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On a general variational framework for existence and uniqueness in Differential Equations

PABLO PEDREGAL

Abstract

Starting from the classic contraction mapping principle, we establish a general, flexible, variational setting that turns out to be applicable to many situations of existence in Differential Equations. This unifying feature is quite appealing and motivated our analysis. We show its potentiality with some selected examples including initial-value, Cauchy problems for ODEs; non-linear, monotone PDEs; linear and non-linear hyperbolic problems; and steady Navier–Stokes systems. Though the paper has the structure of a survey, we would like to explore in the future how this perspective could help in advancing for some new situations in PDEs.

1. Introduction

Possibly the most fundamental result yielding existence and uniqueness of solution of an equation is the classic Banach contraction mapping principle.

Theorem 1.1. *Let $T : H \rightarrow H$ be a mapping from a Banach space H into itself that is contractive in the sense*

$$\|Tx - Ty\| \leq k\|x - y\|, \quad k \in [0, 1), x, y \in H.$$

Then T admits a unique fixed point $\bar{x} \in H$,

$$T\bar{x} = \bar{x}.$$

The proof is well-known, elementary, and independent of dimension. The most fascinating issue is that this basic principle is at the heart of many existence and uniqueness results in Applied Analysis and Differential Equations. Our aim is to stress this fact from a variational stand-point in such a way that could be used in a much more flexible manner in quite diverse settings. This means that we would like to rephrase the previous principle into a variational form that could be directly and flexibly used in many of the situations where uniqueness of solutions is known or expected. Our basic principle is the following.

Proposition 1.2. *Let $E : H \rightarrow \mathbb{R}^+$ be a non-negative, lower semi-continuous functional in a Banach space H , such that*

$$\|x - y\| \leq C(E(x) + E(y)), \quad C > 0, x, y \in H. \tag{1.1}$$

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Suppose, in addition, that

$$\inf_{\mathbf{z} \in \mathbf{H}} E(\mathbf{z}) = 0. \quad (1.2)$$

Then there is a unique minimizer, i.e. a unique $\bar{\mathbf{x}} \in \mathbf{H}$ such that $E(\bar{\mathbf{x}}) = 0$, and

$$\|\mathbf{x} - \bar{\mathbf{x}}\| \leq CE(\mathbf{x}), \quad \mathbf{x} \in \mathbf{H}. \quad (1.3)$$

The proof again is elementary, because every minimizing sequence $\{\mathbf{x}_j\}$ with $E(\mathbf{x}_j) \searrow 0$ must be a Cauchy sequence in \mathbf{H} , according to (1.1), and so it converges to some $\bar{\mathbf{x}} \in \mathbf{H}$. The lower semicontinuity implies that

$$0 \leq E(\bar{\mathbf{x}}) \leq \liminf_{j \rightarrow \infty} E(\mathbf{x}_j) = 0,$$

and $\bar{\mathbf{x}}$ is a minimizer. Condition (1.1) implies automatically that such minimizer is unique, and leads to (1.3).

Condition (1.3) is a very clear statement that functional E in Proposition 1.2 is a measure of how far we are from $\bar{\mathbf{x}}$, the unique point where E vanishes. Indeed, this consequence already points in the direction in which to look for functionals E in specific situations: they should be setup as a way to measure departure from solutions sought. This will be taken as a guiding principle in concrete examples. The usual least-square method (see [2], [6], for example), suitably adapted to each situation, stands as a main, natural possibility for E .

It is not surprising that Proposition 1.2 is more general than Theorem 1.1, in the sense that the latter is a consequence of the former by considering the natural functional

$$E(\mathbf{x}) = \|\mathbf{T}\mathbf{x} - \mathbf{x}\|. \quad (1.4)$$

Indeed, for an arbitrary pair $\mathbf{x}, \mathbf{y} \in \mathbf{H}$,

$$\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{T}\mathbf{x}\| + \|\mathbf{T}\mathbf{x} - \mathbf{T}\mathbf{y}\| + \|\mathbf{T}\mathbf{y} - \mathbf{y}\|,$$

and

$$\|\mathbf{x} - \mathbf{y}\| \leq E(\mathbf{x}) + E(\mathbf{y}) + k\|\mathbf{x} - \mathbf{y}\|.$$

From here, we immediately find (1.1)

$$\|\mathbf{x} - \mathbf{y}\| \leq \frac{1}{1-k}(E(\mathbf{x}) + E(\mathbf{y})).$$

Along every sequence of iterates, we have (1.2) if \mathbf{T} is contractive. Of course, minimizers for E in (1.4) are exactly fixed points for \mathbf{T} .

Our objective is to argue that the basic variational principle in Proposition 1.2 is quite flexible, and can be implemented in many of the situations in Differential Equations where uniqueness of solutions is known.

There are two main requisites in Proposition 1.2. The first one (1.1) has to be shown directly in each particular scenario where uniqueness is sought. Note that it is some kind

of enhanced coercivity, and, as such, stronger than plain coercivity. Concerning (1.2), there is, however, a general strategy based on smoothness that can be applied to most of the interesting situations in practice. For the sake of simplicity, we restrict attention to a Hilbert space situation, and regard \mathbf{H} as a Hilbert space henceforth. If a non-negative functional $E : \mathbf{H} \rightarrow \mathbb{R}^+$ is C^1 - with derivative $E' : \mathbf{H} \rightarrow \mathbf{H}$, then

$$\inf_{\mathbf{x} \in \mathbf{H}} \|E'(\mathbf{x})\| = 0.$$

Therefore, it suffices to demand that

$$\lim_{E'(\mathbf{x}) \rightarrow \mathbf{0}} E(\mathbf{x}) = 0$$

to enforce (1.2). This last requirement is taken to mean

$$\lim_{j \rightarrow \infty} E(\mathbf{x}_j) = 0 \text{ whenever } \lim_{j \rightarrow \infty} E'(\mathbf{x}_j) = \mathbf{0}.$$

Proposition 1.2 becomes then:

Proposition 1.3. *Let $E : \mathbf{H} \rightarrow \mathbb{R}^+$ be a non-negative, C^1 - functional in a Hilbert space \mathbf{H} , such that*

$$\|\mathbf{x} - \mathbf{y}\| \leq C(E(\mathbf{x}) + E(\mathbf{y})), \quad C > 0, \mathbf{x}, \mathbf{y} \in \mathbf{H}. \quad (1.5)$$

Suppose, in addition, that

$$\lim_{E'(\mathbf{x}) \rightarrow \mathbf{0}} E(\mathbf{x}) = 0. \quad (1.6)$$

Then there is a unique $\bar{\mathbf{x}} \in \mathbf{H}$ such that $E(\bar{\mathbf{x}}) = 0$, and

$$\|\mathbf{x} - \bar{\mathbf{x}}\| \leq CE(\mathbf{x})$$

for every $\mathbf{x} \in \mathbf{H}$.

Though the following is a simple observation, it is worth to note it explicitly.

Proposition 1.4. *Under the same conditions as in Proposition 1.3, the functional E enjoys the Palais–Smale condition.*

We remind readers that the fundamental Palais–Smale (see, for instance, [12] among many other sources) condition reads:

If the sequence $\{\mathbf{x}_j\}$ is bounded in \mathbf{H} , and $E'(\mathbf{x}_j) \rightarrow \mathbf{0}$ in \mathbf{H} , then, at least for some subsequence, $\{\mathbf{x}_j\}$ converges in \mathbf{H} .

Again, it is not difficult to suspect the proof. Condition (1.6) informs us that Palais–Smale sequences ($\{\mathbf{x}_j\}$, bounded and $E'(\mathbf{x}_j) \rightarrow \mathbf{0}$) are always minimizing sequences for E ($E(\mathbf{x}_j) \rightarrow 0$), while the estimate (1.5) ensures that (the full) such sequence is a Cauchy sequence in \mathbf{H} . Notice, however, that, due to (1.6), 0 is the only possible critical value of

E , and so critical points become automatically global minimizers regardless of convexity considerations.

In view of the relevance of conditions (1.5) and (1.6), we adopt the following definition in which we introduce some simple, helpful changes to broaden its applicability. We also change the notation to stress that vectors in \mathbf{H} will be functions for us.

Definition 1.5. A non-negative, C^1 -functional

$$E(\mathbf{u}) : \mathbf{H} \rightarrow \mathbb{R}^+$$

defined over a Hilbert space \mathbf{H} is called an error functional if

- (1) behavior as $E' \rightarrow \mathbf{0}$:

$$\lim_{E'(\mathbf{u}) \rightarrow \mathbf{0}} E(\mathbf{u}) = 0 \tag{1.7}$$

over bounded subsets of \mathbf{H} , i.e.

$$\lim_{j \rightarrow \infty} E(\mathbf{x}_j) = 0 \text{ whenever } \lim_{j \rightarrow \infty} E'(\mathbf{x}_j) = \mathbf{0},$$

and $\{\mathbf{x}_j\}$ is a bounded sequence in \mathbf{H} ; and

- (2) enhanced coercivity: there is a positive constant C , such that for every pair $\mathbf{u}, \mathbf{v} \in \mathbf{H}$ we have

$$\|\mathbf{u} - \mathbf{v}\|^2 \leq C(E(\mathbf{u}) + E(\mathbf{v})). \tag{1.8}$$

Our basic result Proposition 1.3 remains the same.

