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REAL ALGEBRAIC ACTIONS ON PROJECTIVE SPACES — A SURVEY

by Ted PETRIE

0. Introduction.

Briefly the subject of this paper is the study of compact subgroups of the group of diffeomorphisms of a smooth manifold. My main objective is to provide an excursion through some new ideas of a particular aspect of the subject. The two main questions delt with here are :

(1) Does a given smooth manifold admit a smooth action of a given compact Lie Group?

(2) If a given group does act on a smooth manifold, how can we construct new actions on the manifold starting from the given action?

The central question which must be answered for dealing with these two questions is :

(3) What are the relations among the representations of the group on the tangent spaces at the points fixed by the group and the global invariants of the manifold eg its Pontrjagin classes and its cohomology?

Let me give two examples of the third question :

Example 1. — Global assumption : X is a smooth closed manifold with $H^*(X, Q) = H^*(S^{2n}, Q)$. Suppose that our compact group G acts on X with just 2 fixed points p and q and assume that the action is free outside p and q.

Conclusion. — Atiyah-Bott [1]; *The two real representations of* G *on the tangent space to* X *at* p *and* q *are equal.* Thus a cohomological assumption implies an equality of representations at the tangent spaces at the fixed points.

Example 2. — Global Assumption: X is a closed manifold having the same cohomology ring as complex projective n space. *Suppose* S^1 *acts on* X *and the fixed point set consists of isolated points. Then the collection of representations of* S^1 *on the tangent space at the various fixed points determine all the Pontrjagin classes of* X[4].

See § 3 for applications of this result to the study of Question 1. In particular see the consequence Corollary 3.3.

For dealing with Question 2, we introduce a set $S_G(M)$ associated to the G manifold M. Roughly $S_G(M)$ consists of those smooth G manifolds admitting a G map to M which induces a homotopy equivalence from the underlying manifold to the manifold underlying M but which itself is not a G homotopy equivalence. Briefly $S_G(M)$ consists of the distinct G homotopy types which resemble M.

The construction of non trivial elements in $S_M(M)$ is also intimately related to Question 3. In Example 2.9 and in the discussion of Theorem 4.6, we show how the representations of G on the tangent spaces at the fixed points are related to the construction of non trivial elements in $S_G(M)$.

Section 2 is devoted to motivating the techniques of constructing elements of $S_G(M)$. Section 3 gives a summary of properties of S^1 and two actions on manifolds homotopy equivalent to complex projective n-space. In particular we give relations among the representations of S^1 on the tangent space at the various isolated fixed points in the spirit of Question 3. (see Theorem 3.4). In conclusion we present the example of Theorem 4.6 which constructs an element in $S_{S^1}(P(\Omega))$ and shows that the relations provided by Theorem 3.4 can be realized.

I wish to thank our hosts especially Professors Godbillon and Cerf for the splendid hospitality and administration. I found this conference extremely stimulating and expect that much significant research will be generated by the participants because of this stimulation.

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1. Statement of objective.

Throughout this paper G will be a compact connected Lie Group. We fix notation :

(1) D (resp. D^c) denotes the catagory of smooth manifolds and smooth maps (resp. compact smooth manifolds and smooth maps).

(2) D_G (resp. D_G^c) is the catagory of smooth G manifolds (resp. compact smooth G manifolds) and smooth G maps. An *object* $M \in D_G$ (resp. D_G^c) consists of a smooth manifold $|M| \in D$ (resp. D^c) together with a faithful representation

$$\rho : G \rightarrow \mathrm{Diff}(|M|) \quad \text{i.e.} \quad \mathrm{Ker}\,\rho = \text{identity}$$

If $x \in |M|$, $g \in G$, we write

$$gx = \rho(g)[x]$$

We require the map $G_x|M| \rightarrow |M|$ defined by $(g, m) \rightarrow \rho(g)m$ to be smooth and say that G acts on $|M|$.

A *map* $f : M \rightarrow N$ in D_G is a map $|f| : |M| \rightarrow |N|$ in D such that

$$fg = gf \quad \text{for all} \quad g \in G.$$

One of the most interesting questions in the subject is

QUESTION 1.1. — *Suppose* $X \in D^c$. *Is there an* $M \in D_G^c$ *with* $|M| = X$?

The question as it stands is much too general for study. Experience indicates that the following is a fruitful modification of Question 1.1 :

QUESTION 1.2. — *Suppose* $X \in D^c$, $M \in D_G^c$ *with* $|M| = X$. *If* $X' \in D^c$ *is homotopy equivalent to* X, *written* $X' \sim X$, *is there an* M' *in* D_G^c *with* $|M'| = X'$?

