ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XVII, n° 1 (2008), p. 193-206.

<http://afst.cedram.org/item?id=AFST_2008_6_17_1_193_0>

Annales de la Faculté des Sciences de Toulouse Vol. XVII, n° 1, 2008
pp. 193–206

Poincaré-Hopf index and partial hyperbolicity(*)

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Abstract. — We use the theory of partially hyperbolic systems [HPS] in order to find singularities of index 1 for vector fields with isolated zeroes in a 3-ball. Indeed, we prove that such zeroes exists provided the maximal invariant set in the ball is partially hyperbolic, with volume expanding central subbundle, and the strong stable manifolds of the singularities are unknotted in the ball.

Résumé. — Nous utilisons des systèmes partiellement hyperboliques [HPS] pour trouver des singularités d'indice 1 pour les champs de vecteurs avec singularités isolées sur la boule tridimensionelle. En fait, on trouvera de telles singularités lorsque l'ensemble maximal invariant dans la boule est partiellement hyperbolique, à sous-fibré central volume-dilatant, et les variétés stables fortes sur les singularités sont toutes non nouées.

1. Introduction

The existence of fixed points of index 1 for homeomorphisms has been considered elsewhere in the literature. For example, by work of Eliashberg [E] every volume-preserving diffeomorphism homotopic to the identity of a closed, oriented surface of genus 0 with zero Calabi class has at least one fixed point of index 1. Dancer and Ortega proved in [DO] that every stable isolated fixed point of a local homeomorphism of $\mathbb{R}^2$ has index 1 (a result which is false in $\mathbb{R}^n, n \geqslant 3$). Le Calvez [L] proved that the index of every isolated fixed point of an orientation-preserving homeomorphisms without wanderting points on an orientable surface must be less than or equal to

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Partially supported by CNPq, FAPERJ and PRONEX/DYN-SYS. from Brazil.

one. Franks [F] studied C^1 area-preserving diffeomorphisms f isotopic to the identity and with zero mean rotation vectors for some admisible lift $\tilde{f}$ on closed orientable surfaces of genus g. He proved that if the projected image $Fix(f, \tilde{f})$ of the fixed points of $\tilde{f}$ is finite, then there are at least two fixed points of index 1 in $Fix(f, \tilde{f})$. Matsumoto [M] generalized Franks's to the homeomorphism case.

In this paper we consider the existence of singularities of index 1 for vector fields X in a neighborhood of a 3-ball B inwardly transverse to the boundary.

To motivate our hypotheses let us mention that if X_t denote the flow of X and the maximal invariant set $\bigcap_{t\geqslant 0} X_t(B)$ of X in B is a hyperbolic set of X ([HK]), then no such singularities exist since the maximal invariant set reduces to a single equilibrium of index -1. Alternatively we can assume that the maximal invariant set is partially hyperbolic according to the definition below [HPS]:

Definition 1.1. — *A compact invariant set* Λ *of a* C^1 *vector field* X *defined in a manifold* M *is* partially hyperbolic *if there are an invariant splitting* $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$*, with* $E^s_x, E^u_x \neq 0$ *for all* $x \in \Lambda$*, and positive constants* K, λ *such that:*

1. E^s_Λ is contracting*, i.e.,*

$$\| DX_t/E^s_x \| \leqslant Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$

2. E^s_Λ dominates E^c_Λ*, i.e.,*

$$\| DX_t/E^s_x \| \cdot \| DX_{-t}/E^c_{X_t(x)} \| \leqslant Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$

However this hypothesis does not work too because of the following counterexample (see Figure 1):

Example 1.2. — There is a C^∞ vector field X in B inwardly transverse to the boundary such that $\bigcap_{t\geqslant 0} X_t(B)$ is partially hyperbolic but X has no singularities of index 1 in B.

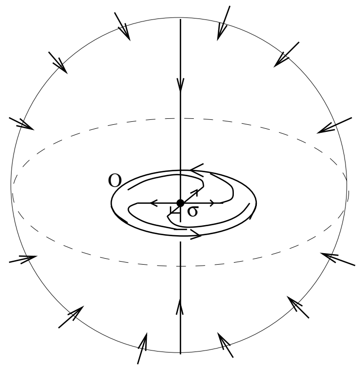

Figure 1

Therefore we need extra hypotheses even if the maximal invariant set in the ball is partially hyperbolic. The ones we shall handle here are related to the central subbundle of the maximal invariant set and the strong stable manifolds of the singularities in B. More precisely, we shall assume that they are *volume expanding* and *unknotted* respectively. Example 1.2 shows that not only the unknotted but also the volume expanding condition is necessary to obtain the result. See [MPP2] where the volume expanding condition is used to characterize robustly transitive sets for three-dimensional vector fields is given. Let us present our result in a precise way.