Proposition 1.6. *Let $E : \mathbf{H} \rightarrow \mathbb{R}^+$ be an error functional according to Definition 1.5. Then there is a unique minimizer $\mathbf{u}_\infty \in \mathbf{H}$ such that $E(\mathbf{u}_\infty) = 0$, and*

$$\|\mathbf{u} - \mathbf{u}_\infty\|^2 \leq CE(\mathbf{u}), \tag{1.9}$$

for every $\mathbf{u} \in \mathbf{H}$.

It is usually said that the contraction mapping principle Theorem 1.1, though quite helpful in ODEs, is almost inoperative for PDEs. We simply mean that this classical result is the basis for the main existence and uniqueness result for ODEs though it seems of little, direct use in PDEs. We will try to make an attempt at convincing readers that, on the contrary, Proposition 1.3 is equally helpful for ODEs and PDEs. To this end, we will examine several selected examples as a sample of the potentiality of these ideas. Specifically, we will look at the following situations though none of our existence results is essentially new at this stage:

- (1) Cauchy, initial-value problems for ODEs [16];

- (2) linear hyperbolic examples [3], [17];
- (3) non-linear, monotone PDEs [4], [5];
- (4) non-linear wave models [7], [14];
- (5) steady Navier–Stokes system [13], [15].

We have included some basic or recent references, as sample sources, for each one of these areas that could serve as a helpful guide for readers to check. We systematically will have to show the two basic properties (1.8) and (1.7) in each situation treated. We can be dispensed with condition (1.7), and replace it by (1.2) if more general results not requiring smoothness are sought. On the other hand, in many regular situations linearization may lead in a systematic way to the following.

Proposition 1.7. *Let*

$$E(\mathbf{u}) : \mathbf{H} \rightarrow \mathbb{R}^+$$

be a C^1 -functional verifying the enhanced coercivity condition (1.8). Suppose there is $\mathbf{T} : \mathbf{H} \rightarrow \mathbf{H}$, a locally Lipschitz operator, such that

$$\langle E'(\mathbf{u}), \mathbf{T}\mathbf{u} \rangle = -dE(\mathbf{u}) \tag{1.10}$$

for every $\mathbf{u} \in \mathbf{H}$, and some constant $d > 0$. Then E is an error functional (according to Definition 1.5), and, consequently, there is a unique global minimizer \mathbf{u}_∞ with $E(\mathbf{u}_\infty) = 0$, and (1.9) holds

$$\|\mathbf{u} - \mathbf{u}_\infty\|^2 \leq CE(\mathbf{u}), \tag{1.11}$$

for some constant C , and every $\mathbf{u} \in \mathbf{H}$.

Note how condition (1.10) leads immediately to (1.7). In this contribution, we will assume smoothness in all of our examples.

Typically our Hilbert spaces \mathbf{H} will be usual Sobolev spaces in different situations, so standard facts about these spaces will be taken for granted. In particular, the following Hilbert spaces will play a basic role for us in those various situations mentioned above

$$H^1(0, T; \mathbb{R}^N), \quad H_0^1(\mathbb{R}_+^N), \quad H^1(\mathbb{R}_+^N), \quad H_0^1(\Omega), \quad H_0^1(\Omega; \mathbb{R}^N),$$

for a domain $\Omega \subset \mathbb{R}^N$ as regular as we may need it to be.

If one is interested in numerical or practical approximation of solutions \mathbf{u}_∞ , note how (1.11) is a clear invitation to seek approximations to \mathbf{u}_∞ by minimizing $E(\mathbf{u})$. The standard way to take a functional to its minimum value is to use a steepest descent algorithm or some suitable variant of it. It is true that such procedure is designed, in fact, to lead the derivative $E'(\mathbf{u})$ to zero; but precisely, condition (1.7) is guaranteeing that in

doing so we are also converging to \mathbf{u}_∞ always. We are not pursuing this direction here, though it has been implemented in some scenarios ([10], [11]).

Definition 1.5 is global. A local parallel concept may turn out necessary for some situations. We will show this in our final example dealing with the steady Navier–Stokes system. The application to parabolic problems, though still feasible, is, in general, more delicate.

Finally, we would like to emphasize that the enhanced coercivity condition is responsible for the uniqueness of global minimizers with $E = 0$ for an error functional E . Without this condition, uniqueness does not follow, in general. It is a fundamental issue to be explored if in situations in PDEs where this condition fails, one can still show existence, possibly without uniqueness. In particular, it would be relevant to investigate if inequality (1.9), that could not hold for every element of the underlying Hilbert space, might still hold for some distinguished sequences. We will address this issue, possibly in particular situations, in the future.

2. Cauchy problems for ODEs

As a preliminary step, we start testing our ideas with a typical initial-value, Cauchy problem for the non-linear system

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) \text{ in } (0, +\infty), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (2.1)$$

where the map

$$\mathbf{f}(\mathbf{y}) : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

is smooth and globally Lipschitz, and $\mathbf{x}_0 \in \mathbb{R}^N$. Under these circumstances, it is well-known that (2.1) possesses a unique solution. Let us pretend not to know anything about problem (2.1), and see if our formalism could be applied in this initial situation to prove the following classical theorem.

Theorem 2.1. *If the mapping $\mathbf{f}(\mathbf{y})$ is globally Lipschitz, there is unique absolutely continuous solution*

$$\mathbf{x}(t) : [0, \infty) \rightarrow \mathbb{R}^N$$

for (2.1).

According to our previous discussion, we need a functional $E : \mathbf{H} \rightarrow \mathbb{R}^+$ defined on an appropriate Hilbert space \mathbf{H} complying with the necessary properties.

For a fixed, but otherwise arbitrary, positive time T , we will take

$$\mathbf{H} = \{\mathbf{z}(t) : [0, T] \rightarrow \mathbb{R}^N : \mathbf{z} \in H^1(0, T; \mathbb{R}^N), \mathbf{z}(0) = \mathbf{0}\},$$

$$E(\mathbf{z}) = \frac{1}{2} \int_0^T |\mathbf{z}'(s) - \mathbf{f}(\mathbf{x}_0 + \mathbf{z}(s))|^2 ds. \quad (2.2)$$

\mathbf{H} is a subspace of the standard Sobolev space $H^1(0, T; \mathbb{R}^N)$, under the norm (recall that $\mathbf{z}(0) = \mathbf{0}$ for paths in \mathbf{H})

$$\|\mathbf{z}\|^2 = \int_0^T |\mathbf{z}'(s)|^2 ds.$$

Note that paths $\mathbf{x} \in \mathbf{H}$ are absolutely continuous, and hence $E(\mathbf{z})$ is well-defined over \mathbf{H} . We first focus on (1.8).

Lemma 2.2. *For paths \mathbf{y}, \mathbf{z} in \mathbf{H} , we have*

$$\|\mathbf{y} - \mathbf{z}\|^2 \leq C(E(\mathbf{y}) + E(\mathbf{z})), \quad C > 0.$$

Proof. The proof is, in fact, pretty elementary. Suppose that

$$\mathbf{z}(0) = \mathbf{y}(0) = \mathbf{x}_0,$$

so that $\mathbf{y} - \mathbf{z} \in \mathbf{H}$. Then

$$\begin{aligned} \mathbf{y}(t) - \mathbf{z}(t) &= \int_0^t (\mathbf{y}'(s) - \mathbf{z}'(s)) ds \\ &= \int_0^t (\mathbf{y}'(s) - \mathbf{f}(\mathbf{y}(s))) ds + \int_0^t (\mathbf{f}(\mathbf{y}(s)) - \mathbf{f}(\mathbf{z}(s))) ds \\ &\quad + \int_0^t (\mathbf{f}(\mathbf{z}(s)) - \mathbf{z}'(s)) ds. \end{aligned}$$

From here, we immediately find

$$|\mathbf{y}(t) - \mathbf{z}(t)|^2 \leq C(E(\mathbf{y}) + E(\mathbf{z})) + CM^2 \int_0^t |\mathbf{y}(s) - \mathbf{z}(s)|^2 ds,$$

if M is the Lipschitz constant for the map \mathbf{f} , and C is a generic, universal constant we will not care to change. From Gronwall's lemma, we can have

$$|\mathbf{y}(t) - \mathbf{z}(t)|^2 \leq C(E(\mathbf{y}) + E(\mathbf{z}))e^{CM^2T} \quad (2.3)$$

for all $t \in [0, T]$ (check for instance [1], page 145, for this version of Gronwall's inequality).

