_{\alpha:\ \{u,v\}} \sum_{e\in\alpha} x_e.$$

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The study of $d_x(0, nu)$ when n is an integer which goes to infinity is of central importance. Kingman's subadditive ergodic theorem implies the existence, for each fixed u, of a "time constant" $t(u)$ such that:

$$\frac{d_x(0,nu)}{n} \underset{n\to+\infty}{\overset{\nu\text{-}a.s.}{\longrightarrow}} t(u).$$

It is known (see [15], pages 127 and 129) that if $\nu(\{0\})$ is strictly smaller than the critical probability for Bernoulli bond percolation on $\mathbb{Z}^d$, then $t(u)$ is positive for every u distinct from the origin. Under such an assumption, one can say that the random variable $d_x(0, nu)$ is located around $nt(u)$, which is of order $\mathrm{O}(|nu|)$, where we denote by $|\cdot|$ the L^1-norm of vertices in $\mathbb{Z}^d$. In this paper, we are interested in the fluctuations of this quantity. Precisely, we define, for any vertex v,

$$\forall x \in \mathbb{R}_+^E, \quad f_v(x) = d_x(0, v).$$

It is widely believed that the fluctuations of f_v are of order $|v|^{1/3}$ when $d = 2$. Apart from some predictions made by physicists, this faith relies on recent results for related growth models [2,13,14]. Until recently, the best results rigourously obtained for the fluctuations of f_v were some moderate deviation estimates of order $\mathrm{O}(|v|^{1/2})$ (see [16, 24]). In 1993, Kesten [16] proved that

$$\mathsf{Var}_\mu(f_v) = \mathrm{O}\big(|v|\big),$$

provided ν admits a finite second-order moment. If, furthermore, ν admits a finite moment of exponential order, there exist two constants C_1 and C_2 such that for any $t \le |v|$,

$$\nu\big(\big|f_v - \mathbb{E}(f_v)\big| > t\sqrt{|v|}\big) \le C_1 \mathrm{e}^{-C_2 t}. \tag{1}$$

Later, Talagrand improved the right-hand side of the above inequality to $\exp(-C_2 t^2)$. In 2003, Benjamini, Kalai and Schramm [5] proved that for Bernoulli edge times bounded away from 0, the variance of f_v is of order $\mathrm{O}(|v|/\log|v|)$, and therefore, the fluctuations are of order $\mathrm{O}(|v|^{1/2}/(\log|v|)^{1/2})$.

It is natural to ask whether the work of Benjamini, Kalai and Schramm [5] can be extended to other distributions, notably continuous distributions which are not bounded away from zero. This has been done in a preliminary version of the present paper [4] by extending the tools of [5], namely a modified Poincaré inequality due to Talagrand. It is also natural, and even more desirable, to try to improve the result of Benjamini, Kalai and Schramm [5] into an exponential inequality in the spirit of (1), with $\sqrt{|v|/\log|v|}$ instead of $\sqrt{|v|}$. In this article, we show that some different modified Poincaré inequalities arising from the context of "threshold phenomena" for Boolean functions (see [9,21]) may be used successfully instead of Talagrand-type inequalities from [4,23]. This is the main result of this paper stated in Theorem 5.4. Whereas we focused on the percolation setting, the argument is fairly general and we present also an abstract exponential concentration result, Theorem 4.2, which is very likely to have applications outside the setting of percolation.

This article is organized as follows. In Section 2, we extend the modified Poincaré inequalities of Falik and Samorodnitsky [9] to non-Bernoulli and countable settings where logarithmic Sobolev inequalities are available, notably to a countable product of Gaussian measures. Section 3 is devoted to the obtention of similar inequalities for other continuous measures by a simple mean of change of variable. In Section 4, we show how to deduce new general exponential concentration bounds from the modified Poincaré inequalities of Section 2. This allows us to obtain in Section 5 an exponential version of the bound of Benjamini et al. in some continuous and discrete settings.

Notation. *Given a probability space $(\mathbb{X}, \mathcal{X}, \mu)$ and a real valued measurable function f defined on $\mathbb{X}$ we let*

$$\|f\|_{p,\mu} = \left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p} \in [0, \infty],$$

and $L^p(\mu)$ denote the set of f such that $\|f\|_{p,\mu} < \infty$. The mean *of $f \in L^1(\mu)$ is denoted*

$$\mathbb{E}_\mu(f) = \int f \,\mathrm{d}\mu,$$

the variance *of* $f \in L^2(\mu)$ *is*

$$\mathsf{Var}_\mu(f) = \|f - \mathbb{E}_\mu(f)\|_{2,\mu}^2,$$

and the entropy *of any positive measurable function* f *is*

$$\mathsf{Ent}_\mu(f) = \int f \log f \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu \log \int f \,\mathrm{d}\mu.$$

When the choice of μ *is unambiguous we may write* $\|f\|_p$ *(respectively* L^p, $\mathbb{E}(f)$, $\mathsf{Var}(f)$ *and* $\mathsf{Ent}(f)$*) for* $\|f\|_{p,\mu}$ *(respectively* $L^p(\mu)$, $\mathbb{E}_\mu(f)$, $\mathsf{Var}_\mu(f)$ *and* $\mathsf{Ent}_\mu(f)$*).*

2. Logarithmic Sobolev and modified Poincaré inequalities on $\mathbb{R}^{\mathbb{N}}$

The relevance of a "modified Poincaré inequality" due to Talagrand [23] in the context of first passage percolation was shown by Benjamini, Kalai and Schramm [5]. Let us explain this point a little bit more. A classical Poincaré inequality has the following form:

$$\mathsf{Var}_\mu(f) \le C\mathcal{E}_\mu(f),$$

where C is a constant, and $\mathcal{E}_\mu(f)$ is an "energy" of f, that is, usually, the mean against μ of the square of some kind of gradient. There is a good theory for this in the context of Markov semi-groups (see [1,3], for instance). By "modified Poincaré inequality," we mean a functional inequality which improves upon the classical Poincaré inequality for a certain class of functions f. This is usually achieved through a hypercontrativity property (see [4] and [19]).

In this section, we will show how to build a modified Poincaré inequality on a product of probability spaces each of which satisfies a Sobolev logarithmic inequality. This approach was initiated independently by Rossignol [21], Falik and Samorodnitsky [9] in the Bernoulli setting.

We shall need some notation for tensorisation. Suppose that we are given a countable collection of probability spaces $(\mathbb{X}_i, \mathcal{X}_i, \mu_i)_{i\in I}$. If i belongs to I, and x^{-i} is an element of $\prod_{j\in I, j\neq i} \mathbb{X}_j$, then for every x_i in $\mathbb{X}_i$, we denote by (x^{-i}, x_i) the element x of $\prod_{j\in I} \mathbb{X}_j$. For every function f from $\prod_{i\in I} \mathbb{X}_i$ to $\mathbb{R}$, every $j \in I$, and every x^{-j} in $\prod_{i\neq j} \mathbb{X}_i$, we denote by $f_{x^{-j}}$ the function from $\mathbb{X}_j$ to $\mathbb{R}$ obtained from f by keeping x^{-j} fixed:

$$\forall x_j \in \mathbb{X}_j, \quad f_{x^{-j}}(x_j) = f(x).$$

Now, suppose that we are given a collection $(\mathcal{A}_i)_{i\in I}$ of linear subspaces, $\mathcal{A}_i \subset L^2(\mathbb{X}_i, \mu_i)$, containing the constant functions. Then, we introduce

$$\mathcal{A}^I = \left\{ f \in L^2\left(\prod_{i\in I} \mathbb{X}_i, \bigotimes_{i\in I} \mu_i\right) \text{ s.t. } \forall j \in I,\ f_{x^{-j}} \in \mathcal{A}_j \text{ for } \bigotimes_{i\neq j} \mu_i\text{-a.e. } x^{-j} \right\}.$$

An operator R_j from $\mathcal{A}_j$ to $L^2(\mathbb{X}_j, \mu_j)$ is naturally extended on $\mathcal{A}^I$, "acting only on coordinate j":

$$\forall f \in \mathcal{A}^I, \forall x \in \prod_{i\in I} \mathbb{X}_i, \quad R_j(f)(x) := R_j(f_{x^{-j}})(x_j).$$

Proposition 2.1. *Let* $(\mathbb{X}_i, \mathcal{X}_i, \mu_i)_{i\in I}$, *be a sequence of probability spaces. Let* $(\mathcal{A}_i)_{i\in I}$ *be a collection of sets such that for every* i, $\mathcal{A}_i$ *is a linear subspace of* $L^2(\mathbb{X}_i, \mu_i)$ *which contains the constant functions. Suppose that for every* i *in* I, μ_i *satisfies a logarithmic Sobolev inequality of the following form*:

$$\forall f \in \mathcal{A}_i, \quad \mathsf{Ent}_{\mu_i}(f^2) \le \mathbb{E}_{\mu_i}(R_i(f)^2),$$

where R_i is a linear operator from $\mathcal{A}_i$ to $L^2(\mu_i)$ with value zero on any constant function. Furthermore, suppose that the following commutation property holds:

$$\forall i, \forall f \in \mathcal{A}^I, \quad \int f \,\mathrm{d} \bigotimes_{j \neq i} \mu_j \in \mathcal{A}_i \quad \textit{and} \quad R_i\left(\int f \,\mathrm{d} \bigotimes_{j \neq i} \mu_j\right) = \int R_i(f) \,\mathrm{d} \bigotimes_{j \neq i} \mu_j.$$

Then, $\mu^I = \bigotimes_{i \in I} \mu_i$ *satisfies the following modified Poincaré inequality*:

$$\forall f \in \mathcal{A}^{\mathbb{N}}, \quad \mathsf{Var}_{\mu^I}(f) \log \frac{\mathsf{Var}_{\mu^I}(f)}{\sum_{i \in I} \|\Delta_i f\|^2_{\mu^I,1}} \le \sum_{i \in I} \mathbb{E}_{\mu^I}\bigl(R_i(f)^2\bigr),$$

where Δ_i is the following operator on $L^2(\mu^I)$:

$$\forall f \in L^2\bigl(\mu^I\bigr), \quad \Delta_i f = f - \int f \,\mathrm{d}\mu_i.$$

Proof. To shorten the notations, we shall write μ instead of μ^I.

First, suppose that I is finite, $I = \{1, \dots, n\}$. The tensorisation property of the entropy (see [1] or [17], Proposition 5.6, page 98, for instance) states that for every positive measurable function g,

$$\mathsf{Ent}_\mu(g) \le \sum_{i=1}^n \mathbb{E}_\mu\bigl(\mathsf{Ent}_{\mu_i}(g)\bigr).$$

Thus, the logarithmic Sobolev inequalities for each μ_i imply that:

$$\forall g \in \mathcal{A}^n, \quad \mathsf{Ent}_\mu\bigl(g^2\bigr) \le \sum_{i=1}^n \mathbb{E}_\mu\bigl(R_i(g)^2\bigr). \tag{2}$$

Now, let f be a function in $\mathcal{A}^n$. Following Rossignol [21], Falik and Samorodnitsky [9], we write $f - \mathbb{E}_\mu(f)$ as a sum of martingale increments, and apply the logarithmic Sobolev inequality (2) to each increment:

$$\sum_{j=1}^n \mathsf{Ent}_\mu\bigl(V_j^2\bigr) \le \sum_{j=1}^n \sum_{i=1}^n \mathbb{E}_\mu\bigl(R_i(V_j)^2\bigr), \tag{3}$$

where

$$f - \mathbb{E}_\mu(f) = \sum_{j=1}^n V_j,$$

and

$$V_j = \int f \,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_{j-1} - \int f \,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_j = \int \Delta_j f \,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_{j-1}.$$

The following inequality, which is a clever application of Jensen's inequality, is shown in [9] and is cleaner than the corresponding one in [21]:

$$\sum_{j=1}^n \mathsf{Ent}_\mu\bigl(V_j^2\bigr) \ge \mathsf{Var}_\mu(f) \log \frac{\mathsf{Var}_\mu(f)}{\sum_{j=1}^n \|V_j\|^2_{\mu,1}}.$$

Jensen's inequality implies that:

$$\sum_{j=1}^n \mathsf{Ent}_\mu\bigl(V_j^2\bigr) \ge \mathsf{Var}_\mu(f) \log \frac{\mathsf{Var}_\mu(f)}{\sum_{j=1}^n \|\Delta_j f\|^2_{\mu,1}}. \tag{4}$$

On the other hand, for every i, the term $\mathbb{E}_\mu(R_i(g)^2)$ in (2) is called an "energy" for g, and we claim that the sum of the energies of the increments of f equals the energy of f:

$$\forall i \in \{1,\dots,n\}, \quad \sum_{j=1}^n \mathbb{E}_\mu\big(R_i(V_j)^2\big) = \mathbb{E}_\mu\big(R_i(f)^2\big). \tag{5}$$

Indeed, since R_i is linear, using the commutation hypothesis, and the fact that $R_i(f)$ is zero on any function f which is constant on coordinate i, we get:

$$\forall i < j, \quad R_i(V_j) = 0,$$
$$R_i(V_i) = \int R_i(f)\,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_{i-1},$$

and

$$\forall i > j, \quad R_i(V_j) = \int R_i(f)\,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_{j-1} - \int R_i(f)\,\mathrm{d}\mu_1 \otimes \cdots \otimes \mathrm{d}\mu_j.$$

Therefore,

$$\sum_{j=1}^n \mathbb{E}_\mu\big(R_i(V_j)^2\big) = \sum_{j=1}^{i-1} \mathbb{E}_\mu\big(R_i(V_j)^2\big) + \mathbb{E}_\mu\big(R_i(V_i)^2\big) = \mathbb{E}_\mu\big(R_i(f)^2\big).$$

Now, claim (5) is proved and the result follows from (3), (4) and (5), at least when I is finite.

