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THEOREMS OF KOROVKIN TYPE FOR ADAPTED SPACES

by Heinz BAUER

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*Dedicated
to Professor Karl Stein, Munich,
on the occasion
of his* 60^{th} *birthday.*

Introduction.

Korovkin's theorem [7, 8], sometimes also called Bohman-Korovkin theorem, arose from the study of the role of Bernstein polynomials in the proof of the Weierstrass approximation theorem. For sequences $(L_n)_{n\in N}$ of positive linear maps $L_n : \mathscr{C}([0, 1]) \to \mathscr{C}([0, 1])$ the theorem of Korovkin states : $(L_n f)$ converges uniformly on $[0, 1]$ to f for all $f \in \mathscr{C}([0, 1])$ provided that $(L_n f)$ converges uniformly to f for the three functions $f(x) = 1, = x, = x^2$ defined on $[0, 1]$.

Several generalizations of this theorem have been obtained by different authors in recent years. The most typical ones are due to Šaškin [10, 11, 12] and Wulbert [15] where $[0, 1]$ is replaced by a compact (Hausdorff) space X and where the role of the three functions id^ν, $\nu = 0, 1, 2$, is taken over by a subset $\mathscr{F}$ of $\mathscr{C}(X)$. One always assumes that the constant function 1 is in $\mathscr{F}$ and that $\mathscr{F}$ separates the points of X. The problem is to find conditions under which for every sequence or net $(L_i)_{i\in I}$ of positive linear maps L_i : $\mathscr{C}(X) \to \mathscr{C}(X)$, uniform convergence of $(L_i f)$ to f for all f in $\mathscr{F}$ or, equivalently, for all functions f in the linear subspace $\mathscr{H}$ of $\mathscr{C}(X)$ spanned by $\mathscr{F}$ implies uniform convergence of $(L_i f)$ to f for all $f \in \mathscr{C}(X)$. It follows from the work of Šaškin and Wulbert that the condition " X is the Choquet boundary of X with respect to $\mathscr{H}$ " is both necessary and sufficient. Wulbert shows in addition that the operators $L_i : \mathscr{C}(X) \to \mathscr{C}(X)$ may be assumed to be only linear and of norm $\leqslant 1$. In the classical situation of Korovkin the above condition can be easily verified.

For positive linear maps L_i of $\mathscr{C}(X)$ into the space $\mathscr{B}(X)$ of all bounded real-valued functions on X, also the more general problem of characterizing the Korovkin closure $\mathscr{H}^k$ of $\mathscr{H}$ in $\mathscr{C}(X)$ was treated by Baskakov [1] and Šaškin [10]. The Korovkin closure $\mathscr{H}^k$ is by definition the set of all functions f in $\mathscr{C}(X)$ such that $(L_i f)$ converges uniformly to f for all nets of positive linear maps $L_i : \mathscr{C}(X) \to \mathscr{B}(X)$ provided that $(L_i h)$ converges uniformly to h for all h in $\mathscr{H}$. Šaškin [10, 12] observed that this problem is closely related to the author's work [2] on an abstract Dirichlet problem and that $\mathscr{H}^k$ coincides with the space $\hat{\mathscr{H}}$ of all $\mathscr{H}$-harmonic functions introduced in [2].

The present paper has mainly two intentions. First, in all known proofs of Korovkin-type theorems linearity of the operators L_i is used in an essential way. We will show that the methods developed in [2] lead in a very simple and natural way to generalizations of Korovkin's theorem for increasing, possibly non-linear maps $L_i : \mathscr{C}(X) \to \mathscr{B}(X)$. These generalizations reproduce with new and extremely easy proofs all the previously known results for positive linear maps. Secondly, we will show that all these generalizations can be obtained for locally compact (Hausdorff) spaces. The key to this second direction of study is Choquet's notion of an adapted vector space of continuous real-valued functions. In view of the results of Mokobodzki-Sibony [9] on adapted cones of continuous functions and their fine boundary, this second possibility of extending Korovkin's theorem seems to be quite natural.

The paper is split up into three sections. § 1 shows that Korovkin-type theorems can be obtained with our methods on arbitrary topological spaces without any use of measure theory. § 2 treats the special case of adapted spaces and shows how a limited amount of measure theory connects our results with older ones. § 3 is concerned with characterizations of the Korovkin closure. Special emphasis is given to the different role of increasing (or positive linear) maps of $\mathscr{C}(X)$ into $\mathscr{B}(X)$ and of maps with values in $\mathscr{C}(X)$ for compact spaces X and, with certain previsions, for locally compact spaces X. Some of our results seem to be new even for compact spaces and positive linear maps.

For the reader's convenience the paper is written in a self-contained manner. The author is grateful to Dr. A. Lupaş, Cluj, for having attracted his attention to the work of Šaškin.

1. $\mathscr{H}$-affine functions and Korovkin-type theorems.

In this section we denote by X a topological space and by $\mathscr{H}$ a linear subspace of the linear space $\mathscr{C}(X) = \mathscr{C}(X, \mathbf{R})$ of all real-valued continuous functions on X.