There are two reasons for considering this question. First the method of classification of smooth manifolds begins by fixing a particular manifold $X \in D^c$ and then describing the manifolds $X' \in D^c$ which are homotopy equivalent to X. In particular we have a good understanding of how manifolds $X' \sim X$ are obtained from X. (See § 2). Second if we are given $M \in D^c_G$ with $|M| = X$, we may be able to make geometric constructions on M in D^c_G yielding M′ with $|M'| = X'$ or at least a new action of G on $|M|$ i.e. $M' \neq M$ in D^c_G but $|M'| = |M|$.

Having settled on Question 1.2 for study, we consider the following setting.

DEFINITION 1.3. — *Let* $S_G(M)$ *for* $M \in D^c_G$ *denote the set of equivalence classes of pairs* (M', f) *where* M′ *and* f *are in* D^c_G *and*

$$|f| = |M'| \to |M|$$

is a homotopy equivalence. Two pairs (M_i, f_i) $i = 0,1$ *are equivalent if there is a map* $\varphi : M_0 \to M_1$ *in* D^c_G *which is a* « G *homotopy equivalence* » *such that* $f_1 \circ \varphi$ *is* G *homotopic to* f_0. *The equivalence class of* (M', f) *will be denoted by* $[M', f] \in S_G(M)$. *The element* $[M, \text{Identity}] \in S_G(M)$ *is called the trivial element.*

If we can describe the set $S_G(M)$, we obtain information about which manifolds homotopy equivalent to $|M|$ admit G actions as well as a description of new G actions on $|M|$.

Example 1.4. — If $M \in D^c_G$ and G acts *freely* on $|M|$ then $S_G(M)$ has only one element. Any G map $f\colon M' \to M$ with $|f| = |M'| \to |M|$ a homotopy equivalence is a G homotopy equivalence because the induced map on the orbit spaces $\bar{f} : M'/G \to M/G$ is a homotopy equivalence. Take a homotopy inverse for $\bar{f}$ and lift it to a G map from M to M′. This will be a G homotopy inverse for f.

On-the-other hand when G acts on $|M|$ with non trivial isotropy groups, the set $S_G(M)$ can be non trivial and quite interesting. In fact when $G = S^1$, $M = P(\Omega)$ with

$$|P(\Omega)| = P(C^{n+1})$$

complex projective n-space, we produce non trivial elements in $S_{S^1}(P(\Omega))$ which arise from real algebraic action of S^1 on real algebraic varieties which are diffeomorphic to $P(C^{n+1})$ (See § 4).

2. Motivation and discussion of constructive techniques.

For the purpose of motivation, let me recall a relevant situation in D^c. If $X' \sim X$, then X' is obtained from X as follows: there is a stable vector bundle ξ over X of fiber dim k for some large integer k and a map t from the total space of ξ, $E(\xi)$, to R^k with these properties:

(1) t is proper.

(2) $t \pitchfork 0$ i.e. t is transverse regular to $0 \in R^k$.

(3) t restricted to each fibre of ξ has degree 1.

Moreover, $X' = t^{-1}(0)$ and the map of X' to X defined by inclusion of X' in $E(\xi)$ followed by projection on X is a homotopy equivalence.

In analogy with the above discussion, we might try to construct elements $[M', f] \in S_G(M)$ for $M \in D_G^c$ like this: Let A be a real representation of G i.e. A is a real vector space $|A| = R^l$ for some l together with a representation of G in $O(l)$ (orthogonal group.) We seek a stable G vector bundle η over M whose fiber dimension is l and a map $t: E(\eta) \to A$ in D_G such that

(1′) $|t|$ is proper

(2′) $|t| \pitchfork 0$

(3′) $|t|$ has degree 1 on each fibres of η.

Under these conditions $t^{-1}(0) = M' \in D_G^c$ and if we're lucky, the map f defined as the composition $M' \subset E(\eta) \to M$ has the property that $|f|$ is a homotopy equivalence. Then $[M', f] \in S_G(M)$.

There are quite interesting difficulties involved in carrying out this procedure. Sometimes it's possible and sometimes not. The three hypothesis on the map $t \in D_G^c$ (1′), (2′) and (3′) impose stringent relations among η, A and the representations TM_p of G on the tangent space of M at p for every fixed point p in M. Since it is easy to illustrate these relations, we do so. The appropriate tool to use is the functor K_G, equivariant complex K theory.