Now, suppose that I is strictly countable, let us say $I = \mathbb{N}$, and let $\mathcal{F}_n$ be the σ-algebra generated by the first n coordinate functions in $\mathbb{R}^{\mathbb{N}}$. Let $f \in \mathcal{A}^{\mathbb{N}}$ and $f_n = \mathbb{E}(f|\mathcal{F}_n)$ be the conditional expectation of f with respect to $\mathcal{F}_n$. Then, the commutation property tells us that f_n belongs to $\mathcal{A}^{\mathbb{N}}$, and $R_i(f_n) = \mathbb{E}(R_i(f)|\mathcal{F}_n)$. Therefore, we can apply the first part of Proposition 2.1, the one that we just proved:

$$\mathsf{Var}_{\mu^n}(f_n) \log \frac{\mathsf{Var}_{\mu^n}(f_n)}{\sum_{i=1}^n \|\Delta_i f_n\|_{\mu^n,1}^2} \le \sum_{i=1}^n \mathbb{E}_{\mu^n}\big(R_i(f_n)^2\big).$$

This may be written as:

$$\mathsf{Var}_{\mu^{\mathbb{N}}}(f_n) \log \frac{\mathsf{Var}_{\mu^{\mathbb{N}}}(f_n)}{\sum_{i=1}^n \|\Delta_i f_n\|_{\mu^n,1}^2} \le \sum_{i=1}^n \mathbb{E}_{\mu^{\mathbb{N}}}\big(\mathbb{E}\big(R_i(f)|\mathcal{F}_n\big)^2\big).$$

Obviously, $\Delta_i f_n = \mathbb{E}(\Delta_i(f)|\mathcal{F}_n)$. Therefore, Jensen's inequality implies:

$$\mathsf{Var}_{\mu^{\mathbb{N}}}(f_n) \log \frac{\mathsf{Var}_{\mu^{\mathbb{N}}}(f_n)}{\sum_{i=1}^n \|\Delta_i f\|_{\mu^n,1}^2} \le \sum_{i=1}^n \mathbb{E}_{\mu^{\mathbb{N}}}\big(\big(R_i(f)\big)^2\big).$$

Of course, f_n converges to f in $L^2(\mu^{\mathbb{N}})$, and we may let n tend to infinity in the last inequality to get the desired result. □

Remark 1. *Actually, a logarithmic Sobolev inequality associated to a probability measure μ_i which is reversible with respect to an operator $\mathbf{L}$ may always be written in the form of Proposition* 2.1. *Indeed, such an inequality may be written as*:

$$\forall f \in \mathcal{A}_i, \quad \mathsf{Ent}_{\mu_i}\big(f^2\big) \le c\mathbb{E}_{\mu_i}(-f\mathbf{L}f),$$

where c is a positive constant. Since $\mathbf{L}$ is a self-adjoint operator in $L^2(\mu_i)$, it admits a spectral representation (*see* [27], *page* 313). *It is easy to show that its eigenvalues are nonnegative* (*see*, *for instance* [3], *page* 7). *The spectral decomposition of* $-\mathbf{L}$ *may*, *therefore*, *be written as*:

$$-\mathbf{L} = \int_0^{\infty} \lambda \,\mathrm{d}E(\lambda),$$

and a suitable candidate for R_i may be deduced from it:

$$R_i = \int_0^{\infty} \sqrt{c\lambda}\,\mathrm{d}E(\lambda). \tag{6}$$

Nevertheless, *in the applications which follow*, *it is essential that the operator R_i is nice enough to allow the quantity* $\sum_{i=1}^{n} R_i(f)^2$ *to be easily controlled*, *and the one given in* (6) *may not be appropriate for this. Another candidate*, *which we shall see to be the right one for certain continuous probability measures*, *is the square root of the "carré du champ" operator* $\Gamma^{1/2}(f,f)$, *where*:

$$\Gamma(f,g) = \frac{1}{2}(\mathbf{L}fg - f\mathbf{L}g - g\mathbf{L}f).$$

But in the discrete case, *this is not the most natural choice. Therefore*, *we prefer not to try to generalize any longer*, *and rather give some examples*.

2.1. Examples

Not surprisingly, we start to illustrate Proposition 2.1 with the Bernoulli and Gaussian cases. Our choice to present them "mixed" might look a little weird at first sight, but this will prove to be useful in the percolation context (see Section 5).

Example 1 (The Bernoulli and Gaussian cases). *We let*

$$\beta_p = (1-p)\delta_0 + p\delta_1$$

be the Bernoulli measure with parameter p on $\{0,1\}$. *If p belongs to* $]0,1[$, β_p (*see*, *for instance* [22], *Theorem* 2.2.8, *page* 336, *or* [1]) *satisfies the following logarithmic Sobolev inequality*: *for any function f from* $\{0,1\}$ *to* $\mathbb{R}$,

$$\mathrm{Ent}_{\beta_p}\big(f^2\big) \le c_{LS}(p)\mathbb{E}_{\beta_p}\big((\Delta f)^2\big),$$

where

$$c_{LS}(p) = \frac{\log p - \log(1-p)}{p-(1-p)}$$

and

$$\Delta f = f - \int f \,\mathrm{d}\beta_p.$$

If S is a countable set, *for any s in S*, *let $\mathbb{X}_s$ be a copy of* $\{0,1\}$, *$\mathcal{A}_s$ be the set of functions from $\mathbb{X}_s$ to $\mathbb{R}$*, *and R_s be the operator Δ acting on $\mathcal{A}_s$. We denote also a product measure λ_p^S on* $\{0,1\}^S$: $\lambda_p^S = \beta_p^{\otimes S}$.

Now we introduce the Gaussian setting. Let

$$\gamma(\mathrm{d}y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2}\,\mathrm{d}y$$

denote the standard Gaussian measure on $\mathbb{R}$. *A map* f *is said to be* weakly differentiable *provided there exists a locally integrable function denoted* $f'(x)$ *such that*

$$\int f'(x)g(x)\,\mathrm{d}x = -\int g'(x)f(x)\,\mathrm{d}y$$

for every smooth function $g:\mathbb{R}\mapsto\mathbb{R}$ *with compact support. The* weighted Sobolev *space* $H_1^2(\gamma)$ *is defined to be the space of weakly differentiable functions* f *on* $\mathbb{R}$ *such that*

$$\|f\|_{H_1^2}^2 = \|f\|_2^2 + \|f'\|_2^2 < \infty.$$

It is well known that γ (*see*, *for instance* [18], *Theorem* 5.1, *page* 92) *satisfies the following logarithmic Sobolev inequality*: *for any function* f *in* $H_1^2(\gamma)$,

$$\mathsf{Ent}_\gamma\big(f^2\big) \le 2\mathbb{E}_\gamma\big(\big(f'(x)\big)^2\big).$$

For any i *in* $\mathbb{N}$, *let* $\mathbb{X}_i$ *be a copy of* $\mathbb{R}$, $\mathcal{A}_i$ *a copy of* $H_1^2(\gamma)$ *and* R_i *the derivation operator on* $\mathcal{A}_i$. *We let* $\gamma^{\mathbb{N}} = \gamma^{\otimes\mathbb{N}}$ *denote the standard Gaussian measure on* $\mathbb{R}^{\mathbb{N}}$.

The set $\mathcal{A}^{S\cup\mathbb{N}}$ *thus defined is the so-called weighted Sobolev space* $H_1^2(\lambda_p^S\otimes\gamma^{\mathbb{N}})$, *which contains the functions* $f\in L^2(\lambda_p^S\otimes\gamma^{\mathbb{N}})$ *verifying the following condition. For all* $i\in\mathbb{N}$, *there exists a function* h_i *in* $L^2(\lambda_p^S\otimes\gamma^{\mathbb{N}})$ *such that*

$$-\int_{\mathbb{R}} g'(y_i)f(x,y)\,\mathrm{d}y_i = \int_{\mathbb{R}} g(y_i)h_i(x,y)\,\mathrm{d}y_i, \quad \lambda_p^S\otimes\gamma^{\mathbb{N}}\ a.s.$$

for every smooth function $g:\mathbb{R}\mapsto\mathbb{R}$ *having compact support. The function* h_i *is called the partial derivative of* f *with respect to* y_i, *and is denoted by* $\frac{\partial f}{\partial y_i}$.

Thus, *we deduce from Proposition* 2.1 *the following result.*

Corollary 2.2. *For any* $p\in\,]0,1[$, *and any* $f\in H_1^2(\lambda_p^S\otimes\gamma^{\mathbb{N}})$,

$$\mathsf{Var}(f)\log\frac{\mathsf{Var}(f)}{\sum_{s\in S}\|\Delta_s f\|_1^2 + \sum_{i\in\mathbb{N}}\|\Delta_i f\|_1^2} \le c_{LS}(p)\sum_{s\in S}\mathbb{E}\big((\Delta_s f)^2\big) + 2\sum_{i\in\mathbb{N}}\mathbb{E}\bigg(\bigg(\frac{\partial f}{\partial x_i}\bigg)^2\bigg).$$

Example 2 (The gamma case (associated to the Laguerre generator)). *We let*

$$\nu_{a,b}(\mathrm{d}y) = \frac{b^a}{\Gamma(a)}y^{a-1}\mathrm{e}^{-by}\mathbb{1}_{y>0}\,\mathrm{d}y$$

denote the gamma probability measure with parameters a *and* b. *This measure is the invariant distribution of the Laguerre semi-group*, *with generator*:

$$\mathbf{L}_{a,b}f(x) = bxf''(bx) - (a-bx)f'(bx).$$

When $a\ge 1/2$ *and* $b=1$, *it can be easily seen that this generator satisfies the* $CD(\rho,\infty)$ *curvature inequality*:

$$\Gamma_2(f) \ge \frac{1}{2}\Gamma(f).$$

This implies that $\nu_{a,b}$ *satisfies the following logarithmic Sobolev inequality* (*see Definition* 3.1, *page* 28 *and Theorem* 3.2, *page* 29 *in* [3], *see also* [1]). *For any weakly differentiable function* $f\in L^2(\nu_{a,b})$, *if* $a\ge 1/2$,

$$\mathsf{Ent}_{\nu_{a,b}}\big(f^2\big) \le \frac{4}{b}\mathbb{E}_{\nu_{a,b}}\big(\big(\sqrt{x}f'(x)\big)^2\big).$$

Therefore, *we deduce the following result from Proposition* 2.1.

