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GLOBAL STABILITY FOR DIAGRAMS OF DIFFERENTIABLE APPLICATIONS

by L.A. FAVARO and C.M. MENDES(*)

Introduction.

The theory for diagrams of differentiable applications put together by J.P. Dufour in [5] presents two essentially distinct cases.

The convergent case is the study of sequences of type $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots M_{n-1} \xrightarrow{f_{n-1}} M_n$ and is similar than that of just one application $M \xrightarrow{f} N$ studied by J. Mather. One of the reasons for this analogy is the existence of the Preparation Theorem for convergent diagrams. The global theory is obtained by gluing local solutions. This problem was studied and solved by Buchner [1] and J.P. Dufour [5].

The divergent case is the study of sequence which contains subdiagrams of type $Q \xleftarrow{g} M \xrightarrow{f} N$. In this case, the non-existence of a Preparation Theorem makes the study much more complicated. Some progress was achieved for the local problem in certain dimensions, see [5]. This doesn't happen for the global situation, since in the first type to be considered $R \xleftarrow{f_1} M \xrightarrow{f_2} R$ the characterization theorem :

"The application $f : V \longrightarrow R^2$ (dim $V \geqslant 2$) is bi-stable if and only if its only singular points are transversal or tangent folds

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or transversal cusps and f restrict to the critical set $\Sigma(f)$ is injective", stated by J.P. Dufour [6] is incorrect as shown by the following examples.

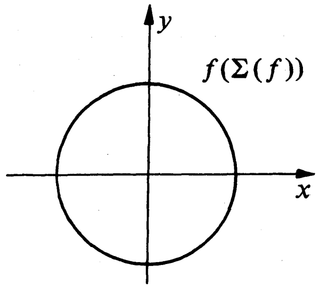
Exemple 1. – Consider

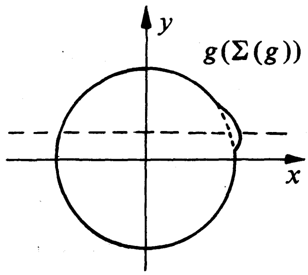
$M = S^2$

$\pi : R^3 \longrightarrow R^2$, $\pi(x, y, z) = (x, y)$

$f = \pi/M$

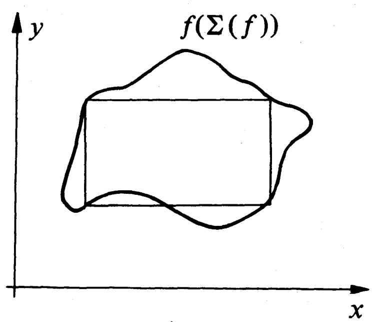
$f = (f_1, f_2)$





satisfies the conditions of the theorem and f_1, f_2 are stable. Otherwise f is not bi-stable since near f we have g with $g(\Sigma(g))$ like figure. Observe that f has two tangent folds in the same horizontal line, but g no. Then f is not bi-equivalent to g, since allowed conjugations preserves tangent folds and diffeomorphism on the target preserve horizontal and vertical lines.

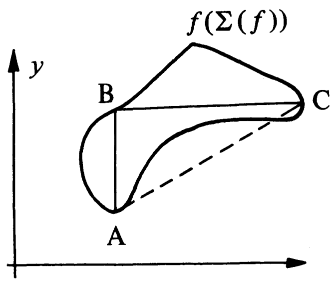
Examples 2 (Due to Leslie Wilson). – Let $f : M \longrightarrow R^2$ be such that $f(\Sigma(f))$ admits a inscribed rectangle with sides parallel to the axes, see figure(*). Then f is not bi-stable.



(*)

Wilson's proof is long and we omit it. He also proves that the bi-stable mappings are not dense. However he observes that there exist closed curves C in R^2 with inscribed rectangles and such that all nearby curve have inscribed rectangles. Also Wilson's argument can be extend for inscribed circuit with sides parallel to the axes, instead of rectangles.

Example 3. – Let $f : M \longrightarrow R^2$ be such that there is a triangule rectangle ABC as in figure(**), where A and C are images of tangent folds and $B \in f(\Sigma(f))$. Then f is not bi-stable.



(**)

The above examples and the theorem of Dufour give us a set of necessary conditions for the C^∞-bi-stability. Are they sufficient ? Does exist C^∞-bi-stable mappings if M is compact ?

The stability for divergent diagrams is very often applied to study family of manifolds and their envelopes, see [3] and [12], stability of maps between foliated manifolds and restrictions of maps to submanifolds. We wish to introduce a notion of stability which admits a version of the Preparation Theorem, the global theory works and is natural with respect to the above applications.

The material is presented in three parts. In the first we introduce the notion of ψ-stability, in the second a survey on the theory of globalization and in the third the characterization theorems of ψ-stability for low dimensions.

All manifolds and applications considered are C^∞ and the notation is the usual in Theory of Singularities.

