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# ENVELOPING ALGEBRAS OF PRE-LIE ALGEBRAS OF ROOTED TREES

MOHAMED BELHAJ MOHAMED

**Abstract.** In this article, we study the insertion pre-Lie algebra of rooted trees $(\mathcal{T}, \rhd)$ and we construct a pre-Lie structure on its doubling space $(\tilde{V}, \blacktriangleright)$. We prove that $\tilde{V}$ is a left pre-Lie module on $\mathcal{T}$. Moreover, we describe the enveloping algebras of the two pre-Lie algebras denoted respectively by $(\mathcal{K}, \diamondsuit, \Upsilon)$ and $(\mathcal{W}, \blacklozenge, \Theta)$ and we show that $(\mathcal{W}, \blacklozenge, \Theta)$ is a module-bialgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$. Finally, we find some relations between the enveloping algebras of the insertion and the grafting pre-lie algebras of rooted trees.

## Contents



## 1. Introduction

The insertion pre-Lie algebra was studied for the first time by A. Connes and D. Kreimer [6, 7] in the context of Feynman graph and thereafter by F. Chapoton and M. Livernet [5] to study the pre-Lie operad, and by A. Dzhumadl'daev, C. Löfwall [8], L. Foissy [9], D. Manchon and A. Saidi [12] in the context of rooted trees.

D. Calaque, K. Ebrahimi-Fard and D. Manchon [3] have studied the Hopf algebra of rooted forest $\mathcal{H}$ where the coproduct $\Delta$ is given by contraction of trees:

$$\Delta(t) = \sum_{s \subseteq t} s \otimes t/s.$$

They showed that the primitive part of the graded dual of this right sided Hopf algebra is endowed with a pre-Lie product defined by insertion of a tree inside another. They also established a relation between the Hopf algebra $\mathcal{H}$ and the Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}_{CK}$ by means of a $\mathcal{H}$-bicomodule structure on $\mathcal{H}_{CK}$.

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*Keywords:* Insertion pre-Lie algebras, Rooted Forest, Bialgebras, Enveloping algebras of pre-Lie algebras, Module-bialgebras.

In a joint work with Dominique Manchon [1], we have studied the doubling of the two bialgebras of rooted trees $\mathcal{H}$ and $\mathcal{H}_{CK}$, and we have established relations similar to those found by D. Calaque, K. Ebrahimi-Fard and D. Manchon in [3].

We have studied in [2] the enveloping algebra of the grafting pre-Lie algebra of rooted trees $(\mathcal{H}', \star, \Gamma)$ and another enveloping algebra of pre-Lie algebra structure on its doubling space denoted by $(\mathcal{D}', \bigstar, \chi)$, and we have proved that $(\mathcal{D}', \bigstar, \chi)$ is a module-bialgebra on $(\mathcal{H}', \star, \Gamma)$.

In this work, we study the insertion pre-Lie algebra of rooted trees. The pre-Lie product $\triangleright$ is defined for all $t, s \in \mathcal{T}$, by:

$$t \triangleright s = \sum_{v \in \mathcal{V}(s)} t \triangleright_v s.$$

We consider the vector space $\tilde{V}$ spanned by the couples $(t, s)$, where $t$ is a tree and $s$ is a subforest of $t$. We define a pre-Lie product $\blacktriangleright$ on $\tilde{V}$ by:

$$(t_1, s_1) \blacktriangleright (t_2, s_2) := \sum_{v \in \mathcal{V}(t_2 - s_2)} (t_1 \triangleright_v t_2, s_1 s_2).$$

Thereafter, we use the method of Oudom and Guin to construct the associated enveloping algebras of the two pre-Lie algebras $\mathcal{T}$ and $\tilde{V}$ denoted by $(\mathcal{K}, \diamondsuit, \Upsilon)$ and $(\mathcal{W}, \blacklozenge, \Theta)$ respectively. We prove that $\tilde{V}$ is a left pre-Lie module on $\mathcal{T}$ and we find some relations between the two pre-Lie structures defined on $\tilde{V}$ and $\mathcal{T}$. Also we show that $(\mathcal{W}, \blacklozenge, \Theta)$ is a module-bialgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$.

In the last section, we give results relating the grafting pre-Lie structures studied in [2] and the insertion pre-Lie structures defined in this article. More precisely, we show that $(\mathcal{H}', \star, \Gamma)$ is a module-coalgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$, which results in the commutativity of this diagram:

$$\begin{array}{ccc}
\mathcal{H}' \otimes \mathcal{K} & \xrightarrow{\quad\gamma\quad} & \mathcal{H}' \\
\Big\downarrow{\scriptstyle I \otimes \Upsilon} & & \Big\downarrow{\scriptstyle \Gamma} \\
\mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{K} & & \mathcal{H}' \otimes \mathcal{H}' \\
\Big\downarrow{\scriptstyle \Gamma \otimes I \otimes I} & & \Big\uparrow{\scriptstyle \gamma \otimes \gamma} \\
\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{K} & \xrightarrow[\tau^{23}]{\qquad} & \mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{H}' \otimes \mathcal{K}
\end{array}$$

where $\gamma$ is an action of $\mathcal{H}'$ on $\mathcal{K}$, defined for all $t_1 \in \mathcal{H}'$ and $t_2 \in \mathcal{K}$ by:

$$\gamma(t_1 \otimes t_2) = t_1 \star t_2.$$

In addition, if we define a map: $\rho : \mathcal{K} \longrightarrow \mathcal{K} \otimes \mathcal{K}$ by: $\rho(t) = \mathbf{1} \otimes t$, we show that $\gamma$ and $\rho$ satisfy the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{K} & \xrightarrow{\star \otimes I} & \mathcal{H}' \otimes \mathcal{K} \\
\downarrow{\scriptstyle I\otimes I \otimes \rho} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{K} & & \mathcal{H}' \\
\downarrow{\scriptstyle \tau^{23}} & & \uparrow{\scriptstyle \star} \\
\mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{H}' \otimes \mathcal{K} & \xrightarrow[\gamma \otimes \gamma]{} & \mathcal{H}' \otimes \mathcal{H}'
\end{array}$$

The commutativity of the last diagram is similar to the fact that $(\mathcal{H}', \star, \Gamma)$ is a module-algebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$. The difference between them is that the map $I \otimes I \otimes \Upsilon$ is replaced by the map $I \otimes I \otimes \rho$.

Similarly, we show that $(\mathcal{D}', \bigstar, \chi)$ and $(\mathcal{W}, \blacklozenge, \Theta)$ satisfy the same results obtained for $(\mathcal{H}', \star, \Gamma)$ and $(\mathcal{K}, \diamondsuit, \Upsilon)$. The two maps $\gamma$ and $\rho$ are replaced here by $\alpha$ and $\sigma$ defined as follows:

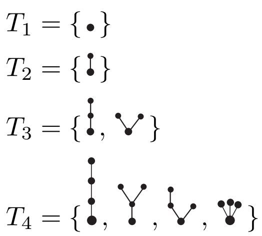
$$\begin{array}{ll}
\alpha : \mathcal{D}' \otimes \mathcal{W} \to \mathcal{D}', & \sigma : \mathcal{W} \to \mathcal{W} \otimes \mathcal{W} \\
(t_1, s_1) \otimes (t_2, s_2) \mapsto (t_1, s_1) \bigstar (t_2, s_2) & (t, s) \mapsto (\mathbf{1}, \mathbf{1}) \otimes (t, s).
\end{array}$$

The Connes-Kreimer Hopf algebra and the Hopf algebra of D. Calaque, K. Ebrahimi-Fard and D. Manchon of rooted trees are two important examples of Hopf algebras in the study of renormalization in quantum field theory. In the same context, we try to find some relations connecting these two Hopf algebras and others resulting from the insertion and the grafting pre-Lie algebras.

