

ANNALES DE L'INSTITUT FOURIER

GILLES GODEFROY

D. LI

**Banach spaces which are M -ideals in their
bidual have property (u)**

Annales de l'institut Fourier, tome 39, n° 2 (1989), p. 361-371

http://www.numdam.org/item?id=AIF_1989__39_2_361_0

© Annales de l'institut Fourier, 1989, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

BANACH SPACES WHICH ARE M -IDEALS IN THEIR BIDUAL HAVE PROPERTY (u)

by G. GODEFROY and D. LI

1. Introduction.

A Banach X is an M -ideal in its bidual if the relation

$$\|y+t\| = \|y\| + \|t\|$$

holds for every $y \in X^*$ and every $t \in X^\perp \subseteq X^{***}$. The spaces $c_0(I) - I$ any set-equipped with their canonical norm belong to this class, which also contains e.g. certain spaces $K(E,F)$ of compact operators between reflexive spaces (see [11]) and certain spaces of the form $C(G)/C_\Lambda(G)$ where G is an abelian compact group and Λ is a subset of the discrete dual group (see [5]). This class has been carefully investigated, in particular by A. Lima and by the « West-Berlin school », since the notion of M -ideal was introduced by Alfsen and Effros in 1972 [1].

We will show in this paper that these spaces somehow « look like » c_0 ; more precisely, that they share the property (u) with this latter space. This solves affirmatively a question that was pending for several years, and provides improvements of some results of [6] and [10].

Our proof uses non-linear arguments. The key lemma is actually a special case of a fundamental lemma ([1], lemma 1.4.) of the original article of Alfsen and Effros.

Notation. — The closed unit ball of a Banach space Z is denoted by Z_1 , and its dual by Z^* . The topology defined on Z^* by the pointwise convergence on Z is denoted by ω^* . The canonical injection from a Banach space X into its bidual X^{**} is denoted by i . A sequence (x_m) in X is said to be a weakly unconditionally convergent series —

Key-words : M -ideals - Weakly unconditionally convergent series - Concave upper-semi-continuous functions.

AMS Classification : 46 B 20.

in short, w.u.c. series — if for every $y \in X^*$,

$$\sum_{m=0}^{\infty} |y(x_m)| < \infty.$$

If (x_m) is a w.u.c. series, then clearly the sequence

$$s_k = \sum_{m=0}^k x_m \quad (k \geq 1)$$

is weakly Cauchy and thus it converges in (X^{**}, w^*) ; we note $\Sigma^* x_m = \lim_{k \rightarrow \infty} (s_k)$ in (X^{**}, ω^*) . A Banach space X has the property (u) ([14]; see [12], p. 32) if every $z \in X^{**}$ which is in the sequential closure of X in (X^{**}, ω^*) may be written

$$z = \Sigma^* x_m$$

for some w.u.c. series (x_m) in X .

If $\tau : Z_1^* \rightarrow \mathbf{R}$ is a real-valued function defined on a dual unit ball Z_1^* , we denote by $\hat{\tau}$ the smallest concave ω^* -u.s.c. function which is greater than τ on Z_1^* . The function $\hat{\tau}$ is the infimum of the affine continuous functions on (Z_1^*, ω^*) which maximize τ on Z_1^* . The reader should consult [2] for a presentation of the basic facts about M -ideals. Similar ideas to those we use in this work are to be found e.g. in [15].

Acknowledgments. — This work was completed while the first-named author was Visiting Professor at the University of Missouri-Columbia during the Academic Year 1987/1988. It is his pleasure to thank the Department of Mathematics of U.M.C. for its hospitality and support, and for the stimulating atmosphere of this visit. The authors are also thankful to J. Saint-Raymond for useful conversations.

2. The main result.

We will now prove :

THEOREM 1. — *Let X be a Banach space which is an M -ideal in its bidual. Then X has the property (u).*

Proof. — If i denotes the canonical injection from X into X^{**} , j^{**} is an isometric injection from X^{**} into X^{****} which is distinct from

the canonical injection if X is not reflexive. We will use a simplified notation that we now define: i^* denotes the canonical projection from X^{***} onto X^* of kernel $i(X)^\perp = X^\perp$, and then of course $i^{**}(z) = z \circ i^*$.