This means

$$\begin{aligned} \|\mathbf{y} - \mathbf{z}\|_{L^\infty(0, T; \mathbb{R}^N)} &\leq e^{CM^2T/2} \sqrt{C(E(\mathbf{y}) + E(\mathbf{z}))}, \\ \|\mathbf{y} - \mathbf{z}\|_{L^2(0, T; \mathbb{R}^N)}^2 &\leq CT e^{CM^2T} (E(\mathbf{y}) + E(\mathbf{z})). \end{aligned} \quad (2.4)$$

But once we can rely on this information, the above decomposition allows us to write in a similar manner

$$\int_0^t |\mathbf{y}'(s) - \mathbf{z}'(s)|^2 ds \leq C(E(\mathbf{y}) + E(\mathbf{z})) + CM^2 \int_0^t |\mathbf{y}(s) - \mathbf{z}(s)|^2 ds$$

and

$$\|\mathbf{y}' - \mathbf{z}'\|_{L^2(0,T;\mathbb{R}^N)}^2 \leq C(E(\mathbf{y}) + E(\mathbf{z})) + CM^2 \|\mathbf{y} - \mathbf{z}\|_{L^2(0,T;\mathbb{R}^N)}^2,$$

and thus, taking into account (2.4),

$$\|\mathbf{y}' - \mathbf{z}'\|_{L^2(0,T;\mathbb{R}^N)}^2 \leq (C + C^2M^2Te^{CM^2T})(E(\mathbf{y}) + E(\mathbf{z})).$$

Our estimate (1.8) is then a consequence that the norm in \mathbf{H} can be taken to be the L^2 -norm of the derivative. \square

The second basic ingredient is (1.7). We will be using Proposition 1.7. We assume further that the mapping \mathbf{f} is smooth with a derivative uniformly bounded to guarantee the uniform Lipschitz condition. For the operator \mathbf{T} , we will put $\mathbf{Z} = \mathbf{Tz}$ for $\mathbf{z} \in \mathbf{H}$, and linearize (2.1) at the path $\mathbf{x}_0 + \mathbf{z}(t)$ to write

$$\begin{aligned} \mathbf{Z}'(t) &= \mathbf{f}(\mathbf{x}_0 + \mathbf{z}(t)) + \nabla \mathbf{f}(\mathbf{x}_0 + \mathbf{z}(t))\mathbf{Z}(t) - \mathbf{z}'(t) \text{ in } [0, T], \\ \mathbf{Z}(0) &= \mathbf{0}. \end{aligned} \tag{2.5}$$

This is a linear, differential, non-constant coefficient system for \mathbf{Z} with coefficients depending on \mathbf{z} . Due to the smoothness condition assumed on \mathbf{f} , such operator \mathbf{T} is locally Lipschitz because the image $\mathbf{Z} = \mathbf{Tz}$ is defined through a linear initial-value, Cauchy problem with coefficients depending continuously on \mathbf{z} .

The important property to be checked, concerning \mathbf{T} , is (1.10). It is elementary to see, under smoothness assumptions which, as indicated, we have taken for granted, that

$$\langle E'(\mathbf{z}), \mathbf{Z} \rangle = \int_0^T (\mathbf{z}'(s) - \mathbf{f}(\mathbf{x}_0 + \mathbf{z}(s)))(\mathbf{Z}'(s) - \nabla \mathbf{f}(\mathbf{x}_0 + \mathbf{z}(s))\mathbf{Z}(s)) ds$$

Hence, for $\mathbf{Z} = \mathbf{Tz}$ coming from (2.5), we immediately deduce that

$$\langle E'(\mathbf{z}), \mathbf{Z} \rangle = -2E(\mathbf{z}).$$

We are, then, entitled to apply Proposition 1.7 to conclude that functional E in (2.2) is an error functional after Definition 1.5, and we are entitled to utilize Proposition 1.6 to conclude the following.

Theorem 2.3. *If the mapping $\mathbf{f}(\mathbf{y}) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is C^1 - with a globally bounded gradient, then, for arbitrary $\mathbf{x}_0 \in \mathbb{R}^N$ and $T > 0$, problem (2.1) admits a unique C^1 -solution*

$$\bar{\mathbf{x}}(t) : [0, T] \rightarrow \mathbb{R}^N,$$

and there is a positive constant C such that

$$\|\mathbf{x} - \bar{\mathbf{x}}\|_{\mathbf{H}}^2 \leq C \int_0^T |\mathbf{x}'(s) - \mathbf{f}(\mathbf{x}_0 + \mathbf{x}(s))|^2 ds$$

for every $\mathbf{x} \in \mathbf{H}$.

There is no difficulty in showing a local version of this result by using the same ideas.

3. Linear hyperbolic example

Since most likely readers will not be used to think about hyperbolic problems in these terms, we will treat the most transparent example of a linear, hyperbolic problem from this perspective, and later apply the method to a non-linear wave equation.

We seek a (weak) solution $u(t, \mathbf{x})$ of some sort of the problem

$$\begin{aligned} u_{tt}(t, \mathbf{x}) - \Delta u(t, \mathbf{x}) - u(t, \mathbf{x}) &= f(t, \mathbf{x}) \text{ in } \mathbb{R}_+^N, \\ u(0, \mathbf{x}) = 0, u_t(0, \mathbf{x}) &= 0 \text{ on } t = 0, \end{aligned} \tag{3.1}$$

for $f \in L^2(\mathbb{R}_+^N)$. Here \mathbb{R}_+^N is the upper half hyperspace $[0, +\infty) \times \mathbb{R}^N$. We look for

$$u(t, \mathbf{x}) \in H_0^1(\mathbb{R}_+^N)$$

(jointly in time and space) such that

$$\int_{\mathbb{R}_+^N} [u_t(t, \mathbf{x})w_t(t, \mathbf{x}) - \nabla u(t, \mathbf{x}) \cdot \nabla w(t, \mathbf{x}) + (f(t, \mathbf{x}) + u(t, \mathbf{x}))w(t, \mathbf{x})] \, d\mathbf{x} \, dt = 0 \tag{3.2}$$

for every test function

$$w(t, \mathbf{x}) \in H^1(\mathbb{R}_+^N).$$

Note how the arbitrary values of the test function w for $t = 0$ imposes the vanishing initial velocity $u_t(0, \mathbf{x}) = 0$.

To setup a suitable error functional

$$E(u) : H_0^1(\mathbb{R}_+^N) \rightarrow \mathbb{R}^+$$

for every $u(t, \mathbf{x})$, and not just for the solution we seek, we utilize a natural least-square concept as indicated in the Introduction. Define an appropriate defect or residual function

$$U(t, \mathbf{x}) \in H^1(\mathbb{R}_+^N),$$

for each such $u \in H_0^1(\mathbb{R}_+^N)$, as the unique variational solution of

$$\int_{\mathbb{R}_+^N} [(u_t + U_t)w_t - (\nabla u - \nabla U) \cdot \nabla w + (f + u + U)w] \, d\mathbf{x} \, dt = 0 \tag{3.3}$$

valid for every $w \in H^1(\mathbb{R}_+^N)$. This function U is indeed the unique minimizer over $H^1(\mathbb{R}_+^N)$ of the strictly convex, quadratic functional

$$I(w) = \int_{\mathbb{R}_+^N} \left(\frac{1}{2} [(w_t + u_t)^2 + |\nabla w - \nabla u|^2 + (u + w)^2] + fw \right) \mathbf{d}\mathbf{x} dt$$

for each fixed u . The size of U is regarded as a measure of the departure of u from being a solution of our problem

$$E : H_0^1(\mathbb{R}_+^N) \rightarrow \mathbb{R}^+,$$

$$E(u) = \int_{\mathbb{R}_+^N} \frac{1}{2} [U_t^2(t, \mathbf{x}) + |\nabla U(t, \mathbf{x})|^2 + U^2(t, \mathbf{x})] \mathbf{d}\mathbf{x} dt. \quad (3.4)$$

We can also put, in a short form,

$$E(u) = \frac{1}{2} \|U\|_{H^1(\mathbb{R}_+^N)}^2; \quad (3.5)$$

or even

$$E(u) = \frac{1}{2} \|u_{tt} - \Delta u - u - f\|_{H^{-1}(\mathbb{R}_+^N)}^2,$$

though we will stick to (3.5) to better manipulate E . Note that the definition of U in terms of u amounts to

$$U_{tt} + \Delta U = u_{tt} - \Delta u - u - f \text{ in } \mathbb{R}_+^N.$$

We would like to apply Proposition 1.6 in this situation, and hence, we set to ourselves the task of checking the two main assumptions in Definition 1.5. Our functional E is definitely smooth and non-negative to begin with.

It is not surprising that in order to work with the wave equation the following two linear operators

$$\mathbb{S} : H^1(\mathbb{R}_+^N) \mapsto H^1(\mathbb{R}_+^N), \quad \mathbb{S}w(t, \mathbf{x}) = w(t, -\mathbf{x}),$$

$$\mathcal{S} : H^1(\mathbb{R}_+^N) \mapsto H^1(\mathbb{R}_+^N)^*,$$

$$\mathcal{S}u(t, \mathbf{x}) = (u(t, -\mathbf{x}), u_t(t, -\mathbf{x}), \nabla u(t, -\mathbf{x})),$$

will play a role. $H^1(\mathbb{R}_+^N)^*$ is here the dual space of $H^1(\mathbb{R}_+^N)$, not to be mistaken with $H^{-1}(\mathbb{R}_+^N)$. Put $\mathbf{H} = \mathcal{S}(H_0^1(\mathbb{R}_+^N))$. The following fact is elementary. Check for instance [3].

Lemma 3.1. (1) *The map \mathbb{S} is an isometry.*

(2) *\mathbf{H} is a closed subspace of $H^1(\mathbb{R}_+^N)^*$, and*

$$\mathcal{S} : H_0^1(\mathbb{R}_+^N) \rightarrow \mathbf{H}$$

is a bijective, continuous mapping. In fact, we clearly have

$$\|u\|_{H_0^1(\mathbb{R}_+^N)} \leq \|\mathcal{S}u\|_{H^1(\mathbb{R}_+^N)^*}. \quad (3.6)$$

We can now proceed to prove inequality (1.8) in this new context.