## 2. Hopf algebra of rooted forests

A *rooted tree* is a finite connected simply connected oriented graph such that every vertex has exactly one incoming edge, except for a distinguished vertex (the root) which has no incoming edge. The set of rooted trees is denoted by $T$ and the set of rooted trees with $n$ vertices is denoted by $T_n$.

*Example 2.1.* —




Let $\mathcal{T}$ be the vector space spanned by the elements of $T$ and $\tilde{\mathcal{H}} = S(\mathcal{T})$ be the algebra of rooted trees. D. Calaque, K. Ebrahimi-Fard and D. Manchon showed that the space $\tilde{\mathcal{H}}$ generated by the rooted forests, graded according to the number of edges, admits a structure of graded bialgebra [3]. The unit is the empty forest, the product is the concatenation, and the coproduct is defined for any non empty forest $t$ by:

$$\Delta(t) = \sum_{s \subseteq t} s \otimes t/s,$$

where $s$ is a covering subforest of a rooted tree $t$ and $t/s$ is the tree obtained by contracting each connected component of $s$ onto a vertex.

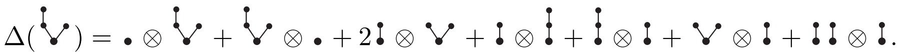
*Example 2.2.* — The coproduct $\Delta$ applied to the tree :

The Hopf algebra $\mathcal{H}$ is given by identifying all elements of degree zero to unit $\mathbf{1}$:

$$\mathcal{H} = \widetilde{\mathcal{H}}/\mathcal{J} \tag{2.1}$$

where $\mathcal{J}$ is the ideal generated by the elements $\mathbf{1} - t$ where $t$ is a forest of degree zero.

The example of coproduct above becomes by identifying the unit to $\bullet$:



Let $\tilde{V}$ be the vector space spanned by the couple $(t, s)$ where $t$ is a tree, and $s$ is a subforest of $t$. We have defined in [1] the doubling of the bialgebra $\tilde{\mathcal{H}}$ by $\tilde{D} := S(\tilde{V})$ and the coproduct $\Lambda$ for all $(t, s) \in \tilde{D}$ by:

$$\Lambda(t, s) = \sum_{s' \subseteq s} (t, s') \otimes (t/s', s/s').$$

The unit $(\mathbf{1}, \mathbf{1})$ is identified to the couple of empty graphs, the counit $\varepsilon$ is given by $\varepsilon(t, s) = \varepsilon(s)$ and the graduation is given by the number of vertices of $s$:

$$|(t, s)| = |s|.$$

The product is given by:

$$(t, s)(t', s') = (tt', ss').$$

We showed that $\tilde{D}$ is a graded bialgebra. Moreover we have $\Lambda(\tilde{V}) \subset \tilde{V} \otimes \tilde{V}$, so we can restrict the coassociative product $\Lambda$ to $\tilde{V}$.

## 3. Enveloping algebra of pre-Lie algebra

In this section, we describe the method of Oudom and Guin [14] to find the enveloping algebra of a pre-Lie algebra.

Definition 3.1. — [4, 10] A left pre-Lie algebra over a field $k$ is a $k$-vector space $\mathcal{A}$ with a binary composition $\rhd$ that satisfies the left pre-Lie identity:

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c), \tag{3.1}$$

for all $a$, $b$, $c \in \mathcal{A}$. Analogously, a right pre-Lie algebra is a $k$-vector space $\mathcal{A}$ with a binary composition $\lhd$ that satisfies the right pre-Lie identity:

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b). \tag{3.2}$$

As any right pre-Lie algebra $(\mathcal{A}, \lhd)$ is also a left pre-Lie algebra with product $a \rhd b := b \lhd a$, we will only consider left pre-Lie algebras for the moment. The left pre-Lie identity rewrites as:

$$L_{[a,b]} = [L_a, L_b], \tag{3.3}$$

where $L_a : A \longrightarrow A$ is defined by $L_a b = a \rhd b$, and where the bracket on the left-hand side is defined by $[a, b] := a \rhd b - b \rhd a$. As a consequence this bracket satisfies the Jacobi identity.

Definition 3.2. — [14] Let $(A, \triangleright)$ be a pre-Lie algebra. We consider the Hopf symmetric algebra $\mathcal{S}(A)$ equipped with its usual coproduct $\Delta$. We extend the product $\triangleright$ to $\mathcal{S}(A)$. Let $a, b$ and $c \in \mathcal{S}(A)$, and $x \in A$. We put:

$$\begin{aligned}
\mathbf{1} \triangleright a &= a \\
a \triangleright \mathbf{1} &= \varepsilon(a)\mathbf{1} \\
(xa) \triangleright b &= x \triangleright (a \triangleright b) - (x \triangleright a) \triangleright b \\
a \triangleright (bc) &= \sum_a (a^{(1)} \triangleright b)(a^{(2)} \triangleright c).
\end{aligned}$$

On $\mathcal{S}(A)$, we define a product $\star$ by:

$$a \star b = \sum_a a^{(1)}(a^{(2)} \triangleright b).$$

Proposition 3.3. — [14] *Let $a, b$ and $c \in \mathcal{S}(A)$. We have:*

$$\varepsilon(a \triangleright b) = \varepsilon(a)\varepsilon(b) \tag{3.4}$$

$$\Delta(a \triangleright b) = \sum_{(a),(b)} (a^{(1)} \triangleright b^{(1)}) \otimes (a^{(2)} \triangleright b^{(2)}) \tag{3.5}$$

$$a \triangleright (b \triangleright c) = (a \star b) \triangleright c. \tag{3.6}$$

Theorem 3.4. — [14] *The product $\star$ is associative and $(\mathcal{S}(A), \star, \Delta)$ is a Hopf algebra.*

*Proof.* — The associativity of $\star$ follows from Definition 3.2 and Proposition 3.3, and the compatibility between $\Delta$ and $\star$ follows from formula (3.5). □

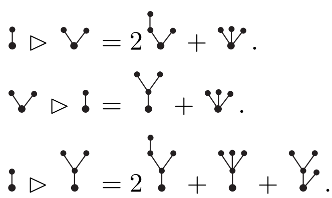
## 4. Insertion pre-Lie algebras of rooted trees

In this section, we study the insertion pre-Lie algebra of rooted trees. Let $\mathcal{T}$ be the vector space spanned by the elements of $T$. The product $\triangleright$ is defined for all $t, s \in \mathcal{T}$, by:

$$t \triangleright s = \sum_{v \in V(s)} t \triangleright_v s,$$

where $t \triangleright_v s$ is the tree obtained by inserting the root of $t$ on the vertex $v$ of $s$. More explicitly, the operation $t \triangleright s$ consists of inserting the root of $t$ on every vertex of $s$.

*Example 4.1.* —



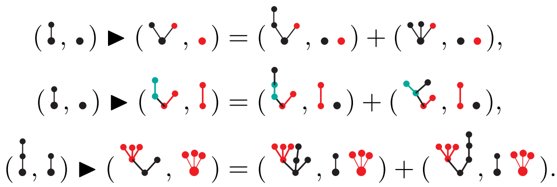
Theorem 4.2. — [3, 5] *Equipped with $\triangleright$, the space $\mathcal{T}$ is a pre-Lie algebra.*

Definition 4.3. — Let $(t_1, s_1)$ and $(t_2, s_2)$ be two elements of $\tilde{V}$. We define the map $\blacktriangleright$ by:

$$(t_1, s_1) \blacktriangleright (t_2, s_2) := \sum_{v \in \mathcal{V}(t_2 - s_2)} (t_1 \rhd_v t_2, s_1 s_2), \tag{4.1}$$

where the notation $v \in \mathcal{V}(t_2 - s_2)$ denotes that $v$ is a vertex of $t_2$ but it is not a vertex of $s_2$.

