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Pluripotential theory on compact Hermitian manifolds

Sławomir Dinew[(1)]

Résumé. — Dans cet article nous collectons des résultats fondamentaux de la théorie du potentiel sur des variétés hermitiennes compactes. En particulier, nous discutons en détail la théorie de la capacité, plusieurs principes de comparaison, et la résolution de l'équation de Calabi-Yau sur les variétés hermitiennes compactes.

Abstract. — In this survey article we collect the basic results in pluripotential theory in the setting of compact Hermitian manifolds. In particular we discuss in detail the corresponding capacity theory, various comparison principles, and the solution of the Hermitian counterpart of the Calabi-Yau equation.

Introduction

Pluripotential theory in the setting of compact Kähler manifolds has proven to be a very effective tool in the study of degeneration of metrics in geometrically motivated problems (see [39, 40, 18, 42], which is by far incomplete list of the literature on the subject). Usually in such a setting *singular* Kähler metrics do appear as limits of smooth ones. Then pluripotential theory provides a natural background for defining the *singular* volume forms associated to such metrics. More importantly it provides useful information on the behavior of the Kähler potentials exactly along the singularity locus, which is hard to achieve by standard PDE techniques. On the other hand

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the theory does not rely on strong geometric assumptions, as most of the results are either local in nature or are modelled on local ones. It is therefore natural to expect that at least some of the methods and applications carry through in the more general Hermitian setting. Indeed any compact, complex and connected manifold X admits a Hermitian metric, hence if developed, such a theory would apply on any specific manifold regardless of the complex and Riemannian structure.

Of course there is inevitably some price to pay. Arguably the most important difference is the lack of invariance of the total volume $\int_X (\omega + dd^c u)^n$ for an admissible function u (which is easily seen after two applications of Stokes theorem). As the Reader will soon verify this leads to troublesome additional terms involving $d\omega$ and/or $d\omega \wedge d^c\omega$ and controlling these in a suitable sense is the main technical difficulty in the whole theory.

A necessary remark should be added here. As the results presented follow closely their Kählerian counterparts one might be tempted to treat the Hermitian theory as a simple generalization of the Kählerian one modulo technical issues. In author's opinion on a deeper level these two theories should be necessarily different. While in the elliptic setting there is no such striking difference available in the current literature, the parabolic counterpart of the theory has provided some examples partially justifying such a belief (we refer to the penultimate section for the details).

The interest towards Hermitian versions of the complex Monge-Ampère equation has grown rapidly in the recent years. The first steps were laid down by the French school most notably by Cherrier [9] and Delanoë [15]. In these papers the Authors followed Aubin and Yau's arguments ([62]) to get existence of smooth solutions of the Monge-Ampère equation in the case of smooth data. The Authors were successful only in particular cases (that is under geometric assumptions on the background metric). The main problem to overcome were the a priori esimates needed to establish the closedness part in the continuity method. Then there seems to be no activity on the subject for quite some time up until the renewed interest and important breakthroughs by Guan-Li [25] and especially Tosatti-Weinkove [55, 56]. Guan and Li were able to solve the equation assuming geometric conditions different than these from [9] and [15], while the missing uniform estimate was finally established without any assumptions in [56]. Parallel to these recent advances foundations of the corresponding pluripotential theory were laid down (see [16],[5] and [41]). The theory is still in an infant state, and the techniques are technical modifications of their Kählerian counterparts.

The author believes that such a theory might have considerable applications in Hermitian geometry. Some of these are already listed in the article-see Section 5. It should be emphasized that in Hermitian geometry there are many conditions imitating Kählerness. These are motivated by various geometric considerations. Some of these conditions have consequences that are relevant to pluripotential theory. Arguably the most interesting condition is the one studied by Guan and Li ([25]). The Authors assumed that $dd^c\omega = 0$ and $dd^c(\omega^2) = 0$. The following properties of metrics satisfying such a condition follow from a simple direct computations but will be crucial in the sequel:

Observation 0.1. — Let (X, ω) be a compact Hermitian manifold of complex dimension $n \geqslant 2$. If the form ω satisfies $dd^c\omega = 0,\ dd^c(\omega^2) = 0$ then

(1) $d\omega \wedge d^c\omega = 0$;

(2) $dd^c(\omega^k) = 0$ for all $k \in \{1, \cdots, n-1\}$.

Under this condition almost every pluripotential argument from the Kähler setting carries through verbatim. On the other hand if this condition is not satisfied then the proofs are considerably more complicated.

The aim of the current survey is to collect the basic notions and results in pluripotential theory which can be found in the existent literature. The note does not contain new material. Instead we have tried to gather some geometric examples spread in the literature, so that a potential Reader with an analytical background might get some basic intuition. Readers familiar with Hermitian geometry in turn can compare the Hermitian and Kähler pluripotential theories and analyze the main differences.

In the analytic part the paper is reasonably self-contained. On the other hand the author has decided to skip some of the definitions of the geometrical notions that appear in the article and are not relevant to the theory itself.

The paper is organized as follows. We start with some basic notation and later we define some of the "close-to-Kähler" conditions which can be found in the literature. Each of these has also its drawbacks, and it serves as yet another evidence how strong and natural the Kählerian condition is. In the next section we describe some explicit examples of Hermitian manifolds and "canonical" metrics on them. This list is of course only a glimpse into the vast world of Hermitian geometry. The existence of suitable adapted coordinates (due to [25]) is shown in Section 4. Such a coordinate system will turn out to be very useful in the proof of higher order a priori estimates for the Dirichlet problem. The main pluripotential tools are discussed afterwards. In particular we show following [16] that "most" pluripotential Kählerian notions have their Hermitian counterparts. It is shown that the complex Monge-Ampère operator is well denfined on bounded ω-plurisubharmonic functions and it shares the convergence properties known from the Kähler case. Special attention is being paid to the most important tool in the whole theory- the comparison principle. As explained it differs considerably from the one known in the Kähler setting unless the form ω satisfies some restrictive additional conditions. In the next section the solution of the Dirichlet problem is presented in detail. For the openness part we follow [55], while for the $\mathcal{C}^2$ estimates we borrow the main idea from [25]. The uniform estimate is taken from [16]. Then we solve the Monge-Ampère equation with right hand side being an L^p function with $p > 1$ following [41]. In the penultimate section we sketch the parabolic version of the theory- the Chern-Ricci flow. As already mentioned we shall discuss here some examples showing significant difference from the Kähler case. Finally we list some open problems which hopefully can be attacked in the recent future.

I lectured on the subject during the final MACK meeting in Toulouse 22-25.10.2014. It is a great pleasure to thank the members of the complex analysis group in Toulouse for the kind hospitality, and especially professor V. Guedj for his invitation.

1. Notation

Throughout the paper X will denote a compact, complex and connected manifold. Unless otherwise specified n will always be the complex dimension of X. By J we shall denote the complex structure on X.

Given a Hermitian metric g on X we identify it with the positive definite (**not necessarily closed!**) $(1,1)$ form ω defined by

$$\forall X, Y \in T_zX \quad \omega(z)(X,Y) := g(z)(JX,Y). \tag{1.1}$$

This form is called the Kähler form of g in the literature, but we shall not use this terminology in order to avoid confusion with the Kähler condition.

As usual d will denote the exterior differentiation operator, while ∂ and $\bar{\partial}$ will be the $(1,0)$ and $(0,1)$ part of it under the standard splitting. In some arguments involving integration by parts it is more convenient to use the operator $d^c := i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$. These will be used interchangeably. We shall also make use of the standard notation ω_u standing for $\omega + dd^c u$.

δ_{ij} will denote the Kronecker delta symbol. We shall make use of Einstein summation convention unless otherwise stated.

Throughout the paper we shall use the common practice of denoting constants independent of the relevant quantities by C. In particular these constants may vary line-to-line. If special distinction between the constants is needed in some arguments these will be further distinguished by $\tilde{C},\ \bar{C}, C_i$ and so on.

A special constant that controls the geometry of (X,ω) (see the next section for details) is denoted by B- it is the infimum over all positive numbers b satisfying

$$-b\omega^2 \leqslant ni\partial\bar{\partial}\omega \leqslant b\omega^2 \quad \text{and} \tag{1.2}$$

$$-b\omega^3 \leqslant n^2 i\partial\omega \wedge \bar{\partial}\omega \leqslant b\omega^3.$$

It should be emphasized that this constant measures how far our metric is from satisfying a special condition studied by Guan and Li ([25]). Of course if ω is Kähler then $B = 0$.

2. Kähler type conditions

Given a fixed Hermitian manifold X it is natural to search for the "best" metric that X admits. The reason is at least twofold: nice metrics usually significantly simplify computations and more importantly it is sometimes possible to deduce geometric or topological information from the existence of these.

Unlike the Kähler case there is a large number of mutually different "Kähler type" conditions. Below we list the most common ones:

Definition 2.1. *(Balanced metric). — Let (X,ω) be a n-dimensional Hermitian manifold. The form ω is said to be balanced if it satisfies*

$$d(\omega^{n-1}) = 0.$$

Of course this definition differs from the Kähler condition only if $n \geqslant 3$. The motivation behind such a condition partially comes from string theory (see [1, 19, 21] and the references therein). There are various constructions of explicit examples of non-Kähler, balanced manifolds in the literature. For example using *conifold transitions* Fu, Li and Yau in [21] proved that such a metric exists on the connected sum $\sharp_k S^3 \times S^3$ of k copies of the product of two three dimensional spheres. Another example is the *Iwasawa manifold* which will be given in the next section.

Balanced metrics impose some geometric restrictions on the underlying manifold (for example it follows from Stokes theorem that no smooth 1-codimensional complex subvariety can be homologous to zero) and hence not every manifold can be endowed with such a metric.

From potential theoretic point of view the most important property of such metrics is that the Laplacian of any admissible (or even merely smooth) function u on X integrates to zero. Namely if we choose the canonical Laplacian associated to the Chern connection on X then we get

$$\int_X \Delta_\omega u\omega^n = n\int_X i\partial\bar{\partial}u\wedge\omega^{n-1} = -n\int_X \bar{\partial}u\wedge\partial(\omega^{n-1}) = 0.$$

An interesting exercise, left to the Reader, is to check that in the intermediate cases between the balanced and Kähler conditions we do not get anything besides Kählerness:

Exercise 2.2. — Suppose $1 < k < n-1$. If ω is a form such that

$$d(\omega^{n-k}) = 0,$$

then $d\omega = 0$ i.e. ω is Kähler.

A second family that we consider are the so-called Gauduchon metrics [23].

Definition 2.3. *(Gauduchon metric). — Let (X, ω) be a n-dimensional Hermitian manifold. The form ω is said to be Gauduchon if it satisfies*

$$dd^c(\omega^{n-1}) = 0.$$

Unlike balanced ones, these exist on **any** compact Hermitian manifold. Moreover a theorem of Gauduchon [23] states that given any Hermitian form ω there exists a conformal factor e^{ϕ_ω} such that the new form $e^{\phi_\omega}\omega$ is Gauduchon. Gauduchon metrics are useful in many geometric contexts, for example the notion of a degree of a line bundle over a Gauduchon manifold is well defined via the formula

$$deg_\omega(L) = \int c_1(L)\wedge\omega^{n-1},$$

where $c_1(L)$ is the first Chern class of L. This is the starting point for a *stability theory* for vector bundles in the Hermitian setting (see [43]).

Yet another difference is that after the exchange of the power $n-1$ to a lesser power we do get nontrivial new conditions. This is in fact how Astheno-Kähler metrics are defined.

Definition 2.4. *(Astheno-Kähler metric). — Let (X,ω) be a n-dimensional Hermitian manifold ($n \geqslant 2$). The form ω is said to be Astheno-Kähler if it satisfies*

$$dd^c(\omega^{n-2}) = 0.$$

This condition was used by Jost and Yau [32] in their study of harmonic maps from Hermitian manifolds to general Riemmanian manifolds.

Unlike the Gauduchon metrics Astheno-Kähler metrics impose some constraints on the underlying manifold. It can be shown that any holomorphic 1 form on such a manifold must be closed. Explicit examples of Astheno-Kähler but non-Kähler manifolds can be found in dimension 3 where they coincide with the *pluriclosed* metrics to be defined below. Another type of examples are the so-called *Calabi-Eckmann* manifolds. These are topologically products $S^{2n-1} \times S^{2m-1}$, $(m > 1,\ n > 1)$ of odd dimensional spheres. Any such manifold admits families of complex structures which can be constructed using Sasakian geometry. In [44] it was shown that a special choice of such a complex structure yields an Astheno-Kähler manifold. Since $H^2(S^{2n-1} \times S^{2m-1}) = 0$ such manifolds are never Kähler.

Much more information regarding Astheno-Kähler geometry can be found in [22].

Finally the important notion of the aforementioned *pluriclosed* metrics is defined as follows:

Definition 2.5. *(pluriclosed metric). — Let (X,ω) be a n-dimensional Hermitian manifold. The form ω is said to be pluriclosed if it satisfies*

$$dd^c\omega = 0.$$

The pluriclosed metrics are also known as SKT (strong Kähler with torsion) in the literature ([20]). Of course in dimension 2 this notion coincides with the Gauduchon condition, hence any complex surface admits pluriclosed metrics. In complex dimension 3 some nontrivial examples of non-Kähler *nilmanifolds* admitting pluriclosed metrics were constructed by Fino, Parton and Salamon in [20]. An interested Reader is referred to [53] for more background on these.

As is easily verified, Gauduchon metrics also have the property that the Laplacian of a smooth function integrates to zero. This is not the case for Astheno-Kähler and pluriclosed metrics in general. A significant drawback is that the Gauduchon condition is not preserved under Ricci flow (unless

$dim_{\mathbb{C}}X = 2$ i.e. we deal with pluriclosed metrics)- we shall disuss this in Section 10.

