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# IDELE CHARACTERS IN SPECTRAL SYNTHESIS ON R/2πZ

**by John J. BENEDETTO**

## Introduction.

The starting point for this paper is Malliavin's construction of real-valued absolutely convergent Fourier series $\varphi$ on $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ having a non-synthesizable zero set, $Z\varphi$ (§ 0.1 will contain relevant definitions and background in spectral synthesis). The construction of such a $\varphi$ by Richards [7] has led us to consider families of functions parameterized by $s = \sigma + i\tau \in \mathbf{C}$, $\sigma > 1$, and having the form

$$(1) \qquad F(s, x) = \sum_{n=1}^{\infty} e^{ik_n x}/n^s$$

that is, for fixed $s$, $\sigma > 1$, the corresponding $\varphi = \varphi_s$ is $\varphi_s(x) = \operatorname{Re} F(s, x)$. The Dirichlet series (1) are discussed in § 1 and the results concerning the corresponding non-synthetizable $\varphi$ are proved in § 2. Because of these results we pose the « abscissa of spectral synthesis » problem at the end of § 2.

Since the above construction of non-synthesizable $\varphi_s$ involves no arithmetic properties of the $k_n$, it seemed reasonable to investigate properties of $\varphi_s$ when the corresponding F was generated from an idele character. We've only considered $\hat{J}_Q$ (see § 0.2), and have shown that those $\varphi_s$ generated by « slow growing » idele characters have synthesizable zero sets (§ 3). We could have expressed the arithmetic properties for ideles over $\mathbf{Q}$ in a less idelic way, but our procedure produces analogous results in a much more general algebraic number theoretic setting; and we hope that $\hat{\mathbf{J}}_{\mathbf{Q}}$ will serve as a prototype technique to generate some examples in synthesis.

The next problem we pose is that of « analytic continuation ». We move left across $\sigma = 1$ and construct pseudo-measures $T_s$, $0 < \sigma \leqslant 1$, associated with certain $F(s, x)$ generated from idele characters. The method to construct $T_s$ involves counting solutions to diophantine equations, being careful on the one hand in estimating upper bounds to ensure that $T_s$ is a pseudo-measure, and on the other hand providing specific lower bounds (when this is possible) to guarantee that $T_s$ is not a measure. The spectral synthesis properties of such pseudo-measures are the subject of forthcoming work, but generally the following types of results evolve :

*a*) $T_s$ generated by « fast growing » idele characters are synthesizable;

*b*) $T_s$ generated by « slow growing » idele characters are non-synthesizable;

*c*) $\varphi_s$ generated by « fast growing » idele characters are non-synthesizable.

(The terminology « fast growing », etc. is clarified in § 3.)

Note that with our original non-synthesizable $\varphi_s$, the pseudo-measures $T_s$ we obtain in $\sigma \in \left(\frac{1}{2}, 1\right)$ are not only synthesizable but $L^2(\mathbf{T})$ functions since $n \longmapsto k_n$ is injective for this case.

*Acknowledgement.*

I thank W. Adams and R. Holzsager for their ingenious technical assistance which supported my fuzzy intuition on several occasions. Professor Holzsager has made substantial further progress on certain of the arithmetic problems that I have posed here, and he will publish his results separately in a number theoretic format.

## 0. Preliminaries.

**0.1.** *Preliminaries from spectral synthesis.*

A(**T**) is the Banach space of absolutely convergent Fourier series

$$\varphi(x) = \Sigma a_n e^{inx}$$

where $\|\varphi\| = \Sigma|a_n|$. The dual of A(**T**) is A′(**T**) the space of pseudo-measures. A′(**T**) is the subspace of distributions with bounded Fourier coefficients and the canonical dual norm $\| \ \|_{A'}$ on A′(**T**) is

$$\|T\|_{A'} = \sup|\hat{T}(n)|.$$

The Radon measures M(**T**) are canonically contained in A′(**T**). If $E \subseteq \mathbf{T}$ is closed, $A'(E) = \{T \in A'(\mathbf{T}) : \operatorname{supp} T \subseteq E\}$. $\varphi \in A(\mathbf{T})$ (resp., $T \in A'(\mathbf{T})$) is *synthesizable* if for all $S \in A'(Z\varphi)$ (resp., for all $\psi \in A(\mathbf{T})$ with $\operatorname{supp} T \subseteq Z\psi$) $\langle S, \varphi\rangle = 0$ (resp., $\langle T, \psi\rangle = 0$). E is a *synthesis* (S) *set* if for all $\varphi \in A(\mathbf{T})$ with $\varphi = 0$ on E and for all $T \in A'(E)$, $\langle T, \varphi\rangle = 0$.

We say that a real-valued $\varphi \in A(\mathbf{T})$ satisfies condition $(M_r)$ if

$$\forall k < r + 1 \qquad \exists C_k > 0 \qquad \text{such that} \qquad \forall u \in \mathbf{R}$$
$$\|e^{iu\varphi}\|_{A'} \leqslant C_k|u|^{-k}.$$

Clearly $(M_r)$ implies $(M_t)$ if $r \geqslant t$. The following is Malliavin's operational calculus technique and we refer to [2, § B.1] for details, remarks and generalizations.

PROPOSITION 0.1. — *If real-valued* $\varphi \in A(\mathbf{T})$ *satisfies* $(M_r)$ *for some* $r > 2$ *then* $Z\varphi$ *is not an* S *set.*

*Remark.* — In the proof of the above result we use $(M_r)$ to construct $T \in A'(\mathbf{T})$ such that

$$\langle T, \varphi\rangle \neq 0, \qquad T\varphi^2 = 0. \tag{0.1}$$

Then by an argument which uses Wiener's result on the reciprocal of $\psi \in A(\mathbf{T})$ we show [1, Theorem 3.15 *d*] that

$S\theta = 0$ implies $\theta = 0$ on supp S; thus $\varphi = 0$ on supp T and so $Z\varphi$ is non-S by (0.1).