From now on, we assume that X is a real Banach space. By ([2], p. 22), we can do so without loss of generality. The notation $\mathbf{1}_{X_1^\perp}$ denotes the characteristic function of the subset X_1^\perp of the unit ball X_1^{***} of X^{***} . Thus for any $z \in X^{**}$, $(\mathbf{1}_{X_1^\perp} \cdot z \vee 0)$ denotes the supremum of 0 and of the pointwise product of z and $\mathbf{1}_{X_1^\perp}$.

With this notation, we have the following crucial lemma.

LEMMA 2. — *If X is an M -ideal in its bidual X^{**} , then for every $z \in X^{**}$ and every $t \in X_1^{***}$ one has*

$$\langle z - i^{**}(z), t \rangle = [\widehat{\mathbf{1}_{X_1^\perp} \cdot z \vee 0}](t) - [\widehat{\mathbf{1}_{X_1^\perp} \cdot z \vee 0}](-t).$$

This lemma is actually a special case of ([1], lemma I.4). For sake of completeness, we give a simplified proof of this special case.

Proof. — If Ψ is a function from X_1^{***} to \mathbf{R}^+ , we define

$$\mathfrak{G}^-(\Psi) = \{(t, \lambda) \in X_1^{***} \times \mathbf{R}^+ \mid 0 \leq \lambda \leq \Psi(t)\}$$

we let $\tau = (\mathbf{1}_{X_1^\perp} \cdot z \vee 0)$, and

$$B = \{(t, z(t)) \in X_1^\perp \times \mathbf{R}^+ \mid z(t) \geq 0\}.$$

We clearly have

$$(1) \quad \mathfrak{G}^-(\hat{\tau}) = \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)).$$

On the other hand,

$$(2) \quad X_1^{***} = \text{conv}(X_1^* \cup X_1^\perp) \text{ since } X \text{ is an } M\text{-ideal in } X^{**}$$

$$(3) \quad \text{if } 0 \leq \lambda \leq z(t), (t, \lambda) \in \text{conv} \{(t, 0); (t, z(t))\}.$$

From (1), (2) and (3) follows

$$\begin{aligned} \mathfrak{G}^-(\hat{\tau}) &= \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)) \\ &= \text{conv} \{(X_1^{***} \times \{0\}) \cup B\} \\ &= \text{conv} \{(X_1^* \times \{0\}) \cup (X_1^\perp \times \{0\}) \cup B\} \end{aligned}$$

since by w^* -compactness we don't need to take w^* -closures.

For every $t \in X_1^{***}$, $(t, \hat{\tau}(t)) \in \mathfrak{G}^-(\hat{\tau})$, hence we may write $(t, \hat{\tau}(t)) = \alpha_1(t_1, 0) + \alpha_2(t_2, 0) + \alpha_3(t_3, z(t_3))$ with :

$$\begin{cases} t_1 \in X^* \\ t_2 \in X_1^\perp \\ t_3 \in X_1^\perp; \quad z(t_3) \geq 0 \\ \alpha_1, \alpha_2, \alpha_3 \geq 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1. \end{cases}$$

Since $t = \alpha_1 t_1 + (\alpha_2 t_2 + \alpha_3 t_3)$ is the unique decomposition of t on the direct sum $X^* \oplus X^\perp$ one has

$$t - i^*(t) = \alpha_2 t_2 + \alpha_3 t_3.$$

Since $z(t_3) \geq 0$, one has $\tau(t_3) = z(t_3)$ and $\tau(-t_3) = 0$. Since $\hat{\tau}$ is concave, one has

$$\begin{aligned} \hat{\tau}(t) &= \alpha_3 z(t_3) \\ &\geq \sum_{i=1}^3 \alpha_i \hat{\tau}(t_i) \\ &\geq \sum_{i=1}^3 \alpha_i \tau(t_i) \\ &= \alpha_2 \tau(t_2) + \alpha_3 \tau(t_3) \\ &= \alpha_2 \tau(t_2) + \alpha_3 z(t_3) \end{aligned}$$

hence $\alpha_2 \tau(t_2) \leq 0$; if $\alpha_2 = 0$ we may take $t_2 = 0$ as well; if $\alpha_2 > 0$ this implies $\tau(t_2) \leq 0$, hence $z(t_2) \leq 0$. In both cases, we have $z(-t_2) \geq 0$ and thus $z(-t_2) = \tau(-t_2)$.