Proposition 3.2. *There is a constant $K > 0$ such that*

$$\|u - v\|_{H_0^1(\mathbb{R}_+^N)}^2 \leq K(E(u) + E(v)),$$

for every pair $u, v \in H_0^1(\mathbb{R}_+^N)$.

Proof. Let $U, V \in H^1(\mathbb{R}^N)$ be the respective residual functions associated with u and v . Because we are in a linear situation, if we replace

$$u - v \mapsto u, \quad U - V \mapsto U,$$

we would have

$$\int_{\mathbb{R}_+^N} [(u_t + U_t)w_t - (\nabla u - \nabla U) \cdot \nabla w + (u + U)w] \, dx \, dt = 0, \quad (3.7)$$

for every $w \in H^1(\mathbb{R}_+^N)$. If we use $\mathbb{S}w$ in (3.7) instead of w , we immediately find

$$\int_{\mathbb{R}_+^N} [(u_t + U_t)w_t(t, -\mathbf{x}) + (\nabla u - \nabla U) \cdot \nabla w(t, -\mathbf{x}) + (u + U)w(t, -\mathbf{x})] \, dx \, dt = 0.$$

The terms involving U can be written in compact form as

$$\langle U, \mathbb{S}w \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N) \rangle}$$

while a natural change of variables in the terms involving u leads to writing these in the form

$$\langle w, \mathcal{S}u \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle}.$$

Hence, for every $w \in H^1(\mathbb{R}_+^N)$, we find

$$\langle U, \mathbb{S}w \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N) \rangle} + \langle w, \mathcal{S}u \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle} = 0.$$

Bearing in mind this identity, we have, through the Lemma 3.1,

$$\begin{aligned} \|u\|_{H_0^1(\mathbb{R}_+^N)} &\leq \|\mathcal{S}u\|_{H^1(\mathbb{R}_+^N)^*} \\ &= \sup_{\|w\|_{H^1(\mathbb{R}_+^N)} \leq 1} \langle w, \mathcal{S}u \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle} \\ &\leq \|U\|_{H^1(\mathbb{R}_+^N)} \|w\|_{H^1(\mathbb{R}_+^N)} \\ &\leq \|U\|_{H^1(\mathbb{R}_+^N)}. \end{aligned}$$

If we go back to

$$u \mapsto u - v, \quad U \mapsto U - V,$$

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we are led to

$$\begin{aligned} \|u - v\|_{H_0^1(\mathbb{R}_+^N)}^2 &\leq \|U - V\|_{H^1(\mathbb{R}_+^N)}^2 \\ &\leq C \left(\|U\|_{H^1(\mathbb{R}_+^N)}^2 + \|V\|_{H^1(\mathbb{R}_+^N)}^2 \right) \\ &\leq C (E(u) + E(v)), \end{aligned}$$

for some constant $C > 0$. □

The second main ingredient to apply Proposition 1.6 is to show that E defined in (3.4) complies with (1.7) too. To this end, we need to compute the derivative $E'(u)$, and so we perform a perturbation

$$u + \epsilon v \mapsto U + \epsilon V,$$

in (3.3) to write

$$\begin{aligned} \int_{\mathbb{R}_+^N} [(u_t + \epsilon v_t + U_t + \epsilon V_t)w_t - (\nabla u + \epsilon \nabla v - \nabla U - \epsilon \nabla V) \cdot \nabla w \\ + (f + u + \epsilon v + U + \epsilon V)w] \, dx \, dt = 0. \end{aligned}$$

The term to order 1 in ϵ should vanish

$$\int_{\mathbb{R}_+^N} [(v_t + V_t)w_t - (\nabla v - \nabla V) \cdot \nabla w + (v + V)w] \, dx \, dt = 0 \quad (3.8)$$

for every $w \in H^1(\mathbb{R}_+^N)$. On the other hand, by differentiating

$$E(u + \epsilon v) = \int_{\mathbb{R}_+^N} \frac{1}{2} ((U + \epsilon V)_t^2 + |\nabla U + \epsilon \nabla V|^2 + (U + \epsilon V)^2) \, dx \, dt,$$

with respect to ϵ , and setting $\epsilon = 0$, we arrive at

$$\langle E'(u), v \rangle = \int_{\mathbb{R}_+^N} (U_t V_t + \nabla U \cdot \nabla V + UV) \, dx \, dt.$$

By taking $w = U$ in (3.8), we can also write

$$\begin{aligned} \langle E'(u), v \rangle &= \int_{\mathbb{R}_+^N} (\nabla v \cdot \nabla U - v_t U_t - vU) \, dx \, dt \\ &= - \langle \mathbb{S}v, \mathcal{S}U \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle}. \end{aligned}$$

From this identity, which ought to be valid for every $v \in H_0^1(\mathbb{R}_+^N)$, we clearly conclude that if $E'(u) \rightarrow \mathbf{0}$ then $\mathcal{S}U \rightarrow \mathbf{0}$ as well, because \mathbb{S} preserves the norm. Realizing that

$$E(u) = \frac{1}{2} \|U\|_{H^1(\mathbb{R}_+^N)}^2 \leq \frac{1}{2} \|\mathcal{S}U\|_{H^1(\mathbb{R}_+^N)^*}^2,$$

by estimate (3.6), we conclude the following.

Proposition 3.3. *The functional E in (3.4) is an error functional in the sense of Definition 1.5.*

Our main abstract result Proposition 1.6 applies in this situation too, and we can conclude

Theorem 3.4. *Problem (3.1) admits a unique weak solution $u \in H_0^1(\mathbb{R}_+^N)$ in the sense (3.2), and for every other $v \in H_0^1(\mathbb{R}_+^N)$, we have*

$$\|u - v\|_{H_0^1(\mathbb{R}_+^N)}^2 \leq KE(v),$$

for some positive constant K .

4. Non-linear monotone problems

Suppose we would like to solve, or approximate the solution of, a certain non-linear elliptic system of PDEs of the form

$$\operatorname{div}[\Phi(\nabla u)] = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

for a non-linear, smooth map

$$\Phi(\mathbf{a}) : \mathbb{R}^N \rightarrow \mathbb{R}^N.$$

$\Omega \subset \mathbb{R}^N$ is assumed to be a regular, bounded domain. One can set up a natural, suitable, non-negative, smooth functional based on the least-squares idea, as already introduced,

$$E(v) : H_0^1(\Omega) \rightarrow \mathbb{R} \tag{4.1}$$

by putting

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla U(\mathbf{x})|^2 \, d\mathbf{x}$$

where

$$\operatorname{div}[\Phi(\nabla v + \nabla u_0) + \nabla U] = 0 \text{ in } \Omega$$

and $U \in H_0^1(\Omega)$. We can also put

$$E(v) = \frac{1}{2} \|\operatorname{div}[\Phi(\nabla v + \nabla u_0)]\|_{H^{-1}(\Omega)}^2.$$

Our goal is to apply again Proposition 1.6, or, since we are now in a non-linear situation, Proposition 1.7. Anyhow, (1.8) is necessary.

Lemma 4.1. *Let $\Phi(\mathbf{a}) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a smooth-map with linear growth at infinity, i.e.*

$$|\Phi(\mathbf{a})| \leq C_1 |\mathbf{a}| + C_0, \tag{4.2}$$

with $C_1 > 0$, and strictly monotone in the sense

$$(\Phi(\mathbf{a}_1) - \Phi(\mathbf{a}_0)) \cdot (\mathbf{a}_1 - \mathbf{a}_0) \geq c |\mathbf{a}_1 - \mathbf{a}_0|^2, \quad c > 0, \tag{4.3}$$

for every pair of vectors \mathbf{a}_i , $i = 0, 1$. Then there is a positive constant C such that

$$\|u - v\|_{H_0^1(\Omega)}^2 \leq C(E(u) + E(v)),$$

for every pair $u, v \in H_0^1(\Omega)$.