A strengthened version of the Gauduchon condition was considered by Popovici in [47]:

Definition 2.6. *(strongly Gauduchon metric). — If (X, ω) is n-dimensional Hermitian manifold, the form ω is said to be strongly Gauduchon if $\partial(\omega^{n-1})$ is $\bar{\partial}$ exact.*

Of course strongly Gauduchon implies Gauduchon and these notions coincide if the $\partial\bar{\partial}$-lemma holds on X (see [47]) but in general the inclusion is strict. Note also that any balanced metric is strongly Gauduchon.

The strongly Gauduchon condition was introduced by Popovici in [47] in connection with studies of deformation limits of projective or Kähler manifolds. We refer to [47] for the geometric conditions imposed by this structure. In particular a necessary and sufficient condition of existence of such a metric on a manifold X is the nonexistence of a positive d-exact $(1,1)$ current on X.

None of the conditions above actually guarantee the invariance of the total volume of the perturbed metric. More precisely the value $\int_X(\omega+dd^c u)^n$ does depend on u and this is the main source of troubles in pluripotential theory. Still a condition weaker that being Käher can be imposed so that the total volume remains invariant. This condition has been investigated by Guan and Li [25]:

Definition 2.7. — *A metric satisfies the condition imposed by Guan and Li if $dd^c\omega = 0$ and $dd^c(\omega^2) = 0$.*

Observe that this is weaker than Kähler yet by twofold application of Stokes theorem it can be shown that the total volume remains invariant. Let us also stress once again that the constant B introduced in the previous section measures how far our metric is from satisfying the above condition.

Remark 2.8. — Non Kähler metrics satisfying the above property do exist. A trivial example, taken from [55] is simply the product of a compact complex curve equipped with a Kähler metric and a non-Kähler complex surface equipped with a Gauduchon metric.

We refer the interested reader to the article [49], for more explicit examples and interactions between the notions above.

3. Explicit examples of non-Kähler hermitian manifolds

We begin this section by defining the most classical examples of non-Kähler manifolds- the Hopf manifolds. These were historically the first ones and were discovered by Hopf in 1948 [29].

DEFINITION 3.1. *(Hopf manifold). — Let t be any nonzero complex number satisfying $|t| \neq 1$. Then it induces a $\mathbb{Z}$ action on $\mathbb{C}^n \setminus \{0\}$ by scaling i.e.*

$$(k, w) \to t^k w,$$

for any $k \in \mathbb{Z}, w \in \mathbb{C}^n \setminus \{0\}$. The action is discrete and properly discontinuous, hence the quotient manifold $\mathbb{C}^n \setminus \{0\}/_{\mathbb{Z}}$ is a smooth manifold.

Remark 3.2. — In the literature more general definitions are being considered. In particular some Authors define Hopf manifolds as above but with the $\mathbb{Z}$ action induced by any contracting-to-zero biholomorphic mapping of $\mathbb{C}^n \setminus \{0\}$ into itself.

It can be proved that the Hopf manifolds are all diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$, hence the first Betti numbers are odd- in particular these are never Kähler. Another obstruction is that $H^2(X, \mathbb{R})$ vanishes which also shows that X cannot be Kähler. In fact it can be proven that Hopf manifolds do not admit even balanced metrics.

On the bright side a Gauduchon metric is explicitly computable in the simplest case. Indeed, suppose that $n = 2$, then the metric

$$\omega = \frac{idz \wedge d\bar{z} + idw \wedge d\bar{w}}{|z|^2 + |w|^2}$$

is clearly invariant under the group action, hence descends onto the quotient manifold. Moreover it is easy to check that $dd^c\omega = 0$, so this metric is pluriclosed (or Gauduchon).

In the two dimensional case Hopf manifolds do belong to the special class of the so-called class VII surfaces, named after the original Kodaira classification list [34, 35, 36, 37]. These are characterized by the two conditions that the first Betti number $b_1(X)$ is equal to 1, while the Kodaira dimension $\kappa(X)$ is minus infinity. Class VII minimal surfaces are the only remaining class of two dimensional manifolds that is not fully classified yet. More precisely the classification was obtained by the works of Kato, Nakamura and most notably Teleman [33], [46], [54] in the cases when the second Betti number $b_2(X)$ is small. Classifiaction is complete in the case $b_2(X) \leqslant 2$ (see [54]). In the remaining cases a theorem of Dloussky-Oeljeklaus-Toma [17] yields a classification provided one can find $b_2(X)$ rational curves (possibly singular) on X. Conjecturally this is always the case and indeed this holds in the classified cases $b_2(X) \leqslant 2$. Hence the classification problem boils down to the construction of rational curves. We shall discuss this in Section 5.

Let us now present one of the simplest examples of a class VII manifold, called Inoue surface [31] (in this case $b_2(X) = 0$).

Definition 3.3. *(Inoue surface). — Let M be a 3×3 integer valued matrix with determinant equal to 1. Suppose that it has a positive eigenvalue α and two complex eigenvalues β and $\bar{\beta}$. Let also (a_1, a_2, a_3) and (b_1, b_2, b_3) be eigenvectors corresponding to α and β respectively. The Inoue surface is defined as the quotient $\mathbb{H} \times \mathbb{C}$, $\mathbb{H}$ being the upper half plane by a group G generated by the following four automorphisms:*

$$g_0(w, z) := (\alpha w, \beta z),$$

$$g_i(w, z) = (w + \alpha_i, z + \beta_i) \quad i = 1, 2, 3.$$

Remark 3.4. — It can be proven that the action is discrete and properly discontinuous, hence the quotient is a smooth manifold. An important property of G in this construction is that it is not an Abelian group but is a solvable one. There are two other classess of surfaces defined by Inoue, also being quotients of $\mathbb{H} \times \mathbb{C}$ by a solvable group.

On Inoue surfaces one can also find an explicit pluriclosed/Gauduchon metric:

Definition 3.5. *(Tricerri metric). — Let $\omega(z, w) := \frac{idw \wedge d\bar{w}}{Im^2(w)} + Im(w) idz \wedge d\bar{z}$. This metric is invariant under the action of G and hence descends to the Inoue surface. It can be computed that $dd^c\omega = 0$.*

Our last example is known as Iwasawa threefold. It is not Kähler for it admits a non closed holomorphic 1-form:

Definition 3.6. *(Iwasawa manifold). — Let*

$$M := \{A \in GL_3(\mathbb{C}) |\ A = \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix}, \quad z_i \in \mathbb{C}, i = 1, 2, 3\}.$$

The Iwasawa threefold is defined as quotient of M by the lattice of such matrices with coefficients being Gaussian integers acting on M by a left multiplication.

It is easily observed that dz_1, dz_2 and $dz_3 - z_1dz_2$ are invariant holomorphic one forms on M. As $d(dz_3 - z_1dz_2) = -dz_1 \wedge dz_2$ is also invariant, it descends to a non zero two form. Thus $dz_3 - z_1dz_2$ is a non closed holomorphic one form on M. It can be shown that

$$idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2 + i(dz_3 - z_1dz_2) \wedge \overline{(dz_3 - z_1dz_2)}$$

descends to a balanced (hence strongly Gauduchon) metric on the Iwasawa threefold.

4. Canonical coordinates

In the Kähler setting many local computations are significantly simplified by the use of canonical coordinates. More specifically such coordinates not only diagonalize the metric at a given point (which we assume to be the center of the associated coordinate chart) but also yield vanishing of all third order derivative terms while the fourth order terms are the coefficients of the curvature tensor.

Of course in the general Hermitian setting one cannot expect vanishing of all third order terms. Yet getting more information than pointwise diagonalization is crucial in some laborious computations. Hence a question appears whether some milder "interpolating" conditions on third order terms are achievable. As observed by Guan and Li [25] this is indeed possible:

Theorem 4.1 (Guan-Li). — *Given a Hermitian manifold (X, ω) and a point $p \in X$ it is possible to choose coordinates near p, such that $g_{i\bar{j}}(p) = \delta_{ij}$ and for any pair i, k one has $\frac{\partial g_{i\bar{i}}}{\partial z_k}(p) = 0$.*

Proof. — Choose first local coordinates z_i around p (identified with 0 in the coordinate chart), such that at this point the metric is diagonalized. Then rechoose coordinates by adding some quadratic terms:

$$w_r = z_r + \sum_{m \neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m} z_m z_r + \frac{1}{2} \frac{\partial g_{r\bar{r}}}{\partial z_r} z_r^2.$$

Observe that

$$\frac{\partial z_r}{\partial w_i} = \delta_{ri} \quad \text{at } p; \tag{4.1}$$

$$\frac{\partial^2 z_r}{\partial w_i \partial w_k} = -\sum_{m \neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m} \left(\frac{\partial z_m}{\partial w_i} \frac{\partial z_r}{\partial w_k} + \frac{\partial z_m}{\partial w_k} \frac{\partial z_r}{\partial w_i}\right) - \frac{\partial g_{r\bar{r}}}{\partial z_r} \frac{\partial z_r}{\partial w_i} \frac{\partial z_r}{\partial w_k}. \tag{4.2}$$

Computing now $\tilde{g}_{i\bar{j}} := g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j})$, one gets

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_k} = \sum_{r,s=1}^{n} g_{r\bar{s}} \frac{\partial^2 z_r}{\partial w_i \partial w_k} \frac{\partial \bar{z}_s}{\partial \bar{w}_j}$$
$$+ \sum_{r,s,p=1}^{n} \frac{\partial g_{r\bar{s}}}{\partial z_p} \frac{\partial z_p}{\partial w_k} \frac{\partial z_r}{\partial w_i} \frac{\partial \bar{z}_s}{\partial \bar{w}_j}.$$

Plugging now (4.1) and (4.2) into the formula above we get

$$\frac{\partial \tilde{g}_{i\bar{i}}}{\partial w_k} = \sum_{r=1}^{n}(-\sum_{m\neq r} -\frac{\partial g_{r\bar{r}}}{\partial z_m}(\delta_{mi}\delta_{rk} + \delta_{mk}\delta_{ri})\delta_{ri} - \frac{\partial g_{r\bar{r}}}{\partial z_r}\delta_{ri}\delta_{rk})$$
$$+ \sum_{r,s,p=1}^{n} \frac{\partial g_{r\bar{s}}}{\partial z_p}\delta_{pk}\delta_{ri}\delta_{si} = 0.$$

□

Remark 4.2. — On the other hand using the change of variables

$$w_r = z_r + \frac{1}{4}\sum_{jk}(\frac{\partial g_{r\bar{j}}}{\partial z_k} + \frac{\partial g_{r\bar{k}}}{\partial z_j})z_j z_k$$

and a computation similar to the one above Streets and Tian [52] were able to construct coordinates satisfying at a fixed point p

i) $g_{i\bar{j}}(p) = \delta_{ij}$;

ii) $\frac{\partial g_{j\bar{k}}}{\partial z_i}(p) + \frac{\partial g_{i\bar{k}}}{\partial z_j}(p) = 0$ for any triple i, j, k.

In general it is impossible to find coordinates satisfying both Guan-Li and Streets-Tian third order conditions.

5. Motivation

In this section we shall briefly list a panorama of potential applications of pluripotential theory on Hermitian manifolds. As the theory is still developing it is expected that this list will grow rapidly in the near future.

To begin with we recall how the Monge-Ampère equation prescribes the Ricci curvature of a metric in the *Kähler* setting.

Given a Kähler metric ω_0 and a representative α of the first Chern class on a manifold X the Calabi problem boils down to finding a metric ω cohomologous to ω_0, such that $Ric(\omega) = \alpha$. By the dd^c lemma any such ω can be written as $\omega_0 + dd^c\phi$ for some smooth potential ϕ. Furthermore $Ric(\omega_0) = \alpha + dd^c h$, where the Ricci potential h is a function uniquely defined modulo an additive constant (which can be fixed if we assume the normalization $\int_X e^h\omega_0^n = \int_X \omega^n$). Recall that in the Kähler setting one has $Ric(\omega) = -dd^c log((\omega)^n)$ with ω^n denoting n-th wedge product of ω (modulo the identification of the coefficient of the volume form with the volume form itself). Hence $Ric(\omega) = \alpha$ is equivalent to

$$Ric(\omega_0 + dd^c\phi) = Ric(\omega_0) - dd^c h \Leftrightarrow -dd^c log\frac{(\omega_0 + dd^c\phi)^n}{(\omega_0)^n} = -dd^c h$$
$$\Leftrightarrow (\omega_0 + dd^c\phi)^n = e^{h+c}\omega_0^n$$

for some constant c. Exploiting the kählerness of ω_0 and integration by parts one easily sees that under our normalization $c = 0$ and we end up with the standard Monge-Ampère equation

$$(\omega_0 + dd^c\phi)^n = e^h\omega_0^n \tag{5.1}$$

with prescribed right hand side.

This equation for smooth h and ω_0 was solved in the celebrated paper of Yau [62]. In modern Kähler geometry it is of crucial importance to understand the behavior ot the potential ϕ (or the form $\omega_0 + dd^c\phi$ itself) if we drop the smoothness assumptions on h and/or the strict positivity of ω_0. Such a situation occurs if we work on *mildly singular* Kähler varieties (see for example [18]) or when one tries to understand the limiting behavior of the Kähler-Ricci flow (see [42] and references therein). It is exactly the setting where pluripotential theory can be applied an indeed in such settings the uniform estimate for the potential ϕ (a starting point for the regularity analysis) is usually obtained in this way (compare [18, 42]).