1. Some Notions of Stability.

1. DEFINITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$. f *is* ψ*-infinitesimally stable if for a given* $w \in \theta(f)$, *there are* $u \in \theta(M)$, $v \in \theta(Q)$ *and* $\eta \in \theta(\pi)$ ($\pi : N \times Q \longrightarrow N$ *is the projection) such that :*

$$\begin{cases} w = (df)u + \eta \circ (f, \psi) \\ 0 = (d\psi)u + v \circ \psi . \end{cases}$$

Observation. – If ψ is infinitesimally stable, then f is ψ-infinitesimally stable if and only if for given $w_1 \in \theta(f)$ and $w_2 \in \theta(\psi)$ there are $u \in \theta(M)$, $v \in \theta(Q)$ and $\eta \in \theta(\pi)$ ($\pi : N \times Q \longrightarrow N$ is the projection) such that :

$$\begin{cases} w_1 = (df)u + \eta \circ (f, \psi) \\ w_2 = (d\psi)u + v \circ \psi . \end{cases}$$

2. DEFINITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$. f *is* ψ*-stable if there exists a nbhd* V_f of f (C^∞ *Whitney Topology*), *such that, for each* g *in* V_f *there are diffeomorphisms* h, k *and* ℓ *such that the diagram below commutes* (π*-usual projection*)

$$\begin{array}{ccccc} M & \xrightarrow{(g,\psi)} & N \times Q & \xrightarrow{\pi} & Q \\ \downarrow h & & \downarrow k & & \downarrow \ell \\ M & \xrightarrow{(f,\psi)} & N \times Q & \xrightarrow{\pi} & Q \end{array}$$

If there exists such that diagram we say that f *is* ψ*-equivalent to* g.

3. DEFINITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$. (f, ψ) *is* D*-stable if the diagram* $M \xrightarrow{(f,\psi)} N \times Q \xrightarrow{\pi} Q$ *is stable, see* [4], *this is, there are nbhds* V_f, V_ψ *and* V_π, *such that if* $g \in V_f$, $\varphi \in V_\psi$ *and* $\rho \in V_\pi$, *there are diffeomorphisms* h, k *and* ℓ *such that the diagram below commutes*

$$\begin{array}{ccccc} M & \xrightarrow{(g,\varphi)} & N \times Q & \xrightarrow{\rho} & Q \\ \downarrow h & & \downarrow k & & \downarrow \ell \\ M & \xrightarrow{(f,\psi)} & N \times Q & \xrightarrow{\pi} & Q \end{array}$$

4. PROPOSITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$ *with* ψ *stable, proper and* (f, ψ) *proper. Then* f *is* ψ*-stable iff* (f, ψ) *is* D-*stable.*

We wish to thank the referee for having pointed out how to suppress the hypothesis N compact, in the above proposition.

Finally we show that the ψ-stability is not equivalent to the D-stability in general.

Example. – Let $M = (-1, 0) \cup (0, 1)$, $N = (-1, 1)$, $Q = R$,
$f: M \longrightarrow N$, $f(x) = x$,
$\psi : M \longrightarrow Q$, $\psi(x) = x^3$.

Observations. – (i) ψ is not proper ;

(ii) ψ is infinitesimally stable ;

(iii) ψ is stable. There exists nbhd W_ψ of ψ such that if $\varphi \in W_\psi$, $\varphi'(x) > 0$, $\forall x \in M$;

(iv) f is C^1 ψ-stable. Let $\epsilon(x)$ be C^∞, flat at 0 with $\epsilon(0) = 0$, $\epsilon(x) > 0$ for $x \neq 0$, describing a nbhd V_f in M such that $\forall g \in V_f$, $g'(x) > 0$.

$$\text{Let } \bar{g} \text{ such that } \bar{g}(x) = \begin{cases} g(x), x \neq 0 \\ 0, \quad x \neq 0 \end{cases}$$

$$|\bar{g}(x) = x| \leqslant \epsilon(x), \ \forall x \in (-1, 1). \ (\bar{g} \text{ is } C^1).$$

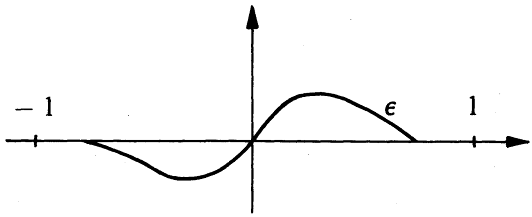
In the diagram take $h = \mathrm{Id}_M$ and $k(u, v) = (\bar{g}(u), v)$.

(v) The diagram $M \xrightarrow{(f,\psi)} N \times Q \xrightarrow{\pi} Q$, where $\pi(u, v) = v$, is infinitesimally stable.

Finally we show that the diagram $M \xrightarrow{(f,\psi)} N \times Q \xrightarrow{\pi} Q$ is not stable.

Take $\epsilon : R \longrightarrow R$ arbitrarily near the null application whose graph has the aspect shown in figure.

The equation $x^3 - \epsilon(x) = 0$ has two solutions (x_1, x_2) in M.



Take now $\rho : R \times R \longrightarrow R, \rho(u, v) = v - \epsilon(u)$. Therefore ρ is sufficiently near π.

Since

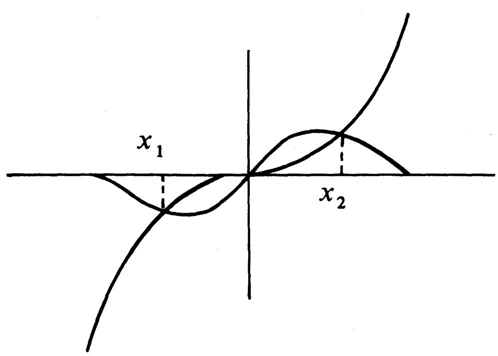
$$[\pi \circ (f, \psi)]^{-1}\{p\} = \{q\}, \forall p \in R - \{0\}$$

$$[\pi \circ (f, \psi)]^{-1}\{0\} = \phi$$

and

$$[\rho \circ (f, \psi)]^{-1}\{0\} = \{x_1, x_2\}$$

the diagram is not stable.



