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WEAKLY SEMIBOUNDED BOUNDARY PROBLEMS AND SESQUILINEAR FORMS

by Gerd GRUBB

In this paper and its successor [8] we study boundary value problems for systems A of linear partial differential operators on a manifold $\overline{\Omega}$ with boundary Γ.

Let A be a $q \times q$-system of differential operators of order $2m$, let ρu denote the Cauchy data $\{\gamma_0 u, \dots, \gamma_{2m-1} u\}$ of u with respect to A, and let B be a system of differential operators on Γ; then A_B denotes the realization of A with domain

$$D(A_B) = \{u \in H^{2m}(\overline{\Omega}) | B\rho u = 0\}.$$

(A and B actually operate on sections in vector bundles over $\overline{\Omega}$, resp. Γ.) The boundary condition $B\rho u = 0$ is assumed to be normal in an appropriate sense. One is interested in the coerciveness inequalities

$$(0.1) \quad \operatorname{Re}(Au, u) \geqslant c_s \|u\|_s^2 - c_0 \|u\|_0^2, \qquad u \in D(A_B),$$

for $s \in [0, m]$ (Sobolev norms); they all require the validity of a weaker inequality

$$(0.2) \qquad \operatorname{Re}(Au, u) \geqslant -c\|u\|_m^2, \qquad u \in D(A_B),$$

which we call *weak semiboundedness*. It was shown in [6] how, for the case where A is scalar and *elliptic*, (0.1) with $s = m$ (Gårding's inequality) is characterized by two conditions on A, B: (i) a condition on the full operators B and A at Γ, necessary and sufficient for (0.2); (ii) a condition on the principal symbols of A and B, related to the condition by Agmon [1].

The present paper is devoted to a thorough study of (0.2) and its analogue for systems of « mixed order », without any *a priori* assumptions on A; e.g. A may degenerate at Γ. The results and notations will be applied to elliptic systems in [8], where we treat (0.1) and other properties along the lines of [5], [6].

In Chapter 1 we introduce notations, and set up a Green's formula and the « halfways » Green's formulae associating A with sesquilinear forms. Furthermore we define normal boundary conditions; here a class of triangular systems of differential operators on Γ play a central rôle.

In Chapter 2, (0.2) is characterized by an explicit condition on A and B, and it is proved that (0.2) is necessary and sufficient for the existence of a sesquilinear form $a(u, v)$ on $H^m(\overline{\Omega}) \times H^m(\overline{\Omega})$, for which

$$(Au, v) = a(u, v), \qquad \text{all} \qquad u, v \in D(A_B);$$

(Theorem 2.4), this determines the boundary problems entering in variational theory. A number of alternative explicit conditions for (0.2) are given, in particular for the case where Γ is noncharacteristic for A; these will be of use in [8]. They are finally used to show that when the « total number of boundary conditions » equals mq, then (0.2) holds precisely when the space of Dirichlet data for A_B equals the space of Dirichlet data for the formally adjoint realization $A'_{B'}$; and in that case $A'_{B'}$ is also weakly semibounded.

Chapter 3 treats the systems $A = (A_{st})_{s,t=1,\ldots,q}$, where A_{st} is of order $m_s + m_t$; $\{m_1, \ldots, m_q\}$ denoting a set of not necessarily equal *nonnegative* integers. Let

$$m = \max\{m_1, \ldots, m_q\}, \quad \text{and} \quad \overline{m} = m_1 + \cdots + m_q.$$

For such systems, a workable definition of Green's formula and of normal boundary conditions does not seem to have been available (cf. [11, p. 241]), the trouble being, roughly speaking, that there are $\overline{m} + mq$ Cauchy data, on which one usually wants to impose $\overline{m}$ boundary conditions (less than half). We here present such definitions, and proceed to characterize the analogue of (0.2):

$$(0.3) \qquad \operatorname{Re}(Au, u) \geqslant -c(\|u_1\|^2_{m_1} + \cdots + \|u_q\|^2_{m_q})$$

(for $u = \{u_1, \ldots, u_q\}$ satisfying a normal boundary condition). The whole discussion in Chapter 2 is shown to generalize to these systems. (In particular, this determines the normal boundary problems to which de Figueiredo [9] can be applied in the study of coerciveness.)

As an extra benefit we find a Green's formula

$$(\mathrm{A}u, v) - (u, \mathrm{A}'v) = \langle \varkappa u, \beta^0 v\rangle - \langle \beta^0 u, \varkappa' v\rangle$$

(valid for all smooth u and v), where $\beta^0 u$ consists of the $\overline{m}$ Dirichlet data of u, and where, when Γ is noncharacteristic for A, $\varkappa$ and $\varkappa'$ are *surjective* trace operators each consisting of $\overline{m}$ more data. Boundary conditions for A that can be expressed by differential operators on $\beta^0 u$ and $\varkappa u$ (the « reduced Cauchy data ») can be treated much like the $2m$-order case. (For instance, it is possible to extend techniques of [11] and of [10] to such boundary conditions.) We show that the normal boundary conditions for which (0.3) holds, i.e. all normal boundary conditions arizing in connection with sesquilinear forms, are indeed differential boundary conditions on $\{\beta^0 u, \varkappa u\}$.

The author is grateful to G. Geymonat for having called our attention to the above systems of mixed order.

Plan of the paper.

CHAPTER 1

NOTATIONS AND PRELIMINARIES

1.1. Green's formula.

Let $\overline{\Omega}$ be an n-dimensional compact riemannian [1] manifold with boundary Γ and interior $\Omega = \overline{\Omega}\setminus\Gamma$. Let E be a C^∞ complex hermitian vector bundle over $\overline{\Omega}$ with fiber dimension $q \geqslant 1$. Then the spaces of square integrable sections $L^2(E)$, $L^2(E|_\Gamma)$, and the Sobolev spaces $H^s(E)$, $H^s(E|_\Gamma)$ ($s \in \mathbf{R}$), $H^s_0(E)$ ($s \geqslant 0$) may be defined (cf. e.g. [11]), and we denote inner products over $\overline{\Omega}$ by $(\ ,\ )$ and inner products (and dualities) over Γ by $\langle\ ,\ \rangle$. The space of C^∞ sections with compact support in Ω will be denoted $C_0^\infty(E)$.

We now introduce trace operators etc., essentially following Hörmander [10, p. 192-193]. Assume, as we may, that $\overline{\Omega}$ is imbedded in an n-dimensional riemannian manifold Σ without boundary, so that E is the restriction of a vector bundle $\tilde{E}$ on Σ. Moreover, let $n(x)$ denote the vector field consisting of the unit tangent vectors to the geodesics normal to Γ and oriented towards Ω; it is defined in a neighbourhood Σ_ε of Γ consisting of the points in Σ with geodesic distance $-\varepsilon < t < \varepsilon$ from Γ, ε sufficiently small. Then one may choose a first order differential operator D_n in $\tilde{E}$ whose symbol equals $n(x)\cdot\xi$ for $x \in \Sigma_\varepsilon$ ($\xi \in$ the cotangent bundle $T^*(\Sigma)$); the so-called *normal derivative*. We then define the trace operators

$$\gamma_k : u \to (D_n^k u)|_\Gamma, \qquad k = 0, 1, 2, \ldots,$$

[1] When $\overline{\Omega}$ is an n-dimensional C^∞ manifold, it may always be provided with an appropriate riemannian structure; we assume this has been done on beforehand, since we want to include the case $\overline{\Omega} \subset \mathbf{R}^n$. The compactness of $\overline{\Omega}$ is not used in any essential way; all estimates are local.

for $u \in C^\infty(\tilde{E})$ or $C^\infty(E)$; recall that γ_k is continuous from $H^s(E)$ into $H^{s-k-\frac{1}{2}}(E|_\Gamma)$ for all $s > k + \frac{1}{2}$ (cf. e.g. [11]).

First order differential operators P in $\tilde{E}$ with principal symbol $a(x)\cdot\xi$ satisfying $a(x)\perp n(x)$ for $x \in \Sigma_\varepsilon$, can be said to *act along* the parallel surfaces Γ_t of Γ (Γ_t consisting of the points in Σ with geodesic distance t from Γ; $t \in]-\varepsilon, \varepsilon[$), since for such operators, $P\tilde{\varphi}|_{\Gamma_t}$ is independent of the choice of the extension $\tilde{\varphi} \in C^\infty(\tilde{E})$ of $\varphi \in C^\infty(E|_{\Gamma_t})$. We then denote $P\tilde{\varphi}|_{\Gamma_t} = P\varphi$. Higher order operators acting along the Γ_t are obtained as sums of products of first order operators acting along Γ_t.

For $f \in C^\infty(\tilde{E})$ denote by f^0 the section that equals f over $\bar{\Omega}$ and equals 0 over $\Sigma\setminus\bar{\Omega}$. Let δ denote the distribution $f \longmapsto \int_\Gamma \gamma_0 f\, d\sigma$. Then one has the formulae

$$(1.1)\quad D_n(f^0) = (D_n f)^0 - i\gamma_0 f\,\delta, \quad \text{and} \quad P(f^0) = (Pf)^0,$$

for $f \in C^\infty(\tilde{E})$, when P acts along the Γ_t, $|t| < \varepsilon$. We shall mainly use these formulae on the following forms:

$$(1.2)\quad (D_n u, v) - (u, D'_n v) = i\langle\gamma_0 u, \gamma_0 v\rangle, \quad u, v \in C^\infty(E),$$

where D'_n is the formal adjoint of D_n; note that $D_n - D'_n$ is of order zero (for the symbol of D_n is real).

$$(1.3)\qquad (Pu, v) - (u, P'v) = 0, \qquad u, v \in C^\infty(E),$$

where the formal adjoint P' of P again acts along the Γ_t.

Let A be a C^∞ differential operator in E of order $r > 0$. In Σ_ε it may be decomposed uniquely

$$(1.4)\qquad A = \sum_{l=0}^{r} A_l D_n^l$$

where the A_l are differential operators of order $r - l$ acting along the Γ_t, $|t| < \varepsilon$; this is seen e.g. by induction from the first order case. Note that A_r is of order 0, so is locally multiplication with a $q \times q$-matrix; globally it may be viewed as a vector bundle morphism in E. We shall identify zero order differential operators with morphisms in this way throughout the paper. Clearly, one has

Remark 1.1. — Γ is non-characteristic for A at a point $x \in \Gamma$ if and only if $A_r(x)$ is bijective.

Let M be the set of integers

$$M = \{0, 1, \ldots, r-1\}, \tag{1.5}$$

then the Cauchy boundary operator ρ for A is defined as

$$\rho = \{\gamma_0, \ldots, \gamma_{r-1}\} = \{\gamma_k\}_{k \in M}, \tag{1.6}$$

usually considered as a column vector. With A′ denoting the formal adjoint of A, we have Green's formula

Lemma 1.2. — *For all* u *and* $v \in H^r(E)$,

$$(Au, v) - (u, A'v) = \langle \mathcal{A}\rho u, \rho v \rangle \tag{1.7}$$

where $\mathcal{A} = (\mathcal{A}_{jk})_{j,k \in M}$ *is a system of differential operators* $\mathcal{A}_{jk}$ *in* $E|_\Gamma$ *of orders* $r - j - k - 1$, *with*

$$\mathcal{A}_{jk} = iA_{j+k+1} + \text{lower order operator}$$

for $r - j - k - 1 \geqslant 0$, *and* $\mathcal{A}_{jk} = 0$ *for* $r - j - k - 1 < 0$.

Proof. — It follows from (1.1) that for each l, each $f \in C^\infty(\tilde{E})$

$$A_l D_n^l(f^0) = (A_l D_n^l f)^0 - iA_l \sum_{k=0}^{l-1} D_n^{l-1-k}(\gamma_k f\, \delta),$$

i.e., with $u = f|_{\overline{\Omega}}$, $v \in C^\infty(E)$,

$$(u, (A_l D_n^l)' v) = (A_l D_n^l u, v) - i \sum_{k=0}^{l-1} \langle \gamma_k u, \gamma_0 (D_n')^{l-1-k} A_l' v \rangle.$$

This gives

$$(A_l D_n^l u, v) - (u, (A_l D_n^l)' v) = i \sum_{k=0}^{l-1} \langle (A_l + S_{lk}) \gamma_k u, \gamma_{l-1-k} v \rangle,$$

where the S_{lk} are differential operators of order $< r - l$ in $E|_\Gamma$, stemming from commutation and taking adjoints. Collecting the terms we obtain (1.7).

$\mathcal{A}$ is *of type* $(-k, -r+1+j)_{j,k \in M}$ in the terminology of Hörmander [10, p. 135] (which we shall use throughout); i.e. it is continuous from $\prod_{k \in M} H^{\alpha-k}(E|_\Gamma)$ into $\prod_{j \in M} H^{\alpha-r+1+j}(E|_\Gamma)$ for all $\alpha \in \mathbf{R}$. Note that it is skew-triangular, the entries in the second diagonal being equal to the zero order operator A_r. By Remark 1.1, $\mathfrak{A}$ is thus invertible if and only if Γ is non-characteristic for A; $\mathfrak{A}^{-1}$ is then also a skew-triangular system of differential operators.

1.2. The even order case; sesquilinear forms.

In this section we assume $r = 2m$ (m integer $\geqslant 1$), and establish an alternative version of (1.7), and the « halfways » Green's formulae.

Define now the subsets M_0 and M_1 of

$$M = \{0, 1, \ldots, 2m-1\}$$

by

$$(1.8) \quad M_0 = \{0, \ldots, m-1\}, \quad M_1 = \{m, \ldots, 2m-1\},$$

so

$$M = M_0 \cup M_1.$$

The Cauchy boundary operator is split into the Dirichlet and the Neumann boundary operators γ and ν

$$(1.9) \quad \gamma = \{\gamma_k\}_{k \in M_0}, \quad \nu = \{\nu_k\}_{k \in M_1}, \qquad \text{so} \qquad \rho = \{\gamma, \nu\}.$$

The matrix $\mathfrak{A}$ is split in four blocks

$$(1.10) \qquad \mathfrak{A} = \begin{pmatrix} \mathfrak{A}^{00} & \mathfrak{A}^{01} \\ \mathfrak{A}^{10} & 0 \end{pmatrix},$$

where $\mathfrak{A}^{\delta\varepsilon} = (\mathfrak{A}_{jk})_{j \in M_\delta,\, k \in M_\varepsilon}$, clearly $\mathfrak{A}^{11} = 0$. Then (1.7) takes the form

$$(1.11) \quad (Au, v) - (u, A'v) = \langle \mathfrak{A}^{00}\gamma u, \gamma v\rangle + \langle \mathfrak{A}^{01}\nu u, \gamma v\rangle + \langle \mathfrak{A}^{10}\gamma u, \nu v\rangle,$$

for $u, v \in H^{2m}(E)$.

Definition 1.3. — *By a sesquilinear form $a(u, v)$ on $H^m(E)$ we shall understand an integro-differential form*

$$(1.12) \qquad a(u, v) = \sum_{i \in I} (Q_i u, P_i v),$$

where the Q_i and P_i are C^∞ differential operators in E of orders $\leqslant m$, indexed by a finite index set I; $a(u, v)$ is defined and continuous for $\{u, v\} \in H^m(E) \times H^m(E)$.

$a(u, v)$ *is said to be* associated with A *if*

(1.13) $a(u, v) = (\mathrm{A}u, v)$, *for all* $u, v \in \mathrm{C}_0^\infty(\mathrm{E})$,

i.e., if $\mathrm{A} = \sum_{i \in \mathrm{I}} \mathrm{P}_i' \mathrm{Q}_i$.

When $\overline{\Omega} \subset \mathbf{R}^n$, (1.12) may of course be written in the usual way: $a(u, v) = \Sigma_{|\alpha|, |\beta| \leq m}(a_{\alpha\beta}\mathrm{D}^\beta u, \mathrm{D}^\alpha v)$.