Again by concavity of $\hat{\tau}$, one has

$$\begin{aligned} \hat{\tau}(-t) &\geq \sum_{i=1}^3 \alpha_i \hat{\tau}(-t_i) \\ &\geq \sum_{i=1}^3 \alpha_i \tau(-t_i) \\ &= \alpha_2 \tau(-t_2) + \alpha_3 \tau(-t_3) \\ &= \alpha_2 z(-t_2) \end{aligned}$$

hence

$$-\hat{\tau}(-t) \leq \alpha_2 z(t_2)$$

and therefore

$$\hat{\tau}(t) - \hat{\tau}(-t) \leq \alpha_3 z(t_3) + \alpha_2 z(t_2)$$

and

$$\begin{aligned} \alpha_3 z(t_3) + \alpha_2 z(t_2) &= \langle z, \alpha_2 t_2 + \alpha_3 t_3 \rangle \\ &= \langle z, t - i^*(t) \rangle \\ &= \langle z - i^{**}(z), t \rangle. \end{aligned}$$

Now the functions $\Phi(t) = \hat{\tau}(t) - \hat{\tau}(-t)$ and $(z - i^{**}(z))$ are both odd functions on X_1^{***} and they satisfy $\Phi \leq z - i^{**}(z)$; hence necessarily $\Phi = z - i^{**}(z)$ on X_1^{***} . □

We now come back to the proof of theorem 1. By lemma 2, for every $z \in X^{**}$, we can write

$$\forall t \in X_1^{***}, \quad \langle i^{**}(z), t \rangle = (z(t) - \hat{\tau}(t)) + \hat{\tau}(-t)$$

hence if we let

$$\begin{aligned} h_1(t) &= z(t) - \hat{\tau}(t) \\ h_2(t) &= -\hat{\tau}(-t) \end{aligned}$$

we have $i^{**}(z) = h_1 - h_2$ and h_1, h_2 are both l.s.c. on (X_1^{***}, w^*) .

We need now a topological argument for going down to (X_1^*, w^*) .

LEMMA 3 (Saint-Raymond). — *Let K be a compact topological space and $S : K \rightarrow K'$ be a continuous surjection. Let f be a function from K' to \mathbf{R} which is such that $(f \circ S)$ is the difference of two l.s.c. functions on K . Then f is the difference of two l.s.c. functions on K' .*

Proof. — Write $f \circ S = g_1 - g_2$ where g_1, g_2 are l.s.c. on K ; we define for $y \in K'$

$$\begin{aligned} \tilde{g}_1(y) &= \inf \{g_1(t) \mid S(t) = y\} \\ \tilde{g}_2(y) &= \inf \{g_2(t) \mid S(t) = y\} \end{aligned}$$

the functions $\tilde{g}_i (i=1,2)$ are l.s.c. on K' . Indeed, pick $\alpha < \tilde{g}_i(y)$; this means

$$(1) \quad \forall t \in S^{-1}(y), \quad g_i(t) > \alpha.$$

Since g_i is l.s.c. and $S^{-1}(y)$ is compact, (1) implies that there exists $\varepsilon > 0$ and an open neighbourhood V of $S^{-1}(y)$ such that

$$(2) \quad \forall t \in V, \quad g_i(t) > \alpha + \varepsilon.$$

Again by compactness, there exists a neighbourhood W of y such

that $S^{-1}(W) \subseteq V$; by (2) and the definition of \tilde{g}_i , this implies

$$\forall y' \in W, \quad \tilde{g}_i(y') \geq \alpha + \varepsilon > \alpha$$

and thus \tilde{g}_i is l.s.c.