We record some routine properties of the $\| \ \|_{A'}$ norm.

(0.2) $\forall \varphi \in A(\mathbf{T}) \subseteq A'(\mathbf{T}), \quad \|\varphi\|_{A'} \leqslant \|\varphi\|_A$

(0.3) $\forall \varphi \in A(\mathbf{T}), \quad |\varphi| \leqslant 1, \quad$ either $\quad \|\varphi\|_{A'} < 1$

or

$$\varphi(x) = a \exp ikx, \qquad k \in \mathbf{Z} \qquad \text{and} \qquad |a| = 1.$$

(0.4) The map $A(\mathbf{T}) \to \mathbf{R}$

$$T \longmapsto \|T\|_{A'}$$

is continuous.

(0.5) $\forall \varphi \in A(\mathbf{T}), \quad \forall k \in \mathbf{Z} \setminus \{0\}, \quad \|\varphi\|_{A'} = \|\varphi^k\|_{A'}$

where $\varphi^k(x) = \varphi(kx)$.

**0.2.** *Preliminaries from number theory.*

References for the material in this section are [4, Chapters 2 and 15; 5; 8].

For each prime number $p = 2, 3, 5, \ldots$ let $\mathbf{Q}_p^\times$ be the non-zero elements of the $p$-adic completion of $\mathbf{Q}$. Thus

$$\mathbf{Q}_p^\times = \left\{ \sum_{j=-n}^{\infty} a_j p^j : \ 0 \leqslant a_j < p, \ a_j \in \mathbf{Z}, \text{ some } a_j \neq 0, \ n \geqslant 0 \right\}.$$

With the $p$-adic valuation $| \ |_p$, $\mathbf{Q}_p^\times$ is a locally compact group under multiplication and is totally disconnected. The compact and open subgroup of units $U_p$ for $\mathbf{Q}_p^\times$ is

$$U_p = \left\{ \sum_{0}^{\infty} a_j p^j \in \mathbf{Q}_p^\times : \ a_0 \neq 0 \right\}.$$

Let $\mathbf{Q}_0^\times = \mathbf{R} \setminus \{0\}$. Define $\mathbf{J_Q}$ to be the set of all sequences $\alpha = \{\alpha_p : p = 0, 2, 3, 5, \ldots\}$ where $\alpha_p \in \mathbf{Q}_p^\times$ and with the property that, for all but a finite number of $p = 2, 3, 5, \ldots$, $\alpha_p \in U_p$. Next, consider sets

$$B = \prod_p B_p \subseteq \mathbf{J_Q}$$

where $B_p \subseteq \mathbf{Q}_p^\times$ is open for all $p = 0, 2, \ldots$ and $B_p = U_p$ for all but a finite member of $p = 2, 3, \ldots$. Such sets B form a basis for a topology on $\mathbf{J_Q}$. As such, with componentwise multiplication, $\mathbf{J_Q}$ is a locally compact abelian group, the *idele group* corresponding to $\mathbf{Q}$.

It is standard to prove that $c \in \hat{\mathbf{J}}_\mathbf{Q}$, the dual group of $\mathbf{J_Q}$, if for each $p = 0, 2, \ldots$ there is $c_p \in \hat{\mathbf{Q}}_p^\times$ such that for all $\alpha = \{\alpha_p\} \in \mathbf{J_Q}$

$$c(\alpha) = \prod_p c_p(\alpha_p)$$

and $c_p(\mathrm{U}_p) = 1$ for all but a finite number of $p$. When $c_p(\mathrm{U}_p) = \{1\}$ we say that $c_p$ is *unramified.*

For each $p = 2, 3, \ldots$ define $\mathrm{P}_p \subseteq \mathbf{Q}_p^\times$ to be

$$\mathrm{P}_p = \{\alpha_p \in \mathbf{Q}_p^\times : |\alpha_p|_p < 1\}$$

and $p_p = \mathrm{P}_p \cap \mathbf{Z}$. Clearly

$$\mathrm{P}_p = \left\{ \sum_1^\infty a_j p^j \in \mathbf{Q}_p^\times \right\}.$$

and so $p_p$ is the multiplicative ideal in $\mathbf{Z}\diagdown\{0\}$ each of whose elements is divisible by $p$. A neighborhood basis of $1 \in \mathbf{Q}_p^\times$ in $\mathrm{U}_p$ is $\{1 + \mathrm{P}_p^n\}$ where $1 + \mathrm{P}^0 = \mathrm{U}_p$. It is standard to check that if $c_p \in \hat{\mathbf{Q}}_p^\times$ then there is a smallest integer $n_p \geqslant 0$ for which

$$c_p(1 + \mathrm{P}_p^{n_p}) = \{1\}.$$

Now, if $c \in \hat{\mathbf{J}}_\mathbf{Q}$ with corresponding « projections » $c_p \in \hat{\mathbf{Q}}_p^\times$ then there are only finitely many primes $p = 2, 3, \ldots$ such that the corresponding $n_p > 0$; this follows since $c_p(\mathrm{U}_p) = 1$ for all but finitely many $p$. As such, given $c$ we form a multiplicative ideal $f_c$ in $\mathbf{Z}$, called the *conductor of* $c$, defined by

$$f_c = \prod_{n_p > 0} \mathrm{P}_p^{n_p}.$$

Recall that the fractional ideals in $\mathbf{Q}$ are singly generated. Then, for example, if $p_1, \ldots, p_r$ are the primes for which $n_{p_j} > 0$, $f_c$ is generated by $p_1^{n_{p_1}} \ldots p_r^{n_{p_r}} = h$. We let $\mathrm{G}_c$ be the set of multiplicative fractional ideals $\mathrm{A} \subseteq \mathbf{Q}$ generated by $n = q_1^{m_1} \ldots q_d^{m_d}$, $m_j \in \mathbf{Z}$, such that each $q_j$ is prime and for all $j, k$, $q_j \neq p_k$. For $\mathrm{A} \in \mathrm{G}_c$ generated by $n$ as above, we define $\alpha = \{\alpha_p\} \in \mathbf{J}_{\mathbf{Q}}^{\times}$ by

$$\alpha_p = \begin{cases} 1, & p \neq q_j \\ q_j^{m_j}, & p = q_j. \end{cases}$$

Then the *Hecke character* associated with $c$ is a multiplicative homomorphism

$$\chi_c : \mathrm{G}_c \to \mathbf{T}$$

given by

$$\chi_c(\mathrm{A}) = c_{q_1}(q_1^{m_1}) \ldots c_{q_d}(q_d^{m_d}).$$

The mapping $c \to \chi_c$ is injective.