Corollary 2.3. *Suppose that $a \geq 1/2$, $b > 0$, and let ${\mathbb{R}_*^+}^{\mathbb{N}}$ be equipped with the product measure $\nu_{a,b}^{\mathbb{N}}$. For any weakly differentiable function f in $L^2(\nu_{a,b}^{\mathbb{N}})$, define*

$$\nabla_i f(x) = \frac{\partial f}{\partial x_i}(x)\sqrt{x_i}.$$

Suppose that,

$$\forall i \in \mathbb{N}, \quad \nabla_i f \in L^2\big(\nu_{a,b}^{\mathbb{N}}\big).$$

Then,

$$\mathsf{Var}(f) \log \frac{\mathsf{Var}(f)}{\sum_{i\in\mathbb{N}} \|\Delta_i f\|_1^2} \leq \frac{4}{b} \sum_{i\in\mathbb{N}} \mathbb{E}\big((\nabla_i f)^2\big).$$

Remark that when $a \in]0, 1/2[$, $\nu_{a,b}$ still satisfies a logarithmic Sobolev inequality with a positive constant $C_{a,b}$ instead of $4/b$, but the precise value of $C_{a,b}$ is not known; *see* [20]. *This gives the analogue of Corollary* 2.3 *for $a \in]0, 1/2[$, with $C_{a,b}$ instead of $4/b$.*

Example 3 (The uniform case). *We let*

$$\lambda(\mathrm{d}y) = \mathbb{1}_{0\leq y\leq 1}\,\mathrm{d}y$$

denote the uniform probability measure on $[0, 1]$. *It is known that λ satisfies the following logarithmic Sobolev inequality* (*it is a direct consequence of the logarithmic Sobolev inequality on the circle* [8]). *For any weakly differentiable function f in $L^2(\lambda)$,*

$$\mathsf{Ent}_\lambda\big(f^2\big) \leq \frac{2}{\pi^2}\mathbb{E}_\lambda\big(\big(f'(x)\big)^2\big).$$

Corollary 2.4. *Let $[0, 1]^{\mathbb{N}}$ be equipped with the product measure $\lambda^{\mathbb{N}}$. Suppose that,*

$$\forall i \in \mathbb{N}, \quad \frac{\partial f}{\partial x_i} \in L^2\big(\lambda^{\mathbb{N}}\big).$$

For any weakly differentiable function f in $L^2(\lambda^{\mathbb{N}})$,

$$\mathsf{Var}(f) \log \frac{\mathsf{Var}(f)}{\sum_{i\in\mathbb{N}} \|\Delta_i f\|_1^2} \leq \frac{2}{\pi^2} \sum_{i\in\mathbb{N}} \mathbb{E}\left(\left(\frac{\partial f}{\partial x_i}\right)^2\right).$$

We shall see in Section 3 *that λ satisfies another logarithmic Sobolev inequality with an energy whose form "looks like" the energy appearing in the gamma case.*

3. Extension from the Gaussian case to other measures

As usual, we can deduce from Corollary 2.2 other inequalities by mean of change of variables. To make this precise, let Ω be a measurable space and $\Psi\colon \mathbb{R}^{\mathbb{N}} \mapsto \Omega$ a measurable isomorphism (meaning that Ψ is one to one with Ψ and Ψ^{-1} measurables). Let $\Psi^*\gamma^{\mathbb{N}}$ denote the image of $\gamma^{\mathbb{N}}$ by Ψ. That is, $\Psi^*\gamma^{\mathbb{N}}(A) = \gamma^{\mathbb{N}}(\Psi^{-1}(A))$. For $g : S \times \Omega \mapsto \mathbb{R}$ such that $g \circ (Id, \Psi) \in H_1^2(\lambda \otimes \gamma^{\mathbb{N}})$, one obviously has

$$\mathsf{Var}_{\lambda\otimes\Psi^*\gamma^{\mathbb{N}}}(g) = \mathsf{Var}_{\lambda\otimes\gamma^{\mathbb{N}}}(g \circ \Psi)$$

and

$$\|\partial_{i,\Psi} g\|_{p,\Psi^*\gamma^{\mathbb{N}}} = \left\| \frac{\partial g \circ \Psi}{\partial y_i} \right\|_{p,\gamma^{\mathbb{N}}},$$

where $\partial_{i,\Psi} g$ is defined as

$$\partial_{i,\Psi} g(x,\omega) = \frac{\partial(x \circ \Psi)}{\partial y_i}\big(q, \Psi^{-1}(\omega)\big).$$

Hence inequality in Corollary 2.2 for $f = g \circ (Id, \Psi)$ transfers to the same inequality for g provided $\frac{\partial f}{\partial y_i}$ is replaced by $\partial_{i,\Psi} g$.

Example 4. *Let $k \geq 2$ be an integer, $\mathbb{S}^{k-1} \subset \mathbb{R}^k$ the unit $k-1$ dimensional sphere, $\Omega = (\mathbb{R}_*^+ \times \mathbb{S}^{k-1})^{\mathbb{N}}$, and let $E = (\mathbb{R}_*^k)^{\mathbb{N}}$. A typical point in Ω will be written as $(\rho,\theta) = (\rho^i, \theta^i)$ and a typical point in E as $y = (y^j)$. Now consider the change of variables $\Psi : E \mapsto \Omega$ given by $\Psi(y) = (\Psi(y)^j)$ with*

$$\Psi^j(y) = \left(\|y^j\|^2, \frac{y^j}{\|y^j\|} \right).$$

The image of $(\gamma^k)^{\mathbb{N}} = \gamma^{\mathbb{N}}$ by Ψ is the product measure $\tilde{\gamma}^{\mathbb{N}}$ where $\tilde{\gamma}$ is the probability measure on $\mathbb{R}_*^+ \times S^{k-1}$ defined by

$$\tilde{\gamma}(\mathrm{d}t\,\mathrm{d}v) = \frac{1}{r_k} \mathrm{e}^{-t/2} t^{k/2-1} \mathbf{1}_{t>0}\,\mathrm{d}t\,\mathrm{d}v.$$

Here $r_k = \int_0^\infty \mathrm{e}^{-t/2} t^{k/2-1}\,\mathrm{d}t$, and $\mathrm{d}v$ stands for the uniform probability measure on $\mathbb{S}^{k-1}$. For $g : \{0,1\}^S \times \Omega \mapsto \mathbb{R}$ with $g \circ (Id, \Psi) \in H_1^2(\lambda \otimes \gamma^{\mathbb{N}})$, let

$$\left(\frac{\partial g}{\partial \rho^j}(x,\rho,\theta), \nabla_{\theta^j} g(x,\rho,\theta) \right) \in \mathbb{R} \times T_{\theta^i}\mathbb{S}^{k-1} \subset \mathbb{R} \times \mathbb{R}^k$$

denote the partial gradient of g with respect to the variable (ρ^i, θ^i), where $T_{\theta^i} S^{k-1} \subset \mathbb{R}^k$ stands for the tangent space of $\mathbb{S}^{k-1}$ at θ_i. It is not hard to verify that for all $i \in \mathbb{N}$ and $j \in \{1,\ldots,k\}$,

$$\partial_{i,j,\Psi} g(x,\rho,\theta) = 2\frac{\partial g}{\partial \rho^i}(x,\rho,\theta)\sqrt{\rho^i}\theta^i_j + \frac{1}{\sqrt{\rho^i}}\big[\nabla_{\theta^i} g(x,\rho,\theta)\big]_j. \tag{7}$$

As a consequence, we may recover in this way Corollary 2.3 *when the parameter a equals $k/2-1$, with k an integer. Indeed, this follows from* (7) *applied to the map $(x,\rho,\theta) \to g(\frac{\rho}{2\alpha})$. Concentrating on the angular part instead of the radial one, we obtain the following modified Poincaré inequality on the sphere.*

Corollary 3.1 (Uniform distribution on $\mathbb{S}^n$). *Let dv_n denote the normalized Riemannian probability measure on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. For $g \in H_2^1(dv_n)$ and $i = 1,\ldots,n+1$ let $\nabla_i g(\theta)$ denote the ith component of $\nabla g(\theta)$ in $\mathbb{R}^{n+1}$ (we see $T_\theta \mathbb{S}^n$ as the vector space of $\mathbb{R}^{n+1}$ consisting of vector that are orthogonal to θ). Then, for $n \geq 2$,*

$$\mathsf{Var}(g) \log \frac{\mathsf{Var}(g)}{\sum_{i=1}^{\mathbb{N}} \|\Delta_i g\|_1^2} \leq \frac{1}{n-1} \sum_{i \in \mathbb{N}} \mathbb{E}\big((\nabla_i g)^2\big). \tag{8}$$

Proof. follows from (7) applied to the map $(x,\rho,\theta) \to g(\theta)$. Details are left to the reader. □

Example 5. *If one wants to get a result similar to Corollary* 2.2 *with* γ *replaced by another probability measure* ν, *one may of course perform the usual change of variables through inverse of repartition function. In the sequel, we denote by*

$$g(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \tag{9}$$

the density of the normalized Gaussian distribution, and by

$$G(x) = \int_{-\infty}^{x} g(u)\,\mathrm{d}u \tag{10}$$

its repartition function. For any function ϕ *from* $\mathbb{R}$ *to* $\mathbb{R}$, *we shall note* $\tilde{\phi}$ *the function from* $\mathbb{R}^{\mathbb{N}}$ *to* $\mathbb{R}^{\mathbb{N}}$ *such that* $(\tilde{\phi}(x))_j = \phi(x_j)$.

Corollary 3.2 (Unidimensional change of variables). *Let* ν *be a probability on* $\mathbb{R}^+$ *absolutely continuous with respect to the Lebesgue measure, with density h and repartition function*

$$H(t) = \int_0^t h(u)\,\mathrm{d}u.$$

Let $\{0,1\}^S \times \mathbb{R}^n$ *be equipped with the probability measure* $\lambda_p^S \otimes \nu^{\otimes\mathbb{N}}$. *Then, for every function* f *on* $\{0,1\}^S \times \mathbb{R}^n$ *such that* $f \circ (Id, \widetilde{H^{-1} \circ G}) \in H_1^2(\lambda_p^S \otimes \gamma^{\mathbb{N}})$,

$$\mathsf{Var}(f) \log \frac{\mathsf{Var}(f)}{\sum_{s\in S} \|\Delta_s f\|_1^2 + \sum_{i\in\mathbb{N}} \|\Delta_i f\|_1^2} \le c_{LS}(p) \sum_{s\in S} \mathbb{E}\big((\Delta_s f)^2\big) + 2\sum_{i\in\mathbb{N}} \mathbb{E}\big((\nabla_i f)^2\big),$$

where for every integer i,

$$\nabla_i f(x,y) = \psi(y_i) \frac{\partial f}{\partial y_i}(x,y),$$

and ψ *is defined on* $I = \{t \ge 0 \text{ s.t. } h(t) > 0\}$:

$$\forall t \in I, \quad \psi(t) = \frac{g \circ G^{-1}(H(t))}{h(t)}.$$

Proof. It is a straightforward consequence of Corollary 2.2, applied to $f \circ (Id, \widetilde{H^{-1} \circ G})$. □

4. A general exponential concentration inequality

In this section, we show how one can deduce from Proposition 2.1 an exponential concentration inequality for a function F of independent variables. We shall prove in Section 5, in the context of first passage percolation, that this new general concentration inequality may in certain cases improve on the ones due to Talagrand [24–26], or Boucheron et al. [7]. The reason why we can get stronger results is that Proposition 2.1 is generally stronger than a simple Poincaré inequality, and it is well known (see [17], Corollary 3.2, page 49 and Theorem 3.3, page 50) that a Poincaré inequality for a measure μ implies an exponential concentration inequality for any Lipschitz function of a random variable with distribution μ. This can be achieved through applying the Poincaré inequality to $\exp(\theta f)$, and then performing some recurrence. This last step is essentially contained in the following simple version, adapted to our case, of Corollary 3.2, page 49 in [17].

Lemma 4.1. *Let f be a measurable real function on a probability space $(\mathbf{X}, \mathcal{X}, \mu)$, and K a positive constant. Suppose that for any real number $\theta < \frac{1}{2\sqrt{K}}$, the function $x \mapsto \mathrm{e}^{\theta f(x)}$ is in $L^1(\mu)$, and*:

$$\mathsf{Var}\big(\mathrm{e}^{\theta f/2}\big) \le K\theta^2 \mathbb{E}\big(\mathrm{e}^{\theta f}\big).$$

Then,

$$\forall t \ge 0, \quad \mu\left(f - \int f \,\mathrm{d}\mu > t\sqrt{K}\right) \le 4\mathrm{e}^{-t}$$

and

$$\forall t \ge 0, \quad \mu\left(f - \int f \,\mathrm{d}\mu < -t\sqrt{K}\right) \le 4\mathrm{e}^{-t}.$$

Now, we can state our general concentration inequality. For any function F on a product space $(\mathbb{X}_i, \mathcal{X}_i, \mu_i)_{i\in I}$, we define the following quantities, which play an important role in Theorem 4.2 (the notation is that of Section 2).

$$W_{i,+}(x) = \int \big(F\big(x^{-i}, y_i\big) - F(x)\big)_+ \,\mathrm{d}\mu_i(y_i),$$

where $h_+ = \sup\{h, 0\}$.

$$W_+(x) = \sum_{i\in I} W_{i,+}.$$

Remark that similar quantities are involved in the work of Boucheron et al. [6,7].