To simplify the discussion, we assume that η is a *complex* G vector bundle over M and A is a *complex* representation of G such that

$$A^G = \{a \in A \mid Ga = a\} = 0.$$

Then we have this commutative diagramm:

$$\begin{array}{ccccc} \eta_p & \xrightarrow{j_p} & E(\eta) & \xrightarrow{t} & A \\ & \underset{i_p}{\nwarrow\!\searrow} & & \underset{i_A}{\swarrow\!\nearrow} & \\ & & p & & \end{array} \qquad 2.1$$

Here η_p is the fiber of η over $p \in M^G$, A is naturally a G vector bundle over p, j_p is the inclusion and i_p and i_A are the zero sections of these G bundles over trivial G space consisting of p.

Let us recall one of the basic facts of K_G theory [3]. Let X be a compact G space and N a complex vector bundle over X. Then there is an element $\lambda_N \in K_G^*(N)$ which generates $K_G^*(N)$ as a free module over $K_G^*(X)$. Moreover if i is the zero section of N we have

$$i^*\lambda_N = \lambda_{-1}(N) = \Sigma(-1)^i\lambda^i(N).$$

Here $\lambda^i(N)$ is the i th exterior power of N.

We can now exploit the hypothesis (1′) and (2′) for t. Since $t_p = tj_p$ is proper there is an induced homomorphism

$$t_p^* : K_G^*(A) \to K_G^*(\eta_p).$$

Using the above facts for the complex G vector bundles A and η_p, we have

$$t_p^*\lambda_A = a_p\lambda_{\eta_p} \qquad 2.2$$

for some $a_p \in K_G(p) = R(G)$ (the complex representation ring of G). Since $i_p^*t_p^* = i_A^*$ by 2.1, we have from 2.2,

$$\lambda_{-1}(A) = i_A^*\lambda_A = i_p^*t_p^*\lambda_A = a_p\lambda_{-1}(\eta_p) \qquad 2.3$$

for every $p \in M^G$.

Since G is a connected Lie Group, R(G) is an integral domain [3]. Thus

$$a_p = \lambda_{-1}(A)/\lambda_{-1}(\eta_p) \in R(G) \qquad \text{for} \qquad p \in M^G. \qquad 2.4$$

Note $a_p = \lambda_{-1}(A)/\lambda_{-1}(\eta_p) \in R(G)$. Viewing R(G) as the character ring of G, we can regard a_p as a complex valued function on G, say $g \to a_p(g)$ for $g \in G$. In particular, we can evaluate a_p at $1 \in G$. Let $E : K_G \to K$ denote the forgetful functor from equivariant K theory to ordinary K theory. Then

$$a_p(1) = E(a_p).$$

On the other hand, a_p is defined by the equation

$$t_p^*\lambda_A = a_p . \lambda_{\eta_p} \quad \text{so applying} \quad E$$
$$|t_p|^*\lambda_{|A|} = a_p(1) . \lambda_{|\eta_p|}.$$

Here $\lambda_{|A|}$ and $\lambda_{|\eta_p|}$ are generators for $K^*(|A|)$ and $K(|\eta_p|)$ over $K^*(p) = Z$. Since $|A| = |\eta_p| = C^k$ it is an easy topological exercise to show that

$$|t_p|^*\lambda_{|A|} = \text{degree } |t_p| . \lambda_{\eta_p}$$

Hence $a_p(1) = \text{degree } |t_p| = 1$.

We have

$$a_p(1) = 1 = \lim_{g \to 1} \frac{\lambda_{-1}(A)(g)}{\lambda_{-1}(\eta_p)(g)} \qquad 2.5$$

Here $\lambda_{-1}(A)(g)$ denotes the value of the character $\lambda_{-1}(A)$ at $g \in G$.

We record these facts in the

PROPOSITION 2.6. — *Let* $M \in D_G^c$, n *a complex* G *vector bundle over* M *of complex fiber dimension* k. *Let* A *be a complex* k *dimensional representation of* G *with* $A^G = 0$. *Suppose there is a map* $t: E(\eta) \to A$ *in* D_G *such that* $|t|$ *is proper and has degree* 1 *on each fiber of* η. *Then*

$$\lambda_{-1}(A)/\lambda_{-1}(\eta_p) = a_p \in R(G) \qquad \textit{for all} \qquad p \in M^G \quad (1)$$
$$a_p(1) = 1. \qquad (2)$$

We can also draw a useful conclusion from the hypothesis that $t \in D_G$ and $|t| \pitchfork 0$.