Remark. – In fact it can be shown that f is C^r ψ-stable.

Envelopes. – Let X be a n-dimensional manifold and take $R^q \xleftarrow{\pi} X \xrightarrow{f} R^m$, where π is a fibration and f restrict to the fibers $\pi^{-1}(y)$ is one to one immersion. Following R. Thom [12] we have a family of manifolds in R^m of codimension q, and the envelope of this family is $E = f(\Sigma(f))$, with $\Sigma(f) = \{x \in X / df_x$ is not surjective$\}$.

The equivalence between families follows from that of divergent diagrams. Such an equivalence assigns the correspondent manifolds and the correspondent envelopes of the families.

The ψ-equivalence defined is weaker than the concept of divergent diagram, but it is still good enough to define an equivalence between envelopes, as we sketch below for families of plane curves.

Let $\varphi(x, y, t) = 0$ be one-parameter family in the $0xy$ plane. Classically the envelope is given by elimination of the parameter t from the equations $\varphi(x, y, t) = 0$ and $\varphi_t(x, y, t) = 0$. Take $R \xleftarrow{\varphi} R^3 \xrightarrow{\pi} R^2$, with $\pi(x, y, t) = (x, y)$ and define $X = \varphi^{-1}(0)$, $f = \pi/X$ and $p : R^3 \longrightarrow R$ by $p(x, y, t) = t$. Then the given family is $R \xleftarrow{p} X \xrightarrow{f} R^2$, with the envelope $E(\varphi) = \pi(\Sigma(f))$, where $\Sigma(f) = \Sigma(\pi, \varphi) \cap \varphi^{-1}(0)$.

If $\varphi(x, y, t) = 0$ and $\psi(x, y, t) = 0$ are one parameters families in the plane, we say that they are equivalent if there exist diffeomorphisms $h : R^3 \longrightarrow R^3$ and $k : R^2 \times R \longrightarrow R^2 \times R$, such that the diagram

$$\begin{array}{ccc} R^3 & \xrightarrow{(\pi, \varphi)} & R^2 \times R \\ \downarrow & & \downarrow \\ R^3 & \xrightarrow{(\pi, \psi)} & R^2 \times R \end{array}$$

commutes. Here $k(u, v, w) = (k_1(u, v, w), k_2(u, v, w), k_3(w))$ and $k_3(0) = 0$. Observe that such an equivalence preserves the envelopes.

The φ-stability of π in the restricted sense used here is equivalent to the stability of $\pi_\varphi = \pi/\varphi^{-1}(0)$, and following Martinet [10], this is equivalent to v-versality of φ. Then the φ-stability corresponds to the versality as is studied by Bruce [2].

For illustration consider $r : R, 0 \longrightarrow R^2, 0$ an arc lenght immersion, see [2]. The family of normal lines defined by $\psi(x_1, x_2, s) = (x - r(s)).\dot{r}(s) = 0$, with $x = (x_1, x_2)$. The envelope is $x = r(s) + k^{-1}(s)\eta(s)$. If $\frac{\partial^2 \psi}{\partial s^2} \neq 0$ we have a regular point of the envelope as a transverse fold of $(x_1, x_2, \psi(x_1, x_2, s))$. The singularities of the envelope occur when $\frac{\partial^2 \psi}{\partial s^2} = 0$. When $\frac{\partial^3 \psi}{\partial s^3} \neq 0$, such a singularity is a transverse cusp. These singularities are the only with stability, see [7].

If $k(s) \neq 0$ is the curvature, $r(0) = 0$ we get $\frac{\partial^2 \psi}{\partial s^2} = \frac{\dot{k}(s)}{k(s)}$ and $\psi(0, k^{-1}(0), s) = \frac{\ddot{k}}{6k} s^3 + o(4)$.

Then if $k(s) \neq 0$ and $\dot{k}(s) \neq 0$ we obtain a transverse fold. If $k(s) \neq 0$, $\dot{k}(s) = 0$ and $\ddot{k}(s) \neq 0$ we have a transverse cusp.

2. ψ-Infinitesimal Stability and ψ-Homotopy Stability.

In this paragraph, we will suppose M is a compact manifold.