Proof. Let $u, v \in H_0^1(\Omega)$, and let $U, V \in H_0^1(\Omega)$ be their respective residuals in the sense

$$\operatorname{div}[\Phi(\nabla u + \nabla u_0) + \nabla U] = 0, \quad \operatorname{div}[\Phi(\nabla v + \nabla u_0) + \nabla V] = 0 \quad (4.4)$$

in Ω , and

$$E(u) = \frac{1}{2} \|\nabla U\|_{L^2(\Omega; \mathbb{R}^N)}^2, \quad E(v) = \frac{1}{2} \|\nabla V\|_{L^2(\Omega; \mathbb{R}^N)}^2.$$

If we use $u - v$ as test field in (4.4), we find

$$\begin{aligned} \int_{\Omega} \Phi(\nabla u + \nabla u_0) \cdot (\nabla u - \nabla v) \, d\mathbf{x} &= - \int_{\Omega} \nabla U \cdot (\nabla u - \nabla v) \, d\mathbf{x}, \\ \int_{\Omega} \Phi(\nabla v + \nabla u_0) \cdot (\nabla u - \nabla v) \, d\mathbf{x} &= - \int_{\Omega} \nabla V \cdot (\nabla u - \nabla v) \, d\mathbf{x}. \end{aligned}$$

The monotonicity condition, together with these identities, takes us, by subtracting one from the other, to

$$c \int_{\Omega} |\nabla u - \nabla v|^2 \, d\mathbf{x} \leq \int_{\Omega} (\nabla V - \nabla U) \cdot (\nabla u - \nabla v) \, d\mathbf{x}.$$

The standard Cauchy–Schwarz inequality implies that

$$c \|\nabla u - \nabla v\|_{L^2(\Omega; \mathbb{R}^N)} \leq \|\nabla U - \nabla V\|_{L^2(\Omega; \mathbb{R}^N)},$$

and thus, taking into account the triangular inequality, we have

$$c^2 \|\nabla u - \nabla v\|_{L^2(\Omega; \mathbb{R}^N)}^2 \leq 4E(u) + 4E(v).$$

The use of Poincaré’s inequality yields our statement. \square

The second ingredient, to apply Theorem 1.7, is the operator \mathbf{T} which comes directly from linearization (or from Newton’s method) as follows. Given an approximation of the solution $v + u_0$, $v \in H_0^1(\Omega)$, we seek a better approximation $V \in H_0^1(\Omega)$ in the form

$$\operatorname{div}[\Phi(\nabla v + \nabla u_0) + \nabla \Phi(\nabla v + \nabla u_0) \nabla V] = 0 \text{ in } \Omega, \quad (4.5)$$

as a linear approximation of

$$\operatorname{div}[\Phi(\nabla v + \nabla u_0) + \nabla V] = 0 \text{ in } \Omega.$$

We therefore define

$$\mathbf{T} : H_0^1(\Omega) \rightarrow H_0^1(\Omega), \quad \mathbf{T}v = V, \quad (4.6)$$

where V is the solution of (4.5). The fact that \mathbf{T} is well-defined is a direct consequence of the standard Lax–Milgram lemma and the identification

$$\mathbf{A} = \nabla\Phi(\nabla v + \nabla u_0), \quad \mathbf{a} = \Phi(\nabla v + \nabla u_0),$$

provided

$$|\nabla\Phi(\mathbf{v})| \leq M, \quad \mathbf{u} \cdot \nabla\Phi(\mathbf{v})\mathbf{u} \geq c|\mathbf{u}|^2, \quad M, c > 0.$$

The first bound is compatible with linear growth at infinity, (4.2), while the second one is a consequence of monotonicity (4.3). On the other hand, the smoothness of \mathbf{T} depends directly on the smoothness of Φ , specifically, we assume Φ to be C^2 . Since \mathbf{T} comes from Newton's method, condition (1.10) is guaranteed. We are hence entitled to apply Proposition 1.7 and conclude that

Theorem 4.2. *Let $\Phi(\mathbf{a}) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a C^2 -mapping such that*

$$|\nabla\Phi(\mathbf{a})| \leq M,$$

$$(\Phi(\mathbf{a}_1) - \Phi(\mathbf{a}_0)) \cdot (\mathbf{a}_1 - \mathbf{a}_0) \geq c|\mathbf{a}_1 - \mathbf{a}_0|^2,$$

for constants $M, c > 0$, and every $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_0$ in \mathbb{R}^N . There is a unique weak solution $u \in u_0 + H_0^1(\Omega)$, for arbitrary $u_0 \in H^1(\Omega)$, of the equation

$$\operatorname{div}[\Phi(\nabla u)] = 0 \text{ in } \Omega, \quad u - u_0 \in H_0^1(\Omega).$$

Moreover

$$\|u - v\|_{H_0^1(\Omega)}^2 \leq CE(v)$$

for every other $v \in u_0 + H_0^1(\Omega)$.

It is not hard to design appropriate sets of assumptions to deal with more general equations of the form

$$\operatorname{div}[\Phi(\nabla v(\mathbf{x}), v(\mathbf{x}), \mathbf{x})] = 0.$$

5. Non-linear waves

We would like to explore non-linear equations of the form

$$u_{tt}(t, \mathbf{x}) - \Delta u(t, \mathbf{x}) - f(\nabla u(t, \mathbf{x}), u_t(t, \mathbf{x}), u(t, \mathbf{x})) = 0 \text{ in } (t, \mathbf{x}) \in \mathbb{R}_+^N,$$

subjected to initial conditions

$$u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x})$$

for appropriate data u_0 and u_1 belonging to suitable spaces to be determined. Dimension N is taken to be at least two. Though more complicated situations could be considered allowing for a monotone main part in the equation, as in the previous section, to better

understand the effect of the term incorporating lower-order terms, we will restrict ourselves to the equation above. Conditions on the non-linear term

$$f(\mathbf{z}, z, u) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

will be specified along the way as needed.

Our ambient space will be $H^1(\mathbb{R}_+^N)$ so that weak solutions u are sought in $H^1(\mathbb{R}_+^N)$. If we assume sufficient regularity on data u_0 and u_1 , the standard change of variables

$$u(t, \mathbf{x}) \mapsto u(t, \mathbf{x}) - u_0(\mathbf{x}) - tu_1(\mathbf{x}),$$

permits us to stick to the situation where both u_0 and u_1 identically vanish and $u \in H_0^1(\mathbb{R}_+^N)$, at the expense of permitting

$$f(\mathbf{z}, z, u, t, \mathbf{x}) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}.$$

We will therefore stick to the problem

$$u_{tt} - \Delta u - f(\nabla u, u_t, u, t, \mathbf{x}) = 0 \text{ in } (t, \mathbf{x}) \in \mathbb{R}_+^N, \quad (5.1)$$

subjected to initial conditions

$$u(0, \mathbf{x}) = 0, \quad u_t(0, \mathbf{x}) = 0. \quad (5.2)$$

A weak solution $u \in H_0^1(\mathbb{R}_+^N)$ of (5.1) is such that

$$\int_{\mathbb{R}_+^N} [-u_t w_t + \nabla u \cdot \nabla w - f(\nabla u, u_t, u, t, \mathbf{x})w] \, dx \, dt = 0 \quad (5.3)$$

for every $w \in H^1(\mathbb{R}_+^N)$. This weak formulation asks for the non-linear term recorded in the function f to comply with

$$|f(\mathbf{z}, z, u, t, \mathbf{x})| \leq C(|\mathbf{z}| + |z| + |u|^{(N+1)/(N-1)}) + f_0(t, \mathbf{x}) \quad (5.4)$$

for a function $f_0 \in L^2(\mathbb{R}_+^N)$, in such a way that the composition

$$f(\nabla u, u_t, u, t, \mathbf{x}) \in L^2(\mathbb{R}_+^N)$$

for every $u \in H^1(\mathbb{R}_+^N)$. As expected, for every $u \in H_0^1(\mathbb{R}_+^N)$ we define its residual $U \in H^1(\mathbb{R}_+^N)$ through

$$\int_{\mathbb{R}_+^N} [(U_t + u_t)w_t - (\nabla u - \nabla U) \cdot \nabla w + (U + f(\nabla u, u_t, u, t, \mathbf{x}))w] \, dx \, dt = 0 \quad (5.5)$$

which ought to be correct for every test $w \in H^1(\mathbb{R}_+^N)$; and the functional

$$E(u) : H_0^1(\mathbb{R}_+^N) \rightarrow \mathbb{R}^+, \quad E(u) = \frac{1}{2} \|U\|_{H^1(\mathbb{R}_+^N)}^2, \quad (5.6)$$

as a measure of departure of u from being a weak solution of (5.1). Note how (5.5) determines U in a unique way. Indeed, such U is the unique minimizer of the strictly convex, quadratic functional

$$\frac{1}{2} \int_{\mathbb{R}_+^N} [(U_t + u_t)^2 + |\nabla U - \nabla u|^2 + (U + f(\nabla u, u_t, u, t, \mathbf{x}))^2] \, d\mathbf{x} \, dt$$

defined for $U \in H^1(\mathbb{R}_+^N)$. We claim that under appropriate additional hypotheses, we can apply Proposition 1.7 to this situation. To explain things in the most affordable way, however, we will show that Proposition 1.6 can also be applied directly. This requires to check that E in (5.6) is indeed an error functional in the sense of Definition 1.5.

We will be using the operators and the formalism right before Lemma 3.1, as well as bound (3.6) in this lemma.

Lemma 5.1. *Suppose the function $f(\mathbf{z}, z, u, t, \mathbf{x})$ is such that*

$$(1) \, f(\mathbf{0}, 0, 0, t, \mathbf{x}) \in L^2(\mathbb{R}_+^N);$$

(2) *the difference $f(\mathbf{z}, z, u, t, \mathbf{x}) - u$ is globally Lipschitz with respect to triplets (\mathbf{z}, z, u) in the sense*

$$|f(\mathbf{z}, z, u, t, \mathbf{x}) - u - f(\mathbf{y}, y, v, t, \mathbf{x}) + v| \leq M \left(|\mathbf{z} - \mathbf{y}| + |z - y| + \frac{1}{D} |u - v|^{(N+1)/(N-1)} \right),$$

where D is the constant of the corresponding embedding

$$H^1(\mathbb{R}_+^N) \subset L^{2(N+1)/(N-1)}(\mathbb{R}_+^N),$$

and $M < 1$.