Let $i_M : M \to E(\eta)$ denote the zero section and $p \in M^G$. Then

$$TE(\eta)_{i_M(p)} = TM_p \oplus \eta_p$$

This is an equality as real representations. Since $i_M(p) \in E(\eta)^G$ and $A^G = 0$,

$$ti_M(p) = 0.$$

Since $|t| \pitchfork 0$, $d\tau i_M(p) : TM_p \oplus \eta_p \to TA_0 = A$ is surjective and this means that the *real* representation defined by A is a *real* factor of $TM_p \oplus \eta_p$. We state this as

PROPOSITION 2.7. — *Let* $M \in D_G$, n *a complex* G *bundle over* M, A *a complex representation of* G *with* $A^G = 0$. *Suppose there is a* $t : E(\eta) \to A$ *in* D_G *such that* $|t| \pitchfork 0$. *Then for every* $p \in M^G$, *the representation* A *is a real factor of* $\eta_p \oplus TM_p$.

Example 2.8. — Let $G = S^1 = \{\lambda \in C \,|\, |\lambda| = 1\}$, $M =$ *a* point with trivial G action. Identify $R(S^1)$ with the ring $Z[t, t^{-1}]$. Let p, q be relatively prime integers and $\eta_0 = t^p \oplus t^q$, $A = t^1 \oplus t^{pq}$ denote the indicated complex two dimensional representations of S^1 i.e. S^1 vector bundles over M. For example, for η_0, the point $t = e^{i\theta} \in S^1$ acts on the point with complex coordinates $(z_0, z_1) \in |\eta_0|$ via the rule

$$t(z_0, z_1) = (t^p . z_0, t^q . z_1).$$

Let $\omega : \eta_0 \to A$ be the map defined by

$$\omega(z_0, z_1) = (\bar{z}_0^a z_1^b, z_0^q + z_1^p)$$

where a, b are positive integers with

$$-ap + bq = 1.$$

Then $\omega \in D_{S^1}$, $|\omega|$ is proper and degree $|\omega| = 1$. Moreover

$$a_p = \lambda_{-1}(A)/\lambda_{-1}(\eta_p) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)} \in Z[t, t^{-1}]$$
$$a_p(1) = 1$$

Example 2.9. — Let $S(t^p \oplus t^q)$ and $S(t^1 \oplus t^{pq})$ denote the S^1 manifolds with $|S^p(t^p \oplus t^q)| = S^4 = |S(t^1 \oplus t^{pq})|$ obtained by regarding S^4 as the one point compactification of C^2. The representations $t^p \oplus t^q$ and $t^1 \oplus t^{pq}$ define smooth actions on the compactification and the resulting S^1 manifolds are $S(t^p \oplus t^q)$ and $S(t^1 \oplus t^{pq})$.

The map ω of example 2.8 defines a map in D^c_S

$$\hat{\omega} : S(t^p \oplus t^q) \to S(t^1 \oplus t^q)$$

and $|\hat{\omega}|$ is a homotopy equivalence being of degree 1. Thus

$$X = [S(t^p \oplus t^q), \hat{\omega}] \in S_{S^1}(S(t^1 \oplus t^q)).$$

I claim that it's not the trivial element because the *algebras* $K^*_{S^1}(S(t^p \oplus t^q))$ and $K^*_{S^1}(t \oplus t^q))$ are distinct.

Let ∞ denote the point at infinity in the one point compactification. Note that

$$\begin{aligned} S(t^p \oplus t^q)^{S^1} &= O \cup \infty = S(t^1 \oplus t^{pq})^{S1} \\ TS(t^p \oplus t^q)_0 &= t^p \oplus t^q = TS(t^p \oplus t^q)_\infty \\ TS(t \oplus t^{pq})_0 &= t^1 \oplus t^{pq} = TS(t^1 \oplus t^{pq})_\infty \end{aligned}$$

The map ω was constructed from $\omega : \eta_0 \to A$; $\eta_0 = t^p \oplus t^q$, $A = t^1 \oplus t^{pq}$. So in a very precise sense the element $[S(t^p \oplus t^q), \hat{\omega}]$ is obtained by altering the representations of S^1 on $TS(t \oplus t^{pq})_x$ for $x \in S(t \oplus t^{pq})^{S^1}$.

This is a very brief glimpse at the importance of the role played by the collection of representations $\{TM_p | p \in M^G\}$ when $M \in D^c_G$.