LEMMA 1.4. — *Let* $a(u, v)$ *be a sesquilinear form on* $\mathrm{H}^m(\mathrm{E})$ *associated with* A. *Then for all* $u \in \mathrm{H}^{2m}(\mathrm{E})$, $v \in \mathrm{H}^m(\mathrm{E})$,

(1.14) $(\mathrm{A}u, v) = a(u, v) + \langle \mathcal{A}^{01}\nu u, \gamma v\rangle + \langle \mathscr{S}\gamma u, \gamma v\rangle$

where $\mathscr{S}$ *is an* $m \times m$*-system of differential operators in* $\mathrm{E}|_\Gamma$, *of type* $(-k, -2m + 1 + j)_{j,k \in \mathrm{M}_0}$.

Proof. — Applying Green's formula (1.7) to each P_i' and (1.4) to each Q_i, we find

$$\begin{aligned}(\mathrm{A}u, v) - a(u, v) &= \sum_{i \in \mathrm{I}} [(\mathrm{P}_i'\mathrm{Q}_i u, v) - (\mathrm{Q}_i u, \mathrm{P}_i v)] \\ &= \sum_{i \in \mathrm{I}} \langle \mathscr{P}_i \gamma \mathrm{Q}_i u, \gamma v\rangle \\ &= \langle \mathscr{R}\rho u, \gamma v\rangle = \langle \mathscr{R}_1 \nu u, \gamma v\rangle + \langle \mathscr{R}_2 \gamma u, \gamma v\rangle\end{aligned}$$

where $\mathscr{R}_1$ is of type $(-k, -2m + 1 + j)_{j \in \mathrm{M}_0, k \in \mathrm{M}_1}$ and $\mathscr{R}_2$ is of type $(-k, -2m + 1 + j)_{j, k \in \mathrm{M}_0}$. In a similar way

$$(u, \mathrm{A}'v) - a(u, v) = \langle \gamma u, \mathscr{R}_3 \nu v\rangle + \langle \gamma u, \mathscr{R}_4 \gamma v\rangle.$$

For any given $\varphi, \psi \in \mathrm{C}^\infty(\mathrm{E}|_\Gamma)^m$ there exist $u, v \in \mathrm{C}^\infty(\mathrm{E})$ with $\nu u = \varphi$, $\gamma u = 0$, $\gamma v = \psi$. Inserting these, we get, by comparison with (1.11)

$$\langle \mathscr{R}_1 \varphi, \psi\rangle = (\mathrm{A}u, v) - (u, \mathrm{A}'v) = \langle \mathcal{A}^{01}\varphi, \psi\rangle,$$

whence $\mathscr{R}_1 = \mathcal{A}^{01}$.

We shall now show that the operator $\mathscr{S}$ in (1.14) can take any value.

LEMMA 1.5. — *Let* $\mathscr{S}$ *be a first order differential operator in* $\mathrm{E}|_\Gamma$. *Then there exists a sesquilinear form* $s(u, v)$ *on* $\mathrm{H}^1(\mathrm{E})$ *such that*

(1.15) $s(u, v) = \langle \mathscr{S}\gamma_0 u, \gamma_0 v\rangle$ *for* $u, v \in \mathrm{H}^1(\mathrm{E})$:

for any such form the operator S *in* E *associated with* $s(u, v)$ *is zero.*

Proof. — Let $\tilde{\mathscr{S}}$ be a first order operator in $\tilde{E}$ that acts along the Γ_t for $|t| < \varepsilon$, so that $\tilde{\mathscr{S}}$ acts like $\mathscr{S}$ on $\Gamma = \Gamma_0$. Then by (1.2) and (1.3),

$$\begin{aligned}\langle \mathscr{S}\gamma_0 u, \gamma_0 v\rangle &= \langle \gamma_0 \tilde{\mathscr{S}} u, \gamma_0 v\rangle \\ &= - i(D_n \tilde{\mathscr{S}} u, v) + i(\tilde{\mathscr{S}} u, D_n' v) \\ &= - i(\tilde{\mathscr{S}} D_n u, v) - i([D_n, \tilde{\mathscr{S}}]u, v) + i(\tilde{\mathscr{S}} u, D_n' v) \\ &= (- iD_n u, \tilde{\mathscr{S}}' v) - (i[D_n, \tilde{\mathscr{S}}]u, v) + (i\tilde{\mathscr{S}} u, D_n' v),\end{aligned}$$

which is a sesquilinear form on $H^1(E)$, since the commutator $[D_n, \tilde{\mathscr{S}}] = D_n\tilde{\mathscr{S}} - \tilde{\mathscr{S}}D_n$ is of first order. Since any form $s(u, v)$ satisfying (1.15) vanishes for $u \in C_0^\infty(E)$, the associated operator S in E is zero.

PROPOSITION 1.6. — *Let* $\mathscr{S} = (\mathscr{S}_{jk})_{j,k\in M_0}$ *be a system of differential operators in* $E|_\Gamma$, *of type* $(-k, -2m+1+j)_{j,k\in M_0}$. *Then there exists a sesquilinear form* $s(u, v)$ *on* $H^m(E)$ *so that*

$$(1.16) \quad s(u, v) = \langle \mathscr{S}\gamma u, \gamma v\rangle \qquad \textit{for} \qquad u, v \in H^m(E),$$

and the associated operator S *in* E *is zero.*

Proof. — The proof is reduced to the preceding case as follows:

Let $\{j, k\} \in M_0 \times M_0$. $\mathscr{S}_{jk}$ is of order $2m - 1 - j - k$ and it may be written as a finite sum

$$\mathscr{S}_{jk} = \sum_{i\in I} P_i Q_i R_i,$$

where the P_i are of order $m - 1 - j$, Q_i of order 1, R_i of order $m - 1 - k$. Now, with notation as in the preceding proof,

$$\begin{aligned}\langle \mathscr{S}_{jk}\gamma_k u, \gamma_j v\rangle &= \sum_{i\in I} \langle P_i Q_i R_i \gamma_k u, \gamma_j v\rangle \\ &= \sum_{i\in I} \langle Q_i R_i \gamma_k u, P_i' \gamma_j v\rangle \\ &= \sum_{i\in I} \langle Q_i \gamma_0 \tilde{R}_i D_n^k u, \gamma_0 \tilde{P}_i' D_n^j v\rangle,\end{aligned}$$

where $\tilde{R}_i D_n^k$ and $\tilde{P}_i' D_n^k$ are differential operators in E of order $m - 1$.

Corollary 1.7. — *For any system* $\mathcal{S}$ *of type*

$$(-k, \; -2m + 1 + j)_{j, k \in M_0}$$

of differential operators in $E|_\Gamma$, *there exists a sesquilinear form* $a(u, v)$ *on* $H^m(E)$ *such that*

$$(1.17) \quad (Au, v) = a(u, v) + \langle \mathcal{A}^{01} \nu u, \gamma u \rangle + \langle \mathcal{S} \gamma u, \gamma v \rangle,$$

for all $u \in H^{2m}(E)$, $v \in H^m(E)$.

Remark 1.8. — The results of this section generalize immediately to the following situation : Let A be of order $r = s + t$, s and t nonnegative integers. Let $M_{0t} = \{0, \ldots, t - 1\}$,

$$M_{1t} = \{t, \ldots, s + t - 1\}, \qquad M_{0s} = \{0, \ldots, s - 1\},$$
$$M_{1s} = \{s, \ldots, s + t - 1\},$$

and set

$$\rho_{0s} = \{\gamma_k\}_{k \in M_{0s}}, \qquad \mathcal{A}^{0s,1t} = (\mathcal{A}_{jk})_{j \in M_{0s},\, k \in M_{1t}},$$

etc. Then (1.7) may be written

$$\begin{aligned}(Au, v) &- (u, A'v) \\ &= \langle \mathcal{A}^{0s,0t} \rho_{0t} u, \rho_{0s} v \rangle + \langle \mathcal{A}^{0s,1t} \rho_{1t} u, \rho_{0s} v \rangle \\ &\qquad + \langle \mathcal{A}^{1s,0t} \rho_{0t} u, \rho_{1s} v \rangle.\end{aligned}$$

By a sesquilinear form on $H^t(E) \times H^s(E)$ we understand an expression (1.12) where the Q_i are of order $\leqslant t$ and the P_i are of order $\leqslant s$, it is associated with A when (1.13) holds. One finds that for such forms,

$$(1.18) \quad (Au, v) = a(u, v) + \langle \mathcal{A}^{0s,1t} \rho_{1t} u, \rho_{0s} v \rangle + \langle \mathcal{S} \rho_{0t} u, \rho_{0s} v \rangle$$

for all $u \in H^r(E)$, $v \in H^s(E)$; where $\mathcal{S}$ can be any $s \times t$-system of differential operators in $E|_\Gamma$, of type

$$(-k, \; -r + 1 + j)_{j \in M_{0s},\, k \in M_{0t}}.$$

Note that $\mathcal{A}^{0s,1t}$ is a quadratic submatrix of $\mathcal{A}$, its second diagonal being contained in the second diagonal of $\mathcal{A}$; so it is invertible if and only if Γ is non-characteristic for A.

1.3. Triangular differential operators.

In this section we study a class of differential operators that are fundamental in our treatment of boundary conditions.

Let N be a finite subset of $\mathbf{N} \cup \{0\}$, the non-negative integers; the number of elements in N is denoted $|N|$. For each $j \in N$ there are given two hermitian vector bundles F_j and E_j over Γ, F_j of fiber dimension $p_j \geqslant 0$ and E_j of fiber dimension $q_j \geqslant 0$. (We shall use some elementary facts about vector bundles, for which we refer e.g. to Atiyah [3].) For each pair $\{j, k\} \in N \times N$ there is given a differential operator B_{jk} from E_k into F_j, of order $j - k$; the convention that differential operators of negative order are zero is used throughout; of course B_{jk} is also zero if p_j or q_k is 0. The B_{jk} form a matrix (or system) of differential operators

$$B = (B_{jk})_{j,k \in N}$$

of type $(-k, -j)_{j,k\in N}$, i.e. B is continuous from $\prod_{k\in M} H^{\alpha-k}(E_k)$ to $\prod_{j\in M} H^{\alpha-j}(F_j)$ for all $\alpha \in \mathbf{R}$.

B is *triangular*, since $B_{jk} = 0$ for $j < k$. We define its *diagonal part* B_d and its *subtriangular part* B_s by

$$B_d = (\delta_{jk}B_{jk})_{j,k\in N}, \qquad B_s = B - B_d; \tag{1.19}$$

a matrix will be said to be *subtriangular* when the diagonal and all elements to one side of it are zero. The elements B_{kk} in the diagonal are differential operators of order 0, so they may, as previously remarked, be regarded as vector bundle morphisms (from E_k to F_k); B_d is also a morphism, from $\bigoplus_{k\in N} E_k$ into $\bigoplus_{k\in N} F_k$.

PROPOSITION 1.9. — *Assume that* B_d *is surjective (so in particular,* $p_k \leqslant q_k$, *all* $k \in N$*). Then the morphism*

$$C_d = B_d^*(B_dB_d^*)^{-1} \tag{1.20}$$

is a right inverse of B_d. *Moreover, the differential operator*

$$C = C_d \sum_{k=0}^{|N|} (-B_sC_d)^k \tag{1.21}$$

is a right inverse of B; *it is a system* $C = (C_{jk})_{j,k \in N}$ *of type* $(-k, -j)_{j,k\in N}$, *the* C_{jk} *being differential operators from* F_k *into* E_j. *In particular,* B *is surjective from* $\prod_{k\in N} H^{\alpha-k}(E_k)$ *to* $\prod_{j\in N} H^{\alpha-j}(F_j)$ *for all* $\alpha \in \mathbf{R}$.

Proof. — The first statement follows from the corresponding statement for vector spaces. (Since B_d is a diagonal matrix, one actually treats each B_{kk} separately, and C_d is a diagonal matrix with $C_{kk} = B_{kk}^*(B_{kk}B_{kk}^*)^{-1}$.) Now observe that B_sC_d is a subtriangular differential operator in $\bigoplus_{j\in N} F_j$, since B_s is subtriangular and C_d is diagonal. Thus B_sC_d is *nilpotent*, its $|N|$ — th power being zero. Defining C by (1.21), we then have

$$\begin{aligned} BC &= (B_d + B_s)(C_d - C_dB_sC_d + C_dB_sC_dB_sC_d - \cdots) \\ &= B_dC_d - B_dC_dB_sC_d + B_dC_dB_sC_dB_sC_d - \cdots \\ &\qquad\qquad + B_sC_d - B_sC_dB_sC_d + \cdots \\ &= I, \end{aligned}$$

since $B_dC_d = I$ $\left(\text{the identity in } \bigoplus_{j\in N} F_j\right)$. Clearly C is a system of the described type; its continuity properties imply the surjectiveness of B.

LEMMA 1.10. — *Assumptions of Proposition* 1.9. *For each* $k \in N$, *the kernel and image of the morphisms* B_{kk} *resp.* B_{kk}^*,

$$(1.22) \qquad Z_k = \ker B_{kk} \qquad \textit{and} \qquad R_k = \operatorname{im} B_{kk}^*$$

are orthogonal subbundles of E_k, *of dimension* $q_k - p_k$ *resp.* p_k. *Moreover,*

$$(1.23) \quad C_{kk}B_{kk} = B_{kk}^*(B_{kk}B_{kk}^*)^{-1}B_{kk} = B_{kk}^*C_{kk}^* = (C_{kk}B_{kk})^*,$$

and it is the orthogonal projection of E_k *onto* R_k; *and* $I - C_{kk}B_{kk}$ *is the orthogonal projection of* E_k *onto* Z_k. *Altogether,* $C_dB_d = B_d^*C_d^*$ *and is the orthogonal projection of* $\bigoplus_{k\in N} E_k$ *onto* $\operatorname{im} B_d^* = \bigoplus_{k\in N} R_k$, *and* $I - C_dB_d$ *is the orthogonal projection of* $\bigoplus_{k\in N} E_k$ *onto* $\ker B_d = \bigoplus_{k\in N} Z_k$.

Proof. — Follows from the corresponding evident statements for vector spaces. (I denotes the identity in various spaces, its meaning should be clear from the context.)

When B_d is surjective (so B_d^* is injective) it follows from Proposition 1.9 that B^* is injective with left inverse C^*. However, we do not have equality between CB and B^*C^* since CB is lower triangular and B^*C^* is upper triangular (unless of course $(CB)_s = 0$). We may define $\tilde{C} = B^*(BB^*)^{-1}$, which does satisfy $\tilde{C}B = B^*\tilde{C}^*$ in better analogy with Lemma 1.10. But $\tilde{C}$ will usually not be a differential operator but a *pseudo-differential* operator, and not triangular, so we prefer to work with C as right inverse of B. Note that the subtriangular part of C is $C - C_d = C_d \sum_{k=1}^{|N|} (-B_sC_d)^k$.

We introduce the spaces (for $\alpha \in \mathbf{R}$):

$$Z^\alpha(B) = \Big\{\varphi \in \prod_{k\in N} H^{\alpha-k-\frac{1}{2}}(E_k) \,|\, B\varphi = 0\Big\};\tag{1.24}$$

$$Z(B) = \bigcup_{\alpha\in \mathbf{R}} Z^\alpha(B);$$

$$R^\alpha(I - CB) = (I - CB)\prod_{k\in N} H^{\alpha-k-\frac{1}{2}}(E_k);\tag{1.25}$$

$$R(I - CB) = \bigcup_{\alpha\in\mathbf{R}} R^\alpha(I - CB);$$

$$R^\alpha(B^*) = B^*\prod_{j\in N} H^{\alpha-j-\frac{1}{2}}(F_j);\quad R(B^*) = \bigcup_{\alpha\in\mathbf{R}} R^\alpha(B^*);\tag{1.26}$$

$$Z^\alpha(I - B^*C^*) = \Big\{\varphi \in \prod_{j\in N} H^{\alpha-j-\frac{1}{2}}(F_j) \,|\, \varphi - C^*B^*\varphi = 0\Big\};$$

$$Z(I - B^*C^*) = \bigcup_{\alpha\in\mathbf{R}} Z^\alpha(I - B^*C^*).\tag{1.27}$$

These definitions apply similarly to B_d and C_d, viewed as differential operators. We also have, with the notation (1.22)

$$Z^\alpha(B_d) = i_Z \prod_{k\in N} H^{\alpha-k-\frac{1}{2}}(Z_k),\quad Z(B_d) = i_Z\, \mathscr{D}'\Big(\bigoplus_{k\in N} Z_k\Big),\tag{1.28}$$

$$R^\alpha(B_d^*) = i_R \prod_{k\in N} H^{\alpha-k-\frac{1}{2}}(R_k),\quad R(B_d^*) = i_R\, \mathscr{D}'\Big(\bigoplus_{k\in N} R_k\Big),\tag{1.29}$$

where i_Z and i_R denote the injections $\bigoplus_{k\in N} Z_k \subsetneq \bigoplus_{k\in N} E_k$ resp. $\bigoplus_{k\in N} R_k \subsetneq \bigoplus_{k\in N} E_k$ (they may be omitted in less precise statements).

