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TOPICS ON KRONECKER SETS

by R. KAUFMAN

In the first part of this note we consider relations between classes of differentiable functions and linear Kronecker sets. The problem in each of three theorems is to find a set E of some narrow type, and a differentiable map φ, so that $\varphi(E)$ is a Kronecker set. The first theorem focusses on functions φ in a prescribed non-quasi-analytic class $C(M_n)$, as described in [2, V] and [6, Ch. 19]. The second deals with a qualitative study of φ', and the third uses van der Corput's inequality to impose a very strong condition on E.

The second part illustrates the use of our method for constructing Kronecker sets; the lesson is that many special phenomena of exceptional sets are present in each set of multiplicity. Here we have in mind the work of Körner [5], and apply our method to strengthen a theorem on the union of two Kronecker sets.

1.

Let E be a compact subset of $[0, 1]$; the exact condition on E is known, that there be a function φ of class $C^1[0, 1]$ so that $\varphi' > 0$ and $\varphi(E)$ is a Kronecker set [3; 1, VII; 4]. There exist, however, sets E of this type such that $\varphi(E)$ is an M_0-set whenever $\varphi' > 0$ and $\varphi \in C^2$. Thus there is some interest in sets E for which $\varphi \in C^\infty$ can be chosen as before; in the next theorem, we prove what is perhaps an extremal result on the possible smoothness of φ. Let (λ_n) be an increasing sequence of positive numbers and $M_n = \lambda_1 \ldots \lambda_n$.

THEOREM 1. — *Let* F *be an* M_0*-set, and let* $\Sigma\lambda_n^{-1} < \infty$. *Then there is a* M_0*-set* $E \subseteq F$, *a function* φ *of class* $C^\infty(R)$ *so that*

(*a*) $\varphi' \geqslant 1$ *everywhere,* $\varphi^{(n)} = 0(M_n)$ *uniformly* $(n \geqslant 1)$.

(*b*) $\varphi(E)$ *is a Kronecker set.*

To explain the significance of the condition $\Sigma\lambda_n^{-1} < \infty$, let ψ be C^∞ of compact support and $P_n = \|\psi^{(n)}\|_2$ so that by the Plancherel theorem (P_n) is log-convex [2] and by the Denjoy-Carleman theorem $\Sigma P_n P_{n+1}^{-1} < \infty$ [2, 6]. Thus the condition on (λ_n) is essential for the technique of partitions of unity, and Theorem 1 may be best-possible among results valid for *all* M_0-sets E.

In the proof of Theorem 1 we take a sequence (a_n) decreasing to 0 so that $\Sigma(a_n\lambda_n)^{-1} < \infty$. Then there is a function ψ of compact support, $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on a neighbourhood of 0, and $|\psi^{(n)}| = 0(a_1\lambda_1 \ldots a_n\lambda_n)$ [2, 6]. Next we can expand the interval on which $\psi = 1$ at will, preserving the inequality $0 \leqslant \psi \leqslant 1$ and the inequalities on $\psi^{(n)}$. Because $\lim a_n = 0$, each homothety of ψ, say $\varphi(x) = \psi(Ax + B)$ also satisfies the inequalities $|\psi_2^{(n)}| \leqslant C_A M_n$ $(0 \leqslant n < \infty)$, with C_A a function of A alone. In particular, if $r < r_1 < s_1 < s$, one of the homotheties equals 1 on (r_1, s_1) and 0 outside (r, s).

After a few reductions we can suppose that F is a closed, totally disconnected subset of $[0, 1]$ and μ is a probability measure in F with $\hat{\mu}$ in $C_0(R)$. Let $(f_m)_1^\infty$ be a dense sequence in the real Banach space $C[0, 1]$, and let $(h_m)_1^\infty$ be a sequence of 2π-periodic C^∞-functions of mean 1, such that $h_m \geqslant 0$ and $h_m(t) = 0$ when $m^{-1} \leqslant |t| \leqslant \pi - m^{-1}$. Thus for any function g and real number y, $h_m(yg - f_m) = 0$ except on the set $(|\exp iyg - \exp if_m| \leqslant m^{-1})$.