In this example we can't regard the construction as giving anything new since the S^1 action on $S(t^p \oplus t^q)$ comes from a representation of S^1 and is among our list of well understood S^1 manifolds. However, we shall see later that by changing the representations $\{TM_p | p \in M^G\}$ we can sometimes produce from M in D^c_G interesting new G manifolds.

3. The central example P(Ω) — a survey.

We now come to the example which is the central point of the study. Let $P(\mathbf{C}^{n+1})$ denote the space of complex lines in $\mathbf{C}^{n+1}$ i.e. complex projective n space. Let $PGL(n+1, \mathbf{C})$ denote the projective linear group and observe that $PGL(n+1, \mathbf{C})$ is a subgroup of Diff $(P(\mathbf{C}^{n+1}))$; hence, any representation $\Omega : G \to PGL(n+1, \mathbf{C})$ defines an action on $P(\mathbf{C}^{n+1})$ and gives a manifold $P(\Omega) \in D^c_G$ with $|P(\Omega)| = P(\mathbf{C}^{n+1})$.

We are interested in the set $S_G(P(\Omega))$. Since much can be said about the more general situation where we have $M \in D^c_G$ with $|M| \sim P(\mathbf{C}^{n+1})$ *without assuming some map* $f: M \to P(\Omega)$ with $|f|: |M| \to |P(\Omega)| = P(\mathbf{C}^{n+1})$ a homotopy equivalence, we describe the results for this situation.

The fundamental fact is this.

THEOREM 3.1 [4]. — *Suppose* $M \in D^c_{S^1}; |M| \sim P(\mathbf{C}^{n+1})$ *and* M^{S^1} *consists of isolated points, then the collection of real representations* $\{TM_p | p \in M^G\}$ *determine the Pontrjagin classes of* $|M|$.

This is a more striking illustration of the importance of $\{TM_p | p \in M^G\}$ than provided in Example 2.9.

From Theorem 3.1 follows.

THEOREM 3.2 [6]. — *Suppose* $M \in D^c_{T^n}$ *where* T^n *is the* n torus *and* $|M| \sim P(\mathbf{C}^{n+1})$. *Then for any homotopy equivalence*

$$g: |M| \to P(\mathbf{C}^{n+1}).$$

we have

$$g^*\mathbf{P}(P(\mathbf{C}^{n+1})) = \mathbf{P}(|M|)$$

$\mathbf{P}(|M|)$ *is the total Pontryagin class of* $|M|$.

COROLLARY 3.3 [6]. — *At most a finite number of* $X \in D^c$ *with* $X \sim P(\mathbf{C}^{n+1})$ *admit an action of* T^n.

Having emphasized the importance of the representations $\{TM_p | p \in M^{S^1}\}$ when $|M| \sim P(\mathbf{C}^{n+1})$, we should determine all relations among these representations. They are by no means independent. The global restrictions

$$|M| \sim P(\mathbf{C}^{n+1})$$

imposes stringent relations among the TM_p $p \in M^G$.

Suppose $M \in D^c_{S^1}$ with $|M| \sim P(\mathbf{C}^{n+1})$ is given. From homological considerations we can invent a representation

$$\Omega: S^1 \to PGL(n+1, \mathbf{C})$$

which depends only on the S^1 action on $|M|$ i.e. on M and we can compare M with $P(\Omega)$. (Note: If we had a map $f: M \to P(\Omega)$ in D_G with $|f|$ a homotopy equivalence, we would have $[M, f] \in S_{S^1}(P(\Omega))$. We have don't assume

f_*). The point is that all the data of the S^1-manifold

$$P(\Omega) \text{ Eg}. \{TP(\Omega)_p | p \in P(\Omega)^{S^1}\}$$

is easily determined from Ω and is a function of the S^1 manifold M.

Let us assume that M^{S^1} consists of isolated points. Then we have these relations among the representations

$$\{TM_p | p \in M^{S^1}\} :$$

THEOREM 3.4. — *There is a* $1 - 1$ *correspondance*

$$\alpha : M^{S^1} \to P(\Omega)^{S^1}$$

such that for every $x \in M^{S^1}$

(i) $a_x = \lambda_{-1}(TP(\Omega)_{\alpha(x)})/\lambda_{-1}(TM_x) \in R(S^1)$,

(ii) $a_p(1) = \pm 1$. *Here* $a_p(1)$ *is the value of the character* a_p *at* $1 \in S^1$.