LEMMA 1.11. — *Assumptions of Proposition* 1.9. *For any* $\alpha \in \mathbf{R}$,

$$(1.30)\quad Z^\alpha(B) = R^\alpha(I - CB), \qquad Z(B) = R(I - CB),$$
$$(1.31)\quad R^\alpha(B^*) = Z^\alpha(I - B^*C^*), \quad R(B^*) = Z(I - B^*C^*).$$

Proof. — When $B\varphi = 0$, $\varphi = \varphi - CB\varphi$. When $\varphi = (I - CB)\psi$, $B\varphi = B\psi - BCB\psi = B\psi - B\psi = 0$. This proves (1.30).

When $\varphi = B^*\psi$,

$$(I - B^*C^*)\varphi = B^*\psi - B^*C^*B^*\psi = B^*\psi - B^*\psi = 0.$$

When $(I - B^*C^*)\varphi = 0$, $\varphi = B^*(C^*\varphi)$. This proves (1.31).

LEMMA 1.12. — *Assumptions of Proposition* 1.9. $I - CB_s$ *and* $I + C_dB_s$ *are each others inverses. Moreover, for any* $\alpha \in \mathbf{R}$

$$(1.32)\quad Z^\alpha(B) = (I - CB_s)Z^\alpha(B_d), \quad Z(B) = (I - CB_s)Z(B_d).$$

$$(1.33)\quad \begin{aligned} R^\alpha(B^*) &= (I + B_s^*C_d^*)R^\alpha(B_d^*),\\ R(B^*) &= (I + B_s^*C_d^*)R(B_d^*). \end{aligned}$$

Proof. — Since C_dB_s is subtriangular, $I + C_dB_s$ has the inverse

$$I - C_dB_s + (C_dB_s)^2 - \cdots = I - C_d \sum_{k=0}^{|N|} (-B_sC_d)^k B_s = I - CB_s,$$

cf. (1.21). Now

$$\begin{aligned} B_d &= B - B_s = B - BCB_s = B(I - CB_s),\\ B &= B_d + B_s = B_d + B_dC_dB_s = B_d(I + C_dB_s), \end{aligned}$$

from which (1.32) immediately follows. (1.33) follows from the adjoint identities

$$B_d^* = (I - B_s^*C^*)B^*, \qquad B^* = (I + B_s^*C_d^*)B_d^*.$$

Combining this lemma with (1.28) and (1.29) we see how $Z^\alpha(B)$ and $R^\alpha(B^*)$ may be « parametrized » by full Sobolev spaces over bundles :

$$(1.34)\quad Z^\alpha(B) = (I - CB_s)i_Z \sum_{k \in N} H^{\alpha-k-\frac{1}{2}}(Z_k),$$

$$(1.35)\quad R^\alpha(B^*) = (I + B_s^*C_d^*)i_R \sum_{k \in N} H^{\alpha-k-\frac{1}{2}}(R_k),$$

(similar statements for $Z(B)$ and $R(B^*)$), where $(I - CB_s)i_Z$ and $(I + B_s^* C_d^*)i_R$ are injective differential operators. Note that $R^\alpha(B^*)$ may also be parametrized by

$$R^\alpha(B^*) = B^* \sum_{j \in N} H^{\alpha - j - \frac{1}{2}}(F_j), \tag{1.36}$$

where B^* is injective.

Surjectiveness of B does not in general imply surjectiveness of B_d. However, it does so in a special case:

LEMMA 1.13. — *Assume that* $F_k = E_k$ *for all* $k \in N$. *Then if* B *is surjective from* $\prod_{k \in N} H^{\alpha-k}(E_k)$ *to* $\prod_{j \in N} H^{\alpha-j}(E_j)$ *for some* $\alpha \in \mathbf{R}$, *the diagonal part* B_d *is an isomorphism* (*so* B *is bijective for all* α).

Proof. — We have

$$N = \{k_1, \ldots, k_p\} \quad (\text{where } 0 \leqslant k_1 < \cdots < k_p).$$

For $1 \leqslant q \leqslant p$ we denote by B^q the submatrix of B, $(B_{jk})_{j,k \in \{k_1, \ldots, k_q\}}$. Since B is lower triangular and surjective, all the submatrices B^q are surjective. In particular, $B^1 = B_{k_1 k_1}$ is a surjective morphism from E_{k_1} to E_{k_1}, thus an isomorphism. We proceed by induction: Assume that B^{q-1} has bijective diagonal part; by Proposition 1.9, B^{q-1} is bijective. Let $\psi = \{0, \ldots, 0, \psi_{k_q}\}$ (q elements), $\psi_{k_q} \in H^{\alpha - k_q}(E_{k_q})$. Since B^q is surjective, there exists

$$\varphi^q = \{\varphi_{k_1}, \ldots, \varphi_{k_q}\} \in \prod_{r=1}^{q} H^{\alpha - k_r}(E_{k_r}),$$

for which $B^q \varphi^q = \psi$. But since B^{q-1} is injective,

$$\varphi_{k_1} = \ldots = \varphi_{k_{q-1}} = 0.$$

Then $\psi_{k_q} = B_{k_q k_q} \varphi_{k_q}$. This proves that $B_{k_q k_q}$ is a surjective morphism from E_{k_q} to E_{k_q}, and thus bijective. So B^q has bijective diagonal part.

Remark 1.14. — All calculations generalize immediately to systems B, where the B_{jk} with $j > k$ are *pseudo-differential* operators of orders $j - k$, but where we still have that the B_{kk} are morphisms and the B_{jk} with $j < k$ are zero.

1.4. Normal boundary conditions.

We shall now define the boundary value problems to be studied in Chapter 2 : Let A and E be as in section 1.1. For each $j \in M = \{0, \ldots, r-1\}$ there is given a hermitian bundle F_j over Γ of dimension $p_j \geqslant 0$. There is given a matrix $B = (B_{jk})_{j,k \in M}$ of differential operators B_{jk} from $E|_\Gamma$ to F_j, of type $(-k, -j)_{j,k \in M}$ (as in section 1.3). Then B defines the homogeneous *boundary condition*

$$B\rho u = 0 \tag{1.37}$$

or, equivalently: $\sum_{k \leqslant j} B_{jk}\gamma_k u = 0$ for all $j \in M$. We shall study the boundary value problem

$$Au = f, \qquad B\rho u = 0,$$

or rather the *realization* A_B of A defined by

$$A_B: \quad u \longmapsto Au, \quad D(A_B) = \{u \in H^r(E) | B\rho u = 0\}. \tag{1.38}$$

The systems of boundary conditions usually studied can be put in the form (1.37); we have just grouped together the conditions of the *same normal order* (like Seeley [12]) and permitted the range space for each normal order j to be a nontrivial bundle. Moreover, we have included zero bundles as ranges (those where $p_j = 0$) for convenience, so that we do not have to distinguish between M and the set $J = \{ j | p_j > 0\}$ that entered in the announcement of results [7]. For elliptic A it is usually assumed that $\sum_{j \in M} p_j = mq$; we shall not assume that on beforehand.

DEFINITION 1.15. — *The boundary condition* $B\rho u = 0$ — *or the differential operator* B — *will be said to be normal when* $B_d = (\delta_{jk} B_{jk})_{j,k \in M}$ *is a surjective vector bundle morphism. (Then in particular* $p_j \leqslant q$ *for all* $j \in M$.)

The definition is a vector bundle version of that of Seeley [12] (cf. also Remark 2.2 below). It extends the wellknown definition of Aronszajn and Milgram for scalar operators.

Remark 1.16. — Let us compare the present definition of normality with that of Geymonat [4, Definizione 2.2]. He considers the case where A and B are matrices of scalar differential operators (i.e., E and the F_j are trivial bundles), and his definition of normality requires that one can supplement B with $rq - \sum_{j\in M} p_j$ rows to obtain a system $\tilde{B} = (\tilde{B}_{jk})_{j,k\in M}$ of $q \times q$-matrices, of type $(-k, -j)_{j,k\in M}$ and with bijective diagonal part. In our framework this means exactly that *trivial* bundles G_j of dimension $q - p_j\,(j \in M)$ may be found, together with morphisms $P_{jj} : E|_\Gamma \to G_j$, such that the morphisms $\tilde{B}_{jj} = B_{jj} \oplus P_{jj} : E|_\Gamma \to F_j \oplus G_j$ are isomorphisms. In comparison, the present definition of normality merely requires that the B_{jj} be surjective (which is satisfied under Geymonat's requirement); then if we let P^0_{jj} denote the orthogonal projections of $E|_\Gamma$ onto $Z_j = \ker B_{jj}$, the $\tilde{B}^0_{jj} = B_{jj} \oplus P^0_{jj}$ are isomorphisms of $E|_\Gamma$ onto $F_j \oplus Z_j$, for $j \in M$. When both requirements are satisfied, P_{jj} defines an isomorphism of Z_j onto G_j, for $j \in M$. So, when E and the F_j are trivial and the B_{jj} are surjective, Geymonat's normality holds *if and only if the* Z_j *are trivial bundles.* This is a global condition that will not in general be satisfied for surjective B_{jj}. (Example : Let

$$\overline{\Omega} = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leqslant 1\},$$

let $E = \overline{\Omega} \times \mathbf{R}^3$, and let B_{jj} be the 1×3-matrix $(x_1\, x_2\, x_3)$, for $(x_1, x_2, x_3) \in \Gamma = S^2$. Then $\ker B_{jj}$ is the tangent bundle of S^2, which is nontrivial.) The present definition of normality is local and at the same time more general than Geymonat's.

The « Lions-Magenes theory » of Geymonat [4] can easily be extended to the present normal boundary condition, on the basis of the Green's formula

$$\begin{aligned}(1.39)\quad (Au, v) - (u, A'v) &= \langle \mathcal{A}(\tilde{B}^0)^{-1}\tilde{B}^0\rho u, \rho v\rangle \\ &= \langle B\rho u, pr_F(\tilde{B}^{0*})^{-1}\mathcal{A}^*\rho v\rangle + \langle P^0\rho u, pr_Z(\tilde{B}^{0*})^{-1}\mathcal{A}^*\rho v\rangle,\end{aligned}$$

pr_F and pr_Z denoting the projections of $\bigoplus_{j\in M}(F_j \oplus Z_j)$ onto $\bigoplus_{j\in M} F_j$ resp. $\bigoplus_{j\in M} Z_j$.

CHAPTER 2

WEAKLY SEMIBOUNDED REALIZATIONS OF OPERATORS OF EVEN ORDER

2.1. Characterization of weak semiboundedness.

Throughout this chapter we assume (with the notations of Chapter 1):

Assumption 2.1. — A is an arbitrary C^∞ differential operator in E of order $r = 2m$, m integer > 0. A_B is the realization defined by a *normal* boundary condition

$$B\rho u = 0. \tag{2.1}$$

We shall study the problem of determining those B for which A_B satisfies the inequality

$$\operatorname{Re} e^{i\theta}(Au, u) \leqslant c\|u\|_m^2, \qquad \text{all} \qquad u \in D(A_B), \tag{2.2}$$

for some $c > 0$, $\theta \in \mathbf{R}$. The inequality is always satisfied for $u \in C_0^\infty(E)$, so depends essentially on the boundary condition and the behaviour of A *at the boundary*. However, it will be seen that it depends on the full operators B and A at Γ, not just on part of (e.g. the principal part of) their symbol. (2.2) is necessary (with $\theta = \pi$) for any of the « coerciveness inequalities »

$$\operatorname{Re}(Au, u) \geqslant c_s\|u\|_s^2 - c_0\|u\|_0^2, \quad u \in D(A_B), \tag{2.3}$$

$s \in]0, m]$, or just semiboundedness

$$\operatorname{Re}(Au, u) \geqslant -c_0\|u\|_0^2, \quad u \in D(A_B). \tag{2.4}$$

These other properties will be treated in [8], under further assumptions on A. We shall here concentrate on the special aspects of (2.2), called weak semiboundedness for lack of a better name.

Remark 2.2. — The assumption of normality is partly justified by the observation of Seeley [12] that for *elliptic* boundary value problems, Agmon's necessary and sufficient condition for the existence of a ray of minimal growth *implies* normality. When e.g. (2.4) holds, there are many rays of minimal growth. The investigations given below have led us to believe that, at least when Γ is noncharacteristic for A and $\sum_{j\in M} p_j = mq$, normality is also necessary for (2.2) (cf. Remark 2.21).

Since $r = 2m$, the notations of section 1.2 apply (cf. in particular (1.8), (1.9)). We split B and its right inverse C accordingly:

$$B = \begin{pmatrix} B^{00} & 0 \\ B^{10} & B^{11} \end{pmatrix}, \qquad B^{\delta\varepsilon} = (B_{jk})_{j\in M_\delta,\, k\in M_\varepsilon};$$

$$C = \begin{pmatrix} C^{00} & 0 \\ C^{10} & C^{11} \end{pmatrix}, \qquad C^{\delta\varepsilon} = (C_{jk})_{j\in M_\delta,\, k\in M_\varepsilon};$$

where B^{01} and C^{01} are zero since $j < k$ in $M_0 \times M_1$. Note that C^{00} and C^{11} are the right inverses by Proposition 1.9 to B^{00} resp. B^{11}. The boundary condition (2.1) may now be formulated as

$$(2.5) \qquad B^{00}\gamma u = 0, \qquad B^{10}\gamma u + B^{11}\nu u = 0.$$

It is wellknown that ρ is surjective from $H^{2m}(E)$ onto $\prod_{k\in M} H^{2m-k-\frac{1}{2}}(E|_\Gamma)$. We therefore have two other formulations of (2.5) (recall notations (1.24), (1.25)):

Lemma 2.3. — *A section* $u \in H^{2m}(E)$ *is in* $D(A_B)$ *if and only if* $\{\gamma u, \nu u\}$ *satisfies either of the equivalent conditions* (i), (ii):

(i) $\gamma u \in Z^{2m}(B^{00}), \qquad \nu u + C^{11}B^{10}\gamma u \in Z^{2m}(B^{11})$;

(ii) $\gamma u \in R^{2m}(I - C^{00}B^{00}), \ \nu u + C^{11}B^{10}\gamma u \in R^{2m}(I - C^{11}B^{11})$.

Proof. — (i) is equivalent with (2.5) since $B^{11}C^{11} = I$. (ii) is equivalent with (i) by Lemma 1.11.