We shall construct a sequence of measures μ_m, beginning with $\mu_0 = \mu$ and

$$(1) \qquad \mu_m = h_m(y_m g_m - f_m) \cdot \mu_{m-1}$$

where $1 < y_1 < \cdots < y_m < \cdots$ and g_m converges so rapidly to a limit φ that $y_m|g_m - \varphi| \leqslant 2m^{-1}$. Also, $|\hat{\mu}_m - \hat{\mu}_{m-1}| < 3^{-m}$ so that the weak limit σ of the sequence

(μ_m) has norm in $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\hat{\sigma}$ is in $C_0(R)$. On the support of σ, which is contained in the support of μ_m, we have $|\exp iy_m\varphi - \exp if_m| \leqslant 2m^{-1}$, and Theorem 1 will be proved by constructing the limit φ with the special properties listed.

Each h_m has an absolutely convergent Fourier expansion $h_m(t) = \Sigma a_k \exp ikt$, $a_0 = 1$. Thus

$$\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u) = \Sigma' a_k \int \exp ik(y_m g_m - f_m) . \exp - iut . d\mu_{m-1}.$$

Now each g_m shall be constructed so that $g_m'' = 0$ on a neighbourhood of F; hence there are disjoint intervals I_j, covering F, on which $g_m' = b_j$, say. We express the integral above as a sum of integrals over I_j's. The j-th term in the k-th integral has the same modulus as $\hat{\nu}(u - ky_m b_j)$, where $\nu \equiv \nu(k, j)$ is absolutely continuous with respect to μ_0 and $\sum_j \|\nu(k, j)\| = \|\mu_{m-1}\|$ for all $k \neq 0$. Each $\hat{\nu} \in C_0(R)$, whence to each $\varepsilon > 0$ there is a T so that for all u

$$|\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u)| \leqslant \varepsilon + \Sigma''|a_k| . \|\nu(k, j)\|,$$

where Σ'' means summation over the set of indices

$$k \neq 0, \qquad |u - ky_m b_j| < T.$$

Next, let us suppose that the values b_j, of $g_{m'}$ on I_j, are distinct, differing among each other by at least $r > 0$. Each inequality $|u - kyb_j| < T$ has at most one solution j for each $k \neq 0$, as soon as $yr > 2T$. When this condition is imposed, the sum Σ'' does not exceed $\max \mu_{m-1}(I_j) . \Sigma|a_k|$.

From this point is clear how to proceed. Let $g_0 = x$ and suppose that g_{m-1} is a known function of class C^∞, with $g_{m-1}'' = 0$ on a neighbourhood of F. Let (I_j) be a covering of F by disjoint intervals so that $\mu_{m-1}(I_j)\Sigma|a_k| < 4^{-m}$, say. Then there is a function $p \geqslant 0$ of compact support, so that $p = c_j > 0$ on I_j, where the numbers c_j are distinct; the indefinite integral q of p, with $q(-\infty) = 0$, has the property that $|q| = 0(1)$ while

$$|q^{(n+1)}| = |p^{(n)}| = 0(M_n) = 0(M_{n+1}).$$

For all $\delta > 0$ sufficiently small, the level sets of $(g_{m-1} + \delta q)$, form a covering of F finer than the covering (I_j) and we choose $g_m = g_{m-1} + \delta q$. Because δ can be arbitrarily small it is clear that the sequence (g_m) can be made to converge to a function φ fulfilling the inequalities $|\varphi^{(n)}| = 0(M_n)$ for $n \geqslant 1$, and the inequality $|g_m - \varphi|y_m < m^{-1}$. This completes the proof of Theorem 1.