Observe that we can imbed $\mathbf{Q}^{\times}$ into $\hat{\mathbf{J}}_{\mathbf{Q}}$ by the map $q_1 \to (q, q, \ldots)$. $c \in \hat{\mathbf{J}}_{\mathbf{Q}}$ is an *idele class character* if $c(\mathbf{Q}^{\times}) = 1$. Idele class characters are needed in algebraic number theory to obtain functional equations.

Preserving the above notation between $n$ and A we define the *Hecke* L *series* associated with $\chi_c$ to be

$$\mathrm{L}(s, c) = \sum_{(n, h)=1} \frac{\chi_c(\mathrm{A})}{n^s}$$

(for the definition of L we consider only $n \in \mathbf{Z}$, recalling that $n$ could be rational in the definition of $\mathrm{G}_c$). Now, $f_c$ is trivial (i.e., $n_p = 0$ for all $p$) if and only if for each $p = 2, 3, \ldots$

$$c_p(\alpha_p) = |\alpha_p|_p^{it_p}$$

where $t_p$ is determined $\mathrm{mod}\, 2\pi/\log p$. Thus for each $c \in \hat{\mathbf{J}}_{\mathbf{Q}}$ with trivial conductor there is a unique set of integers $\{k_p : p = 2, \ldots\}$ such that

$$\mathrm{L}(s, c) = \sum_{1}^{\infty} \frac{1}{n^2} \exp i \sum_{q} m_q k_q \tag{0.6}$$

where $n = \Pi q^{m_q}$, the prime decomposition of $n$. Conversely given $\{k_p : p = 2, \ldots\} \subseteq \mathbf{Z}$ there is a unique $c \in \hat{\mathbf{J}}_{\mathbf{Q}}$ defined by

$$c(n) = \Pi \exp i m_q k_q$$

where $\Pi q^{m_q}$ is the prime decomposition of $n$.

Finally, if we are given $\{k_p : p = 2, \dots\} \subseteq \mathbf{Z}$ and extend additively by

$$k_n = k_p + k_q \qquad \text{if} \qquad n = pq$$

then (0.6) becomes

$$\text{(0.7)} \qquad L(s, c) = \sum_1^\infty \frac{1}{n^s} \exp ik_n.$$

## 1. Pseudo-measure norms and Dirichlet series.

The real part of F defined in (1), considered as a function of $x$ for fixed $s$, $\sigma > 1$, is

$$\varphi_s(x) = F_r(s, x) = \sum_{n=1}^\infty \frac{1}{n^\sigma} [\cos k_n x \cos (\tau \log n) + \sin k_n x \sin (\tau \log n)].$$

Because of condition $(M_r)$ in § 0.1 we shall consider

$$\exp iu\varphi_s(x) = \prod_{n=1}^\infty \exp \frac{iu}{n^\sigma} [\cos k_n x \cos (\tau \log n) + \sin k_n x \sin (\tau \log n)].$$

As such we define the auxiliary functions

$$\text{(1.1)} \quad \psi_{u, m, s}(x) = \prod_{n=1}^m \exp \frac{iu}{n^\sigma} [\cos k_n x \cos (\tau \log n) + \sin k_n x \sin (\tau \log n)]$$

and

$$\text{(1.2)} \quad \theta_{u, m, s}(x) = \exp \frac{iu}{(m+1)^\sigma} [\cos x \cos (\tau \log (m+1)) + \sin x \sin (\tau \log (m+1))].$$

Observe that

$$\psi_{u, m+1, s} = \psi_{u, m, s} \theta_{u, m, s}^{k_{m+1}}.$$

Using (0.4) and (0.5) Richards [7, Lemma 1; 2, Theorem B.6] has made the following key observation concerning the growth of $\| \ \|_{A'}$.

**Proposition 1.1.** — *Given* $\{\psi_u, \theta_u : u \in \mathbf{R}\} \subseteq A(\mathbf{T}) \setminus \{0\}$ *and assume* $u \longmapsto \psi_u$, $u \longmapsto \theta_\alpha$ *are continuous functions. Then*

$$\forall R > 0, \qquad \forall \varepsilon > 0, \qquad \exists k \in \mathbf{Z}, \qquad k > 0,$$

*such that*

$$\forall |u| \leqslant R$$
$$\|\psi_u \theta_u^k\|_{A'} < (1+\varepsilon)\, \|\psi_u\|_{A'}\, \|\theta_u\|_{A'}.$$

Fix $\{\varepsilon_k\}$ so that $\varepsilon_k > 0$ and $\Pi(1+\varepsilon_k) \leqslant 2$. Also for each $c > (1+\sqrt{5})/2$ let $B_c \subseteq \mathbf{C}$ be the closed rectangle $[1+1/c,\ c] \times [-c,\ c]$. Define an increasing function $c : \mathrm{N} \to \mathbf{R}$, $c(m) = \mathrm{R}$, such that

$$\bigcup_{1}^{\infty} B_c = \{s \in \mathbf{C} : \sigma > 1\}.$$

Using *Proposition* 1.1 and a uniformity argument we obtain:

Proposition 1.2. — *Given* $\{\varepsilon_k\}$ *and* $c$ *as above. There is a sequence of integers* $k_n$, $n = 1, 2, \ldots$, *increasing to infinity, such that*

$$\forall s \in \mathbf{C}, \qquad \sigma > 1, \qquad \exists K_\sigma > 0$$

*such that*

$$\forall m \geqslant K_\sigma \qquad \forall |u| \leqslant c(m)$$
$$\|\psi_{u,m+1,s}\|_{A'} < (1+\varepsilon_m)\, \|\psi_{u,m,s}\|_{A'}\, \|\theta_{u,m,s}\|_{A'}. \tag{1.3}$$

Naturally, it may happen that for some $s$, $\sigma > 1$, and some $m$, (1.3) is not true for all $|u| \leqslant c(m)$.