We shall now prove the fundamental result

Theorem 2.4. — *Let* A *be a differential operator in* E *of order* $2m$, *and* A_B *the realization defined by a normal boundary* *condition* (2.5). *The following statements* (i)-(iv) *are equivalent*:

(i) *There exist* $\theta \in \mathbf{R}$, $c > 0$ *such that*

$$\operatorname{Re} e^{i\theta}(\mathrm{A}u, u) \leqslant c\|u\|_m^2, \qquad \textit{all} \qquad u \in \mathrm{D}(\mathrm{A_B})$$

(i.e., $\mathrm{A_B}$ *is weakly semibounded).*

(ii) *The following identity holds*

$$(2.6) \qquad (\mathrm{I} - \mathrm{C^{00}B^{00}})^* \mathcal{A}^{01}(\mathrm{I} - \mathrm{C^{11}B^{11}}) = 0.$$

(iii) *There exists a sesquilinear form* $a_{\mathrm{B}}(u, v)$ *on* $\mathrm{H}^m(\mathrm{E})$ *associated with* A, *such that*

$$(2.7) \qquad (\mathrm{A}u, v) = a_{\mathrm{B}}(u, v), \qquad \textit{all} \qquad u, v \in \mathrm{D}(\mathrm{A_B}).$$

(iv) *There exists* $c > 0$ *such that*

$$(2.8) \qquad |(\mathrm{A}u, v)| \leqslant c\|u\|_m\|v\|_m, \qquad \textit{all} \qquad u, v \in \mathrm{D}(\mathrm{A_B}).$$

Proof. — Clearly (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i), since $a_{\mathrm{B}}(u, v)$ is continuous on $\mathrm{H}^m(\mathrm{E}) \times \mathrm{H}^m(\mathrm{E})$. We shall now show that (i) $\Longrightarrow$ (ii). Let $a(u, v)$ be any sesquilinear form on $\mathrm{H}^m(\mathrm{E})$ associated with A. Then

$$(2.9) \qquad (\mathrm{A}u, v) = a(u, v) + \langle \mathcal{A}^{01}\nu u, \gamma v\rangle + \langle \mathcal{S}\gamma u, \gamma v\rangle$$

for some $\mathcal{S}$ of type $(-k, -2m + 1 + j)_{j,k \in \mathrm{M_0}}$, cf. Lemma 1.4. By Lemma 2.3 we have

$$(2.10) \qquad \begin{aligned} \gamma u &= (\mathrm{I} - \mathrm{C^{00}B^{00}})\varphi_0, \\ \nu u &= (\mathrm{I} - \mathrm{C^{11}B^{11}})\varphi_1 - \mathrm{C^{11}B^{10}}\gamma u, \end{aligned}$$

where $\{\varphi_0, \varphi_1\}$ runs through

$$\prod_{k \in \mathrm{M_0}} \mathrm{H}^{2m-k-\frac{1}{2}}(\mathrm{E}|_\Gamma) \times \prod_{k \in \mathrm{M_1}} \mathrm{H}^{2m-k-\frac{1}{2}}(\mathrm{E}|_\Gamma),$$

This gives by insertion

$$(2.11) \qquad \langle \mathcal{A}^{01}\nu u, \gamma v\rangle = \langle \mathcal{A}^{01}(\mathrm{I} - \mathrm{C^{11}B^{11}})\varphi_1, \gamma v\rangle - \langle \mathcal{A}^{01}\mathrm{C^{11}B^{10}}\gamma u, \gamma v\rangle,$$

where also $\mathcal{A}^{01}\mathrm{C^{11}B^{10}}$ is of type $(-k, -2m + 1 + j)_{j,k \in \mathrm{M_0}}$. Then $a(u, v)$, $\langle \mathcal{S}\gamma u, \gamma v\rangle$ and $\langle \mathcal{A}^{01}\mathrm{C^{11}B^{10}}\gamma u, \gamma v\rangle$ are all continuous on $\mathrm{H}^m(\mathrm{E}) \times \mathrm{H}^m(\mathrm{E})$, so that (i) is equivalent with the existence of $\theta \in \mathbf{R}$, $c_1 > 0$ for which

$$\text{(2.12)}\quad \operatorname{Re} e^{i\theta}\langle \mathcal{A}^{01}(\mathrm{I} - \mathrm{C}^{11}\mathrm{B}^{11})\varphi_1, \gamma u\rangle \leqslant c_1\|u\|_m^2, \quad \text{all} \quad u \in \mathrm{D}(\mathrm{A_B}).$$

We now observe that for $w \in \mathrm{C}_0^\infty(\mathrm{E})$, $u + w \in \mathrm{D}(\mathrm{A_B})$ and $\gamma(u + w) = \gamma u$; $\nu(u + w) = \nu u$. Then (2.12) implies

$$\text{(2.13)}\quad \begin{aligned} \operatorname{Re} e^{i\theta}\langle \mathcal{A}^{01}(\mathrm{I} - \mathrm{C}^{11}\mathrm{B}^{11})\varphi_1, \gamma u\rangle &\leqslant c_1 \inf_{w \in \mathrm{C}_0^\infty(\mathrm{E})} \|u + w\|_m^2 \\ &\leqslant c_2 \sum_{k \in \mathrm{M}_0} \|\gamma_k u\|^2_{m-k-\frac{1}{2}}, \quad \text{all} \quad u \in \mathrm{D}(\mathrm{A_B}) \end{aligned}$$

by a well known theorem (cf. e.g. [11]). Inserting $\gamma u = (\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00})\varphi_0$ and using the continuity of $\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00}$ we conclude from (2.13)

$$\begin{aligned} \operatorname{Re} e^{i\theta}\langle \mathcal{A}^{01}(\mathrm{I} - \mathrm{C}^{11}\mathrm{B}^{11})\varphi_1, (\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00})\varphi_0\rangle \\ = \operatorname{Re} e^{i\theta}\langle (\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00})^*\mathcal{A}^{01}(\mathrm{I} - \mathrm{C}^{11}\mathrm{B}^{11})\varphi_1, \varphi_0\rangle \\ \leqslant c_3 \sum_{k \in \mathrm{M}_0} \|\varphi_{0k}\|^2_{m-k-\frac{1}{2}}, \end{aligned}$$

valid for all the pairs $\{\varphi_0, \varphi_1\}$. That can only hold if $(\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00})^*\mathcal{A}^{01}(\mathrm{I} - \mathrm{C}^{11}\mathrm{B}^{11}) = 0$.

Finally we show that (ii) $\Longrightarrow$ (iii). When (ii) holds, we have by (2.9), (2.11), using that also $\gamma v \in \mathrm{R}^{2m}(\mathrm{I} - \mathrm{C}^{00}\mathrm{B}^{00})$,

$$(\mathrm{A}u, v) = a(u, v) + \langle(\mathcal{S} - \mathcal{A}^{01}\mathrm{C}^{11}\mathrm{B}^{10})\gamma u, \gamma v\rangle,$$

for $u, v \in \mathrm{D}(\mathrm{A_B})$. By Proposition 1.6 there exists a sesquilinear form $s(u, v)$ on $\mathrm{H}^m(\mathrm{E})$ satisfying

$$s(u, v) = \langle(\mathcal{S} - \mathcal{A}^{01}\mathrm{C}^{11}\mathrm{B}^{10})\gamma u, \gamma v\rangle, \quad \text{for} \quad u, v \in \mathrm{H}^m(\mathrm{E}).$$

Let $a_\mathrm{B}(u, v) = a(u, v) + s(u, v)$. Then $a_\mathrm{B}(u, v)$ satisfies (iii). This completes the proof of the theorem.

Remark 2.5. — In the proof that (ii) implies (iii) we have in fact constructed a_B such that (2.7) (and thus also (2.8)) is valid for all $u \in \mathrm{D}(\mathrm{A_B})$, *all* $v \in \mathrm{H}^m(\mathrm{E})$ *with* $\mathrm{B}^{00}\gamma v = 0$.

Remark 2.6. — A somewhat analogous theory can be set up for operators A of arbitrary order r, connecting the inequality (for an integer $t \in [0, r]$)

$$|(\mathrm{A}u, v)| \leqslant c\|u\|_t\|v\|_{r-t},$$

with sesquilinear forms on $\mathrm{H}^t(\mathrm{E}) \times \mathrm{H}^{r-t}(\mathrm{E})$, cf. Remark 1.8.

Remark 2.7. — It is also easy to prove *without use of* (iii) that (ii) implies (i). The equivalence of (i), (ii) and (iv) extends to the case where the B_{jk} with $j > k$ are replaced by pseudo-differential operators B_{jk} from $E|_\Gamma$ to F_j of order $j - k$ (cf. Remark 1.14).

2.2. Discussion of (2.6).

We shall now look more closely at what (2.6) stands for.

LEMMA 2.8. — *The identity* (2.6) *is equivalent with each of the following statements* (2.15)-(2.18)

$$Z(B^{11}) \subset Z((1 - C^{00}B^{00})^*\mathcal{A}^{01}); \tag{2.15}$$
$$\mathcal{A}^{01}Z(B^{11}) \subset R(B^{00*}); \tag{2.16}$$
$$(I - C^{11}B^{11})^*\mathcal{A}^{01*}(I - C^{00}B^{00}) = 0; \tag{2.17}$$
$$\mathcal{A}^{01*}Z(B^{00}) \subset R(B^{11*}). \tag{2.18}$$

Proof. — (2.6) may be written

$$\mathcal{A}^{01}R(I - C^{11}B^{11}) \subset Z((I - C^{00}B^{00})^*)$$

which is equivalent with (2.15) and (2.16) by Lemma 1.11. (2.6) is equivalent with its adjoint equation (2.17), and thus with (2.18) by Lemma 1.11.

Remark 2.9. — Because of the continuity properties of the operators involved, each of the inclusions (2.15), (2.16) and (2.18) is equivalent with the inclusion between the spaces intersected with $\Pi H^{\alpha-k-\frac{1}{2}}(E|_\Gamma)$, any $\alpha \in \mathbf{R}$ (the spaces $Z^\alpha(\ldots)$, $R^\alpha(\ldots)$ in (1.24)-(1.27)). Similar statements hold for the following results.

For any normal boundary condition we shall define the operator

$$Q = (I - C^{00}B^{00})^*\mathcal{A}^{01}(I - C^{11}B^{11}), \tag{2.19}$$

it is an $m \times m$-system of differential operators in $E|_\Gamma$, of type $(-k, -2m + 1 + j)_{j\in M_0, k\in M_1}$, just like $\mathcal{A}^{01}$; in particular it has zeroes below the second diagonal. Define the second-diagonal parts

$$Q_d = (\delta_{j,2m-1-k}Q_{jk})_{j\in M_0, k\in M_1},$$
$$\mathcal{A}_d^{01} = (\delta_{j,2m-1-k}\mathcal{A}_{jk})_{j\in M_0, k\in M_1};$$

they are vector bundle morphisms. It is easily seen that

$$(2.20) \qquad Q_d = (I - C_d^{00}B_d^{00})^*\mathcal{A}_d^{01}(I - C_d^{11}B_d^{11}).$$

Since $Q = 0$ implies $Q_d = 0$, we get immediately

LEMMA 2.10. — *The identity* (2.6) *implies the following equivalent statements* (2.21)-(2.24)

$$(2.21) \qquad (I - C_d^{00}B_d^{00})^*\mathcal{A}_d^{01}(I - C_d^{11}B_d^{11}) = 0;$$
$$(2.22) \qquad Z(B_d^{11}) \subset Z((I - C_d^{00}B_d^{00})^*\mathcal{A}_d^{01});$$
$$(2.23) \qquad \mathcal{A}_d^{01}Z(B_d^{11}) \subset R(B_d^{00*});$$
$$(2.24) \qquad \mathcal{A}_d^{01*}Z(B_d^{00}) \subset R(B_d^{11*}).$$

(2.21)-(2.24) actually express certain properties of the bundle $\bigoplus_{j\in M} F_j$ in relation to $\bigoplus_{j\in M} E|_\Gamma$. Let us make this explicit in the case where $\mathcal{A}^{01}$ is invertible, i.e. Γ is noncharacteristic for A:

THEOREM 2.11. — *Assume that* Γ *is noncharacteristic for* A. *Then* (2.6) *implies that* $Z_j = \ker B_{jj}$ *is isomorphic to a subbundle of* F_{2m-1-j}, *for all* $j \in M$. *In particular,*

$$(2.25) \qquad \sum_{j\in M} p_j \geqslant mq.$$

When furthermore $\sum_{j\in M} p_j = mq$, *then*

$$(2.26) \qquad Z(B_d^{11}) = (\mathcal{A}_d^{01})^{-1}R(B_d^{00*}),$$
$$(2.27) \qquad Z(B_d^{00}) = (\mathcal{A}_d^{01*})^{-1}R(B_d^{11*}),$$

and $Z_j \cong F_{2m-1-j}$ *for all* $j \in M$.

Proof. — Since $\mathcal{A}_d^{01}$ is skew-diagonal and invertible, (2.23) may be written

$$Z(B_{jj}) \subset (\mathcal{A}_{2m-1-j,\,j})^{-1}R(B^*_{2m-1-j,\,2m-1-j}), \quad \text{for all} \quad j \in M_1.$$

This is equivalent with the statement for bundles (cf. (1.22))

$$Z_j \subset (\mathcal{A}_{2m-1-j,j})^{-1}B^*_{2m-1-j,\,2m-1-j}F_{2m-1-j},$$

where $(\mathcal{A}_{2m-1-j,j})^{-1}B^*_{2m-1-j,\,2m-1-j}$ is an injective morphism. This shows the first statement for $j \in M_1$; for $j \in M_0$ it follows similarly from (2.24).

Regarding dimensions, (2.23) and (2.24) imply

$$(2.28)\quad \sum_{j\in M_1}(q-p_j)\leqslant\sum_{j\in M_0}p_j,\qquad \sum_{j\in M_0}(q-p_j)\leqslant\sum_{j\in M_1}p_j,$$

respectively; both statements are equivalent with (2.25). When equality holds in (2.28), (2.23) and (2.24) represent inclusions between vector bundles of the same dimension, these must be identities, so (2.26) and (2.27) hold.

Remark 2.12. — When Γ is characteristic for A, (2.6) may be satisfied with $\sum_{j\in M} p_j < mq$, and (2.6) is in a sense less restrictive on B. We refrain from a systematic treatment here.

Also the inclusions in Lemma 2.8 can now be improved, when $\sum_{j\in M} p_j = mq$, and Γ is noncharacteristic.

THEOREM 2.13. — *Assume that* Γ *is noncharacteristic for* A, *and that* $\sum_{j\in M} p_j = mq$. *Then* (2.6) *is equivalent with each of the statements* (2.29)-(2.32)

$$(2.29)\qquad Z(B^{11})\subset(\mathcal{A}^{01})^{-1}R(B^{00*}),$$
$$(2.30)\qquad Z(B^{11})=(\mathcal{A}^{01})^{-1}R(B^{00*}),$$
$$(2.31)\qquad Z(B^{11})\supset(\mathcal{A}^{01})^{-1}R(B^{00*}),$$
$$(2.32)\qquad B^{11}(\mathcal{A}^{01})^{-1}B^{00*}=0.$$

Proof. — We have from Lemma 2.8 that (2.6) is equivalent with (2.29). Clearly (2.31) and (2.32) are equivalent. Since (2.30) implies (2.29) and (2.31), it remains to show that (2.29) implies (2.30), and that (2.31) implies (2.30).

Assume (2.29). Since we are now dealing with differential operators and not just morphisms, the dimension argument in the previous proof is not directly applicable. We have however, using Theorem 2.11 and Lemma 1.12

$$(2.33)\quad \begin{aligned} R(B^{00*}) &\supset \mathcal{A}^{01}Z(B^{11})\\ &= \mathcal{A}^{01}(I-C^{11}B^{11}_s)Z(B^{11}_d)\\ &= \mathcal{A}^{01}(I-C^{11}B^{11}_s)(\mathcal{A}^{01}_d)^{-1}R(B^{00*}_d)\\ &= \mathcal{A}^{01}(I-C^{11}B^{11}_s)(\mathcal{A}^{01}_d)^{-1}(I-B^{00*}_sC^{00*})R(B^{00*})\\ &= (I+K)R(B^{00*}),\end{aligned}$$

where K is a subtriangular differential operator in $\bigoplus_{k \in M_0} E|_\Gamma$, thanks to the subtriangular character of B_s^{11} and B_s^{00*}. Denoting $R(B^{00*})$ by R, we have found

$$R \supset (I + K)R. \tag{2.34}$$

Now K is a nilpotent operator on $\prod_{k \in M_0} \mathscr{D}'(E|_\Gamma)$. Moreover, since $I + K$ maps R into R, K itself maps R into R. Then $(I + K)|_R$ has the inverse

$$I|_R - K|_R + (K|_R)^2 - \cdots + (-K|_R)^m,$$

so it maps R onto R, and the inclusion in (2.34) must be the identity. Then also the inclusion in (2.33) is the identity, and we have proved (2.30).