Then there is a positive constant K with

$$\|u - v\|_{H_0^1(\mathbb{R}_+^N)}^2 \leq K(E(u) + E(v)),$$

for every pair $u, v \in H_0^1(\mathbb{R}_+^N)$, where E is given in (5.6).

Proof. Note how our hypotheses on the nonlinearity f imply the bound (5.4) by taking

$$\mathbf{y} = \mathbf{0}, \quad y = v = 0, \quad f_0(t, \mathbf{x}) = f(\mathbf{0}, 0, 0, t, \mathbf{x}).$$

If u, v belong to $H_0^1(\mathbb{R}_+^N)$, and U, V in $H^1(\mathbb{R}_+^N)$ are their respective residuals, then

$$\int_{\mathbb{R}_+^N} [(U_t + u_t)w_t - (\nabla u - \nabla U) \cdot \nabla w + (U + f(\nabla u, u_t, u, t, \mathbf{x}))w] \, d\mathbf{x} \, dt = 0,$$

$$\int_{\mathbb{R}_+^N} [(V_t + v_t)w_t - (\nabla v - \nabla V) \cdot \nabla w + (V + f(\nabla v, v_t, v, t, \mathbf{x}))w] \, d\mathbf{x} \, dt = 0,$$

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for every $w \in H^1(\mathbb{R}_+^N)$. By subtracting one from the other, and letting

$$s = u - v, \quad S = U - V,$$

we find

$$\begin{aligned} \int_{\mathbb{R}_+^N} [(S_t + s_t)w_t - (\nabla s - \nabla S) \cdot \nabla w \\ + (S + f(\nabla u, u_t, u, t, \mathbf{x}) - f(\nabla v, v_t, v, t, \mathbf{x}))w] \, d\mathbf{x} \, dt = 0, \end{aligned}$$

for every $w \in H^1(\mathbb{R}_+^N)$. We can recast this identity, by using the formalism in the corresponding linear situation around Lemma 3.1, as

$$\begin{aligned} \langle S, \mathbb{S}w \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N) \rangle} + \langle w, \mathcal{S}s \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle} \\ = - \int_{\mathbb{R}_+^N} [f(\nabla u, u_t, u, t, \mathbf{x}) - f(\nabla v, v_t, v, t, \mathbf{x}) - s]w \, d\mathbf{x} \, dt. \end{aligned}$$

The same manipulations as in the proof of Proposition 3.2, together with the assumed Lipschitz property on f , lead immediately to

$$\begin{aligned} \|s\|_{H_0^1(\mathbb{R}_+^N)} &\leq \| \mathcal{S}s \|_{H^1(\mathbb{R}_+^N)^*} \\ &= \sup_{\|w\|_{H^1(\mathbb{R}_+^N)} \leq 1} |\langle w, \mathcal{S}s \rangle_{\langle H^1(\mathbb{R}_+^N), H^1(\mathbb{R}_+^N)^* \rangle}| \\ &= \sup_{\|w\|_{H^1(\mathbb{R}_+^N)} \leq 1} \left| \langle S, \mathbb{S}w \rangle + \int_{\mathbb{R}_+^N} (f^u - f^v - s)w \, d\mathbf{x} \, dt \right| \\ &\leq \|S\|_{H^1(\mathbb{R}_+^N)} \|w\|_{H^1(\mathbb{R}_+^N)} + M \|s\|_{H_0^1(\mathbb{R}_+^N)} \|w\|_{H^1(\mathbb{R}_+^N)} \\ &\leq \|S\|_{H^1(\mathbb{R}_+^N)} + M \|s\|_{H_0^1(\mathbb{R}_+^N)}. \end{aligned}$$

We are putting

$$f^u = f(\nabla u, u_t, u, t, \mathbf{x}), \quad f^v = f(\nabla v, v_t, v, t, \mathbf{x}),$$

for the sake of notation. Note also the use of the embedding constant. The resulting final inequality, and the relative sizes of these constants, show our claim. \square

We turn to the second important property for E to become an error functional, namely,

$$\lim_{E'(\mathbf{u}) \rightarrow \mathbf{0}} E(\mathbf{u}) = 0$$

over bounded subsets of $H_0^1(\mathbb{R}_+^N)$. We assume that the non-linearity f is C^1 - with respect to (\mathbf{z}, z, u) , and its partial derivatives are uniformly bounded. To compute the derivative $E'(u)$ at an arbitrary $u \in H_0^1(\mathbb{R}_+^N)$, we perform, as usual, the perturbation to first-order

$$u \mapsto u + \epsilon v, \quad U \mapsto U + \epsilon V,$$

and introduce them in (5.5). After differentiation with respect to ϵ , and setting $\epsilon = 0$, we find

$$\int_{\mathbb{R}_+^N} [(V_t + v_t)w_t - (\nabla v - \nabla V) \cdot \nabla w + (V + \bar{f}_z \cdot \nabla v + \bar{f}_z v_t + \bar{f}_u v)w] \, d\mathbf{x} \, dt = 0, \quad (5.7)$$

for all $w \in H^1(\mathbb{R}_+^N)$, where

$$\bar{f}_z(t, \mathbf{x}) = f_z(\nabla u(t, \mathbf{x}), u_t(t, \mathbf{x}), u(t, \mathbf{x}), t, \mathbf{x}),$$

and the same for $\bar{f}_z(t, \mathbf{x})$ and $\bar{f}_u(t, \mathbf{x})$. On the other hand,

$$\langle E'(u), v \rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (E(u + \epsilon v) - E(u))$$

is clearly given by

$$\langle E'(u), v \rangle = \int_{\mathbb{R}_+^N} (\nabla U \cdot \nabla V + U_t V_t + U V) \, d\mathbf{x} \, dt.$$

If we use $w = U$ in (5.7), we can write

$$\langle E'(u), v \rangle = \int_{\mathbb{R}_+^N} \left[-v_t U_t + \nabla v \cdot \nabla U - (\bar{f}_z \cdot \nabla v + \bar{f}_z v_t + \bar{f}_u v)U \right] \, d\mathbf{x} \, dt.$$

The validity of this representation for every $v \in H_0^1(\mathbb{R}_+^N)$ enables us to identify $E'(u)$ with the triplet

$$(-U_t - \bar{f}_z U, \nabla U - U \bar{f}_z, -\bar{f}_u U),$$

in the sense $E'(u) = \mathcal{S}U$ where the linear operator

$$\mathcal{S} : H^1(\mathbb{R}_+^N) \mapsto H^{-1}(\mathbb{R}_+^N)$$

is precisely determined by

$$\langle \mathcal{S}U, v \rangle = \int_{\mathbb{R}_+^N} \left[-v_t U_t + \nabla v \cdot \nabla U - (\bar{f}_z \cdot \nabla v + \bar{f}_z v_t + \bar{f}_u v)U \right] \, d\mathbf{x} \, dt$$

for every $v \in H_0^1(\mathbb{R}_+^N)$. Notice how this operator \mathcal{S} is well-defined because the non-linearity f has been assumed to be globally Lipschitz with partial derivatives uniformly bounded. To conclude that $E'(u) \rightarrow \mathbf{0}$ implies $U \rightarrow 0$ and, hence $E(u) = 0$, we need to ensure that this operator \mathcal{S} is injective. We conjecture that this is so, without further requirements; but to simplify the argument here, we add the assumption that

$$|f_u(\mathbf{z}, z, u, t, z)| \geq \epsilon > 0$$

for every (\mathbf{z}, z, u, t, z) . Under this additional hypothesis, the condition

$$(-U_t - \bar{f}_z U, \nabla U - U \bar{f}_z, -\bar{f}_u U) = \mathbf{0}$$

automatically implies

$$U = \nabla U = U_t = 0$$

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and hence $E(u) = 0$.

Theorem 5.2. *Suppose the non-linearity $f(\mathbf{z}, z, u, t, \mathbf{x})$ is C^1 - with respect to variables (\mathbf{z}, z, u) , and:*

$$(1) f(\mathbf{0}, 0, 0, t, \mathbf{x}) \in L^2(\mathbb{R}_+^N);$$

(2) *the difference $f(\mathbf{z}, z, u, t, \mathbf{x}) - u$ is Lipschitz with respect to triplets (\mathbf{z}, z, u) in the sense*

$$|f(\mathbf{z}, z, u, t, \mathbf{x}) - u - f(\mathbf{y}, y, v, t, \mathbf{x}) + v| \leq M \left(|\mathbf{z} - \mathbf{y}| + |z - y| + \frac{1}{D} |u - v|^{(N+1)/(N-1)} \right),$$

where D is the constant of the corresponding embedding

$$H^1(\mathbb{R}_+^N) \subset L^{2(N+1)/(N-1)}(\mathbb{R}_+^N),$$

and $M < 1$;

(3) *non-vanishing of f_u : there is some $\epsilon > 0$ with*

$$|f_u(\mathbf{z}, z, u, t, \mathbf{x})| \geq \epsilon > 0.$$

Then the problem

$$u_{tt} - \Delta u + f(\nabla u, u_t, u, t, \mathbf{x}) = 0 \text{ in } (t, \mathbf{x}) \in \mathbb{R}_+^N$$

under vanishing initial conditions

$$u(0, \mathbf{x}) = u_t(0, \mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^N,$$

admits a unique weak solution $u \in H_0^1(\mathbb{R}_+^N)$ in the sense (5.3), and

$$\|v - u\|_{H_0^1(\mathbb{R}_+^N)}^2 \leq KE(v),$$

for any other $v \in H_0^1(\mathbb{R}_+^N)$.