We show now that $f = \tilde{g}_1 - \tilde{g}_2$; for every $y \in K'$ and $t \in S^{-1}(y)$, one has

$$\begin{aligned} \tilde{g}_1(y) &\leq g_1(t) = f \circ S(t) + g_2(t) \\ &= f(y) + g_2(t) \end{aligned}$$

hence by definition of \tilde{g}_2

$$\tilde{g}_1(y) \leq f(y) + \tilde{g}_2(y).$$

On the other hand,

$$\begin{aligned} f(y) + \tilde{g}_2(y) &\leq f(y) + g_2(t) \\ &= f \circ S(t) + g_2(t) \\ &= g_1(t) \end{aligned}$$

and thus by definition of \tilde{g}_1 ,

$$f(y) + \tilde{g}_2(y) \leq \tilde{g}_1(y)$$

and this concludes the proof of lemma 3. □

Let us now conclude the proof of the theorem. Since

$$i^{**}(z) = z \circ i^* = h_1 - h_2$$

with h_1 and h_2 l.s.c. on (X_1^{***}, w^*) , we may apply lemma 3 with $f = z$, $S = i^*$ and $K' = (X_1^*, w^*)$; this lemma provides us with the l.s.c. functions \tilde{h}_1 and \tilde{h}_2 on (X_1^*, w^*) such that $z = \tilde{h}_1 - \tilde{h}_2$.

If now $z = \lim_{n \rightarrow \infty} x_n$ in (X^{**}, w^*) , where (x_n) is a sequence in X , we let

$$Y = \overline{\text{span}} \{x_n | n \geq 1\}$$

and we call Q the canonical quotient map from X^* onto Y^* ; since $z \in Y^{\perp\perp} = Q^*(Y^{**})$, there is $z' \in Y^{**}$ such that $z = z' \circ Q$; again by lemma 3, there exist two l.s.c. functions \tilde{h}_1 and \tilde{h}_2 on (Y_1^*, w^*) such that

$$z' = \tilde{h}_1 - \tilde{h}_2.$$

But since Y is separable, the w^* -topology on Y_1^* is defined by a metric d , and then classically the sequences $f_n^i (i=1,2)$ defined for $y \in Y_1^*$ and $n \geq 1$ by

$$f_n^i(y) = \inf \{ \tilde{h}_i(y') + nd(y, y') \mid y' \in Y_1^* \}$$

are increasing sequences of continuous functions on (Y_1^*, w^*) which converge pointwise to \tilde{h}_i . Now the sequence $u_n (n \geq 0)$ of continuous functions on (Y_1^*, w^*) defined by

$$\begin{aligned} u_0 &= f_1^1 - f_1^2 \\ u_n &= f_{n+1}^1 + f_n^2 - f_n^1 - f_{n+1}^2 \quad (n \geq 1) \end{aligned}$$

satisfies

$$\sum_{n=0}^{\infty} |u_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} u_n(y) = z'(y), \quad \forall y \in Y_1^*.$$

But we still have

$$z'(y) = \lim_{n \rightarrow \infty} x_n(y), \quad \forall y \in Y_1^*$$

in this situation, a classical lemma of Pelczynski [14] (see [12], p. 32), which relies on a convex combination argument, shows that there is a sequence $(c_n)_{n \geq 0}$ in Y with

$$\sum_{n=0}^{\infty} |c_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} c_n(y) = z'(y), \quad \forall y \in Y_1^*$$

and since $z = Q^*(z')$ and $c_n = Q^*(c_n)$, this shows that

$$z = \Sigma^* c_n$$

and (c_n) is a w.u.c. series in X . □

Before mentioning a few applications of our result, we would like to mention that the proof provides an explicit expression of $z \in X^{**}$

as a difference of two l.s.c. functions on (X_1^*, w^*) ; indeed, if we define for $y \in X_1^*$

$$v(y) = \inf \{z(t) - [\widehat{1_{X_1^*} \cdot z} \vee 0](t) \mid t \in X_1^{***}, i^*(t) = y\}$$

then the functions v and $(v-z)$ are both l.s.c. on (X_1^*, w^*) .