1. Definition. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$. *A deformation* $F : M \times J \longrightarrow M \times J$ *of* f *is* ψ*-trivial if there exist diffeomorphisms* h, k *and* ℓ *such that* $h(x, 0) = (x, 0)$, $\ell(z, 0) = (z, 0)$, $k(y, 0; z, 0) = (y, 0; z, 0)$, h, k *and* ℓ *keeping level and such that the diagram below commutes* ($\Psi = \psi \times \mathrm{Id}$)

$$\begin{array}{ccccc} M \times I & \xrightarrow{(F, \Psi)} & (N \times I) \times (Q \times I) & \xrightarrow{\pi} & Q \times I \\ \downarrow h & & \downarrow k & & \downarrow \ell \\ M \times I & \xrightarrow{(f \times \mathrm{Id}, \psi)} & (N \times I) \times (Q \times I) & \xrightarrow{\pi} & Q \times I \end{array}$$

(*where* $I = (-\delta, \delta) \subset J$).

f *is* ψ*-homotopically stable if every deformation of* f *is* ψ*-trivial.*

Observation. – Let ψ be proper and stable, in the C^∞ Whitney Topology. Then f is ψ-homotopically stable if and only if for given deformations F of f and Φ of ψ there exist diffeomorphisms h, k and ℓ such that $h(x, 0) = (x, 0)$, $\ell(z, 0) = (z, 0)$ and $k(y, 0; z, 0) = (y, 0; z, 0)$, h, k and ℓ keeping level and such that the diagram below commutes.

$$\begin{array}{ccccc} M \times I & \xrightarrow{(F\,,\,\Phi)} & (N \times I) \times (Q \times I) & \xrightarrow{\pi} & Q \times I \\ \downarrow h & & \downarrow k & & \downarrow \ell \\ M \times I & \xrightarrow{(f \times Id\,,\,\psi \times Id)} & (N \times I) \times (Q \times I) & \xrightarrow{\pi} & Q \times I \end{array}$$

Let $g : M \longrightarrow P$ and G be a deformation of g. Define the vector field along G

$$\tau_G = (dG)\left(\frac{\partial}{\partial t}\right) - G^*\left(\frac{\partial}{\partial t}\right)$$

where $\dfrac{\partial}{\partial t}$ is the usual vector field on $M \times R$ or $P \times R$ as required.

2. PROPOSITION (Thom-Levine). – *Let* F *be a deformation of* f. *Then* F *is* ψ-*trivial if and only if there exist* $I = (-\delta\,,\delta)$, $\xi \in \theta(M \times I)$, $\eta \in \theta(Q \times I)$ *and* $\gamma \in \theta(\pi)$ ($\pi : (N \times I) \times (Q \times I) \longrightarrow N \times I$ *usual projection) satisfying*

(i) ξ, η *and* γ *with* R-*component zero ;*

(ii) $\tau_F = (dF)(\xi) + \gamma \circ (F\,,\Psi)$;

(iii) $0 = (d\Psi)(\xi) + \eta \circ \gamma$

on $M \times I$, *where* $\Psi = \psi \times Id$.

This proposition enables us to obtain the equivalence between the ψ-homotopy stability and the ψ-infinitesimal stability as we show below.

3. PROPOSITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$. *If* f *is* ψ-*homotopically stable then* f *is* ψ-*infinitesimally stable.*

4. PROPOSITION. – *If* f *is* ψ-*infinitesimally stable at* $p \in M$ *and* ψ *is infinitesimally stable then there exist germs of vector fields* ξ, η *and* γ *with* R-*component zero such that*

$$\begin{cases} \tau_F = (dF)\,\xi + \gamma \circ (F\,,\Phi) \\ \tau_\Phi = (d\Phi)\,\xi + \eta \circ \Phi \end{cases}$$

on the germ level at $(p, 0)$, *where* F *is a deformation of* f *and* Φ *is a deformation of* ψ. *In this case, we say that* (F, Φ) *is a deformation of* (f, ψ).

Proof. – Let

$$Z = \left\{ \text{germs of vector fields } w : M \times R \longrightarrow T((N \times R) \times (Q \times R)) \text{ along } (F, \Phi) \text{ at } (p, 0) \text{ with R-component zero} \right\}.$$

$$K = \left\{ d(F, \Phi) \,|\xi|_{(p,0)}/\xi \text{ is a vector field on } M \times R \text{ with R-component zero} \right\}$$

$A = Z/K$ is a finitely generated module over $C^{\infty}_{(p,0)}(M \times R)$.

A is a module over $C^{\infty}_{(\psi(p),0)}(Q \times R)$, via Φ^*. We claim that A is also a finitely generated $C^{\infty}_{(\psi(p),0)}(Q \times R)$-module with generators given by :

$$e_i = \text{projection of } \left(\frac{\partial}{\partial y_i}\bigg|_F , 0 \right) \text{ in A}$$

$$\bar{e}_j = \text{projection of } \left(0 , \frac{\partial}{\partial z_j}\bigg|_\Phi \right) \text{ in A}$$

$$(i = 1, \ldots, n ; j = 1, \ldots, q).$$

The claim follows from the Malgrange Preparation Theorem. Now we show that the claim is sufficient to prove the proposition. In A,

$$(\tau_F , \tau_\Phi) = \left(\sum_{i=1}^{n} (\gamma_i \circ \Phi) \frac{\partial}{\partial y_i}\bigg|_F ; \sum_{j=1}^{q} (\eta_j \circ \Phi) \frac{\partial}{\partial z_j}\bigg|_\Phi \right).$$

Thus in Z,

$$\tau_F = (dF)\,\xi + \sum_{i=1}^{n} (\gamma_i \circ \Phi) \frac{\partial}{\partial y_i}\bigg|_F$$

$$\tau_\Phi = (d\Phi)\,\xi + \left(\sum_{j=1}^{q} \eta_j \frac{\partial}{\partial z_j} \right) \circ \Phi .$$

Now $\eta = \sum_{j=1}^{q} \eta_j \frac{\partial}{\partial z_j}$ has R-component equal to zero and $\tau_\Phi = (d\Phi)\,\xi + \eta \circ \Phi$ on the germ level at $(p, 0)$.