Theorem 4.2. *Let $(\mathbb{X}_i, \mathcal{X}_i, \mu_i)_{i\in I}$, $(\mathcal{A}_i)_{i\in I}$ and $(R_i)_{i\in I}$ be as in Proposition* 2.1, *and satisfying all the hypotheses therein. Let F be a function in $L^2(\prod_{i\in I} \mathbb{X}_i)$. Define*

$$r = \sup_{i\in I} \sqrt{\mathbb{E}\big(W_{i,+}^2\big)},$$
$$s = \sqrt{\mathbb{E}\big(W_+^2\big)}.$$

Define, for every real number $K > ers$:

$$l(K) = \frac{K}{\log(K/(rs\log(K/(rs))))}.$$

Suppose that there exists a real number $K > ers$ such that, for every θ such that $|\theta| \le \frac{1}{2\sqrt{l(K)}}$, $\mathrm{e}^{(\theta/2)F}$ belongs to $\mathcal{A}^I$, and:

$$\sum_{i\in I} \mathbb{E}\big(R_i\big(\mathrm{e}^{(\theta/2)F}\big)\big) \le K\theta^2 \mathbb{E}\big(\mathrm{e}^{\theta F}\big). \tag{11}$$

Then, denoting $\mu = \bigotimes_{i\in I} \mu_i$, for every $t > 0$:

$$\mu\big(F - \mathbb{E}(F) \ge t\sqrt{l(K)}\big) \le 4\mathrm{e}^{-t},$$
$$\mu\big(F - \mathbb{E}(F) \le -t\sqrt{l(K)}\big) \le 4\mathrm{e}^{-t}.$$

Proof. For any function f in $L^1(\prod_{i\in I}\mathbb{X}_i)$, any $i\in I$, $x\in\prod_{i\in I}\mathbb{X}_i$ and $y_i\in\mathbb{X}_i$,

$$\begin{aligned}\|\Delta_i f\|_1 &= \int\left|\int\big(f(x)-f\big(x^{-i},y_i\big)\big)\,\mathrm{d}\mu_i(y_i)\right|\mathrm{d}\mu(x)\\ &\le \iint\big|f(x)-f\big(x^{-i},y_i\big)\big|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_i(y_i)\\ &= 2\iint\big(f(x)-f\big(x^{-i},y_i\big)\big)_+\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_i(y_i)\\ &= 2\iint\big(f(x)-f\big(x^{-i},y_i\big)\big)_-\,\mathrm{d}\mu(x)\,\mathrm{d}\mu_i(y_i),\end{aligned}$$

where $h_-=\sup\{-h,0\}$. On the other hand,

$$\begin{aligned}\big(\mathrm{e}^{(\theta/2)F(x^{-i},y_i)}-\mathrm{e}^{(\theta/2)F(x)}\big)_+ &= \mathrm{e}^{(\theta/2)F(x)}\big(\mathrm{e}^{(\theta/2)(F(x^{-i},y_i)-F(x))}-1\big)_+\\ &\le \mathrm{e}^{(\theta/2)F(x)}\left(\frac{\theta}{2}\big(F\big(x^{-i},y_i\big)(x)-F(x)\big)\right)_+\\ &= \begin{cases}\frac{|\theta|}{2}\mathrm{e}^{(\theta/2)F(x)}\big(F\big(x^{-i},y_i\big)-F(x)\big)_+ & \text{if } \theta>0,\\ \frac{|\theta|}{2}\mathrm{e}^{(\theta/2)F(x)}\big(F\big(x^{-i},y_i\big)-F(x)\big)_- & \text{if } \theta<0.\end{cases}\end{aligned}$$

Therefore,

$$\big(\mathrm{e}^{(\theta/2)F(x^{-i},y_i)}-\mathrm{e}^{(\theta/2)F(x)}\big)_+ \le \begin{cases}\frac{|\theta|}{2}\mathrm{e}^{(\theta/2)F(x)}\big(F\big(x^{-i},y_i\big)-F(x)\big)_+ & \text{if } \theta>0,\\ \frac{|\theta|}{2}\mathrm{e}^{(\theta/2)F(x^{-i},y_i)}\big(F(x)-F\big(x^{-i},y_i\big)\big)_+ & \text{if } \theta<0.\end{cases}$$

Since $F(x)$ and $F(x^{-i},y_i)$ have the same distribution under $\mu\otimes\mu_i$, we get, for any real number θ,

$$\big\|\Delta_i\mathrm{e}^{\theta F/2}\big\|_1 \le |\theta|\iint\big(F\big(x^{-i},y_i\big)-F(x)\big)_+\,\mathrm{d}\mu_i(y_i)\mathrm{e}^{(\theta/2)F(x)}\,\mathrm{d}\mu(x).$$

And, using Cauchy–Schwarz inequality,

$$\sum_{i\in I}\big\|\Delta_i\mathrm{e}^{\theta F/2}\big\|_1 \le |\theta|\sqrt{\mathbb{E}\big(W_+^2\big)\mathbb{E}\big(\mathrm{e}^{\theta F}\big)}.$$

But we also have, again using Cauchy–Schwarz inequality,

$$\big\|\Delta_i\mathrm{e}^{\theta F/2}\big\|_1 \le |\theta|\sqrt{\mathbb{E}\big(W_{i,+}^2\big)\mathbb{E}\big(\mathrm{e}^{\theta F}\big)}.$$

Therefore,

$$\sum_{i\in I}\big\|\Delta_i\mathrm{e}^{\theta F/2}\big\|_1^2 \le \theta^2 rs\,\mathbb{E}\big(\mathrm{e}^{\theta F}\big). \tag{12}$$

Inequality (12), the Poincaré inequality for $\mathrm{e}^{\theta F}$ (Proposition 2.1) and hypothesis (11) imply that:

$$\forall|\theta|\le\frac{1}{2\sqrt{l(K)}},\quad \mathsf{Var}\big(\mathrm{e}^{(\theta/2)F}\big)\log\frac{\mathsf{Var}(\mathrm{e}^{(\theta/2)F})}{\theta^2 rs\mathbb{E}(\mathrm{e}^{\theta F})} \le K\theta^2\mathbb{E}\big(\mathrm{e}^{\theta F}\big). \tag{13}$$

Therefore, we are left in front of the following alternative:

- either $\mathsf{Var}(\mathrm{e}^{\theta F/2}) \le \theta^2\frac{K}{\log(K/(rs))}\mathbb{E}(\mathrm{e}^{\theta F})$,

- or $\mathsf{Var}(e^{\theta F/2}) > \theta^2 \frac{K}{\log(K/(rs))} \mathbb{E}(e^{\theta F})$. But in this case, plugging this minoration into the logarithm of inequality (13) leads to:

$$\mathsf{Var}\big(e^{\theta F/2}\big) \le \theta^2 \frac{K}{\log(K/(rs \log(K/(rs))))} \mathbb{E}\big(e^{\theta \tilde{f}}\big).$$

In any case, for any $|\theta| \le \frac{1}{2\sqrt{l(K)}}$,

$$\mathsf{Var}\big(e^{\theta F/2}\big) \le \theta^2 l(K) \mathbb{E}\big(e^{\theta \tilde{f}}\big).$$

The result follows from Lemma 4.1. □

Remark 2. *It is well known, through Herbst's argument (see, e.g.,* [18], *Theorem* 5.3, *page* 95), *that condition* (11) *implies a subexponential concentration inequality of the form*:

$$\mu\big(\big|F - \mathbb{E}(F)\big| \ge 2t\sqrt{K}\big) \le 2e^{-t^2}.$$

In the applications to follow, $K/(rs)$ is big, and therefore $l(K)$ is small compared to K. Therefore, at the price of trading the sub-Gaussian behavior against a subexponential one, Theorem 4.2 *shows that when $K/(rs)$ is big; the fluctuations of F are lower than $\sqrt{l(K)}$, which is small compared to $\sqrt{K}$.*

Let us give a closer look at the case where for every i, μ_i is the invariant measure of a diffusion process with carré du champ Γ_i. Naturally associated with this diffusion process, a Sobolev logarithmic inequality for μ_i has the form (if it exists):

$$\mathsf{Ent}_{\mu_i}\big(f^2\big) \le c_i \mathbb{E}_{\mu_i}\big(\Gamma_i(f, f)\big),$$

where c_i is a positive constant. Furthermore, we have the following property:

$$\Gamma_i\big(\Phi(f), g\big) = \Phi'(f)\Gamma_i(f, g),$$

which leads to:

$$R_i\big(e^{(\theta/2)F}\big)^2 = c_i \Gamma_i\big(e^{(\theta/2)F}, e^{(\theta/2)F}\big) = c_i \frac{\theta^2}{4} e^{\theta F} \Gamma_i(F, F).$$

Therefore, condition (11) becomes:

$$\mathbb{E}\left(e^{(\theta/2)F} \sum_{i \in I} c_i \Gamma_i(F, F)\right) \le 4K \mathbb{E}\big(e^{\theta F}\big).$$

The main work to satisfy condition (11) is to bound from below, and somewhat independently from $e^{(\theta/2)F}$, the quantity $\sum_{i \in I} \Gamma_i(F, F)$. In some particular cases, and notably percolation, this quantity is upperbounded by F itself. And it is possible to show, following Boucheron et al. [7], that, at least for small θ, $\mathbb{E}(F e^{\theta F})$ is upper bounded by a constant times $\mathbb{E}(F)\mathbb{E}(e^{\theta F})$. More generally, one can state the following result.

Corollary 4.3. *Let $(\mathbb{X}_i, \mathcal{X}_i, \mu_i)_{i \in I}$, $(\mathcal{A}_i)_{i \in I}$ and $(R_i)_{i \in I}$ be as in Proposition* 2.1, *and satisfying all the hypotheses therein. Let F be a function in $L^2(\prod_{i \in I} \mathbb{X}_i)$. Define*

$$r = \sup_{i \in I} \sqrt{\mathbb{E}\big(W_{i,+}^2\big)},$$

$$s = \sqrt{\mathbb{E}\big(W_+^2\big)}.$$

Define, for every real number $K > ers$:

$$l(K) = \frac{K}{\log(K/(rs\log(K/(rs))))}.$$

Suppose that there exists two constants C *and* D *such that, denoting*:

$$A_{CD} = 4C\mathbb{E}(F) + D\left(1 + \frac{2}{C}\right),$$

we have

(i) $C \le \sqrt{l(A_{CD})}$,
(ii) $A_{CD} 4C\mathbb{E}(F) + D(1 + \frac{2}{C}) \ge ers$,
(iii) *for every* θ *such that* $|\theta| \le \frac{1}{2\sqrt{l(A_{CD})}}$, $\mathrm{e}^{\theta F}$ *belongs to* $\mathcal{A}^I$, *and*:

$$\sum_{i\in I} \mathbb{E}\big(R_i\big(\mathrm{e}^{(\theta/2)F}\big)\big) \le C\theta^2\mathbb{E}\big(F\mathrm{e}^{\theta F}\big) + D\theta^2\mathbb{E}\big(\mathrm{e}^{\theta F}\big). \tag{14}$$

Then, denoting $\mu = \bigotimes_{i\in I}\mu_i$, *for every* $t > 0$:

$$\mu\big(F - \mathbb{E}(F) \ge t\sqrt{l(A_{CD})}\big) \le 4\mathrm{e}^{-t},$$
$$\mu\big(F - \mathbb{E}(F) \le -t\sqrt{l(A_{CD})}\big) \le 4\mathrm{e}^{-t}.$$

Proof. The only thing to prove is that condition (11) holds with $K = 4C\mathbb{E}(F) + D(1 + \frac{2}{C})$. This will follow from condition (14) and a variation on the theme of Herbst's argument due to Boucheron et al. [7]. Indeed, recall that using the tensorisation of entropy, the logarithmic Sobolev inequalities for each μ_i imply that:

$$\forall g \in \mathcal{A}^I, \quad \mathsf{Ent}_\mu\big(g^2\big) \le \sum_{i\in I}\mathbb{E}_\mu\big(R_i(g)^2\big).$$

Let us apply this inequality to $g = \mathrm{e}^{(\theta/2)F}$, and use condition (14). For every θ such that $|\theta| \le \frac{1}{2\sqrt{l(4C\mathbb{E}(F))}}$,

$$\mathsf{Ent}_\mu\big(\mathrm{e}^{\theta F}\big) \le C\theta^2\mathbb{E}\big(F\mathrm{e}^{\theta F}\big).$$

This may be written as:

$$\theta\mathbb{E}\big(F\mathrm{e}^{\theta F}\big) - \mathbb{E}\big(\mathrm{e}^{\theta F}\big)\log\mathbb{E}\big(\mathrm{e}^{\theta F}\big) \le C\theta^2\mathbb{E}\big(F\mathrm{e}^{\theta F}\big) + D\theta^2\mathbb{E}\big(\mathrm{e}^{\theta F}\big). \tag{15}$$

First, suppose that θ is positive. The proof of Theorem 5 in [7] shows that, for every $\theta < \frac{1}{C}$,

$$\log\mathbb{E}\big(\mathrm{e}^{\theta F}\big) \le \frac{\theta}{1-\theta C}\mathbb{E}(F) + D\frac{\theta^2}{1-\theta C},$$

and Eq. (15) implies that, for every $\theta < \frac{1}{C}$,

$$\mathbb{E}\big(F\mathrm{e}^{\theta F}\big) \le \frac{\mathbb{E}(F) + D\theta}{(1-\theta C)^2}\mathbb{E}\big(\mathrm{e}^{\theta F}\big).$$

If θ is negative, $\mathrm{e}^{\theta F}$ is decreasing in F, and it follows from Chebyshev's association inequality that (see, e.g., [11], page 43):

$$\mathbb{E}\big(F\mathrm{e}^{\theta F}\big) \le \mathbb{E}(F)\mathbb{E}\big(\mathrm{e}^{\theta F}\big).$$

Now, we gather the case where θ is positive and the case where it is negative. Condition (i) implies that $\frac{1}{2C} \geq \frac{1}{2\sqrt{l(4C\mathbb{E}(F))}}$, and therefore, for every θ such that $|\theta| \leq \frac{1}{2\sqrt{l(4C\mathbb{E}(F))}}$,

$$\sum_{i \in I} \mathbb{E}\big(R_i\big(\mathrm{e}^{(\theta/2)F}\big)\big) \leq C\theta^2 \mathbb{E}\big(F\mathrm{e}^{\theta F}\big) + D\theta^2 \mathbb{E}\big(\mathrm{e}^{\theta F}\big) \leq \left(4C\mathbb{E}(F) + D\left(1 + \frac{2}{C}\right)\right)\theta^2 \mathbb{E}\big(\mathrm{e}^{\theta F}\big),$$

and the result follows from Theorem 4.2. □

The main lesson that we can remember from Corollary 4.2 is the following (very) informal statement.