Actually to make sense of (*i*), one needs to choose a complex representation of S^1 whose underlying real representation is TM_p. Then $\lambda_{-1}(TM_p) \in R(S^1)$, the complex representation ring of S^1. This involves a choice; so a_p is only well defined up to multiplication by $\pm t^{N_p}$ for some integer N_p.

The unusual relations given by (i) and (ii) are extremely difficult to achieve without $a_x = \pm t^{N_x}$ for all x Eg. the case $M = P(\Omega)$. Let us discuss some invariants for distinguishing elements of $S_G(M)$ for $M \in D^c_G$. Denote by $\mathbf{C}_G$ the catagory of $R(G)$ algebras which are closed under the exterior power operations $\{\lambda^i | i = 0, 1, \dots\}$. A morphism is an algebra morphism compatible with the λ^i. To each $[M', f] \in S_G(M)$ we can associate $f^* : K^*_G(M) \to K^*_G(M')$ and $f^* \in C_G$. In short we have a function

$$F : S_G(M) \to C_G$$

defined by $F[M, f] = f^*$.

The values of F are not arbitrary. If $[M, f] \in \mathcal{S}_G(M)$, then $|f|$ is a homotopy equivalence. It follows from the Atiyah-Segal Completion theorem [2] that the map induced by f^* on the completions

$$\hat{f}^* : \hat{K}^*_G(M) \to \hat{K}^*_G(M')$$

is an isomorphism. Here $\hat{K}^*_G$ denotes the completion of K^*_G at the augmentation ideal I of R(G).

In the case of $S_{S^1}(P(\Omega))$ where Ω is a representation of S^1, much more can be said. Suppose $[M, f] \in S_{S^1}(P(\Omega))$ and let $\Gamma = K^*_{S^1}(M)/_T$ where T is the $R(S^1)$ torsion subgroup of $K^*_{S^1}(M)$. We agree to let f^* denote the map $K^*_{S^1}(P(\Omega)) \xrightarrow{f^*} K^*_{S^1}(M) \to \Gamma$. Then if we set $\Lambda = K^*_{S^1}(P(\Omega))$ and $\Theta = K^*_{S^1}(M^{S^1})$, then $f^* : \Lambda \to \Gamma$ is a monomorphism and the inclusion $M^{S^1} \to M$ induces a monomorphism $\Gamma \to \Theta$. This situation can be algebraically stated like this

$$\Lambda \subset \Gamma \subset \Theta$$

are $R(S^1)$ orders closed under the operations λ^i in the semisimple $F(S^1)$ (field of fractions of $R(S^1)$) algebra

$$\Theta \underset{R(S^1)}{\otimes} F(S^1).$$

Let $\mathfrak{p}_m$ denote the ideal of $R(S^1)$ generated by the m th cyclotomic polynomial $\Phi_m(t) \in Z[t, t^{-1}] = R(S^1)$.

THEOREM 3.5 [4]. — *Let* $[M, f] \in S_{S^1}(P(\Omega))$; *then* f^* *induces an isomorphism at all localizations* $\Lambda_{\mathfrak{p}_m} \to \Gamma_{\mathfrak{p}_m}$ *where* m *is a prime power.*

Actually the theorem stated in [4] is much stronger. The assumption of a map $f: M \to P(\Omega)$ is irrelevant. One can manufacture a map $f^* : \Lambda \to \Gamma$ without assuming that it arises geometrically.

Remark. — The assumption that m be a prime power is necessary. The fact that it is false for composite m leads to the existence of non trivial elements in $S_{S^1}(P(\Omega))$.

4. Realizing elements in $S_{S^1}(P(\Omega))$.

Let us now use the geometric discussion of § 2 to construct non trivial elements in $S_{S^1}(P(\Omega))$ and illustrate the properties of the preceeding section.

Let η be the S^1 bundle over $P(\Omega)$ whose total space is $P(\Omega) \times \eta_0$ (η_0 is the representation $t^p \oplus t^q$ of § 2). For simplicity we assume $P(\Omega)^{S^1}$ consists of isolated points.

Let A be the representation $t^1 \oplus t^{pq}$ of § 2. We have seen that the assumption that there exists a map $t: E(\eta) \to A$ in D_{S^1} with $|t| \pitchfork 0$ implies $TP(\Omega)_p \oplus \eta_0$ has A as a real factor for every fixed point $p \in P(\Omega)^{S^1}$. Since A and η_0 have no common irreducible factor, $TP(\Omega)_p$ has A as a real factor for every such p. It is easy to determine the representation $TP(\Omega)_p$ from Ω and we find that this condition that A be a real factor of $TP(\Omega)_p$ for all $p \in P(\Omega)^{S^1}$, implies that Ω must have the form

$$\Omega = \lambda(A) \underset{C}{\otimes} R$$

as a complex representation of S^1.