*Example 4.4.* —



Theorem 4.5. — *The pair $(\tilde{V}, \blacktriangleright)$ is a pre-Lie algebra.*

*Proof.* — Let $(t_1, s_1), (t_2, s_2)$ and $(t_3, s_3)$ be three elements of $\tilde{V}$, we have:

$$\begin{aligned}
&(t_1, s_1) \blacktriangleright \big[(t_2, s_2) \blacktriangleright (t_3, s_3)\big] - \big[(t_1, s_1) \blacktriangleright (t_2, s_2)\big] \blacktriangleright (t_3, s_3) \\
&\quad = (t_1, s_1) \blacktriangleright \Big( \sum_{v \in \mathcal{V}(t_3 - s_3)} (t_2 \rhd_v t_3, s_2 s_3)\Big) - \sum_{r \in \mathcal{V}(t_2 - s_2)} (t_1 \rhd_r t_2, s_1 s_2) \blacktriangleright (t_3, s_3) \\
&\quad = \sum_{\substack{v \in \mathcal{V}(t_3 - s_3) \\ r \in \mathcal{V}(t_2 \rhd_v t_3 - s_2 s_3)}} \big(t_1 \rhd_r (t_2 \rhd_v t_3), s_1 s_2 s_3\big) - \sum_{\substack{v \in \mathcal{V}(t_3 - s_3) \\ r \in \mathcal{V}(t_2 - s_2)}} \big((t_1 \rhd_r t_2) \rhd_v t_3, s_1 s_2 s_3\big) \\
&\quad = \sum_{\substack{r\,,\, v \in \mathcal{V}(t_3 - s_3) \\ r \neq v}} \big(t_1 \rhd_r (t_2 \rhd_v t_3), s_1 s_2 s_3\big) + \sum_{\substack{v \in \mathcal{V}(t_3 - s_3) \\ r \in \mathcal{V}(t_2 - s_2)}} \big(t_1 \rhd_r (t_2 \rhd_v t_3), s_1 s_2 s_3\big) \\
&\qquad\qquad - \sum_{\substack{v \in \mathcal{V}(t_3 - s_3) \\ r \in \mathcal{V}(t_2 - s_2)}} \big(t_1 \rhd_r (t_2 \rhd_v t_3), s_1 s_2 s_3\big) \\
&\quad = \sum_{\substack{r\,,\, v \in \mathcal{V}(t_3 - s_3) \\ r \neq v}} \big(t_1 \rhd_r (t_2 \rhd_v t_3), s_1 s_2 s_3\big).
\end{aligned}$$

The last term is symmetric on $(t_1, s_1)$ and $(t_2, s_2)$, which means that the product $\blacktriangleright$ is pre-Lie. □

## 5. Enveloping algebras of the insertion pre-Lie algebras of rooted trees

We consider the Hopf symmetric algebra $\mathcal{K} := \mathcal{S}(\mathcal{T})$ of the pre-Lie algebra $(\mathcal{T}, \rhd)$, equipped with its usual unshuffling coproduct $\Upsilon$. We extend the product $\rhd$ to $\mathcal{K}$ by the same method used in Definition 3.2 and we define a product $\diamondsuit$ on $\mathcal{K}$ by:

$$t \diamondsuit t' = \sum_{(t)} t^{(1)} (t^{(2)} \rhd t').$$

By Theorem 3.4, the product $\diamondsuit$ is associative, $\diamondsuit$ and $\Upsilon$ are compatible ($\Upsilon$ is a morphism of algebras) and consequently the space $(\mathcal{K}, \diamondsuit, \Upsilon)$ is a Hopf algebra. The unit $\mathbf{1}$ is the empty forest, the counit $\varepsilon_{\mathcal{K}}$ takes value 1 on the empty forest and 0 on the other elements of $\mathcal{K}$, and the graduation is given by the number of connected components of forest.

Similarly, we showed that $(\tilde{V}, \blacktriangleright)$ is a pre-Lie algebra, so we consider the Hopf symmetric algebra $\mathcal{W} := \mathcal{S}(\tilde{V})$ equipped with its usual unshuffling coproduct $\Theta$. We extend the product $\blacktriangleright$ to $\mathcal{W}$ by using Definition 3.2 and we define a product $\blacklozenge$ on $\mathcal{W}$ by:

$$(t,s)\blacklozenge(t',s') = \sum_{(t,s)} (t,s)^{(1)}\big((t,s)^{(2)} \blacktriangleright (t',s')\big).$$

By Theorem 3.4, the product $\blacklozenge$ is associative, $\blacklozenge$ and $\Theta$ are compatible and consequently the space $(\mathcal{W}, \blacklozenge, \Theta)$ is a Hopf algebra. The unit is $(\mathbf{1}, \mathbf{1})$, the counit $\varepsilon_{\mathcal{W}}$ takes value 1 on $(\mathbf{1}, \mathbf{1})$ and 0 on the other elements of $\mathcal{W}$, and the graduation is given by the number of connected components of $s$:

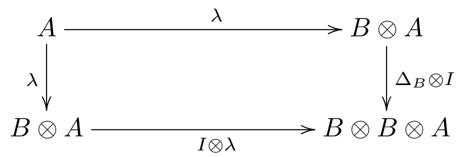
$$|(t,s)| = |s|.$$

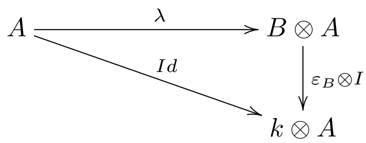
It is clear that, $|(t,s)\blacklozenge(t',s')| = |(t,s)| + |(t',s')|$.

## 6. Module-bialgebra and comodule-bialgebra

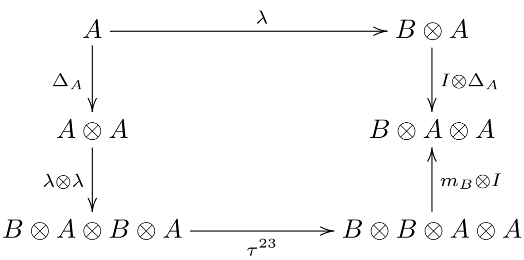
In this section, we recall the definitions of module-bialgebra and comodule-bialgebra using the works of R. K. Molnar [13], and D. Manchon [11].

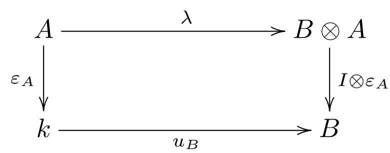
Definition 6.1. — Let $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ and $(A, m_B, \Delta_B, u_B, \varepsilon_B)$ be two unital counital bialgebras over some field $k$. We assume that there exists a coaction $\lambda : A \longrightarrow B \otimes A$, i.e. the following diagrams commute:




(1) $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is a comodule-coalgebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$ if $\Delta_A$ and $\varepsilon_A$ are morphisms of left $A$-comodules. This amounts to the commutativity of the following diagrams:

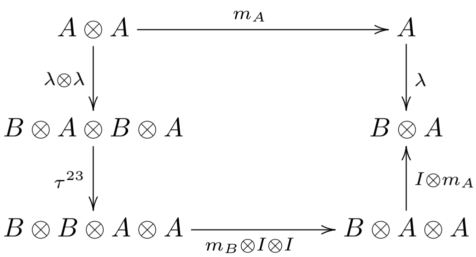


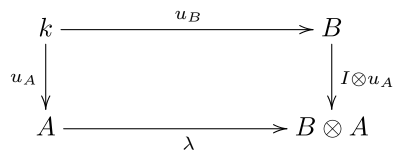




i.e. $(I \otimes \Delta_A) \circ \lambda = (m_B \otimes I) \circ \tau^{23} \circ (\lambda \otimes \lambda) \circ \Delta_A$, and $(I \otimes \varepsilon_A) \circ \lambda = u_B \circ \varepsilon_A$.