A function $f\colon X \to \mathbf{R}$ will be called *$\mathscr{H}$-bounded* if f is minorized and majorized by functions $h_1, h_2 \in \mathscr{H}$, i.e. $h_1 \leqslant f \leqslant h_2$. The set $\mathscr{H}^b$ of all $\mathscr{H}$-bounded functions on X is a linear subspace of the linear space $\mathbf{R}^X$ of all real-valued functions on X. A linear subspace of $\mathscr{H}^b$ is

$$\mathscr{H}_0 := \mathscr{H}^b \cap \mathscr{C}(X). \tag{1.1}$$

For every $\mathscr{H}$-bounded function f we define the two envelopes

$$\hat{f} := \inf \{h \in \mathscr{H} | h \geqslant f\} \tag{1.2}$$

and

$$\check{f} := \sup \{h \in \mathscr{H} | h \leqslant f\} = -\widehat{-f}. \tag{1.3}$$

Then $\hat{f}$ and $\check{f}$ are upper resp. lower semi-continuous $\mathscr{H}$- bounded functions satisfying $\check{f} \leqslant f \leqslant \hat{f}$. For f, $g \in \mathscr{H}^b$ we have

$$\widehat{f+g} \leqslant \hat{f} + \hat{g} \qquad \text{and} \qquad \widehat{\lambda f} = \lambda \hat{f},\ \lambda \in \mathbf{R}_+. \tag{1.4}$$

Hence the set

$$\hat{\mathscr{H}} := \{f \in \mathscr{H}^b | \check{f} = \hat{f}\} = \{f \in \mathscr{H}_0 | \check{f} = \hat{f}\} \tag{1.5}$$

is a linear subspace of $\mathscr{H}_0$ satisfying

$$\mathscr{H} \subset \hat{\mathscr{H}} \subset \mathscr{H}_0 \subset \mathscr{H}^b. \tag{1.6}$$

The functions in $\hat{\mathscr{H}}$ will be called $\mathscr{H}$*-affine* (or, as in [2], $\mathscr{H}$-harmonic) functions on X. They can be characterized as follows :

1.1 Lemma. — *A function* $f: X \to \mathbf{R}$ *is* $\mathscr{H}$*-affine if and only if for every quasi-compact set* $K \subset X$ *and for every* $\varepsilon > 0$ *there exist finitely many functions* $h'_1, \ldots, h'_n$ *and* $h''_1, \ldots, h''_n$ *in* $\mathscr{H}$, $n \in \mathbf{N}$, *satisfying*

$$h'_j \leqslant f \leqslant h''_j \quad (j = 1, \ldots, n) \tag{1.7}$$

and

$$\bar{h}(x) - \underline{h}(x) < \varepsilon \quad \textit{for all } x \in K, \tag{1.8}$$

where

$$\bar{h} := \inf(h''_1, \ldots, h''_n) \quad \textit{and} \quad \underline{h} := \sup(h'_1, \ldots, h'_n). \tag{1.9}$$

Proof. — For $f \in \hat{\mathscr{H}}$ the existence of $h'_1, \ldots, h'_n$ and $h''_1, \ldots, h''_n$ can be proved as in [2; p. 102]. Conversely, for a singleton $K = \{x\}$ the conditions (1.7) and (1.8) imply the existence of functions $h', h'' \in \mathscr{H}$ such that $h' \leqslant f \leqslant h''$ and $h''(x) - h'(x) < \varepsilon$ for given $\varepsilon > 0$. Therefore,

$$\hat{f}(x) - \check{f}(x) < \varepsilon,$$

i.e. f is $\mathscr{H}$-affine.

We will now study the behavior of $\mathscr{H}$-affine functions in settings similar to the classical Korovkin theorem. In the remaining part of this section $(L_i)_{i\in I}$ will denote a *net* of *increasing* maps $L_i : H_0 \to \mathbf{R}^X$. Hence L_i maps each $f \in \mathscr{H}_0$ into a real-valued function $L_i f$ on X such that for f_1, $f_2 \in \mathscr{H}_0$ the relation $f_1 \leqslant f_2$ implies $L_i f_1 \leqslant L_i f_2$. I is the directed index set of the net. More generally, I can be an arbitrary index set with a given filter in it.

1.2 Proposition. — *Let* x_0 *be a point in* X. *If*

$$\lim_{i \in I} L_i h(x_0) = h(x_0) \quad \textit{for all } h \in \mathscr{H},$$

then

$$\check{f}(x_0) \leqslant \liminf_{i \in I} L_i f(x_0) \leqslant \limsup_{i \in I} L_i f(x_0) \leqslant \hat{f}(x_0)$$

for all $f \in \mathscr{H}_0$.

Proof. — Consider functions $g, h \in \mathscr{H}$ satisfying $g \leqslant f \leqslant h$. Since every L_i is increasing we obtain

$$L_i g(x_0) \leqslant L_i f(x_0) \leqslant L_i h(x_0)$$

for all $i \in I$. This implies

$$\begin{aligned} g(x_0) = \lim L_i g(x_0) &\leqslant \liminf L_i f(x_0) \leqslant \limsup L_i f(x_0) \\ &\leqslant \lim L_i h(x_0) = h(x_0); \end{aligned}$$

from this and the definition of $\hat{f}$ and $\check{f}$ the result follows.