Here $\lambda(A) = \sum_{i=0}^{2} \lambda^i(A)$ is the total exterior algebra of A and R is an arbitrary representation of S^1 say of dimension n. (Actually R can't be entirely arbitrary if we insist that $P(\Omega)^{S^1}$ consists of isolated points). In particular $\dim_C \Omega = 4n$; so $\dim_C |P(\Omega)| = 4n - 1$.

LEMMA 4.1. — *A necessary condition for a* $t: E(\eta) \to A$ *in* D_{S^1} *with* $|t| \pitchfork 0$ *is that*

(1) $\Omega = \lambda(A) \otimes R$ *as a complex representation of* S^1,

(2) $\dim_C |P(\Omega)| = 4n - 1 \quad n = \dim_C R$ (This is a consequence of (1)).

It turns out that the condition is sufficient. Namely there is a map $g: P(\Omega) \to A$ in D_{S^1} such that the map

$$t: P(\Omega) \times \eta_0 \to A$$

defined by $t(x, z) = g(x) + \omega(z)$ is in D_{S^1}. Moreover $|t|$ is proper, has degree one on each fiber and $|t| \pitchfork 0$. More is true. If $X(\Omega) = t^{-1}(0)$ then $X(\Omega) \in D^c_{S^1}$ and the map f from $X(\Omega)$ to $P(\Omega)$ defined by the inclusion of $X(\Omega)$ in $P(\Omega) \times \eta_0$ followed by projection on $P(\Omega)$ is in $D^c_{S^1}$ and $|f|$ is a homotopy equivalence.

Thus

$$[X(\Omega), f] \in S_{S^1}(P(\Omega)) \qquad 4.2$$

This element is not the trivial element. The algebra $K^*_{S^1}(X(\Omega))$ is not isomorphic to $K^*_{S^1}(P(\Omega))$. [7].

Let us view the properties of $[X(\Omega), f]$ in the light of the facts of § 2. For simplicity let p and q be prime. Then $f^*: K^*_{S^1}(P(\Omega)) \to K^*_{S^1}(X(\Omega))$ induces an isomorphism

$$(f^*)_{\mathfrak{p}_m}: K^*_{S^1}P(\Omega))_{\mathfrak{p}_m} \to K^*_{S^1}(X(\Omega))_{\mathfrak{p}_m}$$

at *all localizations* $\mathfrak{p}_m$ *except for* $m = p.q$. Compare Theorem 3.5.

Note that $X(\Omega) \subset P(\Omega) \times \eta_0$ and $X(\Omega)^{S^1} = P(\Omega)^{S^1}$ so in this case the correspondence α of Theorem 3.4 is the identity. Since $t^{-1}(O) = X(\Omega)$ and since $|t| \pitchfork 0$, the total space of the normal bundle of $X(\Omega)$ in $P(\Omega) \times \eta_0$ is $X(\Omega) \times A$. From this we deduce that for $x \in X(\Omega)^{S^1} = P(\Omega)^{S^1}$ we have

$$TX(\Omega)_x \oplus A = TP(\Omega)_x \oplus \eta_0 \Longrightarrow \qquad 4.3$$

$$\lambda_{-1}(TX(\Omega)_x).\lambda_{-1}(A) = \lambda_{-1}(TP(\Omega)_x).\lambda_{-1}(\eta_0) \qquad 4.4$$

$$a_x = \lambda_{-1}(TP(\Omega)_x)/\lambda_{-1}(TX(\Omega)_x) = \lambda_{-1}(A)/\lambda_{-1}(\eta_0) \in R(S^1) \qquad 4.5$$

Compare Theorem 3.4 and note that a_x is independant of x. Note also that $a_x = \Phi_{p.q}(t) \in Z[t, t^{-1}] = R(S^1)$ when p and q are prime. This is the reason that $(f^*)_{\mathfrak{p}_{m_{p.q}}}$ is not an isomorphism.

As a final remark, the function $g: P(\Omega) \to A$ can be taken to be real algebraic. This means that $|X(\Omega)| = |t^{-1}(0)|$ is a real algebraic manifold and the action of S^1 on $X(\Omega)$ is real algebraic. Moreover one can show that $|X(\Omega)|$ is diffeomorphic to $|P(\Omega)| = P(\mathbf{C}^{n+1})$, $n + 1 = 4 \dim R$. I don't know whether $|X(\Omega)|$ is isomorphic to $P(\mathbf{C}^{n+1})$ as a real algebraic manifold.