Returning to the Hermitian background the picture described above has to be modified. The obvious obstacles are that a Hermitian metric ω_0 need not define a cohomology class and the dd^c lemma may fail. On the bright side the first Chern class can still be reasonably defined in the *Bott-Chern* cohomology that is the cohomology given by

$$H^{p,q}_{BC} = \frac{ker\{d : C_{p,q}(X) \to C_{p,q+1}(X) \oplus C_{p+1,q}(X)\}}{Im\{dd^c C_{p-1,q-1}(X)\}}, \tag{5.2}$$

where $C_{p,q}(X)$ denotes the space of smooth (p, q)-forms.

Given a Hermitian metric ω_0 its *first Chern form* can be defined analogously to the Kähler setting by

$$Ric^{BC}(\omega_0) := -dd^c log(\omega_0^n).$$

It turns out that the first Ricci forms represent the first Bott-Chern cohomology class $c_1^{BC}(X)$ in the Bott-Chern cohomology. Hence a natural question arises whether any form α in $c_1^{BC}(X)$ is representable as the Ricci form of some metric $\omega_0 + dd^c\phi$. A computation analgous to the one above shows that such a ϕ has to satisfy the equation

$$(\omega_0 + dd^c\phi)^n = e^{h+c}\omega_0^n, \tag{5.3}$$

with a function h as above and some constant $c > 0$. Contrary to the Kähler case, however, the constant need not be equal to zero and thus the Hermitian Monge-Ampère equation has one more degree of freedom. As we shall see later this adds some technical difficulties into the solution of the equation.

The discussion above resulted in the fact that solutions to Hermitian Monge-Ampère equation prescribe the Ricci form in the Bott-Chern cohomology. Thus weakening of the smoothness assumptions on f and/or strict positivity of ω_0 is helpful in situations analogous to the ones in the Kähler setting above. Below we shall list some concrete problems where pluripotential methods may apply.

Arguably one of the most exciting problems in Hermitian geometry is the classification of slass VII surfaces. To this end the conjectural picture, as already explained, reduces the problem to finding rational curves on such a surface. An interested Reader with a background in Kähler geometry will easily realize that existence of such rational curves (albeit only with negative self intesection) is detectable by using the Kähler-Ricci flow which is the parabolic version of the complex Monge-Ampère equation. It is thus plausible to expect (this is explicitly stated in [53]) that a suitable version of the flow in the Hermitian setting would do similar job.

It is however important to mention some serious differences in the Hermitian setting. In the projective case a well known bend-and-break method invented by Mori [45] can be used to construct abundance of rational curves on Fano manifolds. More precisely Mori theorem gives the following result:

THEOREM 5.1. *(Mori theorem). — Let X be a projective manifold and let C be a smooth curve such that $K_X.C < 0$. Then for any point $c \in C$ there is a rational curve containing c.*

Actually Mori theorem yields much more precise information- we refer to [45] for the details.

When the projective setting is changed to Kähler the analogue of the above theorem, to author's knowledge, remains unknown. It is however known that the bend-and-break method fails in the Hermitian case: this follows from an old example by Blanchard [4] (see also the survey [8], Section 2).

It is also worth mentioning that another approach to construction of rational curves exploting some *singularity magnifying* Monge-Ampère equations has been proposed by Y. T. Siu ([50]).

It is thus quite intriguing to investigate the relationships between Monge-Ampère equations and the existence of rational curves. In author's opinion if such a link exists it will reveal substantial differences between Hermitian and Kähler pluripotential theories.

A second direction where pluripotential theory may apply is linked with the transcendental holomorphic Morse inequalities. To explain these we need a bit of notation. Let X be any compact complex connected manifold and α be a real and closed $(1,1)$-form (not necessarily positive!). The *volume* of the class $\{\alpha\}$ is defined as follows:

$$Vol(\{\alpha\}) = \begin{cases} 0 \ \text{ if } \alpha \text{ is not pseudoeffective;} \\ sup_T \int_X T_{ac}^n, \end{cases} \tag{5.4}$$

where the supremum is taken over all positive closed currents cohomologous to α and T_{ac} denotes their absolutely continuous part.

If $\{\alpha\}$ happens to be an integral class i.e. $\{\alpha\} = c_1(L)$ for some line bundle on X then it can be proven (see [11] for details) that this notion agrees with the standard definition of a volume of a line bundle i.e.

$$Vol(c_1(L)) = limsup_{k\to\infty}\frac{n!}{k^n}h^0(X,kL).$$

Let us finally denote by $X(\alpha,\leqslant 1)$ the set where α has at most 1 negative eigenvalue.

The basic conjectural holomorphic Morse inequality reads

$$Vol(\{\alpha\}) \geqslant \int_{X(\alpha,\leqslant 1)} \alpha^n.$$

Such an inequality is known in the integral class case (see [11]) and is a very useful tool in deep problems in algebraic geometry (see for example [13]). It was observed in [6] that the holomorphic Morse inequality implies a related orthogonality result called weak holomorphic Morse inequality:

If α and β are nef cohomology classes such that

$$\{\alpha\}^n - n\{\alpha\}^{n-1}\{\beta\} > 0,$$

then $\{\alpha - \beta\}$ contains a Kähler current.

This second inequality was proved in the transcendental setting by Xiao [61] (with a worse constant $4n$) and then in the stated form by Popovici [48] under the assumption that X admits a metric ω satisfying $i\partial\bar{\partial}\omega = 0,\ \partial\omega \wedge \bar{\partial}\omega = 0$ i.e. the unnamed Kähler type condition studied by Guan and Li. The main technical tool in Xiao's and Popovici approach is to solve a Hermitan complex Monge-Ampère equation with good control on the total volumes. Thus it seems possible that further developments of pluripotential theory may lead to deeper applications to such inequalities.

Yet another direction of applications was initiated in the paper [57]. Motivated by the fundamental paper [12], the Authors' goal was to construct ω-plurisubharmonic functions with prescribed logarithmic singularities. These are very important tools since they can be applied as weights in various Ohsawa-Takegoshi type L^2 extension problems or $\bar{\partial}$ problems. Explicit constructions of such pluricomplex Green type functions are unknown, but J. P. Demailly in [12] introduced a technique of constructing them by solving a family of Monge-Ampère equations with right hand sides converging to Dirac delta measures. More specifically in the Kähler case a family of Monge-Ampère equations

$$\begin{cases} \phi_\epsilon \in \mathcal{C}^\infty(X), sup_X \phi_\epsilon = 0 \\ \omega + dd^c\phi_\epsilon > 0 \\ (\omega + dd^c\phi_\epsilon)^n = \chi_\epsilon \omega^n \end{cases} \tag{5.5}$$

is considered, where for each $\epsilon > 0$ χ_ϵ is a smooth strictly positive function with suitably normalized total integral. Moreover it is required that χ_ϵ converge weakly to a combination $\sum c_j\delta_j$ of weighted Dirac delta measures as ϵ tends to zero. Then the weak limit of the solutions (which exist by the Calabi-Yau theorem [62]) is the required function.

In the Hermitian setting such a technique requires a modification of the approximating equations:

$$\begin{cases} \phi_\epsilon \in \mathcal{C}^\infty(X), sup_X \phi_\epsilon = 0 \\ \omega + dd^c\phi_\epsilon > 0 \\ (\omega + dd^c\phi_\epsilon)^n = e^{c_\epsilon} \chi_\epsilon \omega^n, \end{cases} \tag{5.6}$$

where c_ϵ is some constant (the equations are then solvable by [56]). Successful repetition of the argument relies crucially on controlling total volumes, that is on the uniform control of the constants c_ϵ. This is why the results in [57] are complete only in dimension 2 and 3.

One more field of research where Gauduchon type metrics are used is the investigation of invariance properties under holomorphic deformations. To be more specific we recall that a holomorphic family with a central fiber X is a proper holomorphic submersion $\pi : \mathcal{X} \to \Delta$ from a complex manifold $\mathcal{X}$ onto an open ball $\Delta \in \mathbb{C}^m$ around the origin for some $m \in \mathbb{N}$. The fibres $X_t := \pi^{-1}(t)$ are then all diffeomorphic but need not be biholomorphic. The fibre X_0 is then called a central fibre and the fibres X_t are called deformations of X_0.

It is interesting to study the behavior of geometric structures under deformations. For instance it is a classical result that having a Kähler metric is an *open* property in terms of deformations i.e. if the central fiber is a Kähler manifold then so are all X_t's for t sufficiently small (see [38]). To the contrary *Moishezon* manifolds are examples proving that being Kähler is not a *closed* condition, i.e. all X_t's $t \in \Delta \setminus \{0\}$ being Kähler does not imply that X_0 is a Kähler manifold. In the Hermitian world one can ask whether the notions introduced in the previous sections are deformation open/closed. Some of these questions are already answered (for example balanced condition is not open but the strongly Gauduchon condition is- we refer to [49] for a detailed discussion) but the complete list is still missing. Needless to say better understanding of these conditions is thus highly desirable.

6. Basic notions of pluripotential theory: currents and capacities

In this section we shall define all the basic tools in the Hermitian pluripotential theory. A good reference for classical plurisubharmonic functions is [30]. The pluripotential theory in the local setting was developed by Bedford and Taylor in [3]. For Kählerian counterparts of the discussed notions we refer to [40, 26].

We begin this Section by recalling the definition of the basic object of study: the ω-plurisubharmonic functions:

Definition 6.1. — *The ω plurisubharmonic functions are the elements of the function class*

$$PSH_\omega(X) := \{u \in L^1(X,\omega) : dd^c u \geqslant -\omega,\ u \in \mathcal{C}^\uparrow(X)\},$$

where $\mathcal{C}^{\uparrow}(X)$ denotes the space of upper semicontinuous functions and the inequality is understood in the weak sense of currents.

We call the functions that belong to $PSH_\omega(X)$ either ω-plurisubharmonic or ω-psh for short. We shall often use the handy notation $\omega_u := \omega + dd^c u$.

Note that the definition coincides with the usual one in the Kähler setting. In particular ω-psh functions are locally standard plurisubharmonic functions plus some smooth function. Thus in local coordinates in a chart $\Omega_1 \leqslant \omega \leqslant \Omega_2$ for some (local) Kähler forms $\Omega_1,\ \Omega_2$. In particular local properties of ω-psh functions are the same as in the Kähler setting.

This allows the use of some local results from pluripotential theory developed by Bedford and Taylor in [3]. In particular the Monge-Ampère operator

$$\omega_u^n := \omega_u \wedge \cdots \wedge \omega_u$$

is well defined for bounded ω-psh functions.

Furthermore, if $u_j \in PSH(\omega) \cap L^\infty$ is either uniformly convergent or monotonely convergent (in decreasing or increasing manner) almost everywhere to u, then

$$(dd^c u_j + \omega)^n \to (dd^c u + \omega)^n$$

in the sense of currents. This follows from the convergence theorems in [3] via the following argument. Suppose ω is a Hermitian form in a ball B, and Ω a Kähler form such that $\omega < \Omega$. Write

$$dd^c u_j + \omega = dd^c u_j + \Omega - T, \quad T = (\Omega - \omega).$$

Then by the Newton expansion

$$(dd^c u_j + \omega)^n = (dd^c u_j + \Omega)^n - n(dd^c u_j + \Omega)^{n-1} \wedge T + ... + (-1)^n T^n. \tag{6.1}$$

By the convergence theorem for psh functions [3] all the terms on the right converge as currents, and the sum of their limits is

$$(dd^c u + \Omega)^n - n(dd^c u + \Omega)^{n-1} \wedge T + ... + (-1)^n T^n = (dd^c u + \omega)^n.$$

We note that *all* functions u in $PSH_\omega(X)$, normalized by the condition $sup_X u = 0$ are uniformly integrable. This follows from classical results in potential theory (see [39]). Since such results are important in the Hermitian setting (compare [61]) we give here a short argument following quite closely the one in [26], where the Authors treat the Kähler case.

PROPOSITION 6.2. — *Let $u \in PSH_\omega(X)$ be a function satisfying $sup_X u = 0$. Then there exists a constant C dependent only on X, ω such that*

$$\int_X |u|\omega^n \leqslant C.$$

Proof. — Consider a double covering of X by coordinate balls $B_s^1 \subset\subset B_s^2 \subset X$, $s = 1, \cdots, N$. In each B_s^2 there exists a strictly plurisubharmonic potential ρ_s satisfying the following properties:

$$\begin{cases} \rho_s|_{\partial B_s^2} = 0 \\ inf_{B_s^2} \rho_s \geqslant -C \\ dd^c \rho_s = \omega_{2,s} \geqslant \omega, \end{cases}$$

where C is a constant dependent only on the covering and ω. Note that plurisubharmonicity coupled with the first condition above yields the inequality $\rho_s \leqslant 0$ on B_s^2.

Suppose now that there exists a sequence $u_j \in PSH_\omega(X), sup_X u_j = 0$ satisfying $lim_{j\to\infty} \int_X |u_j|\omega^n = \infty$. After choosing subsequence (which for the sake of brevity we still denote by u_j) we may assume that

$$\int_X |u_j|\omega^n \geqslant 2^j \tag{6.2}$$

and moreover a sequence of points x_j where u_j attains maximum is contained in some fixed ball B_s^1.