The proof that (2.31) implies (2.30) follows similarly from

$$\begin{aligned} Z(B^{11}) &\supset (\mathcal{A}^{01})^{-1}R(B^{00*}) \\ &= (\mathcal{A}^{01})^{-1}(I + B_s^{00*}C_d^{00*})\mathcal{A}_d^{01}(I + C_d^{11}B_s^{11})Z(B^{11}) \\ &= (I + K_1)Z(B^{11}). \end{aligned}$$

Corollary 2.14. — *When Γ is noncharacteristic for A, and $\sum_{j \in M} p_j = mq$, then* (2.6) *is equivalent with each of the statements* (2.35)-(2.38)

$$Z(B^{00}) \subset (\mathcal{A}^{01*})^{-1}R(B^{11*}), \tag{2.35}$$
$$Z(B^{00}) = (\mathcal{A}^{01*})^{-1}R(B^{11*}), \tag{2.36}$$
$$Z(B^{00}) \supset (\mathcal{A}^{01*})^{-1}R(B^{11*}), \tag{2.37}$$
$$B^{00}(\mathcal{A}^{01*})^{-1}B^{11*} = 0. \tag{2.38}$$

Proof. — Follows from Theorem 2.13, using that the identities are pairwise equivalent (adjoint).

Theorem 2.4 together with Corollary 2.14 prove Theorem 1 in [7].

2.3. Existence and uniqueness of B^{11} for given B^{00}.

For the case where Γ is noncharacteristic for A, and $\sum_{j \in M} p_j = mq$, we shall consider the problem of how B^{10} and B^{11} may look, when B^{00} is given, and B shall satisfy (2.6).

(The question on how B^{00} and B^{10} depend on B^{11} is treated similarly.)

We shall denote by $I^{\times}$ the skew-unit matrix

$$I^{\times} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{bmatrix}, \tag{2.39}$$

indexed according to its use. Denote $\bigoplus_{k \in M_0} E|_\Gamma$ by E^0 and $\bigoplus_{k \in M_1} E|_\Gamma$ by E^1. Given $F^0 = \bigoplus_{j \in M_0} F_j$, and B^{00} going from E^0 to F^0.

1° *Existence.* — B^{11} is required to have surjective diagonal part and satisfy (cf. Theorem 2.13)

$$Z(B^{11}) = (\mathcal{A}^{01})^{-1} R(B^{00*}). \tag{2.40}$$

Now we find by Lemmas 1.11-1.12

(2.41)
$$\begin{aligned}(\mathcal{A}^{01})^{-1} R(B^{00*}) &= (\mathcal{A}^{01})^{-1}(I + B_s^{00*} C_d^{00*}) R(B_d^{00*}) \\ &= (\mathcal{A}^{01})^{-1}(I + B_s^{00*} C_d^{00*}) Z((I - C_d^{00} B_d^{00})^*) \\ &= Z((I - C_d^{00} B_d^{00})(I - B_s^{00*} C^{00*}) \mathcal{A}^{01}),\end{aligned}$$

where $(I - C_d^{00} B_d^{00})^* = I - C_d^{00} B_d^{00}$ defines the orthogonal projection of E^0 onto $Z^0 = \bigoplus_{k \in M_0} Z_k$ ($Z_k = \ker B_{kk}$, cf. Lemma 1.10), let us denote it pr_{Z^0}. Then if we define

$$\overline{F}_j = Z_{2m-1-j} \qquad \text{for} \qquad j \in M_1,$$

i.e. $\overline{F}^1 = \bigoplus_{j \in M_1} \overline{F}_j = I^{\times} Z_0$ with $I^{\times} = (\delta_{2m-1-j,k})_{j \in M_1, k \in M_0}$; and

$$\overline{B}^{11} = I^{\times} pr_{Z^0}(I - B_s^{00*} C^{00*}) \mathcal{A}^{01},$$

then $\overline{B}^{11}$ is a differential operator from E^1 to $\overline{F}^1$ of type $(-k, -j)_{j,k \in M_1}$ satisfying (2.40), and its diagonal part $\overline{B}_d^{11} = I^{\times} pr_{Z^0} \mathcal{A}_d^{01}$ is surjective. Also

$$\dim \Big(\bigoplus_{j \in M_0} F_j \Big) + \dim \Big(\bigoplus_{j \in M_1} \overline{F}_j \Big) = mq$$

as required. B^{10} can be any differential operator from E^0 to $\overline{F}^1$ of type $(-k, -j)_{j \in M_1, k \in M_0}$.

2° *Uniqueness.* — Now let $F^1 = \bigoplus_{j \in M_1} F_j$ and B^{10} and B^{11} satisfying (2.6) etc. be given. By Theorem 2.11, $F_j \cong Z_{2m-1-j}$ for $j \in M_1$, so $F^1 = \Phi\overline{F}^1$ for some diagonal bijective morphism $\Phi = (\Phi_{jk})_{j,k \in M_1}$. Moreover, we have $Z(B^{11}) = Z(\overline{B}^{11})$. Using the decomposition

$$\mathcal{D}'(E^1) = Z(\overline{B}^{11}) \dotplus R(\overline{C}^{11}) \quad \text{(direct sum)},$$

written $\varphi = \varphi_0 + \varphi_1$, we find that

$$\begin{aligned} B^{11}\varphi &= B^{11}(\varphi_0 + \varphi_1) = B^{11}\varphi_1 = B^{11}\overline{C}^{11}\overline{B}^{11}\varphi_1 \\ &= B^{11}\overline{C}^{11}\overline{B}^{11}\varphi = \Phi\Phi^{-1}B^{11}\overline{C}^{11}\overline{B}^{11}\varphi \\ &= \Phi\Psi\overline{B}^{11}\varphi, \qquad \text{for all} \qquad \varphi \in \mathcal{D}'(E^1), \end{aligned}$$

where $\Psi = \Phi^{-1}B^{11}\overline{C}^{11}$ is a bijective differential operator of type $(-k, -j)_{j,k \in M_1}$ in $\overline{F}^1$; by Lemma 1.13 it has bijective diagonal part. Conversely, if Ψ is a bijective differential operator in $\overline{F}^1$, then $Z(\Phi\Psi\overline{B}^{11}) = Z(\overline{B}^{11})$.

So there is existence and uniqueness of B^{11} up to isomorphisms. More precisely, we have found:

THEOREM 2.15. — *Assume that* Γ *is noncharacteristic for* A, *and consider normal boundary conditions with* $\sum_{j \in M} p_j = mq$. *Let* $\bigoplus_{j \in M_0} F_j$ *and* B^{00} *be given. Then the operators* B^{10} *and* B^{11}, *for which* (2.6) *is satisfied, are characterized by*

(i) $\bigoplus_{j \in M_1} F_j = \Phi\Big(\bigoplus_{k \in M_1} Z_{2m-1-k}\Big)$, *where* $\Phi = (\Phi_{jk})_{j,k \in M_1}$ *is any diagonal vector bundle isomorphism.*

(ii) B^{10} *is any differential operator from* $\bigoplus_{k \in M_0} E|_\Gamma$ *to* $\bigoplus_{j \in M_1} F_j$ *of type* $(-k, -j)_{j \in M_1, k \in M_0}$.

(iii) $B^{11} = \Phi\Psi\overline{B}^{11}$, *where* Ψ *is any differential operator in* $\bigoplus_{k \in M_1} Z_{2m-1-k}$ *with bijective diagonal part, and*

$$\overline{B}^{11} = I^{\times} \operatorname{pr}_{Z^0}(I - B_s^{00*}C^{00*})\mathfrak{A}^{01}, \tag{2.42}$$

as defined above.

2.4. The adjoint boundary condition.

We shall finally study the (formally) adjoint realization; it represents a boundary condition that may be determined by methods analogous to those of section 2.3. This leads to a more illuminating criterion for weak semiboundedness. The determination of the adjoint realization is independent of whether A is of even or odd order, so for that part, the order will again be denoted r.

DEFINITION 2.16. — *The formally adjoint realization* $(A_B)'$ *of* A_B *is the operator sending* v *into* $A'v$, *with domain*

$$(2.43)\quad D((A_B)') = \{v \in H^r(E) \mid (Au, v) - (u, A'v) = 0 \text{ for all } u \in D(A_B)\}.$$

We shall now show that there exists a differential operator B' so that $(A_B)'$ is the realization of A' determined by the boundary condition $B'\rho v = 0$.

PROPOSITION 2.17. — $D((A_B)') = D(A'_{B'})$, *where*

$$(2.44)\qquad B' = I^{\times}\,\mathrm{pr}_Z(I - B_s^*C^*)\mathfrak{A}^*,$$

here pr_Z *denotes the orthogonal projection of* $\bigoplus_{k\in M} E|_\Gamma$ *onto* $\bigoplus_{k\in M} Z_k$, *and* $I^{\times} = (\delta_{2m-1-j,\,k})_{j,\,k\in M}$. B' *is a system* $(B'_{jk})_{j,\,k\in M}$ *of differential operators* B'_{jk} *from* $E|_\Gamma$ *to* Z_{2m-1-j} *(for* $j, k \in M$), *of type* $(-k, -j)_{j,\,k\in M}$. *When* Γ *is noncharacteristic for* A, B' *is normal.*

Proof. — When $u \in D(A_B)$, ρu runs through

$$Z^r(B) = R^r(I - CB),$$

so

$$(Au, v) - (u, A'v) = \langle \mathfrak{A}(I - CB)\varphi, \rho v\rangle = \langle \varphi, (I - CB)^*\mathfrak{A}^*\rho v\rangle.$$

Thus

$$D((A_B)') = \{v \in H^r(E) \mid (I - CB)^*\mathfrak{A}^*\rho v = 0\} = \{v \in H^r(E) \mid \rho v \in Z^r((I - CB)^*\mathfrak{A}^*)\}.$$

Like in (2.41) we find

$$\begin{aligned} Z^r((I - CB)^*\mathcal{A}^*) &= (\mathcal{A}^*)^{-1}R^r(B^*) = (\mathcal{A}^*)^{-1}(I + B_s^*C_d^*)R^r(B_d^*) \\ &= Z^r((1 - C_dB_d)^*(I - B_s^*C^*)\mathcal{A}^*) \\ &= Z^r(B'), \end{aligned}$$

with B' defined by (2.44), since $(1 - C_dB_d)^* = 1 - C_dB_d$ is the orthogonal projection of $\bigoplus_{k\in M} E|_\Gamma$ onto $\bigoplus_{k\in M} Z_k$, cf. Lemma 1.10. (As in section 2.3 we insert $I^\times$ to get the correct type.) The diagonal part is $B'_d = I^\times \, \mathrm{pr}_Z \, \mathcal{A}_d^*$, which is surjective when $\mathcal{A}$ is invertible (i.e., when Γ is noncharacteristic).

Next, we consider the uniqueness question. Given a bundle $\bigoplus_{j\in M} F'_j$ and a normal system $\tilde{B}' = (\tilde{B}'_{jk})_{j,\,k\in M}$ of differential operators $\tilde{B}'_{jk}$ from $E|_\Gamma$ to F'_j, of type $(-k, -j)_{j,\,k\in M}$, for which $(A_B)' = A'_{\tilde{B}'}$. This means that

$$Z^r(\tilde{B}') = Z^r(B') \tag{2.45}$$

in view of Proposition 2.17. Now $Z^r(\tilde{B}') = R^r(I - \tilde{C}'\tilde{B}')$, so (2.45) implies

$$B'(I - \tilde{C}'\tilde{B}') = 0.$$

In particular, the diagonal part must be zero, so we conclude

$$Z(B'_d) \supset R(I - \tilde{C}'_d\tilde{B}'_d) = Z(\tilde{B}'_d).$$

Assume in the rest of this proof that also B' is normal; then an analogous argument gives the opposite inclusion, so in fact

$$Z(\tilde{B}'_d) = Z(B'_d).$$

This gives for the bundles:

$$\bigoplus_{j\in M} F'_j \simeq \bigoplus_{j\in M} (E|_\Gamma \ominus \ker \tilde{B}'_{jj}) = \bigoplus_{j\in M} (E|_\Gamma \ominus \ker B'_{jj}) \simeq \bigoplus_{j\in M} Z_{2m-1-j},$$

the range space for B'. So $\bigoplus_{j\in M} F'_j = \Phi\left(\bigoplus_{j\in M} Z_{2m-1-j}\right)$ for some diagonal bijective morphism. It is finally seen from (2.45), like in the proof of Theorem 2.15, that $\tilde{B}' = \tilde{B}'C'B'$ where $\tilde{B}'C'$ is a differential operator from $\bigoplus_{k\in M} Z_{2m-1-k}$ to $\bigoplus_{j\in M} F'_j$

with bijective diagonal part. We have proved

THEOREM 2.18. — *Assume that* B', *defined by* (2.44), *is normal. A normal system* $\tilde{B}' = (\tilde{B}'_{jk})_{j,k\in M}$ *of differential operator* $\tilde{B}'_{jk}$ *from* $E|_\Gamma$ *to* F'_j, *of type* $(-k, -j)_{j,k\in M}$, *satisfies*

$$D((A_B)') = D(A'_{\tilde{B}'})$$

if and only if

(i) $F'_j \cong Z_{2m-1-j}$ *for each* $j \in M$;

(ii) $\tilde{B}' = \Psi B'$, *where* Ψ *is a bijective differential operator from* $\bigoplus_{k\in M} Z_{2m-1-k}$ *to* $\bigoplus_{j\in M} F'_j$, *of type* $(-k, -j)_{j,k\in M}$.

Now let $r = 2m$. We then have, with obvious notations,

$$(2.47)\quad B' = \begin{bmatrix} 0 & I^\times \\ I^\times & 0 \end{bmatrix} \begin{bmatrix} pr_{Z^0} & 0 \\ 0 & pr_{Z^1} \end{bmatrix} \times$$
$$\times \begin{bmatrix} I - B_s^{00*}C^{00*} & S_1 \\ 0 & I - B_s^{11*}C^{11*} \end{bmatrix} \begin{bmatrix} \mathcal{A}^{00*} & \mathcal{A}^{10*} \\ \mathcal{A}^{01*} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I^\times pr_{Z^1}(I - B_s^{11*}C^{11*})\mathcal{A}^{01*} & 0 \\ S_2 & I^\times pr_{Z^0}(I - B_s^{00*}C^{00*})\mathcal{A}^{10*} \end{bmatrix}$$

(S_1 and S_2 not worth calculating). Thus, using Lemma 1.12,

$$\begin{aligned}(2.48)\quad Z(B'^{00}) &= Z(I^\times pr_{Z^1}(I - B_s^{11*}C^{11*})\mathcal{A}^{01*}) \\ &= (\mathcal{A}^{01*})^{-1}(I + B_s^{11*}C_d^{11*})Z(I - B_d^{11*}C_d^{11*}) \\ &= (\mathcal{A}^{01*})^{-1}R(B^{11*}).\end{aligned}$$

This formula is valid whether $\mathcal{A}^{01*}$ is invertible or not, if we by $(\mathcal{A}^{01*})^{-1}$ understand the mapping of a set into its inverse image by $\mathcal{A}^{01*}$. Connecting this with Lemma 2.8 (in particular (2.18), cf. also Theorem 2.4), we find

THEOREM 2.19. — A_B *is weakly semibounded* (*or, equivalently, satisfies* (2.6)) *if and only if*

$$(2.49)\qquad Z^{2m}(B^{00}) \subset Z^{2m}(B'^{00}),$$

that is, if and only if $\gamma D(A_B) \subset \gamma D((A_B)')$.

For the noncharacteristic case this has the consequence

COROLLARY 2.20. — *Let* Γ *be noncharacteristic for* A. *If* A_B *is weakly semibounded, then* $(A_B)'$ *is weakly semibounded* *if and only if* $\sum_{j\in M} p_j = mq$. *When* $\sum_{j\in M} p_j = mq$, (2.6) *is equivalent with*

$$Z(B^{00}) = Z(B'^{00}), \tag{2.50}$$

i.e., $\gamma D(A_B) = \gamma D(A'_{B'})$.