If F is a Lipschitz function of a large number of variables, each of which contributes at most to an amount δ, then F has fluctuations of order $\mathrm{O}(\sqrt{\mathbb{E}(F)/\log\frac{1}{\delta}})$, and there is an exponential control for these fluctuations.

5. Application to first passage percolation

5.1. Continuous edge-times distributions

It turns out that Corollary 4.3 is particularly well suited to adapt the argument of Benjamini, Kalai and Schramm [5] to show that the passage time from the origin to a vertex v satisfies an exponential concentration inequality at the rate $\mathrm{O}(\sqrt{|v|/\log|v|})$ when the edges have a $\Gamma(a, b)$ distribution with $a \geq 1/2$. This includes the important case of exponential distribution, for which first passage percolation becomes equivalent to a version of Eden growth model (see for instance [15], page 130). We do not want to restrict ourselves to those distributions. Nevertheless, due to the particular strategy that we adopt, we can only prove our result for some continuous edge times distributions which behave roughly like a gamma distribution. Please note that the definition given below differs (one assumption is removed) from the definition of a nearly gamma distribution that was stated in the preliminary paper [4].

Definition 5.1. *Let ν be a probability on $\mathbb{R}^+$ absolutely continuous with respect to the Lebesgue measure, with density h and repartition function*

$$H(t) = \int_0^t h(u)\,\mathrm{d}u.$$

Define:

$$I = \big\{t \geq 0 \text{ such that } h(t) > 0\big\},$$

and $\psi : I \mapsto \mathbb{R}$ the map:

$$\psi(y) = \frac{g \circ G^{-1}(H(y))}{h(y)}.$$

Let A be a positive real number. The probability measure ν will be said to be nearly gamma *provided it satisfies the following set of conditions*:

(i) *I is an interval*;
(ii) *h restricted to I is continuous*;
(iii) *There exists a positive real number A such that*

$$\forall y \in I, \quad \psi(y) \leq A\sqrt{y}.$$

If we want to emphasize the dependance on A in the above definition, we shall say that ν is *nearly gamma with bound A*. In Definition 5.1, Condition (iii) is of course the most tedious to check. A simple sufficient condition for a probability measure to be nearly gamma will be given in Lemma 5.3, the proof of which relies on the following asymptotics for the Gaussian repartition function G.

Lemma 5.2. *As x tends to $-\infty$,*

$$G(x) = g(x)\left(\frac{1}{|x|} + \mathrm{o}\left(\frac{1}{x}\right)\right),$$

and as x tends to $+\infty$,

$$G(x) = 1 - g(x)\left(\frac{1}{x} + \mathrm{o}\left(\frac{1}{x}\right)\right).$$

Consequently,

$$g \circ G^{-1}(y) \overset{y\to 0}{\sim} y\sqrt{-2\log y},$$

and

$$g \circ G^{-1}(y) \overset{y\to 1}{\sim} (1-y)\sqrt{-2\log(1-y)}.$$

Proof. A simple change of variable $u = x - t$ in G gives:

$$G(x) = g(x)\int_0^{+\infty} \mathrm{e}^{-t^2/2+xt}\,\mathrm{d}t.$$

Integrating by parts, we get:

$$G(x) = g(x)\left(-\frac{1}{x} + \frac{1}{x}\int_0^{+\infty} t\mathrm{e}^{-t^2/2+xt}\,\mathrm{d}t\right) = g(x)\left(\frac{1}{|x|} + \mathrm{o}\left(\frac{1}{x}\right)\right),$$

as x goes to $-\infty$. Since $G(-x) = 1 - G(x)$, we get that, as x goes to $+\infty$:

$$G(x) = 1 - g(x)\left(\frac{1}{x} + \mathrm{o}\left(\frac{1}{x}\right)\right).$$

Let us turn to the asymptotic of $g \circ G^{-1}(y)$ as y tends to zero. Let $x = G^{-1}(y)$, so that "y tends to zero" is equivalent to "x tends to $-\infty$." One has therefore,

$$G(x) = \frac{g(x)}{|x|}\big(1 + \mathrm{o}(1)\big),$$

$$\log G(x) = \log g(x) - \log|x| + \mathrm{o}(1) = -\frac{x^2}{2} - \log|x| + \mathrm{O}(1),$$

$$\log G(x) = -\frac{x^2}{2}\big(1 + \mathrm{o}(1)\big),$$

$$|x| = \sqrt{-2\log G(x)}.$$

Since $g(x) = |x|G(x)(1 + \mathrm{o}(1))$,

$$g(x) = G(x)\sqrt{-2\log G(x)}\big(1 + \mathrm{o}(1)\big),$$

and therefore,

$$g \circ G^{-1}(y) = y\sqrt{-2\log y}\big(1 + \mathrm{o}(1)\big),$$

as y tends to zero. The asymptotic of $g \circ G^{-1}(y)$ as y tends to 1 is derived in the same way. □

Given two functions r and l, we write $l(x) = \Theta(r(x))$ as x goes to $x*$ provided there exist positive constants $C_1 \leq C_2$ such that

$$C_1 \leq \liminf_{x \to x*} \frac{r(x)}{l(x)} \leq \limsup_{x \to x*} \frac{r(x)}{l(x)} \leq C_2.$$

Lemma 5.3. *Assume that conditions* (i) *and* (ii) *of Definition* 5.1 *hold. Let* $0 \leq \underline{\nu} < \overline{\nu} \leq \infty$ *denote the endpoints of* I. *Assume furthermore condition* (iii) *is replaced by conditions* (iv) *and* (v) *below.*

(iv) *There exists* $\alpha > -1$ *such that as* x *goes to* $\underline{\nu}$,

$$h(x) = \Theta\big((x - \underline{\nu})^{\alpha}\big),$$

(v) $\overline{\nu} < \infty$ *and there exists* $\beta > -1$ *such that as* x *goes to* $\overline{\nu}$,

$$h(x) = \Theta\big((\overline{\nu} - x)^{\beta}\big),$$

or $\overline{\nu} = \infty$ *and*

$$\exists A > \underline{\nu}, \forall t \geq A, \quad C_1 h(t) \leq \int_t^{\infty} h(u)\,\mathrm{d}u \leq C_2 h(t),$$

where C_1 *and* C_2 *are positive constants.*

Then, ν *is nearly gamma.*

Proof. Since h is a continuous function on $]\underline{\nu}, \overline{\nu}[$, it attains its minimum on every compact set included in $]\underline{\nu}, \overline{\nu}[$. The minimum of h on $[a, b]$ is, therefore, strictly positive as soon as $\underline{\nu} < a \leq b < \overline{\nu}$. In order to show that condition (iii) holds, we thus have to concentrate on the behaviour of the function ψ near $\underline{\nu}$ and $\overline{\nu}$. Condition (iv) implies that, as x goes to $\underline{\nu}$,

$$H(x) = \Theta\big((x - \underline{\nu})^{\alpha+1}\big). \tag{16}$$

This, via Lemma 5.2, leads to

$$\psi(x) = \Theta\big((x - \underline{\nu})\sqrt{-\log(x - \underline{\nu})}\big), \tag{17}$$

as x goes to $\underline{\nu}$. Similarly, if $\overline{\nu} < \infty$, condition (v) implies that, as x goes to $\overline{\nu}$,

$$H(x) = \Theta\big((\overline{\nu} - x)^{\beta+1}\big), \tag{18}$$

which leads via Lemma 5.2 to

$$\psi(x) = \Theta\big((\overline{\nu} - x)\sqrt{-\log(\overline{\nu} - x)}\big), \tag{19}$$

as x goes to $\overline{\nu}$. Therefore, if $\overline{\nu} < \infty$, condition (iii) holds.

Now, suppose that $\overline{\nu} = \infty$. Condition (v) implies:

$$\forall t \geq A, \quad \frac{1}{C_2} \leq \frac{h(t)}{\int_t^{\infty} h(u)}\,\mathrm{d}u \leq \frac{1}{C_1}.$$

Integrating this inequality between A and y leads to the existence of three positive constants B, C_1' and C_2' such that:

$$\forall y \geq B, \quad C_1' y \leq \log \frac{1}{1 - H(y)} \leq C'2y.$$

Thus,

$$\forall y \geq B, \quad C_1\sqrt{C_1' y} \leq \psi(y) \leq C_2\sqrt{C_2' y}. \tag{20}$$

This, combined with Eq. (17) proves that condition (iii) holds and concludes the proof of Lemma 5.3. □

Remark 3. *With the help of Lemma* 5.3, *it is easy to check that most usual distributions are nearly gamma. This includes all gamma and beta distributions, as well as any probability measure whose density is bounded away from* 0 *on its support, and notably the uniform distribution on* $[a, b]$, *with* $0 \leq a < b$. *Nevertheless, remark that some distributions which have a sub-exponential upper tail may not satisfy the assumptions of Lemma* 5.3, *and be nearly gamma, though. For example, this is the case of the distribution of* $|N|$, *where* N *is a standard Gaussian random variable.*

Now, we can state the main result of this article.

Theorem 5.4. *Let* ν *be a nearly gamma probability measure with an exponential moment, i.e. we suppose that there exists* $\delta > 0$ *such that*:

$$\int e^{\delta x}\,d\nu(x) < \infty.$$

Let μ *denote the measure* $\nu^{\otimes E}$. *Then, there exist two positive constants* C_1 *and* C_2 *such that, for any* $|v| \geq 2$, *and any positive real number* $t \leq |v|$,

$$\mu\left(\left|d_x(0,v) - \int d_x(0,v)\,d\mu(x)\right| > t\sqrt{\frac{|v|}{\log|v|}}\right) \leq C_1 e^{-C_2 t}.$$

Proof. What we present here borrows many ideas from Kesten [16] and of course Benjamini et al. [5]. We would like to apply Corollary 4.3 to the function $e^{\theta f_v}$, for $\theta \leq \sqrt{\frac{\log|v|}{|v|}}$. In fact, we will be able to use Corollary 4.3, but not exactly for f_v, and not exactly for any nearly gamma distribution. The first step is indeed to work with a version of ν with bounded support. Precisely, we shall use the following lemma which is an easy adaptation of Kesten's Lemma 1, page 309 in [16].

Lemma 5.5. *Let* ν *be a nearly gamma distribution with bound* A. *Suppose that* ν *admits an exponential moment, that is, there exists* $\delta > 0$ *such that*:

$$\int e^{\delta x}\,d\nu(x) < \infty.$$

Then there exists a sequence of probability measures $(\nu_k)_{k\geq 2}$, *positive constants* C_3, C_4, C_5 *and a positive integer* k_ν *with the following properties*:

(i) *For every* k, *the support of* ν_k *is included in* $[0, C_5 \log k]$.
(ii) *If* $k \geq k_\nu$, ν_k *is a nearly gamma distribution with bound* A.
(iii) *If* $k = |v|$ *and* $k \geq 2$, *for every* t *greater than* $2C_3\sqrt{\frac{\log|v|}{|v|}}$,

$$\nu\left(\left|d_x(0,v) - \mathbb{E}_\nu\big(d_x(0,v)\big)\right| > t\sqrt{\frac{|v|}{\log|v|}}\right)$$
$$\leq 3e^{-C_3|v|} + C_4 e^{-(\gamma/8)t\sqrt{|v|/\log|v|}} + \tilde{\nu}\left(\left|d_x(0,v) - \mathbb{E}_{\tilde{\nu}}\big(d_x(0,v)\big)\right| > \frac{t}{4}\sqrt{\frac{|v|}{\log|v|}}\right).$$

(iv) *If* $k \geq 2$, ν_k *is stochastically smaller than* ν_{k+1} *and* ν.