Note that $\rho_s + u_j$ is an ordinary plurisubharmonic function in B_s^2 and by the sub mean value property one has

$$\rho_s(x_j) = \rho_s(x_j) + u_j(x_j) \leqslant C \int_{B_s^2} \rho_s(z) + u_j(z) dV \leqslant C \int_{B_s^2} u_j(z) dV + C, \tag{6.3}$$

where dV is the Lebesgue measure in the local coordinate chart, while C denotes constants dependent only on B_s^1 and B_s^2. Thus (6.3) implies that for some constant C one has

$$\int_{B_s^2} |u_j(z)| dV \leqslant C. \tag{6.4}$$

Consider the function $v := \sum_{j=1}^{\infty} \frac{u_j}{2^j}$. By classical potential theory this is again a ω-psh function or constantly $-\infty$. By (6.4), however, the integral of v over B_s^2 is finite, thus it is a true ω-psh function. By the same reasoning we easily obtain that $v \in L^1(B_t^1)$ for any $t \in 1, \cdots, N$ and hence $v \in L^1(X)$. This contradicts (6.2), and thus the existence of a uniform bound is established. □

Recall that the Monge-Ampère capacity associated to (X, ω) is the function defined on Borel sets by

$$Cap_\omega(E) := \sup\{\int_E (\omega + dd^c u)^n \,/\, u \in PSH(X, \omega) \text{ and } 0 \leqslant u \leqslant 1\}.$$

(an elementary induction ([16]) shows that the introducted quantity is finite.)

We refer the reader to [40, 26] for the basic properties of this capacity in the Kähler setting. In the Hermitian case one can repeat much of the Kählerian picture. Below we list some basic properties of cap_ω that will be useful later on:

Proposition 6.3. —

i) If $E_1 \subset E_2 \subset X$ *then* $cap_\omega(E_1) \leqslant cap_\omega(E_2)$,
ii) If U *is open then* $cap_\omega(U) = sup\{cap_\omega(K)|\ K - compact,\ K \subset U\}$,
iii) If $U_j \nearrow U,\ U_j - open$ *then* $cap_\omega(U) = lim_{j\to\infty} cap_\omega(U_j)$.

Proof. — The first property follows from the very definition of cap_ω. To prove the second fix $\varepsilon > 0$ and a competitor u for the supremum, such that

$$cap_\omega(U) \leqslant \int_U \omega_u^n + \varepsilon.$$

Since ω_u^n is a regular Borel measure by inner regularity there is a compact set $K \subset U$ satisfying

$$\int_U \omega_u^n \leqslant \int_K \omega_u^n + \varepsilon \leqslant cap_\omega(K) + \varepsilon.$$

Coupling the above facts and letting ε converge to zero we end up with $cap_\omega(U) \leqslant sup\{cap_\omega(K)|\ K - compact,\ K \subset U\}$, and the reverse inequality follows from the first property.

Finally the third one can be proved as follows. Fix once more $\varepsilon > 0$ and a compact set $K \subset U$, such that

$$cap_\omega(U) \leqslant cap_\omega(K) + \varepsilon.$$

Observe that for j large enough $K \subset U_j$ and hence $cap_\omega(K) \leqslant cap_\omega(U_j) \leqslant lim_{j\to\infty} cap_\omega(U_j)$, and hence

$$cap_\omega(U) \leqslant lim_{j\to\infty} cap_\omega(U_j),$$

while the reverse inequality is obvious. □

For ω-Kähler the patched local Bedford-Taylor capacity was studied in [40]. That is for a fixed double covering $B_s^1 \subset\subset B_s^2 \subset X$, we define the capacity cap'_ω on a Borel set E by

$$cap'_\omega(E) := \sum_{s=1}^{n} cap(E \cap B_s^1, B_s^2),$$

with $cap(E \cap B_s^1, B_s^2)$ denoting the classical Bedford-Taylor capacity [3]. It was shown in [40] that cap_ω and cap'_ω are equicontinuous in the Kähler case. Observe that the latter can also be defined on non-Kähler manifolds. Following the proof in [40] it can be proven that cap_ω and cap'_ω are equicontinuous also in the Hermitian case (except that in each strictly pseudoconvex domain V_s one considers two local Kähler forms $\omega_{1,s}$ and $\omega_{2,s}$ satisfying $\omega_{1,s} \leqslant \omega \leqslant \omega_{2,s}$ and works with the potentials of those metrics.)

Coupling this fact with the argument from [40] (Lemma 4.3) one obtains the following corollary:

Corollary 6.4. — *Let $p > 1$ and f be a non negative function belonging to $L^p(\omega^n)$. Then for some absolute constant C dependent only on (X, ω) and any compact $K \subset X$ one has*

$$\int_K f\omega^n \leqslant C(p,X)||f||_p cap_\omega(K) exp(-C cap_\omega^{-1/n}(K)) \leqslant C(p,X)||f||_p cap_\omega(K)^2,$$

with $C(p, X)$ a constant dependent on p and (X, ω).

Note that the second inequality is a simple consequence of the following elementary fact:

Observation 6.5. — Given any two positive constants C_0, C_1, there is a positive constant C_2, such that for all $x \in [0, C_0]$ $exp(-C_1/x^{1/n}) \leqslant C_2 x$.

As yet another consequence of psh-like property of ω-psh functions one gets the capacity estimate of sublevel sets of those functions.

Proposition 6.6. — *Let $u \in PSH_\omega(X), sup_X u = 0$. Then there exists an independent constant C such that for any $t > 1$ $cap_\omega(\{u < -t\}) \leqslant \frac{C}{t}$.*

Proof. — We shall use the double covering introduced in Proposition 6.2. Fix a function $v \in PSH_\omega(X), 0 \leqslant v \leqslant 1$. Then we obtain

$$\int_{\{u<-t\}} \omega_v^n \leqslant \frac{1}{t}\int_X -u\omega_v^n \leqslant \frac{1}{t}(\sum_{s=1}^{N}\int_{B_s^1} -u(\omega_{2,s} + dd^c v)^n)$$
$$\leqslant \frac{1}{t}(\sum_{s=1}^{N}\int_{B_s^1} -(u+\rho_s)(dd^c(\rho_s + v))^n).$$

Now by the generalized L^1 Chern-Levine-Nirenberg inequalities (see, for example [14], Proposition 3.11) applied to each pair $B_s^1 \subset\subset B_s^2$ one obtains that the last quantity can be estimated by

$$\begin{aligned}&\frac{1}{t}\sum_{s=1}^{N} C_{B_s^1,B_s^2}||u+\rho_s||_{L^1(B_s^2)}||\rho_s+v||_{L^\infty(B_s^2)}\\&\leqslant \frac{1}{t}max_s\{C_{B_s^1,B_s^2}\}(CN\int_X -u\omega^n + C)(C+1)^n,\end{aligned}$$

where constants $C_{B_s^1,B_s^2}$ depend on the covering, while C - only on (X,ω). By Proposition 6.2 this quantity is uniformly bounded and the statement follows. □

We finish this Section with a lemma which shall be used throughout the note. It follows from the proof of the comparison principle by Bedford and Taylor in [2].

Lemma 6.7. — *Let u, v be bounded $PSH_\omega(X)$ functions and T a (positive but non necessarily closed) current of the form $\omega_{u_1} \wedge \cdots \wedge \omega_{u_{n-1}}$ for bounded functions u_i belonging to $PSH_\omega(X)$. Then*

$$\int_{\{u<v\}} dd^c(u-v)\wedge T \geqslant \int_{\{u<v\}} d^c(u-v)\wedge dT.$$

Proof. — Suppose first that u, v and the boundary of the set $\{u < v\}$ are smooth. If ρ is a smooth defining function of $\{u < v\}$, then $u - v = \alpha\rho$ for some positive function α on the closure of $\{u < v\}$.

Given any smooth positive $(n-1, n-1)$ form θ we thus get the equality

$$\int_{\partial\{u<v\}} d^c(u-v)\wedge\theta = \int_{\partial\{u<v\}} \alpha d^c\rho\wedge\theta.$$

On the other hand if σ denotes the surface area element on $\partial\{u < v\}$ induced by ω then $\sigma = \frac{*d\rho}{||d\rho||}$, where $*$ stands for the Hodge star operator with respect to ω.

Now if $d^c\rho\wedge\theta = fd\sigma$ for some function f we end up with the equality

$$\alpha d\rho\wedge d^c\rho\wedge\theta = \alpha f d\rho\wedge\frac{*d\rho}{||d\rho||}.$$

But $d\rho\wedge d^c\rho\wedge\theta \geqslant 0$, which yields that $\alpha f \geqslant 0$ and thus

$$\int_{\{u<v\}}(dd^c(u-v)\wedge\theta - d^c(u-v)\wedge d\theta) = \int_{\partial\{u<v\}} d^c(u-v)\wedge\theta$$

$$= \int_{\partial\{u<v\}} \alpha f d\sigma \geqslant 0.$$

The case of a current T of the given form is done by approximation of each u_j by a decreasing sequence of smooth ω-psh functions.

Finally if either u, v or $\partial\{u < v\}$ is not smooth we consider approximating sequence of smooth ω-psh functions $u^j,\ v^j$. By the Sard theorem for almost every t the sets $\{u^j < v^j + t\}$ have smooth boundary. Thus we can apply the argument above to the pair $(u^j, v^j + t)$ and then let t to zero. Finally we let $j \to \infty$ and the desired inequality follows. □

7. Comparison principle in Hermitian setting

Comparison principle is the most efficient tool in pluripotential theory. Let us recall that in the Kähler setting it says that for any $u,\ v \in PSH_\omega(X) \cap L^\infty(X)$ we have

$$\int_{\{u<v\}} \omega_v^n \leqslant \int_{\{u<v\}} \omega_u^n.$$

Such an inequality is in general impossible on Hermitian manifolds due to the following proposition:

Proposition 7.1. — *A necessary condition for the comparison principle to hold is that*

$$\forall u \in PSH_\omega(X) \cap L^\infty(X) \quad \int_X (\omega + dd^c u)^n = \int_X \omega^n.$$

Proof. — Note that for any bounded ω-psh function u we can find a constant C such that $u - C < 0 < u + C$. Then applying the comparison principle to the pairs $(u - C, 0)$ and $(0, u + C)$ (the integration takes place over the whole of X) one gets that $\int_X \omega_u^n = \int_X \omega^n$, whence the result. □

Thus unless ω is of special type we have to allow some additional error terms into the inequality. The next theorem shows that such a result indeed holds. Below we present a weaker form of a comparison principle with "error terms" which will be useful in obtaining a priori estimates:

Theorem 7.2. *([16]). — Let ω be a Hermitian metric on a complex compact manifold X and let $u, v \in PSH_\omega(X) \cap L^\infty(X)$. Then there exists a polynomial P_n of degree $n-1$ and zeroth degree coefficient equal to* 0*, such that*

$$\int_{\{u<v\}} \omega_v^n \leqslant \int_{\{u<v\}} \omega_u^n + P_n(BM) \sum_{k=0}^{n} \int_{\{u<v\}} \omega_u^k \wedge \omega^{n-k},$$

where B is defined by (1.2) and $M = sup_{\{u<v\}}(v-u)$. The coefficients of the polynomial are nonnegative and depend only on the dimension of X.

This claim says that provided the product of B and the supremum of $v-u$ is small enough the error terms are small. Of course these error terms are bounded anyway and can be incorporated in the coefficients of the polynomial P_n but here it is empahsized that P_n is independent of the functions u and v and also that the error terms involve lower order Hessians of ω_u. In general it is impossible to control these pointwise but it will turn ou later that these can be cotrolled by ω_u^n in the integral sense over specific subdomains.

Proof. — Note that

$$\int_{\{u<v\}} \omega_v^n = \int_{\{u<v\}} \omega \wedge \omega_v^{n-1} + \int_{\{u<v\}} dd^c v \wedge \omega_v^{n-1}$$
$$\leqslant \int_{\{u<v\}} \omega \wedge \omega_v^{n-1} + \int_{\{u<v\}} dd^c u \wedge \omega_v^{n-1} + \int_{\{u<v\}} d^c(v-u) \wedge d(\omega_v^{n-1}),$$

where we have used Lemma 6.7. Again by (1.2) we have

$$dd^c(\omega_v^{n-1}) \leqslant B[\omega^2 \wedge \omega_v^{n-2} + \omega^3 \wedge \omega_v^{n-3}].$$

Thus by Stokes theorem

$$\int_{\{u<v\}} \omega_v^n \leqslant \int_{\{u<v\}} \omega_u \wedge \omega_v^{n-1} - \int_{\{u<v\}} d(v-u) \wedge d^c(\omega_v^{n-1})$$
$$\leqslant \int_{\{u<v\}} \omega_u \wedge \omega_v^{n-1} + \int_{\{u<v\}} (v-u) \wedge dd^c(\omega_v^{n-1})$$
$$\leqslant \int_{\{u<v\}} \omega_u \wedge \omega_v^{n-1} + sup_{\{u<v\}}(v-u)B \int_{\{u<v\}} (\omega^2 \wedge \omega_v^{n-2} + \omega^3 \wedge \omega_v^{n-3}).$$

Repeating the above procedure of replacing ω_v by ω and ω_u in the end one obtains the statement. □

In the computations above it is easy to see that the term $\int_{\{u<v\}} \omega_u^{n-1} \wedge \omega$ will never appear on the right hand side but we shall not use this fact. Also for small n the polynomials P_n are explicitly computable: in particular one can take $P_2(x) = 2x, P_3(x) = 2x^2 + 4x$. In general we can use the following (very) crude count: In the process we exchange a term $\int_{\{u<v\}} \omega_v^k \wedge \omega_u^l \wedge \omega^{k-l}$ for the term $\int_{\{u<v\}} \omega_v^{k-1} \wedge \omega_u^{l+1} \wedge \omega^{k-l}$ and $\int_{\{u<v\}} (v-u) dd^c(\omega_v^{k-1} \wedge \omega_u^l \wedge \omega^{k-l})$. The latter term splits into six pieces and each of them contains ω_v with power no higher than $k-1$. Of course there are special cases when some

of these terms coincide or do not appear, but the upshot is that there will be at most 7^n terms in the very end. Thus one can take P_n as $P_n(x) = 7^n(x + x^2 + \cdots + x^{n-1})$.