Proof. — Let A_B be weakly semibounded. Then by Theorem 2.11, $\sum_{j\in M} p_j \geqslant mq$. When $\sum_{j\in M} p_j > mq$, the fiber dimension of the range space for B' is

$$\sum_{j\in M} (q - p_{2m-1-j}) = 2mq - \sum_{k\in M} p_k < mq,$$

so, by Theorem 2.11, $A'_{B'}$ cannot be weakly semibounded.

Now assume $\sum_{j\in M} p_j = mq$. Then by Theorem 2.13 and (2.48), (2.6) is equivalent with (2.50). Since $(A'_{B'})' = A_B$, and (2.50) *is symmetric in* $\{A, B\}$ *and* $\{A', B'\}$, (2.50) must also be necessary and sufficient for the weak semiboundedness of $A'_{B'}$.

Remark 2.21. — It was shown in [6, Theorem 3.4] for the case where A is elliptic, that also for *general* realizations $\tilde{A}$ of A, $\gamma D(\tilde{A}) \subset \overline{\gamma D(\tilde{A}^*)}$ (closure in $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$) is necessary for weak semiboundedness (the above proof was inspired from that theory). This is the reason for our conjecture that normality is necessary under much more general circumstances than those accounted for in Remark 2.2; for the lack of normality tends to enlarge $Z(B^{00})$ and diminish $\gamma D((A_B)')$.

CHAPTER 3

SYSTEMS OF TYPE $(m_t, -m_s)_{s,t=1,\dots,q}$.

3.1. Green's formulae.

In this chapter we describe how the results of Chapters 1 and 2 extend to systems A that are of « mixed order », and of a symmetric type.

Let $\{m_1, \dots, m_q\}$ be a set of nonnegative integers, and let $A = (A_{st})_{s,t=1,\dots,q}$ be a $q \times q$-matrix of differential operators on $\overline{\Omega}$, of type $(m_t, -m_s)_{s,t=1,\dots,q}$; i.e., A_{st} is of order $m_s + m_t$. Among the systems of this type are the strongly elliptic systems, cf. [2]. Denote

(3.1)
$$N = \{1, \dots, q\}, \quad m = \max_{t \in N} m_t, \quad \overline{m} = m_1 + \cdots + m_q,$$

and assume $m > 0$.

For such systems one usually studies boundary conditions of the following kind: There is given a set of p integers $\{\mu_1, \dots, \mu_p\}$ and a $p \times q$-matrix of differential operators $\mathscr{B} = (\mathscr{B}_{st})_{s=1,\dots,p;\, t=1,\dots,q}$ on $\overline{\Omega}$, of type

$$(m_t, \mu_s)_{s=1,\dots,p;\, t=1,\dots,q};$$

it defines the boundary value problem (3.2)-(3.3)

$$(3.2)\quad Au = f, \quad \text{i.e.,} \quad \sum_{t=1}^{q} A_{st} u_t = f_s, \quad s = 1, \dots, q;$$

$$(3.3)\quad \gamma_0 \mathscr{B} u = 0, \quad \text{i.e.,} \quad \sum_{t=1}^{q} \gamma_0 \mathscr{B}_{st} u_t = 0, \quad s = 1, \dots, p;$$

here $f = \{f_1, \dots, f_q\}$, $u = \{u_1, \dots, u_q\}$.

(3.3) determines a realization $A_{\mathscr{B}}$ by

$$A_{\mathscr{B}} : u \longmapsto Au, \; D(A_{\mathscr{B}}) = \Big\{ u \in \prod_{t \in N} H^{m_t + m}(\overline{\Omega}) \,|\, \gamma_0 \mathscr{B} u = 0 \Big\}.$$

We shall say that $A_{\mathcal{B}}$ is weakly semibounded if there exist $c > 0$, $\theta \in \mathbf{R}$ such that

$$\operatorname{Re} e^{i\theta}(Au, u) \leqslant c \sum_{t \in N} \|u_t\|^2_{m_t}, \qquad \text{for all} \qquad u \in D(A_{\mathcal{B}});$$

like in Chapter 2, the inequality depends only on A and $\mathcal{B}$ at Γ, but involves the exact operators, not just for instance principal symbols. When A is elliptic, one usually assumes $p = \overline{m}$; we do not assume that on beforehand.

If for some s, $m_t - \mu_s < 0$ for all t, then the terms $\mathcal{B}_{st}$ are all zero, so $\sum_{t=1}^{q} \gamma_0 \mathcal{B}_{st} u_t = 0$ is trivially satisfied; we can therefore assume that $\mu_s \leqslant \max m_t = m$ for all s. We shall furthermore assume that the μ_s are $\geqslant -m+1$, hereby we exclude boundary conditions of very high order, just like boundary conditions of order $\geqslant 2m$ were excluded in Chapter 2. Our boundary conditions still include those that arize in connection with sesquilinear forms as in Guedes de Figueiredo [9].

To apply our techniques we shall set up a Green's formula and reformulate (3.3), such that *differential operators on* Γ *of the same order are grouped together.*

Let $\{s, t\} \in N \times N$. We have by Lemma 1.2 for u_t, $\varphi_s \in C^\infty(\overline{\Omega})$

$$[I_{st} \equiv](A_{st}u_t, \varphi_s) - (u_t, A'_{st}\varphi_s) = \sum_{j', k'=0}^{m_s+m_t-1} \langle \mathcal{C}_{stj'k'} \gamma_{k'} u_t, \gamma_{j'} \varphi_s \rangle,$$

where $\mathcal{C}_{stj'k'}$ is of order $m_s + m_t - j' - k' - 1$. Set $j = j' - m_s + m$ and $k = k' - m_t + m$, and set

$$\tilde{\mathcal{C}}_{stjk} = \mathcal{C}_{st,\, j+m_s-m,\, k+m_t-m}$$

for

$$j \in \{-m_s + m, \ldots, 2m-1\}$$

and

$$k \in \{-m_t + m, \ldots, 2m-1\},$$

where we put $\tilde{\mathcal{C}}_{stjk} = 0$ for $j \geqslant m_t + m$ or $k \geqslant m_s + m$. Then $\tilde{\mathcal{C}}_{stjk}$ is of order

$$m_s + m_t - (j + m_s - m) - (k + m_t - m) - 1 = 2m - j - k - 1$$

for all j, k. (Here, for $j \geqslant m_t + m$ or $k \geqslant m_s + m$, the order is negative in accordance with $\tilde{\mathcal{C}}_{stjk} = 0$. We have actually just augmented the usual boundary matrix by some zero rows and columns.) Now

$$I_{st} = \sum_{j=-m_s+m}^{2m-1} \sum_{k=-m_t+m}^{2m-1} \langle \tilde{\mathcal{C}}_{stjk} \gamma_{k+m_t-m} u_t, \gamma_{j+m_s-m} v_s \rangle.$$

For $u, v \in \sum_{t \in N} C^\infty(\overline{\Omega})$ one has

$$(3.4) \qquad (Au, v) - (u, A'v) = \sum_{s,\, t \in N} I_{st},$$

and we shall now regroup the terms in $\sum_{s,\, t \in N} I_{st}$. Define as usual

$$(3.5) \qquad M = \{0, \ldots, 2m-1\}, \quad M_0 = \{0, \ldots, m-1\}, \quad M_1 = \{m, \ldots, 2m-1\}.$$

For each $k \in M$, define

$$(3.6) \qquad N_k = \{t \in N \mid k + m_t - m \geqslant 0\},$$

and denote $|N_k| = q_k$. Clearly,

$$(3.7) \qquad \varnothing \neq N_0 \subset \cdots \subset N_{m-1} \subset N_m = N_{m+1} = \cdots = N_{2m-1} = N,$$

using that $N_m = \{t \mid m_t \geqslant 0\} = N$ (and *all* N_k equal N if and only if all m_t equal m as in Chapter 2). Note that $q_m = \cdots = q_{2m-1} = q$. Moreover, it is easily seen that

$$(3.8) \qquad q_0 + \cdots + q_{m-1} = \overline{m}.$$

Denote the trivial bundles $\Gamma \times \prod_{t \in N_k} \mathbf{C}$ by E_k.

Now define for each $\{j, k\} \in M \times M$ the $N_j \times N_k$ matrix $\tilde{\mathcal{C}}_{jk}$ by

$$(3.9) \qquad \tilde{\mathcal{C}}_{jk} = (\tilde{\mathcal{C}}_{stjk})_{s \in N_j,\, t \in N_k};$$

it is a differential operator from E_k to E_j of order $2m - j - k - 1$. Altogether the $\tilde{\mathcal{C}}_{jk}$ from a system

$$(3.10) \qquad \tilde{\mathcal{C}} = (\tilde{\mathcal{C}}_{jk})_{j,k \in M}$$

of type $(-k, -2m+1+j)_{j,k \in M}$.

Introduce the vector valued trace operators β_k, for $k \in M$, by

$$\beta_k u = \{\gamma_{k+m_t-m} u_t\}_{t \in N_k}; \tag{3.11}$$

they are continuous and surjective from $\prod_{t \in N_k} H^{\alpha+m_t+m}(\overline{\Omega})$ onto $H^{\alpha+2m-k-\frac{1}{2}}(E_k)$, for $\alpha + 2m > k + \frac{1}{2}$, respectively. Altogether they form a vector (of vectors) β:

$$\beta u = \{\beta_k u\}_{k \in M}. \tag{3.12}$$

Actually, βu consists of a rearrangement of the traces $\{\gamma_0 u_1, \gamma_1 u_1, \ldots, \gamma_{m_1+m-1} u_1; \cdots; \gamma_0 u_q, \gamma_1 u_q, \cdots, \gamma_{m_q+m-1} u_q\}$, and it has a total number of $\sum_{t \in N} (m_t + m) = \sum_{k \in M} q_k$ elements. These are exactly all that enter in (3.4). It is thus reasonable to call βu the *Cauchy data* of u. We now find

$$\begin{aligned}
\sum_{s,t \in N} I_{st} &= \sum_{s,t \in N} \sum_{j=-m_s+m}^{2m-1} \sum_{k=-m_t+m}^{2m-1} \langle \tilde{\mathfrak{A}}_{stjk} \gamma_{k+m_t-m} u_t, \gamma_{j+m_s-m} v_s \rangle \\
&= \sum_{j=0}^{2m-1} \sum_{k=0}^{2m-1} \sum_{s \in N_j} \sum_{t \in N_k} \langle \tilde{\mathfrak{A}}_{stjk} \gamma_{k+m_t-m} u_t, \gamma_{j+m_s-m} v_s \rangle \\
&= \sum_{j,k \in M} \langle \tilde{\mathfrak{A}}_{jk} \beta_k u, \beta_j v \rangle = \langle \tilde{\mathfrak{A}} \beta u, \beta v \rangle,
\end{aligned}$$

and have hereby proved *Green's formula*

$$(Au, v) - (u, A'v) = \langle \tilde{\mathfrak{A}} \beta u, \beta v \rangle, \tag{3.13}$$

it is valid for all $u, v \in \prod_{t \in N} H^{m_t+m}(\overline{\Omega})$. (A similar formula holds when the functions u_t are replaced by sections in bundles, we omit this aspect for simplicity.)

Let us now consider the case where Γ is noncharacteristic for A. This means that the $N \times N$-matrix A^0, whose entries are the functions A_{st, m_s+m_t} stemming from the decompositions $A_{st} = \sum_{l=0}^{m_s+m_t} A_{st,l} D_n^l$, is bijective. The elements $\tilde{\mathfrak{A}}_{j,2m-1-j}$ are $N_j \times N_{2m-1-j}$-*submatrices* of A^0, so that when the m_t are not all equal, $\tilde{\mathfrak{A}}$ can never be invertible (in view of (3.7), cf. also below). This is the main reason for the trouble with setting up e.g. a Lions-Magenes theory for boundary value problems for systems of mixed order. However, it is possible to treat a particular class of boundary value problems, as indicated below.

Define (cf. (3.5))

$$(3.14)\qquad \beta^0 u = \{\beta_k u\}_{k\in M_0}, \qquad \beta^1 u = \{\beta_k u\}_{k\in M_1},$$

and note that $\beta^0 u$ is a rearrangement of the *Dirichlet data*

$$\{\gamma_0 u_1, \ldots, \gamma_{m_1-1}u_1; \ldots; \gamma_0 u_q, \ldots, \gamma_{m_q-1}u_q\};$$

$\beta^0 u$ has $\sum_{k\in M_0} q_k = \overline{m}$ entries and $\beta^1 u$ has mq entries. With the usual decomposition of $\tilde{\mathfrak{A}}$

$$\tilde{\mathfrak{A}} = \begin{bmatrix} \tilde{\mathfrak{A}}^{00} & \tilde{\mathfrak{A}}^{01} \\ \tilde{\mathfrak{A}}^{10} & 0 \end{bmatrix}, \quad \tilde{\mathfrak{A}}^{\delta\varepsilon} = (\tilde{\mathfrak{A}}_{jk})_{j\in M_\delta,\, k\in M_\varepsilon},$$

(3.13) takes the form

$$(3.15)\quad (\mathrm{A}u, v) - (u, \mathrm{A}'v) = \langle \tilde{\mathfrak{A}}^{00}\beta^0 u, \beta^0 v\rangle + \langle \tilde{\mathfrak{A}}^{01}\beta^1 u, \beta^0 v\rangle + \langle \tilde{\mathfrak{A}}^{10}\beta^0 u, \beta^1 v\rangle.$$

Because of (3.7), the second diagonal in $\tilde{\mathfrak{A}}^{01}$, resp. $\tilde{\mathfrak{A}}^{10}$, consists of $\mathrm{N}_j \times \mathrm{N}$-submatrices, resp. $\mathrm{N} \times \mathrm{N}_k$-submatrices, of A^0. Then we obtain, by application of Proposition 1.9 to $\mathrm{I}^\times\tilde{\mathfrak{A}}^{01}$ and to $\mathrm{I}^\times(\tilde{\mathfrak{A}}^{10})^*$ (cf. (2.39)):

THEOREM 3.1. — *When* Γ *is noncharacteristic for* A, $\tilde{\mathfrak{A}}^{01}$ *is* surjective *with a right inverse* $\mathfrak{D}^{01}$ *of type*

$$(-2m + 1 + k, -j)_{j\in M_1,\, k\in M_0},$$

and $\tilde{\mathfrak{A}}^{10}$ *is* injective *with a left inverse* $\mathfrak{D}^{10}$ *of type*

$$(-2m + 1 + k, -j)_{j\in M_0,\, k\in M_1}.$$

It will be seen below that these properties suffice to generalize the results of Chapter 2 in a very satisfactory way. Moreover, (3.15) may be viewed as a Green's formula for some special boundary operators, for Theorem 3.1 clearly implies

COROLLARY 3.2. — *Define* $\varkappa$ *and* $\varkappa'$ *by*

$$(3.16)\qquad \begin{aligned} \varkappa u &= \tilde{\mathfrak{A}}^{01}\beta^1 u \\ \varkappa' u &= -\tilde{\mathfrak{A}}^{10*}\beta^1 u - \tilde{\mathfrak{A}}^{00*}\beta^0 u. \end{aligned}$$

Then we have for all $u, v \in \prod_{t\in N} H^{m_t+m}(\overline{\Omega})$

$$(3.17) \quad (Au, v) - (u, A'v) = \langle \varkappa u, \beta^0 v\rangle - \langle \beta^0 u, \varkappa' v\rangle,$$

where, if Γ *is noncharacteristic for* A, $\{\beta^0, \varkappa\}$ *and* $\{\beta^0, \varkappa'\}$ *are surjective continuous mappings of* $\prod_{t\in N} H^{\alpha+m_t+m}(\overline{\Omega})$ *onto* $\prod_{k\in M_0} H^{\alpha+2m-k-\frac{1}{2}}(E_k) \times \prod_{k\in M_0} H^{\alpha+k+\frac{1}{2}}(E_k)$ *for all* $\alpha > -\frac{1}{2}$.