Consider a compact manifold M with boundary ∂M (possibly empty). Denote by X a C^1 vector field in M whose flow X_t is inwardly transverse to ∂M (if non-emtpy).

It follows from the Invariant Manifold Theory [HPS] that if Λ is a partially hyperbolic set of X then through each point $x \in \Lambda$ passes a unique *strong stable manifold* $W^{ss}_X(x)$ tangent at x to the subspace E^s_x. These manifolds are invariant in the sense that $X_t(W^{ss}_X(x)) = W^{ss}_X(X_t(x))$ for all $(x,t) \in \Lambda \times \mathbb{R}$. In particular, $W^{ss}_X(\sigma)$ is formed by solutions of X when $\sigma \in \Lambda$ is a singularity of X. Note that the dimension of $W^{ss}_X(x)$ is precisely the dimension of E^s_x for all $x \in \Lambda$.

The next definition is motivated by the definition of cube with knotted hole ([BMo] p. 218) and the definition of trivially embedded stable separatrices ([GMZ] p. 980). We denote by ∂A the boundary of A. A curve is called *simple* if it has no self-intersection points.

Definition 1.3. — *Let c be a simple* non-closed *compact curve in a* 3-*ball B satisfying $\partial B \cap c = \partial c$. We say that c is* unknotted in B *if there is* *a simple compact curve $\beta \in \partial B$ with $\partial\beta = \partial c$ such that the simple* closed *curve $\beta \cup c$ is unknotted in B (see Figure 2).*

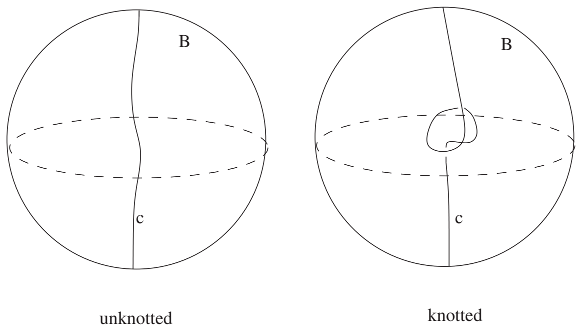

Figure 2

We shall use this definition in the following context. Let X be a C^1 vector field defined in a neighborhood of a 3-ball B such that $\bigcap_{t\geqslant 0} X_t(B)$ is a partially hyperbolic set with one-dimensional contracting subbundle E^s. Then each strong stable manifold $W^{ss}_X(x)$ is a one-dimensional submanifold and if $\sigma \in B$ is a singularity of X satisfying

$$\left(\bigcap_{t\geqslant 0} X_t(B)\right) \cap W^{ss}_X(\sigma) = \{\sigma\}, \tag{1.1}$$

then $c = W^{ss}_X(\sigma) \cap B$ is a simple closed curve satisfying $\partial B \cap c = \partial c$.

With this in mind we can state our third definition.

Definition 1.4. — *Let X be a C^1 vector field defined in a neighborhood of a 3-ball B inwardly transverse to ∂B such that $\bigcap_{t\geqslant 0} X_t(B)$ is a partially hyperbolic set with one-dimensional contracting subbundle E^s. We say that X has* unknotted singular manifolds *in B if for every singularity σ of X satisfying (1.1) the curve $c = W^{ss}_X(\sigma) \cap B$ is unknotted in B.*

The motivation for the definition above comes from the following example.

Example 1.5. — There is a C^1 vector field X defined in a neighborhood of a 3-ball B having a hyperbolic singularity $\sigma \in B$ with one-dimensional stable manifold $W^s_X(\sigma)$ (hence of index -1) such that $\left(\bigcap_{t\geqslant 0} X_t(B)\right) \cap W^s_X(\sigma) = \{\sigma\}$ and $W^s_X(\sigma) \cap B$ is *not* unknotted in B.

This example can be constructed in the following way: Take the vector field in Figure 3-(a) (which is that of Example 1.2) and the small tubular neighborhood described in Figure 3-(b). Remove this neighborhood from the ball and inserts the tubular flow depicted in Figure 1 p. 26 of [C] (or in Figure 3-(c)) instead. The resulting vector field in Figure 3-(d) is the one in Example 1.5.