Proposition 1.3. — *Given* $\{\varepsilon_k\}$ *and* $c$ *as above. Then*

$$\forall s \in \mathbf{C}, \qquad \sigma > 1, \qquad \exists K_\sigma > 0$$

*such that*

$$\forall m \geqslant K_\sigma \qquad \forall |u| \leqslant c(m) \qquad \forall n \geqslant m$$
$$\|e^{iu\varphi_s}\|_{A'} \leqslant 2 \prod_{j=m}^{n} \|\theta_{u,j,s}\|_{A'}. \tag{1.4}$$

*Proof.* — Take $K_\sigma$ as in *Proposition* 1.2 and fix $m \geqslant K_\sigma$; thus we have (1.3) for all $|u| \leqslant c(m)$.

Arguing iteratively, and using (0.3) and the definition of $\varepsilon_k$,

$$\|\psi_{u,k,s}\|_{A'} \leqslant 2 \prod_{j=m}^{k} \|\theta_{u,j,s}\|_{A'} \tag{1.5}$$

for all $k \geqslant m$ and all $|u| \leqslant c(m)$.

(1.4) follows by taking $\overline{\lim}$'s in (1.5) and invoking (0.3) again.

q.e.d.

PROPOSITION 1.4. — *Given* $\{\varepsilon_k\}$ *and* $c$ *as above and form the corresponding* F *of* (1). *Take any closed interval* $I \subseteq \mathbf{R}\setminus\{0\}$. *Then*

$$\exists \rho \in (0, 1)$$

*such that*

$$\forall s \in \mathbf{C},\quad \sigma > 1,\quad \forall n \in \mathbf{Z}\setminus\{0\} \quad \forall u/(m+1)^\sigma \in \pm \mathrm{I},$$
$$\|\theta^n_{u,m,s}\|_{A'} \leqslant \rho. \tag{1.6}$$

*Proof.* — From (0.3) and (0.5) there is $\rho_{u,m,s} \in (0, 1)$ such that

$$\forall n \in \mathbf{Z}\setminus\{0\}, \qquad \|\theta^n_{u,m,s}\|_{A'} \leqslant \rho_{u,m,s}.$$

For $n$ fixed, the function $\alpha \longmapsto \theta^n_{u,m,s}$ from $\mathrm{I} \to \mathrm{A}(\mathbf{T})$ is continuous, where $\alpha = u/(m+1)^\sigma$.

(1.6) follows since continuous functions achieve their maxima on compact sets.

q.e.d.

## 2. Examples of non-S sets.

In light of *Proposition* 1.3 and *Proposition* 1.4 we shall see that the key to *Theorem* 2.1 is to choose $c$ so that for all $n$

$$\begin{aligned} \eta^{-1/\sigma}c(n)^{1/\sigma} - c(n+1)^{1/\sigma} &\geqslant f(c(n+1)) \\ \eta^{-1/\sigma}c(n)^{1/\sigma} - \quad (n+2) \quad &\geqslant f(c(n+1)) \end{aligned} \tag{2.1}$$

for some $\eta \in (0, 1)$ and some « relatively quickly increasing » $f$.

THEOREM 2.1. — *Let* $c(n) = e^n$ *and form the corresponding* F *of* (1). *Then for all* $s \in \mathbf{C}, \sigma > 1$, *there is* $\mathrm{M}_\sigma > 0$ *and* $\delta_\sigma > 0$ *such that*

$$\forall u \in \mathbf{R}, \qquad \|e^{iu\varphi_s}\|_{A'} \leqslant \mathrm{M}_\sigma e^{-\delta_\sigma |u|^{1/\sigma}}. \tag{2.2}$$

In particular $(\mathrm{M}_r)$ is satisfied for all $r > 2$ and $\mathrm{Z}\varphi_s$ is non-S.

*Proof.* — Take $\mathrm{K}_\sigma$ as in *Proposition* 1.2 for a fixed $s$. Let $\mathrm{I} = [\eta, 1]$ where $0 < \eta < 1/e$.

Choose the corresponding $0 < \rho < 1$ from *Proposition* 1.4. Without loss of generality we do the calculation for $u \geqslant 0$.

Take $u \geqslant e^{n_\sigma}$ where $n_\sigma \geqslant K_\sigma$ and $e^{(n_\sigma+1)/\sigma} \geqslant n_\sigma + 2$; and let $n \geqslant n_\sigma$ have the property that

$$c(n+1) \geqslant u \geqslant c(n).$$

From *Proposition* 1.3

$$\|e^{iu\varphi_s}\|_{A'} \leqslant 2 \prod_{n+1}^{m} \|\theta_{u,j,s}^{k_{j+1}}\|_{A'} \tag{2.3}$$

for all $|u| \leqslant c(n+1)$ and $m \geqslant n+1$.

For our fixed $u$ and $\sigma$ we now want to count which $j$'s have the property that

$$u/(1+j)^\sigma \in I. \tag{2.4}$$

If $c(n+1)^{1/\sigma} \leqslant 1 + j$ then $u/(1+j)^\sigma \leqslant 1$ and if

$$\eta^{1/\sigma}(1+j) \leqslant c(n)^{1/\sigma} \qquad \text{then} \qquad \eta \leqslant u/(1+j)^\sigma.$$

Because of (2.3) and these inequalities the number of $j$'s for which (2.4) holds is estimated by

$$\eta^{-1/\sigma}c(n)^{1/\sigma} - 1 - \max\,(n+1,\ c(n+1)^{1/\sigma} - 1) \tag{2.5}$$

(note the resemblance to (2.1)).