(2) $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is a comodule-algebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$ if $\lambda$ is a unital algebra morphism. This amounts to say that $m_A$ and $u_A$ are

morphisms of left $A$-comodules. In other words, the two following diagrams commute:

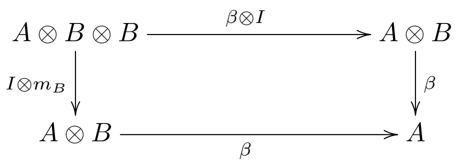


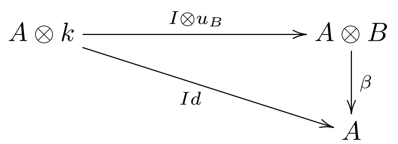




i.e., $\lambda\circ m_A = (I\otimes m_A)\circ(m_B\otimes I\otimes I)\circ\tau^{23}\circ(\lambda\otimes\lambda)$, and $(I\otimes u_A)\circ u_B = \lambda\circ u_A$.

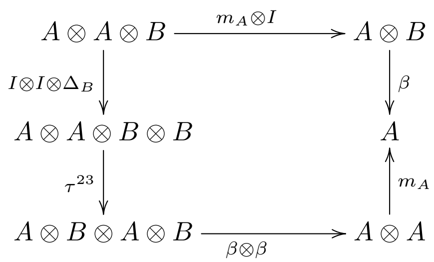
(3) $(A,\, m_A,\, \Delta_A,\, u_A,\, \varepsilon_A)$ is a comodule-bialgebra on $(B,\, m_B,\, \Delta_B,\, u_B,\, \varepsilon_B)$ if $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is both a comodule-algebra and a comodule-coalgebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$.

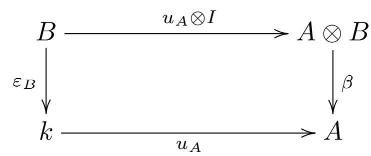
Definition 6.2. — Let $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ and $(A, m_B, \Delta_B, u_B, \varepsilon_B)$ be two unital counital bialgebras over some field $k$. We assume that there exists an action $\beta : A\otimes B \longrightarrow A$, i.e. the following diagrams commute:



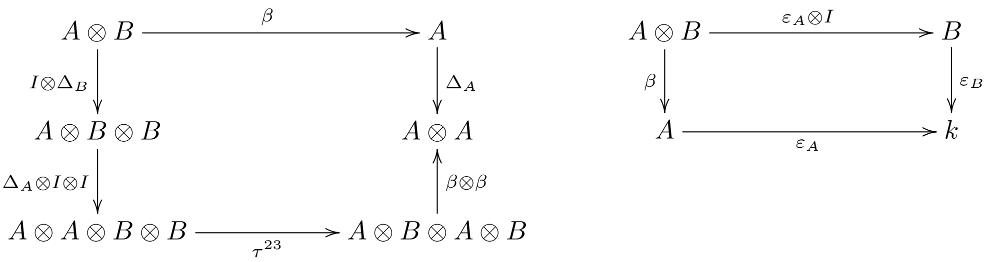

(1) $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is a module-algebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$ if the product $m_A$ and the unit $u_A$ are morphisms of left $A$-modules. This amounts to the commutativity of the following diagrams:




i.e. $\beta\circ(m_A\otimes I) = m_A\circ(\beta\otimes\beta)\circ\tau^{23}\circ(I\otimes I\otimes\Delta_B)$, and $\beta\circ(u_A\otimes I) = u_A\circ\varepsilon_B$.

(2) $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is a module-coalgebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$ if $\beta$ is a unital algebra morphism. This amounts to say that $\Delta_A$ and $\varepsilon_A$ are morphisms of left $A$-modules. In other words, the two following diagrams commute:




i.e., $\Delta_A \circ \beta = (\beta \otimes \beta) \circ \tau^{23} \circ (\Delta_A \otimes I \otimes I) \circ (I \otimes \Delta_B)$, and $\varepsilon_B \circ (\varepsilon_A \otimes I) = \varepsilon_A \circ \beta$.

(3) $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is a module-bialgebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$ if $(A, m_A, \Delta_A, u_A, \varepsilon_A)$ is both a module-algebra and a module-coalgebra on $(B, m_B, \Delta_B, u_B, \varepsilon_B)$.

## 7. Relations between the two pre-Lie structures

In this section, we find some relations connecting the pre-Lie structures as well as the bialgebra structures defined previously. We prove that $\tilde{V}$ is a left pre-Lie module on $\mathcal{T}$ and we find some relations between the two pre-Lie structures defined on $\tilde{V}$ and $\mathcal{T}$. We also show that $(\mathcal{K}, m, \Upsilon)$ is a comodule-coalgebra on $(\widetilde{\mathcal{H}}, m, \Delta)$ and that $(\mathcal{W}, m, \Theta)$ is a comodule-coalgebra on $(\widetilde{\mathcal{D}}, m, \Lambda)$. Moreover, we prove that $(\mathcal{W}, \blacklozenge, \Theta)$ is a module-bialgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$.

Definition 7.1. — Let $(\mathcal{A}, \circ)$ be a pre-Lie algebra. A left $\mathcal{A}$-module is a vector space $\mathcal{M}$ provided with a bilinear law denoted by $\succ: \mathcal{A} \otimes \mathcal{M} \longrightarrow \mathcal{M}$ such that for all $x, y \in \mathcal{A}$ and $m \in \mathcal{M}$, we have:

$$x \succ (y \succ m) - (x \circ y) \succ m = y \succ (x \succ m) - (y \circ x) \succ m. \tag{7.1}$$

Definition 7.2. — Let $t_1 \in \mathcal{T}$ and $(t_2, s_2) \in \tilde{V}$. We define the map $\heartsuit$ by:

$$t_1 \heartsuit (t_2, s_2) := \sum_{v \in \mathcal{V}(s_2)} (t_1 \rhd_v t_2, t_1 \rhd_v s_2). \tag{7.2}$$

Theorem 7.3. — *Equipped with $\heartsuit$, the space $\tilde{V}$ is a left $\mathcal{T}$-module. In other words for any $t_1, t_2 \in \mathcal{T}$ and $(t_3, s_3) \in \tilde{V}$, we have:*

$$t_1 \heartsuit \Big[ t_2 \heartsuit (t_3, s_3) \Big] - (t_1 \rhd t_2) \heartsuit (t_3, s_3) = t_2 \heartsuit \Big[ t_1 \heartsuit (t_3, s_3) \Big] - (t_2 \rhd t_1) \heartsuit (t_3, s_3).$$

*Proof.* — Let $t_1, t_2$ be two elements of $\mathcal{T}$ and let $(t_3, s_3)$ be an element of $\tilde{V}$, we have:

$$\begin{aligned} t_1 \heartsuit \Big[ t_2 \heartsuit (t_3, s_3) \Big] &- (t_1 \rhd t_2) \heartsuit (t_3, s_3) \\ &= t_1 \heartsuit \Big[ \sum_{r \in \mathcal{V}(s_3)} (t_2 \rhd_r t_3, t_2 \rhd_r s_3) \Big] - \sum_{l \in \mathcal{V}(t_2)} t_1 \rhd_l t_2 \Big) \heartsuit (t_3, s_3) \\ &= \sum_{\substack{r \in \mathcal{V}(s_3) \\ l \in \mathcal{V}(t_2 \rhd_r s_3)}} [t_1 \rhd_l (t_2 \rhd_r t_3), t_1 \rhd_l (t_2 \rhd_r s_3)] \\ &\qquad - \sum_{\substack{l \in \mathcal{V}(t_2) \\ r \in \mathcal{V}(s_3)}} \big[ (t_1 \rhd_l t_2) \rhd_r t_3, (t_1 \rhd_l t_2) \rhd_r s_3 \big]. \end{aligned}$$