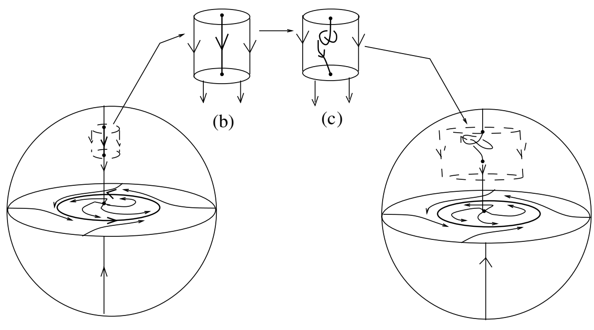

Figure 3

Our last definition is the following.

Definition 1.6. — *Let* Λ *be a partially hyperbolic set of* X. *We say that the central subbundle* E^c_Λ *above is* volume expanding *if the constants* K, λ *in Definition 1.1 satisfies the following additional property:*

$$| J(DX_t/E^c_x) | \geqslant K^{-1}e^{\lambda t}, \quad \forall x \in \Lambda,\ \forall t > 0,$$

where $J(L)$ *denotes the jacobian of a linear operator* L.

Examples of partially hyperbolic sets with volume expanding central subbundle are the non-trivial hyperbolic sets and the *geometric Lorenz attractor* ([ABS], [GW]). More examples can be found elsewhere [B], [M1], [MPP1], [MPu]. See Chapter 9 in [BDV] for some background. It is not difficult to see that if $\bigcap_{t\geqslant 0} X_t(B)$ is a partially hyperbolic with volume expanding central subbundle, then the contracting subbundle E^s is one-dimensional and so Definition 1.4 applies.

With the above definitions in mind we can state our main result.

Theorem. — *Let* X *be a* C^1 *vector field with isolated singularities in a* 3*-ball* B *inwardly transverse to* ∂B. *If* $\bigcap_{t\geqslant 0} X_t(B)$ *is a partially hyperbolic* *set with volume expanding central subbundle and X has unknotted singular manifolds in B, then X has a singularity of index 1 in B.*

Let us present an example where the hypotheses of the Theorem are satisfied. It is a minor modification of the geometric Lorenz attractor (compare with [GT] p. 2).

Example 1.7. — Let X be the C^∞ vector field in $\mathbb{R}^3$ depicted in Figure 4. Then, X is inwardly transverse to the boundary of B, $\bigcap_{t\geqslant 0} X_t(B)$ is partially hyperbolic with volume expanding central subbundle and X has unknotted singular manifolds in B.

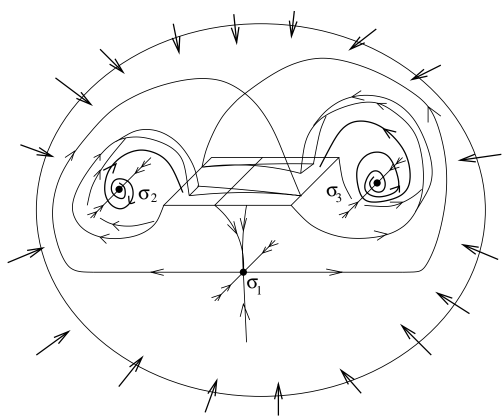

Figure 4

A natural question is if the conclusion of the Theorem holds without the unknotted assumption. Note that Example 1.5 does not give negative answer for such a question because $\bigcap_{t\geqslant 0} X_t(B)$ in that example may not be partially hyperbolic. Indeed, $\bigcap_{t\geqslant 0} X_t(B)$ intersects the tubular neighborhood in Figure 3-(c) due to the Wazewski Principle (see p. 26 in [C]). In particular, we don't know if the partially hyperbolicity of $\bigcap_{t\geqslant 0} X_t(B)$ with one-dimensional contracting subbundle E^s does imply that X has unknotted singular manifolds in B.

This paper is organized as follows. In Section 2 we give two lemmas related to dynamical systems in the 3-ball and the solid torus. In Section 3 we give some properties of the singular-hyperbolic sets introduced in [MPP2]. In Section 4 we prove the Theorem using the results in sections 2 and 3.

2. Some dynamics on the 3-ball and the solid torus

In this section we prove two lemmas concerning the dynamics in the 3-ball and the solid torus. We start with some basic notations and definitions.