3. Applications.

We gather in this section a few consequences of our result.

3.1. P. Saab and the first-named author showed in ([6], Theorem 1) that if X is an M -ideal in its bidual then X has the property (V) of Pelczynski; the proof uses « pseudo-balls » ([3]) and the local reflexivity principle. Since such an X does not contain $\ell^1(N)$, our result is an improvement of ([6], Theorem 1), and of course also of the fact ([10]) that non-reflexive M -ideals in their bidual contain $c_0(N)$.

Another result of [6] is a structural result (Corollary 6) for certain spaces E such that $K(E)$ is an M -ideal in $L(E)$. The proof uses Banach algebras techniques that require to work with complex Banach spaces. This is not needed any more, and our result together with the proofs of ([6], Theorem 4 and Corollary 6) implies for instance the

PROPOSITION 4. — *Let E be a separable reflexive space with $A.P.$ such that $K(E)$ is an M -ideal in $L(E)$. Then E is complemented in a reflexive space with an unconditional finite dimensional decomposition.*

There are some similarities between the techniques of [6] and of the present work; the main difference is that instead of using l.s.c. affine functions on a non-symmetric convex set — namely, the state space of a Banach algebra — we employ l.s.c. convex functions on a symmetric convex set — namely, a dual unit ball.

3.2. A Banach space Y is said to have to property (X) [7] if the following holds: $z \in Y^{**}$ belongs to Y if and only if for every w.u.c. series (y_n) in Y^* ,

$$z(\Sigma_* y_n) = \Sigma z(y_n)$$

where $(\Sigma_* y_n)$ denotes the limit of the sequence $\left\{ s_k = \sum_{n=1}^k y_n \mid k \geq 1 \right\}$ in (Y^*, w^*) . This condition roughly means that an abstract Radon-

Nikodym theorem is available for deciding which elements of Y^{**} belong to Y . Property (X) is equivalent to saying that $Y < \ell^1(N)$ for Edgar's ordering of Banach spaces [4]. For more details about this property, the reader may consult the recent survey [8].

Let us recall now the following easy

Claim. — If X is separable, does not contain $\ell^1(N)$ and has the property (u), then X^* has the property (X).

Proof of the claim. — We must show that every $t \in X^{***}$ such that $t(\Sigma_* z_n) = \Sigma t(z_n)$ for every w.u.c. series in X^{**} belongs to X^* . We can write $t = y + t'$ with $y \in X^*$ and $t' \in X^\perp$; since $y(\Sigma_* z_n) = \Sigma y(z_n)$ by w^* -continuity of y , we also have $t'(\Sigma_* z_n) = \Sigma t'(z_n)$.

Since X is separable, does not contain $\ell^1(N)$ and has (u), every $z \in X^{**}$ can be written $z = \Sigma^* x_n = \Sigma_* i(x_n)$ for some w.u.c. series in X ; but since $t'(x_n) = 0$ for every n , this implies $t'(z) = 0$, hence $t' = 0$ and $t = y \in X^*$.

Now this claim, together with theorem 1, shows :

PROPOSITION 5. — *If a separable Banach space X is an M -ideal in its bidual, then X^* has the property (X).*

By ([8], Theorem VII.8) this implies the following

COROLLARY 6. — *Let X be a separable Banach space X which is an M -ideal in its bidual, and let Y be an arbitrary Banach space. Let $T : X^{**} \rightarrow Y^*$ be a bounded linear operator. The following are equivalent :*

- (1) *there is an operator $T_0 = Y \rightarrow X^*$ such that $T_0^* = T$,*
- (2) *$\ker(T)$ and $T(X_1^{**})$ are w^* -closed,*
- (3) *T is $(w^* - w^*)$ -Borel,*
- (4) *T is $(w^* - w^*)$ -strongly Baire measurable.*

Let us conclude this work with a few natural questions.