Without the global Lipschitzianity condition on f in the previous statement, but only the smoothness with respect to triplets (\mathbf{z}, z, u) , only a local existence result is possible. This is standard.

6. The steady Navier–Stokes system

For a bounded, Lipschitz, connected domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, we are concerned with the steady Navier–Stokes system

$$-\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla u = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad (6.1)$$

for a vector field $\mathbf{u} \in H_0^1(\Omega; \mathbb{R}^N)$, and a scalar, pressure field $u \in L^2(\Omega)$. The external force field \mathbf{f} is assumed to belong to the dual space $H^{-1}(\Omega; \mathbb{R}^N)$. The parameter $\nu > 0$ is viscosity. Because of the incompressibility condition, the system can also be written in the form

$$-\nu \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla u = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega.$$

A weak solution is a divergence-free vector field $\mathbf{u} \in H_0^1(\Omega; \mathbb{R}^N)$, and a scalar field $u \in L^2(\Omega)$, normalized by demanding vanishing average in Ω , such that

$$\int_{\Omega} [\nu \nabla \mathbf{u}(\mathbf{x}) : \nabla \mathbf{v}(\mathbf{x}) - \mathbf{u}(\mathbf{x}) \nabla \mathbf{v}(\mathbf{x}) \mathbf{u}(\mathbf{x}) - u(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x})] \, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle$$

where the right-hand side stands for the duality pairing

$$H^{-1}(\Omega; \mathbb{R}^N) - H_0^1(\Omega; \mathbb{R}^N).$$

We propose to look at this problem incorporating the incompressibility constraint into the space as part of feasibility as is usually done; there is also the alternative to treat the same situation incorporating a penalization on the divergence into the functional, instead of including it into the class of admissible fields (see [9], and also [8]). The pressure field rises as the multiplier corresponding to the divergence-free constraint.

Let

$$\mathbb{D} \equiv H_{0,div}^1(\Omega; \mathbb{R}^N) = \{\mathbf{u} \in H_0^1(\Omega; \mathbb{R}^N) : \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega\}.$$

For every such \mathbf{u} , we determine its residual \mathbf{U} , in a unique way, as the solution of the restricted variational problem

$$\text{Minimize in } \mathbf{V} \in \mathbb{D} : \int_{\Omega} \left[\frac{1}{2} |\nabla \mathbf{V}|^2 + (\nu \nabla \mathbf{u} - \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{V} \right] \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{V} \rangle.$$

The pressure ν comes as the corresponding multiplier for the divergence-free constraint, in such a way that the unique minimizer \mathbf{U} is determined through the variational equality

$$\int_{\Omega} (\nabla \mathbf{U} : \nabla \mathbf{V} + (\nu \nabla \mathbf{u} - \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{V} + \nu \operatorname{div} \mathbf{V}) \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{V} \rangle$$

valid for every test field $\mathbf{V} \in H_0^1(\Omega; \mathbb{R}^N)$. This is the weak form of the optimality condition associated with the previous variational problem

$$-\Delta \mathbf{U} - \nu \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \mathbf{f} + \nabla \nu = \mathbf{0} \text{ in } \Omega, \quad (6.2)$$

for $\mathbf{U} \in \mathbb{D}$. The multiplier $\nu \in L_0^2(\Omega)$ (square-integrable fields with a vanishing average) is the pressure. We define

$$E(\mathbf{u}) : \mathbb{D} \rightarrow \mathbb{R}^+, \quad E(\mathbf{u}) = \frac{1}{2} \int_{\Omega} |\nabla \mathbf{U}(\mathbf{x})|^2 \, d\mathbf{x}. \quad (6.3)$$

For $\mathbf{u}, \mathbf{v} \in \mathbb{D}$, let $\mathbf{U}, \mathbf{V} \in \mathbb{D}$ be their respective residuals, and u, v their respective pressure fields. Put

$$\mathbf{w} = \mathbf{u} - \mathbf{v} \in \mathbb{D}, \quad \mathbf{W} = \mathbf{U} - \mathbf{V} \in \mathbb{D}, \quad w = u - v \in L^2(\Omega).$$

It is elementary to find, by subtraction of the corresponding system (6.2) for \mathbf{u} and \mathbf{v} , that

$$-\Delta \mathbf{W} - \nu \Delta \mathbf{w} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u} - \mathbf{v} \otimes \mathbf{v}) + \nabla w = \mathbf{0} \text{ in } \Omega. \quad (6.4)$$

It is the presence of the non-linear term $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$, so fundamental to the Navier–Stokes system, what makes the situation different compared to a linear setting.

We write

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{u} - \mathbf{v} \otimes \mathbf{v}) = \operatorname{div}(\mathbf{w} \otimes \mathbf{u}) + \operatorname{div}(\mathbf{v} \otimes \mathbf{w}),$$

and bear in mind the well-know fact

$$\int_{\Omega} (\mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{u} + \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{v}) \, d\mathbf{x} = \int_{\Omega} \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{u} \, d\mathbf{x} = 0 \quad (6.5)$$

for every $\mathbf{u}, \mathbf{v} \in \mathbb{D}$. If we use \mathbf{w} as a test function in (6.4), we find that

$$\int_{\Omega} [\nabla \mathbf{W} : \nabla \mathbf{w} + \nu |\nabla \mathbf{w}|^2 - (\mathbf{w} \otimes \mathbf{u}) : \nabla \mathbf{w} - (\mathbf{v} \otimes \mathbf{w}) : \nabla \mathbf{w}] \, d\mathbf{x} = 0. \quad (6.6)$$

Note that the integral involving w vanishes because \mathbf{w} is divergence-free. By (6.5),

$$\begin{aligned} \int_{\Omega} (\mathbf{w} \otimes \mathbf{u}) : \nabla \mathbf{w} \, d\mathbf{x} &= 0, \\ \int_{\Omega} (\mathbf{v} \otimes \mathbf{w}) : \nabla \mathbf{w} \, d\mathbf{x} &= - \int_{\Omega} (\mathbf{w} \otimes \mathbf{w}) : \nabla \mathbf{v} \, d\mathbf{x}. \end{aligned}$$

Identity (6.6) becomes

$$\int_{\Omega} [\nabla \mathbf{W} : \nabla \mathbf{w} + \nu |\nabla \mathbf{w}|^2 + (\mathbf{w} \otimes \mathbf{w}) : \nabla \mathbf{v}] \, d\mathbf{x} = 0. \quad (6.7)$$

We can use \mathbf{v} as a test function in the corresponding system (6.2) for \mathbf{v} to have

$$\int_{\Omega} (\nabla \mathbf{V} : \nabla \mathbf{v} + \nu |\nabla \mathbf{v}|^2 - \mathbf{f} \cdot \mathbf{v}) \, d\mathbf{x} = 0.$$

Again we have utilized that fields in \mathbb{D} are divergence-free, and the second identity in (6.5). This last identity implies, in an elementary way, that

$$\nu \|\mathbf{v}\|_{H_0^1(\Omega; \mathbb{R}^N)} \leq \|\mathbf{f}\|_{H^{-1}(\Omega; \mathbb{R}^N)} + \sqrt{2E(\mathbf{v})}. \quad (6.8)$$

Recall that

$$2E(\mathbf{v}) = \|\mathbf{V}\|_{H_0^1(\Omega; \mathbb{R}^N)}^2.$$

We now have all the suitable elements to exploit (6.7). If $C = C(n)$ is the constant of the Sobolev embedding of $H^1(\Omega)$ into $L^4(\Omega)$ for $N \leq 4$, then (6.7) leads to

$$\nu \|\mathbf{w}\|^2 \leq \|\mathbf{W}\| \|\mathbf{w}\| + C^2 \|\mathbf{w}\|^2 \|\mathbf{v}\|$$

where all norms here are in $H_0^1(\Omega; \mathbb{R}^N)$. On the other hand, if we replace the size of \mathbf{v} by the estimate (6.8), we are carried to

$$\nu \|\mathbf{w}\|^2 \leq \|\mathbf{W}\| \|\mathbf{w}\| + \frac{C^2}{\nu} \|\mathbf{w}\|^2 \left(\|\mathbf{f}\|_{H^{-1}(\Omega; \mathbb{R}^N)} + \sqrt{2E(\mathbf{v})} \right),$$

or

$$\left(\nu - \frac{C^2}{\nu} \left(\|\mathbf{f}\| + \sqrt{2E(\mathbf{v})} \right) \right) \|\mathbf{w}\|^2 \leq \|\mathbf{W}\| \|\mathbf{w}\|.$$

Since

$$\|\mathbf{W}\| = \|\mathbf{U} - \mathbf{V}\| \leq \|\mathbf{U}\| + \|\mathbf{V}\|,$$

we would have

$$\left(\nu - \frac{C^2}{\nu} \left(\|\mathbf{f}\| + \sqrt{2E(\mathbf{v})} \right) \right)^2 \|\mathbf{u} - \mathbf{v}\|^2 \leq 4(E(\mathbf{u}) + E(\mathbf{v})).$$

The form of this inequality leads us to the following interesting generalization of Definition 1.5.