The interior and the boundary of a set A will be denoted by $Int(A)$ and ∂A respectively.

Let X be a vector field with flow X_t on a manifold M. The *omega-limit set* of a point p is the set $\omega(p)$ defined by:

$$\omega_X(p) = \left\{ x : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}.$$

The *α-limit set* $\alpha_X(p)$ of p is the ω-limit set of p with respect to the time-reversed vector field $-X$. A compact invariant set A of X is an *attracting set* if it has an *isolating block*, i.e. a compact neighborhood U of it such that

$$A = \bigcap_{t\geqslant 0} X_t(U).$$

It follows from the definition that if U is an isolating block of an attracting set A, then $\omega_X(x) \subset A$ for all $x \in U$.

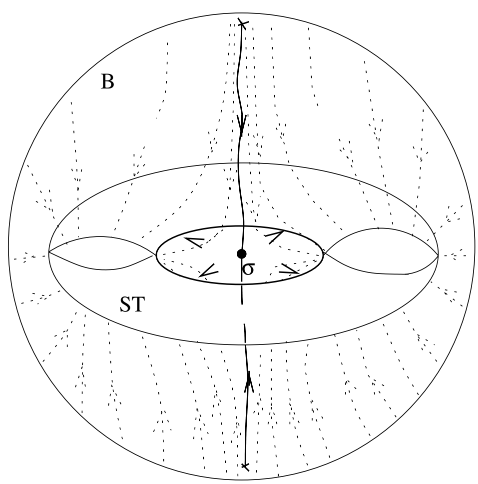

Figure 5

A compact invariant set H of X is *hyperbolic* if there is a continuous invariant splitting $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ over H consisting of a contracting subbundle E^s_H, an expanding subbundle E^u_H and the flow direction E^X_H. The

Stable Manifold Theory [HPS] asserts that through each point $x \in H$ there are a stable manifold $W^s_X(x)$, tangent to $E^s_x \oplus E^X_x$, and an unstable manifold $W^u_X(x)$, tangent to $E^X_x \oplus E^u_x$. An *attracting closed orbit* is a closed orbit O which is a hyperbolic set with zero dimensional E^u (equivalently O is a hyperbolic closed orbit which is also an attracting set).

The lemma below allows us to construct invariant solid torus using the unknotted assumption in the Theorem.

LEMMA 2.1. — *Let X be a C^1 vector field in a 3-ball B inwardly transverse to the boundary. Assume also that X has a unique singularity σ in B which is hyperbolic of index -1 (thus $dim(W^s_X(\sigma)) = 1$). If $(\bigcap_{t\geqslant 0} X_t(B)) \cap W^s_X(\sigma) = \{\sigma\}$ and $W^s_X(\sigma)$ is unknotted in B, then there is a solid torus $ST \subset Int(B)$ such that X is inwardly transverse to $\partial(ST)$ and X has no singularities in ST.*

Proof. — Since $(\bigcap_{t\geqslant 0} X_t(B)) \cap W^s_X(\sigma) = \{\sigma\}$ we have that the separatrices of $W^s_X(\sigma) \setminus \{\sigma\}$ exit B in the past as in Figure 5. Then, by using the flow of X we can construct a torus T transverse to X in the interior of B by removing an small tubular neighborhood in B of the curve $c = W^s_X(\sigma) \cap B$. Note that T is the boundary of a compact manifold ST contained in the interior of B. Moreover, X points inward to ST in $T = \partial(ST)$. The hypothesis that $W^s_X(\sigma)$ is unknotted in B implies that ST is a solid torus. The result follows. □

The second lemma allows us to construct attracting periodic orbits from invariant solid torus.

LEMMA 2.2. — *Let Z be a C^1 vector field defined in a neighborhood of a solid torus ST inwardly transverse to the boundary. If $\bigcap_{t\geqslant 0} Z_t(ST)$ is a hyperbolic set of Z, then $\bigcap_{t\geqslant 0} Z_t(ST)$ is an attracting periodic orbit of Z.*

Proof. — Denote $H = \bigcap_{t\geqslant 0} Z_t(ST)$. To prove the result it suffices to prove that the unstable subbundle E^u_H of H is zero dimensional. Assume by contradiction that this is not so. Then, $E^u_x \neq 0$ for all $x \in H$ since H is connected. As H is also an attracting set we have that $dim(E^s_x) \neq 0$ for all $x \in H$ as well. On the other hand, there is no singularity in H (since the hyperbolic splitting is continuous) therefore $dim(E^s_x) = 1$ for all $x \in \bigcap_{t\geqslant 0} Z_T(ST)$. Then, the stable manifolds $\{W^s_Z(x)\}_{x\in H}$ induce a codimension one foliation $\mathcal{F}$ on ST transverse to $\partial(ST)$.