Theorem 2 is a variation on the idea of building jumps into the derivative, but using Lebesgue measure for the initial measure μ, and differential calculus, we give a more precise conclusion. Let φ be of class $C^1[0, 1]$, let F be a closed subset of $[0, 1]$, $m(F) > 0$, and let

$$H(y) = m(x \in F,\ \varphi'(x) < y), \qquad -\infty < y < \infty.$$

be the relative distribution function of φ'.

Theorem 2. — *F contains an M_0-set E such that $\varphi(E)$ is a Kronecker set, provided H is continuous.*

An equivalent statement is that such a set $E \subseteq F$ can be found, provided H is not a pure saltus-function. Theorem 2 is proved by the same inductive process as before, beginning now with the Lebesgue measure restricted to F, i.e. $\mu_0 = \chi_F m$. In place of arguments on the sequence of functions g_m we use

Lemma. — *Let H, φ, and F be as in Theorem 2. Then* $\lim_T \sup_u \left|\int_F \exp - iut . \operatorname{expi} T\varphi(t) . dt\right| = 0$ *as* $T \to +\infty$.

Proof. — Let k be a positive integer, $\delta > 0$, and u real, and let $S(k, \delta, u)$ be the union of all intervals

$$[pk^{-1}, (p+1)k^{-1}], \qquad (p = 0, \ldots, k-1)$$

containing a point x at which $|\varphi' - u| < \delta$. Using the uniform continuity of φ' on $[0, 1]$ and of H on $(-\infty, \infty)$, we see that to each $\varepsilon > 0$ there exist k, δ so that $m(F \cap S(k, \delta, u)) < \varepsilon$ for all real u. When u and T are specified let us denote by G any intersection $F \cap (a, b)$ where $|-u + T\varphi'| \geqslant \delta T$ throughout (a, b). We shall give a uniform method of estimating $\int_G \operatorname{expi} - iut . \exp T\varphi(t) . dt$ and this will prove the lemma.

For definiteness we suppose $-u + T\varphi'(t) > 0$ on $[a, b]$ and construct the sequence $a = a_0, a_1, \ldots$ such that

$-ut + T\varphi$ increases by exactly $2\pi T^{-1}$ between each pair a_n, a_{n+1}. Thus $a_{n+1} - a_n \leqslant 2\pi(\delta T)^{-1}$ and (a, b) is covered by intervals (a_n, a_{n+1}) and a remainder $< 2\pi(\delta T)^{-1}$. The polygonal interpolation $\tilde{\varphi}$ of φ, with nodes $a_0, a_1, \ldots, b$ has the property $|\tilde{\varphi} - \varphi| = o(T^{-1})$ by the mean-value theorem, and the estimation

$$\int_G \exp - iut.\exp i\, T\tilde{\varphi}(t).dt \to 0 \qquad \text{as} \qquad T \to +\infty$$

follows from the Lebesgue density theorem. This concludes the proof of the lemma.

THEOREM 3. — *Let* φ *have an absolutely continuous derivative on* $[0, 1]$, *and* $C_1 \leqslant \varphi'' \leqslant C_2$ *almost everywhere* $(0 < C_1 < C_2 < \infty)$. *Let* $w(u)$ *be positive on* $[0, \infty)$, *increasing to* $+\infty$. *Then there is a subset* E *so that* $\varphi(E)$ *is a Kronecker set, and a measure* $\mu \geqslant 0$ *in* E *such that* $\hat{\mu}(u) = 0\left(|u|^{-\frac{1}{2}}\right) w(|u|)$.

As in the two previous proofs, all depends on a suitable estimate of an exponential integral. The sequence (f_m), dense in $C[0, 1]$, is now supposed to contain functions of class C^2; thus beginning with the Lebesgue measure m on $[0, 1]$, all measures constructed by the inductive process have the form $\mu_m = p_m m$, with $p_m \in C^2$. Thus there is a constant C_m so that

$$\left|\int f\, d\mu_m\right| \leqslant C_m \sup_x \left|\int_0^x f(t)\, dt\right| \quad (0 \leqslant t \leqslant 1).$$

Thus the following estimation enables us to complete the proof.