Also

$$\tau_F = (dF)\,\xi + \Big(\sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial y_i}\Big|_{\pi} \Big) \circ (F , \Phi)$$

where $\tilde{\gamma}_i(\bar{x}, \bar{y}) = \gamma_i(\bar{y})$.

Now $\gamma = \sum_{i=1}^{n} \tilde{\gamma}_i \frac{\partial}{\partial y_i}\Big|_{\pi}$ is such that $\tau_F = (dF)\,\xi + \gamma \circ (F , \Phi)$ on the germ level at $(p, 0)$, completing the proof.

This proposition is true for germs at $S = \{p_1, \ldots, p_2\} \subset \psi^{-1}(q)$.

Now we introduce two singular sets in M, and we show how to solve the equations near then.

5. DEFINITION. – *For the diagram* $Q \xleftarrow{\psi} M \xrightarrow{f} N$ *we consider :* Σ *the set of all* $p \in M$ *such that we can not solve the system of equations* $w_1 = (df)u$, $w_2 = (d\psi)u$ *as germ at* p.

Σ_1 *the set of all* $p \in M$ *such that we can not solve the system of equations* $w_1 = (df)u + v \circ (f, \psi)$, $w_2 = (d\psi)u$ *as germ at* p.

6. LEMMA. – *If* $\Sigma = \emptyset$ *and* $(F, \Phi) : M \times R \longrightarrow (N \times R) \times (Q \times R)$ *is a deformation of* (f, ψ), *then for given* $w_1 \in \theta(F)$ *and* $w_2 \in \theta(\Phi)$, *with* R*-component zero, there exists* $\xi \in \theta(M \times I)$, *with* R*-component zero such that*

$$\begin{cases} w_1 = (dF)\,\xi \\ w_2 = (d\Phi)\,\xi . \end{cases}$$

7. PROPOSITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$, *where* ψ *is infinitesimally stable and* f *is* ψ*-infinitesimally stable. Then* Σ_1 *is closed and* $\Sigma_1 \cap \psi^{-1}(q)$ *is a finite set,* $\forall q \in Q$.

Proof. – Let P_2^q be the set of all $(y, q) \in N \times \{q\}$ such that we can not solve the system of equations $w_1 = d(f, \psi)\xi + \gamma \circ (f, \psi)$, $w_2 = (d\pi)\gamma$ as germs at $(f, \psi)^{-1}(y, q)$ and at (y, q).

This is a finite set, see [4].

We have $\Sigma_1 \cap \psi^{-1}(q) \subset \bigcup_{(y,q) \in P_2^q} \Sigma \cap (f, \psi)^{-1}(y, q)$ as a finite union of finite sets.

Now we show that Σ_1 is closed. Take $x_0 \in M - \Sigma_1$ and let $(y_0, q_0) = (f, \psi)(x_0)$. Given $(w, \overline{w}) \in \theta(f, \psi)_{x_0}$ we have :

$$(w, \overline{w}) = d(f, \psi)\, u + (v(f, \psi), 0).$$

From the Malgrange Preparation Theorem and following [8], this is equivalent to

$$J^k \theta(f, \psi)_{x_0} = d(f, \psi)_{x_0} J^k \theta(M)_{x_0} + (f, \psi)^* J^k(\theta(N) \times 0)_{(y_0, q_0)}.$$

Then, consider the application

$$\widetilde{(f, \psi)} : J^k \theta(M)_{x_0} \oplus J^k(\theta(N) \times 0)_{(y_0, q_0)} \longrightarrow J^k \theta(f, \psi)_{x_0},$$

given by $[d(f, \psi) + (f, \psi)^*]^k$. It is surjective.

In local coordinates $\widetilde{(f, \psi)}$ is a linear map of $B^k_{m,n} \oplus B^k_{n+k,n} \longrightarrow B^k_{m,n+k}$, where $B^k_{r,s}$ is the vector space of polynomials maps from R^r to R^s of degree at most k. Since $\widetilde{(f, \psi)}$ is continuous in x, it follows that $\widetilde{(f, \psi)}$ is onto at every point in the neighbourhood of x_0. Then $M - \Sigma_1$ is open and the proof is complete.

8. PROPOSITION. – *Let ψ be infinitesimally stable, f ψ-infinitesimally stable and (F, Φ) a deformation of (f, ψ). Then there exist $\xi \in \theta(M \times R)$, $\eta \in \theta(Q \times R)$ and $\gamma \in \theta(\pi)$ $(\pi : (N \times R) \times (Q \times R) \longrightarrow N \times R$, usual projection with R-component zero such that*

$$\begin{cases} \tau_F = (dF)\xi + \gamma \circ (F, \Phi) \\ \tau_\Phi = (d\Phi)\xi + \eta \circ \Phi \end{cases}$$

on a nbhd $W \subset M \times R$ of $\Sigma_1 \times \{0\}$.

The proof is achieved using a usual partition of unity argument and the fact that $\Sigma_1 \cap \psi^{-1}(q)$ is a finite set.