Next we apply an argument in [B] based on the following definition: A *half-Reeb component* of $\mathcal{F}$ is a saturated subset $H \subset ST$, bounded by an annulus leaf A and an annulus $K \subset \partial(ST)$ with $\partial K = \partial A$, such that the double $2H$ is a Reeb component [G] of the double foliation $2\mathcal{F}$ (see Figure 6).

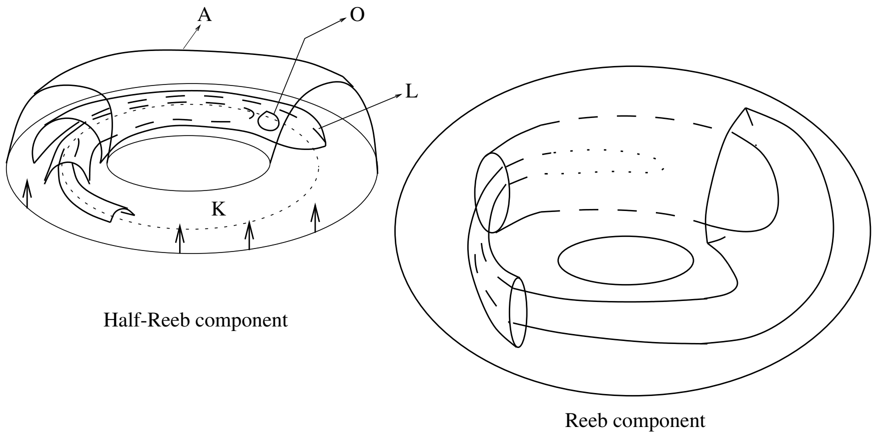

Figure 6

We claim that $\mathcal{F}$ has neither Reeb nor half-Reeb components. Indeed, since $\mathcal{F}$ is induced by stable manifolds we have that $\mathcal{F}$ has no compact leaves in $Int(ST)$. Consequently $\mathcal{F}$ has no Reeb components. Now suppose by contradiction that there is a half-Reeb component H of $\mathcal{F}$. Let A, K be the boundary annuli of H with $K \subset \partial(ST)$. Pick $x \in Int(H)$. Note that the positive trajectory of x does not intersect A. As Z points inward to ST as indicated in Figure 6 we have that $\omega_Z(x) \subset Int(H)$. Since $\bigcap_{t\geqslant 0} Z_t(ST)$ is an attracting set with isolating block ST of Z and $x \in ST$ we have $\omega_Z(x) \subset \bigcap_{t\geqslant 0} Z_t(ST)$.

Now, $\omega_Z(x)$ is contained in $\bigcap_{t\geqslant 0} Z_t(ST)$ which is a hyperbolic set. By using the orbit of x we can construct a periodic pseudo-orbit close to $\omega_Z(x)$. By the Shadowing Lemma for flows (Theorem 18.1.6 p. 569 in [HK]) we have that such a pseudo-orbit is shadowed by a periodic orbit $O \subset Int(H)$. We have that O is contained in a leaf L of $\mathcal{F}$ and $L \neq A$. The last property implies that L is a half-plane, and so, it is simply connected as well. Consequently, O bounds a disk in L. Applying the Poincaré-Bendixon Theorem [PdM] to this disk we could find a singularity in H which is absurd. This contradiction proves the claim.