The applications which we intend to give suggest the following terminology: The net $(L_i)_{i \in I}$ will be said to *converge* pointwise on X (resp. uniformly on the quasi-compact subsets of X) (1) *to the identity on a subset* $\mathscr{P}$ *of* $\mathscr{H}_0$ if

$$\lim_{i \in I} L_i p(x) = p(x) \tag{1.10}$$

for all $p \in \mathscr{P}$ and all $x \in X$ (resp. if for all $p \in \mathscr{P}$ the convergence (1.10) is uniform on every quasi-compact subset K of X).

With the definition of $\mathscr{H}$-affine functions we thus obtain:

1.3 Corollary. — *The net* $(L_i)_{i \in I}$ *converges pointwise on* X *to the identity on* $\hat{\mathscr{H}}$ *if it converges pointwise to the identity on* $\mathscr{H}$.

We now replace pointwise convergence by uniform convergence on quasi-compact subsets of X.

1.4 Proposition. — *Let* K *be a quasi-compact subset of* X, *and suppose that* $(L_i h)_{i \in I}$ *converges uniformly on* K *to* h *for all* $h \in \mathscr{H}$. *Then* $(L_i f)_{i \in I}$ *converges uniformly on* K *to* f *for all functions* $f \in \mathscr{H}_0$ *satisfying* $\check{f}(x) = \hat{f}(x)$ *for all* $x \in K$.

(1) For a locally compact (Hausdorff) space X, uniform convergence on the compact subsets of X will be also called "locally uniform convergence on X".

Proof. — Let f be a function in $\mathscr{H}_0$ satisfying $\check{f} = \hat{f}$ on K. As in [2; p. 102] one proves for every $\varepsilon > 0$ the existence of functions $h'_1, \ldots, h'_n$ and $h''_1, \ldots, h''_n$ in $\mathscr{H}$, $n \in \mathbf{N}$, such that $h'_j \leqslant f \leqslant h''_j$, $j = 1, \ldots, n$, and

$$\bar{h}(x) - \underline{h}(x) < \varepsilon$$

for all $x \in \mathrm{K}$. Here $\underline{h}$ and $\bar{h}$ are defined as in (1.9). By assumption there exists a section $\mathrm{S} = \{i \in \mathrm{I} | i \geqslant i_0\}$ of I such that for all $i \in \mathrm{S}$, all $j = 1, \ldots, k$ and for all $x \in \mathrm{K}$ we have

$$|\mathrm{L}_i h'_j(x) - h'_j(x)| \leqslant \varepsilon$$

and

$$|\mathrm{L}_i h''_j(x) - h''_j(x)| \leqslant \varepsilon.$$

Furthermore, $\mathrm{L}_i h'_j \leqslant \mathrm{L}_i f \leqslant \mathrm{L}_i h''_j$ for all $i \in \mathrm{I}$ and $j = 1, \ldots, n$ since all L_i are increasing maps. From this we conclude

$$h'_j(x) - \varepsilon \leqslant \mathrm{L}_i h'_j(x) \leqslant \mathrm{L}_i f(x) \leqslant \mathrm{L}_i h''_j(x) \leqslant h''_j(x) + \varepsilon$$

for $i \in \mathrm{S}, j = 1, \ldots, n$, and $x \in \mathrm{K}$; hence

$$\underline{h}(x) - \varepsilon \leqslant \mathrm{L}_i f(x) \leqslant \bar{h}(x) + \varepsilon$$

holds on K for all $i \in \mathrm{S}$. Together with $\underline{h} \leqslant f \leqslant \bar{h}$ this leads to

$$|\mathrm{L}_i f(x) - f(x)| \leqslant \bar{h}(x) - \underline{h}(x) + \varepsilon < 2\varepsilon$$

for all $x \in \mathrm{K}$ and $i \in \mathrm{S}$. This proves the uniform convergence of $(\mathrm{L}_i f)$ on K to f.

1.5 Corollary. — *The net* $(\mathrm{L}_i)_{i \in \mathrm{I}}$ *converges uniformly on the quasi-compact subsets of* X *to the identity on* $\hat{\mathscr{H}}$ *if it converges uniformly on the quasi-compact subset of* X *to the identity on* $\mathscr{H}$.

2. Case of an adapted space $\mathscr{H}$.

From now on X will be a *locally compact* (Hausdorff) space. Furthermore, the linear subspace $\mathscr{H}$ of $\mathscr{C}(\mathrm{X})$ will be assumed to be *adapted* in the sence of Choquet [3; p. 274].