In the term $\sum_{\substack{r\in\mathcal{V}(s_3)\\ l\in\mathcal{V}(t_2\rhd_r s_3)}}[t_1 \rhd_l (t_2 \rhd_r t_3), t_1 \rhd_l (t_2 \rhd_r s_3)]$, the vertex $r$ is common between $s_3$ and $t_2$. Then the condition $l \in \mathcal{V}(t_2 \rhd_r s_3)$ is equivalent to $l \in \mathcal{V}(s_3),\ l \neq r$ or $l \in \mathcal{V}(t_2)$. We then get:

$$\begin{aligned}
t_1\heartsuit\Big[t_2\heartsuit(t_3,s_3)\Big] - (t_1 \rhd t_2)\heartsuit(t_3,s_3)
&= \sum_{\substack{l\ ,\ r\in\mathcal{V}(s_3)\\ l\neq r}} [t_1 \rhd_l (t_2 \rhd_r t_3), t_1 \rhd_l (t_2 \rhd_r s_3)]\\
&\quad + \sum_{\substack{l\in\mathcal{V}(t_2)\\ r\in\mathcal{V}(s_3)}} [t_1 \rhd_l (t_2 \rhd_r t_3), t_1 \rhd_l (t_2 \rhd_r s_3)]\\
&\quad - \sum_{\substack{l\in\mathcal{V}(t_2)\\ r\in\mathcal{V}(s_3)}} \big[(t_1 \rhd_l t_2) \rhd_r t_3, (t_1 \rhd_l t_2) \rhd_r s_3\big]\\
&= \sum_{\substack{l\ ,\ r\in\mathcal{V}(s_3)\\ l\neq r}} [t_1 \rhd_l (t_2 \rhd_r t_3), t_1 \rhd_l (t_2 \rhd_r s_3)].
\end{aligned}$$

The last term is symmetric in $t_1$ and $t_2$, therefore:

$$t_1\heartsuit\Big[t_2\heartsuit(t_3,s_3)\Big] - (t_1 \rhd t_2)\heartsuit(t_3,s_3) = t_2\heartsuit\Big[t_1\heartsuit(t_3,s_3)\Big] - (t_2 \rhd t_1)\heartsuit(t_3,s_3). \qquad \square$$

Theorem 7.4. — *The law $\heartsuit$ is a derivation of the algebra $(\tilde{V}, \blacktriangleright)$. In other words, for any $t_1 \in \mathcal{T}$ and $(t_2,s_2),(t_3,s_3) \in \tilde{V}$, we have:*

$$t_1\heartsuit\big((t_2,s_2) \blacktriangleright (t_3,s_3)\big) = \big(t_1\heartsuit(t_2,s_2)\big) \blacktriangleright (t_3,s_3) + (t_2,s_2) \blacktriangleright \big(t_1\heartsuit(t_3,s_3)\big).$$

*Proof.* — Let $t_1 \in \mathcal{T}$ and $(t_2,s_2),(t_3,s_3) \in \tilde{V}$, we have:

$$\begin{aligned}
t_1\heartsuit\big((t_2,s_2) \blacktriangleright (t_3,s_3)\big) &= t_1\heartsuit\Big(\sum_{v\in\mathcal{V}(t_3-s_3)} (t_2 \rhd_v t_3, s_2s_3)\Big)\\
&= \sum_{r\in\mathcal{V}(s_2s_3)\ \ v\in\mathcal{V}(t_3-s_3)} \big(t_1 \rhd_r (t_2 \rhd_v t_3), t_1 \rhd_r (s_2s_3)\big)\\
&= \sum_{r\in\mathcal{V}(s_2)\ \ v\in\mathcal{V}(t_3-s_3)} \big((t_1 \rhd_r t_2) \rhd_v t_3, (t_1 \rhd_r s_2)s_3\big)\\
&\quad + \sum_{r\in\mathcal{V}(s_3)\ \ v\in\mathcal{V}(t_3-s_3)} \big(t_2 \rhd_v (t_1 \rhd_r t_3), s_2(t_1 \rhd_r s_3)\big)\\
&= \big(t_1\heartsuit(t_2,s_2)\big) \blacktriangleright (t_3,s_3) + (t_2,s_2) \blacktriangleright \big(t_1\heartsuit(t_3,s_3)\big). \qquad \square
\end{aligned}$$

Theorem 7.5. — *We denote by $P_2$ the projection on the second component. The following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{T}\otimes\tilde{V} & \xrightarrow{\ \heartsuit\ } & \tilde{V}\\
\Big\downarrow{\scriptstyle I\otimes P_2} & & \Big\downarrow{\scriptstyle P_2}\\
\mathcal{T}\otimes\mathcal{K} & \xrightarrow[\ \rhd\ ]{} & \mathcal{K}
\end{array}$$

*In other words, $P_2$ is a morphism of pre-Lie modules.*

*Proof.* — Let $t_1 \in \mathcal{T}$ and $(t_2, s_2) \in \tilde{V}$, we have:

$$\begin{aligned}
P_2\big(t_1 \heartsuit (t_2, s_2)\big) &= P_2\Big(\sum_{v \in \mathcal{V}(s_2)} (t_1 \rhd_v t_2, t_1 \rhd_v s_2)\Big)\\
&= \sum_{v \in \mathcal{V}(s_2)} t_1 \rhd_v s_2\\
&= t_1 \rhd s_2\\
&= t_1 \rhd P_2(t_2, s_2)\\
&= (I \otimes P_2)\big(t_1 \otimes (t_2, s_2)\big).
\end{aligned}$$

□

THEOREM 7.6. — (1) $(\mathcal{K}, m, \Upsilon)$ *is a comodule-coalgebra on* $(\widetilde{\mathcal{H}}, m, \Delta)$.
(2) $(\mathcal{W}, m, \Theta)$ *is a comodule-coalgebra on* $(\widetilde{\mathcal{D}}, m, \Lambda)$.

*Proof.* —

(1) It is clear that $\Delta : \mathcal{K} \longrightarrow \widetilde{\mathcal{H}} \otimes \mathcal{K}$ is a coaction. It follows from the coassociativity of $\Delta$. We now show that $\Upsilon$ and $\varepsilon_{\mathcal{K}}$ are morphisms of left $\mathcal{K}$-comodules. Let $t$ be an element of $\mathcal{K}$.

$$\begin{aligned}
(I \otimes \Upsilon) \circ \Delta(t) &= (I \otimes \Upsilon)\Big(\sum_{s \subseteq t} s \otimes t/s\Big)\\
&= \sum_{s \subseteq t} s \otimes \Upsilon(t/s)\\
&= \sum_{s \subseteq t\ ,\ (t/s)} s \otimes (t/s)^{(1)} \otimes (t/s)^{(2)}\\
&= \sum_{s_1 \subseteq t^{(1)}\ ,\ s_2 \subseteq t^{(2)}\ ,\ (t)} s_1 s_2 \otimes t^{(1)}/s_1 \otimes t^{(2)}/s_2.
\end{aligned}$$

We use the shorthand notation: $m^{13} := (m \otimes I) \circ \tau^{23}$.

$$\begin{aligned}
(m^{13} \otimes I) \circ (\Delta \otimes \Delta) \circ \Upsilon(t) &= (m^{13} \otimes I)\Big(\sum_{(t)} \Delta(t^{(1)}) \otimes \Delta(t^{(2)})\Big)\\
&= (m^{13} \otimes I)\Big(\sum_{s_1 \subseteq t^{(1)}\ ,\ s_2 \subseteq t^{(2)}\ ,\ (t)} s_1 \otimes t^{(1)}/s_1 \otimes s_2 \otimes t^{(2)}/s_2\Big)\\
&= \sum_{s_1 \subseteq t^{(1)}\ ,\ s_2 \subseteq t^{(2)}\ ,\ (t)} s_1 s_2 \otimes t^{(1)}/s_1 \otimes t^{(2)}/s_2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(I \otimes \varepsilon_{\mathcal{K}}) \circ \Delta(t) &= (I \otimes \varepsilon_{\mathcal{K}})\Big(\sum_{s \subseteq t} s \otimes t/s\Big)\\
&= \sum_{s \subseteq t} s \otimes \varepsilon_{\mathcal{K}}(t/s)\\
&= \sum_{s \subseteq t} \varepsilon_{\mathcal{K}}(s) \otimes t/s\\
&= \varepsilon_{\mathcal{K}}(t)\mathbf{1}\\
&= u_{\widetilde{\mathcal{H}}} \circ \varepsilon_{\mathcal{K}}(t),
\end{aligned}$$

which proves the first part of the theorem.