Proof. Kesten's argument in [16] is simply to consider the truncated edge times at $C_5 \log|v|$. We cannot use this directly because we have to deal with continuous distribution. Instead, we can repatriate the mass beyond $2C_5 \log|v|$, and spread it continuously over $[C_5 \log|v|, 2C_5 \log|v|]$. This mass is small, of course. Precisely, thanks to the exponential moment assumption, for every positive number c,

$$\nu\bigl(\bigl[c\log|v|, +\infty\bigl[\bigr) \le \int e^{\delta x}\, d\nu(x) \frac{1}{|v|^{\delta c}}.$$

Let u be a continuous density on the real line with support included in $[0, 1]$ and C_5 a positive constant to be fixed later. We define ν_k to be the continuous distribution on the real line with density:

$$\forall x \in \mathbb{R}, \quad h_k(x) = \left(h(x) + u\left(\frac{x - C_5\log k}{C_5 \log k}\right)\frac{\nu([2C_5\log k, +\infty[)}{C_5\log k}\right)\mathbb{1}_{x \le 2C_5 \log k}.$$

Statements (i) and (iv) are obvious. To see that (ii) holds, let H_k be the repartition function of ν_k. Obviously,

$$\begin{aligned}
&\forall x \le C_5\log|v|, && h_k(x) = h(x),\\
&\forall x \le 2C_5\log|v|, && h_k(x) \ge h(x),\\
&\forall x \ge C_5 c\log|v|, && h_k(x) = 0,
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\forall x \le C_5\log|v|, && H_k(x) = H(x),\\
&\forall x \in \mathbb{R}, && H_k(x) \ge H(x),\\
&\forall x \ge 2C_5\log|v|, && H_k(x) = 1.
\end{aligned}$$

Observe now that $g \circ G^{-1}$ is decreasing on $[1/2, 1]$, and that

$$\forall x \ge C_5 \log k, \quad H_k(x) \ge H(x) \ge 1 - \int e^{\delta x}\, d\nu(x)\frac{1}{k^{\delta C_5}}.$$

Therefore, let $k_\nu = \lceil (2\int e^{\delta x}\, d\nu(x))^{1/(\delta C_5)}\rceil$,

$$\forall k \ge k_\nu, \forall x \le 2C_5 \log k, \quad \frac{g \circ G^{-1}(H_k(x))}{h_k(x)} \le \psi(x).$$

This implies that the distributions $(\nu_k)_{k \ge k_\nu}$ are all nearly gamma with the same bound A.

It remains to prove (iv). We define the following coupling π_k of (ν, ν_k):

$$\int g(x, y)\, d\pi_k(x, y) = \int g\bigl(x, H_k^{-1}\bigl(H(x)\bigr)\bigr)\, d\nu(x).$$

Denote by $\gamma = \gamma(\tilde{x})$ the ($\nu_k^{\otimes E}$ a.s. unique) $\tilde{x}$-geodesic from 0 to v. The following inequalities hold for π_k-almost every $(x, \tilde{x})$.

$$0 \le d_x(0, v) - d_{\tilde{x}}(0, v) \le \sum_{e\in\gamma} x_e - \sum_{e\in\gamma}\tilde{x}_e \le \sum_{e\in\gamma} x_e \mathbb{1}_{x_e > C_5 \log k}.$$

Now, if $k = |v|$, we choose to take $C_5 = \frac{4d}{\delta}$, and the end of the proof follows exactly Kesten's Lemma 1, page 309 in [16]. □

Now, we suppose that $k = |v| \ge k_\nu$ and we shall work with ν_k, whose support is included in $[0, 2C_5\log|v|]$. Let us define $\mu_k = \nu_k^{\otimes E}$. In the whole proof, Y shall denote a random variable with distribution ν. Remark that, thanks to part (iv) of Lemma 5.5

$$\int e^{\delta x}\,d\nu_k(x) \le \int e^{\delta x}\,d\nu(x) = \mathbb{E}\big(e^{\delta Y}\big),$$

and, for any positive real number α,

$$\int x^{\alpha}\,d\nu_k(x) \le \int x^{\alpha}\,d\nu(x) = \mathbb{E}\big(Y^{\alpha}\big).$$

A crucial idea in the work of Benjamini Kalai and Schramm is to work with a randomised version of f_v in order to take a full benefit of Corollary 3.2. This randomisation trick relies on the following lemma:

Lemma 5.6. *There exists a constant $c > 0$, such that, for every $m \in \mathbb{N}^*$, there exists a function g_m from $\{0,1\}^{m^2}$ to $\{0,\dots,m\}$ such that*:

$$\max_{y\in\{0,\dots,m\}} \lambda\big(x \text{ s.t. } g_m(x) = y\big) \le \frac{c}{m},$$

and

$$\forall q \in \big\{1,\dots,m^2\big\},\quad \nabla_q g_m \in \{0,1\},$$

where

$$\nabla_q g(x) = g(x_1,\dots,x_{q-1},1,x_{q+1},\dots,x_{m^2}) - g(x_1,\dots,x_{q-1},0,x_{q+1},\dots,x_{m^2}).$$

Since Benjamini et al. do not give a full proof for this lemma, we offer the following one.

Proof. From Stirling's formula,

$$\binom{m^2}{\lfloor m^2/2\rfloor}\cdot\frac{m}{2^{m^2}} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}},$$

and this implies that the following supremum is finite:

$$c_1 = \sup\left\{2\binom{m^2}{\lfloor m^2/2\rfloor}\cdot\frac{m}{2^{m^2}} \text{ s.t. } m \in \mathbb{N}^*\right\}.$$

Notice also that $c_1 \ge 1$. Now, let $\preceq$ denote the alphabetical order $\{0,1\}^{m^2}$, and let us list the elements in $\{0,1\}^{m^2}$ as follows:

$$(0,0,\dots,0) = x_1 \preceq x_1 \preceq \dots \preceq x_{2^{m^2}} = (1,1,\dots,1).$$

For any m in $\mathbb{N}^*$, we define the following integer:

$$k(m) = \left\lceil \frac{2^{m^2}}{m} \right\rceil,$$

and the following function on $\{0,1\}^{m^2}$:

$$\forall i \in \big\{1,\dots,2^{m^2}\big\},\quad g_m(x_i) = \left\lfloor \frac{i}{k(m)} \right\rfloor.$$

Remark that $g_m(x_{2^{m^2}}) \le m/c_1 \le 1$. Therefore, g is a function from $\{0,1\}^{m^2}$ to $\{0,\dots,m\}$. Now, suppose that x_i and

x_l differ from exactly one coordinate. Then,

$$|i-l| \le \binom{m^2}{i} + \binom{m^2}{l} \le 2\binom{m^2}{\lfloor m^2/2\rfloor} \le c_1 \frac{2^{m^2}}{m} \le k(m).$$

Consequently,

$$g_m(x_i) - g_m(x_l) \le \left\lfloor \frac{l}{k(m)} + 1 \right\rfloor - \left\lfloor \frac{l}{k(m)} \right\rfloor = 1,$$

which implies that $\nabla_q g_m \in \{0,1\}$. Finally, for any $y \in \{0,\dots,m\}$, g takes the value y at most $k(m)$ times, and

$$\lambda\big(x \text{ s.t. } g_m(x) = y\big) \le \frac{k(m)}{2^{m^2}} \le \frac{c_1}{m} + \frac{1}{2^{m^2}} \le \frac{2c_1}{m}.$$

So the lemma holds with $c = 2c_1$. □

Now, we define our randomised version of f_v as follows. Let m be a positive integer, to be fixed later, and $S = \{1,\dots,d\} \times \{1,\dots,m^2\}$. Let $c > 0$ and g_m be as in Lemma 5.6. As in [5], for any $a = (a_{i,j})_{(i,j)\in S} \in \{0,1\}^S$, let

$$z = z(a) = \sum_{i=1}^{d} g_m(a_{i,1},\dots,a_{i,m^2})\mathbf{e}_i,$$

where $(\mathbf{e}_1,\dots,\mathbf{e}_d)$ denotes the standard basis of $\mathbb{Z}^d$. We now equip the space $\{0,1\}^S \times \mathbb{R}_+^E$ with the probability measure $\lambda \otimes \mu_k$, where $\lambda := \lambda_{1/2}^S$ is the uniform measure on $\{0,1\}^S$, and we define the following function $\tilde f$ on $\{0,1\}^S \times \mathbb{R}_+^E$:

$$\forall (a,x) \in \{0,1\}^S \times \mathbb{R}_+^E, \quad \tilde f(a,x) = d_x\big(z(a), v + z(a)\big).$$

When m is not too big, f and $\tilde f$ are not too far apart.

Lemma 5.7. *For any positive real number t,*

$$\mu_k\big(\big|f - \mathbb{E}(f)\big| > t\big) \le \lambda \otimes \mu_k\left(\big|\tilde f - \mathbb{E}(\tilde f)\big| > \frac{t}{2}\right) + \mathrm{e}^{-\delta t/(4m)}\mathbb{E}\big(\mathrm{e}^{\delta Y}\big).$$

Proof. Let $\alpha(a)$ be a path from 0 to $z(a)$, such that $|\alpha(a)| = |z(a)|$ (here, $|\alpha|$ is the number of edges in α). Let $\beta(a)$ denote a path disjoint from $\alpha(a)$, which goes from v to $v + z(a)$. Then,

$$\big|\tilde f(a,x) - f(x)\big| \le d_x\big(0, z(a)\big) + d_x\big(v, v+z(a)\big) \le \sum_{e\in\alpha(a)} x_e + \sum_{e\in\beta(a)} x_e,$$

which is stochastically dominated by a sum of $2m$ independent variables $Y_1,\dots,Y_{2m}$ with distribution ν. Remark that, due to the translation invariance of the distribution of f under μ_k, f and $\tilde f$ have the same mean against $\lambda \otimes \mu_k$. Thus, using $|z| \le m$, we have:

$$\mu_k\big(\big|f - \mathbb{E}(f)\big| > t\big) \le \lambda \otimes \mu_k\left(\big|\tilde f - \mathbb{E}(\tilde f)\big| > \frac{t}{2}\right) + \lambda \otimes \mu_k\left(|f - \tilde f| > \frac{t}{2}\right).$$

Now, by Markov's inequality, we get that for any positive real number t,

$$\begin{aligned}\lambda \otimes \mu_k\left(|f - \tilde f| > \frac{t}{2}\right) &\le \mathbb{P}\left(\sum_{i=1}^{2m} Y_i > \frac{t}{2}\right) = \mathbb{P}\left(\frac{\delta}{2m}\sum_{i=1}^{2m} Y_i > \frac{\delta t}{4m}\right) \\ &\le \mathrm{e}^{-\delta t/(4m)}\mathbb{E}\big(\mathrm{e}^{(\delta/(2m))Y}\big)^{2m} \le \mathrm{e}^{-\delta t/(4m)}\mathbb{E}\big(\mathrm{e}^{\delta Y}\big).\end{aligned}$$

This concludes the proof of this lemma. □

It remains to bound $\lambda \otimes \mu_k(|\tilde{f} - \mathbb{E}(\tilde{f})| > t)$. To this end, we will use an adaptation of Corollary 4.3, applied to $F = \tilde{f}$. Denote, for any s in S and any e in E,

$$W_{s,+} = \int \big(F\big(x^{-s}, y_s\big) - F(x)\big)_+ \,\mathrm{d}\beta_{1/2}(y_s),$$

and

$$W_{S,+} = \sum_{s \in S} W_{s,+},$$

$$W_{e,+} = \int \big(F\big(x^{-e}, y_e\big) - F(x)\big)_+ \,\mathrm{d}\nu_k(y_e),$$

and

$$W_{E,+} = \sum_{e \in E} W_{e,+}.$$

Applying Corollary 3.2 with $p = 1/2$ (note that $c_{LS}(1/2) = 2$), we can get the following minor adaptation of Corollary 4.3. The notations are those of Corollary 3.2 and Definition 5.1.