9. PROPOSITION. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$, *where ψ is infinitesimally stable, f is ψ-infinitesimally stable and U is a nbhd of $\Sigma_1 \times \{0\}$. Let still (F, Φ) be a deformation of (f, ψ). Given $\tau \in \theta(F)$ and $\mu \in \theta(\Phi)$ with R-component zero, there exist $\xi \in \theta(M \times R)$ and $\gamma \in \theta(\pi)$, with R-component zero such that*

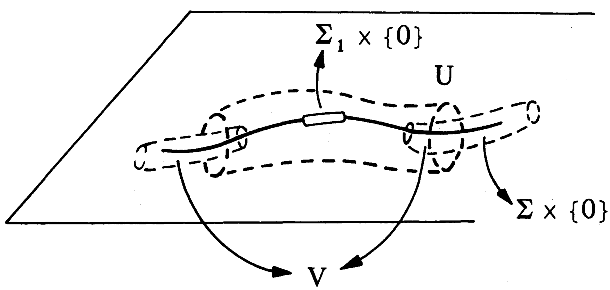
$$\begin{cases} \tau = (dF)\,\xi + \gamma \circ (F\,,\Phi) \\ \mu = (d\Phi)\,\xi \end{cases}$$

on a nbhd V *of* $(\Sigma \times \{0\}) - U$.

Note that if $(p\,,0) \in (\Sigma \times \{0\}) - U$ then the equations

$$\begin{cases} w_1 = (df)\,\xi + \gamma \circ (f\,, \psi) \\ w_2 = (d\,\psi)\,\xi \end{cases}$$

have solutions on the germ level at p , $\forall\, w_1$, w_2 .



Using the Malgrange Preparation Theorem and a usual partition of unity argument we get the result.

10. THEOREM. – *Let* $Q \xleftarrow{\psi} M \xrightarrow{f} N$, *where* ψ *is infinitesimally stable,* f *is* ψ*-infinitesimally stable and* $(f\,,\psi)(\Sigma - \Sigma_1) \cap (f\,,\psi)(\Sigma_1) = \phi$. *Let* $(F\,,\Phi)$ *be a deformation of* $(f\,,\psi)$. *Then, there exist* $\xi \in \theta(M \times I)$, $\eta \in \theta(Q \times I)$ *and* $\gamma \in \theta(\pi)$ $(\pi : (N \times I) \times (Q \times I) \longrightarrow N \times I$, *usual projection) with* R*-component zero, such that*

$$\begin{cases} \tau_F = (dF)\,\xi + \gamma \circ (F\,,\Phi) \\ \tau_\Phi = (d\Phi)\,\xi + \eta \circ \Phi \end{cases} \quad on\ M \times I.$$

Proof. – From proposition 8, there are ξ', η' and γ' satisfying

(i) ξ', η' and γ' with R-component zero

$$\text{(ii)}\ \begin{cases} \tau_F = (dF)\,\xi' + \gamma' \circ (F\,,\Phi) \\ \tau_\Phi = (d\Phi)\,\xi' + \eta' \circ \Phi \end{cases}$$

on a nbhd W of $\Sigma_1 \times \{0\}$.

Then

$$\begin{cases} \tau_F - (dF)\,\xi' - \gamma' \circ (F, \Phi) = (dF)\,\xi_0 + \gamma_0 (F, \Phi) \\ \tau_\Phi \quad - (d\Phi)\,\xi' - \eta' \circ \Phi = (d\Phi)\,\xi_0 \end{cases}$$

on the nbhd W *of* $\Sigma_1 \times \{0\}$.

Take a countable collection of open sets U_i such that :

$$\forall i \in N,\ \overline{U}_i \subset W,\ U_i \text{ nbhd of } \Sigma_1 \times \{0\},$$

$$U_{i+1} \subset U_i \text{ and } \bigcap_{i \in N} \overline{U}_i = \Sigma_1 \times \{0\}.$$

We have

$$\begin{cases} \tau_F - (dF)\,\xi' - \gamma' \circ (F, \Phi) = (dF)\,\xi_i + \gamma_i \circ (F, \Phi) \\ \tau_\Phi - (d\Phi)\,\xi' - \eta' \circ \Phi = (d\Phi)\,\xi_i \end{cases}$$

on the nbhd V_i of $(\Sigma \times \{0\}) - U_i$ (apply successively the proposition 9).

Let

$$W' = (N \times R) \times (Q \times R) - (F, \Phi)\,((\Sigma \times \{0\}) - W)$$

$$V_i' = (N \times R) \times (Q \times R) - (F, \Phi)\,((\Sigma \times \{0\}) - V_i)$$

$\{W', V_1', V_2', \ldots\}$ is an open covering of $(N \times R) \times (Q \times R)$.

Let $\rho_0, \rho_1, \rho_2, \ldots$ be a C^∞ partition of unity on $(N \times R) \times (Q \times R)$ which is subordinate to covering $W', V_1', V_2', \ldots$

Then

$$\begin{cases} \tau_F - (dF)\,\xi' - \gamma' \circ (F, \Phi) = (dF)\,\tilde{\xi} + \tilde{\gamma} \circ (F, \Phi) \\ \tau_\Phi - (d\Phi)\,\xi' - \eta' \circ \Phi = (d\Phi)\,\tilde{\xi} \end{cases}$$

on $\widetilde{W} = W_0 \cap [\bigcap_{i \in N} V_i]$ where

$$W_0 = W \cup (F, \Phi)^{-1}\,[(N \times R) \times (Q \times R) - \operatorname{supp} \rho_0]$$

$$\widetilde{V}_i = V_i \cup (F, \Phi)^{-1}\,[(N \times R) \times (Q \times R) - \operatorname{supp} \rho_i]\ ;\ i = 1, 2, \ldots$$

$$\tilde{\xi} = \Sigma\, \rho_i \circ (F, \Phi)\, \xi_i$$

$$\tilde{\gamma} = \Sigma\, \rho_i\, \gamma_i\,.$$

Thus

$$\begin{cases} \tau_F = (dF)\,\xi_1 + \gamma \circ (F\,,\Phi) \\ \tau_\Phi = (d\Phi)\,\xi_1 + \eta \circ \Phi \end{cases}$$

on $\widetilde{W}$, where ξ_1, η and γ have R-component zero.