Since $c(n) = e^n$, (2.5) is

$$e^{n/\sigma}(\eta^{-1/\sigma} - e^{1/\sigma}) = re^{n/\sigma} \geqslant re^{-1/\sigma}u^{1/\sigma}. \tag{2.6}$$

Combining (2.3), (2.6), and *Proposition* 1.4 we have (2.2) since $p = e^{-P}$, $p > 0$.

q.e.d.

*Remark.* — Let us see how much generality we have in choosing $c$ so that *Theorem* 2.1 is valid. $c(k)$ must grow faster than $k$ so that (2.5) is positive. If $c(k)$ grows like a polynomial the technique of *Theorem* 2.1 will not work for all $\sigma$ and this leads to the abscissa of convergence problem below. If $c(k)$ grows too fast (e.g., $c(k) = \exp k \log k$) the procedure again fails since (2.5) again becomes negative.

*Example* 2.1. — For the choice of $\{k_n\}$ in *Proposition* 1.1 and *Proposition* 1.2 it is desirable to choose the smallest possible $k_{m+1} > 0$ given $k_1, \ldots, k_m > 0$. Thus, given $\varepsilon_1$, $c(1)$, and

$k_1 = 1$ we find $k_2$. Using Schäfli's integral form for Bessel functions we compute

$$\forall \mathrm{N}, \quad \sum_{|m|>\mathrm{N}} |\hat{\psi}_{u,1,s}(m)| = 2 \sum_{m>\mathrm{N}} |\mathrm{J}_m(u)|$$
$$\leqslant 2 \sum_{m>\mathrm{N}} \left|\frac{c(1)}{2}\right|^m \left(\sum_{r=0}^{\infty} \frac{c(1)^{2r}}{2^{2r} r!(m+r)!}\right)$$
$$\leqslant 2 \exp\left(\frac{c(1)^2}{4}\right) \sum_{m>\mathrm{N}} \left(\frac{c(1)}{2}\right)^m \Big/ m!.$$

From the proof of *Proposition* 1.1 we now take $k_2 = 2\mathrm{N}_{\varepsilon_1}$ where $\mathrm{N}_{\varepsilon_1}$ is the smallest N for which

$$\sum_{m>\mathrm{N}} \left(\frac{c(1)}{2}\right)^m \Big/ m! < \frac{\varepsilon_1}{2} \exp\left(-\frac{c(1)^2}{4}\right).$$

*Abscissa of Spectral Synthesis Problem.* — Given specified F of (1) our calculations [3] indicate that $\mathrm{Z}\varphi_s$ becomes more non-spectral as $\sigma \to 1+$. We would like to determine those F for which there is an abscissa $\sigma = \sigma_0$ of spectral synthesis, i.e.,

$$\forall \sigma > \sigma_0, \qquad \mathrm{Z}\varphi_s \text{ is S}$$
$$\forall 1 < \sigma < \sigma_0, \qquad \mathrm{Z}\varphi_s \text{ is non-S}.$$

## 3. Spectral synthesis functions and idele characters.

For the remainder of the paper assume for convenience that $c \in \hat{\mathbf{J}}_{\mathrm{Q}}$ has trivial conductor, and consider the Hecke L-series characterized by $\{k_p\} \subseteq \mathbf{Z}$, where $p = 2, 3, 5, \ldots$. Given $\mathrm{L}(s, c)$ we associate the Fourier series $\mathrm{F}(s, x)$ (of (1)) for fixed $s, \sigma > 1$; and $\mathrm{ZF} \subseteq \mathbf{T}$ is $\{x : \mathrm{F}(s, x) = 0\}$, where $s$ is fixed.

*Remarks.* — 1. If $c$ is an idele class character then the corresponding Hecke character $\chi_c$ is precisely a classical Dirichlet character (for Dirichlet L-series); consequently the corresponding Fourier series $\mathrm{F}(s, x)$ is a trigonometric polynomial for each $s, \sigma > 1$. In particular $\mathrm{Z}\varphi_s$ is finite and thus S.

2. The terminology « fast growing » etc., from the introduction indicates that $\{k_p\}$ tends to infinity at certain rates.

Proposition 3.1. — *Given* $c \in \hat{\mathbf{J}}_Q$ *with trivial conductor and corresponding* $\{k_p\} \subseteq \mathbf{Z}$.

a) *If* $k_p = 0(p^\alpha)$, $p \to \infty$, *for some* $\alpha > 0$ *then*

$$\forall \sigma > 1 + \alpha, \qquad F(s, x) \in C^1(\mathbf{T}).$$

*b*) *If* $k_p = 0\,(\log p)$, $p \to \infty$, *then*

$$\forall \sigma > 1, \qquad F(s, x) \in C^1(\mathbf{T}).$$

*c*) *For each of the above cases*, ZF *is* S.

*Proof.* — $c$ is clear from the Beurling-Pollard result.

*a*) Differentiating F with respect to $x$

$$\left| \sum_1^\infty \frac{ik_n}{n^s} e^{ik_n x} \right| \leqslant \sum_1^\infty \frac{|k_n|}{n^\sigma}$$

we need only check that $|k_n|/n^\alpha$ is bounded for $\alpha > 0$. If $n = \Pi p^r$

$$|k_n| \leqslant K_1 \Sigma r p^\alpha \leqslant K_2 \Sigma p^{\alpha r} \leqslant K_3 n^\alpha.$$

*b*) Arguing in the same way we need only check that for $\beta > 0$, $|k_n|/n^\beta$ is bounded.

If $n = \Pi p^r$

$$|k_n| \leqslant K \Sigma r \log p = K \log n$$

and $\log n/n^\beta$ is bounded.

q.e.d.