Now we finish the proof of the lemma. Take the double foliation $2\mathcal{F}$ defined on the double manifold $M = 2ST$. On the one hand, ST is a solid torus so M is diffeomorphic to $S^2 \times S^1$. Consequently, $\pi_2(M) \neq 0$. On the other hand, the claim says that $\mathcal{F}$ has neither Reeb nor half-Reeb components. Therefore, $2\mathcal{F}$ has no Reeb components. Then, standard results in foliation theory (e.g. Theorem 1.10-(iii) p. 92 in [G]) imply that $2\mathcal{F}$ is the product foliation $S^2 \times *$ of $M = S^2 \times S^1$. Then, $\mathcal{F}$ is the product foliation $D \times *$

by meridian disks on ST, and so, the leaves of $\mathcal{F}$ are invariant disks. But applying Poincare-Bendixon's to one of such disks as before we could find a singularity of Z in $Int(ST)$ which is absurd. This contradiction proves the result. $\square$

3. Some singular-hyperbolic dynamics

The results of this section resemble ones in [BMo]. Let X be a C^1 vector field defined in a 3-manifold. A compact invariant subset of X is called *singular-hyperbolic* if it is partially hyperbolic with volume expanding central subbundle and its singularities are hyperbolic. A compact invariant subset without singularities of a singular-hyperbolic set is hyperbolic and satisfies $E^s_x \neq 0$ and $E^u_x \neq 0$ for all x on it [BDV]. It follows that a singular-hyperbolic set has no attracting closed orbits. We denote by $Sing(X)$ the set of singular points of X.

In this section Λ denotes a connected singular-hyperbolic set of X and $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ denotes the corresponding partially hyperbolic splitting.

LEMMA 3.1. — *$X(x) \notin E^s_x$ for every $x \in \Lambda \setminus Sing(X)$.*

Proof. — Suppose by contradiction that $X(x_0) \in E^s_{x_0}$ for some $x_0 \in \Lambda \setminus Sing(X)$. The invariance of E^s implies $X(x) \in E^s_x$ for all x in the orbit of x_0. So, $X(x) \in E^s_x$ for all $x \in \alpha_X(x_0)$ since E^s is continuous. As E^s is contracting we conclude that $\omega_X(x)$ is a singularity for all $x \in \alpha_X(x_0)$. In particular, $\alpha_X(x_0)$ contains a singularity σ which is necessary saddle-type (i.e. neither attracting nor repelling).

Now we arrive to a contradiction depending on whether $\alpha_X(x_0) = \{\sigma\}$ or not. In the first case (i.e. $\alpha_X(x_0) = \{\sigma\}$) we would have $x_0 \in W^u_X(\sigma)$. Define the unitary vectors

$$v^t = \frac{DX_t(x_0)(X(x_0))}{\| DX_t(x_0)(X(x_0)) \|}, \qquad \forall t \in \mathbb{R}.$$

It follows that

$$v^t \in T_{X_t(x_0)} W^u_X(\sigma) \cap E^s_{X_t(x_0)}, \qquad \forall t \in \mathbb{R}.$$

Take a sequence $t_n \to \infty$ such that the sequence v^{-t_n} converges to v^∞ (say). Clearly v^∞ is an unitary vector. As $X_{-t}(x_0) \to \sigma$ and E^s is continuous we obtain

$$v^\infty \in T_\sigma W^u_X(\sigma) \cap E^s_\sigma.$$

Therefore v^∞ is an unitary vector which is simultaneously expanded and contracted by DX_t a contradiction.

In the second case (i.e. $\alpha_X(x_0) \neq \{\sigma\}$) we would have $(W^u_X(\sigma) \setminus \{\sigma\}) \cap \alpha_X(x_0) \neq \emptyset$. Pick $x_1 \in (W^u_X(\sigma) \setminus \{\sigma\}) \cap \alpha_X(x_0)$. As $x_1 \in \alpha_X(x_0)$ we have $X(x_1) \in E^s_{x_1}$ and then we get a contradiction as in the first case replacing x_0 by x_1. The lemma is proved. $\square$

We denote by $Sing(X)$ the set of singularities of X.

Lemma 3.2. — *$X(x) \in E^c_x$ for every $x \in \Lambda$.*

Proof. — Fix an open neighborhood U of $Sing(X) \cap B$. Therefore $\Lambda \setminus U$ is compact, and so, Lemma 3.1 implies that the angle between E^s_x and $X(x)$ is uniformly bounded away from 0 for $x \in \Lambda \setminus U$. On the other hand, if we take U formed by linearizing coordinates around the singularities in $Sing(X) \cap B$, we can see that the angle between E^s_x and $X(x)$ is also uniformly bounded away from 0 for $x \in \Lambda \cap U$. We conclude that the angle between E^s_x and $X(x)$ is uniformly bounded away from 0 for all $x \in \Lambda \setminus Sing(X)$.