This means by definition that $\mathscr{H}$ has the following three properties:

$$\mathscr{H} = \mathscr{H}^+ - \mathscr{H}^+; \qquad (2.1)$$

$$\text{for all } x \in X \text{ there is an } f \in \mathscr{H}^+ \text{ such that } f(x) > 0; \qquad (2.2)$$

$$\mathscr{H}^+ \subset o(\mathscr{H}^+). \qquad (2.3)$$

Recall that $\mathscr{H}^+ = \{h \in \mathscr{H} | h \geqslant 0\}$ and that a real-valued function f on X is said to be *dominated* by a real-valued function $g \geqslant 0$ on X if for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| \leqslant \varepsilon g(x)$ holds on $X \setminus K$. We will then write $f \in o(g)$. By $o(\mathscr{H}^+)$ we denote the set $\bigcup_{h \in \mathscr{H}^+} o(h)$. Therefore (2.3) states that every function in $\mathscr{H}^+$ is dominated by some other function in $\mathscr{H}^+$. Observe that $o(\mathscr{H}^+)$ is a linear subspace of $\mathbf{R}^X$. Therefore (2.1) and (2.3) imply $\mathscr{H} \subset o(\mathscr{H}^+)$.

Under these assumptions we have

$$\mathscr{H}^b = \{f \in \mathbf{R}^X | \ |f| \leqslant h \text{ for some } h \in \mathscr{H}^+\}; \qquad (2.4)$$

$$\mathscr{H}^b \subset o(\mathscr{H}^+); \qquad (2.5)$$

$$\mathscr{H}_0 = o(\mathscr{H}^+) \cap \mathscr{C}(X); \qquad (2.6)$$

$$\mathscr{C}_c(X) \subset \mathscr{H}_0, \qquad (2.7)$$

where $\mathscr{C}_c(X)$ is the linear space of all $f \in \mathscr{C}(X)$ with *compact support.* — Obviously (2.4) follows from (2.1) since $\mathscr{H}^+$ is a convex cone. (2.5) is a consequence of (2.3) and (2.4). Since $\mathscr{H}_0 = \mathscr{H}^b \cap \mathscr{C}(X)$ it suffices to prove that

$$o(\mathscr{H}^+) \cap \mathscr{C}(X) \subset \mathscr{H}_0$$

in order to obtain (2.6). Every $f \in o(\mathscr{H}^+) \cap \mathscr{C}(C)$ is dominated by some $h \in \mathscr{H}^+$. Therefore, $|f(x)| \leqslant h(x)$ holds on the complement of a compact set $K \subset X$. From (2.2) follows the existence of a function $g \in \mathscr{H}^+$ which is strictly positive on K. The continuous function f is bounded on K, hence $|f(x)| \leqslant \lambda g(x)$ on K for some $\lambda > 0$. But then $|f| \leqslant \lambda g + h$ on X, that is $f \in \mathscr{H}_0$. (2.7) is an immediate consequence of (2.6) since every $f \in \mathscr{C}_c(X)$ is dominated by even every $h \in \mathscr{H}^+$.

2.1 *Remark.* — All our assumptions are satisfied when X is compact and $\mathscr{H}$ is a linear subspace of $\mathscr{C}(X)$ containing the constant functions. In this case $\mathscr{H}_0 = \mathscr{C}(X)$ and $\mathscr{H}^b$ is the space of all bounded real-valued functions on X. Some of the results in [2] are now extended to the more general situation of this section.

For a point $x \in X$ a positive Radon measure μ on X will be called an *$\mathscr{H}$-representing measure* if all functions in $\mathscr{H}$ are μ-integrable and if $\int h\, d\mu = h(x)$ holds for all $h \in \mathscr{H}$. The set of all $\mathscr{H}$-representing measures for x will be denoted by $\mathscr{M}_x = \mathscr{M}_x(\mathscr{H})$. For $\mu \in \mathscr{M}_x$ all functions in $\mathscr{H}_0$ are μ-integrable by (2.4).

2.2 Lemma. — *For all $x \in X$ and all $f \in \mathscr{H}_0$ the set*

$$\left\{\int f\, d\mu \,|\, \mu \in M_x\right\}$$

is the compact interval $[\check{f}(x), \hat{f}(x)]$.

Proof (compare with [2; p. 101] and [9; p. 5-18]). — Obviously $\int f\, d\mu \in [\check{f}(x), \hat{f}(x)]$ for all $\mu \in \mathscr{M}_x$. Conversely, let γ be a number in $[\check{f}(x), \hat{f}(x)]$. On $\mathscr{H}_0$ the map $g \longmapsto \hat{g}(x)$ is a sublinear functional p. This functional majorizes the linear form $\lambda f \longmapsto \lambda\gamma$ defined on the linear subspace $\mathbf{R}f$ of $\mathscr{H}_0$ generated by f. The Hahn-Banach theorem hence implies the existence of a linear form T on $\mathscr{H}_0$ satisfying $T \leqslant p$ and $T(f) = \gamma$. T is positive because of the first condition: $g \in \mathscr{H}_0$, $g \leqslant 0$ implies $T(g) \leqslant p(g) = g(x) \leqslant 0$. It follows from (2.6) that together with $\mathscr{H}$ also $\mathscr{H}_0$ is an adapted linear subspace of $\mathscr{C}(X)$. Therefore, by Choquet [3; p. 276] there exists a positive Radon measure μ on X such that $\mathscr{H}_0 \subset \mathscr{L}^1(\mu)$ and $T(g) = \int g\, d\mu$ for all $g \in \mathscr{H}_0$. Then $\mu \in \mathscr{M}_x$ and $\int f\, d\mu = \gamma$ since

$$\int h\, d\mu = T(h) \leqslant p(h) = \hat{h}(x) = h(x)$$

for all $h \in \mathscr{H}$ implies $\int h\, d\mu = h(x)$ for all $h \in \mathscr{H}$.