In order to generalize the synthesis result in *Proposition* 3.1 we say that $\{k_p\} \subseteq \mathbf{Z}^+$ or the corresponding $c \in \hat{\mathbf{J}}_Q$ is *r-bounded*, $r \geqslant 0$, if

$$\forall \beta > r \qquad \exists M_\beta \qquad \text{such that} \qquad \forall n, \qquad k_n/n^\beta \leqslant M_\beta.$$

Thus, for example, $k_p = O(p^\alpha)$, $p \to \infty$, $\alpha > 0$, is 0-bounded.

Proposition 3.2. — *Given* $c \in \hat{\mathbf{J}}_Q$ *with trivial conductor and* $\{k_p\} \subseteq \mathbf{Z}^+$. *If* $c$ *is r-bounded then for each* $\sigma > 1 + r$, $F(s, x)$ *is a function of bounded variation and* ZF *is* S.

*Proof.* — The fact that ZF is S if F has bounded variation is standard [6, p. 62]. The bounded variation follows classically once we observe that

$$\int_{\mathbf{T}} |F(s, x + h) - F(s, x)|\, dx = O(h), \qquad |h| \to 0.$$

By direct computation the integral is bounded by $K|h|\Sigma|k_n|/n^\sigma$; and so setting $\sigma = 1 + r + \gamma$ we let $\beta = r + \gamma/2$ and apply the $r$-boundedness.

q.e.d.

In order to generate non-S $Z\varphi$ in § 1 and § 2 the sequence $\{k_n\}$ was chosen to have a certain lacunarity. We now observe that no matter how fast $\overline{\lim}\, |k_p|$ tends to infinity the sequence $\{k_n\}$, generated by the corresponding $c \in \hat{\mathbf{J}}_{\mathrm{Q}}$, has no lacunarity properties.

*Example* 3.1. — Given $c \in \hat{\mathbf{J}}_{\mathrm{Q}}$ with corresponding $\{k_p\}$. Let $k_p > 0$, $k_p$ increasing to infinity. If $\{k_m\}$ were lacunary then $k_{m+1}/k_m \geqslant \delta > 1$ for all $m$. Suppose this is the case. Then for all $m$

$$\frac{k_{2m}}{k_m} = \frac{k_{2m}}{k_{2m-1}} \frac{k_{2m-1}}{k_{2m-2}} \cdots \frac{k_{m+1}}{k_m} > \delta^{2m}.$$

On the other hand

$$\frac{k_{2m}}{k_m} = 1 + \frac{k_2}{k_m}$$

so that since $k_p \to \infty$, $\{k_m\}$ can not be lacunary.

## 4. Idelic pseudo-measures.

Given $c \in \hat{\mathbf{J}}_{\mathrm{Q}}$ with trivial conductor, and corresponding $\{k_n\} \subseteq \mathbf{Z}$ and F. Our problem is to find conditions in order that

$$T_s \sim \sum_n \frac{1}{n^s} e^{ik_n x}, \qquad 0 < \sigma \leqslant 1$$

represent an element of $A'(\mathbf{T})$ for a fixed $s$, $0 < \sigma \leqslant 1$; by this we mean that we wish to find conditions on $\{k_p\}$ for which

$$(4.1) \quad b_s(n) = \sum_{m \in H(n)} \frac{1}{m^s}, \qquad H(n) = \{m : k_m = n\},$$

is a bounded sequence. When the sequence $\{b_s(n) : n \in \mathbf{Z}\}$ is bounded we say that $c$ determines the pseudo-measure $T_s$.

Clearly Dirichlet characters do not yield pseudo-measures in this way. As a generalization of this fact, we have.

PROPOSITION 4.1. — *Given* $c \in \hat{J}_Q$ *with corresponding* $\{k_p\} \subseteq \mathbf{Z}^+$. *If* $\{k_p\}$ *is bounded* $c$ *does not determine a pseudo-measure for any* $s$, $0 < \sigma \leqslant 1$; *further, there is* $n \geqslant 1$ *such that*

$$\sum_{m \in H(n)} \frac{1}{m}$$

*diverges.*

*Proof.* — Assuming $1 \leqslant k_p \leqslant B$ we find $n$ such that

$$\sum_{k_m = n} \frac{1}{m}$$

diverges. Let $C = 1/(B + 1)$.

*a*) We first observe that

$$\exists n \in [1, B] \quad \text{such that} \quad \forall k \geqslant 1, \tag{4.2}$$
$$N_j/(j - k) \geqslant C$$

for infinitely many $j \geqslant k$, where

$$N_j = \text{card}\,\{k_p = n : p_k \leqslant p \leqslant p_j\}$$

and $p_j$ is the $j$-th prime.

*b*) Choose $n$ from *a*. Then

$$\sum_{\substack{m=a \\ p_m \in H(n)}}^{j} \frac{1}{p_m} \geqslant \sum_{m=j-N_j+1}^{j} \frac{1}{p_m} \sim \log p_j/p_{j-N_j+1}$$

by the integral test. Consequently from the prime number theorem

$$\sum_{\substack{m=a \\ p_m \in H(n)}}^{j} \frac{1}{p_m} \geqslant \log \frac{j \log j}{(j - N_j + 1) \log (j - N_j + 1)} \tag{4.3}$$
$$\geqslant \log \frac{j}{j - N_j + 1}.$$

We apply (4.2) which yields $j - N_j + 1 \leqslant j(1 - C) + 1 + Ca$

and so the right hand side of (4.3) is greater than or equal to

$$\log\left(1\Big/\left[(1-C)+\frac{1}{j}+C\frac{a}{j}\right]\right).$$

Now choose $K, \alpha > 0$ such that $\log(1/[(1-C)+K]) > \alpha$; and then take $j$ large enough so that

$$(1-C)+\frac{1}{j}+C\frac{a}{j} < 1-C+K.$$

Letting $a = a_i$ we define $a_{i+1} = j+1$.