We shall call $\{\beta^0 u, \varkappa u\}$ (resp. $\{\beta^0 u, \varkappa' u\}$) the *reduced Cauchy data* of u with respect to A (resp. A′). (The $\tilde{\mathcal{A}}^{00}$-term may of course be distributed in other ways). Boundary conditions for A that can be expressed as normal conditions on the reduced Cauchy data (i.e., « factor through $\tilde{\mathcal{A}}^{01}$ ») can be treated much like those of Chapter 2; in particular one may set up a Lions-Magenes theory (details will be given elsewhere). Note that the Dirichlet operator β^0 belongs to this class. One of the main theorems of this chapter (Theorem 3.11) will be that the boundary conditions defining weakly semibounded realizations are indeed conditions on the reduced Cauchy data. A few more comments on this class are given in section 3.4.

We conclude this section by establishing the « halfways » Green's formulae. By a sesquilinear form on $\prod_{t\in N} H^{m_t}(\overline{\Omega})$ we shall understand an integro-differential form

$$(3.18) \qquad a(u, v) = \sum_{s,t\in N} \sum_{i\in I(s,t)} (Q_{sti}u_t, P_{sti}v_s),$$

where the Q_{sti} and P_{sti} are differential operators on $\overline{\Omega}$ of order $\leqslant m_t$, resp. $\leqslant m_s$, and the $I(s, t)$ are finite index sets. $a(u, v)$ is defined and continuous on $\prod_{t\in N} H^{m_t}(\overline{\Omega}) \times \prod_{s\in N} H^{m_s}(\overline{\Omega})$, and it is associated with A if and only if

$$(3.19) \qquad A_{st} = \sum_{i\in I(s,t)} P'_{sti}Q_{sti}, \qquad \text{all} \quad s, t \in N.$$

Applying Remark 1.8 to each A_{st} and collecting the terms one finds, just as in section 1.2

Theorem 3.3. — *When* $a(u, v)$ *is a sesquilinear form on* $\prod_{t\in N} H^{m_t}(\overline{\Omega})$ *associated with* A, *then for all* $u \in \prod_{t\in N} H^{m_t+m}(\overline{\Omega})$, *all* $v \in \prod_{s \in N} H^{m_s}(\overline{\Omega})$

$$(3.20) \quad (Au, v) = a(u, v) + \langle \tilde{\mathfrak{A}}^{01}\beta^1 u, \beta^0 v\rangle + \langle \mathcal{S}\beta^0 u, \beta^0 v\rangle;$$

where $\mathcal{S} = (\mathcal{S}_{jk})_{j, k \in M_0}$ *is a system of differential operators from* E_k *to* E_j, *of type* $(-k, -2m+1+j)_{j, k \in M_0}$. *Conversely, for any such system* $\mathcal{S}$ *there exists a sesquilinear form* $a(u, v)$ *on* $\prod_{t \in N} H^{m_t}(\overline{\Omega})$, *fitting together with* A *in* (3.20).

3.2. Normal boundary conditions; weakly semibounded realizations.

We shall now reformulate (3.3). By (1.4) we have

$$\gamma_0 \mathcal{B}_{st} u_t = \sum_{l=0}^{m_t - \mu_s} \mathcal{B}_{st, l} \gamma_l u_t,$$

where each $\mathcal{B}_{st, l}$ is a differential operator in Γ of order $m_t - \mu_s - l$. Letting $k = l - m_t + m$, we can write

$$\gamma_0 \mathcal{B}_{st} u_t = \sum_{k=-m_t+m}^{2m-1} \mathcal{B}_{st, k+m_t-m} \gamma_{k+m_t-m} u_t$$

where we have added on some zero terms (recall $-m \leqslant -\mu_s \leqslant m-1$), and $\mathcal{B}_{st, k+m_t-m}$ is of order $m - \mu_s - k$. Introduce the index sets

$$L = \{1, \ldots, p\}, \quad L_j = \{s \in L | m - \mu_s = j\} \quad \text{for } j \in M,$$

and denote $|L_j| = p_j$. Clearly, L equals the disjoint union $\bigcup_{j \in M} L_j$, and $\sum_{j \in M} p_j = p$. Denote the trivial bundles $\Gamma \times \prod_{s \in L_j} \mathbf{C}$ by F_j (with F_j being the zero bundle $\Gamma \times \{0\}$ when $L_j = \emptyset$). For each $\{j, k\} \in M \times M$ we now define the $p_j \times q_k$-matrix

$$B_{jk} = (\mathcal{B}_{st, k+m_t-m})_{s \in L_j, t \in N_k}$$

it is a differential operator from E_k to F_j of order $j - k$. Altogether the B_{jk} form a system $B = (B_{jk})_{j, k \in M}$ of type

$(-k, -j)_{j,k\in M}$. With β as defined above, the boundary condition (3.3) may then be written in the form

$$\text{(3.21)} \qquad B\beta u = 0,$$

$\left(\text{or, equivalently, } \sum_{k\leqslant j} B_{jk}\beta_k u = 0 \text{ for all } j\in M\right)$.

Our considerations in the following will be valid also when the F_j are nontrivial vectorbundles. From now on we study boundary conditions (3.21) where $B = (B_{jk})_{j,k\in M}$ goes from $\bigoplus_{k\in M} E_k$ to $\bigoplus_{j\in M} F_j$ and is of type $(-k, -j)_{j,k\in M}$, and the F_j are any bundles over Γ of dimension p_j. Definition 1.15 can now be generalized:

Definition 3.4. — *The boundary condition* $B\beta u = 0$ — *or the operator* B — *will be said to be normal when the diagonal part* $B_d = (\delta_{jk}B_{jk})_{j,k\in M}$ *is a surjective morphism. (Then in particular* $p_j \leqslant q_j$ *for all* $j\in M$).

Assume from now on that the boundary condition is normal. We split B in blocks as usual

$$\text{(3.22)} \qquad B = \begin{bmatrix} B^{00} & 0 \\ B^{10} & B^{11}\end{bmatrix}, \qquad B^{\delta\varepsilon} = (B_{jk})_{j\in M_\delta,\, k\in M_\varepsilon},$$

and the considerations in section 1.3 now apply to B, B^{00} and B^{11}, which have the right inverses C, C^{00} resp. C^{11}. (3.21) may be written

$$\text{(3.23)} \qquad B^{00}\beta^0 u = 0, \qquad B^{10}\beta^0 u + B^{11}\beta^1 u = 0.$$

Define $Z^\alpha(B^{00}) = \left\{\varphi\in\prod_{k\in M_0} H^{\alpha-k-\frac{1}{2}}(E_k) \,|\, B^{00}\varphi = 0\right\}$, etc., then Lemma 2.3 generalizes to the present case.

Theorem 3.5. — *Let* $B = (B_{jk})_{j,k\in M}$ *be a system of differential operators from* E_k *to* F_j, *of type* $(-k, -j)_{j,k\in M}$, *defining a normal boundary condition* $B\beta u = 0$. *Let* A_B *be the realization of* A *defined by*

$$\text{(3.24)} \quad A_B : u \longmapsto Au,\ D(A_B) = \left\{u\in\prod_{t\in N} H^{m_t+m}(\overline{\Omega}) \,|\, B\beta u = 0\right\}.$$

The following statements (i)-(iv) *are equivalent*:

(i) *There exist* $\theta \in \mathbf{R}$, $c > 0$ *such that*

(3.25) $\operatorname{Re} e^{i\theta}(Au, u) \leqslant c \sum_{t \in N} \|u_t\|^2_{m_t}$, *all* $u \in D(A_B)$

(*i.e.*, A_B *is weakly semibounded*).

(ii) *The following identity holds*

(3.26) $$(I - C^{00}B^{00})^* \tilde{\mathfrak{A}}^{01}(I - C^{11}B^{11}) = 0.$$

(iii) *There exists a sesquilinear form* $a_B(u, v)$ *on* $\prod_{t \in N} H^{m_t}(\overline{\Omega})$ *associated with* A, *such that*

(3.27) $(Au, v) = a_B(u, v)$, *all* $u, v \in D(A_B)$.

(iv) *There exists* $c > 0$ *such that*

(3.28) $|(Au, v)| \leqslant c\Big(\sum_{t \in N} \|u_t\|^2_{m_t}\Big)^{\frac{1}{2}}\Big(\sum_{t \in N} \|v_t\|^2_{m_t}\Big)^{\frac{1}{2}}$, *all* $u, v \in D(A_B)$.

Proof. — With the notations introduced above, the proof goes in complete analogy with the proof of Theorem 2.4 (γ and ν being replaced by β^0 and β^1, $\mathfrak{A}^{01}$ replaced by $\tilde{\mathfrak{A}}^{01}$).

The Remarks 2.5 and 2.7 extend immediately to the present case. Remark 2.6 extends as follows : For systems $A = (A_{st})_{s,t \in N}$ of type $(m_t, -l_s)_{s,t \in N}$ where $\{m_t\}_{t \in N}$ and $\{l_s\}_{s \in N}$ are sets of nonnegative integers, one can set up Green's formulae generalizing (3.15) and (3.20), just like the formulae in Remark 1.8 generalize (1.11) and (1.14), and one can again define Cauchy data for A and for A', and normal boundary conditions. Then the inequality

$$|(Au, v)| \leqslant c\Big(\sum_{t \in N} \|u_t\|^2_{m_t}\Big)^{\frac{1}{2}}\Big(\sum_{s \in N} \|v_s\|^2_{l_s}\Big)^{\frac{1}{2}}$$

may be set in relation to sesquilinear forms on

$$\prod_{t \in N} H^{m_t}(\overline{\Omega}) \times \prod_{s \in N} H^{l_s}(\overline{\Omega}),$$

generalizing Theorem 3.5. We refrain from details in order to limit notations. [Not all systems of mixed order are of this type.]

3.3. Discussion of (3.26).

All the statements in sections 2.2 and 2.3, that do not depend on whether $\mathfrak{A}^{01}$ is invertible, extend immediately. We shall use the convention that when an operator S is not invertible, S^{-1} denotes the mapping sending sets into their inverse images by S, i.e. S^{-1} is viewed as a relation.

Lemmas 2.8 and 2.10 thus generalize to

Lemma 3.6. — *The identity* (3.26) *is equivalent with each of the following statements*

$$Z(B^{11}) \subset Z((1 - C^{00}B^{00})^*\tilde{\mathfrak{A}}^{01}), \tag{3.29}$$
$$Z(B^{11}) \subset (\tilde{\mathfrak{A}}^{01})^{-1}R(B^{00*}), \tag{3.30}$$
$$Z(B^{00}) \subset (\tilde{\mathfrak{A}}^{01*})^{-1}R(B^{11*}), \tag{3.31}$$

and it implies each of the equivalent statements

$$Z(B^{11}_d) \subset Z((1 - C^{00}_dB^{00}_d)^*\tilde{\mathfrak{A}}^{01}_d), \tag{3.32}$$
$$Z(B^{11}_d) \subset (\tilde{\mathfrak{A}}^{01}_d)^{-1}R(B^{00*}_d), \tag{3.33}$$
$$Z(B^{00}_d) \subset (\tilde{\mathfrak{A}}^{01*}_d)^{-1}R(B^{11*}_d). \tag{3.34}$$

Here, the right sides in (3.29) and (3.30) are identical, by Lemma 1.11. When (3.29) holds, we already have

$$B^{10}\beta^0 u + B^{11}\beta^1 u = 0 \Longleftrightarrow \beta^1 u + C^{11}B^{10}\beta^0 u \in Z(B^{11})$$
$$\Longrightarrow \beta^1 u + C^{11}B^{10}\beta^0 u \in Z((I - C^{00}B^{00})^*\tilde{\mathfrak{A}}^{01})$$
$$\Longleftrightarrow (I - C^{00}B^{00})^*\tilde{\mathfrak{A}}^{01}\beta^1 u + (I - C^{00}B^{00})^*\tilde{\mathfrak{A}}^{01}C^{11}B^{10}\beta^0 u = 0,$$

showing that our boundary condition *implies* a condition on the reduced Cauchy data. A more precise statement will be obtained in Theorem 3.11 below.

We shall now show how Theorems 2.11 and 2.13 may be generalized.

Theorem 3.7. — *Assume that* Γ *is noncharacteristic for* A. *Then* (3.26) *implies that* $Z_j = \ker B_{jj}$ *is a subbundle of* $(\tilde{\mathfrak{A}}_{2m-1-j,j})^{-1}B^*_{2m-1-j,\,2m-1-j}F_{2m-1-j}$ *for* $j \in M_1$, *resp. is isomorphic to a subbundle of* F_{2m-1-j} *for* $j \in M_0$. *In particular,*

$$\sum_{j \in M} p_j \geqslant \sum_{k \in M_0} q_k \quad [\equiv \overline{m}]. \tag{3.35}$$

When furthermore $\sum_{j \in M} p_j = \sum_{k \in M_0} q_k$,

$$(3.36) \qquad Z(B_d^{11}) = (\tilde{\mathfrak{A}}_d^{01})^{-1} R(B_d^{00*}),$$
$$(3.37) \qquad Z(B_d^{00}) = (\tilde{\mathfrak{A}}_d^{01*})^{-1} R(B_d^{11*});$$

so $Z_j = (\tilde{\mathfrak{A}}_{2m-1-j,\, j})^{-1} B^*_{2m-1-j,\, 2m-1-j} F_{2m-1-j}$ *for* $j \in M_1$, *and* $Z_j = (\tilde{\mathfrak{A}}^*_{j,\, 2m-1-j})^{-1} B^*_{2m-1-j,\, 2m-1-j} F_{2m-1-j} \cong F_{2m-1-j}$ *for* $j \in M_0$.

Proof. — When Γ is noncharacteristic, then $\tilde{\mathfrak{A}}_d^{01}$ is a *surjective* morphism from $\bigoplus_{k \in M_1} E_k$ (of dimension mq) to $\bigoplus_{j \in M_0} E_j$ $\left(\text{of dimension } \sum_{j \in M_0} q_j\right)$, and $\tilde{\mathfrak{A}}^{01*}$ is an *injective* morphism in the other direction. Then (3.33) resp. (3.34) imply the first statement of the theorem; in particular we have for the dimensions in (3.33) resp. (3.34)

$$(3.38) \qquad \sum_{j \in M_1} (q - p_j) \leqslant mq - \sum_{j \in M_0} q_j + \sum_{j \in M_0} p_j;$$
$$(3.39) \qquad \sum_{j \in M_0} (q_j - p_j) \leqslant \sum_{j \in M_1} p_j.$$

Each of these inequalities is equivalent with (3.35). Now assume there is identity in (3.35) and thus in (3.38) and (3.39). Then (3.33) resp. (3.34) are inclusions between vector bundles of the same dimension, so they are identities, and we have proved (3.36) and (3.37).

To prove the analogue of Theorem 2.13 we shall first extend some considerations from section 2.3. Denote, for $\varepsilon = 0,1$, $E^\varepsilon = \bigoplus_{k \in M_\varepsilon} E_k$, $Z^\varepsilon = \bigoplus_{k \in M_\varepsilon} Z_k$ and $F^\varepsilon = \bigoplus_{k \in M_\varepsilon} F_k$. Then $I - C_d^{00} B_d^{00}$ defines the orthogonal projection of E^0 onto Z^0, denote it pr_{Z^0}.

PROPOSITION 3.8. — *Define the system* $\overline{B}^{11} = (\overline{B}_{jk})_{j,k \in M_1}$ *of differential operators* $\overline{B}_{jk}$ *from* E_k *into* Z_{2m-1-j}, *of type* $(-k, -j)_{j,k \in M_1}$, *by*

$$(3.40) \qquad \overline{B}^{11} = I^\times \mathrm{pr}_{Z^0}(I - B_s^{00*} C^{00*}) \tilde{\mathfrak{A}}^{01},$$

its diagonal part $I^\times \mathrm{pr}_{Z^0} \tilde{\mathfrak{A}}_d^{01}$ *is surjective when* Γ *is noncharacteristic for* A. *Then*

$$(3.41) \quad (\tilde{\mathfrak{A}}^{01})^{-1} R(B^{00*}) = Z((I - C^{00} B^{00})^* \tilde{\mathfrak{A}}^{01}) = Z(\overline{B}^{11}).$$

Proof. — A direct generalization of the existence proof in section 2.3.