Proposition 5.8. *Let ν be a probability on $\mathbb{R}^+$ absolutely continuous with respect to the Lebesgue measure, with density h and repartition function*

$$H(t) = \int_0^t h(u)\,\mathrm{d}u.$$

Let $\{0,1\}^S \times \mathbb{R}^n$ be equipped with the probability measure $\lambda_p^S \otimes \nu^{\otimes \mathbb{N}}$. Let F be a function from $\{0,1\}^S \times \mathbb{R}^n$ to $\mathbb{R}$. Define

$$r_S = \sup_{s \in S} \sqrt{\mathbb{E}\big(W_{s,+}^2\big)},$$

$$s_S = \sqrt{\mathbb{E}\big(W_{S,+}^2\big)},$$

$$r_E = \sup_{e \in E} \sqrt{\mathbb{E}\big(W_{e,+}^2\big)},$$

$$s_E = \sqrt{\mathbb{E}\big(W_{E,+}^2\big)},$$

and

$$K_{ES} = r_S s_S + r_E s_E.$$

Define, for every real number $K > eK_{ES}$:

$$l(K) = \frac{K}{\log(K/(K_{ES}\log(K/K_{ES})))}.$$

Suppose that there exists three positive real numbers C, D and A_{CD} such that:

(i) $C \le \sqrt{l(A_{CD})}$,

(ii) $A_{CD} \geq \sup\{eK_{ES}, 4C\mathbb{E}(F) + D(1+\frac{2}{C})\}$,

(iii) *for every* θ *such that* $|\theta| \leq \frac{1}{2\sqrt{l(A_{CD})}}$, $\mathrm{e}^{\theta F} \circ (Id, \widetilde{H^{-1} \circ G}) \in H_1^2(\lambda_p^S \otimes \gamma^{\mathbb{N}})$ *and*:

$$\sum_{s\in S} \bigl\| \Delta_s\bigl(\mathrm{e}^{(\theta/2)F}\bigr)\bigr\|_2^2 \leq D\theta^2 \mathbb{E}\bigl(\mathrm{e}^{\theta F}\bigr) \tag{21}$$

and

$$\sum_{e\in E} \bigl\| \nabla_e\bigl(\mathrm{e}^{(\theta/2)F}\bigr)\bigr\|_2^2 \leq C\theta^2 \mathbb{E}\bigl(F\mathrm{e}^{\theta F}\bigr), \tag{22}$$

where for every e *in* E,

$$\nabla_e f(x,y) = \psi(y_e)\frac{\partial f}{\partial y_e}(x,y),$$

and ψ *is defined on* $I = \{t \geq 0 \text{ s.t. } h(t) > 0\}$:

$$\forall t \in I, \quad \psi(t) = \frac{g \circ G^{-1}(H(t))}{h(t)}.$$

Then, denoting $\mu = \lambda_S \otimes \gamma^E$, *for every* $t > 0$:

$$\mu\bigl(F - \mathbb{E}(F) \geq t\sqrt{l(A_{CD})}\bigr) \leq 4\mathrm{e}^{-t},$$
$$\mu\bigl(F - \mathbb{E}(F) \leq -t\sqrt{l(A_{CD})}\bigr) \leq 4\mathrm{e}^{-t}.$$

First, we need to prove that $\mathrm{e}^{\theta \tilde{f}} \circ (Id, \widetilde{H^{-1} \circ G})$ belongs to $H_1^2(\lambda \otimes \gamma^{\mathbb{N}})$ when ν is nearly gamma. This is the aim of the following lemma.

Lemma 5.9. *If* ν *is nearly gamma, and has bounded support, for any positive number* θ, *the function* $\mathrm{e}^{\theta f_v} \circ \widetilde{H^{-1} \circ G}$ *belongs to* $H_1^2(\gamma^{\mathbb{N}})$, $\mathrm{e}^{\theta \tilde{f}} \circ (Id, \widetilde{H^{-1} \circ G})$ *belongs to* $H_1^2(\lambda \otimes \gamma^{\mathbb{N}})$. *Furthermore, conditionally to* z, *there is almost surely only one* x*-geodesic from* z *to* $z+v$, *denoted by* $\gamma_x(z)$, *and*:

$$\frac{\partial \tilde{f}}{\partial x_e}(a,x) = \mathbb{1}_{e\in\gamma_x(z(a))}.$$

Proof. The fact that $\mathrm{e}^{\theta f_v} \circ \widetilde{H^{-1} \circ G}$ and $\mathrm{e}^{\theta \tilde{f}} \circ (Id, \widetilde{H^{-1} \circ G})$ are in L^2 is obvious since ν has bounded support. We shall prove that $\mathrm{e}^{\theta f_v} \circ \widetilde{H^{-1} \circ G}$ satisfies the integration by part formula (a) of the definition of H_1^2. The similar result for $\mathrm{e}^{\theta \tilde{f}}$ is obtained in the same way. Now, we fix x^{-e} in $(\mathbb{R}^+)^{E(\mathbb{Z}^d)\setminus\{e\}}$. We denote by g_e the function defined on $\mathbb{R}^+$ by:

$$g(y) = f_v\bigl(x^{-e}, y\bigr).$$

We will show that there is a nonnegative real number y_∞ such that:

$$\begin{cases} \forall y \leq y_\infty, & g(y) = g(0) + y \quad \text{and} \\ \forall y > y_\infty, & g(y) = g(y_\infty). \end{cases} \tag{23}$$

For any $n \geq |v|$, let us denote by Γ_n the set of paths from 0 to v whose number of edges is not greater than n. We have:

$$g(y) = \inf_{n\geq|v|} g_n(y),$$

where

$$g_n(y) = \inf_{\gamma \in \Gamma_n} \sum_{e' \in \gamma} \left(x^{-e}, y\right)_{e'}.$$

The functions g_n form a nonincreasing sequence of nondecreasing functions:

$$\forall n \geq |v|, \forall y \in \mathbb{R}^+, \forall y' \geq y, \quad g_{n+1}(y) \leq g_n(y) \leq g_n\left(y'\right).$$

In particular, this implies that for every y in $\mathbb{R}^+$,

$$g(y) = \lim_{n \to \infty} g_n(y).$$

Now, we claim that, for every $n \geq |v| + 3$, there exists $y_n \in \mathbb{R}^+$ such that:

$$\begin{cases} \forall y \leq y_n, \quad g_n(y) = g_n(0) + y \quad \text{and} \\ \forall y > y_n, \quad g_n(y) = g_n(y_n), \end{cases} \tag{24}$$

and furthermore,

$$\text{the sequence } (y_n)_{n \geq |v|+3} \text{ is nonincreasing.} \tag{25}$$

Indeed, since Γ_n is a finite set, the infimum in the definition of g_n is attained. Let us call a path which attains this infimum an (n, y)-geodesic and let $\tilde{\Gamma}(n, y, e)$ be the set of (n, y)-geodesics which contain the edge e. Remark that as soon as $n \geq |v| + 3$, there exists a real number A such that e does not belong to any (n, A)-geodesic: it is enough to take A greater than the sum of the length of three edges forming a path between the end-points of the edge e. Therefore, the following supremum is finite:

$$y_n = \sup\left\{y \in \mathbb{R}^+ \text{ s.t. } \tilde{\Gamma}(n, y, e) \neq \emptyset\right\}.$$

Now, if e belongs to an (n, y)-geodesic γ, for any $y' \leq y$, γ is an (n, y')-geodesic to which e belongs, and $g_n(y) - g_n(y') = y - y'$. If $\tilde{\Gamma}(n, y, e)$ is empty, then for any $y' \geq y$, e does not belong to any (n, y')-geodesic, and $g_n(y) = g_n(y')$. This proves that:

$$\begin{aligned} &\forall y < y_n, \qquad g_n(y) = g_n(0) + y, \\ &\forall y, y' > y_n, \quad g_n(y) = g_n\left(y'\right). \end{aligned}$$

Since g_n is continuous, we have proved claim (24). Now remark that if e does not belong to any (n, y)-geodesic, then e does not belong to any $(n + 1, y)$-geodesic, since $\Gamma_n \subset \Gamma_{n+1}$. Therefore, $y_{n+1} \leq y_n$, and this proves claim (25). Since $(y_n)_{n \geq |v|+3}$ is nonnegative, it converges to a nonnegative number y_∞ as n tends to infinity. Now, let n be a integer greater than $|v| + 3$:

$$\forall n \geq N, \forall y, y' > y_n, \quad g_n(y) = g_n\left(y'\right).$$

Since $y_n \leq y_N$,

$$\forall n \geq N, \forall y, y' > y_N, \quad g_n(y) = g_n\left(y'\right).$$

Letting n tend to infinity in the last equation, we get:

$$\forall N \geq |v| + 3, \forall y, y' > y_N, \quad g(y) = g\left(y'\right).$$

Therefore,

$$\forall y, y' > y_\infty, \quad g(y) = g\left(y'\right).$$

On the other side,

$$\forall n \geq |v| + 3, \forall y \leq y_n, \quad g_n(y) = g_n(0) + y.$$

Since $y_n \geq y_\infty$,

$$\forall n \geq |v| + 3, \forall y \leq y_\infty, \quad g_n(y) = g_n(0) + y.$$

Letting n tend to infinity in the last expression, we get:

$$\forall y \leq y_\infty, \quad g(y) = g(0) + y.$$

Finally, g is continuous. Indeed, the convergent sequence (g_n) is uniformly equicontinuous, since all these functions are 1-Lipschitz, and the continuity of g follows from Arzel–Ascoli theorem. We have proved claim (23). Remark that $y_\infty = y_\infty(x^{-e})$ depends on x^{-e}. We define, for any x^{-e},

$$h_e(x^{-e}, x_e) = \begin{cases} 1 & \text{if } x_e \leq y_\infty(x^{-e}), \\ 0 & \text{if } x_e > y_\infty(x^{-e}). \end{cases}$$

It is easy to see that, for any smooth function $F : \mathbb{R} \to \mathbb{R}$ having compact support, for any x^{-e},

$$-\int_{\mathbb{R}} F'(x_e) \mathrm{e}^{\theta f_v(x^{-e}, x_e)} \, \mathrm{d}x_e = \theta \int_{\mathbb{R}} F(x_e) h_e(x^{-e}, x_e) \mathrm{e}^{\theta f_v(x^{-e}, x_e)} \, \mathrm{d}x_e. \tag{26}$$

It is known that there is almost surely a geodesic from 0 to v (see [12] for instance), that is, the infimum in the definition of f_v is attained with probability 1. Furthermore, in this setting, where the distribution of the lengths is continuous, there is almost surely only one unique x-geodesic from 0 to v. For any x, we shall denote by $\gamma_x(0)$ the unique x-geodesics from 0 to $0 + v$. Then, with ν-probability 1, one can see from the definitions of y_n and y_∞ that:

$$h_e(x^{-e}, x_e) = \mathbb{1}_{e \in \gamma_x(0)}. \tag{27}$$

Performing the change of variable $x \mapsto \widetilde{H^{-1} \circ G}$ in Eq. (26), one gets the integration by parts formula (a) for $\mathrm{e}^{\theta f_v} \circ \widetilde{H^{-1} \circ G}$, with the following partial derivative with respect to x_e:

$$x \mapsto \theta \psi(x_e) h_e(\widetilde{H^{-1} \circ G}(x)) \mathrm{e}^{\theta f_v}.$$

The expression of $\frac{\partial \tilde{f}}{\partial x_e}(a, x)$ is derived in the same way than (27). □

Now, we want to apply Proposition 5.8 to $F = \tilde{f}$.

Bound on $\sum_{s \in S} \|\Delta_s(\mathrm{e}^{(\theta/2)F})\|_2^2$. Here, we can perform a quite rough upper bound, since there are not many elements in S. For any $a \in \{0,1\}^S$, and any q in S, denote by $\tau_q a$ the element of $\{0,1\}^S$ obtained from a by flipping the coordinate q. Then, for any function g on $\{0,1\}^S$,

$$\begin{aligned} \|\Delta_q \mathrm{e}^{\theta g/2}\|_p^p &= \frac{1}{4} \sum_{a \in \{0,1\}^S} |\mathrm{e}^{(\theta/2)g(a)} - \mathrm{e}^{(\theta/2)g(\tau_q a)}|^p \lambda(a) \\ &= \frac{1}{2} \sum_{a:\, \theta g(a) > \theta g(\tau_q a)} \mathrm{e}^{(\theta p/2)g(a)} (1 - \mathrm{e}^{(\theta/2)(g(\tau_q a) - g(a))})^p \lambda(a) \\ &\leq \frac{|\theta|^p}{2^{p+1}} \sum_{a:\, \theta g(a) > \theta g(\tau_q a)} \mathrm{e}^{(\theta p/2)g(a)} |g(a) - g(\tau_q a)|^p \lambda(a) \end{aligned}$$

$$\leq \frac{|\theta|^p}{2^{p+1}} \sum_{a \in \{0,1\}^S} \mathrm{e}^{(\theta p/2) g(a)} \left| g(a) - g(\tau_q a) \right|^p \lambda(a)$$

$$= \frac{|\theta|^p}{2} \left\| \mathrm{e}^{\theta g/2} \Delta_q g \right\|_p^p.$$

According to Lemma 5.6, for any $q \in \{0,1\}^{m^2}$, $\nabla_q g_m \in \{0,1\}$. Therefore, for any $s = (i, q) \in S$,

$$|\Delta_s \tilde{f}| \leq \frac{1}{2} (x_{(z,z+\mathbf{e_1})} + x_{(z+v,z+v+\mathbf{e_1})}).$$

Therefore, we get the following bounds:

$$\sum_{s \in S} \left\| \Delta_s \mathrm{e}^{\theta \tilde{f}/2} \right\|_2^2 \leq \theta^2 C_5^2 m \left(\log |v| \right)^2 \mathbb{E}\left(\mathrm{e}^{\theta \tilde{f}} \right). \tag{28}$$

Bound on r_S.

$$r_S \leq 2\sqrt{\mathbb{E}\left(Y^2\right)}. \tag{29}$$

Bound on s_S.