Let us summarize these facts in the

THEOREM 4.6. — *Let* $(p, q) = 1$ *be positive integers,* $A = t^1 \oplus t^{pq}$, $A = t^1 \oplus t^{pq}$ *the indicated complex* 2 *dimensional representation of* S, $\wedge(A)$ *the total exterior algebra of* A *and* $\Omega = \wedge(A) \otimes_{\mathbf{C}} R$ *where* R *is an arbitrary complex representation of* S *of dimension* n. *Then* $S_{S^1}(P(\Omega))$ *has at least one non trivial element* $[X(\Omega), f]$ *and*

(i) $X(\Omega)^{S^1} = P(\Omega)^{S^1}$

(ii) $\lambda_{-1}(TP(\Omega)_x)/\lambda_{-1}(TX(\Omega)_x) = \lambda_{-1}(A)/\lambda_{-1}(\eta_0) \in R(S^1)$

For all $x \in X(\Omega)^{S^1}$.

(iii) $F[X(\Omega), f] = f^*$ *induces an isomorphism at all localizations* $(f^*)_{\mathfrak{p}_m}: K^*(P(\Omega))_{\mathfrak{p}_m} \to K^*(X(\Omega))_{\mathfrak{p}_m}$ *m prime to p.q.*

(iv) $(f^*)_{\mathfrak{p}_m}$ *is not an isomorphism when* $m = pq$,

(v) $$|X(\Omega)| = P(\mathbf{C}^{n+1}) \text{ in } D^c\ [7].$$

In summary we've indicated the importance of the representations $\{TM_p | p \in M^G\}$ in studying Diff $(|M|)$ for $|M|$ in a fixed homotopy type. In particular when $G = S^1$, $|M| \sim P(\mathbf{C}^{n+1})$ and M^{S^1} consists of isolated fixed points, we showed that the representations $\{TM_p | p \in M^{S^1}\}$ had to satisfy the relations given in Theorem 3.4. Note that in this case M^{S^1} must consist of $n+1$ points.

It is probably not the case that if we are given $n+1$ representations $\{R_p | p \in P(\Omega)^{S^1}\}$ of complex dimension n satisfying for all $p \in M^{S^1}$

(i) $$a_p = \lambda_i(TP(\Omega)_p)/\lambda_{-1}(R_p) \in R(S^1)$$
(ii) $$a_p(1) = \pm 1,$$

that there is an element $[M, f] \in \mathcal{S}_{S^1}(P(\Omega))$ with

$$\lambda_{-1}(TP(\Omega)_p)/\lambda_{-1}(TM_p) = \lambda_{-1}(TP(\Omega)_p)/\lambda_{-1}(R_p)$$

That is, I suspect that there are more relations among the $\{TM_p | p \in M^{S^1}\}$ then those given in Theorem 3.4. This is certainly true when n is even.

On-the-other-hand, the above $M = X(\Omega)$ provide examples where non trivial a_p actually occur. For every pair of relatively prime integers p, q. The example $X(\Omega)$ (Ω depends on p and q) gives for $s \in X(\Omega)^{S^1}$

$$a_s = \lambda_{-1}(TP(\Omega)_s)/\lambda_{-1}(TX(\Omega)_s) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)} \in R(S^1)$$

In particular $a_s \neq t^{N_s}$ for any integer N_S and the representations $\{TX(\Omega)_S\}$ are distinct from the representations $\{TP(\Omega)_S\}$. These are the first and only known examples of this phenomena.

REFERENCES

[1] M. F. Atiyah and R. Bott, A Lefschitz fixed point formula for elliptic complexes II, Applications, Vol. 88 (1968), 451-491.

[2] M. F. Atiyah and G. Segal, Equivarient K-theory and completion, *J. Diff. Geom.* (3) (1968), 1-18.
[3] M. F. Atiyah and I. Singer, The index of elliptic operators I, III. *Annals of Math.* (2) 87 (1968), 484-530 and 546-604.
[4] T. Petrie, S^1 actions on homotopy complex projective spaces. *Bull. AMS* (78) 2, March 1972, 105-153.
[5] T. Petrie, Exotic S^1 actions on CP^3 and related to appear.
[6] T. Petrie, Torus actions on homotopy complex projective spaces to appear.
[7] T. Petrie, Real algebraic S′ actions on complex projective spaces. to appear.

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