2.3 Corollary 1. — *A function $f : X \to \mathbf{R}$ is $\mathscr{H}$-affine if and only if $f \in \mathscr{H}_0$ and $\int f\, d\mu = f(x)$ for all $\mu \in \mathscr{M}_x$.*

Indeed, for $x \in X$ and $f \in \mathscr{H}_0$, $\check{f}(x) = \hat{f}(x)$ is equivalent to $\{f(x)\} = \left\{\int f\, d\mu : \mu \in M_x\right\}$ by the preceding lemma.

We are now able to connect propositions 1.2 and 1.4 with the notion of the Choquet boundary. Recall that

$$\partial_{\mathscr{H}} X := \{x \in X : \mathscr{M}_x = \{\varepsilon_x\}\} \tag{2.8}$$

is the Choquet boundary of X with respect to $\mathscr{H}$.
From lemma 2.2. we then deduce :

2.4 Corollary 2. — *A point $x \in X$ lies in the Choquet boundary $\partial_{\mathscr{H}} X$ if and only if $\check{f}(x) = \hat{f}(x)$ — or equivalently, $\hat{f}(x) = f(x)$ — holds for all $f \in \mathscr{H}_0$.*

Proof. — $\check{f}(x) = \hat{f}(x)$ for all $f \in \mathscr{H}_0$ is equivalent to

$$\int f\, d\mu = f(x)$$

for all $\mu \in \mathscr{M}_x$. Since, by (2.7), $\mathscr{C}_c(X) \subset \mathscr{H}_0$ this means that $\mathscr{M}_x$ reduces to $\{\varepsilon_x\}$. The rest of the statement follows from $-\check{f} = \widehat{-f}$ and $\check{f} \leqslant f \leqslant \hat{f}$.
In particular, we have (cf. [2]) :

2.5 Corollary 3. — $\partial_{\mathscr{H}} X = X$ *if and only if* $\hat{\mathscr{H}} = \mathscr{H}_0$.

2.6 *Remark.* — By a result of Mokobodzki-Sibony [9; p. 5-26] $\partial_{\mathscr{H}} X$ is non-empty if X is countable at infinity and if in addition $\mathscr{H}$ is linearly separating, i.e. for every pair of points $x \neq y$ of X there exist functions $h_1, h_2 \in \mathscr{H}$ satisfying

$$h_1(x)\, h_2(y) \neq h_1(y)\, h_2(x).$$

Furthermore, $h \geqslant 0$ on $\partial_{\mathscr{H}} X$ implies $h \geqslant 0$ (on X) for all $h \in \mathscr{H}$.

2.7 Theorem. — *Let $(L_i)_{i \in I}$ be a net of increasing maps $L_i : \mathscr{H}_0 \to \mathbf{R}^X$, and let K be a compact subset of $\partial_{\mathscr{H}} X$. Assume that $(L_i h)_{i \in I}$ converges uniformly on K to h for all $h \in \mathscr{H}$. Then $(L_i f)_{i \in I}$ converges uniformly on K to f for all $f \in \mathscr{H}_0$.*

This is an immediate consequence of proposition 1.4 and corollary 2.4. By choosing a singleton for K it follows in particular, that pointwise convergence on $\partial_{\mathscr{H}}X$ of $(L_i h)$ to h for all $h \in \mathscr{H}$ implies pointwise convergence on $\partial_{\mathscr{H}}X$ of $(L_i f)$ to f for all $f \in \mathscr{H}_0$.

2.8 Corollary. — *Let $(L_i)_{i \in I}$ be a net of increasing maps $L_i : \mathscr{H}_0 \to \mathbf{R}^X$ and assume that $\partial_{\mathscr{H}}X = X$. Then pointwise (resp. locally uniform) convergence on X of the net (L_i) to the identity on $\mathscr{H}$ implies pointwise (resp. locally uniform) convergence on X of the net (L_i) to the identity on $\mathscr{H}_0$.*

We finally prove a Korovkin-type result which in its formulation is closest to the formulation of the classical Korovkin theorem. To simplify the statement of the result we define for a non-empty set $\mathscr{F} \subset \mathscr{C}(X)$

$$\mathscr{F}^\nu := \{f^\nu | f \in \mathscr{F}\} \quad \text{for } \nu = 0, 1, 2.$$

Hence $\mathscr{F}^0$ only contains the constant function 1, $\mathscr{F}^1 = \mathscr{F}$, and $\mathscr{F}^2$ is the set of all squares of functions in $\mathscr{F}$.