Question 3.4. — Does there exist a separable Asplund space with property (u) which is not isomorphic to an M -ideal in its bidual? It looks reasonable to believe that this question has a positive answer ; a candidate example is the space $K(L^p)$ ($1 < p < \infty, p \neq 2$) which has (u) ([12], Th. 3) but is not M -ideal of its bidual for its canonical norm [11].

Let us also mention that a separable \mathcal{L}^∞ -space which is isomorphic to an M -ideal in its bidual is in fact isomorphic to $c_0(N)$ [9]. We do not know whether any isomorphic predual of $\ell^1(N)$ which has property (u) is isomorphic to $c_0(N)$.

Question 3.5. — A reformulation of Proposition 5 is that if Y is a separable space such that there exists a projection $\pi : Y^{**} \rightarrow Y$ with :

$$(a) \|z\| = \|\pi(z)\| + \|z - \pi(z)\|, \forall z \in Y^{**}$$

(b) $(\ker \pi)$ w^* -closed,

then Y has the property (X). It is not known whether the assumption (b) can be removed, or whether (a) alone implies the weaker property (V*) (see [14]), or at least that Y contains a complemented copy of $\ell^1(N)$ if it is not reflexive.

BIBLIOGRAPHY

- [1] E. M. ALFSEN, E. G. EFFROS, Structure in real Banach spaces I , Ann. of Math., 96 (1972), 98-128.
- [2] E. BEHREND, M -structure and the Banach-Stone theorem, Lecture Notes in Mathematics 736, Springer-Verlag (1977).
- [3] E. BEHREND, P. HARMAND, Banach spaces which are proper M -ideals, Studia Mathematica, 81 (1985), 159-169.
- [4] G. A. EDGAR, An ordering of Banach spaces, Pacific J. of Maths, 108, 1 (1983), 83-98.
- [5] G. GODEFROY, On Riesz subsets of abelian discrete groups, Israel J. of Maths, 61, 3 (1988), 301-331.
- [6] G. GODEFROY, P. SAAB, Weakly unconditionally convergent series in M -ideals, Math. Scand., to appear.
- [7] G. GODEFROY, M. TALAGRAND, Nouvelles classes d'espaces de Banach à predual unique, Séminaire d'Ana. Fonct. de l'École Polytechnique, Exposé n° 6 (1980/1981).
- [8] G. GODEFROY, Existence and uniqueness of isometric preduals : a survey, in Banach space Theory, Proceedings of a Research workshop held July 5-25, 1987, Contemporary Mathematics vol. 85 (1989), 131-194.
- [9] G. GODEFROY, D. LI, Some natural families of M -ideals, to appear.
- [10] P. HARMAND, A. LIMA, On spaces which are M -ideals in their biduals, Trans. Amer. Math. Soc., 283-1 (1984), 253-264.
- [11] A. LIMA, M -ideals of compact operators in classical Banach spaces, Math. Scand., 44 (1979), 207-217.
- [12] J. LINDENSTRAUSS, L. TZAFRIRI, Classical Banach spaces, Vol. II, Springer-Verlag (1979).

- [13] F. LUST, Produits tensoriels projectifs d'espaces de Banach faiblement sequentiellement complets, Coll. Math., 36-2 (1976), 255-267.
- [14] A. PELCZYNSKI, Banach spaces on which every unconditionally convergent operator is weakly compact, Bull. Acad. Pol. Sciences, 10 (1962), 641-648.
- [15] R. R. SMITH, J. D. WARD, Applications of convexity and M -ideal theory to quotient Banach algebras, Quart. J. of Maths. Oxford, 2-30 (1978), 365-384.

Manuscrit reçu le 24 juin 1988.

G. GODEFROY,
Université de Paris VI
Équipe d'Analyse
Tour 46-0, 4^e étage
4, place Jussieu
75252 Paris Cedex 05.

D. LI,
Université de Paris-Sud
Mathématiques
Bâtiment 425
91405 Orsay Cedex.