Below we shall state a technical refined version of the above theorem. It works only for *special* sublevel domains but has the advantage that all the lower order Hessian terms are incorporated into the ω_u^n-term at the cost of enlarging the constant 1 in front of it. This inequality was proven by Cuong and Kolodziej in [41]:

THEOREM 7.3. *(Comparison principle-refined version). — Let X, ω, u and v be as above. Take $0 < \varepsilon < 1$ and let $m(\varepsilon) = inf_X(u - (1-\varepsilon)v)$. Then for any small constant $0 < s < \frac{\varepsilon^3}{16B}$*

$$\int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega^n_{(1-\varepsilon)v} \leqslant (1 + n^2 14^n \frac{sB}{\varepsilon^n}) \int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega_u^n$$

for some universal constant C dependent only on X, n and ω.

Observe that this comparison principle works only for sublevel sets very close to the empty set $\{u < (1-\varepsilon)v + m(\varepsilon)\}$. The bonus is that we control not only ω_v^n but also the (integrals of) lower order Hessians of ω_v.

Proof. — Denote by $a_k = \int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega_u^k \wedge \omega^{n-k}$. Observe that from the assumptions made on s the BM term from the first version of the comparison principle is small here, hence $P_n(Bs) \leqslant n7^n Bs$ (since $x^k \leqslant x$ for $k \geqslant 1, x \in (0,1)$). Then it is enough to get rid of the lower order Hessians of ω_u.

Note that $\varepsilon\omega \leqslant \omega_{(1-\varepsilon)v+m(\varepsilon)+s}$ and hence

$$\varepsilon a_k \leqslant \int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega_u^k \wedge \omega_{(1-\varepsilon)v} \wedge \omega^{n-k-1}.$$

Swapping now $(1-\varepsilon)v + m(\varepsilon) + s$ with u as in the previous proof we get

$$\varepsilon a_k \leqslant a_{k+1} + sB(a_k + a_{k-1} + a_{k-2}) \tag{7.1}$$

(with the understanding that $a_{-1} = a_{-2} = 0$). Now we shall prove inductively that $a_k \leqslant \frac{2}{\varepsilon} a_{k+1}$. Indeed for $k = 0, 1$ this follows from inequality (7.1) and the assumption that $sB \leqslant \frac{\varepsilon^3}{16}$. Suppose now that the inequality is true for $k-2$ and $k-1$ then (7.1) results in

$$\varepsilon a_k \leqslant a_{k+1} + \frac{\varepsilon^3}{16}(a_k + \frac{2}{\varepsilon} a_k + \frac{4}{\varepsilon^2} a_k) \leqslant a_{k+1} + \frac{\varepsilon}{2} a_k,$$

which proves the claim.

Our inductive argument gives us the inequality $a_k \leqslant \frac{2^n}{\varepsilon^n} a_n$, so the integrals of lower order Hessians can be estimated by $\int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega_u^n$ and the result follows. $\square$

Observe that when $B = 0$ (in particular when ω is Kähler) the theorem above gives us the standard comparison principle.

Finally we state two "partial" comparison principles. The first one is for the Laplacian operator with respect to a Gauduchon metric:

Proposition 7.4. — *Let ω be a Gauduchon metric and let $\phi,\ \psi \in PSH_\omega(X) \cap L^\infty(X)$. Then*

$$\int_{\{\phi<\psi\}} \omega_\psi \wedge \omega^{n-1} \leqslant \int_{\{\phi<\psi\}} \omega_\phi \wedge \omega^{n-1}.$$

Since the total integral of the Laplacian is independent of the potential in the Gauduchon case the proof copies the argument from the Kähler setting (see for example [40]). The second one involves mixed Hessian operators. Just like in Proposition 7.1 the necessary condition for the inequality

$$\int_{\{u<v\}} \omega_v^k \wedge \omega^{n-k} \leqslant \int_{\{u<v\}} \omega_u^k \wedge \omega^{n-k}$$

to hold for any pair $u,\ v$ of bounded ω-psh functions is the constancy of the total masses $\int_X \omega_v^k \wedge \omega^{n-k}$. Observe that this is the case if ω satisfies the Guan-Li condition. It turns out that this necessary condition is also a sufficient one.

Proposition 7.5. *Let ω be a metric satisfyig the condition $\int_X \omega_v^k \wedge \omega^{n-k} = \int_X \omega^n$ for any bounded ω-psh function v. Then for any two bounded ω-psh functions u and v the inequality*

$$\int_{\{u<v\}} \omega_v^k \wedge \omega^{n-k} \leqslant \int_{\{u<v\}} \omega_u^k \wedge \omega^{n-k}$$

holds.

Proof. — Recall that the locality of the Monge-Ampère operator (which is independent of the underlying metric) [3] (see also [28] [18]) yields $(\omega + dd^c \max(u,v))^n|_{\{u>v\}} = (\omega + dd^c u)^n|_{\{u>v\}}$. In fact the same argument can be applied to Hessian terms. In particular one also obtains

$$(\omega + dd^c \max(u,v))^k \wedge \omega^{n-k}|_{\{u>v\}} = (\omega + dd^c u)^k \wedge \omega^{n-k}|_{\{u>v\}}.$$

Repeating the argument from [28] we obtain for any $\epsilon > 0$

$$\int_{\{u-\epsilon<v\}} (\omega + dd^c v)^k \wedge \omega^{n-k} = \int_{\{u-\epsilon<v\}} (\omega + dd^c \max(u, v))^k \wedge \omega^{n-k}$$
$$= \int_X (\omega + dd^c \max(u, v))^k \wedge \omega^{n-k} - \int_{\{u-\epsilon\geqslant v\}} (\omega + dd^c \max(u, v))^k \wedge \omega^{n-k}$$
$$\leqslant \int_X (\omega + dd^c u)^k \wedge \omega^{n-k} - \int_{\{u-\epsilon>v\}} (\omega + dd^c u)^k \wedge \omega^{n-k}$$
$$= \int_{\{u-\epsilon\leqslant v\}} (\omega + dd^c u)^k \wedge \omega^{n-k},$$

where we have used the invariance of the total volume and the positivity of the measure in passing from the second line to the last one.

Letting $\epsilon \searrow 0$ and using monotone convergence one obtains the claimed result. □

8. The complex Monge-Ampère equation on compact Hermitian manifolds

In this section we shall discuss in detail the solvability of the Dirichlet problem for the complex Monge-Ampère equation in the Hermitian setting. Our goal will be the following theorem:

THEOREM 8.1. — *Let (X, ω) be a compact Hermitian manifold of complex dimension n. Let also f be any smooth strictly positive function on X. Then the following problem*

$$\begin{cases} u \in \mathcal{C}^\infty(X), \quad \omega + dd^c u > 0, \\ sup_X u = 0, \\ c \in \mathbb{R}, \\ (\omega + dd^c u)^n = e^c f \omega^n, \quad f \in \mathcal{C}^\infty(X), \quad f > 0. \end{cases} \tag{8.1}$$

admits a unique solution (u, c). Furthermore there exist constants C_k, $k = 0, 1, 2, \cdots$ dependent only on X, ω and f, such that the C^k-th norm of the function u is bounded by C_k.

Note that we do not assume compatibility conditions on f (i.e. we do not assume that $\int_X f\omega^n = \int_X \omega^n$) but instead we introduce an additional constant c in the equation.

In the case when ω is Kähler the solvability of this equation was proved by Yau in his seminal paper [62]. The Hermitian case was studied by Cherrier [9], and later by Guan-Li, Tosatti-Weinkove [25, 55] up until the final resolution by Tosatti and Weinkove in [56].

The method of proof will follow the classical continuity method approach. More precisely we consider the family of problems

$$(*)_t \qquad \begin{cases} u \in PSH_\omega(X), \\ sup_X u = 0, \\ (\omega + dd^c u_t)^n = e^{c_t}(1 - t + tf)\omega^n \quad f \in \mathcal{C}^\infty(X), f > 0, \end{cases} \tag{8.2}$$

for $t \in [0, 1]$. Clearly the problem $(*)_0$ is solvable and it is enough to prove that the set

$$A := \{T \in [0, 1] |\ (*)_t \text{ is solvable for every } t \leqslant T\}$$

is open and closed in $[0, 1]$.

To this end we shall first prove uniqueness of the constant c and uniqueness of the solution u. Then we pass to the openness. The hard part (as usual) is the closedness which is achieved by establishing a priori estimates for the solutions.

8.1. Uniqueness

In [56] the authors proved that if u, v are smooth ω-psh functions and their Monge-Ampère measures satisfy $\omega_u^n = e^{c_1} f\omega^n$, $\omega_v^n = e^{c_2} f\omega^n$ for some smooth function f and some constants c_1 and c_2 then in fact $c_1 = c_2$ and u and v differ by a constant. This is the counterpart of the uniqueness of potentials in the Calabi conjecture from the Kähler case.

The equality $u = v$ is easy. Indeed, suppose that we already knew that $c_1 = c_2$. Then we have

$$0 = e^{c_1} f\omega^n - e^{c_1} f\omega^n = \omega_u^n - \omega_v^n = dd^c(u - v) \wedge (\sum_{k=0}^{n-1} \omega_u^k \wedge \omega_v^{n-1-k}).$$

This can be treated as a linear strictly elliptic equation with respect to $u - v$ for the coefficients of the form $\sum_{k=0}^{n-1} \omega_u^k \wedge \omega_v^{n-1-k}$ pointwise give strictly positive definite matrix. But then the strong maximum principle yields that $u - v$ must be a constant.

Now we show that $c_1 = c_2$. The proof is taken from [16] and is in the spirit of pluripotential theory. Suppose, to the contrary, that

$$\omega_u^n = e^{c_1} f\omega^n,\ \omega_v^n = e^{c_2} f\omega^n$$

for some smooth u, v. We can without loss of generality assume that $c_2 > c_1$.

Consider the Hermitian metric $\omega+dd^cu$. Since by the assumptions above it is smooth and strictly positive one finds a unique Gauduchon function ϕ_u, such that

$$inf_X\phi_u = 0,\ dd^c(e^{(n-1)\phi_u}(\omega + dd^cu)^{n-1}) = 0.$$

Then one can apply the comparison principle for the Laplacian with respect to the Gauduchon metric (Proposition 7.4) $e^{\phi_u}(\omega + dd^cu)$ which yields

$$\int_{\{u<v\}} e^{(n-1)\phi_u}(\omega + dd^cu)^{n-1} \wedge \omega_v \leqslant \int_{\{u<v\}} e^{(n-1)\phi_u}\omega_u^n.$$

Exchanging now v with $v + C$ (which does not affect the reasoning above) for big enough C one obtains

$$\int_X e^{(n-1)\phi_u}(\omega + dd^cu)^{n-1} \wedge \omega_v \leqslant \int_X e^{(n-1)\phi_u}\omega_u^n.$$

Note that the left hand side can be estimated from below using (pointwise) the AM-GM inequality:

$$\int_X e^{(n-1)\phi_u}(\omega + dd^cu)^{n-1} \wedge \omega_v \geqslant \int_X e^{(n-1)\phi_u+\frac{(c_2-c_1)}{n}}\omega_u^n.$$

Coupling the above estimates one obtains

$$1 < e^{\frac{(c_2-c_1)}{n}} \leqslant 1,$$

a contradiction.

8.2. Continuity method: openness

The openness part boils down to showing that if $(*)_T$ is solvable then the problem $(*)_t$ is also solvable for t close enough to T. This is achieved by applying the implicit function theorem between well chosen Banach spaces and linearization of the equation. Here the linearized operator is essentially the Laplacian, and we shall prove that this operator is bijective in our setting. The details are taken from [55].

First of all we need the following classical fact:

Proposition 8.2. — *Let ω be a Gauduchon metric on X and let Δ_ω be the Laplacian operator with respect to ω. Then, given any $f \in L^2(X,\omega)$ there is a unique $W^{2,2}$ function u which solves the problem*

$$\Delta_\omega u = f, \quad \int_X v\omega^n = 0$$

if and only if $\int_X f\omega^n = 0$. *Furthermore if* $\alpha \in (0,1)$ *and* $f \in \mathcal{C}^\alpha(X)$, *then* $u \in \mathcal{C}^{2,\alpha}(X)$.

Proof. — Uniqueness of normalized solutions follows from the ellipticity of Δ_ω. The formal computation

$$\begin{aligned}
&\int_X < \Delta_\omega u, g > \omega^n = \int_X g dd^c u \wedge \omega^{n-1} = \int_X u dd^c(g\omega^{n-1}) \\
&= \int_X (u dd^c g \wedge \omega^{n-1} + u dg \wedge d^c(\omega^{n-1}) - u d^c g \wedge d(\omega^{n-1})) \\
&= \int_X < u, \Delta_\omega^* g > \omega^n
\end{aligned}$$

shows that the adjoint operator Δ_ω^* is second order elliptic and moreover it contains no zero order term (note that we use the Gauduchon condition here!) thus it contains only constant functions in its kernel. On the other hand, again by classical elliptic theory the image of Δ_ω in L^2 is perpendicular to the kernel of Δ_ω^* which proves the first assertion. The second assertion is a consequence of the classical Schauder theory of linear elliptic equations. □

Suppose now that at time T we have a smooth solution u to the problem $(*)_T$ (we skip the indice T for the ease of notation). Let ϕ_u denotes the Gauduchon function associated to ω_u. We normalize it by adding a constant if needed so that $\int_X e^{(n-1)\phi_u}(\omega + dd^c u)^n = 1$. We also fix a small positive constant $\alpha < 1$ (dependent on X, ω and n- the dependence will be important in the later stages when we prove higher order a priori estimates).