2.9 Proposition. — *Let $\mathscr{F}$ be a subset of $\mathscr{C}(X)$ separating the points of X and satisfying $\mathscr{F}^0 \cup \mathscr{F}^1 \subset o(\mathscr{F}^+)$. Consider a net $(L_i)_{i \in I}$ of positive linear maps of $\tilde{\mathscr{F}}$ into $\mathbf{R}^X$ where $\tilde{\mathscr{F}}$ is the linear space of all functions $f \in \mathscr{C}(X)$ such that $|f|$ is majorized by a linear combination of functions in $\mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2$. Assume that $(L_i f)$ converges on X pointwise* (resp. *locally uniformly*) *to f for all functions $f \in \mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2$. Then $(L_i f)$ converges on X pointwise* (resp. locally uniformly) *to f for all functions $f \in \tilde{\mathscr{F}}$, in particular for all bounded functions $f \in \mathscr{C}(X)$.*

Proof. — Consider the linear subspace $\mathscr{H}$ of $\mathscr{C}(X)$ generated by $\mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2$. Then $\mathscr{H}$ is adapted : By assumption, for every $f \in \mathscr{F}$ there exists a function $g \in \mathscr{F}^+$ such that $|f(x)| \leqslant g(x)$ holds in the complement $X \setminus K$ of a compact set $K \subset X$. Let $\alpha > 0$ be an upper bound for f on K. Then $|f| \leqslant \alpha + g$ on X, and the equality

$$f = (f + \alpha + g) - (\alpha + g)$$

shows that f is the difference of non-negative functions in the linear span of $\mathscr{F}^0 \cup \mathscr{F}^1$, hence of functions in $\mathscr{H}^+$. From the definition of $\mathscr{H}$ then follows $\mathscr{H} = \mathscr{H}^+ - \mathscr{H}^+$. Furthermore, $\mathscr{F} \subset o(\mathscr{F}^+)$ implies $\mathscr{F}^2 \subset o(\mathscr{F}^2) \subset o(\mathscr{H}^+)$. Since

$$\mathscr{F}^0 \cup \mathscr{F}^1 \subset o(\mathscr{F}^+) \subset o(\mathscr{H}^+),$$

we thus have $\mathscr{H} \subset o(\mathscr{H}^+)$. Property (2.2) of an adapted space follows trivially from $1 \in \mathscr{H}$.

The definition of $\tilde{\mathscr{F}}$ now shows that $\tilde{\mathscr{F}}$ is the space $\mathscr{H}_0$ derived from the adapted space $\mathscr{H}$. Because of the linearity of the maps $L_i : \mathscr{H}_0 \to \mathbf{R}^X$, our assumption implies pointwise (resp. locally uniform) convergence of $(L_i f)$ to f for all functions f in $\mathscr{H}$. Thus the statement of proposition 2.9 is a consequence of corollary 2.8 if $\partial_{\mathscr{H}} X = X$ holds. Therefore we consider $x \in X$ and $\mu \in \mathscr{M}_x$. For every $f \in \mathscr{F}$ the function $(f - f(x))^2$ is in $\mathscr{H}$ and vanishes at x. Hence

$$\int (f - f(x))^2 \, d\mu = 0$$

which implies that the support $S\mu$ of μ is contained in $\{f = f(x)\} : = \{x' \in X \mid f(x') = f(x)\}$, i.e.

$$S\mu \subset \bigcup_{f \in \mathscr{F}} \{f = f(x)\}.$$

This intersection equals $\{x\}$ since $\mathscr{F}$ separates points. We have $\int d\mu = 1$ because of $1 \in \mathscr{H}$. Hence μ must be the unit mass ε_x at x. This proves $\partial_{\mathscr{H}} X = X$.

2.10. *Remarks.* — 1) Let X be the unit interval [0, 1] and let $\mathscr{F}$ consist of the single function $x \longmapsto x$. Then the above proposition yields to the classical Korovkin theorem [7, 8].

2) For a compact space X all our assumptions about $\mathscr{F} \subset \mathscr{C}(X)$ reduce itself to a single one: $\mathscr{F}$ has to separate points. Under the assumption that each L_i is a positive linear map of $\mathscr{C}(X)$ into $\mathscr{C}(X)$ this particular form of proposition 2.9 can be found implicitely in Felbecker-Schempp [4] and explicitely in Grossman [6]. However, the proof is totally different.

3) In the situation of remark 2.1 and under the assumption that each L_i is a positive linear map of $\mathscr{C}(X)$ into the linear space $\mathscr{B}(X)$ of bounded real-valued functions, theorem 2.7 and its corollary are contained in a result of Wulbert [15]. However, Wulbert also treats nets of linear maps $L_i : \mathscr{C}(X) \to \mathscr{B}(X)$ with norm $\|L_i\| \leqslant 1$.