We have that $\Sigma \times \{0\} \subset \widetilde{W}$ and that $\widetilde{W}$ is open, since it is intersection of open sets of a locally finite family.

If $\dim M < \dim (N \times Q)$, then $\Sigma = M$ and the theorem is proved. So assume $\dim M \geqslant \dim (N \times Q)$, applying Lemma 6 for $(f\,,\psi)/M - \Sigma$, we have

$$\begin{cases} \tau_F - (dF)\,\xi_1 - \gamma \circ (F\,,\Phi) = (dF)\,\bar{\xi} \\ \tau_\Phi - (d\Phi)\,\xi_1 - \eta \circ \Phi = (d\Phi)\,\bar{\xi} \end{cases}$$

on $(M - \Sigma) \times I$.

Let $p : M \times I \longrightarrow R$ be differentiable such that $p \equiv 0$ on $\Sigma \times \{0\}$ and $p \equiv 1$ outside of $\widetilde{W}$. Thus $p\bar{\xi}$ is globally defined and we have :

$$\begin{cases} \tau_F = (dF)\,(\xi_1 + p\bar{\xi}) + \gamma \circ (F\,,\Phi) \\ \tau_\Phi = (d\Phi)\,(\xi_1 + p\bar{\xi}) + \eta \circ \Phi \end{cases}$$

on $M \times I$, completing the proof.

Now we get a partial converse of the proposition 3.

11. THEOREM. – *Let* $Q \xleftarrow{\ \psi\ } M \xrightarrow{\ f\ } N$. *If* ψ *is infinitesimally stable,* f *is* ψ*-infinitesimally stable and* $(f, \psi)(\Sigma - \Sigma_1) \cap (f, \psi)(\Sigma_1) = \phi$ *then* f *is* ψ*-homotopically stable.*

Proof. – Note that if $\Phi = \psi \times \mathrm{Id} = \Psi$ then $\tau_\Phi = 0$. Thus applying Theorem 10 we have that

$$\begin{cases} \tau_F = (dF)\,\xi + \gamma \circ (F\,,\Psi) \\ 0 = (d\Psi)\,\xi + \eta \circ \Psi \end{cases} \qquad \text{on } M \times I$$

where ξ, η and γ have R-component zero.

From proposition 2 we have that f is ψ-homotopically stable.

Note. – We don't know if the condition

$$(f, \psi)(\Sigma - \Sigma_1) \cap (f, \psi)(\Sigma_1) = \phi$$

can be eliminated.

3. Characterizations Theorems.

In this paragraph we characterize the applications $f : M \longrightarrow R$ which are ψ-stable, where M is a compact manifold and $\psi : M \longrightarrow R$ is stable.

Case $R \xleftarrow{\psi} M \xrightarrow{f} R$, $\dim M \geqslant 2$.

1. DEFINITION. – *Let p a fold point of $(f, \psi) : M \longrightarrow R^2$ (respectively a cusp point). p is a transversal fold (respectively a transversal cusp) if ψ is regular at p. p is a tangent fold if ψ admits p as nondegenerate critical point.*

Following Dufour [6], it is possible to obtain :

2. LEMMA. – *Let $(f, \psi) : M \longrightarrow R^2$ with p a transversal fold. Then (f, ψ) is locally ψ-equivalent to*

$$(x_1, y, x_3, \dots, x_n) \longrightarrow \Big(\sum_{i \neq 2} \pm x_i^2, y\Big) \text{ at } 0.$$

If p is a tangent fold, then (f, ψ) is locally ψ-equivalent to

$$(x, x_2, \dots, x_n) \longrightarrow \Big(x, x^2 + \sum_{i=2} \pm x_i^2\Big) \text{ at } 0.$$

If p is a transversal cusp, then (f, ψ) is locally ψ-equivalent to

$$(x, y, x_3, \dots, x_n) \longrightarrow \Big(\frac{x^3}{3} + xy + \sum_{i=3}^{n} \pm x_i^2, y\Big) \text{ at } 0.$$

3. LEMMA. – *Let $(f, \psi) : M \longrightarrow R^2$ and $p \in M$.*

– *If p is a transversal cusp then $p \in \Sigma_1$.*

– *If* p *is a transversal fold then* $p \in \Sigma - \Sigma_1$.

– *If* p *is a tangent fold then* $p \in \Sigma_1$.