For all $y > y_0$, $k \geqslant 1$ or $k \leqslant -1$, and real u

$$\left|\int_0^x \exp - iut.\exp ik(y\varphi - f_m).dt\right| \leqslant C'_m |u|^{-\frac{1}{2}},$$

and moreover, the integrals are uniformly $o(1)$ as $y \to \infty$.

To prove this we use the inequality $\varphi'' \geqslant C_1 > 0$ and $f_m \in C^2$ to choose y so large that $y\varphi'' - f''_m \geqslant \frac{1}{2} C_1 y$. By van der Corput's inequality [7, p. 197] the integrals are uniformly $0\left(|ky|^{-\frac{1}{2}}\right)$, so the second part is disposed of. Moreover, the first inequalities are valid on domains of the type $|ky| \geqslant \varepsilon|u|$, for any fixed $\varepsilon > 0$. For the complementary domain $|u|\varepsilon > |ky|$,

$$|k(y\varphi - f_m)'| \leqslant \varepsilon|u| . |\varphi'| + y^{-1}\varepsilon\,|u|\,|f_m'|.$$

Thus for small ε and large y, $-ut + k(y\varphi - f_m)$ has derivative $\geqslant \frac{1}{2}|u|$ or $\leqslant -\frac{1}{2}|u|$. We can then write

$-ut + k(y\varphi - f_m) = up(t)$ where $\frac{1}{2} \leqslant |p'(t)|$ and $|p''(t)| \leqslant C_m''$. The integral then takes the form $\int_a^b \exp - ius\,q(s)\,ds$ where $|q(s)| \leqslant 2$ and $q(s)$ has total variation $\leqslant 2C_m''$. The integral then has modulus $< 8C_m''|u|^{-1} \leqslant C_m'|u|^{-\frac{1}{2}}$, because $|u| > |y| \to +\infty$. This proves the required estimation.

The last theorem about differentiable functions is a complement to the first and second; its proof involves a lemma on interpolation of differentiable functions. The set E of Theorem 1 can be mapped by a diffeomorphism φ of class $C(M_n)$ onto a Kronecker set, and in fact $\varphi'' = 0$ on E; by [1] E can also be mapped by a C^1-diffeomorphism ψ onto a Kronecker set, and here $\psi' = 1$ on E. Quite possibly E could be constructed so that the diffeomorphism φ is smooth and has derivative 1 on E, but the method of Theorem 1 plainly fails to accomplish this. The theorem to be proved shows that the existence of diffeomorphisms φ and ψ by no means implies that their characteristics can be attained simultaneously. A similar property of stability of M_0-sets is obtained in [4] by an entirely different technique.

THEOREM 2'. — *Let* F *and* φ *be as in Theorem* 2, *and let* $w(u)$ *be positive and increasing on* $(0, \infty)$, $w(0+) = 0$. *Then the set* $E \subseteq F$ *can be so chosen that* $\varphi(E)$ *is a Kronecker set, but* $\psi(E)$ *is an* M_0*-set whenever* $\psi \in C^1[0, 1]$, $\psi' = 1$ *on* E, *and* $|\psi'(s) - \psi'(t)| \leqslant w(|s - t|)$ *for* $0 \leqslant s < t \leqslant 1$.

When F is totally disconnected, φ can be constructed in any non-quasi-analytic class $C(M_n)$ so that φ' is strictly increasing and $\varphi'' = 0$ on F; the simple example $\varphi(x) = x^2$ illustrates that analyticity has no obvious consequences about E. The necessity of the lemma needs to be explained.