(2) The coaction is given by $\Lambda : \mathcal{W} \longrightarrow \widetilde{\mathcal{D}} \otimes \mathcal{W}$. Similarly to part (1), we prove that $\Theta$ and $\varepsilon_{\mathcal{W}}$ are morphisms of left $\mathcal{W}$-comodules. □

Lemma 7.7. — *Let $(t_1, s_1)$ and $(t_2, s_2)$ be two elements of $\mathcal{W}$. The product $\blacklozenge$ satisfies the following result:*

$$(t_1, s_1)\blacklozenge(t_2, s_2) = \sum_{\substack{v\in\mathcal{V}(t_2-s_2)\\(t_1)}} \big(t_1^{(1)}(t_1^{(2)} \triangleright_v t_2), s_1 s_2\big).$$

*Proof.* —

$$\begin{aligned}(t_1, s_1)\blacklozenge(t_2, s_2) &= \sum_{(t_1,s_1)} (t_1, s_1)^{(1)}\big((t_1, s_1)^{(2)} \blacktriangleright (t_2, s_2)\big)\\ &= \sum_{\substack{v\in\mathcal{V}(t_2-s_2)\\(t_1,s_1)}} (t_1^{(1)}, s_1^{(1)})(t_1^{(2)} \triangleright_v t_2, s_1^{(2)} s_2)\\ &= \sum_{\substack{v\in\mathcal{V}(t_2-s_2)\\(t_1,s_1)}} \Big(t_1^{(1)}(t_1^{(2)} \triangleright_v t_2), s_1^{(1)} s_1^{(2)} s_2\Big)\\ &= \sum_{\substack{v\in\mathcal{V}(t_2-s_2)\\(t_1)}} \big(t_1^{(1)}(t_1^{(2)} \triangleright_v t_2), s_1 s_2\big).\end{aligned}$$
□

Lemma 7.8. — *Let $(t_1, s_1)$ and $(t_2, s_2)$ be two elements of $\mathcal{W}$. The product $\blacklozenge$ satisfies the following result:*

$$(t_1, s_1)\blacklozenge(t_2, s_2) = (t_1 \diamondsuit t_2 - t_1 \wedge t_2, s_1 s_2),$$

*where:*

$$t_1 \wedge t_2 := \sum_{\substack{v\in\mathcal{V}(s_2)\\(t_1)}} (t_1)^{(1)}(t_1)^{(2)} \triangleright_v t_2.$$

*Proof.* —

$$\begin{aligned}(t_1, s_1)\blacklozenge(t_2, s_2) &= \sum_{\substack{v\in\mathcal{V}(t_2-s_2)\\(t_1)}} \big(t_1^{(1)}(t_1^{(2)} \triangleright_v t_2), s_1 s_2\big)\\ &= \sum_{v\in\mathcal{V}(t_2)\ ,\ (t_1)} t_1^{(1)}(t_1^{(2)} \triangleright_v t_2) - \sum_{v\in\mathcal{V}(s_2)\ ,\ (t_1)} t_1^{(1)}(t_1^{(2)} \triangleright_v t_2), s_1 s_2\\ &= (t_1 \diamondsuit t_2 - t_1 \wedge t_2, s_1 s_2).\end{aligned}$$
□

Theorem 7.9. — *$(\mathcal{W}, \blacklozenge, \Theta)$ is a module-bialgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$.*

*Proof.* — We consider the map: $\varphi : \mathcal{W} \otimes \mathcal{K} \longrightarrow \mathcal{W}$ defined for all $(t, s) \in \mathcal{W}$ and $t' \in \mathcal{K}$ by:

$$\varphi((t, s) \otimes t') = (t\diamondsuit t', s).$$

Firstly, we show that $\varphi$ is an action. Let $(t_1, s_1) \in \mathcal{W}$ and $t_2, t_3 \in \mathcal{K}$, we have:

$$\begin{aligned}\varphi \circ (\varphi \otimes I)[(t_1, s_1) \otimes t_2 \otimes t_3] &= \varphi[(t_1 \diamondsuit t_2, s_1) \otimes t_3]\\ &= \big((t_1 \diamondsuit t_2) \diamondsuit t_3, s_1\big)\\ &= \big(t_1 \diamondsuit (t_2 \diamondsuit t_3), s_1\big)\\ &= \varphi[(t_1, s_1) \otimes t_2 \diamondsuit t_3]\\ &= \varphi \circ (I \otimes \diamondsuit)[(t_1, s_1) \otimes t_2 \otimes t_3].\end{aligned}$$

Moreover,

$$\begin{aligned}\varphi \circ (I \otimes u_{\mathcal{K}})[(t_1, s_1) \otimes 1] &= \varphi[(t_1, s_1) \otimes \mathbf{1}]\\ &= (t_1 \diamondsuit \mathbf{1}, s_1)\\ &= (t_1, s_1)\\ &= I(t_1, s_1).\end{aligned}$$

Secondly, we use Lemma 7.7 and Lemma 7.8 to show that:

$$\blacklozenge \circ (\varphi \otimes \varphi) \circ \Upsilon^{23} = \varphi \circ (\blacklozenge \otimes I), \quad \text{and} \quad \varphi \circ (u_{\mathcal{W}} \otimes I) = u_{\mathcal{W}} \circ \varepsilon_{\mathcal{K}},$$

where: $\Upsilon^{23} = \tau^{23} \circ (I \otimes I \otimes \Upsilon)$. Let $t$ be an element of $\mathcal{K}$.

$$\begin{aligned}\varphi \circ (u_{\mathcal{W}} \otimes I)(t) &= \varphi((\mathbf{1}, \mathbf{1}) \otimes t)\\ &= (\mathbf{1} \diamondsuit t, \mathbf{1})\\ &= (t, \mathbf{1})\\ &= \varepsilon_{\mathcal{K}}(t)(\mathbf{1}, \mathbf{1})\\ &= u_{\mathcal{W}} \circ \varepsilon_{\mathcal{K}}(t).\end{aligned}$$

Let $(t_1, s_1)$ and $(t_2, s_2)$ be two elements of $\mathcal{W}$:

$$\begin{aligned}\blacklozenge \circ (\varphi \otimes \varphi) \circ \Upsilon^{23}\big((t_1, s_1) \otimes (t_2, s_2) \otimes t\big) &= \sum_{(t)} \varphi\big((t_1, s_1) \otimes t^{(1)}\big) \blacklozenge \varphi\big((t_2, s_2) \otimes t^{(2)}\big)\\ &= \sum_{(t)} (t_1 \diamondsuit t^{(1)}, s_1) \blacklozenge (t_2 \diamondsuit t^{(2)}, s_2)\\ &= \sum_{(t)} \big((t_1 \diamondsuit t^{(1)}) \diamondsuit (t_2 \diamondsuit t^{(2)}) - (t_1 \diamondsuit t^{(1)}) \wedge (t_2 \diamondsuit t^{(2)}), s_1 s_2\big)\\ &= \big((t_1 \diamondsuit t_2) \diamondsuit t - (t_1 \wedge t_2) \diamondsuit t, s_1 s_2\big).\end{aligned}$$