Proof. – Let us prove for transversal cusps. The normal form in this case is :

$$(x, y, x_3, \ldots, x_n) \longrightarrow \left(\frac{x^3}{3} + xy + \sum_{i=3}^{n} \pm x_i^2, y\right).$$

Suppose that $p \in M - \Sigma_1$, then for any θ, it is possible to solve the system :

$$\begin{cases} \theta(x, y, x_3, \ldots, x_n) = (x^2 + y)\,X(x, y, x_3, \ldots, x_n) \\ \quad + xY(x, y, \ldots, x_n) + \sum_{i=3}^{n} \pm 2x_i\, X_i(x, y, \ldots, x_n) \\ \quad\quad + U\left(\frac{x^3}{3} + xy + \sum_{i=3}^{n} \pm x_i^2, y\right) \\ 0 = Y(x, y, x_3, \ldots, x_n). \end{cases}$$

This system is equivalent to

$$\theta(x, y) = (x^2 + y)\,X(x, y) + U\left(\frac{x^3}{3} + xy, y\right)$$

and this is equivalent to

$$\mu(x) = \beta\left(-\frac{2}{3}x^3, -x^2\right).$$

But this is impossible.

Following Dufour [6] we can prove :

4. PROPOSITION. – *The set* Ω *of all maps* $(f, \psi) : M \longrightarrow R^2$ *which have as singular point only tangent or transversal folds or transversal cusps is open and dense in* $C^\infty(M, R^2)$ *with the Whitney Topology.*

5. THEOREM. – *Let* $\psi : M \longrightarrow R$ *be stable. Then* $f : M \longrightarrow R$ *is* ψ*-stable if and only if :*

(i) (f, ψ) *have only tangent or transversal folds or transversal cusps as singular points.*

(ii) $(f, \psi)^{-1}(x_0, y_0) \cap \Sigma(f, \psi)$ *is empty or one point or two transversal folds.*

(iii) *Images of fold curves intersect transversally.*

Proof. –

Necessity

Since Ω is invariant and dense in $C^\infty(M, R^2)$, it follows that $(f, \psi) \in \Omega$ and we get the first condition. The last two come from the normal crossing conditions see [8], p. 158.

Sufficiency

The set of all maps satisfying the three conditions is open. Then, to prove that such a map (f, ψ) is ψ-stable is sufficient to prove that it is ψ-homotopically stable.

From proposition 2, § 2, it is enough to solve for each deformation (F, Φ) of (f, ψ) the equations

$$\begin{cases} \tau_F = (dF)\,\xi + \gamma \circ (F, \Phi) \\ \tau_\Phi = (d\Phi)\,\xi + \eta \circ \Phi . \end{cases}$$

Let us prove the case of transversal cusp. In this case the ψ-infinitesimal stability is given by

$$\begin{aligned} \theta(x, y, x_3, \ldots, x_n) = {} & (x^2 + y)\, X(x, y, \ldots, x_n) - x V(y) \\ & + \sum_{i=3}^{n} \pm 2x_i\, X_i(x, y, x_3, \ldots, x_n) \\ & + U\Big(\frac{x^3}{3} + xy + \sum_{i=3}^{n} \pm x_i^2 \,,\, y\Big). \end{aligned}$$

This equation is equivalent to

$$\theta(x, y) = (x^2 + y)\, X(x, y) - x V(y) + U\Big(\frac{x^3}{3} + xy \,,\, y\Big) \qquad (*)$$

which is equivalent to

$$\mu(x) = x\alpha(-x^2) + \beta\Big(-\frac{2}{3}x^3 \,,\, -x^2\Big), \qquad (**)$$

as we show below.

To get (**) from (*) it is enough to take $y = -x^2$.

Reciprocally, given $\theta(x, y)$ we get

$$\theta(x, -x^2) = -xV(-x^2) + U\left(-\frac{2}{3}x^3, -x^2\right).$$

Then

$$\gamma(x, y) = \theta(x, y) + xV(y) - U\left(\frac{x^3}{3} + xy, y\right) \qquad (\Delta)$$

is zero for $y = -x^2$. From the Division Theorem we have :

$$\gamma(x, y) = (x^2 + y)Z(x, y) + \Gamma(x)$$

where $\gamma(x, -x^2) = \Gamma(x) = 0$.

Then $\gamma(x, y) = (x^2 + y)Z(x, y)$.

Combined with (Δ) we have

$$\theta(x, y) = (x^2 + y)Z(x, y) - xV(y) + U\left(\frac{x^3}{3} + xy, y\right)$$

which is the equation (*).

Now the ψ-infinitesimal stability is equivalent to solve the equation (**), which we know to be possible.

Now from proposition 4, § 2, we have the local ψ-homotopy stability.

From Lemma 3 a map satisfying the hypothesis of the theorem is such that $(f, \psi)(\Sigma - \Sigma_1) \cap (f, \psi)(\Sigma_1) = \phi$.

Now, using the same globalization techniques as in Theorem 10, § 2, we have that (f, ψ) is ψ-homotopically stable, and this completes the proof.

Case $R \xleftarrow{\psi} M \xrightarrow{f} R$, $\dim M = 1$.

6. DEFINITION. – *Let* $(f, \psi) : M \longrightarrow R^2$.

$p \in M$ *is of the type* ψ*-singular if it is a singular point of* ψ.

7. THEOREM. – *Let* $\psi : M \longrightarrow R$ *be stable.*

$f : M \longrightarrow R$ *is* ψ*-stable iff* (f, ψ) *is an immersion with normal crossing which avoid* ψ*-singular values.*

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