Beginning from the set F, the subset $E \subseteq F$ is to be defined, and thereby a certain subset of C^1, say S(E). But $S_1(E)$ is then known only in principle and is obviously much larger than S(F). Thus the construction seems to be circular, because it requires some knowledge of S(E) to proceed. To circumvent this obstacle we consider all sets S(E) simultaneously, attempting to replace each function ψ_1 in S(E) by a function ψ_1 in $C^1[0, 1]$, such that $\psi = \psi_1$ in E and $\psi_1' = 1$ in $E \cup F$. This, however, is possible only if $[0, 1] \sim F$ meets each interval $(a, b) \subseteq (0, 1)$ in a subset whose measure is not much smaller than $(b - a) . w(b - a)$, because $\psi_1' = 1$ on F. To solve this problem of interpolation we must therefore replace F_1 by a subset whose complementary intervals are specially constructed.

LEMMA. — *Corresponding to the function* w *there is a closed set* F_1 *with this property: whenever* $E \subseteq [0, 1]$ *and* $\psi \in S_1(E)$, *then* ψ *coincides with a function* $\psi_1 \in C^1[0, 1]$ *whose derivative is* 1 *on* $E \cup F_1$. *Moreover all the derivatives* ψ_1' *so constructed are equicontinuous on* $[0, 1]$. *Finally,* $m(F \cap F_1) > 0$.

Proof. — Let T_n be an increasing sequence of positive numbers and $R \sim F_1$ the set defined by $x \notin F_1$ if $|T_n x - q| \leqslant n^{-2}$ for some integer q and $n \geqslant 1$. When (T_n) increases rapidly, $m(F \cap F_1) > 0$; we specify that $T_1 > 8$ and $w(8T_n^{-1}) \leqslant (n + 1)^{-3}$. Thus, if $b - a > 8T_n^{-1}$, $(R \sim F_1) \cap (a, b)$ contains intervals of length $2n^{-2}T_n^{-1}$, whose total length exceeds $n^{-2}(b - a)$. Let now E be a closed subset of $[0, 1]$ and ψ a function in the class $S_1(E)$, and write $\psi_2(x) = x - \psi(x)$. Now ψ_2 is considered as a function defined only on E; then

$$|\psi_2(s) - \psi_2(t)| \leqslant |s - t| w(|s - t|)$$

by the mean-value theorem. To each interval $[a, b]$ meeting E only in its end-points a and b, there is a least integer n with $b - a > 8T_n^{-1}$. Then (a, b) meets $R \sim F_1$ in a certain set of intervals, of total length $> n^{-2}(b - a)$. The derivative of ψ_2 will have a triangular graph over these intervals, of height h, and will vanish elsewhere in (a, b).

The common height h of these triangles fulfills an inequality $n^{-2}(b-a).|h| \leqslant 2|\psi_2(b)-\psi_2(a)| \leqslant 2(b-a)w(b-a)$. In case $1 \geqslant b-a > 8T_1^{-1}$, we obtain $|h| \leqslant 2w(1)$; when $8T_{n-1}^{-1} \geqslant b-a > 8T_n^{-1}$, we have $w(b-a) \leqslant n^{-3}$ and then $|h| \leqslant 2n^{-1}$. To complete the extension of ψ_2 onto $[0, 1]$, we extend to be constant to the left and right of $[0, 1]$. The equicontinuity of the aggregate $\{\psi_1'\}$ follows from the triangular shape and the fact that $|\psi_1'| \leqslant 2n^{-1}$ on the interval (a, b) provided $b-a < 2n^{-2}T_n^{-1}$. Finally, set

$$\psi_1(x) = x - \psi_2(x)$$

and the lemma is complete.

To prove the main result, we can assume that $F_1 = F$, and construct E as in Theorem 2 so that $\psi_1(E)$ is an M_0-set for each function ψ_1 constructed in the lemma. This can be accomplished with the aid of uniform estimates for integrals of the form

$$\int_F \exp - iu\psi_1(t).\exp iy\varphi(t).dt.$$

To each $\delta > 0$ there is a neighbourhood $V_\delta \supseteq F$ on which $|\psi_1' - 1| < \delta$ for each function ψ_1, and from this point the argument of Theorem 2 is valid, so the integrals tend to 0 uniformly as $y \to \infty$.