Thus starting with $a_1 = 1$ we form $\{a_i\}$ and

$$\sum_{m\in H(n)}\frac{1}{m} \geqslant \sum_{p_m\in H(n)}\frac{1}{p_m} = \sum_{i=1}^{\infty}\sum_{\substack{m=a_i\\ p_m\in H(n)}}^{a_{i+1}}\frac{1}{p_m} > \sum_{1}^{\infty}\alpha = \infty.$$

q.e.d.

Proposition 4.2. — *Given* $c \in \hat{J}_Q$ *with corresponding* $\{k_p\} \subseteq \mathbf{Z}^+$ *increasing to infinity. Then*

$$\forall n, \qquad \text{card } H(n) < \infty.$$

*Proof.* — Given $n = \Pi p^r$; to show card $H(k_n) < \infty$. Choose a prime $q = p_k$ such that $k_q > k_n$, by hypothesis.

For each $j < k$ choose $r_j \in \mathbf{Z}$ such that $r_j k_{p_j} > k_q$, and define

$$m_0 = \prod_{j=1}^{k-1} p_j^{r_j}.$$

Then

$$\forall m \geqslant m_0, \qquad k_m \geqslant k_q > k_n$$

and so $H(k_n)$ is finite.

q.e.d.

We now give a procedure to find $c \in \hat{J}_Q$ which determine $T_s \in A'(\mathbf{T})\setminus M(\mathbf{T})$.

Lemma. — *Let* $T \sim \sum_{1}^{\infty} c_n e^{inx} \in A'(\mathbf{T})$. *If*

$$\sum_{1}^{\infty} c_n/n \quad \textit{diverges}$$

*then* $T \in A'(\mathbf{T})\setminus M(\mathbf{T})$.

*Proof.* — If $T \in M(\mathbf{T})$ then

$$\forall t(x) = \sum_{|n| \leqslant N} a_n e^{inx}, \qquad \left| \sum_{|n| \leqslant N} a_n c_n \right| \leqslant \|T\|_1 \, \|t\|_\infty .$$

Take $a_n \geqslant 0$ for $n > 0$, $\{a_n\}$ decreasing to 0, and set $a_n = - a_{-n}$ for $n < 0$.

Recall, $\left| \sum_1^N a_n \sin nx \right| \leqslant A(\pi + 1)$ if $na_n \leqslant A$.

For such $a_n$,

$$\left| \sum_{n=1}^N a_n c_n \right| = \left| \sum_{|n| \leqslant N} a_n c_n \right| \leqslant \|T\|_1 \, 2A(\pi + 1)$$

and so $\sum_1^\infty a_n c_n$ converges.

q.e.d.

For technical convenience we now let $s = 1$.

PROPOSITION 4.3. — *Given* $c \in \hat{\mathbf{J}}_Q$ *with corresponding* $\{k_p\} \subseteq \mathbf{Z}^+$. *If*

(4.4) $$k_p = O(\log^2 p), \qquad p \to \infty$$

*then* $T_1 \in A'(\mathbf{T})$; *and* $T_1 \notin M(T)$ *if*

$$\sum_1^\infty \hat{T}_1(n)/n$$

*diverges.*

*Proof.* — The fact that $T_1 \notin M(\mathbf{T})$ follows from the *Lemma.* Assume $\{k_p\}$ increases ot infinity and $k_{p_1} > 0$.

Let $\Pi(r)$ be the set of all integers

$$n = \prod_{j=1}^r p_j^{r_j}, \qquad r_j \geqslant 0.$$

We first observe that if

$$a = \sum_1^r 1/p_j,$$
$$b = \sum_{m \in \Pi(r)} 1/m,$$

then

(4.5) $$e^a - 1 < b < e^{2a} - 1.$$

This follows since

$$1 + b = \prod_{j=1}^{r}\left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right)$$

and

$$e^{1/p} < 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots = 1 + \frac{1}{p-1} \leqslant 1 + \frac{2}{p} < e^{2/p}.$$

Set

$$H_{n,r} = \sum_{m \in H(n) \cap \Pi(r)} \frac{1}{m}.$$

We use (4.4), (4.5), and Mertens' estimate

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + C + O(1/\log x), \qquad x \to \infty,$$

to calculate

$$\begin{aligned}(4.6)\quad H_{n,r} &\leqslant \sum_{m \in \Pi(r)} 1/m \leqslant \exp 2 \sum_{1}^{r} 1/p_j \\ &\leqslant C\,(\log p_r)^2 \leqslant \frac{K}{2} k_{p_r} < K k_{p_r}\left(1 - \frac{1}{p_{r+1} - 1}\right), \qquad K \geqslant 1,\end{aligned}$$

for all $n$.

Note that

$$\Pi(r+1) = \Pi(r) \cup p_{r+1}\Pi(r) \cup p_{r+1}^2\Pi(r) \cup \cdots,$$

a disjoint union. Thus,

$$H(n) \cap \Pi(r+1) = \bigcup_{j=0}^{\infty} (H(n) \cap p_{r+1}^j \Pi(r))$$

is a disjoint union.

If $m \in H(n) \cap \Pi(r+1)$ then $k_m = n$ and $m = up_{r+1}^j$, $u \in \Pi(S_r)$. Thus

$$k_u = n - jk, \qquad k = k_{p_{r+1}}.$$

Consequently,

$$\sum_{m \in H(n) \cap p_{r+1}^j \Pi(r)} 1/m = \sum_{\substack{u \in H(n-jk) \\ u \in \Pi(r)}} 1/(up_{r+1}^j) = \frac{1}{p_{r+1}^j} H_{n-jk,r},$$

and so

$$(4.7)\quad H_{n,r+1} = H_{n,r} + \frac{1}{p_{r+1}} H_{n-k,r} + \frac{1}{p_{r+1}^2} H_{n-2k,r} + \cdots.$$

From (4.5) there is $M \geqslant 1$ such that

$$\forall n, \qquad H_{n,1} \leqslant M.$$

Since $b_1(n) = \sup_r H_{n,r}$ we'll prove that if $n$ is fixed then

$$H_{n,r} \leqslant MKk_{p_1} \quad \text{for each } r.$$

Using (4.6) and (4.7),

$$\begin{aligned} H_{n,2} &\leqslant H_{n,1} + M\left(\frac{1}{p_2 - 1}\right) \\ &\leqslant MKk_{p_1}\left(1 - \frac{1}{p_2 - 1}\right) + MKk_{p_1}\left(\frac{1}{p_2 - 1}\right) = MKk_{p_1}. \end{aligned}$$

We argue similarly for any $H_{n,r}$.

q.e.d.