Definition 6.1. A non-negative, C^1 -functional

$$E(\mathbf{u}) : \mathbf{H} \rightarrow \mathbb{R}^+$$

defined over a Hilbert space \mathbf{H} is called an error functional if there is some positive constant c (including $c = +\infty$) such that:

- (1) behavior as $E' \rightarrow \mathbf{0}$:

$$\lim_{E'(\mathbf{u}) \rightarrow \mathbf{0}} E(\mathbf{u}) = 0$$

over bounded subsets of \mathbf{H} ; and

- (2) enhanced coercivity: there is a positive constant C (that might depend on c), such that for every pair \mathbf{u}, \mathbf{v} belonging to the sub-level set $\{E \leq c\}$, we have

$$\|\mathbf{u} - \mathbf{v}\|^2 \leq C(E(\mathbf{u}) + E(\mathbf{v})).$$

It is interesting to note that the sub-level sets $\{E \leq d\}$ for $d < c$, for a functional E verifying Definition 6.1, cannot maintain several connected components.

Because our basic result Proposition 1.6 is concerned with zeros of E , it is still valid under Definition 6.1.

Proposition 6.2. *Let $E : \mathbf{H} \rightarrow \mathbb{R}^+$ be an error functional according to Definition 6.1. Then there is a unique minimizer $\mathbf{u}_\infty \in \mathbf{H}$ such that $E(\mathbf{u}_\infty) = 0$, and*

$$\|\mathbf{u} - \mathbf{u}_\infty\|^2 \leq CE(\mathbf{u}),$$

for every $\mathbf{u} \in \mathbf{H}$ provided $E(\mathbf{u})$ is sufficiently small ($E(\mathbf{u}) \leq c$, the constant in Definition 6.1).

The calculations that motivated this generalization yield the following.

Proposition 6.3. *Let $N \leq 4$, and $\Omega \subset \mathbb{R}^N$, a bounded, Lipschitz, connected domain. If $\nu > 0$ and $\mathbf{f} \in H^{-1}(\Omega; \mathbb{R}^N)$ are such that the quotient $\|\mathbf{f}\|/\nu^2$ is sufficiently small, then the functional E in (6.3) complies with Definition 6.1.*

We now turn to examining the interconnection between E and E' . To this end, we gather here (6.2) and (6.3)

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2} \int_{\Omega} |\nabla \mathbf{U}(\mathbf{x})|^2 \, d\mathbf{x}, \\ -\Delta \mathbf{U} - \nu \Delta \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \mathbf{f} + \nabla u &= \mathbf{0} \text{ in } \Omega, \end{aligned}$$

for $\mathbf{u}, \mathbf{U} \in \mathbb{D}$ and $u \in L_0^2(\Omega)$. If we replace

$$\mathbf{u} \mapsto \mathbf{u} + \epsilon \mathbf{v}, \quad \mathbf{U} \mapsto \mathbf{U} + \epsilon \mathbf{V}, \quad u \mapsto u + \epsilon v,$$

to first-order in ϵ , we would have

$$\begin{aligned} E(\mathbf{u} + \epsilon \mathbf{v}) &= \frac{1}{2} \int_{\Omega} |\nabla \mathbf{U} + \epsilon \nabla \mathbf{V}|^2 \, d\mathbf{x}, \\ -\Delta(\mathbf{U} + \epsilon \mathbf{V}) - \nu \Delta(\mathbf{u} + \epsilon \mathbf{v}) + \operatorname{div}((\mathbf{u} + \epsilon \mathbf{v}) \otimes (\mathbf{u} + \epsilon \mathbf{v})) - \mathbf{f} + \nabla(u + \epsilon v) &= \mathbf{0}. \end{aligned}$$

By differentiating with respect to ϵ , and setting $\epsilon = 0$, we arrive at

$$\begin{aligned} \langle E'(\mathbf{u}), \mathbf{v} \rangle &= \int_{\Omega} \nabla \mathbf{U} \cdot \nabla \mathbf{V} \, d\mathbf{x}, \\ -\Delta \mathbf{V} - \nu \Delta \mathbf{v} + \operatorname{div}(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}) + \nabla v &= \mathbf{0}. \end{aligned}$$

If we use \mathbf{U} as a test function in this last system, we realize that

$$\langle E'(\mathbf{u}), \mathbf{v} \rangle = \int_{\Omega} [-\nu \nabla \mathbf{v} \cdot \nabla \mathbf{U} + \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{U} + \mathbf{v} \otimes \mathbf{u} : \nabla \mathbf{U}] \, d\mathbf{x}$$

for every $\mathbf{v} \in \mathbb{D}$. If we set $\mathbf{w} = E'(\mathbf{u}) \in \mathbb{D}$, then

$$\int_{\Omega} [\nabla \mathbf{w} \cdot \nabla \mathbf{v} + \nu \nabla \mathbf{U} \cdot \nabla \mathbf{v} + (\nabla \mathbf{U} \mathbf{u} + \mathbf{u} \nabla \mathbf{U}) \cdot \mathbf{v}] \, dx,$$

for every $\mathbf{v} \in \mathbb{D}$. In particular, if we plug $\mathbf{v} = \mathbf{U}$ in, bearing in mind that due to (6.5) the last two terms drop out, we are left with

$$\nu \|\nabla \mathbf{U}\|^2 = -\langle \nabla \mathbf{w}, \nabla \mathbf{U} \rangle, \quad \text{or} \quad \nu \|\nabla \mathbf{U}\| \leq \|\nabla \mathbf{w}\|.$$

If the term on the right-hand side, which is $\|E'(\mathbf{u})\|$, tends to zero, so does the one on the left-hand side, which is $\nu \sqrt{2E(\mathbf{u})}$. This shows the second basic property of an error functional. As a result, Proposition 6.2 can be applied.

Theorem 6.4. *If $\Omega \subset \mathbb{R}^N$, $N \leq 4$, is a bounded, Lipschitz, connected domain, and $\nu > 0$ and $\mathbf{f} \in H^{-1}(\Omega; \mathbb{R}^N)$ in the steady Navier–Stokes system (6.1) are such that the quotient $\|\mathbf{f}\|/\nu^2$ is sufficiently small, then there is a unique weak solution \mathbf{u} in \mathbb{D} , and*

$$\|\mathbf{u} - \mathbf{v}\|_{H_0^1(\Omega; \mathbb{R}^N)}^2 \leq CE(\mathbf{v})$$

provided $E(\mathbf{v})$ is sufficiently small.

References

- [1] Kendall Atkinson and Weimin Han. *Theoretical numerical analysis. A functional analysis framework*, volume 39 of *Texts in Applied Mathematics*. Springer, 3rd edition, 2001.
- [2] Pavel B. Bochev and Max D. Gunzburger. *Least-Squares Finite Element Methods*, volume 166 of *Applied Mathematical Sciences*. Springer, 2009.
- [3] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, 2011.
- [4] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.
- [5] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, 2001.
- [6] Roland Glowinski. *Numerical Methods for Nonlinear Variational Problems*. Springer Series in Computational Physics. Springer, 1984.
- [7] Alain Haraux. *Nonlinear vibrations and the wave equation*. SpringerBriefs in Mathematics. Springer, 2018.

- [8] Jérôme Lemoine and Arnaud Münch. A fully space-time least-squares method for the unsteady Navier–Stokes system. *J. Math. Fluid Mech.*, 23(4): article no. 102 (38 pages), 2021.
- [9] Jérôme Lemoine, Arnaud Münch, and Pablo Pedregal. Analysis of continuous H^{-1} -least-squares methods for the steady Navier–Stokes system. *Appl. Math. Optim.*, 83(1):461–488, 2021.
- [10] Arnaud Münch and Pablo Pedregal. About least-squares type approach to address direct and controllability problems. In *Evolution equations: long time behavior and control*, volume 439 of *London Mathematical Society Lecture Note Series*, pages 118–136. Cambridge University Press, 2018.
- [11] Pablo Pedregal. A variational approach to dynamical systems and its numerical simulation. *Numer. Funct. Anal. Optim.*, 31(7-9):814–830, 2010.
- [12] Paul H. Rabinowitz. The mountain pass theorem: theme and variations. In *Differential equations (Sao Paulo, 1981)*, volume 957 of *Lecture Notes in Mathematics*, pages 237–271. Springer, 1982.
- [13] James C. Robinson, José L. Rodrigo, and Witold Sadowski. *The three-dimensional Navier–Stokes equations. Classical theory.*, volume 157 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2016.
- [14] Guido Schneider and Hannes Uecker. *Nonlinear PDEs. A dynamical systems approach*, volume 182 of *Graduate Studies in Mathematics*. American Mathematical Society, 2017.
- [15] Roger Temam. *Navier–Stokes equations. Theory and numerical analysis*. American Mathematical Society, 2001. Reprint of the 1984 edition.
- [16] Lloyd N. Trefethen, Ásgeir Birkisson, and Tobin A. Driscoll. *Exploring ODEs*, volume 157 of *Other Titles in Applied Mathematics*. Society for Industrial and Applied Mathematics, 2018.
- [17] A. Valli. *A compact course on linear PDEs*, volume 126 of *Unitext*. Springer, 2020.

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