4) Corollary 2.8 can be applied in the following situation: $X = \mathbf{R}^n$ and $\mathscr{H}$ = linear space of all polynomials on $\mathbf{R}^n$ with real coefficients. Then $\mathscr{H}$ is adapted [3; p. 285] and $\partial_{\mathscr{H}} X = X$. Indeed, for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ consider the polynomial $y \longmapsto p(y) = \sum_{i=1}^{n} (y_i - x_i)^2$ on $\mathbf{R}^n$; for every $\mu \in \mathscr{M}_x$ we must have $0 = p(x) = \int p \, d\mu_x$ and hence $\mu = \varepsilon_x$ since $S\mu = \{x\}$ and $1 \in \mathscr{H}$. Obviously, $\mathscr{H}_0$ is here the space of all $f \in \mathscr{C}(\mathbf{R}^n)$ satisfying

$$|f(x)| \leqslant \alpha |x_1|^p \cdot \cdots \cdot |x_n|^p + \beta \quad \text{for all } x \in \mathbf{R}^n,$$

where $\alpha, \beta \geqslant 0$ and $p \in \mathbf{N}_0$ are suitably chosen.

5) In proposition 1.4 uniform convergence on K of $(L_i f)$ to f is proved for all functions $f \in \mathscr{H}_0$ satisfying $\check{f}(x) = \hat{f}(x)$ for all $x \in K$. In view of lemma 2.2 the latter condition states that $\int f \, d\mu = f(x)$ holds for all $x \in K$ and all $\mu \in \mathscr{M}_x$. The set $\hat{\mathscr{H}}_K$ of these functions $f \in \mathscr{H}_0$ is a linear subspace of $\mathscr{H}_0$. In the terminology of Franchetti [5] which is developed in the framework of remark 2.1, $\hat{\mathscr{H}}_K$ would be the maximal linear subspace of $\mathscr{H}_0$ such that K is contained in the relative Choquet boundary of $\mathscr{H}$ with respect to $\hat{\mathscr{H}}_K$. Thus, Franchetti's main result (for positive linear operators) is a special case of proposition 1.4.

3. Characterization of the Korovkin closure.

Our assumptions about X and $\mathscr{H}$ will be the same as in preceding paragraph: X is a locally compact space and $\mathscr{H}$ is an adapted linear subspace of $\mathscr{C}(X)$. Corollary 1.3 states that every $\mathscr{H}$-affine function, i.e. every function $f \in \hat{\mathscr{H}}$, has the following *Korovkin property:* For every net $(L_i)_{i \in I}$

of increasing maps $L_i : \mathscr{H}_0 \to \mathbf{R}^X$ pointwise convergence on X to the identity on $\mathscr{H}$ implies that the net $(L_i f)_{i \in I}$ converges pointwise on X to f.

We will now prove that the $\mathscr{H}$-affine functions are the only functions in $\mathscr{H}_0$ having this Korovkin property. In view of corollary 2.3 this is an almost obvious observation which, in the situation of remark 2.1 and for nets of positive linear maps L_i, has been made already by Baskakov [1] and Šaškin [10]. In fact, the following stronger statement holds:

3.1 Proposition. — *Consider the set* **P** *of all positive linear maps* $L : \mathscr{H}_0 \to \mathscr{H}^b$ *which leave invariant all function* $h \in \mathscr{H}$, i.e. $Lh = h$. *Then* $\hat{\mathscr{H}}$ *is the set of all functions* $f \in \mathscr{H}_0$ *which are left invariant by all* $L \in \mathbf{P}$.

Proof. — By corollary 1.3, a function $f \in \hat{\mathscr{H}}$ is left invariant even by all increasing maps $L : \mathscr{H}_0 \to \mathbf{R}^X$ which leave invariant all functions in $\mathscr{H}$. Conversely, let $f \in \mathscr{H}_0$ be a function which is left invariant by all $L \in \mathbf{P}$. Let us choose for each $x \in X$ an arbitrary measure μ_x in $\mathscr{M}_x$. For $q \in \mathscr{H}_0$ define Lq as the function $x \longmapsto \int q \, d\mu_x$ defined on X. Then $Lq \in \mathscr{H}^b$ since $|q| \leqslant h$ for some $h \in H$ implies $|Lq(x)| = \left|\int q \, d\mu_x\right| \leqslant \int h \, d\mu_x = h(x)$ for all $x \in X$. Therefore, L is a positive linear map of $\mathscr{H}_0$ in $\mathscr{H}^b$ which by definition leaves invariant all $h \in \mathscr{H}$. Hence $Lf = f$, i.e. $\int f \, d\mu_x = f(x)$ for all x, and since μ_x was arbitrarily chosen in $\mathscr{M}_x$, for all $\mu_x \in \mathscr{M}_x$. Then by corollary 2.3 f is $\mathscr{H}$-affine.

The preceding proposition states in other words that, in order to characterize $\mathscr{H}$-affine functions by the Korovkin-property, it suffices to consider only *constant* nets of positive linear maps of $\mathscr{H}_0$ into $\mathscr{H}^b$.

The following example shows that one cannot replace $\mathscr{H}^b$ by $\mathscr{H}_0$.