THEOREM 3.9. — *Assume that* Γ *is noncharacteristic for* A, *and that* $\sum_{j\in M} p_j = \overline{m}$. *Then* (3.26) *is equivalent with each of the following statements* (3.42)-(3.44)

$$\text{(3.42)} \qquad Z(B^{11}) \subset Z(\overline{B}^{11}),$$
$$\text{(3.43)} \qquad Z(B^{11}) = Z(\overline{B}^{11}),$$
$$\text{(3.44)} \qquad Z(B^{11}) \supset Z(\overline{B}^{11}),$$

for $\overline{B}^{11}$ *defined in Proposition* 3.8.

Proof. — In view of (3.41) and Lemma 3.6, (3.26) is equivalent with (3.42). Since (3.43) implies (3.42) and (3.44), we have to show (3.42) $\Longrightarrow$ (3.43), and (3.44) $\Longrightarrow$ (3.43). By use of Theorem 3.7 we find, assuming (3.42),

$$\text{(3.45)} \quad Z(B^{11}_d) = (\tilde{\mathcal{C}}^{01}_d)^{-1}R(B^{00*}_d) = Z((I - C^{00}_d B^{00}_d)\tilde{\mathcal{C}}^{01}_d) = Z(\overline{B}^{11}_d).$$

Then (3.42) implies, by use of Lemma 1.12,

$$\begin{aligned} Z(\overline{B}^{11}) \supset Z(B^{11}) &= (I - C^{11}B^{11}_s)\, Z(B^{11}_d) \\ &= (I - C^{11}B^{11}_s)Z(\overline{B}^{11}_d) = (I - C^{11}B^{11}_s)(I + \overline{C}^{11}_d\overline{B}^{11}_s)Z(\overline{B}^{11}) \\ &= (I + K)Z(\overline{B}^{11}), \end{aligned}$$

where K is subtriangular. Now the argument in the proof of Theorem 2.13 applies, showing that the inclusion must be the identity, and we have proved (3.43).

(3.44) implies (3.43) in a similar way.

Part of Corollary 2.14 is immediately generalized (and the remaining part will come out as a corollary at the end of section 3.4) :

COROLLARY 3.10. — *Assumptions of Theorem* 3.9. (3.26) *is equivalent with* (3.46) *and* (3.47)

$$\text{(3.46)} \qquad Z(B^{00}) \subset (\tilde{\mathcal{C}}^{01*})^{-1}R(B^{11*})$$
$$\text{(3.47)} \qquad Z(B^{00}) = (\tilde{\mathcal{C}}^{01*})^{-1}R(B^{11*}).$$

With Theorem 3.9 we can now show that weakly semibounded realizations represent boundary conditions on the reduced Cauchy data.

THEOREM 3.11. — *Assume that* Γ *is noncharacteristic for* A, *and that* A_B *is the realization of a normal boundary condition*

$$B\beta u = 0, \tag{3.48}$$

with $\sum_{j \in M} p_j = \overline{m}$. *If* A_B *is weakly semibounded, then there exists a differential operator* $\tilde{B}^{11} = (\tilde{B}_{jk})_{j\in M_1, k \in M_0}$ *from* $\bigoplus_{k \in M_0} E_k$ *to* $\bigoplus_{j \in M_1} F_j$, *of type* $(-2m+1+k, -j)_{j \in M_1, k \in M_0}$ *and with surjective second-diagonal part, such that*

$$B^{11} = \tilde{B}^{11}\tilde{\mathfrak{A}}^{01}. \tag{3.49}$$

Hereby (3.48) *is equivalent with*

$$B^{00}\beta^0 u = 0, \qquad B^{10}\beta^0 u + \tilde{B}^{11}\varkappa u = 0. \tag{3.50}$$

Proof. — By Theorem 3.9, we have

$$Z(B^{11}) = Z(\overline{B}^{11});$$

moreover, we have by Theorem 3.7 that $F^1 = \Phi I^{\times} Z^0$, with $I^{\times} = (\delta_{j, 2m-1-k})_{j \in M_1, k \in M_0}$ and Φ a diagonal vector bundle isomorphism. Now

$$\mathscr{D}'(E^1) = Z(\overline{B}^{11}) \dotplus R(\overline{C}^{11})$$

so that, using the argument in section 2.3,

$$\begin{aligned} B^{11} &= B^{11}\overline{C}^{11}\overline{B}^{11} \\ &= B^{11}\overline{C}^{11} I^{\times} \operatorname{pr}_{Z^0} (I - B_s^{00*}C^{00*})\tilde{\mathfrak{A}}^{01} = \tilde{B}^{11}\tilde{\mathfrak{A}}^{01}, \end{aligned} \tag{3.51}$$

where $\tilde{B}^{11} = B^{11}\overline{C}^{11} I^{\times} \operatorname{pr}_{Z^0}(I - B_s^{00*}C^{00*})$ is a differential operator from E^0 to F^1 of type

$$(-2m+1+k, -1)_{j \in M_1, k \in M_0}.$$

Note that

$$\tilde{B}^{11} = \Phi\Psi I^{\times} \operatorname{pr}_{Z^0}(I - B_s^{00*}C^{00*}), \tag{3.52}$$

where $\Psi = \Phi^{-1}B^{11}\overline{C}^{11}$ is a bijective differential operator in $\bigoplus_{k \in M_1} Z_{2m-1-k}$ of type $(-k, -j)_{j, k \in M_1}$, so that by Lemma 1.13

its diagonal part is a vector bundle isomorphism. Thus the second-diagonal part of $\tilde{B}^{11}$,

$$\tilde{B}^{11}_d = \Phi\Psi_d I^\times \mathrm{pr}_{Z^0},$$

is a surjective morphism.

This proof not only gives the existence of $\tilde{B}^{11}$, it also serves to discuss the characterization of B^{11}, when B^{00} is given and B shall satisfy (3.26). For we then have by (3.51)-(3.52)

$$B^{11} = \Phi\Psi\overline{B}^{11};$$

and on the other hand, any such operator (with Φ a diagonal vector bundle isomorphism and Ψ a bijective differential operator in $\bigoplus_{k\in M_1} Z_{2m-1-k}$ of type $(-k, -j)_{j,k\in M_1})$ satisfies $Z(B^{11}) = Z(\overline{B}^{11})$. So Theorem 2.15 carries over word for word.

Theorem 3.12. — *The complete analogue of Theorem* 2.15 *holds.*

3.4. The adjoint boundary condition.

Define the formally adjoint realization $(A_B)'$ as the operator sending φ into $A'\varphi$ and with domain

$$(3.53)\quad D((A_B)') = \Big\{\varphi \in \prod_{t\in N} H^{m_t+m}(\overline{\Omega}) \,|\, (Au, \varphi) - (u, A'\varphi) = 0 \text{ for all } u \in D(A_B)\Big\}.$$

Like in section 2.4, we easily find that

$$D((A_B)') = \Big\{\varphi \in \prod_{t\in N} H^{m_t+m}(\overline{\Omega}) \,|\, (I - CB)^*\mathfrak{A}^*\beta\varphi = 0\Big\}$$

so defining

$$(3.54)\qquad B' = I^\times \mathrm{pr}_Z(I - B_s^*C^*)\tilde{\mathfrak{A}}^*$$

we have

$$(3.55)\qquad D((A_B)') = D(A'_{B'}).$$

However, B' need not be normal; in fact (cf. (2.47))

$$B' = \begin{bmatrix} B'^{00} & 0 \\ B'^{10} & B'^{11} \end{bmatrix},$$

where

$$B'^{00} = I^{\times} \operatorname{pr}_{Z^1}(I - B_s^{11*}C^{11*})\tilde{\mathfrak{A}}^{01*};$$
$$B'^{11} = I^{\times} \operatorname{pr}_{Z^0}(I - B_s^{00*}C^{00*})\tilde{\mathfrak{A}}^{10*},$$

where we can only be sure that B'^{11} has surjective diagonal part (when Γ is noncharacteristic), cf. Theorem 3.1. But we have in any case, as in section 2.4

$$Z(B'^{00}) = Z((I - C^{11}B^{11})^*\tilde{\mathfrak{A}}^{01*}) = (\tilde{\mathfrak{A}}^{01*})^{-1}R(B^{11*}),$$

which can be applied to Lemma 3.6 and Corollary 3.10, giving

PROPOSITION 3.13. — *A_B is weakly semibounded if and only if*

$$Z^{2m}(B^{00}) \subset Z^{2m}(B'^{00}), \tag{3.56}$$

that is, if and only if $\beta^0 D(A_B) \subset \beta^0 D(A'_{B'})$. *When* Γ *is non-characteristic and* $\sum_{j\in M} p_j = \overline{m}$, (3.56) *is equivalent with*

$$Z^{2m}(B^{00}) = Z^{2m}(B'^{00}), \tag{3.57}$$

i.e., $\beta^0 D(A_B) = \beta^0 D(A'_{B'})$.

Now A_B and $A'_{B'}$ are no longer analogous, so the trick of Corollary 2.20 cannot be applied to prove weak semiboundedness of $A'_{B'}$. (Computations seem unmanageable.) We shall circumvent this by using that we are in fact dealing with boundary conditions on the reduced Cauchy data $\{\beta^0 u, \varkappa u\}$ (boundary conditions where B^{11} factors through $\tilde{\mathfrak{A}}^{01}$), in view of Theorem 3.11. For such conditions, it is easy to repeat the whole theory in a simpler version based on the Green's formula

$$(Au, v) = a(u, v) + \langle \varkappa u, \beta^0 v\rangle + \langle \mathcal{S}\beta^0 u, \beta^0 v\rangle,$$

cf. (3.20). This gives

THEOREM 3.14. — *Assume that* Γ *is noncharacteristic for* A. *Let* A_B *be the realization defined by a boundary condition*

$$B^{00}\beta^0 u = 0, \quad B^{10}\beta^0 u + \tilde{B}^{11}\varkappa u = 0, \tag{3.58}$$

where the differential operators B^{00}, B^{10} *and* $\tilde{B}^{11}$ *go from*

$\bigoplus_{k\in M_0} E_k$ *to* $\bigoplus_{j\in M_0} F_j$, $\bigoplus_{j\in M_1} F_j$ *and* $\bigoplus_{j\in M_1} F_j$ *(respectively), and are of types* $(-k,\ -j)_{j,k\in M_0}$, $(-k,\ -j)_{j\in M_1,\, k\in M_0}$ and $(-2m+1+k,\ -j)_{j\in M_1,\, k\in M_0}$ *(respectively). Assume that the boundary condition is normal in the sense that the diagonal part* B^{00}_d *and the second-diagonal part* $\tilde{B}^{11}_d$ *are surjective morphisms, and assume that* $\sum_{j\in M} p_j = \overline{m}$. *Let* $I^{\times}\tilde{C}^{11}$ *denote the right inverse of* $\tilde{B}^{11}I^{\times}$ *according to Proposition* 1.9. *Then* A_B *is weakly semibounded if and only if each of the following equivalent conditions hold*:

$$(I - C^{00}B^{00})^*(I - \tilde{C}^{11}\tilde{B}^{11}) = 0; \tag{3.59}$$
$$Z(\tilde{B}^{11}) \subset R(B^{00*}); \tag{3.60}$$
$$Z(\tilde{B}^{11}) \supset R(B^{00*}); \tag{3.61}$$
$$\tilde{B}^{11}B^{00*} = 0; \tag{3.62}$$
$$Z(B^{00}) \subset R(\tilde{B}^{11*}); \tag{3.63}$$
$$Z(B^{00}) = R(\tilde{B}^{11*}); \tag{3.64}$$
$$Z(B^{00}) \supset R(\tilde{B}^{11*}). \tag{3.65}$$

It is used in the proof that (3.58) is equivalent with

$$\begin{aligned} &\beta^0 u \in R(I - C^{00}B^{00}); \\ &\varkappa u + \tilde{C}^{11}B^{10}\beta^0 u \in R(I - \tilde{C}^{11}\tilde{B}^{11}). \end{aligned} \tag{3.66}$$

By Green's formula (3.17):

$$(Au, v) - (u, A'v) = \langle \varkappa u, \beta^0 v\rangle - \langle \beta^0 u, \varkappa' v\rangle$$

we now see, using (3.66), that $v \in D((A_B)')$ if and only if

$$0 = \langle (I - \tilde{C}^{11}\tilde{B}^{11})\varphi_1 - \tilde{C}^{11}B^{10}(I - C^{00}B^{00})\varphi_0,\ \beta^0 v\rangle \\ - \langle (I - C^{00}B^{00})\varphi_0,\ \varkappa' v\rangle$$

for all smooth φ_0, φ_1, i.e. if and only if

$$\begin{gathered} (I - \tilde{C}^{11}\tilde{B}^{11})^*\beta^0 v = 0 \\ (I - C^{00}B^{00})^* B^{10*}\tilde{C}^{11*}\beta^0 v + (I - C^{00}B^{00})^*\varkappa' v = 0. \end{gathered}$$

This may be reformulated by the usual technique to a *normal* boundary condition

$$\tilde{B}'^{00}\beta^0 v = 0, \qquad \tilde{B}'^{10}\beta^0 v + \tilde{B}'^{11}\varkappa' v = 0, \tag{3.67}$$

where $\tilde{B}'^{00}$, $\tilde{B}'^{10}$ and $\tilde{B}'^{11}$ go from $\bigoplus_{k\in M_0} E_k$ to $\tilde{F}^0$, $\tilde{F}^1$ and $\tilde{F}^1$ (respectively), of types as in Theorem 3.14, $\tilde{B}'^{00}$ and $\tilde{B}'^{11}$ having surjective diagonal resp. second-diagonal part; here $\tilde{F}^0 = \bigoplus_{j\in M_0} \ker \tilde{B}_{2m-1-j,j}$ and $\tilde{F}^1 = \bigoplus_{j\in M_1} \ker B_{2m-1-j,2m-1-j}$. So $(A_B)' = A'_{\tilde{B}'}$, the realization of A' defined by the boundary condition (3.67). Moreover, A_B is the formally adjoint realization to $A'_{\tilde{B}'}$ in the analogous way. Observing that

$$Z^{2m}(\tilde{B}'^{00}) = \beta^0 D((A_B)') = Z^{2m}((I - \tilde{C}^{11}\tilde{B}^{11})^*) = R^{2m}(\tilde{B}^{11*}),$$

we may write (3.64) as

$$Z^{2m}(B^{00}) = Z^{2m}(\tilde{B}'^{00}); \tag{3.68}$$

and since A_B and $(A_B)' = A'_{\tilde{B}'}$ *now enter in a symmetric way*, we can conclude that (3.68) must also imply weak semiboundedness of $(A_B)'$. So we have proved

THEOREM 3.15. — *When* Γ *is noncharacteristic for* A *and* $\sum_{j\in M} p_j = \overline{m}$, *then* A_B *is weakly semibounded if and only if* $(A_B)'$ *is weakly semibounded.*

The formulation analogous to Corollary 2.20 is also valid. Let us finally note that the adjoint equation to (3.62),

$$B^{00}\tilde{B}^{11*} = 0, \tag{3.69}$$

and its equivalent statement

$$Z(B^{00}) \supset R(\tilde{B}^{11*}), \tag{3.70}$$

provide the analogues of the last two statements in Corollary 2.14: When $B^{11} = \tilde{B}^{11}\tilde{\mathcal{C}}^{01}$ as in Theorem 3.11, then $B^{11*} = \tilde{\mathcal{C}}^{01*}\tilde{B}^{11*}$, where $\tilde{\mathcal{C}}^{01*}$ is *invertible*, so (3.70) and (3.69) are equivalent with

$$Z(B^{00}) \supset (\tilde{\mathcal{C}}^{01*})^{-1}R(B^{11*}), \tag{3.71}$$

resp.,

$$B^{00}(\tilde{\mathcal{C}}^{01*})^{-1}B^{11*} = 0, \tag{3.72}$$

the perhaps simplest version of (3.26). Corollary 3.10 can now be completed with (3.71) and (3.72).

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