Consider the two Banach manifolds

$$B_1 := \{w \in \mathcal{C}^{2,\alpha}(X) | \int_X w e^{(n-1)\phi_u} \omega_u^n = 0\}$$

and

$$B_2 := \{h \in \mathcal{C}^{\alpha}(X) | \int_X e^{h+(n-1)\phi_u} \omega_u^n = 1\}.$$

Consider the mapping $\mathcal{T} : B_1 \to B_2$ given by

$$\mathcal{T}(v) := log(\frac{(\omega + dd^c u + dd^c v)^n}{(\omega + dd^c u)^n}) - log(\int_X e^{(n-1)\phi_u}(\omega + dd^c u + dd^c v)^n).$$

Observe that $\mathcal{T}(0) = 0$ and that any function v sufficiently close to 0 in $\mathcal{C}^{2,\alpha}$ norm is $\omega + dd^c u$- plurisubharmonic.

By the implicit function theorem the equation $\mathcal{T}(v) = h$ is solvable for any $h \in B_2$ sufficiently close in $\mathcal{C}^\alpha$ norm to zero if the Frechet derivative

$$(D\mathcal{T}) : T_0B_1 = B_1 \to T_0B_2 = \{g \in \mathcal{C}^\alpha(X) | \int_X g e^{(n-1)\phi_u}\omega_u^n = 0\}$$

is an invertible linear mapping.

But a computation shows that

$$(D\mathcal{T})(\eta) = \Delta_{\omega+dd^cu}\eta - n\int_X e^{(n-1)\phi_u}\omega_u^{n-1} \wedge dd^c\eta.$$

Note that the last summand is zero beacuse $e^{\phi_u}(\omega + dd^cu)$ is Gauduchon. The question is thus whether $\Delta_{\omega+dd^cu} : B_1 \to T_0B_2$ is a continuous bijective mapping.

By Proposition 8.2 (recall that $e^{\phi_u}(\omega + dd^cu)$ is Gauduchon metric!) the equation

$$\Delta_{e^{\phi_u}(\omega+dd^cu)}(\eta) = \tau$$

is solvable if and only if $\int_X \tau e^{n\phi_u}(\omega + dd^cu)^n = 0$ and the solution is unique up to an additive constant. Note that $\Delta_{e^{\phi_u}(\omega+dd^cu)}(\eta) = e^{-\phi_u}\Delta_{(\omega+dd^cu)}(\eta)$ thus $(D\mathcal{T})(\eta) = \tau$ is solvable if and only if $\int_X \tau e^{(n-1)\phi_u}(\omega + dd^cu)^n = 0$ i.e. exactly if τ belongs to T_0B_2. This proves the surjectivity of $(D\mathcal{T})$ and injectivity follows from the normalization condition. Finally continuity of $(D\mathcal{T})$ follows from the Schauder $\mathcal{C}^{2,\alpha}$ a priori estimates for the Laplace equation.

8.3. Continuity method: closedness- higher order estimates

Before starting the proofs of a priori estimates let us stress that third and higher order ones follow from standard Schauder elliptic theory as long as $\mathcal{C}^{2,\alpha}$ estimates are proven for some small positive $\alpha < 1$. Thus we are left with proving estimates up to order $2 + \alpha$.

By the complex version of the Evans-Krylov theory (see [59] for a nice overview) there is a constant

$$C = C(X, \omega, n, ||\Delta u||_{\mathcal{C}^0}, ||u||_{\mathcal{C}^0}, ||f||_{\mathcal{C}^1})$$

and $0 < \alpha < 1$ dependent on the same quantities, such that if u solves the equation (8.1) then

$$||u||_{\mathcal{C}^{2,\alpha}} \leqslant C.$$

Thus what remains is to prove uniform bound for the Laplacian of u, and of u itself.

8.4. Continuity method: closedness- second order estimate

The aim of this subsection is to prove the following estimate:

Theorem 8.3. *[25]. — If u is a solution to Equation 8.1 then there exists a constant $C = C(X, \omega, n, ||\Delta f||_{\mathcal{C}^0}, ||u||_{\mathcal{C}^0})$, such that*

$$0 \leqslant n + \Delta u \leqslant C,$$

where the Laplacian is the ordinary Chern Laplacian with respect to the metric ω.

Once we have second order estimates the gradient estimate follows by interpolation. Our proof will differ slightly from the one in [25] but, of course, the main idea remains the same.

Proof. — Consider the function $A(u) := log(n + \Delta u) + h(u)$, where h is an additonal uniformly bounded strictly decreasing function that we shall choose later on. If we can prove that at the point z where A attains maximum we have that $n + \Delta u$ is bounded then we are done since at any other point x we have

$$log(n + \Delta u)(x) \leqslant A(z) - h(u(x)) \leqslant C.$$

Thus let us fix a point of maximum of A and identify it with zero in a local chart. We shall use ordinary partial derivatives in this chart- in particular $g_{i\bar{j},k}$ will denote $\frac{\partial g_{i\bar{j}}}{\partial z_k}$ and so on. Let us also denote by g' the metric $g_{i\bar{j}} + u_{i\bar{j}}$, while $g^{k\bar{l}}, g'^{k\bar{l}}$ will denote the inverse transposed matrices of g and g' respectively.

In order to simplify the computations let us assume that we have chosen coordinates diagonalizing the metric $g_{i\bar{j}}$ and $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ and then rechoose the canonical coordinates so that additionally $g_{i\bar{i},k}(0) = 0$ for any i, k. Observe that the Hessian of u is still diagonal at zero. Moreover we can safely assume that $\Delta u(0) \geqslant 1$, say, for otherwise we are done.

Applying logarithm to both sides of Equation 8.1 and differentiating twice at z we get

$$g'^{p\bar{r}}(g_{p\bar{r},k} + u_{p\bar{r}k}) = log(f)_k + g^{p\bar{r}} g_{p\bar{r},k}; \tag{8.3}$$

$$\begin{aligned} -g'^{p\bar{s}} g'^{h\bar{r}}(g_{h\bar{s},\bar{l}} + u_{h\bar{s}\bar{l}})(g_{p\bar{r},\bar{k}} + u_{p\bar{r}\bar{k}}) + g'^{p\bar{r}}(g_{p\bar{r},k\bar{l}} + u_{p\bar{r}k\bar{l}}) \\ = log(f)_{k\bar{l}} - g^{p\bar{s}} g^{h\bar{r}} g_{h\bar{s},\bar{l}} g_{p\bar{r},k} + g^{p\bar{r}} g_{p\bar{r},k\bar{l}}. \end{aligned} \tag{8.4}$$

Taking trace in the second equation we obtain

$$-g'^{p\bar{p}}g'^{r\bar{r}}|g_{r\bar{p},k}+u_{r\bar{p}k}|^2+g'^{r\bar{r}}(g_{r\bar{r},k\bar{k}}+u_{r\bar{r}k\bar{k}})=\Delta log(f)-|g_{p\bar{r},k}|^2+g_{r\bar{r},k\bar{k}}. \tag{8.5}$$

Let us now investigate the function A at the point of maximum. From the vanishing of the first derivative of A we get the equalities

$$0=\frac{g^{i\bar{j}}_{,k}u_{i\bar{j}}+g^{i\bar{j}}u_{i\bar{j}k}}{\Delta u+n}+h'u_k=\frac{u_{i\bar{i}k}}{\Delta u+n}+h'u_k. \tag{8.6}$$

(The first term in the first summand vanishes because we have chosen the special coordinates!) Taking now the trace of the Hessian of A at the point z with respect to g' we obtain the inequality

$$0\geqslant g'^{k\bar{k}}A_{k\bar{k}}=g'^{k\bar{k}}[\frac{(g^{i\bar{j}}u_{i\bar{j}})_{k\bar{k}}}{\Delta u+n}-\frac{|\sum_i u_{i\bar{i}k}|^2}{(\Delta u+n)^2}+h'u_{k\bar{k}}+h''|u_k|^2]. \tag{8.7}$$

From Equation (8.6) the second term can be exchanged by $-(h')^2g'^{k\bar{k}}|u_k|^2$, while the third one reads $h'(n-\sum_k g'^{k\bar{k}})$. In order to estimate the first term we observe that

$$(g^{i\bar{j}}u_{i\bar{j}})_{k\bar{k}}=g^{i\bar{i}}_{,k\bar{k}}u_{i\bar{i}}+u_{i\bar{i}k\bar{k}}+2Re(g^{i\bar{j}}_{k}u_{i\bar{j}\bar{k}}).$$

The fourth order term, after taking trace with $g'^{k\bar{k}}$ can be exchanged using Equation (8.5).

Note that, exploiting the diagonality of g at z one has

$$g^{i\bar{j}}_{,k}=-g^{i\bar{s}}g^{l\bar{j}}g_{l\bar{s},k}=-g_{j\bar{i},k}.$$

Altogether the first term then reads

$$\begin{aligned}g'^{k\bar{k}}\frac{(g^{i\bar{j}}u_{i\bar{j}})_{,k\bar{k}}}{\Delta u+n}&=g'^{k\bar{k}}\frac{g^{i\bar{i}}_{k\bar{k}}u_{i\bar{i}}}{\Delta u+n}\\&-g'^{k\bar{k}}\frac{2Re(g_{j\bar{i},k}u_{i\bar{j}\bar{k}})}{\Delta u+n}-g'^{k\bar{k}}\frac{g_{k\bar{k},i\bar{i}}-\Delta log f}{\Delta u+n}\\&-\frac{|g_{r\bar{k},i}|^2}{\Delta u+n}+\frac{g'^{r\bar{r}}g'^{k\bar{k}}|g_{r\bar{k},i}+u_{r\bar{k}i}|^2}{\Delta u+n}.\end{aligned}$$

Note that the first summand above is controlled from below by $-C\sum_k g'^{k\bar{k}}$ with the constant C dependent on the sup norm of all second order derivatives of g. The same goes for all the terms in the third and fourth summand (the dependence of C on the relevant quantities is clear- note also that in a sense these terms are even "better" due to the Laplacian in the denominator).

Summing up our computations up to now inequality (8.7) results in

$$0 \geqslant [-h' - C]\sum_k g^{'k\bar{k}} - C + [h'' - (h')^2]\sum_k g^{'k\bar{k}}|u_k|^2$$
$$+ \frac{g^{'r\bar{r}}g^{'k\bar{k}}|g_{r\bar{k},i} + u_{r\bar{k}i}|^2}{\Delta u + n} - g^{'k\bar{k}}\frac{2Re(g_{j\bar{i},k}u_{i\bar{j}\bar{k}})}{\Delta u + n}.$$

The last summand can be rewritten as follows:

$$\begin{aligned} g^{'k\bar{k}}\frac{2Re(g_{j\bar{i},k}u_{i\bar{j}\bar{k}})}{\Delta u + n} &= g^{'k\bar{k}}\frac{2Re(g_{j\bar{i},k}u_{i\bar{k}\bar{j}})}{\Delta u + n} \\ &= g^{'k\bar{k}}\frac{2Re(g_{j\bar{i},k}(g_{i\bar{k},\bar{j}} + u_{i\bar{k}\bar{j}} - g_{i\bar{k},\bar{j}}))}{\Delta u + n} \\ &= g^{'k\bar{k}}\sum_{i\neq j}\sqrt{g^{'i\bar{i}}g^{'}_{i\bar{i}}}\frac{2Re(g_{j\bar{i},k}g^{'}_{i\bar{k},\bar{j}})}{\Delta u + n} - g^{'k\bar{k}}\frac{2Re(g_{j\bar{i},k}g_{i\bar{k},\bar{j}})}{\Delta u + n}. \end{aligned}$$

(We sum only over indices $i \neq j$ for in the special coordinates $g_{i\bar{i},k} = 0$). Applying Schwarz inequality the latter is bounded above by

$$g^{'k\bar{k}}\sum_{i\neq j}g^{'i\bar{i}}\frac{|g^{'}_{i\bar{k},\bar{j}}|^2}{\Delta u + n} + g^{'k\bar{k}}\sum_{i\neq j}\frac{g^{'}_{i\bar{i}}|g_{j\bar{i},k}|^2}{n + \Delta u}$$
$$+ C\sum_k g^{'k\bar{k}} \leqslant \sum_{i\neq j}g^{'k\bar{k}}g^{'i\bar{i}}\frac{|g^{'}_{i\bar{k},\bar{j}}|^2}{\Delta u + n} + C\sum_k g^{'k\bar{k}},$$

where we have also used the elementary inequality $g^{'}_{i\bar{i}} \leqslant \Delta u + n$.

Thus our main inequality reduces to

$$0 \geqslant [-h' - C]\sum_k g^{'k\bar{k}} - C + [h'' - (h')^2]\sum_k g^{'k\bar{k}}|u_k|^2 + \frac{g^{'r\bar{r}}g^{'k\bar{k}}|g^{'}_{r\bar{k},k}|^2}{\Delta u + n}$$

The last term can be handled as follows

$$\begin{aligned} \frac{g^{'r\bar{r}}g^{'k\bar{k}}|g^{'}_{r\bar{k},k}|^2}{\Delta u + n} &= g^{'r\bar{r}}\frac{[(\sum_k g^{'k\bar{k}}|g^{'}_{r\bar{k},k}|^2)(\sum_k g^{'}_{k\bar{k}})]}{(\Delta u + n)^2} \\ &\geqslant g^{'r\bar{r}}\frac{|\sum_k(u_{r\bar{k}k} + g_{r\bar{k},k})|^2}{(\Delta u + n)^2} \\ &= g^{'r\bar{r}}|h'u_r + \frac{\sum_k g_{r\bar{k},k}}{\Delta u + n}|^2, \end{aligned}$$

where in the last equality we have made use of Equation (8.6).