Example (cf. Scheffold [14]). — Consider $X = [0, 1]$ and

$$\mathscr{H} = \left\{ h \in \mathscr{C}([0, 1]) \,\middle|\, \int_0^1 h(x) \, dx = h(1) \right\}.$$

For each $x_0 \in [0, 1[$ there exist (e.g. piecewise linear) functions $h \in \mathscr{H}$ satisfying $h(x) > h(x_0) = 0$ for all $x \neq x_0$ in $[0, 1]$. This proves that $\partial_{\mathscr{H}} X = [0, 1[$ since, by definition, the Lebesgue measure on $[0, 1]$ is a representing measure for $x_0 = 1$.

Corollary 2.3 and the definition of $\mathscr{H}$ yield $\hat{\mathscr{H}} = \mathscr{H}$; furthermore, $\mathscr{H}_0 = \mathscr{C}(X)$. Now let $L: \mathscr{C}([0, 1]) \to \mathscr{C}([0, 1])$ be an arbitrary positive linear map such that $Lh = h$ for all $h \in \mathscr{H}$. Then $f \longmapsto Lf(x)$, $f \in \mathscr{C}([0, 1])$, is an $\mathscr{H}$-representing measure μ_x for every x, hence $\mu_x = \varepsilon_x$ for all $x \in [0, 1[$, and, by continuity, for all $x \in [0, 1]$. This proves that $Lf = f$ holds even for *all* $f \in \mathscr{C}([0, 1])$.

However, one can replace $\mathscr{H}^b$ by $\mathscr{H}_0$ if one continues to work with *nets* of increasing maps $L_i: \mathscr{H}_0 \to \mathscr{H}_0$.

3.2 Definition. — *Korovkin closure of* $\mathscr{H}$ *will be called the set of all functions* $f \in \mathscr{H}_0$ *having the following "continuous Korovkin property": For every net* $(L_i)_{i \in I}$ *of positive linear maps* $L_i: \mathscr{H}_0 \to \mathscr{H}_0$ *which converges pointwise on* X *to the identity on* $\mathscr{H}$, *the net* $(L_i f)_{i \in I}$ *converges pointwise on* X *to* f.

The adapted space $\mathscr{H}$ will be called a *Korovkin space* if $\mathscr{H}_0$ is its Korovkin closure.

3.3 Theorem. — $\hat{\mathscr{H}}$ *is the Korovkin closure of* $\mathscr{H}$.

Proof. — Consider a function $f \in \mathscr{H}_0$ having the continuous Korovkin property. In view of corollary 2.3, we have to prove $\int f \, d\mu_0 = f(x_0)$ for every point $x_0 \in X$ and every measure $\mu_0 \in \mathscr{M}_{x_0}$. We choose a neighborhood base $\mathfrak{B}$ of x_0 consisting of open relatively compact neighborhoods of x_0. For every $V \in \mathfrak{B}$ denote by q_V a function in $\mathscr{C}_c(X)$ satisfying $0 \leqslant q_V \leqslant 1$, $q_V(x_0) = 1$, $q_V(x) = 0$ for all $x \in \complement V$. For $V \in \mathfrak{B}$ and $g \in \mathscr{H}_0$ we define

$$L_V g := \left(\int g \, d\mu_0\right) q_V + g - g q_V.$$

Then $L_V g$ is a function in $\mathscr{H}_0$ since $\mathscr{C}_c(X) \subset \mathscr{H}_0$. $L_V: \mathscr{H}_0 \to \mathscr{H}_0$ is a positive linear map such that the net $(L_V h)_{V \in \mathfrak{B}}$ converges pointwise to h for every $h \in \mathscr{H}$. Therefore, by assumption, $\lim_{V \in \mathfrak{B}} L_V f(x_0) = f(x_0)$. But this proves

$$\int f \, d\mu_0 = f(x_0)$$

since $L_V f(x_0) = \int f \, d\mu_0$ for all $V \in \mathfrak{B}$. — Conversely, by corollary 1.3, all functions $f \in \hat{\mathscr{H}}$ are in the Korovkin closure of $\mathscr{H}$.

3.4 Corollary. — *$\mathscr{H}$ is a Korovkin space if and only if $\partial_{\mathscr{H}} X = X$.*

Proof. — This is an immediate consequence of corollary 3.5 and of the above theorem.

3.5 *Remarks.* — 1) Our proof of theorem 3.3 shows that it suffices to use sequences $(L_n)_{n \in N}$ in the definition of the Korovkin closure if the space X is metrizable.

2) For compact spaces X and finitely generated spaces $\mathscr{H}$, corollary 3.4 was obtained by different methods by Šaškin [11, 12].

3) In the definition of the Korovkin closure and of Korovkin spaces one can replace pointwise convergence by locally uniform convergence on X without changing the results 3.3 and 3.4. This follows from corollary 1.5 and the observation that the net (L_V) from the proof of theorem 3.3 converges even uniformly on X to the identity on $\mathscr{H}$.

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Added in proof. — The results of this paper should also be compared with :

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