On the other hand:

$$\begin{aligned}\varphi \circ (\blacklozenge \otimes I)\big((t_1, s_1) \otimes (t_2, s_2) \otimes t\big) &= \varphi\big((t_1, s_1) \blacklozenge (t_2, s_2) \otimes t\big)\\ &= \varphi\big((t_1 \diamondsuit t_2 - t_1 \wedge t_2, s_1 s_2) \otimes t\big)\\ &= \big((t_1 \diamondsuit t_2 - t_1 \wedge t_2) \diamondsuit t, s_1 s_2\big)\\ &= \big((t_1 \diamondsuit t_2) \diamondsuit t - (t_1 \wedge t_2) \diamondsuit t, s_1 s_2\big).\end{aligned}$$

Finally, we prove that $\Theta$ and $\varepsilon_{\mathcal{W}}$ are morphisms of modules. Let $(t_1, s_1) \in \mathcal{W}$ and $t_2 \in \mathcal{K}$:

$$\begin{aligned}
\varepsilon_{\mathcal{W}} \circ \varphi\big((t_1, s_1) \otimes t_2\big) &= \varepsilon_{\mathcal{W}}\big(t_1 \diamondsuit t_2, s_1\big) \\
&= \big(\varepsilon_{\mathcal{K}}(t_1 \diamondsuit t_2), \varepsilon_{\mathcal{K}}(s_1)\big) \\
&= \big(\varepsilon_{\mathcal{K}}(t_1)\varepsilon_{\mathcal{K}}(t_2), \varepsilon_{\mathcal{K}}(s_1)\big) \\
&= \varepsilon_{\mathcal{W}}(t_1, s_1)\varepsilon_{\mathcal{K}}(t_2) \\
&= \varepsilon_{\mathcal{K}} \circ \big(\varepsilon_{\mathcal{W}} \otimes I\big)\big((t_1, s_1) \otimes t_2\big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Theta \circ \varphi\big((t_1, s_1) \otimes t_2\big) &= \Theta\big(t_1 \diamondsuit t_2, s_1\big) \\
&= \sum_{(t_1)} \Theta\big(t_1^{(1)}(t_1^{(2)} \triangleright t_2), s_1\big) \\
&= \sum_{(t_1)\ ,\ (s_1)} (t_1^{(11)}(t_1^{(12)} \triangleright t_2^{(1)}), {s_1}^{(1)}) \otimes (t_1^{(21)}(t_1^{(22)} \triangleright t_2^{(2)}), s_1^{(2)}) \\
&= \sum_{(t_1, s_1)\ ,\ (t_2)} (t_1^{(1)} \diamondsuit t_2^{(1)}, s_1^{(1)}) \otimes (t_1^{(2)} \diamondsuit t_2^{(2)}, s_1^{(2)}).
\end{aligned}$$

We use the shorthand notation: $(\Theta \otimes I \otimes I) \circ (I \otimes \Upsilon) = \Theta \otimes \Upsilon$.

$$\begin{aligned}
&(\varphi \otimes \varphi) \circ \tau^{23} \circ (\Theta \otimes \Upsilon)\big((t_1, s_1) \otimes t_2\big) \\
&\qquad = (\varphi \otimes \varphi)\big( \sum_{(t_1, s_1)\ ,\ (t_2)} (t_1^{(1)}, s_1^{(1)}) \otimes t_2^{(1)} \otimes (t_1^{(2)}, s_1^{(2)}) \otimes t_2^{(2)}\big) \\
&\qquad = \sum_{(t_1, s_1)\ ,\ (t_2)} \varphi\big((t_1^{(1)}, s_1^{(1)}) \otimes t_2^{(1)}\big) \otimes \varphi\big((t_1^{(2)}, s_1^{(2)}) \otimes t_2^{(2)}\big) \\
&\qquad = \sum_{(t_1, s_1)\ ,\ (t_2)} (t_1^{(1)} \diamondsuit t_2^{(1)}, s_1^{(1)}) \otimes (t_1^{(2)} \diamondsuit t_2^{(2)}, s_1^{(2)}).
\end{aligned}$$

□

## 8. Relation with the grafting pre-Lie algebra

We have studied the grafting pre-Lie algebra $(\mathcal{T}, \rightarrow)$ of rooted trees in [2], and we have introduced a pre-Lie structure on its doubling space $(V, \rightsquigarrow)$. The grafting pre-Lie product is given, for all $t, s \in \mathcal{T}$, by:

$$t \rightarrow s = \sum_{v \in V(s)} t \rightarrow_v s,$$

where $t \rightarrow_v s$ is the tree obtained by grafting the root of $t$ on the vertex $v$ of $s$. The pre-Lie product on the doubling space $V$ is defined for all $(t_1, s_1)$ and $(t_2, s_2)$ in $V$ by:

$$(t_1, s_1) \rightsquigarrow (t_2, s_2) := \sum_{v \in \mathcal{V}(t_2 - s_2)} (t_1 \rightarrow_v t_2, s_1 s_2),$$

where the notation $v \in \mathcal{V}(t_2 - s_2)$ denotes that $v$ is a vertex of $t_2$ but is not a vertex of $s_2$.

We have constructed the enveloping algebra of the grafting pre-Lie algebra of rooted trees $(\mathcal{T}, \rightarrow)$ using the method of Oudom and Guin [14]. We have considered the Hopf symmetric algebra $\mathcal{H}' := \mathcal{S}(\mathcal{T})$ of the pre-Lie algebra $(\mathcal{T}, \rightarrow)$, equipped with its usual unshuffling coproduct $\Gamma$ and a product $\star$ defined on $\mathcal{H}'$ by:

$$t \star t' = \sum_{(t)} t^{(1)}(t^{(2)} \rightarrow t').$$

We have also constructed $(\mathcal{D}', \bigstar, \chi)$ the enveloping algebra of $(V, \rightsquigarrow)$, where $\chi$ is the usual unshuffling coproduct and $\bigstar$ is defined by:

$$(t,s)\bigstar(t',s') = \sum_{(t,s)} (t,s)^{(1)}\big((t,s)^{(2)} \rightsquigarrow (t',s')\big).$$

In this section, we give some relations between the insertion and the grafting pre-Lie algebras of rooted trees.

Definition 8.1. — We consider the two maps: $\gamma : \mathcal{H}' \otimes \mathcal{K} \longrightarrow \mathcal{H}'$ defined for all $t_1 \in \mathcal{H}'$ and $t_2 \in \mathcal{K}$ by:

$$\gamma(t_1 \otimes t_2) = t_1 \star t_2,$$

and $\rho : \mathcal{K} \longrightarrow \mathcal{K} \otimes \mathcal{K}$ defined for all $t \in \mathcal{K}$ by:

$$\rho(t) = \mathbf{1} \otimes t.$$

Theorem 8.2. — (1) *$(\mathcal{H}', \star, \Gamma)$ is a module-coalgebra on $(\mathcal{K}, \diamondsuit, \Upsilon)$.*

(2) *The two maps $\gamma$ and $\rho$ satisfy the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{K} & \xrightarrow{\star \otimes I} & \mathcal{H}' \otimes \mathcal{K} \\
\Big\downarrow {\scriptstyle I \otimes I \otimes \rho} & & \Big\downarrow {\scriptstyle \gamma} \\
\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{K} & & \mathcal{H}' \\
\Big\downarrow {\scriptstyle \tau^{23}} & & \Big\uparrow {\scriptstyle \star} \\
\mathcal{H}' \otimes \mathcal{K} \otimes \mathcal{H}' \otimes \mathcal{K} & \xrightarrow[\gamma \otimes \gamma]{} & \mathcal{H}' \otimes \mathcal{H}'
\end{array}$$