Now take $x \in \Lambda$. If $t > 0$ we define $v_t = X(X_{-t}(x))$. The previous conclusion implies that the angle between v_t and $E^s_{X_{-t}(x)}$ is bounded away from zero. It then follows from the dominance of E^s over E^c in Definition 1.1-(2) that the angle between $DX_t(X_{-t}(x))(v_t)$ and E^c_x goes to 0 as $t \to \infty$. As $X(x) = DX_t(X_{-t}(x))(v_t)$ we conclude that $X(x) \in E^c_x$. As $x \in \Lambda$ is arbitrary we obtain the result. $\square$

Lemma 3.3. — *$\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$ for every singularity $\sigma \in \Lambda$ of X.*

Proof. — By contradiction suppose that there is $x_0 \in \Lambda \cap (W^{ss}_X(\sigma) \setminus \{\sigma\})$. As before we define for all $t \in \mathbb{R}$ the unitary tangent vector

$$v^t = \frac{DX_t(x_0)(X(x_0))}{\| DX_t(x_0)(X(x_0)) \|}.$$

Since $W^{ss}_X(\sigma)$ is invariant and $x_0 \in W^{ss}_X(\sigma)$ we get $v^t \in T_{X_t(x_0)} W^{ss}_X(\sigma)$ for all t. On the other hand, Lemma 3.1 implies that $v^t \in E^c_{X_t(x_0)}$ for all t. Therefore

$$v^t \in T_{X_t(x_0)} W^{ss}_X(\sigma) \cap E^c_{X_t(x_0)}.$$

By taking limit as $t \to \infty$ as before we would obtain an unitary vector

$$v^\infty \in T_\sigma W^{ss}_X(\sigma) \cap E^c_\sigma.$$

But $T_\sigma W^{ss}_X(\sigma) = E^s_\sigma$ so

$$v^\infty \in E^s_\sigma \cap E^c_\sigma$$

which is absurd since v^{∞} is unitary (hence non-zero) and the sum $E^s_\sigma \oplus E^c_\sigma$ is direct. This contradiction proves the lemma. $\square$

4. Proof of the Theorem

Let X be a C^1 vector field with isolated zeroes in a 3-ball B inwardly transverse to the boundary. Assume that $\bigcap_{t\geqslant 0} X_t(B)$ is partially hyperbolic with volume expanding central subbundle and that X has unknotted singular manifolds in B. *Assume by contradiction that X has no singularities of index 1 in B.*

It follows from the volume expanding condition on the central subbundle of $\bigcap_{t\geqslant 0} X_t(B)$ that each singularity of X in B is hyperbolic or saddle-node (i.e. 1 is its unique eigenvalue of modulus 1). As is well known a saddle-node singularity disappears after an small perturbation. Therefore, by making a small perturbation if necessary, we can assume that each singularity of X in B is hyperbolic none of which has index 1. Since B has Euler number 1 and X points inward in ∂B we have from Poincaré-Hopf (e.g. [CMV], [Mi]) that X has only one singularity σ in B which has index -1.

On the other hand, we have that $\Lambda = \bigcap_{t\geqslant 0} X_t(B)$ is partially hyperbolic with volume expanding central subbundle by assumption. As σ is the sole singularity of Λ and σ is hyperbolic, we conclude that Λ is a singular-hyperbolic set. Note that Λ is also connected since B also is. Therefore $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$ by Lemma 3.3. But σ has index -1 so $W^{ss}_X(\sigma) = W^s_X(\sigma)$ therefore

$$\left(\bigcap_{t\geqslant 0} X_t(B)\right) \cap W^s_X(\sigma) = \{\sigma\}.$$

We also have that $W^s_X(\sigma) = W^{ss}_X(\sigma)$ is unknotted in B since X has unknotted singular manifolds. Therefore, by Lemma 2.1, there is a solid torus $ST \subset Int(B)$ such that X is inwardly transverse to $\partial(ST)$ and X has no singularities in ST. In particular, ST is positively invariant.

Note that $\bigcap_{t\geqslant 0} X_t(ST) \subset \bigcap_{t\geqslant 0} X_T(B)$ which is singular-hyperbolic. Since X has no singularities in ST we conclude that $\bigcap_{t\geqslant 0} X_t(ST)$ is hyperbolic. Then, by Lemma 2.2 applied to $Z = X$, we would have that $\bigcap_{t\geqslant 0} X_t(ST)$ is an attracting periodic orbit.

However, this is absurd since a singular-hyperbolic set has no attracting periodic orbits. This contradiction proves the result.

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