*Example* 4.1. — Condition (4.4) determines many pseudo-measures. We note that $k_p = [\log p]$ also determines a pseudo-measure by appropriate technical refinements. In order to give specific examples of $T \in A'(\mathbf{T}) \setminus M(\mathbf{T})$ determined by $c \in \hat{\mathbf{J}}_Q$ we now use $k_p = [\log p]$, $k_2 = 1$ (for technical convenience only). If $n$ is given and $k \leqslant n$ then from the prime number theorem ($p_j \sim j \log j$, $j \to \infty$) the number of primes in $[e^k, e^{k+1})$ is estimated by $e^k(k(e-1)-1)/k(k+1) \equiv N_k$. Thus there are approximately $N_k$ primes $p$ for which $k_p = k$. Therefore we write

$$\text{(4.8-2)} \qquad \begin{array}{rl} \underline{n} & \\ n = & 1 + 1 + \cdots + 1 \\ = & k_3 + 1 + \cdots + 1 \\ & \vdots \\ = & k_{p(n)} \end{array} \qquad\qquad \begin{array}{rl} \underline{m} & (\text{where } k_m = n) \\ m = & 2 \\ = & 3 \cdot 2^{n-k_3} \\ & \vdots \\ = & p(n) \end{array}$$

where $p(n)$ is the largest prime $p$ for which $k_p = n$. Consequently the sum $\Sigma \dfrac{1}{m}$ for those $m$ listed in (4.8) is bounded below approximately by

$$\text{(4.9-2)} \qquad \frac{1}{e} \sum_{\substack{k+a=n+1 \\ a \geqslant 0 \\ k \geqslant 2}} \frac{1}{2^a} \frac{(k-1)(e-1)-1}{(k-1)k}.$$

We rewrite (4.9) as

$$\frac{1}{e}\sum_{a=0}^{n-1}\frac{1}{2^a}\frac{(n-a)(e-1)-1}{(n-a)(n+1-a)}$$

and estimate it by

$$(4.10\text{-}2)\quad \frac{1}{n}+\frac{1}{2(n-1)}+\frac{1}{2^2(n-2)}+\cdots+\frac{1}{2^{n-1}}$$

(we can also estimate the integral

$$e^{-n}\int_1^n \frac{e^x}{x}\,dx\Big).$$

For the next steps we form (4.8-3), ..., (4.8-$p$), ... where if $m$ is listed in (4.8-$p$) and $k_m = n$ then

$$m = q2^{a_2}3^{a_3}\ldots p^{a_p}$$

where $q$ is a prime or 1 and if $a_2 + \cdots + a_{p-1} > 0$ then $a_p > 0$. We form the corresponding sum (4.9-$p$) by again counting the number of primes $q$ in the allowable (that is, $a_p > 0$ and $n = k_q + a_2k_2 + \cdots + a_pk_p$) intervals $[e^k, e^{k+1})$. Consequently, we form a sequence of finite sums (4.10-$p$) whose total sum over $p$ is a lower bound $b(n)$ of $\hat{T}_1(n)$ and check that $\sum_1 b(n)/n$ diverges.

*Example* 4.2. — If $E \subseteq \mathbf{Z}$ is lacunary (e.g., *Example* 3.1) then E is Sidon [6]. Sidon sets are a special case of $\Lambda(t)$ sets for all $t \in (0, \infty)$. $\gamma(t)$ sets E, for $t \in (1, \infty)$, are characterized by the property that

$$\{\mu \in M(\mathbf{T}) : \forall n \notin E,\ \hat{\mu}(n) = 0\} \subseteq L^t(\mathbf{T}).$$

Given $c \in \hat{\mathbf{J}}_Q$ with corresponding $k_p$ (let $\{k_p\}$ increase to infinity with $k_2 \geqslant 1$). $r_t(n, \{k_p\})$ is the number of representations of $n$ as a sum of $t$ elements, possibly repeated, from $\{k_p\}$. When $\{k_p\}$ is not $\Lambda(2t)$ for any $t$ then

$$\forall t, \qquad \sup_n r_t(n, \{k_p\}) = \infty.$$

Consequently, such $c \in \hat{\mathbf{J}}_Q$ which further satisfy (4.4) are a natural source in which to find $T \in A'(\mathbf{T})\diagdown M(\mathbf{T})$.

## BIBLIOGRAPHY

[1] J. Benedetto, Harmonic Synthesis and Pseudo-Measures, U. of Maryland Mathematics Dept. Lecture Notes No. 5 (1968).
[2] J. Benedetto, Harmonic Analysis on Totally Disconnected Sets, Lecture Notes in Mathematics 202, Springer-Verlag, New York (1971).
[3] J. Benedetto, Dirichlet Series, Spectral Synthesis, and Algebraic Number Fields, U. of Maryland Mathematics Dept. TR 71-41 (1971), 1-23.
[4] J. W. S. Cassels and A. Fröhlich, (editors) Algebraic Number Theory, Thompson Book Company, Washington, D.C. (1967).
[5] L. J. Goldstein, Analytic Number Theory Prentice, Hall, N. J. (1971).
[6] J.-P. Kahane, Séries de Fourier absolument convergentes, Springer-Verlag, New York (1970).
[7] I. Richards, « On the Disproof of Spectral Synthesis » *J. of Comb. Theory* 2 (1967), 61-70.
[8] A. Weil, Basic Number Theory, Springer-Verlag, New York (1967).

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