Expanding the squares and applying Schwarz inequality once more we end up with

$$\begin{aligned}&\frac{g^{'r\bar{r}}g^{'k\bar{k}}|g^{'}_{r\bar{k},k}|^2}{\Delta u+n}\\&\geqslant g^{'r\bar{r}}((h')^2+h')|u_r|^2-|h'|g^{'r\bar{r}}\frac{|\sum_k g_{r\bar{k},k}|^2}{(\Delta u+n)^2},\end{aligned}$$

and the last summand is estimated by $C\sum_r g^{'r\bar{r}}$.

Summing up our main inequality now reads

$$0\geqslant[-h'-C]\sum_k g^{'k\bar{k}}-C+[h''-h']\sum_k g^{'k\bar{k}}|u_k|^2.$$

So if we choose the function $h(t)=Ce^{-t}$ for a sufficiently large constant C, and assuming a bound on $osc_X u$ we end up with

$$0\geqslant C\sum_k g^{'k\bar{k}}-C,$$

which shows that $g^{'k\bar{k}}$ are upper bounded and hence $g^{'}_{k\bar{k}}$ are also lower bounded. From the equation we immediately get that $g^{'}_{k\bar{k}}$ are upper bounded at the point z which establishes the desired estimate. □

8.5. Continuity method: closedness- uniform estimate

The last and historically the hardest step is to establish the uniform $\mathcal{C}^0$ estimate. The uniform estimate was proven by Cherrier, Guan-Li and Tosatti-Weinkove ([9, 25, 55]) under various additional assumptions on the metric ω. The general result with no assumptions on ω was first accomplished by Tosatti and Weinkove in [56]. There the Authors used a version of Moser iteration to obtain the following bound:

$$Vol(\{u<inf_X u+\varepsilon\})\geqslant\delta, \tag{8.8}$$

for some fixed constants ε and δ. Roughly speaking such an estimate tells us that there is some control from below on the volume of "small" sublevel sets. This coupled with suitable Sobolev inequality completes the proof, see [56] for details.

Below we prove the uniform estimate using techniques from pluripotential theory taken from [16]. For different approaches we refer also to [5] and [24]. More specifically we shall prove the following result

Theorem 8.4. — *Let u be a solution to the equation 8.1. Then there exists a constant $C > 0$ dependent on $||f||_p, p, X, \omega, n$, such that* $\inf_X u \geqslant -C$.

In the proof we shall prove and exploit a similar bound to (8.8) but we shall use the capacity instead of the volume. Thus our goal is the inequality

$$cap_\omega(\{u < inf_X u + \varepsilon\}) \geqslant \delta.$$

Indeed suppose that such an inequality is already proven. Then exploiting Proposition 6.6 we immediately get a uniform bound of $inf_X u$ and we are done.

Let us first establish an additional capacity inequality which is modelled on an analogous argument from the Kähler setting:

Proposition 8.5. *([16],[41]). — Let u be a ω-psh solution of the equation $\omega_u^n = f\omega^n$, where $f \in L^p(X, \omega)$ for some $p > 1$ and v be any bounded continuous ω-psh function satisfying $-C_0 \leqslant v \leqslant 0$. Take a constant $0 < \varepsilon < 1$ and let $0 < t << \varepsilon$, $0 < s << \varepsilon$ be two sufficiently small constants. Then there is a constant $C = C(n, X, \omega, p, \varepsilon, C_0)$, such that*

$$t^n cap_\omega(\{u < (1-\varepsilon)v + inf_X[u - (1-\varepsilon)v] + s\})$$
$$\leqslant C||f||_{L^p} cap_\omega(\{u < (1-\varepsilon)v + inf_X[u - (1-\varepsilon)v] + s + t\})^2.$$

Proof. — For notational simplicity we denote by $m(\varepsilon)$ the quantity $inf_X[u-(1-\varepsilon)v]$ and by $U(s,\varepsilon)$ the set $\{u < (1-\varepsilon)v + m(\varepsilon) + s\}$. Throughout the proof we shall assume s and t are small enough, so that all technical requrements for the application of Theorem 7.3 are satisfied.

Pick any ω-psh function w such that $0 \leqslant w \leqslant 1$. As w is a competitor for the supremum in the definition of the capacity we need to bound from above the quantity $t^n \int_{\{u<(1-\varepsilon)v+m(\varepsilon)+s\}} \omega_w^n$.

To this end observe that the following inequality holds:

$$m(\varepsilon) - (C_0+1)t \leqslant inf_X[u - (1-\varepsilon)((1-t)v + tw)] \leqslant m(\varepsilon)$$

Thus we get the following string of set inclusions

$$U(s,\varepsilon) = \{u < (1-\varepsilon)v + m(\varepsilon) + s\} \subset \{u < (1-\varepsilon)((1-t)v + tw) + m(\varepsilon) + s\}$$
$$\subset \{u < (1-\varepsilon)((1-t)v+tw) + inf_X[u-(1-\varepsilon)((1-t)v+tw)] + s + (C_0+1)t\} = V$$
$$\subset \{u < (1-\varepsilon)v + m(\varepsilon) + s + 2(C_0+1)t\} = U(s + 2(C_0+1)t, \varepsilon).$$

Note that $(1-t)v+tw$ is a ω-psh function, and the set V is defined so that Theorem 7.3 can be applied for the pair $(u,(1-t)v+tw)$ provided s and t are sufficiently small. Thus

$$((1-\varepsilon)t)^n \int_{U(s,\varepsilon)} \omega_w^n \leqslant ((1-\varepsilon)t)^n \int_V \omega_w^n$$
$$\leqslant \int_V \omega^n_{(1-\varepsilon)((1-t)v+tw)} \leqslant C \int_V \omega_u^n \leqslant C \int_{U(s+2(C_0+1)t,\varepsilon)} \omega_u^n,$$

where we have made use of Theorem 7.3 in the penultimate inequality. Note that the constant C depends on ε but is independent of u and v.

Continuing the string of inequalities we get

$$C \int_{U(s+2(C_0+1)t,\varepsilon)} \omega_u^n \leqslant C||f||_{L^p} cap_\omega(U(s+2(C_0+1)t,\varepsilon))^2,$$

where the last inequality follows from Corollary 6.4. Thus our claim follows after we exchange t with $2(C_0+1)t$. □

Remark 8.6. — Observe that we haven't made use of the continuity of v. This assumption will be used later to guarantee openness of the sets $U(s,\varepsilon)$.

Let us now explain how the above estimate implies that $cap_\omega(\{u < inf_X u+\varepsilon\}) \geqslant \delta$ for some ε and δ. In fact we shall prove the following more genral statement:

Proposition 8.7. — *There exists a small constant s_0, such that for any $s < s_0$ one has $s \leqslant ||f||_{L^p}^{1/n} C cap_\omega(U(s,\varepsilon))^{\frac{1}{n}}$, for a constant C dependent on n, ϵ, X, C_0, p and ω.*

In particular we get our desired bound by plugging $v = 0$ and taking any fixed positive $\varepsilon < 1$.

Proof. — Suppose s_0 is chosen so small that Proposition 8.5 applies for any $s, t \leqslant s_0$. Define inductively s_i to be the supremum of all numbers between 0 and s_{i-1} such that

$$2cap_\omega(U(s,\varepsilon)) < cap_\omega(U(s_{i-1},\varepsilon))\}.$$

Then s_i is clearly a decreasing sequence and any s_i is well defined for the sets shrink to the empty set as s decreases to zero. Observe also that $U(s,\varepsilon)$ are open sets and from the continuity of the capacity for increasing open sets (recall Proposition 6.3) we get $2cap_\omega(U(s_{i+1},\varepsilon)) \leqslant cap_\omega(U(s_i,\varepsilon))$, while by definition $lim_{s\to s_i^+} 2cap_\omega(U(s,\varepsilon)) \geqslant cap_\omega(U(s_{i-1},\varepsilon))$.

Take now an s, such that $s_i \leqslant s < s_{i-1}$. Then from Proposition 8.5 we get

$$(s_{i-1}-s)^n cap_\omega(U(s,\varepsilon)) \leqslant C cap_\omega(U(s_{i-1},\varepsilon))^2.$$

Observe that since $s \geqslant s_i$ we have $2cap_\omega(U(s,\varepsilon)) \geqslant cap_\omega(U(s_{i-1},\varepsilon))$.

Coupling these inequalities we obtain

$$\begin{aligned}(s_{i-1}-s)^n &\leqslant 4C cap_\omega(U(s,\varepsilon))\\ &\leqslant 4C(\frac{1}{2})^{i-1} cap_\omega(U(s_0,\varepsilon)),\end{aligned}$$

where the last inequality follows from iteration.

If we now let s to s_i, then take n-th roots and finally sum up the inequalities over i we will obtain

$$s_0=\sum_{i=1}^{\infty}(s_i-s_{i+1}) \leqslant (4C)^{1/n}\sum_{j=0}^{\infty}(\frac{1}{2})^{j\frac{1}{n}} cap_\omega(U(s_0,\varepsilon))^{\frac{1}{n}},$$

which is the claimed result. □

9. Weak solutions for degenerate right hand side

In this section we shall discuss the solvability of the Dirichlet problem

$$\begin{cases} u \in PSH_\omega(X), sup_X u = 0\\ (\omega+dd^cu)^n = e^c f\omega^n \quad f \in L^p(X,\omega),\ p>1,\ f \geqslant 0.\end{cases} \tag{9.1}$$

Of course the hope is to use the smooth solvability to approximate the singular right hand sides by smooth functions f_j in a suitable way, and then to extract a convergent subsequence of solutions u_j. This approach leads to a problem, namely the behavior of the constants c_j in such an approximation procedure. The technical heart of the matter if we want to extract convergent subsequences is to show that these c_j's are bounded from above and below **independently** of the supremum norms of f_j. This was proven in [41]:

THEOREM 9.1. — *Let $X, \omega, f \neq 0$ and p be as above. Let also f_j be a sequence of smooth strictly positive functions convergent in L^p norm to f. Then the corresponding sequence of contants c_j associated to the problems*

$$(*)_i \begin{cases} u_i \in PSH_\omega(X),\\ sup_X u_i = 0,\\ (\omega+dd^cu_i)^n = e^{c_i} f_i\omega^n\end{cases} \tag{9.2}$$

is uniformly bounded from above and below.

Proof. — Let us first give a lower bound for c_j's. For the sake of brevity we drop the index j in what follows. Recall that from the proof of Proposition 8.5 applied to $\varepsilon = \frac{1}{2}$, say, and $v = 0$ one has

$$t^n cap_\omega(\{u < inf_X u + s\}) \leqslant C cap_\omega(\{u < inf_X u + s + t\})^2$$

for all t, s smaller than a fixed constant ε_0. Taking $t = s$ and estimating the capacity on the right hand side by an uniform constant, which is legitimate since $cap_\omega(\{u < inf_X u + s + t\}) \leqslant cap_\omega(X)$, one gets the inequality

$$cap_\omega(\{u < inf_X u + s\}) \leqslant \frac{C}{s^n}.$$

On the other side from Proposition 8.7 one has

$$s \leqslant (\tilde{C} e^c ||f||_{L^p})^{1/n} cap_\omega(\{u < inf_X u + s_0\})^{\frac{1}{n}}.$$

Coupling these one obtains

$$s^2 \leqslant \bar{C} e^{c/n} ||f||_{L^p}^{1/n},$$

for all $s \leqslant \varepsilon_0$. But then obviously c cannot decrease to minus infinity, hence we get a lower bound.

The upper bound is established as follows: since f_j converge to f in L^p, convergence also holds for $f_j^{1/n}$ towards $f^{1/n}$ in L^1 (we have to use the compactness of X here). Thus for j large enough

$$\int_X f_j^{1/n} \omega^n > \frac{\int_X f^{1/n} \omega^n}{2} > 0.$$

But from the AM-GM inequality one has $(\omega + dd^c u_j) \wedge \omega^{n-1} \geqslant (e^{c_j} f_j)^{1/n} \omega^n$ thus

$$e^{c_j/n} \leqslant \frac{2}{\int_X f^{1/n} \omega^n} \int_X (\omega + dd^c u_j) \wedge \omega^{n-1}.$$

if we multiply ω^{n-1} in the last integral by the Gauduchon function $e^{(n-1)\phi}$ (which is uniformly bounded) we get

$$\begin{aligned} e^{c_j/n} &\leqslant \frac{2}{\int_X f^{1/n} \omega^n} e^{-(n-1) inf \phi} \int_X (\omega + dd^c u_j) \wedge e^{(n-1)\phi} \omega^{n-1} \\ &= \frac{2}{\int_X f^{1/n} \omega^n} e^{-(n-1) inf \phi} \int_X e^{(n-1)\phi} \omega^n \end{aligned}$$

by Stokes theorem. □

Now we are ready for the proof of the existence theorem:

THEOREM 9.2. — *The Dirichlet problem 9.1 admits a continuous solution.*

Proof. — It is enough to show that the sequence of solutions u_j of the problems (9.2) admits a Cauchy subsequence in the uniform topology. Indeed then one can extract a continuous limit. The Monge-Ampère operator is continuous with respect to uniform convergence, thus the limiting function solves the equation.

First of all we can assume that (after passing to a subsequence) the sequence of the constants c_j is convergent to some c. Let us still denote this subsequence by c_j.

Note that the family u_j is normalized by $sup_X u_j = 0$, hence it forms a relatively compact subset in the L^1 topology. Thus we can assume that the u_j converge in L^1 to a ω-psh function u (take another subsequence if necessary).

Observe now that in the sequence of Dirichlet problems (9.2), right hand sides are uniformly bounded in L^p for the chosen subsequence. By theorem 8.4 we get that the sequence u_j is then uniformly bounded. Let then $C_0 > 0$ be a constant such that $u_j \geqslant -C_0$ for every j.

We shall argue by contradiction. To this end consider the quantities $S_{kj} := inf_X(u_k - u_j) \leqslant 0$. Since $sup_X(u_k - u_j) = -inf_X(u_j - u_k)$, it is enough to prove that the numbers S_{kj} converge to zero as k and j tend to infinity.