Theorem 2′ is valid for the weaker inequality

$$|\psi'(s) - \psi'(t)| = 0(w|t-s|),$$

since there is a function g, locally constant on E, so that $g' + \psi'$ has modulus of continuity at most $w^{\frac{1}{2}}$.

2.

THEOREM 4. — *Let λ be a continuous, finite measure on* R, *and* F *an M_0-set. Then there is a Kronecker set* E *and a positive measure $\mu \neq 0$ in* E *such that each set* $\{|\hat{\mu}(u)| \geqslant \delta, |\hat{\lambda}(u)| \geqslant \delta\}$ *is compact. Moreover to each $\delta > 0$ there is a u_0 so that the set* $\{|\hat{\mu}(u+u_0)| \geqslant \delta, |\hat{\lambda}(u)| \geqslant \delta\}$ *is empty.*

Here we set $\varphi(x) = x$ so that the support of the limit measure μ is a Kronecker set. Of course we cannot obtain uniform convergence of $\hat{\mu}_m$, but only pointwise convergence, sufficient to ensure that $\frac{3}{2} > \mu(E) > \frac{1}{2}$. However, we can obtain uniform convergence on each of the subsets $R_\delta = \{|\hat{\lambda}(u)| \geqslant \delta\}$. A classical theorem of Wiener shows that $|\hat{\lambda}|^2$ has mean-value 0, and for each δ there is an $\eta > 0$ such that $R_\delta + (0, \eta) \subseteq R_{\frac{1}{2}\delta}$. Thus $R_\delta + (0, \eta)$ meets $[-a, a]$ in a set of measure $o(a)$. An elementary covering argument shows that this remains valid for $R_\delta + I$, I being a fixed, finite interval, and plainly that property is preserved by dilations and translations of R_δ. Thus we obtain the important property of R_δ: there is a sequence y_m such that $\lim d(ky_m, R_\delta) = +\infty$ for each $\delta > 0$ and each integer $k \neq 0$. Examination of the formula for $\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u)$ shows that it is possible to force the sequence $\hat{\mu}_m$ to converge uniformly on each R_δ. Because each $\hat{\mu}_m \in C_0$, the limit $\hat{\mu} \in C_0(R_\delta)$ and this expresses the first property claimed for $\hat{\mu}$.

To obtain the second we choose a sequence (u_m) along with (μ_m). Now $\hat{\mu}_m \in C_0(R)$ so there is a number $-u_m$ so far from $R_m - 1$ that $|\hat{\mu}_m(u - u_m)| < m^{-1}$ whenever $|\hat{\lambda}(u)| \geqslant m^{-1}$; or $|\hat{\mu}_m| < m^{-1}$ on $R_m - 1 + u_m$. From here the argument is almost as before, except that $\hat{\mu}_m - \hat{\mu}_{m-1}$ must be controlled on a set of the type finite $+ R_\delta$ and this is easily attained. In the limit we have, for example, $|\hat{\mu}| < 2m^{-1}$ on $R_m - 1 + u_m$ so that the inequalities $|\hat{\lambda}| > m^{-1}$ and $|\hat{\mu}(u - u_m)| > 2m^{-1}$ exclude each other.

Here is a simple consequence of Theorem 4. To each uncountable closed set E thereare a Kronecker set E_1, disjoint from E and probability measures μ in E_1, λ in E, so that $\limsup |\hat{\mu}| + |\hat{\lambda}| \leqslant 1$, and a sequence of characters χ_m so that $|\widehat{\chi_m \mu}| + |\hat{\lambda}| < + m^{-1} + 1$. Thus $E \cup E_1$ is at most $H_{\frac{1}{2}}$, in a sense somewhat stronger than in [5]; of course the most interesting case occurs when E is itself a Kronecker set.

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