$$r_S \leq 2m. \tag{30}$$

Bound on $\sum_{e \in E} \|\nabla_e (\mathrm{e}^{(\theta/2)F})\|_2^2$. Let A be as in Definition 5.1. Since ν_k is nearly gamma with bound A (see Lemma 5.5),

$$\sum_{e \in E} \left\| \mathrm{e}^{\theta \tilde{f}/2} \nabla_e \tilde{f} \right\|_2^2 \leq A \mathbb{E}\left(\tilde{f} \mathrm{e}^{\theta \tilde{f}} \right).$$

Bound on s_E. Remark that:

$$\left(\tilde{f}\left(x^{-e}, y_e \right) - \tilde{f}(x) \right)_+ \leq y_e \mathbb{1}_{e \in \gamma_x(z)},$$

and $\gamma_x(z)$ is independent from y_e. Therefore,

$$0 \leq W_{e,+} \leq \mathbb{E}(Y) \mathbb{1}_{e \in \gamma_x(z)}, \tag{31}$$

which leads to:

$$0 \leq W_{E,+} \leq \mathbb{E}(Y) \left| \gamma_x(z) \right|,$$

and

$$s_E \leq \mathbb{E}(Y) \sqrt{\mathbb{E}\left(\left| \gamma_x(z) \right|^2 \right)}.$$

Now, following Kesten [16], page 308, we claim that there exists some constant C_6, depending only on ν (and not on k) such that:

$$\mathbb{E}_{\nu_k}\left(\left| \gamma_x(z) \right|^2 \right) \leq C_6 |v|^2. \tag{32}$$

Indeed, for any $a > 0$ and $y > 0$,

$$\mu_k\left(\left| \gamma_x(0) \right| \geq y|v| \right)$$

$$\leq \mu_k\left(f_v \geq ay|v| \right) + \mu_k\left(\exists \text{ a self-avoiding path } r \text{ starting at 0 of at least } y|v| \text{ steps but with } \sum_{e \in r} x_e < ay|v| \right).$$

Proposition 5.8 of [15] shows that for a suitable $a > 0$, the second term in the right-hand side of the above inequality is at most $C\mathrm{e}^{-C'y|v|}$ for some constants C and C'. Further more, a, C and C' do not depend on k: it suffices to choose them for ν_{k_ν}, and the same constants work for any $k \geq k_\nu$ (see part (iv) of Lemma 5.5 and the remark of [16], page 309). On the other hand, f_v is dominated by the sum of $|v|$ independent variables with distribution ν, $X_1, \ldots, X_{|v|}$. Thus,

$$\begin{aligned}
\mathbb{E}\bigl(\bigl|\gamma_x(z)\bigr|^2\bigr) &= \mathbb{E}\bigl(\bigl|\gamma_x(z)\bigr|^2\bigr) \\
&= |v|^2 \int_0^\infty \mu\bigl(\bigl|\gamma_x(0)\bigr|^2 > y|v|^2\bigr)\,\mathrm{d}y \\
&\leq |v|^2 \int_0^\infty \mu\left(\left(\sum_{i=1}^{|v|} X_i\right)^2 \geq ay|v|^2\right) + |v|^2 C \int_0^\infty \mathrm{e}^{-C'\sqrt{y}|v|}\,\mathrm{d}y \\
&= \frac{1}{a^2}\mathbb{E}\left(\left(\sum_{i=1}^{|v|} X_i\right)^2\right) + 2C\int_0^\infty t\mathrm{e}^{-C't}\,\mathrm{d}t \\
&\leq C_6|v|^2.
\end{aligned}$$

This proves claim (32). Therefore,

$$s_E \leq \sqrt{C_6}\,\mathbb{E}(Y)|v|.$$

Bound on r_E. From inequality (31), we get:

$$r_E \leq \mathbb{E}(Y)\sqrt{\sup_{e\in E}\mathbb{P}\bigl(e \in \gamma_x(z)\bigr)}.$$

Now, we use the fact that for any fixed z, μ is invariant under translation by z.

$$\begin{aligned}
\mathbb{P}\bigl(e \in \gamma_x(z)\bigr) &= \mathbb{E}_\lambda\bigl(\mathbb{E}_\mu(\mathbb{1}_{e-z\in\gamma_x(0)})\bigr) = \mathbb{E}_\mu\left(\sum_{e'\in\gamma_x(0)} \mathbb{E}_\lambda(\mathbb{1}_{e-z=e'})\right) = \mathbb{E}_\mu\left(\sum_{e'\in\gamma_x(0)} \mathbb{P}_\lambda\bigl(z = e - e'\bigr)\right) \\
&\leq \sup_{z_0}\mathbb{P}(z = z_0)\mathbb{E}_\mu\bigl(\bigl|\gamma_x(0) \cap \mathcal{Q}_e\bigr|\bigr) \leq \sup_{z_0}\mathbb{P}(z = z_0)\mathbb{E}_\mu\bigl(\bigl|\gamma_x(0) \cap \mathcal{B}_e\bigr|\bigr),
\end{aligned}$$

where $\mathcal{Q}_e = \{e' \in E(\mathbb{Z}^d) \text{ s.t. } \mathbb{P}(z = e - e') > 0\} \subset \mathcal{B}_e = e + \mathcal{B}(0, dm)$. Using Lemma 5.6,

$$\sup_{z_0}\mathbb{P}(z = z_0) \leq \left(\frac{c}{m}\right)^d.$$

Now, we claim that

$$\mathbb{E}_{\mu_k}\bigl(\bigl|\gamma_x(0) \cap \mathcal{B}_e\bigr|\bigr) \leq C_7 m^{d-1}. \tag{33}$$

We proceed as we did to obtain (32). Indeed, for any $a > 0$ and $y > 0$,

$$\begin{aligned}
&\mu_k\bigl(\bigl|\gamma_x(0) \cap \mathcal{B}_e\bigr| \geq ym\bigr) \\
&\quad\leq \mu_k\left(\sum_{e'\in\gamma_x(0)\cap\mathcal{B}_e} x_{e'} \geq ay|v|\right) \\
&\qquad + \sum_{w\in\partial\mathcal{B}_e} \mu_k\left(\exists \text{ a self-avoiding path } r \text{ starting at } w \text{ of at least } ym \text{ steps but with } \sum_{e\in r} x_e < aym\right).
\end{aligned}$$

We use again the constants a, C and C' arising from Proposition 5.8 of [15], and which depend on ν, but not on k. Remark that there are at most $(dm)^{d-1}$ vertices in $\partial\mathcal{B}_e$. On the other hand, let r be a deterministic path going through every vertex of the surface of the ball $\mathcal{B}_e$, and such that there is a constant C'' (depending only on d) such that $|r| \le C'' m^{d-1}$. From the definition of a geodesic, we get:

$$f_v \le \sum_{e' \in r} x_e.$$

Thus,

$$\begin{aligned}
\mathbb{E}\big(\big|\gamma_x(0) \cap \mathcal{B}_e\big|\big) &= m \int_0^\infty \mu_k\big(\big|\gamma_x(0) \cap \mathcal{B}_e\big| > ym\big)\,\mathrm{d}y \\
&\le m \int_0^\infty \mu_k\bigg(\sum_{e' \in r} x_e \ge aym\bigg) + m(dm)^{d-1} \int_0^\infty \mathrm{e}^{-C' ym}\,\mathrm{d}y \\
&= \frac{1}{a}\mathbb{E}_{\mu_k}\bigg(\sum_{e' \in r} x_e\bigg) + \frac{2C}{C'}(dm)^{d-1} \int_0^\infty t\mathrm{e}^{-C' t}\,\mathrm{d}t \\
&\le C_7 m^{d-1}.
\end{aligned}$$

This proves claim (33). Therefore:

$$\begin{aligned}
r_E &\le \mathbb{E}(Y)\sqrt{\left(\frac{c}{m}\right)^d C_7 m^{d-1}}, \\
r_E &\le \frac{C_8}{m^{1/2}}.
\end{aligned} \tag{34}$$

End of the proof. Now, we choose $m = \lceil |v|^{1/4} \rceil$. Define $C = A$, $D = C_5^2 m (\log|v|)^2$. The bounds obtained before lead to:

$$K_{ES} = \mathrm{O}\big(|v|^{7/8}\big),$$

and:

$$4C\mathbb{E}(F) + D\left(1 + \frac{2}{C}\right) = \mathrm{O}\big(|v|\big).$$

So we can choose $A_{CD} = C_4|v|$, with C_4 a positive constant, such that (ii) of Proposition 5.8 applied to $F = \tilde{f}$ is satisfied. It is clear that, for $|v|$ large enough, conditions (i) and (iii) are also satisfied. Remark also that:

$$l(A_{CD}) = \mathrm{O}\left(\frac{|v|}{\log|v|}\right).$$

Therefore, there exists a constant C_{12} such that for every $t > 0$:

$$\mu_k\left(\tilde{f} - \mathbb{E}(\tilde{f}) > t\sqrt{\frac{|v|}{\log|v|}}\right) \le 4\mathrm{e}^{-C_{12} t} \tag{35}$$

and

$$\mu_k\left(\tilde{f} - \mathbb{E}(\tilde{f}) < -t\sqrt{\frac{|v|}{\log|v|}}\right) \le 4\mathrm{e}^{-C_{12} t}. \tag{36}$$

Lemmas 5.7 and 5.5 conclude the proof of Theorem 5.4. □

Remark 4. *Inequalities* (35), (36) *and Lemma* 5.7 *imply, after integration, that the variance of f_v is of order* $\mathrm{O}(|v|/\log|v|)$. *Of course, we do not need the assumption that ν has a bounded support to obtain such a result. Instead, we just need ν to have a second moment. The proof mimics* [5], *and the ideas presented here. Details may be found in* [4], *which is a preliminary version of the present paper.*

5.2. Bernoulli distributions

The method developed in Section 5.1 applies also to the case where the edge-times are distributed according to a Bernoulli law $\nu = (1-p)\delta_a + p\delta_b$, and a is strictly positive. The proof follows exactly the same pattern as the proof of the nearly gamma case, except that:

(1) one does not need Lemma 5.5, since ν has bounded support,
(2) the geodesic is not almost surely unique anymore,
(3) the energy $\sum_{e\in E}\mathbb{E}(R_e(\mathrm{e}^{(\theta/2)\tilde f})^2)$ is different.

Point (1) is just good news. Point (2) is not a problem: the bounds on s_E, s_S, r_E and r_S remain valid if we choose for $\gamma_x(z)$ one geodesic among all the possible ones. So we shall only show how to circumvent point (3), that is, how one can bound $\sum_{e\in E}\mathbb{E}(R_e(\mathrm{e}^{(\theta/2)\tilde f})^2)$, where

$$R_e(f) = \sqrt{c_{LS}(p)}\Delta_e f.$$

First, imitating the proof of Theorem 4.2, we write:

$$\sum_{e\in E}\mathbb{E}\bigl(R_e\bigl(\mathrm{e}^{(\theta/2\tilde f}\bigr)^2\bigr) \le c_{LS}(p)\frac{\theta^2}{4}\mathbb{E}\bigl(V_{E,+}\mathrm{e}^{\theta\tilde f}\bigr),$$

where:

$$V_{E,+} = \sum_{e\in E}\int \bigl(\tilde f\bigl(x^{-e}, y_e\bigr) - \tilde f(x)\bigr)_+^2 \,\mathrm{d}\nu(y_e).$$

Now,

$$V_{E,+} \le \sum_{e\in E}\int (b-a)^2 \mathbb{1}_{e\in\gamma_x(z)}\,\mathrm{d}\nu(y_e) = (b-a)^2\bigl|\gamma_x(z)\bigr|_1 \le \frac{(b-a)^2}{a}\tilde f.$$

Therefore,

$$\sum_{e\in E}\mathbb{E}\bigl(R_e\bigl(\mathrm{e}^{(\theta/2)\tilde f}\bigr)^2\bigr) \le c_{LS}(p)\frac{\theta^2}{4}\frac{(b-a)^2}{a}\mathbb{E}\bigl(\tilde f\mathrm{e}^{\theta\tilde f}\bigr). \tag{37}$$

The bound (37) allows us to obtain the following equivalent of Theorem 5.4 in the case of Bernoulli distributions.

Proposition 5.10. *Let a and b be two real numbers such that $0 < a < b$. We define $\nu = (1-p)\delta_a + p\delta_p$ and $\mu = \nu^{\otimes E}$. Then, there exist two positive constants C_1 and C_2 such that, for any $|v| \ge 2$, and any positive real number t,*

$$\mu\left(\left|d_x(0,v) - \int d_x(0,v)\,\mathrm{d}\mu(x)\right| > t\sqrt{\frac{|v|}{\log|v|}}\right) \le C_1\mathrm{e}^{-C_2 t}.$$

Remark 5. *When $a = 0$, the previous argument does not work, and it is hard to compare $V_{E,+}$ to $\tilde f$ itself. Although the quantity $V_{E,+}$ may be controlled when $1 - p < p_c(\mathbb{Z}^d)$ via Kesten's work* (*see Proposition* 5.8 *in* [15]), *we do not know how to adapt the entire proof to this case.*

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