Suppose that this is not the case and let $1 > \varepsilon > 0$ be a constant such that $S_{kj} \leqslant -(C_0 + 3)\varepsilon$ for arbitrarily large $j \neq k$ (we can further decrease ε if needed). Then if $m_{kj}(\varepsilon)$ as usual denotes the infimum over X of the quantity $u_k - (1-\varepsilon)u_j$ we obtain the inequality $m_{kj}(\varepsilon) \leqslant S_{kj}$.

As in the proof of Proposition 8.5 suppose that $s, t << \varepsilon$. Then we have a set inclusion

$$\{u_k < (1-\varepsilon)u_j + m_{kj}(\varepsilon) + s + t\} \subset \{u_k < u_j + S_{kj} + \varepsilon C_0 + s + t\},$$

and the last set is in turn contained in

$$\{u_k < u_j - \varepsilon\} \subset \{|u_k - u_j| \geqslant \varepsilon\}$$

by our assumption on the constants S_{kj}.

From the proof of Proposition 8.5 we then know that for all t and s smaller than a (fixed) ε

$$
\begin{aligned}
t^n cap_\omega(\{u_k < (1-\varepsilon)u_j + m_{kj}(\varepsilon) + s\}) &\leqslant C \int_{\{u_k<(1-\varepsilon)u_j+m_{kj}(\varepsilon)+s+t\}} \omega_{u_k}^n \\
&\leqslant C \int_{|u_k-u_j|\geqslant\varepsilon} \omega_{u_k}^n = C \int_{|u_k-u_j|\geqslant\varepsilon} e^{c_k} f_k \omega^n \\
&\leqslant C||e^{c_k} f_k||_{L^p} (Vol(|u_k - u_j| \geqslant \varepsilon))^{p/(p-1)}.
\end{aligned}
$$

The latter quantity converges to zero as $j, k \to \infty$, as u_k converge to u in L^1. But arguing analogously to the proof of Proposition 8.7 the capacity term on the left hand side cannot converge to zero when t and s are fixed, a contradiction. □

10. A glimpse of parabolic o theory: Chern-Ricci flow

Similarly to the Kählerian setting the elliptic theory has its parabolic counterpart. Below we shall briefly sketch the interplay between the parabolic and elliptic theories and emphasize the aspects where pluripotential theory applies. To this end we define the so-called Monge-Ampère flow:

Definition 10.1. — *The Monge-Ampère flow is the parabolic flow defined for the family of (smooth) ω-psh functions $\phi_t(z) = \phi(t, z)$ by*

$$
\begin{cases}
\frac{\partial \phi}{\partial t} = log(\frac{det(g_{j\bar{k}} + \phi_{j\bar{k}})}{det(g_{j\bar{k}})}) - F \\
\phi|_{t=0} = \phi_0,
\end{cases}
\tag{10.1}
$$

with $g_{j\bar{k}}$ being the Hermitian metric associated to ω and F stands for a fixed smooth function.

Note that when we apply dd^c operator to both sides equation (10.1) reads

$$
\frac{d}{dt}(\omega + dd^c\phi_t) = Ric_\omega^{BC} - Ric_{\omega+dd^c\phi_t}^{BC} - dd^c F,
$$

with Ric_ω^{BC} denoting the Chern-Ricci form of ω. In other words the flow deforms the family of metrics $\omega_t = \omega + dd^c\phi_t$ in direction opposite to their Chern-Ricci form.

Assume for a while that the flow exists for all time and converges to a smooth metric $\omega + dd^c\phi_\infty$. Then at time infinity we get the equation

$$
Ric_{\omega+dd^c\phi_\infty}^{BC} = Ric_\omega^{BC} - dd^c F,
$$

which can be rewritten in the form

$$\omega_{\phi_\infty}^n = e^{F+c}\omega^n$$

for some constant c. Thus a proof of long time existence and convergence of the Monge-Ampère flow would result in an alternative proof of existence of solutions to the problem (8.1).

This approach was successfully applied in the Kähler setting by H. D. Cao in [7]. In the Hermitan setting analogus analysis was carried out by M. Gill in [24]. He proved the following result:

THEOREM 10.2. *([24]). — Let g and F be as above. Fix the inital condition $\phi_0 = 0$. Assume that the metric is normalized so that $\int_X \omega^n = 1$. Then the Monge-Ampère flow exists for all time and the normalized solutions $\tilde{\phi}_t := \phi_t - \int_X \phi_t\omega^n$ smoothly converge to a limitng function $\tilde{\phi}$ which solves the equation*

$$log(\frac{det(g_{j\bar{k}} + \tilde{\phi}_{j\bar{k}})}{det(g_{j\bar{k}})}) = F + c$$

for some constant c.

This is a parabolic proof of the solvability of the Dirichlet problem.

All this suggests that Hermitian versions of the Kähler-Ricci flow can serve as a tool for deformation of metrics towards more canonical ones.

Before proceeding further we recall that the classical unnormalized Kähler-Ricci flow is given by

$$\frac{\partial\omega}{\partial t} = -Ric(\omega),\ \omega(0) = \omega_0, \tag{10.2}$$

where ω_0 is a fixed Kähler form on a (Kähler) manifold X. A standard result in the Kähler-Ricci flow theory (see [60]) states that this flow exists up until a time T defined by

$$T = sup\{t > 0|\ [\omega_0] - tc_1(X) \text{ is Kahler class}\},$$

with $[\omega]$ denoting the cohomology class of ω and $c_1(X)$ denoting the first Chern class of X (T may be also infinite). Furthermore the potentials ϕ_t of metrics $\omega_t = \omega_0 - tRic_{\omega_0} + dd^c\phi_t$ converge weakly to a *necessarily singular* $\omega_0 - TRic_{\omega_0}$-psh function ϕ_T solving the degenerate equation

$$\omega_T^n = (\omega_0 - TRic_{\omega_0} + dd^c\phi_T)^n = e^{\frac{\partial\phi_T}{\partial t}}\omega^n.$$

This singular potential can be studied from pluripotential viewpoint. In particular (see [60]) it can be shown that it is a bounded and contnuous function smooth outside an analytic set. Furthermore X equipped with the singular metric ω_T is a length space whose metric completion is a (possibly singular) Kähler space X_1. After a desingularization $\pi : X_1' \to X_1$ we end up with a smooth Kähler manifold X_1' with a degenerate data $\pi^*(\omega_T)$. It turns out (see [53]) that the Kähler-Ricci flow can be started from the data $X_1', \pi^*(\omega_T)$ and then smoothly continued up until the next singular time. The general picture is that the flow deforms the metric spaces (X, ω_t) to this potentially singular Kähler space, then performs a surgery and continues. This is the basic mechanism behind the metric version of the minimal model program proposed by Tian and Song [53].

Returning to the Hermitian case a natural question to ask is to what extent these results do generalize. A natural counterpart of the Kähler-Ricci flow to study is the so-called Chern-Ricci flow ([58]):

Definition 10.3. — *The Chern-Ricci flow is the flow of Hermitian metrics whose evolution equation reads*

$$\frac{\partial \omega}{\partial t} = -Ric^{BC}(\omega),$$

with Ric^{BC} denoting the Chern-Ricci form.

Observe that when ω is a Kähler form the Chern-Ricci flow coincides with the classical (unnormalized) Kähler-Ricci flow. A second observation is that whenever the first Bott-Chern class vanishes and we choose a Ricci flat metric ω_0 the Chern-Ricci flow reduces to the Monge-Ampère flow above.

It is expected that the behavior of the flow should detect subtle geometric information of the ambient manifold. Of course this requires studying the flow with *carefully chosen* initial metric. It is hence natural to ask which of the Kähler like conditions are preserved under the flow. Sadly both balanced and Gauduchon metrics are not in general preserved as a simple computation shows. It is however interesting to observe that the *pluriclosed* metrics as well as the metrics satisfying the unnamed condition studied by Guan-Li are preserved under the Chern-Ricci flow:

Observation 10.4. — *If ω_0 is either pluriclosed or satisfying $dd^c\omega_0 = 0$, $d\omega_0 \wedge d^c\omega_0 = 0$ then the same will hold for all the metrics along the flow.*

Indeed

$$\frac{\partial}{\partial t} dd^c\omega = dd^c \frac{\partial}{\partial t}\omega = dd^c(-Ric(\omega)) = 0.$$

The condition $d\omega \wedge d^c\omega = 0$ is preserved for an analogous reason.

The preservation of the pluriclosedness is especially interesting in complex dimension two, because then it coincides with the Gauduchon condition. Thus given any metric there is a conformal factor turning it into a Gauduchon one and running the flow with the Gauduchon metric as an initial condition will give us a flow of Gauduchon metrics.

The surface case was studied in [58]. Note that on cohomological level (in the Bott-Chern $\partial\bar{\partial}$ cohomology) the right hand side of the flow equation satisfies

$$Ric^{BC}(\omega) \in c_1^{BC}(X),$$

the last term denoting the first Bott-Chern class (represented by Chern-Ricci forms). Thus exactly as in the Kähler case unless c_1^{BC} is of special type it is not expected that the flow will exist for all time. Hence the natural expectation is that in the case of finite time singularity (or after a renormalization at infinity) the flow should deform the manifold into some (possibly singular) complex space. It is extremely interesting to observe that the general Hermitian picture is **completely different**: The limiting space may not have any complex structure anymore!

The relevant example is taken from [58]: Consider the Inoue surface equipped with the Tricerri metric $\omega(z,w) := \frac{idw\wedge d\bar{w}}{Im^2(w)} + Im(w)idz \wedge d\bar{z}$. By explicit computation the solution of the flow with Tricerri initial condition reads (on $\mathbb{H} \times \mathbb{C}$)

$$\omega(t) = (1 + \frac{t}{4})\frac{1}{Im^2(w)}idw \wedge d\bar{w} + Im(w)idz \wedge d\bar{z}.$$

Clearly it exists for all time and after a renormalization we get the convergence $\frac{\omega(t)}{t} \to \frac{t}{4Im^2(w)}idw \wedge d\bar{w}$. The picture downastairs is considerably more complicated: in [58] the following result was proved:

Theorem 10.5. — *The manifold* $(\mathbb{H}\times\mathbb{C})/_G, \frac{\omega(t)}{t})$ *converges in the Gromov-Hausdorff sense to the circle* S^1 *or radius* $\frac{log\alpha}{2\sqrt{2}\pi}$.

In particular there is no complex structure on the limiting space!

Analogous phenomenon also happens in the Hopf surface case. It was observed in [58] that for the standard Hopf surface, namely the one associated with the contraction $(z_1, z_2) \to (\frac{1}{2}z_1, \frac{1}{2}z_2)$ equipped with the metric

$$\omega = \frac{idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2}{|z_1|^2 + |z_2|^2}$$

the evolution of the metrics on the covering space is given by

$$\omega(t) = \sum_{j,k=1}^{2} \frac{1}{|z_1|^2 + |z_2|^2}((1-2t)\delta_{jk} + 2t\frac{z_j \bar{z}_k}{|z_1|^2 + |z_2|^2})idz_j \wedge d\bar{z}_k.$$

Thus the existence time is up until $t = \frac{1}{2}$ and as t converges to $\frac{1}{2}$ the metrics converge to the nonnegative form

$$\sum_{j,k=1}^{2} \frac{z_j \bar{z}_k}{(|z_1|^2 + |z_2|^2)^2} idz_j \wedge d\bar{z}_k.$$

Once again downstairs on the Hopf manifold the convergence in the Gromov-Haudorff sense is towards the circle S^1 of radius $\frac{log(2)}{\sqrt{2}\pi}$.

Thus it is extremely interesting to study the limiting behavior of the potentials associated to the forms $\omega(t)$ in the Hermitian case. In particular the reason of the vanishing of the complex structure is still not well understood.

11. Open problems

We finish this survey by listing some open problems:

Problem 11.1. — *What is the necessary and sufficient condition on the metric so that the invariance of total volumes holds?*

The author is unaware of any example not satisfying the Guan-Li condition for which the invariance holds.

Problem 11.2. — *Are there some "better" versions of the comparison principle in the general sase? What is the optimal value of the constant that appears in the inequality?*

For metrics satisfying the condition studied by Guan-Li there are no error terms. It is thus likely that a satisfactory answer to this question may shed some light on the first problem.

Problem 11.3. — *What is the geometric/potential theoretic meaning of $cap_\omega(X)$? What if we choose a Gauduchon metric ω?*

For a generically chosen metric these constants will presumably not be of much significance. To the contrary for "canonical" metrics it is natural to expect that the total capacity should encode some potential theoretic information. In particular it would be interesting to know is there any link between these numbers and the constant B.

Problem 11.4. — *Is the constant appearing in the Monge-Ampère equation somehow related to the geometry of* (X, ω)*?*

In general understanding the role of these additional constants c, and especially establishing bounds on these under suitable degenerations of the right hand side is a key issue in many applications. In particular the Demailly mass concentration technique will crucially depend on that.

Problem 11.5. — *Suppose that the Chern-Ricci flow* $\frac{\partial \omega_t}{\partial t} = -Ric^{BC}(\omega_t)$ *for pluriclosed metrics has a finite time singularity. What is the limiting behavior of the associated potentials? Under what conditions the limiting space has a complex structure?*

Problem 11.6. — *Construct* ω *plurisubahrmonic functions with prescribed logarithmic singularities (that is for a collection of points* z_j *and weights* τ_j *find a function* ϕ *so that* $\phi(z) \leqslant \tau_j log(||z - z_j||) + O(1)$ *near* z_j*.) This was achieved in [57] under the assumption that the dimension of the underlying manifold is equal to* 2 *or* 3*.*

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