*Proof.* —

(1) Firstly, we show that $\gamma$ is an action. Let $t_1 \in \mathcal{H}'$ and $t_2, t_3 \in \mathcal{K}$:

$$\begin{aligned}
\gamma \circ (\gamma \otimes I)[t_1 \otimes t_2 \otimes t_3] &= \gamma[(t_1 \star t_2) \otimes t_3] \\
&= (t_1 \star t_2) \star t_3 \\
&= t_1 \star (t_2 \star t_3) \\
&= \gamma[t_1 \otimes (t_2 \star t_3)] \\
&= \gamma \circ (I \otimes \star)[t_1 \otimes t_2 \otimes t_3].
\end{aligned}$$

Moreover,

$$\begin{aligned}
\gamma \circ (I \otimes u_{\mathcal{K}})[t_1 \otimes 1] &= (t_1 \star \mathbf{1}) \\
&= t_1. \\
&= I(t_1 \otimes 1).
\end{aligned}$$

Secondly, we prove that $\Gamma$ and $\varepsilon_{\mathcal{H}'}$ are morphisms of modules.

$$\begin{aligned}\varepsilon_{\mathcal{H}'}\circ\gamma\big(t_1\otimes t_2\big) &= \varepsilon_{\mathcal{H}'}\big(t_1\star t_2\big)\\ &= \varepsilon_{\mathcal{H}'}(t_1)\varepsilon_{\mathcal{K}}(t_2)\\ &= \varepsilon_{\mathcal{K}}\circ(\varepsilon_{\mathcal{H}'}\otimes I)\big(t_1\otimes t_2\big).\end{aligned}$$

Otherwise,

$$\begin{aligned}\Gamma\circ\gamma\big(t_1\otimes t_2\big) &= \Gamma\big(t_1\star t_2\big)\\ &= \sum_{(t_1\star t_2)}(t_1\star t_2)^{(1)}\otimes(t_1\star t_2)^{(2)}.\end{aligned}$$

We use the shorthand notation: $(\Gamma\otimes I\otimes I)\circ(I\otimes\Upsilon)=\Gamma\otimes\Upsilon$.

$$\begin{aligned}(\gamma\otimes\gamma)\circ\tau^{23}\circ(\Gamma\otimes\Upsilon)\big(t_1\otimes t_2\big) &= (\gamma\otimes\gamma)\circ\tau^{23}\big(\Gamma(t_1)\otimes\Upsilon(t_2)\big)\\ &= (\gamma\otimes\gamma)\circ\tau^{23}\Big(\sum_{(t_1)\,,\,(t_2)} t_1^{(1)}\otimes t_1^{(2)}\otimes t_2^{(1)}\otimes t_2^{(2)}\Big)\\ &= \sum_{(t_1)\,,\,(t_2)}\gamma(t_1^{(1)}\otimes t_2^{(1)})\otimes\gamma(t_1^{(2)}\otimes t_2^{(2)})\\ &= \sum_{(t_1)\,,\,(t_2)}(t_1^{(1)}\star t_2^{(1)})\otimes(t_1^{(2)}\star t_2^{(2)})\\ &= \sum_{(t_1)\,,\,(t_2)}(t_1\star t_2)^{(1)}\otimes(t_1\star t_2)^{(2)}.\end{aligned}$$

(2) We denote by: $\rho^{23}:=\tau^{23}\circ(I\otimes I\otimes\rho)$. Let $t_1,t_2\in\mathcal{H}'$ and $t\in\mathcal{K}$:

$$\begin{aligned}\star\circ(\gamma\otimes\gamma)\circ\rho^{23}\big(t_1\otimes t_2\otimes t\big) &= \gamma\big(t_1\otimes\mathbf{1}\big)\star\gamma\big(t_2\otimes t\big)\\ &= (t_1\star\mathbf{1})\star(t_2\star t)\\ &= t_1\star(t_2\star t).\end{aligned}$$

Moreover,

$$\begin{aligned}\gamma\circ(\star\otimes I)\big(t_1\otimes t_2\otimes t\big) &= \gamma\big((t_1\star t_2)\otimes t\big)\\ &= (t_1\star t_2)\star t\\ &= t_1\star(t_2\star t).\end{aligned}$$

□

Definition 8.3. — We consider the two maps: $\alpha:\mathcal{D}'\otimes\mathcal{W}\longrightarrow\mathcal{D}'$ defined for all $(t_1,s_1)\in\mathcal{D}'$ and $(t_2,s_2)\in\mathcal{W}$ by:

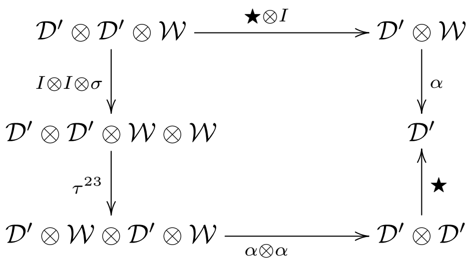
$$\alpha\big((t_1,s_1)\otimes(t_2,s_2)\big)=(t_1,s_1)\bigstar(t_2,s_2),$$

and $\sigma:\mathcal{W}\longrightarrow\mathcal{W}\otimes\mathcal{W}$ defined for all $(t,s)\in\mathcal{W}$ by:

$$\sigma(t,s)=(\mathbf{1},\mathbf{1})\otimes(t,s).$$

Theorem 8.4. — (1) $(\mathcal{D}',\bigstar,\chi)$ *is a module-coalgebra on* $(\mathcal{W},\blacklozenge,\Theta)$.

(2) *The two maps* $\alpha$ *and* $\sigma$ *satisfy the following commutative diagram:*




*Proof.* —

(1) The action is given by $\alpha$ defined above. We prove this part of the theorem similarly to that of the previous theorem.

(2) We denote by $\sigma^{23}$ the following map: $\sigma^{23} := \tau^{23} \circ (I \otimes I \otimes \sigma)$. Let $(t, s) \in \mathcal{D}'$ and $(u_1, r_1), (u_2, r_2) \in \mathcal{W}$, we have:

$$\begin{aligned}
\bigstar \circ (\alpha \otimes \alpha) \circ \sigma^{23}\big((t,s) \otimes (u_1,r_1) \otimes (u_2,r_2)\big)& \\
&= \alpha\big((t,s) \otimes (\mathbf{1},\mathbf{1})\big) \bigstar \alpha\big((u_1,r_1) \otimes (u_2,r_2)\big) \\
&= \big((t,s) \bigstar (\mathbf{1},\mathbf{1})\big) \bigstar \big((u_1,r_1) \bigstar (u_2,r_2)\big) \\
&= (t,s) \bigstar \big((u_1,r_1) \bigstar (u_2,r_2)\big).
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\alpha \circ (\bigstar \otimes I)\big((t,s) \otimes (u_1,r_1) \otimes (u_2,r_2)\big) &= \alpha\big((t,s) \bigstar (u_1,r_1) \otimes (u_2,r_2)\big) \\
&= \big((t,s) \bigstar (u_1,r_1)\big) \bigstar (u_2,r_2) \\
&= (t,s) \bigstar \big((u_1,r_1) \bigstar (u_2,r_2)\big).
\end{aligned}$$

This result is similar to the fact that $(\mathcal{D}', \bigstar, \chi)$ is a module-algebra on $(\mathcal{W}, \blacklozenge, \Theta)$. The map $I \otimes I \otimes \Theta$ in the module-algebra structure is replaced here by the map $I \otimes I \otimes \sigma$. □

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